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Equilibrium Alpha in Asset Pricing in an Ambiguity-averse Economy*

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Abstract

We derive an equilibrium asset pricing relation analogous to the capital asset pricing model (CAPM) for investors whose preferences follow the robust mean-variance preferences introduced by Maccheroni, Marinacci, and Ruffino (2013). Our model defines a precise relation between the value of *alpha* from the market regression and ambiguity: alpha is positive if the asset has greater exposure to market ambiguity than market risk, and vice versa.

Keywords: Ambiguity aversion, asset pricing, capital asset pricing model (CAPM), robust mean-variance preferences

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1 Introduction

As an extension of mean-variance preferences, Maccheroni, Marinacci, and Ruffino (2013) introduce *robust* mean-variance preferences that take into account an aversion to ambiguity in the spirit of Knight (1921). Formally, they consider the situation where an investor is uncertain about the objective probability that governs the occurrence of states, and characterize the newly derived preferences as an approximation for the smooth model of decision making under ambiguity as proposed by Klibanoff, Marinacci, and Mukerji (2005).

To study the scope of robust mean-variance preferences, Maccheroni et al. (2013) consider a static portfolio choice problem with a risk-free asset, a purely risky asset, and an ambiguous asset, which is a risky asset that has exposure to ambiguity. In particular, they find the following relations based on *alpha*, that is, the intercept obtained from the regression of the excess return of the ambiguous asset on the excess return of the purely risky asset: (i) a positive alpha leads to a long position in the ambiguous asset, and (ii) a negative alpha leads to a short position in the ambiguous asset. Maccheroni et al. (2013) also show that (iii) an increase in ambiguity aversion decreases the optimal exposure to the ambiguous asset.

Given the above results, Maccheroni et al. (2013) argue that alpha well captures the extra return arising from the ambiguous nature of investment. However, this argument lacks a proper theoretical foundation because if we apply the standard mean-variance preferences to the three-asset example in Maccheroni et al. (2013), we can easily show that the same conclusion holds for risk-averse investors, where risk aversion replaces ambiguity aversion in the third aforementioned property. Thus, a positive alpha simply shows that the ambiguous asset has a statistically better risk/return trade-off than the purely risky asset.

The purpose of this paper is to consider one of the economies that relates ambiguity to alpha when derived from a particular regression specified via an equilibrium concept. Formally, we consider an economy of investors with robust mean-variance preferences, where we impose the restriction on preference parameters under which investors hold the same composition of risky assets. The equilibrium of this economy leads to the augmented version of the capital asset pricing model (CAPM) known as *robust* CAPM,

$$E_{\overline{Q}}[r_k - r_f] = \phi_k E_{\overline{Q}}[r_M - r_f], \tag{1}$$

where $\overline{Q}$ is the probability induced by the robust mean-variance preferences, r_k is the return on the risky asset, r_f is the risk-free rate, and r_M is the return on the *market portfolio*, that is, the portfolio consisting of all risky assets in the economy.[1,2] In particular, ϕ_k is shown to be a convex combination of a risk beta β_k^R and an ambiguity beta β_k^A, each of which defines the compensation scheme for bearing market risk and market ambiguity, respectively.

In terms of the alpha, when $\overline{Q}$ is equal to the objective probability P, regression analysis of (1) derives the intercept α_k that satisfies

$$\alpha_k = \left(\phi_k - \beta_k^R\right) E_P[r_M - r_f]. \tag{2}$$

Assuming that $E_P[r_M - r_f] > 0$, (2) implies that the intercept α_k captures an ambiguity premium in *excess* of the risk premium. That is, if the ambiguity beta β_k^A is larger than the risk beta β_k^R, the intercept α_k is positive, and vice versa. Hence, the regression alpha relates to the ambiguity of the investment.

[1] A risky asset has nondeterministic payoffs with exposure to both risk and ambiguity.

[2] Alternatively, the market portfolio is the portfolio consisting of all assets in the economy, including the risk-free asset. We employ standard finance terminology in this regard.

