An Infinite-horizon Model
of Nonmonotone Utility Smoothing*

Katsutoshi Wakai†

November 2011

Abstract

We provide an infinite-horizon version of nonmonotone intertemporal preferences suggested by Wakai (2011) that captures a strong dislike of volatility involved in a utility sequence. In addition, as an intermediate result, we derive a nonmonotone version of Gilboa and Schmeidler’s (1989) multiple priors utility defined on an arbitrary state space.

Keywords: discount factor, nonmonotone preferences, utility smoothing

JEL Classification Numbers: D90

---

*Financial support from the Japanese government in the form of research grant, Grant-in-Aid for Scientific Research (C) (20530146), is gratefully acknowledged.

†Graduate School of Economics, Kyoto University, Yoshida-hommachi, Sakyo-ku, Kyoto 606-8501, Japan; E-mail: wakai@econ.kyoto-u.ac.jp
1 Introduction

Nonmonotone preferences have been applied to intertemporal choice, and axiomatic foundations have been provided for some of those models. For example, Rozen (2009) axiomatizes, in an infinite-horizon setting, a popular version of a habit-formation model that violates monotonicity. By relaxing the monotonicity assumption of Wakai’s (2008) model of utility smoothing, Wakai (2011) axiomatizes, in a finite-horizon setting, a model of discount factors that captures a strong desire to smooth a utility distribution implied by a sequence of outcomes. The purpose of this paper is to provide axiomatically an infinite-horizon version of Wakai (2011).

A key result is the derivation of conditions that specify the relation among discount factors. Such a condition is noninnocuous and crucial for an infinite-horizon setting, because the model expresses nonmonotone preferences by assigning a negative discount factor to a certain time period while keeping the summation of all discount factors constant. Another important result is that we derive a nonmonotone version of Gilboa and Schmeidler’s (1989) multiple priors utility defined on an arbitrary state space. This result is obtained as an intermediate result because our axiomatization of the main representation adopts a method developed by Gilboa and Schmeidler (1989) so that we first need to derive a nonmonotone version of their model.

The remainder of the paper presents sets of axioms and representations. We also provide proofs that show the equivalence between these axioms and representations.


2 Representation

We consider a discrete-time model in an infinite-horizon setting, where time varies over \( T = \{0, 1, \ldots\} = \mathbb{N} \). For each \( t \in T \), a continuation of time is denoted by \( T_t = \{t, t+1, \ldots\} \). Let \( X \) be a nonempty consumption set, which can be a convex set in \( \mathbb{R} \) or a finite set of objects. Denote by \( M \) the set of all probability distributions over \( X \) with finite support. Following Wakai (2008), we adapt an intertemporal version of the Anscombe–Aumann (1963) framework by considering acts of the form \( l : T \to M \), where \( l = (l_0, l_1, \ldots) \). A constant act \( \bar{p} \) is a function \( l : T \to M \) such that \( l_t = p \in M \) for all \( t \in T \); \( \bar{p} \) is also identified with \( p \in M \). The collection of all acts is denoted by \( \mathcal{L} \), and the collection of all constant acts is denoted by \( \mathcal{C} \). A mixture operation \( + \) on \( \mathcal{L} \) is defined by a period-wise application of a probability mixture operation \( + \) on \( M \); that is, \( (\alpha l + (1 - \alpha)l')_t = \alpha l_t + (1 - \alpha)l'_t \) for \( \alpha \in [0, 1] \) and \( l, l' \in \mathcal{L} \).

The primitive of the model is the collection of complete and transitive preference orderings \( \{\succeq_t\} \equiv \{\succeq_t \mid t \in T\} \), whose elements are defined on \( \mathcal{L} \). For each \( t \in T \), assume that a conditional ordering \( \succeq_t \) is Archimedean, nondegenerate, and independent of the payoff history \( (l_0, l_1, \ldots, l_{t-1}) \). A conditional ordering on \( M \), also denoted by \( \succeq_t \), is defined as follows: for \( p, q \in M \), \( p \succeq_t q \) if and only if \( \bar{p} \succeq_t \bar{q} \).

First, we maintain the following three axioms assumed in Wakai (2011).

