A Note on Recursive Multiple-Priors*

Katsutoshi Wakai†

The State University of New York at Buffalo

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Abstract

One of the assumptions of recursive multiple-priors utility (Epstein and Schneider, J. Econ. Theory, 113(1) (2003), 1-31) is that conditional preferences at every node satisfy an intertemporal version of the multiple-priors axioms (Gilboa and Schmeidler, J. Math. Econ., 18(2) (1989) 141-153). This note weakens this assumption: Given that conditional preferences depend only on the continuations of acts and satisfy dynamic consistency and completeness, the recursive multiple-priors utility is derived if we assume the multiple-priors axioms for \textit{ex-ante} preferences alone.

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\footnotesize{†}Department of Economics, The State University of New York at Buffalo, 427 Fronczak Hall, Buffalo, NY, 14260; e-mail: kwakai@buffalo.edu; Tel: (716) 645-2121 ext. 430; Fax: (716) 645-2127.
1 Introduction

Ambiguity is a notion distinct from risk. Gilboa and Schmeidler’s [2] multiple-priors utility is one of the models that captures this notion. Epstein and Schneider [1] extend this model to a dynamic setting under a fixed information structure and show that for a multiple-priors decision maker to behave consistently over time, each set of priors at every time-event node needs to be rectangular, and each set is updated by Bayes’ Rule applied prior by prior (i.e., recursive multiple-priors utility). Their analysis is based on the following two crucial assumptions: (i) conditional preferences depend only on the continuations of acts and satisfy dynamic consistency and (ii) conditional preferences at every node satisfy an intertemporal version of the multiple-priors axioms.

Suppose that Assumption (i) is satisfied. This paper asks the following question: “Do we need to assume the multiple-priors properties at ‘every’ node a priori as ‘axioms’ in order to derive the recursive multiple-priors utility?” We find that it is enough to assume that (i) the multiple-priors axioms are satisfied among ex-ante preferences at the initial period and (ii) conditional preferences at every non-initial node are complete. Hence, any distinctive characteristics of the multiple-priors utility need to be assumed only among ex-ante preferences; dynamic consistency and other axioms imply that the same characteristics are satisfied at all other nodes.

2 Model and Results

We follow the set-up of Epstein and Schneider [1]. Time is discrete and its horizon is finite, where \( T ≡ \{0, ..., T\} \). The state space is \( \Omega \). The information structure is fixed and represented by the filtration \( \{\mathcal{F}_t\}^T_0 \), where \( \mathcal{F}_0 \) is trivial and for each time \( t \), \( \mathcal{F}_t \) corresponds to a finite partition; \( \mathcal{F}_t(\omega) \)
is an event in the partition $\mathcal{F}_t$ containing $\omega$. Then, if $\omega$ is the true state, the decision maker at time $t$ knows that $\mathcal{F}_t(\omega)$ is true.

At each $t$ and $\omega$, the outcome space is defined as a space of simple lotteries over $C \subseteq \mathbb{R}_+$, denoted by $\Delta_s(C)$. Let $\mathcal{H}$ be a set of all $\Delta_s(C)$-valued adapted processes $h = (h_t)$ (called acts), where each $h_t : \Omega \rightarrow \Delta_s(C)$ is $\mathcal{F}_t$-measurable. We use the notation $h_t(\omega)$ to refer to a specific lottery assigned at $(t, \omega)$. Also, define a lottery act $l = (l_t)$ as a $\Delta_s(C)$-valued adapted process such that for each $t$, $l_t(\omega) = l_t(\omega')$ for all $\omega, \omega' \in \Omega$; lottery acts involve risk but not ambiguity. In addition, $(l_{t-\tau} - (\tau+k), q, q')$ denotes the lottery act $l'$ in which (i) $l'_t = l_t$ for $t \neq \tau, \tau + k$ and (ii) for all $\omega \in \Omega$, $l'_\tau(\omega) = q$ and $l'_{\tau+k}(\omega) = q'$, where $q, q' \in \Delta_s(C)$. Furthermore, for any given act $h$ and any given $\omega$, $h(\omega)$ is the lottery act $l$ such that $l_\tau(\omega') = h_\tau(\omega)$ in every period $\tau$ and in every state $\omega'$. We also define the following operation: $(\alpha h + (1 - \alpha)h')_t(\omega) = \alpha h_t(\omega) + (1 - \alpha)h'_t(\omega)$ for any $\alpha \in [0, 1]$.