There is a large body of literature concerning asset pricing in an economy comprising ambiguity-averse investors. Of these, our work is most related to Chen and Epstein (2002), who in a continuous-time setting study the economy of the representative agent whose preferences exhibit ambiguity aversion in the way suggested in Gilboa and Schmeidler (1989). They show that the resulting relation among asset returns follows a CAPM-type equation, where the coefficient for the market excess return is a combination of the risk beta and the ambiguity beta, the latter of which captures the relation between asset return and ambiguity aversion. Our result is a discrete-time analogue of that in Chen and Epstein (2002), with the exception that the ambiguity beta describes the relation between asset return and market ambiguity.

The remainder of the paper derives the robust CAPM and the equilibrium alpha. All proofs are presented in the appendix.

2 The Robust CAPM and Equilibrium Alpha

We consider the static portfolio choice problem in Maccheroni et al. (2013). Let $(\Omega, \mathcal{F}, P)$ be a probability space. There is a finite number $K+1$ of assets whose payoffs are defined over Ω, where the first K assets are risky assets. The $(K+1)$th asset is the risk-free asset that pays a gross return of r_f in every state of nature. Let r_k be the random variable that describes the gross return of the kth asset for $k = 1, ...K$, and let $\mathbf{r}$ be a K-dimensional vector of returns on the K risky assets. There are H investors in this economy, where each investor is endowed with positive initial wealth W_h.

We model ambiguity as follows. Each investor believes that there is a finite number L of possible regimes in this economy, and is unsure which regime is faced. Each regime l specifies the probability of state realization, denoted by

Q_l, and investor h's belief of possible regimes is expressed by the investor's subjective prior μ_h defined over L regimes. We then define the probability measure $\overline{Q}_h$ on $\mathcal{F}$, called the reduction of μ_h on Ω, by

$$\overline{Q}_h(A) = \mu_h(1)Q_1(A) + ... + \mu_h(L)Q_L(A) \text{ for all } A \in \mathcal{F}.$$

We assume that the support of μ_h is identical for all investors, that is, the reduction of μ_h must be based on the same set of probability measures $\{Q_1, ..., Q_L\}$. Let $E_{\overline{Q}_h}[r^k]$ be the expected return of the kth asset under $\overline{Q}_h$, and let $\Sigma_{\overline{Q}_h}$ be the $K \times K$-dimensional variance-covariance matrix of the K risky assets under $\overline{Q}_h$. Furthermore, for each $k = 1, ..., K$, we denote by $E[r^k]$ the random variable defined over L regimes, where the value of $E[r^k]$ at regime l is the conditional expected return of the kth asset under Q_l. The $K \times K$-dimensional variance-covariance matrix of $E[r^k]$ under μ_h is denoted by Σ_{μ_h}.

We assume that investors can trade assets without transaction costs and short sell and borrow without restriction. They also invest all of their wealth into $(K+1)$ assets, where the K-dimensional vector of portfolio weights on the K risky assets is denoted by $\mathbf{w}$. Thus, the return of the portfolio $r_{\mathbf{w}}$ is given by

$$r_{\mathbf{w}} = r_f + \mathbf{w}\cdot(\mathbf{r} - r_f\mathbf{1}),$$

where $\mathbf{1}$ is the K-dimensional unit vector. Let $\sigma^2_{\overline{Q}_h}(r_{\mathbf{w}})$ be the variance of $r_{\mathbf{w}}$ under $\overline{Q}_h$, and let $\sigma^2_{\mu_h}(E[r_{\mathbf{w}}])$ be the variance of $E[r_{\mathbf{w}}]$ under μ_h.