**Axiom 1 – Utility equivalence (UE):** For each \( t \in T \) and for all \( l, l' \in \mathcal{L} \), if \( l_\tau \simeq_t l'_\tau \) for all \( \tau \in T \), then \( l \simeq_t l' \).

**Axiom 2 – Constant independence (CI):** For each \( t \in T \) and for all

---

1In Shalev (1997), Axiom 1 is called substitutability. In Gilboa and Schmeidler (1989), Axiom 2 is called certainty independence, and Axiom 3 is called uncertainty aversion.
$l, l' \in L$, $p \in C$, and for all $\alpha \in (0, 1)$, $l \succ_t l'$ if and only if $\alpha l + (1 - \alpha)p \succ_t \alpha l' + (1 - \alpha)p$.

**Axiom 3 – Time-variability aversion (TVA):** For each $t \in T$ and for all $l, l' \in L$, and for all $\alpha \in (0, 1)$, $l \succeq_t l'$ implies $\alpha l + (1 - \alpha)l' \succeq_t l$.

Axioms 2 and 3, together with monotonicity, characterize Gilboa and Schmeidler’s (1989) multiple priors model, where $T$ is interpreted as a state space. We deviate from Gilboa and Schmeidler (1989) by replacing monotonicity with UE.

For an infinite-horizon model, we must impose an additional axiom that ensures that preferences are well defined.

**Axiom 4 – Boundedness (BD):** For each $t \in T$:

(i) There exist $\overline{p}^*, \underline{p}^* \in C$ such that $\overline{p}^* \succeq_t \overline{p} \succeq_t \underline{p}^*$ for all $\overline{p} \in C$.

(ii) There exist $\overline{q}^*, \underline{q}^*, \overline{r}^*, \underline{r}^* \in C$ such that for all $l \in L$, if $\overline{q}^* \succeq_t l \succeq_t \underline{q}^*$ for all $t \in T$, then $\overline{r}^* \succeq_t l \succeq_t \underline{r}^*$.

Gilboa and Schmeidler (1989) show that to extend the finite-horizon version of multiple priors utility to an infinite state space, utility sequences must be bounded. Condition (i) of BD ensures this property. However, without assuming monotonicity, Condition (i) cannot preclude the possibility that a representation is unbounded on $L$. Condition (ii) of BD, together with the Archimedean property, guarantees that the representation is bounded on $L$.

To state the first result, we need to introduce additional notations. Let $\mathbb{B}_t$ be a collection of all bounded and real-valued functions on $T_t$. We adopt $\mathbb{B}_t$ with the sup norm and use the weak$^*$ topology on $\mathbb{B}_t^*$; that is, the dual space of

\footnote{This condition is adapted from Epstein and Schneider (2003).}
$\mathbb{B}_t$. Let $\Sigma_t$ be the $\sigma$-algebra that consists of all subsets of $T_t$. By construction, each element in $\mathbb{B}_t$ is $\Sigma_t$-measurable. Let $1_t \in \mathbb{B}_t$ be the constant function of one. For an affine function $U_t : M \to \mathbb{R}$, define a function $U_t \circ (.)^t : \mathcal{L} \to \mathbb{B}_t$ by $U_t \circ (l)_\tau^t = U_t(l_\tau)$ for each $\tau \in T_t$; $U_t \circ (l)^t$ denotes a utility sequence defined on $T_t$ induced by act $l$. Furthermore, for a set $D_t \subset \mathbb{B}_t^*$, a norm of $D_t$, denoted by $\| D_t \|$, is defined by

$$\| D_t \| \equiv \sup_{b^t \in D_t} \sup_{t} \{ |b^t(a)| : a \in \mathbb{B}_t \text{ satisfying } |a_\tau| \leq 1 \text{ for each } \tau \in T_t \}.$$ 

We say that $D_t$ is bounded if $\| D_t \| < \infty$.

The following is the first result of this paper. The proposition holds when we change $T$ to an arbitrary set with an infinite number of elements. Hence, it can be regarded as an nonmonotone version of Gilboa and Schmeidler’s (1989) multiple priors utility.

**Proposition 1:** The following statements are equivalent:

(i) $\{ \succeq_t \}$ satisfy A1 to A4.

