Kyoto University,
Graduate School of Economics
Discussion Paper Series

Optimal Switching under Ambiguity and Its Applications in Finance

Yuki Shigeta

Discussion Paper No. E-16-005

Graduate School of Economics
Kyoto University
Yoshida-Hommachi, Sakyo-ku
Kyoto City, 606-8501, Japan

July, 2016

Optimal Switching under Ambiguity and Its Applications in Finance

Yuki Shigeta*

Graduate School of Economics, Kyoto University, Japan

July 1, 2016

Abstract

In this paper, we study optimal switching problems under ambiguity. To characterize the optimal switching under ambiguity in the finite horizon, we use multidimensional reflected backward stochastic differential equations (multidimensional RBSDEs) and show that a value function of the optimal switching under ambiguity coincides with a solutions to multidimensional RBSDEs with allowing negative switching costs. Furthermore, we naturally extend the finite horizon problem to the infinite horizon problem. In some applications, we show that ambiguity affects an optimal switching strategy with the different way to a usual switching problem without ambiguity.

Key words: Optimal Switching, Ambiguity Aversion, Reflected Backward Stochastic Differential Equation, Viscosity Solution.
AMS subject classifications: 60G40, 60H30.

1 Introduction

Optimal switching problems are widely used to describe many situations in finance and economics. For example, they are applied to natural resource extractions ([4] and [3]), reversible investments ([20]), and entry and exit decisions of firms ([8]). In plain words, the optimal switching problems are the problems that a decision maker chooses her actions from a discrete state space to maximize her profit (objective function).

In this paper, our aims are to construct optimal switching problems under ambiguity and to derive general properties of solutions to these problems. A concept of ambiguity aversion is one of prominent issues in recent finance and economics. The ambiguity aversion (also known as the Knightian uncertainty aversion or the model uncertainty aversion) is the behavior that an economic agent prefers avoiding the event whose occurrence probability is

*E-mail address: sy46744@gmail.com

unknown. [13] first provides illustrative examples of the ambiguity aversion and [14] and [27] economically axiomatize the ambiguity aversion. After these works, [5] establish a model of the ambiguity aversion in continuous time and [25] and [6] construct optimal stopping problems under ambiguity.

Using a concept of the ambiguity aversion, one can describe the properties not captured by a usual trade-off between returns and risks. Therefore, we can consider a more practical optimal switching problem. In existing literature, [16] mention that their model can be applied to optimal switching problems under ambiguity, but, these problems are relatively unexplored in existing literature. Therefore, it is worth studying optimal switching problems under ambiguity.

To deal with optimal switching problems under ambiguity, we use frameworks of backward stochastic differential equations (hereafter BSDEs). BSDEs are introduced by [2] and [22] establish a general theory of BSDEs. Many researchers (e.g., [12], [26], [5] and [6]) apply the theory of BSDEs to various problems in finance and economics. Recently, a theory of multidimensional reflected BSDEs (hereafter multidimensional RBSDEs) is developed by [16], [17] and [15] to study the optimal switching problems. This approach makes us naturally incorporate ambiguity aversion into the optimal switching problems. Therefore, multidimensional RBSDEs have an important role in this study.

In this paper, our contributions are as follows.

1. We characterize the optimal switching problems under ambiguity in both of the finite horizon and infinite horizon using multidimensional RBSDEs.

2. We show that value functions of the optimal switching problems under ambiguity are viscosity solutions to some system of partial differential equations.

3. Unlike existing literature, we do not assume non-negativity of switching costs.

We first define the optimal switching problems under ambiguity and characterize them using the theory of multidimensional RBSDEs by [16]. [16] assume non-negativity of the switching costs and this assumption has an important role in their study. However, there are optimal switching problems that definitely need negative switching costs (i.e., positive switching benefits) such as the buy low and sell high problem ([28]) and the pair-trading problem ([21]). Therefore, we do not assume the non-negativity of the switching costs, and we need to modify the proof of [16] to allow negative switching costs. In order to allow negative switching costs, we add a weak assumption of the switching costs. Since existing literature usually assumes non-negativity of switching costs (for example, [16], [17] and [15]), our results are more general than those of the existing literature in the sense of allowing negative switching costs. Furthermore, using the results of [15], we show that value functions of the optimal switching problems under ambiguity are viscosity solutions of some system of partial differential equations.

Moreover, we show that under some conditions, the value function in the finite horizon problem converges to the value function in the infinite horizon. [10] studies the infinite horizon problem using multidimensional RBSDEs under a non-negativity assumption of switching costs, but the most of existing studies mainly focus to the finite horizon problem. Therefore, our results may provide new insights in the optimal switching problems using multidimensional RBSDEs.

Finally, we give some examples of optimal switching problems under ambiguity in finance. We show that under certain conditions, the optimal switching problems under ambiguity can be interpreted as the optimal switching problems under a certain probability measure determined a priori. Therefore, the results of existing literature can be used to optimal switching problems under ambiguity. However, the problems not meeting these conditions provide more interesting results. In section 6.3, we consider the buy low and sell high problem under ambiguity, which does not satisfy these conditions. Our results indicate that effects of ambiguity in this problem can not be reproduced by a simple change of the probability measure.

The rest of this paper is organized as follows. Section 2 defines the optimal switching problems under ambiguity in the finite horizon using the concept of multiple priors introduced by [5]. Section 3 introduces multidimensional RBSDEs and proves the existence of their solutions. Section 4 verifies that the value functions of the optimal switching problems under ambiguity are characterized by the solutions to the multidimensional RBSDEs, and derives the system of partial differential equations, which the value functions satisfy. Section 5 considers the infinite horizon problem. Section 6 provides some applications of optimal switching problems under ambiguity in finance. Lengthy proofs are in Appendix.

2 Preliminaries and Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a d-dimensional Brownian motion $W = (W_t)_{t\geq 0}$. Let $T > 0$ be a finite constant time. We first consider an optimal switching problem during $[0, T]$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be an augmentation of the natural filtration generated by W.

We denote by $\alpha = (\alpha_t)_{t\geq 0}$ a control process such that

$$\alpha_t = \sum_{k\geq 0} i_k \mathbb{1}_{[\tau_k, \tau_{k+1})}(t), \tag{1}$$

where $(i_k)_{k\geq 0}$ is a regime process taking values in a discrete state space $\mathcal{I} = \{1, \dots, I\}$, $I > 0$, and $(\tau_k)_{k\geq 0}$ is a non-decreasing sequence of stopping times. $\mathbb{1}_A(x)$ is an indicator function such that for a given set A,

$$\mathbb{1}_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

We suppose that each i_k is $\mathcal{F}_{\tau_k}$-measurable. Under a control α, a decision maker chooses a regime i_k on $[\tau_k, \tau_{k+1})$ for all $k \geq 0$. For convenience, we also write a control as a sequence of pairs of regimes and stopping times: $\alpha = (\tau_k, i_k)_{k\geq 0}$.

Let $X = (X_t)_{0\leq t\leq T}$ be a d-dimensional stochastic process satisfying the following stochastic differential equation (hereafter SDE):

$$\mathrm{d}X_t = b(t, X_t, \alpha_t)\mathrm{d}t + \sigma(t, X_t, \alpha_t)\mathrm{d}W_t, \tag{2}$$

where $\alpha = (\alpha_t)_{0\leq t\leq T}$ is a control process. b and σ are measurable functions satisfying the following.

Hypothesis 1 *$b : [0, T] \times \mathbb{R}^d \times \mathcal{I} \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \times \mathcal{I} \to \mathbb{R}^{d\times d}$ satisfy the following Lipschitz condition and quadratic growth condition:*

$$\begin{aligned} \|b(t, x, i) - b(t, y, i)\| + \|\sigma(t, x, i) - \sigma(t, y, i)\| &\leq L\|x - y\|, \\ \|b(t, x, i)\|^2 + \|\sigma(t, x, i)\|^2 &\leq L^2(1 + \|x\|^2), \end{aligned}$$

for every $t \in [0, T]$, $i \in \mathcal{I}$, and $x, y \in \mathbb{R}^d$, where L is a positive constant and $\|x\|$ is the Euclid norm of $x \in \mathbb{R}^d$.

Let $L^q_t(\mathbb{R}^d)$ be a set of d-dimensional, q-th integrable (that is, an L^q norm on $(\Omega, \mathcal{F}, \mathbb{P})$ is finite), and $\mathcal{F}_t$-measurable random vectors. Let $\mathcal{T}^T_t$ be a set of stopping times taking values in $[t,T]$. Let $\widetilde{\mathcal{I}}_t$ be a set of $\mathcal{F}_t$-measurable random variables taking values in $\mathcal{I}$. We define $\widetilde{\mathcal{K}^q_T}$ and $\overline{\mathcal{K}_T}$ as follows,

$$\widetilde{\mathcal{K}^q_T} := \left\{(\nu,\eta,\iota) \mid \nu \in \mathcal{T}_0^T,\ \eta \in L^q_\nu(\mathbb{R}^d),\ \iota \in \widetilde{\mathcal{I}}_\nu\right\},$$
$$\overline{\mathcal{K}_T} := [0,T]\times\mathbb{R}^d\times\mathcal{I}.$$

By Hypothesis 1, for every $(\nu,\eta,\iota) \in \widetilde{\mathcal{K}^2_T}$ and progressively measurable control α starting from $\alpha_\nu = \iota$, there exists a unique strong solution to the SDE (2) on $[\nu,T]$ starting from $X_\nu = \eta$ and controlled by α. We denote this controlled process by $X^{\nu,\eta,\iota,\alpha} = (X_s^{\nu,\eta,\iota,\alpha})_{\nu\le s\le T}$. Furthermore, it is well known that the moments of X is upper bounded (e.g., Corollary 2.5.12 in [19] and Theorem 5.2.9 in [18]). We shortly summarize the results of the moment estimates of X.

Proposition 2 *Under Hypothesis 1, for every $q>0$, there exist constants $C_{q,X}\ge 1$ and $C_q>0$ such that*

$$\mathbb{E}\left[\max_{t\le s\le T}\|X_s^{t,x,i,\alpha}\|^q\right] \le C_{q,X}(1+\|x\|^q)e^{C_q(T-t)},$$

for all $0\le t\le T$, $x\in\mathbb{R}^d$, $i\in\mathcal{I}$ and control α. Note that $C_{q,X}$ and C_q do not depend on t,T,x,i and α. Furthermore, if a constant ρ is sufficiently large such that $\rho > C_q$, then there exists a positive constant $C^\infty_{q,X}$ such that

$$\mathbb{E}\left[\max_{s\ge t} e^{-\rho s}\left(1+\|X_s^{t,x,i,\alpha}\|^q\right)\right] \le C^\infty_{q,X}(1+\|x\|^q)e^{-(\rho-C_q)t}, \tag{3}$$

for all $0\le t$, $x\in\mathbb{R}^d$, $i\in\mathcal{I}$ and control α. Note that $C^\infty_{q,X}$ does not depend on t,x,i and α.

The proof of Proposition 2 is in appendix Appendix A. Moreover, we can easily show that the results of Proposition 2 hold in the case when the initial time is a stopping time. For every $\nu\in\mathcal{T}_0^T$, $\eta\in L^{2q}_\nu(\mathbb{R}^d)$, $i\in\mathcal{I}$ and control α, we have

$$\mathbb{E}\left[\max_{\nu\le s\le T}\|X_s^{\nu,\eta,i,\alpha}\|^q \;\Big|\; \mathcal{F}_\nu\right] \le C_{q,X}(1+\|\eta\|^q)e^{C_q(T-\nu)}.$$

We first consider an optimal switching problem without ambiguity. An objective function of the optimal switching problem without ambiguity is

$$\begin{aligned} J^{na}(t,x,i,\alpha) := \mathbb{E}\Bigg[&\int_t^T D_s^{t,x,i,\alpha}\psi(s,X_s^{t,x,i,\alpha},\alpha_s)\mathrm{d}s \\ &+ D_T^{t,x,i,\alpha}g(X_T^{t,x,i,\alpha},\alpha_T) - \sum_{t\le\tau_k\le T} D_{\tau_k}^{t,x,i,\alpha}c_{i_{k-1},i_k}(\tau_k,X_{\tau_k}^{t,x,i,\alpha}) \;\Big|\; \mathcal{F}_t\Bigg], \end{aligned} \tag{4}$$

where ψ, g, and c are measurable functions. ψ represents running rewards for the switching problem without ambiguity. g represents a terminal payoff. c is a switching cost function. $c_{i,j}(t,x)$ represents a switching cost from regime i to j at time t and $X_t = x$. $D^{t,x,i,\alpha}$ is a discount factor such that for any $(t,x,i)\in\overline{\mathcal{K}_T}$ and control α,

$$D_s^{t,x,i,\alpha} = \exp\left\{-\int_t^s \rho(t,X_u^{t,x,i,\alpha},\alpha_u)\mathrm{d}u\right\},\quad s\in[t,T], \tag{5}$$

where $\rho(t,x,i)$ is a bounded measurable function. By the definition (5), we allow the discount rate to be random and controllable. Therefore, the objective function (4) represents the expected and discounted total profit on $[t,T]$.