Investors decide their own portfolio weight $\mathbf{w}$ based on the following *robust mean-variance preferences* as introduced by Maccheroni et al. (2013):

$$E_{\overline{Q}_h}[r_{\mathbf{w}}] - \frac{\lambda_h}{2}\sigma^2_{\overline{Q}_h}(r_{\mathbf{w}}) - \frac{\theta_h}{2}\sigma^2_{\mu_h}(E[r_{\mathbf{w}}]),$$

which is equivalent to

$$r_f + \mathbf{w}\cdot E_{\overline{Q}_h}[\mathbf{r} - r_f\mathbf{1}] - \frac{\lambda_h}{2}\mathbf{w}^T\Sigma_{\overline{Q}_h}\mathbf{w} - \frac{\theta_h}{2}\mathbf{w}^T\boldsymbol{\Sigma}_{\mu_\mathbf{h}}\mathbf{w}, \tag{3}$$

where $E_{\overline{Q}_h}[\mathbf{r} - r_f\mathbf{1}]$ is the K-dimensional vector of expected return for $\mathbf{r} - r_f\mathbf{1}$. We assume that both λ_h and θ_h are positive. By the Arrow–Pratt analysis, the first assumption roughly implies that investors are risk averse. Maccheroni et al. (2013) also shows that the second assumption roughly implies that investors are ambiguity averse in the way defined by Klibanoff et al. (2005).

The vector of optimal portfolio weights $\mathbf{w}^*$ is that which maximizes (3). The first-order condition implies that $\mathbf{w}^*$ satisfies

$$\left[\lambda_h \Sigma_{\overline{Q}_h} + \theta_h \Sigma_{\mu_h}\right] \mathbf{w}^{h*} = E_{\overline{Q}_h}[\mathbf{r} - r_f\mathbf{1}]. \tag{4}$$

Compared with the mean-variance preferences (that is, $\theta_h = 0$), ambiguity aversion additionally introduces the term $\theta_h \Sigma_{\mu_h}$ on the left-hand side.

We aim to derive the robust CAPM and the corresponding alpha defined in (2). For this purpose, we must impose the following conditions.

Assumption 1:

(i) $\mu_h = \mu_{h'} = \mu$ for all h, h'.

(ii) $\dfrac{\theta_h}{\lambda_h} = \dfrac{\theta_{h'}}{\lambda_{h'}} = \eta$ for all h, h'.[3]

(iii) $\Sigma_{\overline{Q}_h}$ is positive definite for some h.

(iv) $\mathbf{1}^T \left[\lambda_h \Sigma_{\overline{Q}_h} + \theta_h \Sigma_{\mu_h}\right]^{-1} E_{\overline{Q}_h}[\mathbf{r} - r_f\mathbf{1}] > 0$ for some h.

Condition (i) states that all investors have homogeneous beliefs about the regimes. Condition (ii) allows investors to have different λ_h and θ_h, but the ratio of θ_h to λ_h, that is, the relative ambiguity aversion, must be identical for all investors. Condition (iii) guarantees that $\left[\lambda_h \Sigma_{\overline{Q}_h} + \theta_h \Sigma_{\mu_h}\right]$ is symmetric

[3] Hara and Honda (2013) investigate the detailed conditions for fund separation in an economy where investors' preferences follow a version of that in Klibanoff et al. (2005). They derive the condition for two-fund separation analogous to Assumption 1-(i) and -(ii).

and positive definite. Condition (iv) corresponds to that usually assumed in mean-variance analysis in finance.

Let $\overline{Q}$ be the reduction of μ on Ω, and let $\Sigma_{\overline{Q}}$ and Σ_μ be the corresponding variance-covariance matrices of returns and expected returns, respectively. By Assumption 1-(iii), investor h's optimal portfolio becomes $\mathbf{w}^{h*} = \lambda_h^{-1}\mathbf{w}^*$, where $\mathbf{w}^*$ is defined by

$$\mathbf{w}^* \equiv \left[\Sigma_{\overline{Q}} + \eta\Sigma_\mu\right]^{-1} E_{\overline{Q}}[\mathbf{r} - r_f\mathbf{1}]. \tag{5}$$

Thus, each investor's optimal portfolio is a linear combination of the risk-free asset and the portfolio of risky assets, each of whose weights is defined by

$$\mathbf{w}_k^M \equiv \frac{\mathbf{w}_k^*}{(\mathbf{1}\cdot\mathbf{w}^*)},$$

where the denominator is positive given Assumption 1-(iv).