(ii) For each $t \in T$, there exists an affine function $U_t : M \to \mathbb{R}$ and a non-empty, closed, convex, compact, and bounded set $D_t \subset \mathbb{B}_t^*$, each element of which, $b^t \in D_t$, is a finitely additive weighting function $b^t$ on $\Sigma_t$, satisfying $\int_{T_t} 1_t db^t = 1$ such that $\succeq_t$ is represented by $V_t(\cdot)$, where

$$V_t(l) \equiv \min_{b^t \in D_t} \int_{T_t} U_t \circ (l)^t db^t.$$

Moreover, $D_t$ is unique, and $U_t$ is unique up to a positive affine transformation. Furthermore, $\max_{p \in M} U_t(p)$ and $\min_{p \in M} U_t(p)$ exist.

**Proof.** Necessity of the axioms is routine. For sufficiency, CI implies that there exists an affine function $U_t : M \to \mathbb{R}$ that represents $\succeq_t$ on $M$, which is unique up to a positive affine transformation. Furthermore, it follows
from Condition (i) of BD that $\max_{p \in M} U_t(p)$ and $\min_{p \in M} U_t(p)$ exist. Let $U_t \equiv \{U_t \circ (l)'|l \in \mathcal{L}\}$. Then representation (1) follows from Lemmas A.2–A.4 of Wakai (2011), with the following modifications.

(a) We require that $U_t \subset \mathbb{B}_t$. (i) of BD ensures this condition.

(b) $\mathbb{D}_t$ is defined as a collection of continuous linear functionals that separate a pair of convex subspaces of $\mathbb{B}_t$, each of which satisfies a certain property. To apply the separation theorem of an infinite state space (Royden, 1988, p. 240, Theorem 20), we require (ii) of BD, which guarantees that one of the subspaces has an interior point.

(c) $\mathbb{D}_t$ must be bounded. (ii) of BD induces this condition.

Note that without BD, a finite-horizon model satisfies the two conditions, $U_t \subset \mathbb{B}_t$ and the boundedness of $\mathbb{D}_t$, which are sufficient to prove (1).

The next objective is to derive an infinite-horizon version of Wakai (2011), for which we maintain the following two axioms assumed in Wakai (2011).

**Axiom 5 – Downside monotonicity (DMT):** For each $t \in T$ and for any $p, p' \in M$ satisfying $p \succ_t p'$, and for any nonempty proper subset $A$ of $T_t$, if $l_\tau = p$ for all $\tau \in A$ and $l_{\tau'} = p'$ for all $\tau' \in T_t \setminus A$, then $\bar{p} \succ_t l$.

**Axiom 6 – Dynamic consistency (DC):** For each $t \in T$ and for all $l, l' \in \mathcal{L}$, if $l_\tau = l'_\tau$ for all $\tau \leq t$ and if $l \succeq_{t+1} l'$, then $l \succeq_t l'$; the latter ranking is strict if the former is strict.

First, by imposing DMT, we exclude the possibility that $U_t(c_t)$ is always assigned with a negative weight because the reduction in instantaneous utility from a constant utility sequence always makes the DM worse off. Second, DC specifies the relationship between conditional orderings.
In an infinite-horizon model, if \( \succeq_t \) is nonmonotone, the Archimedean property is too weak to guarantee the consistency in the tail-end behavior. Hence, we must impose an additional condition in a form of continuity. For this purpose, we say that \( \{l^n\}_{n \geq 1} \) converges to \( l \) if for each \( \tau \in T \), there exists \( n_\tau \) such that for all \( n \geq n_\tau \), \( l^n_\tau = l_\tau \). We then restrict the tail-end behavior by imposing the following axiom.

**Axiom 7 - Tail-end continuity (TC):** For each \( t \in T \), suppose that \( \{l^n\}_{n \geq 1} \) converges to \( l \). Then for \( l' \in \mathcal{L} \), (i) if \( l \succ_t l' \), there exists \( N_t \) such that \( l^n \succ_t l' \) for all \( n \geq N_t \) and (ii) if \( l' \succ_t l \), there exists \( N_t \) such that \( l' \succ_t l^n \) for all \( n \geq N_t \).