For all $\nu \in \mathcal{T}_0^T$ and $\iota \in \widetilde{\mathcal{I}}_\nu$, let $\mathbb{A}_\iota[\nu, T]$ be a set of controls such that

$$\mathbb{A}_\iota[\nu,T] := \left\{ \alpha = (\alpha_s)_{\nu\le s\le T} \;\middle|\; \begin{array}{c} \mathbb{E}\left[\left|\sum_{\nu\le\tau_k\le T} c_{i_{k-1},i_k}(\tau_k, X^{\nu,x,\iota,\alpha}_{\tau_k})\right|^2\right] < \infty,\ \forall x\in\mathbb{R}^d, \\ \text{and } \alpha_\nu = \iota. \end{array} \right\}. \tag{6}$$

We call a control in $\mathbb{A}_\iota[\nu, T]$ an admissible control. The optimal switching problem without ambiguity is

$$\sup_{\alpha\in\mathbb{A}_i[t,T]} J^{na}(t,x,i,\alpha), \tag{7}$$

for all $(t, x, i) \in \overline{\mathcal{K}_T}$.

The optimal switching problems expressed as (7) are well studied in many researchers (e.g., [3], [20], [9], and [1]). However, one of the weakness of the optimal switching problem (7) is not to take into account ambiguity. The problem (7) assumes that the decision maker knows the functional form of the distribution parameters b and σ a priori, whereas we do not know them in practice. Therefore, it needs to take into account uncertainty about the distribution of X in order to derive more useful switching strategies. Hence, we consider an optimal switching problem under ambiguity hereafter.

We first define a set of degrees of ambiguity. For $t \in [0, T]$, let Θ_t be a set of d-dimensional $\mathcal{F}_t$-measurable random variables. We assume the form of Θ_t as follows.

Hypothesis 3

1. *There exists a non-negative constant C such that*

$$\mathbb{P}(\|\theta_t\| \le C,\ \forall\theta_t \in \Theta_t,\ t \in [0,T]) = 1.$$

2. Θ_t *is convex and compact valued for all* $t \in [0, T]$.

3. Θ_t *is a progressively measurable correspondence for all* $t \in [0, T]$.

4. $0 \in \Theta_t$ $\mathrm{d}t \otimes \mathbb{P}$*-a.e..*

Let

$$\Theta[t,T] := \left\{ \theta = (\theta_s)_{t\le t\le T} \;\middle|\; \begin{array}{c} \theta \text{ is right-continuous with left limits and} \\ \theta_s \in \Theta_s \text{ for all } s \in [t,T]. \end{array} \right\}.$$

For all $\theta \in \Theta[t, T]$, we define a density process $\zeta^{\theta,t} = (\zeta^{\theta,t}_s)_{t\le s\le T}$ such that

$$\zeta^{\theta,t}_s := \exp\left\{ -\int_t^s \theta'_u \mathrm{d}W_u - \frac{1}{2}\int_t^s \|\theta_u\|^2 \mathrm{d}u \right\}, \quad s \in [t,T],$$

where x' is a transpose of a vector $x \in \mathbb{R}^d$. By Hypothesis 3, for all $\theta \in \Theta[t, T]$, $\zeta^{\theta,t}$ is a martingale with respect to $\mathbb{F}$. Therefore, for all $\theta \in \Theta[t, T]$, we can define a new probability measure such that

$$\mathbb{P}^\theta_T(A) := \mathbb{E}[\mathbb{1}_A \zeta^{\theta,t}_T], \quad A \in \mathcal{F}_T.$$

We denote by $\mathbb{E}^\theta_T$ the expectation operator under the probability measure $\mathbb{P}^\theta_T$.

Under the probability measure $\mathbb{P}^\theta_T$, by the Girsanov theorem, the SDE (2) can be expressed as

$$\mathrm{d}X_t = \Big(b(t,X_t,\alpha_t) - \sigma(t,X_t,\alpha_t)\theta_t\Big)\mathrm{d}t + \sigma(t,X_t,\alpha_t)\mathrm{d}W^\theta_t,\ t\in[0,T],$$

where W^θ is a d-dimensional Brownian motion under $\mathbb{P}^\theta_T$. This implies that we can take account of the ambiguity about the drift of X under $\mathbb{P}^\theta_T$.

Θ represents a set of priors of the decision maker. [5] establish a decision making problem under ambiguity in continuous time, which means that the decision maker would like to avoid the event whose occurrence probability is unknown. To incorporate ambiguity into an optimal switching problem, we use the concept of [5]. In their model, the decision maker chooses her subjective probability measure before choosing her decision as if her expected utility is minimized. They succeed to pose such a decision making problem under Hypothesis 3. They called Hypothesis 3 the rectangular condition.

The objective function under ambiguity is

$$J(t,x,i,\alpha) := \inf_{\theta\in\Theta[t,T]} \mathbb{E}_T^\theta\Bigg[\int_t^T D_s^{t,x,i,\alpha}\Big(\psi(s,X_s^{t,x,i,\alpha},\alpha_s) - \theta_s'\phi(s,X_s^{t,x,i,\alpha},\alpha_s)\Big)\mathrm{d}s + D_T^{t,x,i,\alpha} g(X_T^{t,x,i,\alpha},\alpha_T) - \sum_{t\le\tau_k\le T} D_{\tau_k}^{t,x,i,\alpha} c_{i_{k-1},i_k}(\tau_k, X_{\tau_k}^{t,x,i,\alpha}) \,\Big|\, \mathcal{F}_t\Bigg],$$

where ϕ is a measurable function from $[0,T]\times\mathbb{R}^d\times\mathcal{I}$ onto $\mathbb{R}^d$. ϕ determines a running premium for ambiguity. Our settings allow choices of ambiguity levels to affect the running rewards through the term $\theta_\cdot'\phi(\cdot, X_\cdot^{t,x,i}, \alpha_\cdot)$. The optimal switching problem under ambiguity is

$$\sup_{\alpha\in\mathbb{A}_i[t,T]} J(t,x,i,\alpha),$$

for all $(t,x,i)\in\overline{\mathcal{K}_T}$.

Furthermore, we assume the functions, ρ,ψ,ϕ,g, and c as follows.

Hypothesis 4

1. *$\rho(\cdot,\cdot,i)$ is a continuous, non-negative and upper bounded function for all $i\in\mathcal{I}$.*

2. *Polynomial growth condition*
   $\psi(\cdot,\cdot,i)$, $\phi(\cdot,\cdot,i)$, $g(\cdot,i)$ and $c_{i,j}(\cdot,\cdot)$ are continuous for all $i,j\in\mathcal{I}$, and $c_{i,i}(t,x)=0$ for all $(t,x,i)\in\overline{\mathcal{K}_T}$. Furthermore, there exist positive constants C_f and q such that

   $$|\psi(t,x,i)| + \|\phi(t,x,i)\| + |g(x,i)| + |c_{i,j}(t,x)| \le C_f(1+\|x\|^q),$$

   for all $(t,x,i,j)\in[0,T]\times\mathbb{R}^d\times(\mathcal{I})^2$. Without loss of generality, we assume $q\ge 1$.

3. *Non-free loop conditions*

   (a) *For all finite loop $(i_0,i_1,\dots,i_m)\in\mathcal{I}^{m+1}$ with $i_0=i_m$ and $i_0\ne i_1$ and for all $(t,x)\in[0,T]\times\mathbb{R}^d$, c satisfies*

   $$c_{i_0,i_1}(t,x)+\cdots+c_{i_{m-1},i_m}(t,x) > 0.$$

   (b) *g satisfies the following inequality,*

   $$g(x,i) \ge \max_{j\in\mathcal{I}\setminus\{i\}}\{g(x,j) - c_{i,j}(T,x)\},$$

   for all $(x,i)\in\mathbb{R}^d\times\mathcal{I}$.

4. *Strong triangular condition*
   Let

   $$\mathcal{N} = \left\{ i\in\mathcal{I} \,\middle|\, \exists j\in\mathcal{I},\ j\ne i,\ \int_{[0,T]\times\mathbb{R}^d} \mathbb{1}\{c_{i,j}(t,x)<0\}(t,x)\mathrm{d}t\mathrm{d}x > 0 \right\},$$

   $$C_i = -\min_{j\in\mathcal{I},\ x\in\mathbb{R}^d,\ t\in[0,T]} \frac{c_{i,j}(t,x)}{1+\|x\|^q}, \quad i\in\mathcal{N},$$

where q is defined in Hypothesis 4.2. Then, for all $i \in \mathcal{N}$,

$$c_{k,j}(t,x) \leq c_{k,i}(t,x) - C_i(1 + C_{q,X}(1 + \|x\|^q)e^{C_q(T-t)}), \tag{8}$$

for all $t \in [0,T]$, $x \in \mathbb{R}^d$ and $(j,k) \in \mathcal{I}$ with $j \neq i$ and $k \neq i$, where $C_{q,X}$ and C_q are defined in Proposition 2.

Hypothesis 4.1 implies that the discount rate is upper bounded and non-negative. The non-negativity is usual, and the assumption of upper boundedness guarantees the Lipschitz condition of a generator in the BSDE literature. Hypothesis 4.2 and Proposition 2 guarantee the value function of our optimal switching problem to be finite. Therefore, it is needed in order to consider meaningful problems.

The non-free loop conditions (Hypothesis 4.3) say that whenever one first stands in some regime (call regime A), next instantaneously goes to the other regimes, and finally goes back to the regime A at the same time, then she has to pay a positive cost. Hence, the non-free loop conditions exclude the possibility that one can gain a positive profit by a looping switching strategy at the same time. If the non-free loop conditions are not postulated, then the value function diverges as the decision maker obtains an infinitely large reward by such a looping strategy. Since it is an arbitrage, the non-free loop conditions are natural in the optimal switching problems.

Unlike the previous literature, we do not assume non-negativity of the cost functions. Our specification of ambiguity allows this generalization. However, we need an additional assumption in this case. If some cost function can take a negative value, it needs to satisfy the strong triangular condition (Hypothesis 4.4).

The strong triangular condition means that the switching benefits are not too large to take these benefits. Heuristically speaking, if one first stands in the regime k and if $c_{i,j} < 0$, then the cost that she goes to the regime j via the regime i is at least as large as the cost that she directly goes to the regime j. The strong triangular condition implies the standard triangle inequality. Indeed, by the inequality (8), we have

$$\begin{aligned} c_{k,i}(t,x) + c_{i,j}(t,x) &\geq c_{k,i}(t,x) - C_i(1 + \|x\|^q) \\ &\geq c_{k,i}(t,x) - C_i(1 + C_{q,X}(1 + \|x\|^q)e^{C_q(T-t)}) \geq c_{k,j}(t,x), \end{aligned}$$

for all $i \in \mathcal{N}$, $(j,k) \in \mathcal{I}$, $(t,x) \in [0,T] \times \mathbb{R}^d$ with $k \neq i$ and $j \neq i$. Therefore, our triangular condition (8) is stronger than the standard triangle inequality.

By Proposition 2 and Hypothesis 4, we can show that an expected total cost does not diverge for every admissible control.

Proposition 5 *Under Hypotheses 1 and 4,*

$$\mathbb{E}\left[-\sum_{t \leq \tau_k \leq T} D^{t,x,i,\alpha}_{\tau_k} c_{i_{k-1},i_k}(\tau_k, X^{t,x,i,\alpha}_{\tau_k}) \right] \leq C_f(1 + C_{q,X}(1 + \|x\|^q)e^{C_q(T-t)}), \tag{9}$$

for all $(t,x,i) \in \overline{\mathcal{K}_T}$ and $\alpha = (\tau_k, i_k)_{k \geq 0} \in \mathbb{A}_i[t,T]$.