At the equilibrium, the demand for assets is equal to the supply of assets. Thus, $\mathbf{w}^M = (\mathbf{w}_1^M, ..., \mathbf{w}_K^M)$ becomes the portfolio weights in the market portfolio. Let r_M be the return of the market portfolio defined by $r_M \equiv \mathbf{w}^M \cdot \mathbf{r}$. We also denote by $cov_{\overline{Q}}(r_k, r_M)$ the covariance between r_k and r_M under $\overline{Q}$, and by $cov_\mu(E[r_k], E[r_M])$ the covariance between $E[r_k]$ and $E[r_M]$ under μ. Then, it follows from standard mean-variance analysis that the equilibrium relationship between the return of the risky asset k and the return of the market portfolio satisfies the robust CAPM

$$E_{\overline{Q}}[r_k - r_f] = \phi_k E_{\overline{Q}}[r_M - r_f], \tag{6}$$

where

$$\phi_k \equiv \frac{cov_{\overline{Q}}(r_k, r_M) + \eta cov_\mu(E[r_k], E[r_M])}{\sigma^2_{\overline{Q}}(r_M) + \eta\sigma^2_\mu(E[r_M])}. \tag{7}$$

The CAPM is a special case of (6), where all investors are ambiguity neutral by having $\eta = 0$.[4]

[4] By treating $\Sigma_{\overline{Q}} + \eta\Sigma_\mu$ as a variance-covariance matrix, the derivation of the CAPM

Ambiguity has two effects on (6): the determination of the portfolio weights in the market portfolio, and the determination of ϕ_k.[5] We treat $\overline{Q}$ as subjective risk as in the subjective expected utility model. This allows us to focus on the determination of ϕ_k because the market portfolio is identified as the portfolio consisting of all risky assets. For this analysis, it is informative to define the following two quantities.

$$\beta_k \equiv \frac{cov_{\overline{Q}}(r_k, r_M)}{\sigma^2_{\overline{Q}}(r_M)} \text{ and } \beta_{E[r_k]} \equiv \frac{cov_\mu(E[r_k], E[r_M])}{\sigma^2_\mu(E[r_M])}. \tag{8}$$

We refer to the former as a *risk beta* and the latter as an *ambiguity beta*. Then, the following proposition summarizes the behavior of ϕ_k.

Proposition 1: *Suppose that Assumption 1 holds. Then, for each k,*

(i) *if* $\sigma^2_\mu(E[r_M]) = 0$, $\phi_k = \beta_k$.

(ii) *if* $\sigma^2_\mu(E[r_M]) \neq 0$,

$$\phi_k = \beta_k \left(\frac{\sigma^2_{\overline{Q}}(r_M)}{\sigma^2_{\overline{Q}}(r_M) + \eta\sigma^2_\mu(E[r_M])} \right) + \beta_{E[r_k]} \left(\frac{\eta\sigma^2_\mu(E[r_M])}{\sigma^2_{\overline{Q}}(r_M) + \eta\sigma^2_\mu(E[r_M])} \right). \tag{9}$$

We omit the proof because the derivation is straightforward.[6]

The interpretation of β_k is analogous to that for the CAPM beta, that is, only the risk contributing to market volatility is priced. Thus, a positively contributed asset earns a positive excess return because it bears market risk, whereas a negatively contributed asset earns a negative excess return because

beta implies that $\phi_k \equiv (\mathbf{w}(k)^T \left[\Sigma_{\overline{Q}} + \eta\Sigma_\mu\right] \mathbf{w}^M)/((\mathbf{w}^M)^T \left[\Sigma_{\overline{Q}} + \eta\Sigma_\mu\right] \mathbf{w}^M)$, where $\mathbf{w}(k)$ is a K-dimensional vector of asset weights that assigns one for asset k and zero for all other assets.