For comparison, without assuming monotonicity, Shalev (1997) extends Schmeidler’s (1989) Choquet expected utility to an intertemporal setting and axiomatizes a model that captures a loss-aversion type of behavior documented by Kahneman and Tversky (1979).\(^3\) To derive an infinite-horizon version, Shalev (1997) assumes TC only for a sequence \( \{l^n\}_{n \geq 1} \) that monotonically converges to \( l \) because he models only ex ante preferences, in which an infinite-horizon version of the representation is defined as a limit of the finite-horizon version.\(^4\) By contrast, we separately model each conditional ordering \( \succeq_t \) defined on an infinite-horizon setting. Hence, we need a stronger notion of continuity to establish the consistency among conditional orderings.

The following is the infinite-horizon version of Wakai (2011).

**Proposition 2:** The following statements are equivalent:

\(^3\)In a finite-horizon setting, Waegenaere and Wakker (2001) generalize Shalev’s (1997) model.

(i) \{\succeq_t\} satisfy A1 to A7.

(ii) There exists an affine function \( U: M \to \mathbb{R} \), a collection of nonempty sets \( \{\mathbb{D}_t\}_{t \in T} \), each element of which, \( \mathbb{D}_t \subseteq \mathbb{R}^*_+ \), is a closed, convex, compact, and bounded set of discount functions \( b^t: T \to \mathbb{R} \) satisfying \( \sum_{t=1}^{\infty} b^t_t = 1 \), and a collection of sets of discount factors \( \{[\delta_{\tau+1}, \overline{\delta}_{\tau+1}]\}_{\tau \in T} \) satisfying (a) \( 0 < \delta_{\tau+1} < 1 \) and \( \overline{\delta}_{\tau+1} \leq \delta_{\tau+1} \) for all \( \tau \in T \) and (b) for each \( t \in T \),
\[
\left( \prod_{\tau=t+1}^{t+s} \delta_{\tau} \right) \| \mathbb{D}_{t+s} \| \text{ converges to zero as } s \text{ goes to infinity such that for each } t \in T, \succeq_t \text{ is represented by } V_t(\cdot), \text{ where}
\]
\[
V_t(l) \equiv \min_{b^t \in \mathbb{D}_t} \left\{ \sum_{\tau=t}^{\infty} b^t_\tau U(l_\tau) \right\} 
= \min_{\delta_{t+1} \in [\delta_{t+1}, \overline{\delta}_{t+1}]} \left[ (1 - \delta_{t+1})U(l_t) + \delta_{t+1}V_{t+1}(l) \right].
\]

Moreover, \( \delta_t, \delta_t, \text{ and } \mathbb{D}_t \text{ are unique; } U \text{ is unique up to a positive affine transformation; and } \max_{p \in M} U(p) \text{ and } \min_{p \in M} U(p) \text{ exist.} \) Furthermore, \( \mathbb{D}_t \) is recursively constructed from \( \{[\delta_{\tau+1}, \overline{\delta}_{\tau+1}]\}_{\tau \in T} \), as shown in the proof below.

In an infinite-horizon model, \( \{\mathbb{D}_t\}_{t \in T} \) and \( \{[\delta_{\tau+1}, \overline{\delta}_{\tau+1}]\}_{\tau \in T} \) must additionally satisfy condition (b). In fact, because \( \{\mathbb{D}_t\}_{t \in T} \) is constructed from \( \{[\delta_{\tau+1}, \overline{\delta}_{\tau+1}]\}_{\tau \in T} \), condition (b) is a condition on \( \{[\delta_{\tau+1}, \overline{\delta}_{\tau+1}]\}_{\tau \in T} \).

**Proof.** Necessity of the axioms is routine. For sufficiency, (3) follows from Lemmas A.5 and A.6 of Wakai (2011), where time-independent \( U: M \to \mathbb{R} \) replaces all \( U_t \), and for all \( t \in T \), \( [\delta_{t+1}, \overline{\delta}_{t+1}] \) satisfies \( 0 < \delta_{t+1} < 1 \) and \( \overline{\delta}_{t+1} \leq \delta_{t+1} \). Without a loss of generality, assume that \( U(M) \supset [-1, 1] \). The rest of the proof is divided into three steps.