Proof of Proposition 5. Fix an arbitrary $(t,x,i) \in \overline{\mathcal{K}_T}$ and $\alpha = (\tau_k, i_k)_{k \geq 0} \in \mathbb{A}_i[t,T]$. We first prove

$$\mathbb{E}\left[-\sum_{k=1}^{n} D^{t,x,i,\alpha}_{\tau_k} c_{i_{k-1},i_k}(\tau_k, X^{t,x,i,\alpha}_{\tau_k}) \right] \leq C_f(1 + C_{q,X}(1 + \|x\|^q)e^{C_q(T-t)}),$$

for all $n \geq 1$. If $\mathbb{P}(i_{n-1} \in \mathcal{N} \mid \mathcal{F}_{\tau_{n-1}}) = 0$, then $c_{i_{n-1},i_n}(\tau_n, X_{\tau_n}^{t,x,i,\alpha}) \geq 0$. Hence, we have

$$
\begin{aligned}
(10)\quad & -D_{\tau_{n-1}}^{t,x,i,\alpha} c_{i_{n-2},i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}) - D_{\tau_n}^{t,x,i,\alpha} c_{i_{n-1},i_n}(\tau_n, X_{\tau_n}^{t,x,i,\alpha}) \\
& \qquad\qquad \leq -D_{\tau_{n-1}}^{t,x,i,\alpha} c_{i_{n-2},i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}).
\end{aligned}
$$

If $\mathbb{P}(i_{n-1} \in \mathcal{N} \mid \mathcal{F}_{\tau_{n-1}}) > 0$, then, by Proposition 2, we have

$$
\begin{aligned}
& \mathbb{E}\left[-D_{\tau_{n-1}}^{t,x,i,\alpha} c_{i_{n-2},i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}) - D_{\tau_n}^{t,x,i,\alpha} c_{i_{n-1},i_n}(\tau_n, X_{\tau_n}^{t,x,i,\alpha}) \;\Big|\; \mathcal{F}_{\tau_{n-1}}\right] \\
& \quad \leq -\mathbb{E}\Big[D_{\tau_{n-1}}^{t,x,i,\alpha}\Big(c_{i_{n-2},i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}) - C_{i_{n-1}}\Big(1 + \|X_{\tau_n}^{t,x,i,\alpha}\|^q\Big)\Big)\mathbb{1}_{\{i_{n-1}\in\mathcal{N}\}} \\
& \qquad\qquad + D_{\tau_{n-1}}^{t,x,i,\alpha} c_{i_{n-2},i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha})\mathbb{1}_{\{i_{n-1}\notin\mathcal{N}\}} \;\Big|\; \mathcal{F}_{\tau_{n-1}}\Big] \\
& \quad \leq -\mathbb{E}\Big[D_{\tau_{n-1}}^{t,x,i,\alpha}\Big(c_{i_{n-2},i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}) \\
& \qquad\qquad - C_{i_{n-1}}\Big(1 + C_{q,X}\Big(1 + \|X_{\tau_{n-1}}^{t,x,i,\alpha}\|^q\Big)e^{C_q(T-\tau_{n-1})}\Big)\Big)\mathbb{1}_{\{i_{n-1}\in\mathcal{N}\}} \\
& \qquad\qquad + D_{\tau_{n-1}}^{t,x,i,\alpha} c_{i_{n-2},i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha})\mathbb{1}_{\{i_{n-1}\notin\mathcal{N}\}} \;\Big|\; \mathcal{F}_{\tau_{n-1}}\Big].
\end{aligned}
$$

By Hypothesis 4.4, there exists an $\mathcal{F}_{\tau_{n-1}}$-measurable random variable $\widetilde{i}_{n-1}$ taking values in $\mathcal{I}$ such that

$$
\begin{aligned}
& -\mathbb{E}\Big[D_{\tau_{n-1}}^{t,x,i,\alpha}\Big(c_{i_{n-2},i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}) \\
& \qquad\qquad - C_{i_{n-1}}\Big(1 + C_{q,X}\Big(1 + \|X_{\tau_{n-1}}^{t,x,i,\alpha}\|^q\Big)e^{C_q(T-\tau_{n-1})}\Big)\Big)\mathbb{1}_{\{i_{n-1}\in\mathcal{N}\}} \;\Big|\; \mathcal{F}_{\tau_{n-1}}\Big] \\
& \leq -\mathbb{E}\Big[D_{\tau_{n-1}}^{t,x,i,\alpha} c_{i_{n-2},\widetilde{i}_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha})\mathbb{1}_{\{i_{n-1}\in\mathcal{N}\}} \;\Big|\; \mathcal{F}_{\tau_{n-1}}\Big].
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
(11)\quad & \mathbb{E}\left[-D_{\tau_{n-1}}^{t,x,i,\alpha} c_{i_{n-2},i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}) - D_{\tau_n}^{t,x,i,\alpha} c_{i_{n-1},i_n}(\tau_n, X_{\tau_n}^{t,x,i,\alpha}) \;\Big|\; \mathcal{F}_{\tau_{n-1}}\right] \\
& \quad \leq \quad -\mathbb{E}\Big[D_{\tau_{n-1}}^{t,x,i,\alpha}\Big(c_{i_{n-2},\widetilde{i}_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha})\mathbb{1}_{\{i_{n-1}\in\mathcal{N}\}} \\
& \qquad\qquad + c_{i_{n-2},i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha})\mathbb{1}_{\{i_{n-1}\notin\mathcal{N}\}}\Big) \;\Big|\; \mathcal{F}_{\tau_{n-1}}\Big] \\
& \quad \leq -\mathbb{E}\Big[D_{\tau_{n-1}}^{t,x,i,\alpha} c_{i_{n-2},i_{n-1}^*}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}) \;\Big|\; \mathcal{F}_{\tau_{n-1}}\Big],
\end{aligned}
$$

where

$$
i_{n-1}^* = \arg\min_{j\in\mathcal{I}\setminus\{i_{n-2}\}} \left\{c_{i_{n-2},j}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha})\right\},
$$

and i_{n-1}^* is obviously $\mathcal{F}_{\tau_{n-1}}$-measurable. Therefore, the inequalities (10) and (11) lead to

$$
\begin{aligned}
\mathbb{E}\left[-\sum_{k=1}^{n} D_{\tau_k}^{t,x,i,\alpha} c_{i_{k-1},i_k}(\tau_k, X_{\tau_k}^{t,x,i,\alpha})\right] & \leq \mathbb{E}\left[-D_{\tau_1}^{t,x,i,\alpha} c_{i,i_1^*}(\tau_1, X_{\tau_1}^{t,x,i,\alpha})\right] \\
& \leq C_f\left(1 + \mathbb{E}\left[\|X_{\tau_1}^{t,x,i,\alpha}\|^q\right]\right) \\
& \leq C_f\left(1 + \mathbb{E}\left[\max_{t\leq s\leq T}\|X_s^{t,x,i,\alpha}\|^q\right]\right) \\
& \leq C_f(1 + C_{q,X}(1 + \|x\|^q)e^{C_q(T-t)}).
\end{aligned}
$$

Since $\alpha \in \mathbb{A}_i[t,T]$, by the Lebesgue dominated convergence theorem, we obtain the inequality (9). □

Proposition 5 has an important role in our switching problem. The other studies assuming non-negativity of switching costs naturally derive a lower boundary of the total expected costs, that is 0. However, we do not naturally say that the total costs are non-negative since our switching costs can take a negative value. Therefore, we need to estimate a lower boundary of the total expected costs by Proposition 5.

Remark 6 *Even if the cost functions do not satisfy the strong triangular condition, it is possible that Proposition 5 holds. In this case, the following discussion in this paper also holds. Essentially, we need*

$$\mathbb{E}\left[-\sum_{t\le\tau_k\le T} D^{t,x,i,\alpha}_{\tau_k} c_{i_{k-1},i_k}(\tau_k, X^{t,x,i,\alpha}_{\tau_k})\right] \le C(1+\|x\|^q),$$

for all $(t,x,i)\in\overline{\mathcal{K}_T}$ *and* $\alpha\in\mathbb{A}_i[t,T]$*, where* C *is a positive constant not depending on* (t,x,i) *and* α*.*

3 Multidimensional Reflected BSDEs

Next, we consider a representation of the objective function by BSDEs.

For all $\nu\in\mathcal{T}^T_0$, we denote by $\mathbb{S}^2[\nu,T]$ the set of real-valued progressively measurable processes Y such that

$$\mathbb{E}\left[\sup_{\nu\le t\le T}|Y_t|^2\right]<\infty,$$

and by $\mathbb{H}^2_d[\nu,T]$ the set of $\mathbb{R}^d$-valued progressively measurable processes Z such that

$$\mathbb{E}\left[\int_\nu^T \|Z_t\|^2\mathrm{d}t\right]<\infty.$$

Especially, we denote by $\mathbb{S}^2_c[\nu,T]$ a set of all continuous processes in $\mathbb{S}^2[\nu,T]$ and by $\mathbb{K}^2[\nu,T]$ a set of all non-decreasing processes in $\mathbb{S}^2[\nu,T]$.

We consider the following BSDE: For given $(\nu,\eta,\iota)\in\widetilde{\mathcal{K}^{2q}_T}$, $\theta\in\Theta[\nu,T]$ and $\alpha\in\mathbb{A}_\iota[\nu,T]$,

$$\begin{aligned}
-\mathrm{d}Y^{\nu,\eta,\iota,\theta,\alpha}_t &= \Big(\psi(t,X^{\nu,\eta,\iota,\alpha}_t,\alpha_t)-\rho(t,X^{\nu,\eta,\iota,\alpha}_t,\alpha_t)Y^{\nu,\eta,\iota,\theta,\alpha}_t \\
&\quad -\theta'_t\Big(\phi(t,X^{\nu,\eta,\iota,\alpha}_t,\alpha_t)+Z^{\nu,\eta,\iota,\theta,\alpha}_t\Big)\Big)\mathrm{d}t \\
&\quad -(Z^{\nu,\eta,\iota,\theta,\alpha}_t)'\mathrm{d}W_t-\mathrm{d}A^{\nu,\eta,\iota,\alpha}_t,\ t\in[\nu,T], \\
Y^{\nu,\eta,\iota,\theta,\alpha}_T &= g(X^{\nu,\eta,\iota,\alpha}_T,\alpha_T),\quad A^{\nu,\eta,\iota,\alpha}_t=\sum_{t\le\tau_k\le T} c_{i_{k-1},i_k}(\tau_k,X^{\nu,\eta,\iota,\alpha}_{\tau_k}),\ t\in[\nu,T], \\
&(Y^{\nu,\eta,\iota,\theta,\alpha},Z^{\nu,\eta,\iota,\theta,\alpha})\in\mathbb{S}^2[\nu,T]\times\mathbb{H}^2_d[\nu,T].
\end{aligned}\tag{12}$$

Since $g(X^{\nu,\eta,\iota,\alpha}_T,\alpha_T)\in L^2_T(\mathbb{R})$ and $(\phi(t,X^{\nu,\eta,\iota,\alpha}_t,\alpha_t))_{\nu\le t\le T}$, $(\psi(t,X^{\nu,\eta,\iota,\alpha}_t,\alpha_t))_{\nu\le t\le T}\in\mathbb{H}^2_1[\nu,T]$ and since θ and ρ are uniformly bounded by Hypotheses 1, 3 and 4, the BSDE (12) has a unique solution in $\mathbb{S}^2[\nu,T]\times\mathbb{H}^2_d[\nu,T]$. Furthermore, by Proposition 2.2 in [12], the solution of the BSDE (12), also denoted by $(Y^{\nu,\eta,\iota,\theta,\alpha}_t,Z^{\nu,\eta,\iota,\theta,\alpha}_t)_{\nu\le t\le T}$, can be represented as the following form.

$$Y_t^{\nu,\eta,\iota,\theta,\alpha} = \frac{1}{D_t^{\nu,\eta,\iota,\alpha}\zeta_t^{\theta,\nu}}\mathbb{E}\Bigg[\int_t^T D_s^{\nu,\eta,\iota,\alpha}\zeta_s^{\theta,\nu}\Big(\psi(s,X_s^{\nu,\eta,\iota,\alpha},\alpha_s)-\theta_s'\phi(s,X_s^{\nu,\eta,\iota,\alpha},\alpha_s)\Big)\mathrm{d}s$$
$$+D_T^{\nu,\eta,\iota,\alpha}\zeta_T^{\theta,\nu}g(X_T^{\nu,\eta,\iota,\alpha},\alpha_T)-\sum_{t\le\tau_k\le T}D_{\tau_k}^{\nu,\eta,\iota,\alpha}\zeta_{\tau_k}^{\theta,\nu}c_{i_{k-1},i_k}(\tau_k,X_{\tau_k}^{\nu,\eta,\iota,\alpha})\;\Big|\;\mathcal{F}_t\Bigg]$$
$$=\mathbb{E}_T^\theta\Bigg[\int_t^T\frac{D_s^{\nu,\eta,\iota,\alpha}}{D_t^{\nu,\eta,\iota,\alpha}}\Big(\psi(s,X_s^{\nu,\eta,\iota,\alpha},\alpha_s)-\theta_s'\phi(s,X_t^{\nu,\eta,\iota,\alpha},\alpha_s)\Big)\mathrm{d}s$$
$$+\frac{D_T^{\nu,\eta,\iota,\alpha}}{D_t^{\nu,\eta,\iota,\alpha}}g(X_T^{0,x,i,\alpha},\alpha_T)-\sum_{t\le\tau_k\le T}\frac{D_{\tau_k}^{\nu,\eta,\iota,\alpha}}{D_t^{\nu,\eta,\iota,\alpha}}c_{i_{k-1},i_k}(\tau_k,X_{\tau_k}^{\nu,\eta,\iota,\alpha})\;\Big|\;\mathcal{F}_t\Bigg],\tag{13}$$

where we have used the Bayes rule in the second equality.