The decision maker has a preference ordering on $\mathcal{H}$ at any $(t, \omega) \in T \times \Omega$, denoted by $\succeq_{t,\omega}$.

Epstein and Schneider [1] impose the following axioms on the collection of preference orderings $\{\succeq_{t,\omega} \equiv \{\succeq_{t,\omega} | (t, \omega) \in T \times \Omega\}$:

**Axiom 1 (Conditional Preference - CP).** For each $t$ and $\omega$: (i) $\succeq_{t,\omega} = \succeq_{t,\omega'}$ if $\mathcal{F}_t(\omega) = \mathcal{F}_t(\omega')$. (ii) If $h'_\tau(\omega') = h_\tau(\omega')$ for all $\tau \geq t$ and $\omega' \in \mathcal{F}_t(\omega)$, then $h' \sim_{t,\omega} h$.

**Axiom 2 (Multiple-Priors - MP).** For each $t$ and $\omega$: (i) $\succeq_{t,\omega}$ is complete and transitive. (ii) For all $h, h'$ and lottery acts $l$, and for all $\alpha \in (0, 1)$, $h' \succ_{t,\omega} h$ if and only if $\alpha h' + (1 - \alpha)l \succ_{t,\omega} \alpha h + (1 - \alpha)l$. (iii) If $h'' \succ_{t,\omega} h' \succ_{t,\omega} h$, then $\alpha h'' + (1 - \alpha)h \succ_{t,\omega} h' \succ_{t,\omega} \beta h'' + (1 - \beta)h$ for some $\alpha$ and $\beta \in (0, 1)$. (iv) If $h'(\omega') \succeq_{t,\omega} h(\omega')$ for all $\omega'$, then $h' \succeq_{t,\omega} h$. (v) If $h' \sim_{t,\omega} h$, then $\alpha h' + (1 - \alpha)h \succeq_{t,\omega} h$ for all $\alpha \in [0, 1]$.
\( \alpha \in (0,1) \). (vi) \( h' \succ_{t,\omega} h \) for some \( h' \) and \( h \).

**Axiom 3 (Risk Preference - RP).** For any lottery act \( l \), for all \( p,p',q,q' \in \Delta_s(C) \), if

\[
(l_{-\tau,-(\tau+1)},p,p') \succeq_{t,\omega} (l_{-\tau,-(\tau+1)},q,q')
\]

for some \( \omega,t \) and \( \tau \geq t \), then it is true for every \( \omega,t \) and \( \tau \geq t \).

For the following axioms, we define the usual notion of nullity: for any \( \tau > t \), the event \( A \) in \( \mathcal{F}_\tau \) is \( \succeq_{t,\omega} \)-null if \( h'_\tau(\cdot) = h_\tau(\cdot) \) on \( A^c \) implies that \( h' \sim_{t,\omega} h \).

**Axiom 4 (Dynamic Consistency - DC).** For every \( t \) and \( \omega \) and for all acts \( h' \) and \( h \), if \( h'_\tau(\cdot) = h_\tau(\cdot) \) for all \( \tau \leq t \) and if \( h' \succeq_{t+1,\omega} h \) for all \( \omega' \), then \( h' \succeq_{t,\omega} h \); the latter ranking is strict if the former ranking is strict at every \( \omega' \) in a \( \succeq_{t,\omega} \)-nonnull event.

**Axiom 5 (Full Support - FS).** Each nonempty event in \( \bigcup_{t=0}^T \mathcal{F}_t \) is \( \succeq_0 \)-nonnull.

For each \( t \), \( \Delta(\Omega,\mathcal{F}_t) \) is the set of probability measures on the \( \sigma \)-algebra \( \mathcal{F}_t \). We say that a measure \( p \in \Delta(\Omega,\mathcal{F}_T) \) has full support if \( p(A) > 0 \) for every non-empty \( A \in \mathcal{F}_T \). In addition, for any measure \( p \in \Delta(\Omega,\mathcal{F}_T) \), \( p_t(\omega) = p(\cdot|\mathcal{F}_t)(\omega) \) is its \( \mathcal{F}_t \)-conditional and \( p_t^{+1} \) is the restriction of \( p_t \) to \( \mathcal{F}_{t+1} \). Define the following two sets, \( \mathcal{P}_t(\omega) \) and \( \mathcal{P}_t^{+1}(\omega) \), as