(Step1) For each \( t \in T \),
\[
\left( \prod_{\tau=t+1}^{t+s} \delta_{\tau} \right) \| \mathbb{D}_{t+s} \| \text{ converges to zero as } s \text{ goes to infinity.}
\]
Let \( p[0] \in M \) satisfy \( U(p[0]) = 0 \). Let \( \mathcal{B}([-1, 1]) \) be the collection of all \( l \in \mathcal{L} \) such that \( U(l_{\tau}) \in [-1, 1] \) for all \( \tau \in \mathcal{T} \). Suppose, by way of contradiction, that there exists a sequence \( \{l^n\}_{n \geq 1} \subset \mathcal{B}([-1, 1]) \) such that for some \( \varepsilon > 0 \) and for all \( s > 0 \), there exists \( k(s) \geq s \) that satisfies

\[
\left( \prod_{\tau=t+1}^{t+k(s)} \delta_{\tau} \right) |V_{t+k}(l^{k(s)})| > \varepsilon.
\]

Define a new sequence \( \{l^m\}_{m \geq 1} \) by

\[
l^m_{\tau} = \begin{cases} 
    l^k_{\tau} & \text{if } V_{t+k}(l^k_{\tau}) < 0 \\
    q \in M \text{ satisfying } U(q) = -U(l^k_{\tau}) & \text{if } V_{t+k}(l^k_{\tau}) > 0
  \end{cases}
\]

for \( \tau \geq t + k(m) \).

By construction, \( \{l^m\}_{m \geq 1} \) converges to \( \overline{p}[0] \). In particular, if \( V_{t+k}(l^k_{\tau}) > 0 \), it follows from Proposition 1 that

\[
V_{t+m}(l^m_{\tau}) \leq -V_{t+k}(l^k_{\tau}) < 0.
\]

This shows that for all \( m > 0 \),

\[
\left( \prod_{\tau=t+1}^{t+k(m)} \delta_{\tau} \right) V_{t+m}(l^m_{\tau}) < -\varepsilon < 0. \quad (4)
\]

Moreover, the repeated application of (3) implies that for all \( m > 0 \),

\[
V_t(l^m_{\tau}) = \left( \prod_{\tau=t+1}^{t+k(m)} \delta_{\tau} \right) V_{t+m}(l^m_{\tau}). \quad (5)
\]

Thus, (4) and (5) imply that \( V_t(l^m_{\tau}) \) does not converge to \( V_t(\overline{p}[0]) = 0 \), which contradicts TC.

(Step 2) For each \( t \in \mathcal{T} \),

\[
\sum_{s=0}^{\infty} \left( \prod_{\tau=t+1}^{t+s} \delta_{\tau} \right) |1 - \delta_{t+s+1}| \leq \|D_t\|, \text{ where } \prod_{\tau=t+1}^{t} \delta_{\tau} \equiv 1 \text{ and for each } \tau \geq t + 1, \delta_{\tau} \in [\overline{\delta}_{\tau}, \overline{\delta}_{\tau}].
\]
Suppose, by way of contradiction, that there exists a sequence \( \{\delta_{\tau}\}_{t+1}^{\infty} \) such that
\[
\sum_{s=0}^{\infty} \left( \prod_{\tau=t+1}^{t+s} \delta_{\tau} \right) |1 - \delta_{t+s+1}| \geq \|D_t\| + \varepsilon
\]
for some \( \varepsilon > 0 \), where for each \( \tau \geq t+1, \delta_{\tau} \in [\delta_{\tau}, \overline{\delta_{\tau}}] \). Let \( m > 0 \) be a positive integer such that for all \( m' \geq m \),
\[
\sum_{s=0}^{m'-1} \left( \prod_{\tau=t+1}^{t+s} \delta_{\tau} \right) |1 - \delta_{t+s+1}| > \|D_t\| + \frac{2}{3} \varepsilon. \tag{6}
\]
Let \( p^{-1}, p^{[1]} \in M \) satisfy \( U(p^{-1}) = -1 \) and \( U(p^{[1]}) = 1 \), respectively. Consider \( l \in B([-1, 1]) \) such that \( l_{\tau} = p^{-1} \) if \( (1 - \delta_{t+k''+1}) \geq 0 \) and \( l_{\tau} = p^{[1]} \) if \( (1 - \delta_{t+k''+1}) < 0 \). By (Step 1), there exists a positive integer \( n \) such that for all \( n' \geq n \) and for all \( b^{t+n'} \in D_{t+n'} \),
\[
\left| \left( \prod_{\tau=t+1}^{t+n'} \delta_{\tau} \right) \int_{T_{t+n}} U \circ (l)^{t+n'} db^{t+n'} \right| < \frac{\varepsilon}{3}. \tag{7}
\]
Let \( k \equiv \max[m, n] \). By repeatedly applying (3), (6), and (7),
\[
V_t(l) = \min_{b^t \in D_t} \int_{T_t} U_t \circ (l)^t db^t \\
\leq -\sum_{s=0}^{k-1} \left( \prod_{\tau=t+1}^{t+s} \delta_{\tau} \right) |1 - \delta_{t+s+1}| + \left( \prod_{\tau=t}^{t+k} \delta_{\tau} \right) \min_{b^{t+k} \in D_{t+k}} \int_{T_{t+k}} U \circ (l)^{t+k} db^{t+k} \\
< -\|D_t\| - \frac{\varepsilon}{3}.
\]
This leads to \( |V_t(l)| > \|D_t\| \), which contradicts the definition of \( \|D_t\| \). \( \square \)