[5] If $\Sigma_\mu = \lambda\Sigma_{\overline{Q}}$ for $\lambda > 0$, ambiguity has no impact on the determination of the portfolio weights in the market portfolio because the weights are identical to those under the CAPM.

[6] If $\sigma^2_\mu(E[r_M]) = 0$, then $cov_\mu(E[r_k], E[r_M]) = 0$.

it provides a hedge against market risk. As for the interpretation of $\beta_{E[r_k]}$, $\sigma_\mu^2(E[r_M])$ measures market ambiguity, and $cov_\mu(E[r_k], E[r_M])$ measures the contribution of asset k's expected return to market ambiguity. Thus, $\beta_{E[r_k]}$ defines the compensation scheme for ambiguity, which is analogous to that for market risk.

Proposition 1-(i) shows that $\sigma_\mu^2(E[r_M]) = 0$ corresponds to the situation where ambiguity has no impact on the determination of ϕ_k. Once the market portfolio is identified, only the risk in asset k's return is priced. Proposition 1-(ii) shows that for the situation where market ambiguity is positive, ϕ_k becomes a convex combination of the risk beta β_k and the ambiguity beta $\beta_{E[r_k]}$. The more ambiguity averse investors become (that is, the larger η is), the more the ambiguity beta dominates ϕ_k, and vice versa.

Next, to derive the regression alpha, we assume the following.

Assumption 2: $\overline{Q} = P$.

Regression analysis then leads to

$$E_P[r_k - r_f] = \alpha_k + \beta_k E_P[r_M - r_f],$$

where β_k is equal to the risk beta defined in (8) by replacing $\overline{Q}$ with P. Thus, α_k must satisfy

$$\alpha_k = (\phi_k - \beta_k)\, E_P[r_M - r_f],$$

where ϕ_k is defined in (7) by replacing $\overline{Q}$ with P.

For the main proposition, we introduce a few more notations. Let $\sigma_{Q_l}^2(r_M)$ be the conditional variance of the market return r_M at regime l, and let $cov_{Q_l}(r_k, r_M)$ be the conditional covariance between r_k and r_M at regime l. Then, we define the conditional beta $\beta_k(Q_l)$ by

$$\beta_k(Q_l) \equiv \frac{cov_{Q_l}(r_k, r_M)}{\sigma_{Q_l}^2(r_M)}.$$

We can obtain this term as the coefficient from a regression of asset k' return on the market return under regime l's probability Q_l.

We now show that in the robust CAPM, α_k captures an *excess* ambiguity premium (see Appendix for the proof).

Proposition 2: *Suppose that Assumptions 1 and 2 hold. Then, for each* k,

(i) *if* $\sigma_\mu^2(E[r_M]) = 0$, $\alpha_k = 0$.

(ii) *if* $\sigma_\mu^2(E[r_M]) \neq 0$,

$$\alpha_k = \left(\beta_{E[r_k]} - \beta_k\right)\left(\frac{\eta\sigma_\mu^2(E[r_M])}{\sigma_P^2(r_M) + \eta\sigma_\mu^2(E[r_M])}\right) E_P[r_M - r_f].$$

(iii) *if* $\sigma_\mu^2(E[r_M]) \neq 0$ *and* $\sigma_{Q_l}^2(r_M) > 0$ *for each regime* l,

$$\alpha_k = \left(\beta_k - E_\mu\left[\beta_k(Q_l)\frac{\sigma_{Q_l}^2(r_M)}{E_\mu\left[\sigma_{Q_l}^2(r_M)\right]}\right]\right) \times \left(\frac{\eta E_\mu\left[\sigma_{Q_l}^2(r_M)\right]}{(1+\eta)\sigma_P^2(r_M) - \eta E_\mu\left[\sigma_{Q_l}^2(r_M)\right]}\right) E_P[r_M - r_f].$$

Note that Assumption 1-(iii) and -(iv) as well as (5) imply that $E_P[r_M - r_f] > 0$. Also, $(1+\eta)\sigma_P^2(r_M) - \eta E_\mu\left[\sigma_{Q_l}^2(r_M)\right] = \sigma_P^2(r_M) + \eta\sigma_\mu^2(E[r_M])$.