Now, we also consider another BSDE such that

$$\begin{aligned}-\mathrm{d}Y_t^{\nu,\eta,\iota,\alpha} &= \Big(\psi(t,X_t^{\nu,\eta,\iota,\alpha},\alpha_t)-\rho(t,X_t^{\nu,\eta,\iota,\alpha},\alpha_t)Y_t^{\nu,\eta,\iota,\alpha}\\ &\qquad -\max_{\theta_t\in\Theta_t}\Big\{\theta_t'\Big(\phi(t,X_t^{\nu,\eta,\iota,\alpha},\alpha_t)+Z_t^{\nu,\eta,\iota,\alpha}\Big)\Big\}\Big)\mathrm{d}t\\ &\qquad -(Z_t^{\nu,\eta,\iota,\alpha})'\mathrm{d}W_t-\mathrm{d}A_t^{\nu,\eta,\iota,\alpha},\ t\in[\nu,T],\\ Y_T^{\nu,\eta,\iota,\alpha} &= g(X_T^{\nu,\eta,\iota,\alpha},\alpha_T),\quad A_t^{\nu,\eta,\iota,\alpha}=\sum_{t\le\tau_k\le T}c_{i_{k-1},i_k}(\tau_k,X_{\tau_k}^{\nu,\eta,\iota,\alpha}),\ t\in[\nu,T],\\ &(Y^{\nu,\eta,\iota,\alpha},Z^{\nu,\eta,\iota,\alpha})\in\mathbb{S}^2[\nu,T]\times\mathbb{H}_d^2[\nu,T].\end{aligned}\tag{14}$$

The BSDE (14) also has a unique solution in $\mathbb{S}^2[\nu,T]\times\mathbb{H}_d^2[\nu,T]$. From the comparison theorem, the solution of the BSDE (14) is a minimum value of $Y_t^{\nu,\eta,\iota,\theta,\alpha}$ over $\theta\in\Theta[\nu,T]$, that is, the following inequality holds.

$$Y_t^{\nu,\eta,\iota,\theta,\alpha}\ge Y_t^{\nu,\eta,\iota,\alpha},\tag{15}$$

$\mathbb{P}$-almost surely for all $t\in[\nu,T]$ and $\theta\in\Theta[\nu,T]$.

Combining the inequality (15) with the equality (13), we deduce that

$$\begin{aligned}Y_t^{t,x,i,\alpha} &= \inf_{\theta\in\Theta[t,T]}\mathbb{E}_T^\theta\Bigg[\int_t^T D_s^{t,x,i,\alpha}\Big(\psi(s,X_s^{t,x,i,\alpha},\alpha_s)-\theta_s'\phi(s,X_s^{t,x,i,\alpha},\alpha_s)\Big)\mathrm{d}s\\ &\qquad +D_T^{t,x,i,\alpha}g(X_T^{t,x,i,\alpha},\alpha_T)-\sum_{s\le\tau_k\le T}D_{\tau_k}^{t,x,i,\alpha}c_{i_{k-1},i_k}(\tau_k,X_{\tau_k}^{t,x,i,\alpha})\;\Big|\;\mathcal{F}_t\Bigg]\\ &= J(t,x,i,\alpha),\end{aligned}$$

for all $(t,x,i)\in\overline{\mathcal{K}_T}$ and $\alpha\in\mathbb{A}_i[t,T]$. Therefore, $Y_t^{t,x,i,\alpha}$ is the objective function under ambiguity.

For the sake of brevity, we assume as follows.

Hypothesis 7 *Suppose that Θ_t is measurable with respect to the σ-algebra generated by X_t and α_t for all $t\in[0,T]$. We denote by $\Theta_t^{x,i}$ a Θ_t with $X_t=x$ and $\alpha_t=i$. For all $(t,x,i)\in\overline{\mathcal{K}_T}$ and $z\in\mathbb{R}^d$, let*

$$\varsigma(t,x,i,z):=\max_{\theta_t\in\Theta_t^{x,i}}\Big\{\theta_t'\Big(\phi(t,x,i)+z\Big)\Big\}.$$

Then, suppose that ς is a deterministic and measurable function. Moreover, suppose that $\varsigma(\cdot,\cdot,i,\cdot)$ is continuous for all $i\in\mathcal{I}$.

By Hypothesis 3.1 and 4, ς satisfy the polynomial growth condition with respect to x and z and the Lipschitz condition with respect to z: There exists a positive constant C_ς such that

$$|\varsigma(t,x,i,z)| \le C_\varsigma(1+\|x\|^q+\|z\|), \quad |\varsigma(t,x,i,z)-\varsigma(t,x,i,\widetilde{z})| \le C_\varsigma\|z-\widetilde{z}\|,$$

for all $(t,x,i,z,\widetilde{z}) \in \overline{\mathcal{K}_T} \times (\mathbb{R}^d)^2$.

Under Hypothesis 7, the BSDE (14) can be expressed as

$$\begin{aligned} -\mathrm{d}Y_t^{\nu,\eta,\iota,\alpha} &= \Big(\psi(t,X_t^{\nu,\eta,\iota,\alpha},\alpha_t) - \rho(t,X_t^{\nu,\eta,\iota,\alpha},\alpha_t)Y_t^{\nu,\eta,\iota,\alpha} - \varsigma(t,X_t^{\nu,\eta,\iota,\alpha},\alpha_t,Z_t^{\nu,\eta,\iota,\alpha})\Big)\mathrm{d}t \\ &\qquad - (Z_t^{\nu,\eta,\iota,\alpha})'\mathrm{d}W_t - \mathrm{d}A_t^{\nu,\eta,\iota,\alpha},\ t\in[\nu,T], \\ Y_T^{\nu,\eta,\iota,\alpha} &= g(X_T^{\nu,\eta,\iota,\alpha},\alpha_T), \quad A_t^{\nu,\eta,\iota,\alpha} = \sum_{t\le\tau_k\le T} c_{i_{k-1},i_k}(\tau_k,X_{\tau_k}^{\nu,\eta,\iota,\alpha}),\ t\in[\nu,T], \\ &(Y^{\nu,\eta,\iota,\alpha},Z^{\nu,\eta,\iota,\alpha}) \in \mathbb{S}^2[\nu,T]\times\mathbb{H}_d^2[\nu,T]. \end{aligned} \tag{16}$$

Now, let us consider the multidimensional RBSDE. For given $\nu \in \mathcal{T}_0^T$ and $\eta \in L_\nu^{2q}(\mathbb{R}^d)$ and for all $i \in \mathcal{I}$,

$$\begin{aligned} -\mathrm{d}Y_t^{\nu,\eta,i} &= \Big(\psi(t,X_t^{\nu,\eta,i},i) - \rho(t,X_t^{\nu,\eta,i},i)Y_t^{\nu,\eta,i} - \varsigma(t,X_t^{\nu,\eta,i},i,Z_t^{\nu,\eta,i})\Big)\mathrm{d}t \\ &\qquad - (Z_t^{\nu,\eta,i})'\mathrm{d}W_t + \mathrm{d}K_t^{\nu,\eta,i},\ t\in[\nu,T], \\ Y_T^{\nu,\eta,i} &= g(X_T^{\nu,\eta,i},i), \quad K_\nu^{\nu,\eta,i}=0, \quad Y_t^{\nu,\eta,i} \ge \max_{j\in\mathcal{I}\setminus\{i\}}\{Y_t^{\nu,\eta,j} - c_{i,j}(t,X_t^{\nu,\eta,i})\},\ t\in[\nu,T], \\ &\int_\nu^T \Big(Y_t^{\nu,\eta,i} - \max_{j\in\mathcal{I}\setminus\{i\}}\{Y_t^{\nu,\eta,j} - c_{i,j}(t,X_t^{\nu,\eta,i})\}\Big)\mathrm{d}K_t^{\nu,\eta,i} = 0, \\ &(Y^{\nu,\eta,i},Z^{\nu,\eta,i},K^{\nu,\eta,i}) \in \mathbb{S}^2[\nu,T]\times\mathbb{H}_d^2[\nu,T]\times\mathbb{K}^2[\nu,T], \quad i\in\mathcal{I}, \end{aligned} \tag{17}$$

where $X^{\nu,\eta,i} = (X_t^{\nu,\eta,i})_{\nu\le t\le T}$ is a strong solution to the following SDE,

$$\mathrm{d}X_t = b(t,X_t,i)\mathrm{d}t + \sigma(t,X_t,i)\mathrm{d}W_t,\ t\in[\nu,T], \quad X_\nu = \eta. \tag{18}$$

In the next section, we show that the solution $Y_t^{t,x,i}$ of the multidimensional RBSDE (17) is a value function of the optimal switching problem under ambiguity. In this section, we first prove the existence of solutions to the multidimensional RBSDE (17).

Theorem 8 *Under Hypotheses 1, 3, 4 and 7, the multidimensional RBSDE* (17) *has a solution in* $(\mathbb{S}_c^2[\nu,T]\times\mathbb{H}_d^2[\nu,T]\times\mathbb{K}^2[\nu,T])^I$ *for any* $\nu\in\mathcal{T}_0^T$ *and* $\eta\in L_\nu^{2q}(\mathbb{R}^d)$.

In the case when the switching costs are non-negative, Theorem 8 are proved by Theorem 3.2 in [16] and Theorem 2.1 in [17]. We use the strategy of the proof of Theorem 3.2 in [16], but there is a problem for a priori estimates of Picard's iterations of the multidimensional RBSDE (17). [16] define the process in $\mathbb{S}^2[\nu,T]$ that is larger than all Picard's iterations, however, this process may not be larger than Picard's iterations in our problem since we allow the switching costs to be negative. Therefore, we can not use the results of [16] straightforwardly. However, thanks to Proposition 5, we can define the other process in $\mathbb{S}^2[\nu,T]$ that is larger than all Picard's iterations in our problem.

Proof of Theorem 8. Throughout this proof, we fix an arbitrary $\nu\in\mathcal{T}_0^T$ and $\eta\in L_\nu^{2q}(\mathbb{R}^d)$.

Step.1 Picard's iterations. Let $(Y^{\nu,\eta,i,0},Z^{\nu,\eta,i,0})$ be a solution to the following BSDE.

$$\begin{aligned} -\mathrm{d}Y_t^{\nu,\eta,i,0} &= \Big(\psi(t,X_t^{\nu,\eta,i},i) - \rho(t,X_t^{\nu,\eta,i},i)Y_t^{\nu,\eta,i,0} - \varsigma(t,X_t^{\nu,\eta,i},i,Z_t^{\nu,\eta,i,0})\Big)\mathrm{d}t \\ &\qquad - (Z_t^{\nu,\eta,i,0})'\mathrm{d}W_t,\ t\in[\nu,T], \\ Y_T^{\nu,\eta,i,0} &= g(X_T^{\nu,\eta,i},i), \quad (Y^{\nu,\eta,i,0},Z^{\nu,\eta,i,0}) \in \mathbb{S}^2[\nu,T]\times\mathbb{H}_d^2[\nu,T], \end{aligned}$$

for all $i \in \mathcal{I}$. Then, by Hypotheses 1, 3, 4 and 7, the above BSDE has a unique solution. For any $n \geq 1$, we consider the following RBSDE recursively.

$$
(19)\quad \begin{aligned}
-\mathrm{d}Y_t^{\nu,\eta,i,n} &= \Big(\psi(t, X_t^{\nu,\eta,i}, i) - \rho(t, X_t^{\nu,\eta,i}, i)Y_t^{\nu,\eta,i,n} - \varsigma(t, X_t^{\nu,\eta,i}, i, Z_t^{\nu,\eta,i,n})\Big)\mathrm{d}t \\
&\qquad - (Z_t^{\nu,\eta,i,n})'\mathrm{d}W_t + \mathrm{d}K_t^{\nu,\eta,i,n},\ t \in [\nu, T], \\
Y_T^{\nu,\eta,i,n} &= g(X_T^{\nu,\eta,i}, i), \quad K_\nu^{\nu,\eta,i,n} = 0, \\
Y_t^{\nu,\eta,i,n} &\geq \max_{j \in \mathcal{I}\setminus\{i\}} \{Y_t^{\nu,\eta,j,n-1} - c_{i,j}(t, X_t^{\nu,\eta,i})\},\ t \in [\nu, T], \\
&\int_\nu^T \Big(Y_t^{\nu,\eta,i,n} - \max_{j \in \mathcal{I}\setminus\{i\}} \{Y_t^{\nu,\eta,j,n-1} - c_{i,j}(t, X_t^{\nu,\eta,i})\}\Big)\mathrm{d}K_t^{\nu,\eta,i,n} = 0, \\
&(Y^{\nu,\eta,i,n}, Z^{\nu,\eta,i,n}, K^{\nu,\eta,i,n}) \in \mathbb{S}^2[\nu, T] \times \mathbb{H}_d^2[\nu, T] \times \mathbb{K}^2[\nu, T], \quad i \in \mathcal{I}.
\end{aligned}
$$

Under Hypotheses 1, 3, 4 and 7, by Theorem 5.2 in [11], the RBSDE (19) has a unique solution for all n and i. Furthermore, by the comparison theorem (Theorem 4.1 in [11]), we have $Y_t^{\nu,\eta,i,n-1} \leq Y_t^{\nu,\eta,i,n}$, $\mathbb{P}$-a.s. for all i and n.