\[
\mathcal{P}_t(\omega) = \{p_t(\omega)|p \in \mathcal{P}\} \text{ and } \mathcal{P}_t^{+1}(\omega) = \{p_t^{+1}(\omega)|p \in \mathcal{P}\},
\]

where \( \mathcal{P} \) is a subset of \( \Delta(\Omega,\mathcal{F}_T) \). Then, \( \mathcal{P} \) is said to be \( \{\mathcal{F}_t\} \)-rectangular if for all \( t \) and \( \omega \), \( \mathcal{P}_t(\omega) = \int \mathcal{P}_t^{+1}(\omega) d\mathcal{P}_t^{+1}(\omega) \). Furthermore, \( u : \Delta_s(C) \to \mathbb{R} \) is called mixture linear provided that \( u(\alpha p + (1-\alpha)q) = \alpha u(p) + (1-\alpha)u(q) \) for all \( p,q \in \Delta_s(C) \) and \( \alpha \in [0,1] \).
Given the above terminology, Epstein and Schneider [1] show that these axioms are necessary and sufficient for the representation of preferences to satisfy the recursive multiple-priors utility.

**Theorem (Epstein and Schneider [1]):** \{\succeq_{t,\omega}\} satisfies CP, MP, RP, DC, and FS if and only if there exist \( \mathcal{P} \subset \Delta(\Omega, \mathcal{F}_T) \), closed, convex, and \( \{\mathcal{F}_t\}\)-rectangular, with all measures in \( \mathcal{P} \) having full support, \( \beta > 0 \), and a mixture linear and nonconstant \( u : \Delta_s(C) \to \mathbb{R} \) such that: for every \( t \) and \( \omega \), \( \succeq_{t,\omega} \) is represented by \( V_t(.,\omega) \), where

\[
V_t(h,\omega) = \min_{m \in \mathcal{P}_t(\omega)} \int \Sigma_{\tau \geq t} \beta^{\tau-t} u(h_\tau) dm. \tag{1}
\]

Moreover, \( \beta \) and \( \mathcal{P} \) are unique, and \( u \) is unique up to a positive linear transformation.

We now state the result.\(^1\) By “\( \{\succeq_{0,\omega}\} \) satisfies MP and RP,” we mean that the statement of MP and RP holds only for \( t = 0 \).

**Proposition:** Suppose that \( \{\succeq_{t,\omega}\} \) satisfies CP, DC, and FS. Then, \( \{\succeq_{t,\omega}\} \) satisfies MP and RP if and only if (i) \( \{\succeq_{0,\omega}\} \) satisfies MP and RP and (ii) \( \{\succeq_{t,\omega}\} \) is complete.

DC relates \( \succeq_{t+1,\omega'} \) to \( \succeq_{t,\omega} \). Then, DC does not rule out the situation under which the decision maker can rank \( f \) and \( g \) ex-ante but cannot rank them ex-post. For example (an extreme case), assume that under a two-period model, the decision maker cannot rank any pair of acts at time 1. Then, DC vacuously holds between \( \{\succeq_{1,\omega}\} \) and \( \{\succeq_{0,\omega}\} \). Clearly, consistency is defined only among acts that can be ranked ex-post. Hence, without completeness, DC is ineffective.

\(^1\)To extend our result to an infinite horizon setting, we need to assume Axioms 7 and 8 of Epstein and Schneider [1] at time 0.
Proof. If \( \succeq_{t, \omega} \) satisfies MP and RP, then (i) and (ii) obviously hold. For the converse, we first define the following axiom to narrow the properties implied by RP:

**Axiom 3b (RP').** At \((t, \omega)\), for any lottery act \(l\), for all \(p, p', q, q' \in \Delta s(C)\), if

\[
(l_{-\tau, -(\tau+1)}, p, p') \succeq_{t, \omega} (l_{-\tau, -(\tau+1)}, q, q')
\]

for some \(\tau \geq t\), then it is true for every \(\tau \geq t\).