The first result corresponds to Proposition 1-(i), where ambiguity has no impact on ϕ_k. Thus, in terms of the regression analysis, the economy with ambiguity-averse investors is observationally equivalent to that with ambiguity-neutral investors. The second result corresponds to Proposition 1-(ii), where α_k captures the effect of ambiguity. However, α_k is not equal to the ambiguity premium specified by the second term in (9). It is instead the ambiguity premium in excess of the risk premium because the regression beta captures the combined effect of risk and ambiguity for $\beta_{E[r_k]} = \beta_k$. Thus, α_k is positive if the ambiguity beta $\beta_{E[r_k]}$ is larger than the risk beta β_k, and vice versa.

The third result shows that α_k can be rewritten solely based on the conditional and unconditional betas from the regressions of $r_k - r_f$ on $r_M - r_f$: α_k is positive if the risk beta β_k is larger than the weighted expected value of conditional betas $E_\mu \left[\beta_k(Q_l) \frac{\sigma^2_{Q_l}(r_M)}{E_\mu \left[\sigma^2_{Q_l}(r_M) \right]} \right]$, and vice versa. A conditional beta $\beta_k(Q_l)$ has a greater impact when the conditional variance of the market return $\sigma^2_{Q_l}(r_M)$ is high. Intuitively, the weighted expected value of the conditional betas measure the average level of risk implied by the asset returns. If ambiguity increases the unconditional beta over this average level, it has an effect analogous to raising the risk of asset k's return. Thus, investors require a positive premium from this incremental risk.

Finally, for comparison, we consider the three-asset example in Maccheroni et al. (2013) where they assume that $\sigma^2_P(r_1) = 0$ and $\sigma^2_P(r_2) > 0$. For a risk-averse investor (that is, an ambiguity-neutral investor with $\theta_h = 0$), (4) leads to the optimal portfolio weight for the ambiguity asset ($k = 2$):

$$\mathbf{w}^*_2 = \frac{\sigma^2_P(r_1)\left(E_P[r_2 - r_f]\right) - cov_P(r_1, r_2)\left(E_P[r_1 - r_f]\right)}{\lambda_h \left(\sigma^2_P(r_1)\sigma^2_P(r_2) - cov_P(r_1, r_2)^2\right)}, \tag{10}$$

where we can show the denominator to be positive. Maccheroni et al. (2013) also obtain the following alpha from the regression of the excess return $r_2 - r_f$ on the excess return $r_1 - r_f$:

$$\alpha_2 = \frac{\sigma^2_P(r_1)\left(E_P[r_2 - r_f]\right) - cov_P(r_1, r_2)\left(E_P[r_1 - r_f]\right)}{\sigma^2_P(r_1)}. \tag{11}$$

By comparing (10) with (11), we find that properties (i) to (iii) discussed in the Introduction hold for the risk-averse investor. This confirms the importance of deriving the restriction on the alpha via an equilibrium concept.