Step.2 Non-ambiguity processes. Consider the following BSDE.

$$
\begin{aligned}
-\mathrm{d}U_t^{\nu,\eta,i,0} &= \Big(\psi(t, X_t^{\nu,\eta,i}, i) - \rho(t, X_t^{\nu,\eta,i}, i)U_t^{\nu,\eta,i,0}\Big)\mathrm{d}t - (V_t^{\nu,\eta,i,0})'\mathrm{d}W_t,\ t \in [\nu, T], \\
U_T^{\nu,\eta,i,0} &= g(X_T^{\nu,\eta,i}, i), \quad (U^{\nu,\eta,i,0}, V^{\nu,\eta,i,0}) \in \mathbb{S}^2[\nu, T] \times \mathbb{H}_d^2[\nu, T], \quad i \in \mathcal{I}.
\end{aligned}
$$

Then, the above BSDE has a unique solution. Similarly, we consider the following RBSDE for any $n \geq 1$.

$$
\begin{aligned}
-\mathrm{d}U_t^{\nu,\eta,i,n} &= \Big(\psi(t, X_t^{\nu,\eta,i}, i) - \rho(t, X_t^{\nu,\eta,i}, i)U_t^{\nu,\eta,i,n}\Big)\mathrm{d}t - (V_t^{\nu,\eta,i,n})'\mathrm{d}W_t + \mathrm{d}S_t^{\nu,\eta,i,n},\ t \in [\nu, T], \\
U_T^{\nu,\eta,i,n} &= g(X_T^{\nu,\eta,i}, i), \quad S_\nu^{\nu,\eta,i,n} = 0, \\
U_t^{\nu,\eta,i,n} &\geq \max_{j \in \mathcal{I}\setminus\{i\}} \{U_t^{\nu,\eta,j,n-1} - c_{i,j}(t, X_t^{\nu,\eta,i})\},\ t \in [\nu, T], \\
&\int_\nu^T \Big(U_t^{\nu,\eta,i,n} - \max_{j \in \mathcal{I}\setminus\{i\}} \{U_t^{\nu,\eta,j,n-1} - c_{i,j}(t, X_t^{\nu,\eta,i})\}\Big)\mathrm{d}S_t^{\nu,\eta,i,n} = 0, \\
&(U^{\nu,\eta,i,n}, V^{\nu,\eta,i,n}, S^{\nu,\eta,i,n}) \in \mathbb{S}^2[\nu, T] \times \mathbb{H}_d^2[\nu, T] \times \mathbb{K}^2[\nu, T], \quad i \in \mathcal{I}.
\end{aligned}
$$

Then the above RBSDE has a unique solution and we obtain that $U_t^{\nu,\eta,i,n} \geq U_t^{\nu,\eta,i,n-1}$, $\mathbb{P}$-a.s. for all $(t, i) \in [\nu, T] \times \mathcal{I}$ and $n \geq 1$ by the comparison theorem. By the definition of ς and Hypothesis 3.4, we have

$$
\varsigma(t, x, i, z) \geq 0, \quad \forall (t, x, i, z) \in [0, T] \times \mathbb{R}^d \times \mathcal{I} \times \mathbb{R}^d.
$$

Hence, applying the comparison theorem again to $U_t^{\nu,\eta,i,n}$ and $Y_t^{\nu,\eta,i,n}$, we obtain that $U_t^{\nu,\eta,i,n} \geq Y_t^{\nu,\eta,i,n}$, $\mathbb{P}$-a.s. for all $(t, i) \in [\nu, T] \times \mathcal{I}$ and $n \geq 1$. Furthermore, $U^{\nu,\eta,i,n}$ has a Snell envelope representation such that

$$
\begin{aligned}
U_t^{\nu,\eta,i,n} = \operatorname*{esssup}_{\tau^* \in \mathcal{T}_t^T} \mathbb{E}\Bigg[&\int_t^{\tau^*} \frac{D_s^{\nu,\eta,i}}{D_t^{\nu,\eta,i}} \psi(s, X_s^{\nu,\eta,i}, i)\mathrm{d}s + \frac{D_T^{\nu,\eta,i}}{D_t^{\nu,\eta,i}} g(X_T^{\nu,\eta,i}, i)\mathbb{1}_{\{\tau^* = T\}} \\
&+ \frac{D_{\tau^*}^{\nu,\eta,i}}{D_t^{\nu,\eta,i}} \max_{j \in \mathcal{I}\setminus\{i\}} \Big\{U_{\tau^*}^{\nu,\eta,j,n-1} - c_{i,j}(\tau^*, X_{\tau^*}^{\nu,\xi,i})\Big\} \mathbb{1}_{\{\tau^* < T\}} \;\Big|\; \mathcal{F}_t\Bigg],
\end{aligned}
$$

for all $t \in [\nu, T]$ and $n \geq 1$, where

$$
D_t^{\nu,\eta,i} = \exp\left\{-\int_\nu^t \rho(s, X_s^{\nu,\eta,i}, i)\mathrm{d}s\right\}, \quad t \in [\nu, T].
$$

Step.3 A priori estimates. Fix an arbitrary $t \in [\nu, T]$. Let $(\tau_0, i_0) = (t, i)$ and

$$\tau_k = \inf\left\{ s \in [\tau_{k-1}, T] \;\middle|\; U^{\nu,\eta,i_{k-1},n-(k-1)}_{\tau_n} = \max_{j \in \mathcal{I}\setminus\{i_{n-1}\}} \left\{ U^{\nu,\eta,j,n-k}_{\tau_n} - c_{i_{k-1},j}(\tau_k, X^{\nu,\eta,i,\alpha}_{\tau_k}) \right\} \right\},$$

i_k is such that $U^{\nu,\eta,i_{k-1},n-(k-1)}_{\tau_n} = U^{\nu,\eta,i_k,n-k}_{\tau_n} - c_{i_{k-1},i_k}(\tau_k, X^{\nu,\eta,i,\alpha}_{\tau_k})$,

for all $k = 1, \dots, n$. Then, we define $\alpha^n = (\tau_k, i_k)_{k \geq 0}$ and it holds that

$$\begin{aligned} U^{\nu,\eta,i,n}_t = \mathbb{E}\Bigg[\int_t^T & D_s^{t,X^{\nu,\eta,i}_t,i,\alpha^n} \psi(s, X^{\nu,\eta,i,\alpha^n}_s, \alpha^n_t) \mathrm{d}s + D_T^{t,X^{\nu,\eta,i}_t,i,\alpha^n} g(X^{\nu,\eta,i,\alpha^n}_T, \alpha^n_T) \\ & - \sum_{k=1}^n D_{\tau_k}^{t,X^{\nu,\eta,i}_t,i,\alpha^n} c_{i,j}(\tau_k, X^{\nu,\eta,i,\alpha^n}_{\tau_k}) \mathbb{1}_{\{\tau_k < T\}} \;\Bigg|\; \mathcal{F}_t \Bigg], \end{aligned}$$

by Proposition 2.3 in [11]. Furthermore, by the polynomial growth condition for c, it is easy to check that α^n is in $\mathbb{A}_i[\nu, T]$. Thus, by Proposition 5, we have

$$\mathbb{E}\left[-\sum_{k=1}^n D_{\tau_k}^{t,X^{\nu,\eta,i}_t,i,\alpha^n} c_{i,j}(\tau_k, X^{\nu,\eta,i,\alpha^n}_{\tau_k}) \mathbb{1}_{\{\tau_k < T\}} \;\middle|\; \mathcal{F}_t \right] \leq C_f(1 + C_{q,X}(1 + \|X^{\nu,\eta,i}_t\|^q) e^{C_{2q}T}).$$

On the other hand, by Proposition 2, there exists a constant $C_T > 0$ such that

$$\begin{aligned} & \mathbb{E}\left[\int_t^T D_s^{t,X^{\nu,\eta,i}_t,i,\alpha^n} \psi(s, X^{\nu,\eta,i,\alpha^n}_s, \alpha^n_t) \mathrm{d}s + D_T^{t,X^{\nu,\eta,i}_t,i,\alpha^n} g(X^{\nu,\eta,i,\alpha^n}_T, \alpha^n_T) \;\middle|\; \mathcal{F}_t \right] \\ & \leq \mathbb{E}\left[\int_t^T |\psi(s, X^{\nu,\eta,i,\alpha^n}_s, \alpha^n_t)| \mathrm{d}s + |g(X^{\nu,\eta,i,\alpha^n}_T, \alpha^n_T)| \;\middle|\; \mathcal{F}_t \right] \\ & \leq C_T(1 + \|X^{\nu,\eta,i}_t\|^q). \end{aligned}$$

Finally, there exists a positive constant $C_M > 0$ such that

$$\begin{aligned} U^{\nu,\eta,i,n}_t & = \mathbb{E}\Bigg[\int_t^T D_s^{t,X^{\nu,\eta,i}_t,i,\alpha^n} \psi(s, X^{\nu,\eta,i,\alpha^n}_s, \alpha^n_t) \mathrm{d}s + D_T^{t,X^{\nu,\eta,i}_t,i,\alpha^n} g(X^{\nu,\eta,i,\alpha^n}_T, \alpha^n_T) \\ & \qquad - \sum_{k=1}^n D_{\tau_k}^{t,X^{\nu,\eta,i}_t,i,\alpha^n} c_{i,j}(\tau_k, X^{\nu,\eta,i,\alpha^n}_{\tau_k}) \mathbb{1}_{\{\tau_k < T\}} \;\Bigg|\; \mathcal{F}_t \Bigg] \\ & \leq C_M(1 + \|X^{\nu,\eta,i}_t\|^q), \end{aligned}$$

for all $n \geq 1$. Note that C_M does not depend on n and t. This implies that

$$U^{\nu,\eta,i,n}_t \leq M^{\nu,\eta}_t := C_M \left(1 + \sum_{j \in \mathcal{I}} \|X^{\nu,\eta,j}_t\|^q \right),$$

for all $t \in [\nu, T]$, $i \in \mathcal{I}$ and $n \geq 1$. By Proposition 2, $M^{\nu,\eta}$ is in $\mathbb{S}^2[\nu, T]$. Since $Y^{\nu,\eta,i,0}_t \leq Y^{\nu,\eta,i,n}_t \leq U^{\nu,\eta,i,n}_t \leq M^{\nu,\eta}_t$ for all $t \in [\nu, T]$, $i \in \mathcal{I}$ and $n \geq 1$ and since $Y^{\nu,\eta,i,0} \in \mathbb{S}^2[\nu, T]$ for all $i \in \mathcal{I}$, there exists a finitely positive constant C_a such that

$$\sum_{i \in \mathcal{I}} \mathbb{E}\left[\sup_{\nu \leq t \leq T} |Y^{\nu,\eta,i,n}_t|^2 \right] \leq C_a, \tag{20}$$

for all $n \geq 0$. Furthermore, by the polynomial growth condition for c, Proposition 2 and the inequality (20), there exists a positive constant C_b such that

$$\mathbb{E}\left[\sup_{\nu \leq t \leq T} \left| \Big(\max_{j \in \mathcal{I}\setminus\{i\}} \{ Y^{\nu,\eta,j,n-1}_t - c_{i,j}(t, X^{\nu,\eta,i}_t) \} \Big)^+ \right|^2 \right] \leq C_b,$$

for all $n \geq 0$. Hence, Proposition 3.5 in [11] leads to that there exists a finitely positive constant C_c such that

$$\mathbb{E}\left[\sup_{\nu\le t\le T}|Y_t^{\nu,\eta,i,n}|^2+\int_\nu^T\|Z_t^{\nu,\eta,i,n}\|^2\mathrm{d}t+|K_T^{\nu,\eta,i,n}|^2\right]\le C_c, \tag{21}$$

for all $n \geq 0$ and $i \in \mathcal{I}$.

Step.4 The rest of this proof is exactly the same as step 3-5 in the proof of Theorem 3.2 in [16]. Thanks to the inequality (21), we can use the monotone limit theorem by [23] and show that a limit of $(Y^{\nu,\eta,i,n})_{n\geq 0}$ and associated processes $(Z^{\nu,\eta,i}, K^{\nu,\eta,i})$ satisfy properties of the solution to the multidimensional RBSDE (17). This limit, denoted by $(Y^{\nu,\eta,i})$, and $(K^{\nu,\eta,i})$ are continuous by the non-free loop condition. By the continuity of $(Y^{\nu,\eta,i})$ and $(K^{\nu,\eta,i})$, we conclude that a triplet $(Y^{\nu,\eta,i}, Z^{\nu,\eta,i}, K^{\nu,\eta,i})$ is a $\mathbb{S}^2[\nu,T]\times\mathbb{H}_d^2[\nu,T]\times\mathbb{K}^2[\nu,T]$ limit of the sequence $(Y^{\nu,\eta,i,n}, Z^{\nu,\eta,i,n}, K^{\nu,\eta,i,n})_{n\geq 0}$. □

Remark 9 *According to Corollary 3.3 in [16], the solution* $(Y^{\nu,\eta,i})$ *constructed in Theorem 8 is a minimum solution of the multidimensional RBSDE* (17)*: For any solution* $(\widetilde{Y}^{\nu,\eta,i})$ *of the multidimensional RBSDE* (17),

$$\widetilde{Y}_t^{\nu,\eta,i}\geq Y_t^{\nu,\eta,i}, \mathbb{P}\text{-}a.s.,$$

for all $t\in[\nu,T]$ *and* $i\in\mathcal{I}$.

Theorem 8 provides the existence of the multidimensional RBSDE (17). Other articles prove the uniqueness of the solution after proving the existence. However, we do not prove the uniqueness. Instead, we prove the pathwise uniqueness of the minimal solution of the multidimensional RBSDE (17) since this is a sufficient condition for the verification of the optimal switching problem under ambiguity.