Next, we define two properties implied by CP, DC, FS, and completeness of \(\succeq_{t, \omega}\). For any \(h \in \mathcal{H}\), let \(\mathcal{H}(t, \omega; h)\) be a collection of all acts \(h' \in \mathcal{H}\) such that \(h'_\tau(\cdot) = h_\tau(\cdot)\) for all \(\tau \leq t\) and \(h'_\tau(\omega') = h_\tau(\omega')\) for all \(\tau > t\) and for all \(\omega' \in \Omega \setminus \mathcal{F}_{t+1}(\omega)\). This is a collection of acts that are different from \(h\) only in terms of lotteries assigned on \(\{t + 1, ..., T\} \times \mathcal{F}_{t+1}(\omega)\).

(A) By CP, at each \((t + 1, \omega') \in \{t + 1\} \times \mathcal{F}_{t+1}(\omega)\), for any \(f, g \in \mathcal{H}(t, \omega; h)\),

\[
f \succeq_{t+1, \omega'} g \text{ if and only if } f' \succeq_{t+1, \omega'} g',
\]

where \(f', g' \in \mathcal{H}\) such that \(f'_\tau(\omega') = f_\tau(\omega')\) and \(g'_\tau(\omega') = g_\tau(\omega')\) for all \((\tau, \omega') \in \{t + 1, ..., T\} \times \mathcal{F}_{t+1}(\omega)\). Hence, to prove that \(f' \succeq_{t+1, \omega'} g'\) for \(f', g' \in \mathcal{H}\), we only need to show that \(f \succeq_{t+1, \omega'} g\) for \(f, g \in \mathcal{H}(t, \omega; h)\) at some \(h \in \mathcal{H}\) (i.e., only continuations of acts matter).

(B) By CP, DC, FS, and completeness of \(\succeq_{t, \omega}\), (i) \(\mathcal{F}_{t+1}(\omega)\) is \(\succeq_{t, \omega}\)-nonnull and (ii) at each \((t + 1, \omega') \in \{t + 1\} \times \mathcal{F}_{t+1}(\omega)\), for any \(f, g \in \mathcal{H}(t, \omega; h)\), \(f \succeq_{t+1, \omega'} g\) if and only if \(f \succeq_{t, \omega} g\).

We prove MP and RP' by induction. By “\(\succeq_{t, \omega}\) satisfies MP,” we mean that the statement of MP holds only for a fixed \(t\) and \(\omega\).
By assumption, at any \((0, \omega) \in \{0\} \times \Omega, \succeq_{0,\omega}\) satisfies MP and RP'. Suppose that \(\succeq_{t,\omega}\) satisfies MP and RP' at every \((t, \omega) \in \{0, \ldots, t\} \times \Omega\) for some \(t < T\).

(Step 1: MP) Take some \(h \in \mathcal{H}\). By the induction hypothesis, \(\succeq_{t,\omega}\) satisfies (i) (transitivity), (iii) (continuity), and (iv) (monotonicity) on \(\mathcal{H}(t, \omega; h)\); by (A) and (B), this implies that at each \((t + 1, \omega') \in \{t + 1\} \times \mathcal{F}_{t+1}(\omega), \succeq_{t+1,\omega'}\) satisfies (i), (iii), and (iv).

In terms of (ii) (certainty independence), by the induction hypothesis, for any \(f, g \in \mathcal{H}(t, \omega; h)\) and lottery acts \(l\), and for all \(\alpha \in (0, 1)\), \(f \succeq_{t,\omega} g\) if and only if \(\alpha f + (1 - \alpha)l \succeq_{t,\omega} \alpha g + (1 - \alpha)l\). Then \(\alpha f + (1 - \alpha)l\) and \(\alpha g + (1 - \alpha)l\) belong to \(\mathcal{H}(t, \omega; h')\), where \(h' = \alpha h + (1 - \alpha)l\). By (A) and (B), this shows that at each \((t + 1, \omega') \in \{t + 1\} \times \mathcal{F}_{t+1}(\omega), \succeq_{t+1,\omega'}\) satisfies (ii).