Appendix: Proof of Proposition 2

We only provide the proof for (iii). By simple computation,

$$\sigma_P^2(r_M) = E_\mu\left[\sigma_{Q_l}^2(r_M)\right] + \sigma_\mu^2(E\left[r_M\right]) \tag{A.1}$$

and

$$cov(r_k, r_M) = E_\mu\left[cov_{Q_l}(r_k, r_M)\right] + cov_\mu(E\left[r_k\right], E\left[r_M\right]). \tag{A.2}$$

Then, it follows from (A.1) and (A.2) that

$$\begin{aligned}
\beta_{E[r_k]} &= \frac{cov_\mu(E\left[r_k\right], E\left[r_M\right])}{\sigma_\mu^2(E\left[r_M\right])} \\
&= \frac{cov(r_k, r_M) - E_\mu\left[cov_{Q_l}(r_k, r_M)\right]}{\sigma_\mu^2(E\left[r_M\right])} \\
&= \beta_k \frac{\sigma_P^2(r_M)}{\sigma_\mu^2(E\left[r_M\right])} - E_\mu\left[\beta_k(Q_l)\frac{\sigma_{Q_l}^2(r_M)}{E_\mu\left[\sigma_{Q_l}^2(r_M)\right]}\right] \frac{E_\mu\left[\sigma_{Q_l}^2(r_M)\right]}{\sigma_\mu^2(E\left[r_M\right])}.
\end{aligned}$$

By (A.1),

$$\begin{aligned}
\left(\beta_{E[r_k]} - \beta_k\right) &= \beta_k\left(\frac{\sigma_P^2(r_M) - \sigma_\mu^2(E\left[r_M\right])}{\sigma_\mu^2(E\left[r_M\right])}\right) \\
&\quad - E_\mu\left[\beta_k(Q_l)\frac{\sigma_{Q_l}^2(r_M)}{E_\mu\left[\sigma_{Q_l}^2(r_M)\right]}\right] \frac{E_\mu\left[\sigma_{Q_l}^2(r_M)\right]}{\sigma_\mu^2(E\left[r_M\right])} \\
&= \left(\beta_k - E_\mu\left[\beta_k(Q_l)\frac{\sigma_{Q_l}^2(r_M)}{E_\mu\left[\sigma_{Q_l}^2(r_M)\right]}\right]\right) \frac{E_\mu\left[\sigma_{Q_l}^2(r_M)\right]}{\sigma_\mu^2(E\left[r_M\right])}
\end{aligned}$$

This result, as well as (A.1) and (A.2), implies that

$$\begin{aligned}
\alpha_k &= \left(\beta_{E[r_k]} - \beta_k\right)\left(\frac{\eta\sigma_\mu^2(E[r_M])}{\sigma_Q^2(r_M) + \eta\sigma_\mu^2(E[r_M])}\right) E_P[r_M - r_f] \\
&= \left(\beta_k - E_\mu\left[\beta_k(Q_l)\frac{\sigma_{Q_l}^2(r_M)}{E_\mu\left[\sigma_{Q_l}^2(r_M)\right]}\right]\right) \\
&\quad \times \left(\frac{\eta E_\mu\left[\sigma_{Q_l}^2(r_M)\right]}{(1+\eta)\sigma_P^2(r_M) - \eta E_\mu\left[\sigma_{Q_l}^2(r_M)\right]}\right) E_P[r_M - r_f],
\end{aligned}$$

which is the condition stated in Proposition 2-(iii). ■

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References

1. Chen, Z., and L. Epstein (2002): "Ambiguity, Risk, and Asset Returns in Continuous Time," *Econometrica*, Vol. 70, pp. 1403–1443.

2. Gilboa, I., and D. Schmeidler (1989): "Maxmin Expected Utility with Nonunique Prior," *Journal of Mathematical Economics*, Vol. 18, pp. 141–153.

3. Hara, C., and T. Honda (2013): "Asset Demand and Ambiguity Aversion," working paper.

4. Klibanoff, P., M. Marinacci, and S. Mukerji (2005): "A Smooth Model of Decision Making under Ambiguity," *Econometrica*, Vol. 73, pp. 1849–1892.

5. Knight, F. (1921): *Risk, Uncertainty and Profit*, Houghton Mifflin, Boston.

6. Maccheroni, F., M. Marinacci, and D. Ruffino (2013): "Alpha as Ambiguity: Robust Mean-Variance Portfolio Analysis," *Econometrica*, Vol. 81, pp. 1075–1113.