Proposition 10 *Suppose that Hypotheses 1, 3, 4 and 7 are satisfied. For any* $(\nu,\widetilde{\nu})\in(\mathcal{T}_0^T)^2$ *and* $\eta\in L_\nu^{2q}(\mathbb{R}^d)$ *such that* $\nu\leq\widetilde{\nu}$ $\mathbb{P}$*-a.s., we consider the minimum solutions of the multidimensional RBSDE* (17) $Y^{\nu,\eta,i}$ *and* $Y^{\widetilde{\nu},X_{\widetilde{\nu}}^{\nu,\eta,i},i}$*. Then,*

$$Y_t^{\nu,\eta,i}=Y_t^{\widetilde{\nu},X_{\widetilde{\nu}}^{\nu,\eta,i},i}\ \mathbb{P}\text{-}a.s., \tag{22}$$

for all $i\in\mathcal{I}$ *and* $t\in[\widetilde{\nu},T]$.

Proof of Proposition 10. By Hypothesis 1, the SDE (18) has a strong solution for all $i\in\mathcal{I}$. This implies that

$$X_t^{\nu,\eta,i}=X_t^{\widetilde{\nu},X_{\widetilde{\nu}}^{\nu,\eta,i},i}\ \mathbb{P}\text{-a.s.},$$

for all $i\in\mathcal{I}$ and $t\in[\widetilde{\nu},T]$. Hence, $(Y^{\nu,\eta,i}, Z^{\nu,\eta,i}, \widehat{K}^{\nu,\eta,i}=K^{\nu,\eta,i}-K_{\widetilde{\nu}}^{\nu,\eta,i})$ satisfies the following multidimensional RBSDE on $[\widetilde{\nu},T]$.

$$\begin{aligned}
-\mathrm{d}Y_t^{\nu,\eta,i}&=\Big(\psi(t,X_t^{\widetilde{\nu},X_{\widetilde{\nu}}^{\nu,\eta,i},i},i)-\rho(t,X_t^{\widetilde{\nu},X_{\widetilde{\nu}}^{\nu,\eta,i},i},i)Y_t^{\nu,\eta,i}\\
&\qquad-\varsigma(t,X_{\widetilde{\nu}}^{\nu,\eta,i},i,Z_t^{\nu,\eta,i,n})\Big)\mathrm{d}t-(Z_t^{\nu,\eta,i})'\mathrm{d}W_t+\mathrm{d}\widehat{K}_t^{\nu,\eta,i},\ t\in[\widetilde{\nu},T],\\
Y_T^{\nu,\eta,i}&=g(X_T^{\widetilde{\nu},X_{\widetilde{\nu}}^{\nu,\eta,i},i},i),\quad \widehat{K}_{\widetilde{\nu}}^{\nu,\eta,i}=0,\\
Y_t^{\nu,\eta,i}&\geq\max_{j\in\mathcal{I}\setminus\{i\}}\{Y_t^{\nu,\eta,j}-c_{i,j}(t,X_t^{\widetilde{\nu},X_{\widetilde{\nu}}^{\nu,\eta,i},i})\},\quad t\in[\widetilde{\nu},T],\\
&\int_{\widetilde{\nu}}^T\Big(Y_t^{\nu,\eta,i}-\max_{j\in\mathcal{I}\setminus\{i\}}\{Y_t^{\nu,\eta,j}-c_{i,j}(t,X_t^{\widetilde{\nu},X_{\widetilde{\nu}}^{\nu,\eta,i},i})\}\Big)\mathrm{d}\widehat{K}_t^{\nu,\eta,i}=0,\\
&(Y^{\nu,\eta,i}, Z^{\nu,\eta,i}, \widehat{K}^{\nu,\eta,i})\in\mathbb{S}^2[\widetilde{\nu},T]\times\mathbb{H}_d^2[\widetilde{\nu},T]\times\mathbb{K}^2[\widetilde{\nu},T],\quad i\in\mathcal{I}.
\end{aligned}\tag{23}$$

Since for each i, the multidimensional RBSDE (23) is the same as the multidimensional RBSDE (17) starting from $(\widetilde{\nu}, X_{\widetilde{\nu}}^{\nu,\eta,i}, i)$, it holds that $Y_t^{\nu,\eta,i} \geq Y_t^{\widetilde{\nu}, X_{\widetilde{\nu}}^{\nu,\eta,i}, i}$ $\mathbb{P}$-a.s. for all $i \in \mathcal{I}$ and $t \in [\widetilde{\nu}, T]$ because of the minimality of $Y^{\widetilde{\nu}, X_{\widetilde{\nu}}^{\nu,\eta,i}, i}$ (see Remark 9).

On the other hand, recursively applying the comparison theorem to the Picard's iterations of $Y_t^{\nu,\eta,i}$ constructed in Theorem 8 on $[\widetilde{\nu}, T]$ leads to that

$$Y_t^{\nu,\eta,i,n} \leq Y_t^{\widetilde{\nu}, X_{\widetilde{\nu}}^{\nu,\eta,i}, i} \ \mathbb{P}\text{-a.s.},$$

for all $n \geq 0$, $i \in \mathcal{I}$ and $t \in [\widetilde{\nu}, T]$. Taking a limit of the above inequality, we obtain that $Y_t^{\nu,\eta,i} \leq Y_t^{\widetilde{\nu}, X_{\widetilde{\nu}}^{\nu,\eta,i}, i}$ for all $i \in \mathcal{I}$ and $t \in [\widetilde{\nu}, T]$. Hence, the equality (22) holds. □

4 Verification and a Viscosity Solution

In this section, we show that the minimum solution in Theorem 8 can be interpreted as the value function of the optimal switching problem under ambiguity. Proposition 11 provides a verification of Y. The proof of Proposition 11 is standard, so it is in appendix Appendix B.

Proposition 11 *Suppose that Hypotheses 1, 3, 4 and 7 are satisfied.*

1. *For an arbitrary* $(\nu, \eta, \iota) \in \widetilde{\mathcal{K}_T^{2q}}$, *let* $Y^{\nu,\eta,\iota}$ *be a minimum solution of the multidimensional RBSDE* (17)*. Then,*
$$Y_t^{\nu,\eta,\iota} \geq Y_t^{\nu,\eta,\iota,\alpha}, \quad \forall t \in [\nu, T],$$
for all $\alpha = (\tau_k, i_k)_{k\geq 0} \in \mathbb{A}_\iota[\nu, T]$.

2. *Let* $\alpha^* = (\tau_k^*, i_k^*)_{k\geq 0}$ *be a control such that* $(\tau^*, i_0^*) = (\nu, \iota)$ *and that for all* $n \geq 1$,
$$\tau_n^* := \inf\left\{ s \in [\tau_{n-1}^*, T] \;\middle|\; Y_s^{\tau_{n-1}^*, X_{\tau_{n-1}^*}^*, i_{n-1}^*} = \max_{j \in \mathcal{I}\setminus\{i_{n-1}^*\}} \{ Y_s^{\tau_{n-1}^*, X_{\tau_{n-1}^*}^*, j} - c_{i_{n-1}^*, j}(s, X_s^*) \} \right\},$$
i_n^* *is such that* $Y_{\tau_n^*}^{\tau_{n-1}^*, X_{\tau_{n-1}^*}^*, i_{n-1}^*} = Y_{\tau_n^*}^{\tau_{n-1}^*, X_{\tau_{n-1}^*}^*, i_n^*} - c_{i_{n-1}^*, i_n^*}(\tau_n^*, X_{\tau_n^*}^*)$,
where $X^* = X^{\nu,\eta,\iota,\alpha^*}$*. Then,* α^* *is an admissible control and*
$$Y_t^{\nu,\eta,\iota} = Y_t^{\nu,\eta,\iota,\alpha^*}, \quad \forall t \in [\nu, T].$$

By Proposition 11, we obtain that

$$Y_t^{t,x,i} = \sup_{\alpha \in \mathbb{A}_i[t,T]} Y_t^{t,x,i,\alpha} = \sup_{\alpha \in \mathbb{A}_i[t,T]} J(t, x, i, \alpha),$$

for all $(t, x, i) \in \overline{\mathcal{K}_T}$. Hence, $Y_t^{t,x,i}$ is the value function of the optimal switching problem under ambiguity. Furthermore, α^* defined in Proposition 11.2 is an optimal control of the problem.

We next study a relationship between the multidimensional RBSDE (17) and partial differential equations (hereafter PDEs). Let $u : [0, T] \times \mathbb{R}^d \times \mathcal{I} \to \mathbb{R}$ be a function. Consider the following PDE,

$$\begin{aligned}
&\min\{-u_t(t,x,i) - \mathcal{L}^i u(t,x,i) - \psi(t,x,i) + \rho(t,x,i)u(t,x,i) \\
&\qquad + \varsigma(t,x,i,\sigma'(t,x,i)\nabla u(t,x,i)), \\
&\qquad\qquad u(t,x,i) - \max_{j \in \mathcal{I}\setminus\{i\}} \{u(t,x,j) - c_{i,j}(t,x)\}\} = 0, \quad (t,x,i) \in \overline{\mathcal{K}_T}, \\
&u(T,x,i) = g(x,i),
\end{aligned} \tag{24}$$

where $u_t(t,x,i) = \frac{\partial u(t,x,i)}{\partial t}$, $\nabla u(t,x,i) = \frac{\partial u(t,x,i)}{\partial x}$ and

$$\mathcal{L}^i f(t,x) = (\nabla f(t,x))' b(t,x,i) + \frac{1}{2}\mathrm{tr}\left(\sigma\sigma'(t,x,i)\frac{\partial f(t,x)}{\partial x \partial x'}\right).$$

If the PDE (24) has a classical solution, then we can easily show that this solution is a value function of the optimal switching problem under ambiguity. However, the classical solution does not always exist. We shall consider a more general concept of solutions, i.e., a viscosity solution. Let $C^{1,2}([0,T) \times \mathbb{R}^d \times \mathcal{I})$ be a set of functions that are continuously differentiable with respect to t and twice continuously differentiable with respect x on $[0,T) \times \mathbb{R}^d \times \mathcal{I}$.

Definition 12 (Viscosity solution)

1. *Viscosity supersolution.*
   A lower semi-continuous function $(u(\cdot,\cdot,1),\dots,u(\cdot,\cdot,I))$ *is a viscosity supersolution of the PDE* (24) *if for any* $(t,x,i) \in [0,T) \times \mathbb{R}^d \times \mathcal{I}$ *and any* $\varphi \in C^{1,2}([0,T) \times \mathbb{R}^d \times \mathcal{I})$ *such that* $v(\cdot,\cdot,i) - \varphi(\cdot,\cdot,i)$ *attains a local minimum at* (t,x) *for all* $i \in \mathcal{I}$,

$$\begin{aligned}&\min\{-\varphi_t(t,x,i) - \mathcal{L}^i\varphi(t,x,i) - \psi(t,x,i) + \rho(t,x,i)u(t,x,i) + \varsigma(t,x,i,\sigma'(t,x,i)\nabla\varphi(t,x,i)),\\ &\qquad u(t,x,i) - \max_{j\in\mathcal{I}\setminus\{i\}}\{u(t,x,j) - c_{i,j}(t,x)\}\} \geq 0,\\ &u(T,x,i) \geq g(x,i).\end{aligned}$$

2. *Viscosity subsolution.*
   A upper semi-continuous function $(u(\cdot,\cdot,1),\dots,u(\cdot,\cdot,I))$ *is a viscosity subsolution of the PDE* (24) *if for any* $(t,x,i) \in [0,T) \times \mathbb{R}^d \times \mathcal{I}$ *and any* $\varphi \in C^{1,2}([0,T) \times \mathbb{R}^d \times \mathcal{I})$ *such that* $v(\cdot,\cdot,i) - \varphi(\cdot,\cdot,i)$ *attains a local maximum at* (t,x) *for all* $i \in \mathcal{I}$,

$$\begin{aligned}&\min\{-\varphi_t(t,x,i) - \mathcal{L}^i\varphi(t,x,i) - \psi(t,x,i) + \rho(t,x,i)u(t,x,i) + \varsigma(t,x,i,\sigma'(t,x,i)\nabla\varphi(t,x,i)),\\ &\qquad u(t,x,i) - \max_{j\in\mathcal{I}\setminus\{i\}}\{u(t,x,j) - c_{i,j}(t,x)\}\} \leq 0,\\ &u(T,x,i) \leq g(x,i).\end{aligned}$$

3. *Viscosity solution.*
   A locally bounded function $(u(\cdot,\cdot,1),\dots,u(\cdot,\cdot,I))$ *is a viscosity solution of the PDE* (24) *if its lower semi-continuous envelope is a viscosity supersolution of the PDE* (24)*, and if its upper semi-continuous envelope is a viscosity subsolution of the PDE* (24)*.*

More details of the viscosity solutions are in [7]. We define a set of functions $\mathcal{CP}([0,T]\times\mathbb{R}^d)$ as follows.

$$\mathcal{CP}([0,T]\times\mathbb{R}^d) := \left\{ f : [0,T]\times\mathbb{R}^d \to \mathbb{R} \;\middle|\; \begin{array}{c} f \text{ is jointly continuous and} \\ \text{there exist positive constants } C \text{ and } q \\ \text{such that } |f(t,x)| \leq C(1+\|x\|^q), \\ \text{for all } (t,x) \in [0,T]\times\mathbb{R}^d. \end{array} \right\}.$$

Let

$$v(t,x,i) := Y_t^{t,x,i},$$

for $(t,x,i) \in \overline{\mathcal{K}_T}$, where $Y_t^{t,x,i}$ is a minimum solution of the multidimensional RBSDE (17).