(Step 3) For each \( t \in T, D_t = \Delta_t \), where \( \Delta_t \subset B_t^* \) is constructed from \( \{[\delta_{\tau+1}, \overline{\delta_{\tau+1}}]\}_{\tau \in T} \) as shown below.

Given (3), define a sequence \( \{\gamma_{\tau}\}_{t}^{\infty} \) by
\[
\gamma_t \equiv 1 \text{ and } \gamma_{t+\tau} \equiv \delta_{t+\tau} \gamma_{t+\tau-1} \text{ for } \tau > 0,
\]
\[
\]
where $\delta_\tau \in [\delta_\tau, \bar{\delta}_\tau]$ for each $\tau \geq t + 1$. It follows from (Step 2) that for all $l \in \mathbb{B}([-1,1])$,

$$\left| \sum_{s=0}^{\infty} \left( \prod_{\tau=t+1}^{t+s} \delta_\tau \right) (1 - \delta_{t+s+1}) U(l_{t+s}) \right| \leq \|D_i\|, \quad (8)$$

where the left-hand side converges. Given (8), construct $b^t \in \mathbb{B}_t^*$ from $\{\gamma_\tau\}_{t}^\infty$ as follows: for each $t \in T$,

$$b^t(a) \equiv \sum_{\tau=t}^{\infty} b^t_\tau a_\tau \text{ for each } a \in \mathbb{B}_t, \text{ where } b^t_\tau \equiv \gamma_\tau - \gamma_{\tau+1} \text{ for } s \geq t. \quad (9)$$

Define a nonempty set $\Delta_t \subset \mathbb{B}_t^*$ by

$$\Delta_t \equiv \{ b^t \in \mathbb{B}_t^* | b^t \text{ satisfies (9) for some admissible } \{\gamma_\tau\}_{t}^\infty \}. \quad (9)$$

Each element of $\Delta_t$ is a discrete and countably additive weighting function on $\Sigma_t$. Furthermore, Proposition 1 and (Step 1) imply that $\lim_{s \to \infty} \prod_{\tau=t+1}^{t+s} \delta_\tau \to 1$, which leads to $\sum_{\tau=t}^{\infty} b_\tau = \sum_{\tau=t}^{\infty} (\gamma_\tau - \gamma_{\tau+1}) = \gamma_t = 1$. Moreover, $\Delta_t$ is closed, and, by Lemma A.6 of Wakai (2008), it is also convex. In addition, by (8), $\Delta_t$ is bounded. Then it follows from Theorem 3′ of Lax (2002, p. 121) that $\Delta_t$ is compact. Finally, given (8), the conditions of $\Delta_t$ stated above are sufficient to adapt the proof of Lemma A.7 of Wakai (2008), which shows that $D_i = \Delta_t$. ■

**References**