As for (v) (uncertainty aversion), by the induction hypothesis, for all \(f, g \in \mathcal{H}(t, \omega; h)\) and for all \(\alpha \in (0, 1)\), if \(f \simeq_{t,\omega} g\), then \(\alpha f + (1 - \alpha)g \succeq_{t,\omega} f\). Since \(f, g\), and \(\alpha f + (1 - \alpha)g\) are in \(\mathcal{H}(t, \omega; h)\), (A) and (B) imply that at each \((t + 1, \omega') \in \{t + 1\} \times \mathcal{F}_{t+1}(\omega), \succeq_{t+1,\omega'}\) satisfies (v).

To prove (vi) (non-degeneracy), since \(\mathcal{F}_{t+1}(\omega)\) is \(\succeq_{t,\omega}\)-nonnull, there exist \(h \in \mathcal{H}\) and \(f, g \in \mathcal{H}(t, \omega; h)\) such that \(f \succeq_{t,\omega} g\). Then, by (A) and (B), this proves that at each \((t + 1, \omega') \in \{t + 1\} \times \mathcal{F}_{t+1}(\omega), \succeq_{t+1,\omega'}\) satisfies (vi).

Hence, all of the above imply that under the induction hypothesis, at all \((t + 1, \omega') \in \{t + 1\} \times \Omega, \succeq_{t+1,\omega'}\) satisfies MP.

(Step 2: RP') Assume that \(t + 1 < T - 1\) (otherwise, RP' is trivially true (when \(t + 1 = T - 1\)) or vacuously true (when \(t + 1 = T\)). Take some \(\tau \geq t + 1\). Let \(h\) be a lottery act defined by

\[h \equiv (l_{-\tau, -(\tau+1)}, p, p'), \quad \text{where } p, p' \in \Delta_s(C)\]
Let \( g \in H(t, \omega; h) \) such that \( g_\tau(\omega') = q \) and \( g_{\tau+1}(\omega') = q' \) for all \( \omega' \in \mathcal{F}_{t+1}(\omega) \). Then, by (A) and (B), at each \( (t+1, \omega') \in \{t+1\} \times \mathcal{F}_{t+1}(\omega) \), \( (l_{-\tau,-(\tau+1)}, p, p') \succeq_{t+1,\omega'} (l_{-\tau,-(\tau+1)}, q, q') \) if and only if \( h \succeq_{t,\omega} g \). By (i) of (B) (\( \succeq_{t,\omega} \)-nonnullity of \( \mathcal{F}_{t+1}(\omega) \)) and the induction hypothesis (in particular, \( \text{RP}' \) and (iv) (monotonicity) of MP), \( h \succeq_{t,\omega} g \) if and only if \( (l_{-\tau,-(\tau+1)}, p, p') \succeq_{t,\omega} (l_{-\tau,-(\tau+1)}, q, q') \). This proves that at each \( (t+1, \omega') \in \{t+1\} \times \mathcal{F}_{t+1}(\omega) \), \( \succeq_{t+1,\omega'} \) satisfies \( \text{RP}' \). Hence, under the induction hypothesis, at all \( (t+1, \omega') \in \{t+1\} \times \Omega \), \( \succeq_{t+1,\omega'} \) satisfies \( \text{RP}' \).

By the induction principle, (Step 1) and (Step 2) show that \( \{\succeq_{t,\omega}\} \) satisfies MP and \( \text{RP}' \). Finally, \( \text{RP} \) remains to be shown. Given that \( \{\succeq_{t,\omega}\} \) satisfies MP and \( \text{RP}' \), by Lemma A.1 of Epstein and Schneider [1], \( \succeq_{t,\omega} \) is represented by an intertemporal version of multiple-priors utility (1) without rectangularity, where \( \beta_{t,\omega} \) and \( u_{t,\omega} \) replace \( \beta \) and \( u \) in (1). Take one of \( \omega \in \Omega \), and define \( \beta \equiv \beta_{0,\omega} \) and \( u \equiv u_{0,\omega} \). By CP, let \( \beta_{0,\omega'} \equiv \beta \) and \( u_{0,\omega'} \equiv u \) for all \( \omega' \in \Omega \). By DC, for any \( t > 0 \), we can also define \( \beta_{t,\omega} \equiv \beta \) and \( u_{t,\omega} \equiv u \) for all \( \omega \in \Omega \). This proves that \( \{\succeq_{t,\omega}\} \) satisfies \( \text{RP} \).

References