Now, we will prove that v is a unique viscosity solution of the PDE (24) in $\mathcal{CP}([0,T]\times\mathbb{R}^d)$. [15] study the viscosity solution of the PDE similar to (24). Main differences between our model and the model in [15] are as follows.

1. [15] consider that a generator of RBSDE for Y^i depends on the other Y^j, but we consider the case when it does not depend on the other Y^j.

2. They assume that switching costs are non-negative, but we allow negative switching costs.

3. They assume that a dynamics of the forward variable X does not depend on a control process, but we allow the dynamics of X to depend on the control.

In fact, the results of [15] can be applied to our model. [15] prove the existence and uniqueness of the viscosity solution without using non-negativity of the switching costs. Furthermore, the controllability of X does not affect to their results. Hence, we can provide the existence and uniqueness of the solution to the PDE (17) in the viscosity sense and prove that the value function is a unique viscosity solution to (17).

Proposition 13 *Suppose that Hypotheses 1, 3, 4 and 7 are satisfied. Let*

$$\vec{v} := (v(\cdot,\cdot,1),\dots,v(\cdot,\cdot,I)).$$

Then, $\vec{v}$ is a unique viscosity solution of the PDE (24) in $(\mathcal{CP}([0,T]\times\mathbb{R}^d))^I$.

Proof of Proposition 13. Let $(t,x,i)\in\overline{\mathcal{K}_T}$. Let $(Y^{t,x,i,n})_{n\geq 0}$ be a sequence of the Picard's iterations defined in Theorem 8. Then, by [12], there exists $v_n(\cdot,\cdot,i)\in\mathcal{CP}([0,T]\times\mathbb{R}^d)$ for all $n\geq 0$ and $i\in\mathcal{I}$ such that

$$Y_s^{t,x,i,n} = v_n(t, X_s^{t,x,i}, i),$$

for all $s\in[t,T]$. Furthermore, we define $\overline{v}\in\mathcal{CP}([0,T]\times\mathbb{R}^d)$ as

$$\overline{v}(t,x) := M_t^{t,x},$$

where $M^{t,x}$ is defined in Theorem 8. Recall that $Y^{t,x,i,n}\to Y^{t,x,i}$ in the mean-square sense. Therefore, $\vec{v}$ is a lower semi-continuous function and it satisfies the polynomial growth condition with respect to x since $v_0\leq v_n\leq\overline{v}$ and $v_n\leq v_{n+1}$ for all $n\geq 1$.

On the other hand, Corollary 1 in [15] provides the continuity and uniqueness of a viscosity solution to the PDE (17). Furthermore, by Theorem 1 in [15], $\vec{v}$ is a viscosity solution of the PDE (17). Hence, we conclude that $\vec{v}$ is a unique viscosity solution of the PDE (17) in $(\mathcal{CP}([0,T]\times\mathbb{R}^d))^I$. □

5 The Infinite Horizon Problem

In this section, we consider the infinite horizon optimal switching problem under ambiguity. Let $\mathbb{A}_i[\nu,\infty)$ be a set of admissible controls like (1) but $\tau_k\to\infty$ $\mathbb{P}$-almost surely. Furthermore, we assume as follows.

Hypothesis 14

1. *Time-homogeneity.* $b,\sigma,\psi,\phi,\varsigma,$ *and* c *do not depend on* t. *There exists a positive constant* ρ *such that*

$$\rho(t,x,i)=\rho>0,$$

for all $(t,x,i)\in[0,\infty)\times\mathbb{R}^d\times\mathcal{I}$. Θ_t *only depends on the values of* X_t *and* α_t. *We denote* Θ_t *with* $X_t=x\in\mathbb{R}^d$ *and* $\alpha_t=i\in\mathcal{I}$ *by* $\Theta^{x,i}$.

2. *Sufficiently large discount.* ρ *is sufficiently large in the following sense. There exist constants* $C \geq 0$ *and* $c_\infty > 0$ *such that*

$$\mathbb{E}\left[e^{-\rho t}\zeta_t^{\theta,0}\|X_t^{x,i,\alpha}\|^q\right] \leq C(1+\|x\|^q)e^{-c_\infty t}, \tag{25}$$

$$\mathbb{E}\left[\sup_{s\geq t} e^{-\rho s}\|X_s^{x,i,\alpha}\|^q\right] \leq C(1+\|x\|^q)e^{-c_\infty t}, \tag{26}$$

for all $(t,x,i) \in [0,\infty)\times\mathbb{R}^d\times\mathcal{I}$, $\theta \in \Theta[0,\infty)$ *and* $\alpha \in \mathbb{A}_i[0,\infty)$*, where* $X^{x,i,\alpha}$ *is a solution to the SDE* (2) *starting at* $X_0^{x,i,\alpha} = x$ *and controlled by* $\alpha \in \mathbb{A}_i[0,\infty)$.

3. *Polynomial growth conditions.* ψ, ϕ *and* c *are continuous and satisfy the polynomial growth condition in Hypothesis 4.2.*

4. *Non-negative reward condition.*

$$\psi(x,i) - \varsigma(x,i,0) \geq 0, \tag{27}$$

for all $(x,i) \in \mathbb{R}^d \times \mathcal{I}$.

5. *Temporary terminal condition. There exist polynomial growth functions* $g(x,1), \ldots, g(x,I)$ *such that*

   (a)

$$g(x,i) \leq 0, \tag{28}$$

   for all $i \in \mathcal{I}$ *and* $x \in \mathbb{R}^d$*;*

   (b)

$$g(x,i) \geq \max_{j\in\mathcal{I}\setminus\{i\}}\{g(x,j) - c_{i,j}(x)\}, \tag{29}$$

   for all $i \in \mathcal{I}$ *and* $x \in \mathbb{R}^d$*;*

   (c)

$$\inf_{\theta_t\in\Theta[T,\widetilde{T})} \mathbb{E}\left[e^{-\rho\widetilde{T}}\zeta_{\widetilde{T}}^{\theta,T} g(X_{\widetilde{T}}^{\nu,\eta,i}, i) \,\middle|\, \mathcal{F}_T\right] \geq e^{-\rho T} g(X_T^{\nu,\eta,i}, i), \tag{30}$$

   for all $0 \leq T \leq \widetilde{T}$, $\nu \in \mathcal{T}_0^T$, $\eta \in L_\nu^{2q}(\mathbb{R}^d)$ *and* $i \in \mathcal{I}$.

6. *Non-free loop condition in the infinite horizon. For all finite loop* $(i_0, i_1, \ldots, i_m) \in \mathcal{I}^{m+1}$ *with* $i_0 = i_m$ *and* $i_0 \neq i_1$ *and for all* $x \in \mathbb{R}^d$*,* c *satisfies*

$$c_{i_0,i_1}(x) + \cdots + c_{i_{m-1},i_m}(x) > 0.$$

7. *Strong triangular condition in the infinite horizon.*

$$c_{k,j}(x) \leq c_{k,i}(x) - C_i(1 + C_{q,X}^{\infty}(1+\|x\|^q)),$$

for all $i \in \mathcal{N}$, $(j,k) \in \mathcal{I}$ *and* $x \in \mathbb{R}^d$ *with* $j \neq i$ *and* $k \neq j$*, where* $C_i, C_{q,X}^{\infty}$ *and* q *are defined in Proposition 2 and Hypothesis 4.*

The time-homogeneity (Hypothesis 14.1) is a standard condition. With taking account of the time-homogeneity and the Markov property of X, the starting time does not matter to the optimal switching problem. The sufficiently large discount condition (Hypothesis 14.2) is also standard. If it is not postulated, then the value function can diverge. Therefore, we need this condition to consider meaningful problems. However, the condition (25) is slightly strong. Indeed, it is sufficient to satisfy (25) with $\theta = 0$ and (26) in order to prove the finiteness of the value function (Proposition 15). The condition (25) is needed to prove the convergent property of the value function from the finite horizon to the infinite horizon (Proposition 18).

Under the non-negative reward condition (Hypothesis 14.4), the rewards of the optimal switching problem in the infinite horizon is non-negative. Indeed, by the definition of ς, we have

$$\psi(X_t^{x,i,\alpha}, \alpha_t) - \theta_t' \phi(X_t^{x,i,\alpha}, \alpha_t) \geq \psi(X_t^{x,i,\alpha}, \alpha_t) - \varsigma(X_t^{x,i,\alpha}, \alpha_t, 0) \geq 0,$$

for all $(t, x, i) \in [0, \infty) \times \mathbb{R}^d \times \mathcal{I}$, $\theta_t \in \Theta_t$ and $\alpha \in \mathbb{A}_i[0, \infty)$. The non-negative reward condition guarantees that an optimal switching problem in a longer finite horizon has a large value function. This restriction is needed to exchange the orders of taking limits of Picard's iterations n and time horizons T. This is slightly restrictive, however, it can be replaced to a lower bounded condition (Remark 16).

The temporary terminal conditions (Hypothesis 14.5) are assumed for purely technical reasons. However, they are not so restrictive. If all switching costs are non-negative, then we can choose $g(x, i) = 0$ for all $(x, i) \in \mathbb{R}^d \times \mathcal{I}$ satisfying all the temporary terminal conditions. Once we find the constants $g_1, \ldots, g_I$ satisfying the inequality (29), then $g_1 - \max_{j \in \mathcal{I}} g_j, \ldots, g_I - \max_{j \in \mathcal{I}} g_j$ satisfy all the temporary terminal conditions. If $g(x, i)$ satisfies the inequalities (28) and (29) and if $g(\cdot, i)$ is twice continuously differentiable for all $i \in \mathcal{I}$, then one of sufficient conditions to satisfy the inequality (30) is

$$\mathcal{L}^i g(x, i) - \rho g(x, i) - (\nabla g(x, i))' \sigma(x, i) \theta \geq 0, \tag{31}$$

for all $(x, i) \in \mathbb{R}^d \times \mathcal{I}$ and $\theta \in \Theta^{x,i}$. The condition (31) can be derived by applying the Ito's lemma to $e^{-\rho t} \zeta_t^\theta g(X_t, i)$. If the switching costs are constants, we can easily find the constants satisfying the temporary terminal conditions. On the other hand, in the major applications such as the buy low and sell high problem and the pair-trading problem, we can also find the functions satisfying the temporary terminal conditions. The other assumptions are essentially the same as the finite horizon problem.

The objective function in the infinite horizon is

$$J(x, i, \alpha) = \inf_{\theta \in \Theta[0,\infty)} \mathbb{E}\Bigg[\int_0^\infty e^{-\rho t} \zeta_t^\theta \Big(\psi(X_t^{x,i,\alpha}, \alpha_t) - \theta_t' \phi(X_t^{x,i,\alpha}, \alpha_t)\Big) \mathrm{d}t - \sum_{k=1}^\infty e^{-\rho \tau_k} \zeta_{\tau_k}^\theta c_{i_{k-1}, i_k}(X_{\tau_k}^{x,i,\alpha})\Bigg],$$

for $(x, i) \in \mathbb{R}^d \times \mathcal{I}$ and $\alpha \in \mathbb{A}_i[0, \infty)$. The optimal switching problem under ambiguity in the infinite horizon is

$$v^\infty(x, i) := \sup_{\alpha \in \mathbb{A}_i[0,\infty)} J(x, i, \alpha), \tag{32}$$

for $(x, i) \in \mathbb{R}^d \times \mathcal{I}$. We can easily show that v^∞ is polynomial growth with respect to x.

Proposition 15 *Under Hypotheses 1 and 14, there exists a positive constant C such that*

$$0 \leq v^\infty(x, i) \leq C(1 + \|x\|^q),$$

for all $x \in \mathbb{R}^d$ and $i \in \mathcal{I}$. Thus, v^∞ is polynomial growth with respect to x.

Proof of Proposition 15. It is clear that v^∞ is non-negative by the non-negative reward condition. Fix an arbitrary $x \in \mathbb{R}^d$ and $i \in \mathcal{I}$. Then, by the polynomial growth condition of ψ and c and the strong triangular condition, we have

$$J(x,i,\alpha) \leq \mathbb{E}\left[\int_0^\infty e^{-\rho t}\psi(X_t^{x,i,\alpha},\alpha_t)\mathrm{d}t - \sum_{k=1}^\infty e^{-\rho\tau_k} c_{i_{k-1},i_k}(X_{\tau_k}^{x,i,\alpha})\right] \leq C(1+\|x\|^q),$$

for all $\alpha \in \mathbb{A}_i[0,\infty)$, where C is the positive constant not depending on x, i and α. Hence, we obtain the desired result. □

Remark 16 *Hypothesis 14.6 (the inequality (27)) can be replaced to a lower bounded condition. We assume that there exists some constant $c_{\psi,\varsigma}$ such that*

$$\psi(x,i) - \varsigma(x,i,0) \geq c_{\psi,\varsigma},$$

for all $(x,i) \in \mathbb{R}^d \times \mathcal{I}$. Then,

$$\begin{aligned} J(x,i,\alpha) - \frac{c_{\psi,\varsigma}}{\rho} &= J(x,i,\alpha) - \int_0^\infty e^{-\rho t} c_{\psi,\varsigma}\mathrm{d}t \\ &= \inf_{\theta\in\Theta[0,\infty)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\zeta_t^\theta\Big(\psi(X_t^{x,i,\alpha},\alpha_t) - \theta_t'\phi(X_t^{x,i,\alpha},\alpha_t) - c_{\psi,\varsigma}\Big)\mathrm{d}t\right. \\ &\qquad\left. - \sum_{k=1}^\infty e^{-\rho\tau_k}\zeta_{\tau_k}^\theta c_{i_{k-1},i_k}(X_{\tau_k}^{x,i,\alpha})\right], \end{aligned}$$

for all $(x,i) \in \mathbb{R}^d \times \mathcal{I}$ and $\alpha \in \mathbb{A}_i[0,\infty)$. By the definition ς, we have

$$\psi(X_t^{x,i,\alpha},\alpha_t) - \theta_t'\phi(X_t^{x,i,\alpha},\alpha_t) - c_{\psi,\varsigma} \geq \psi(X_t^{x,i,\alpha},\alpha_t) - \varsigma(X_t^{x,i,\alpha},\alpha_t,0) - c_{\psi,\varsigma} \geq 0,$$

for all $(t,x,i) \in [0,\infty)\times\mathbb{R}^d\times\mathcal{I}$, $\theta_t \in \Theta_t$ and $\alpha \in \mathbb{A}_i[0,\infty)$. Hence, we can replace the original rewards to non-negative rewards. $c_{\psi,\varsigma}$ may be negative, but it is finite.

Remark 17 *Similarly to Remark 6, the strong triangular condition in the infinite horizon is not necessarily needed. Instead of the strong triangular condition, it is sufficient to hold the following inequality*

$$\mathbb{E}\left[-\sum_{k=1}^\infty e^{-\rho\tau_k} c_{i_{k-1},i_k}(X_{\tau_k}^{x,i,\alpha})\right] \leq C(1+\|x\|^q),$$

for all $x \in \mathbb{R}^d$, $i \in \mathcal{I}$ and $\alpha \in \mathbb{A}_i[0,\infty)$, where C is a positive constant not depending on (x,i) and α. Furthermore, if the above inequality is satisfied, then we do not also need the inequality (26).

We consider the following multidimensional RBSDE on $[\nu,T]$ for $\nu \in \mathcal{T}_0^T$ and $\eta \in L_\nu^{2q}(\mathbb{R}^d)$,

$$(33)\qquad \begin{aligned} -\mathrm{d}\widehat{Y}_t^{T,\nu,\eta,i} &= \Big(\psi(X_t^{\nu,\eta,i},i) - \rho\widehat{Y}_t^{T,\nu,\eta,i} - \varsigma(X_t^{\nu,\eta,i},i,\widehat{Z}_t^{T,\nu,\eta,i})\Big)\mathrm{d}t \\ &\qquad - (\widehat{Z}_t^{T,\nu,\eta,i})'\mathrm{d}W_t + \mathrm{d}\widehat{K}^{T,\nu,\eta,i},\ t \in [\nu,T], \\ \widehat{Y}_T^{T,\nu,\eta,i} &= g(X_T^{\nu,\eta,i},i), \quad \widehat{K}_\nu^{T,\nu,\eta,i} = 0, \\ \widehat{Y}_t^{T,\nu,\eta,i} &\geq \max_{j\in\mathcal{I}\setminus\{i\}}\left\{\widehat{Y}_t^{T,\nu,\eta,j} - c_{i,j}(X_t^{\nu,\eta,i})\right\},\ t \in [\nu,T], \\ &\int_0^T \Big(\widehat{Y}_t^{T,\nu,\eta,i} - \max_{j\in\mathcal{I}\setminus\{i\}}\left\{\widehat{Y}_t^{T,\nu,\eta,j} - c_{i,j}(X_t^{\nu,\eta,i})\right\}\Big)\mathrm{d}t = 0, \\ &(\widehat{Y}^{T,\nu,\eta,i},\widehat{Z}^{T,\nu,\eta,i},\widehat{K}^{T,\nu,\eta,i}) \in \mathbb{S}^2[\nu,T]\times\mathbb{H}_d^2[\nu,T]\times\mathbb{K}^2[\nu,T], \quad i \in \mathcal{I}, \end{aligned}$$

where g is the function satisfying the temporary terminal conditions. By Theorem 8 and Proposition 10, there exists a unique minimum solution of the multidimensional RBSDE (33). Now, we show that the solution to the multidimensional RBSDE (33) converges to the value function (32) as $T \to \infty$.

Proposition 18 *Under Hypotheses 1, 3, 7 and 14,* $\widehat{Y}_t^{T,\nu,\eta,\iota} \le \widehat{Y}_t^{\widetilde{T},\nu,\eta,\iota}$ *for all* $\nu \in \mathcal{T}_0^T$, $\nu \le t \le T \le \widetilde{T}$, $\eta \in L_\nu^{2q}(\mathbb{R}^d)$ *and* $\iota \in \widetilde{\mathcal{I}}_\nu$. *Furthermore, for all* $(t,x,i) \in [0,\infty) \times \mathbb{R}^d \times \mathcal{I}$,

$$\lim_{T\to\infty} \widehat{Y}_t^{T,t,x,i} = v^\infty(x,i). \tag{34}$$

Finally, $v^\infty(\cdot,i)$ *is continuous for all* $i \in \mathcal{I}$.

Since the proof of Proposition 18 is too long, we put it in appendix Appendix C.

We next study the relationships between v^∞ and PDE. Consider the following PDE.

$$\begin{aligned}\min\{-\mathcal{L}^i u(x,i) - \psi(x,i) + \rho u(x,i) + \varsigma(x,i,\sigma'(x,i)\nabla u(x,i)),\\ u(x,i) - \max_{j\in\mathcal{I}\setminus\{i\}}\{u(x,j) - c_{i,j}(x)\}\} = 0, \quad (x,i) \in \mathbb{R}^d \times \mathcal{I},\end{aligned} \tag{35}$$

where

$$\mathcal{L}^i f(x) = (\nabla f(x))' b(x,i) + \frac{1}{2}\mathrm{tr}\left(\sigma\sigma'(x,i)\frac{\partial f(x)}{\partial x \partial x'}\right).$$

Then the following proposition holds.

Proposition 19 *Under Hypotheses 1, 3, 7 and 14,* v^∞ *is a viscosity solution of the PDE (35).*

The proof of Proposition 19 is in appendix Appendix D. By Proposition 19, we can study the optimal switching problem under ambiguity through the PDE (35). Moreover, we can easily show the uniqueness of the solution to the PDE (35) using the method of Proposition 3.1 in [15], so we omit the proof of the uniqueness.

6 Financial Applications

6.1 Monotone Conditions

We first prove that under certain conditions, the optimal switching problem under ambiguity can be interpreted as the optimal switching problem with a shift of the drift of X not depending on its value function. We first assume the followings.

Hypothesis 20 *Monotone conditions. We assume* $d = 1$.

1. *κ-ignorance. There exist non-negative constants* $\kappa_1, \dots, \kappa_I$ *such that*

   $$\Theta_t^{x,i} = [-\kappa_i, \kappa_i],$$

   for all $i \in \mathcal{I}$, $x \in \mathbb{R}^d$ *and* $t \in [0,\infty)$.

2. *For every* $x, y \in \mathbb{R}$, X *satisfies,*

   $$x \le y \quad \Rightarrow \quad X_s^{t,x,i} \le X_s^{t,y,i}, \ \mathbb{P}\text{-a.s.},$$

   for all $t, s \in [0,T]$, $i \in \mathcal{I}$ *with* $t \le s$.

3. ρ *does not depend on a value of* x.

4. For every $(t,i)\in[0,T]\times\mathcal{I}$, $\psi(t,\cdot,i)$ *is non-decreasing.*

5. $\phi(t,x,i)=0$ *for every* $(t,x,i)\in\overline{\mathcal{K}_T}$.

6. For every $i\in\mathcal{I}$, $g(\cdot,i)$ *is non-decreasing.*

7. For every $(t,i,j)\in[0,T]\times(\mathcal{I})^2$, $c_{i,j}(t,\cdot)$ *is non-increasing.*

[5] call Hypothesis 20.1 κ-ignorance. The other conditions guarantee the monotonicity of the value function with respect to the initial value of X. Under Hypothesis 20, we can prove the following result.

Proposition 21 *Suppose that Hypotheses 1, 3, 4, 7 and 20 are satisfied. For all* $(t,x,i)\in\overline{\mathcal{K}_T}$ *and* $\alpha\in\mathbb{A}_i[t,T]$, *let* ${}^{-\kappa}X^{t,x,i,\alpha}$ *be a solution to the following SDE,*

$$\begin{aligned}\mathrm{d}^{-\kappa}X_s^{t,x,i,\alpha}&=\Big(b(s,{}^{-\kappa}X_s^{t,x,i,\alpha},\alpha_s)-\kappa_{\alpha_s}|\sigma(s,{}^{-\kappa}X_s^{t,x,i,\alpha},\alpha_s)|\Big)\mathrm{d}s+\sigma(s,{}^{-\kappa}X_s^{t,x,i,\alpha},\alpha_s)\mathrm{d}W_s,\\ {}^{-\kappa}X_t^{t,x,i,\alpha}&=x.\end{aligned}$$

Then, the value function $v(t,x,i)$ *satisfies*

$$\begin{aligned}(36)\quad v(t,x,i)=\sup_{\alpha\in\mathbb{A}_i[t,T]}\mathbb{E}\Big[\int_t^T {}^{-\kappa}D_s^{t,i,\alpha}\psi(s,{}^{-\kappa}X_s^{t,x,i,\alpha},\alpha_s)\mathrm{d}s+{}^{-\kappa}D_T^{t,i,\alpha}g({}^{-\kappa}X_T^{t,x,i,\alpha},\alpha_T)\\ -\sum_{t\le\tau_k\le T}{}^{-\kappa}D_{\tau_k}^{t,i,\alpha}c_{i_{k-1},i_k}(\tau_k,{}^{-\kappa}X_{\tau_k}^{t,x,i,\alpha})\;\Big|\;\mathcal{F}_t\Big],\end{aligned}$$

where

$${}^{-\kappa}D_s^{t,i,\alpha}=\exp\left\{-\int_t^s\rho(u,\alpha_u)\mathrm{d}u\right\},\qquad s\in[t,T].$$

Furthermore, $x\to v(t,x,i)$ *is non-decreasing for all* $(t,i)\in[0,T]\times\mathcal{I}$.

Proof of Proposition 21. By the κ-ignorance and $\phi=0$, we have

$$\varsigma(t,x,i,z)=\kappa_i|z|,$$

for all $(t,x,i,z)\in\overline{\mathcal{K}_T}\times\mathbb{R}$. Now, fix an arbitrary $t\in[0,T]$ and $x,\widetilde{x}\in\mathbb{R}$ with $x\le\widetilde{x}$. Then, by the monotone conditions 2-6, we have

$$\psi(s,X_s^{t,x,i},i)-\rho(s,i)y-\kappa_i|z|\le\psi(s,X_s^{t,\widetilde{x},i},i)-\rho(s,i)y-\kappa_i|z|,\tag{37}$$

$$g(X_T^{t,x,i},i)\le g(X_T^{t,\widetilde{x},i},i),\tag{38}$$

for all $(s,i,y,z)\in[t,T]\times\mathcal{I}\times\mathbb{R}\times\mathbb{R}$. Furthermore, by the monotone conditions 2 and 7, we have

$$\max_{j\in\mathcal{I}\setminus\{i\}}\Big\{y^j-c_{i,j}(s,X_s^{t,x,i})\Big\}\le\max_{j\in\mathcal{I}\setminus\{i\}}\Big\{\overline{y}^j-c_{i,j}(s,X_s^{t,\widetilde{x},i})\Big\},\tag{39}$$

for all $(s,i)\in[t,T]\times\mathcal{I}$ and $(y^1,\dots,y^I),(\overline{y}^1,\dots,\overline{y}^I)\in\mathbb{R}^I$ with $y^k\le\overline{y}^k$ for all $k\in\mathcal{I}$. Let $(Y^{t,x,i,n})_{i\in\mathcal{I},\,n\ge0}$ and $(Y^{t,\widetilde{x},i,n})_{i\in\mathcal{I},\,n\ge0}$ be the Picard's iterations defined in Theorem 8 with starting x and $\widetilde{x}$, respectively. Then, by the inequalities (37) to (39), recursively applying the comparison theorem leads to that

$$Y_s^{t,x,i,n}\le Y_s^{t,\widetilde{x},i,n},$$

for all $i \in \mathcal{I},\ s \in [t,T]$ and $n \geq 0$. Taking a limit of the above inequality, we have

$$v(t,x,i) = Y_t^{t,x,i} \leq Y_t^{t,\widetilde{x},i} = v(t,\widetilde{x},i), \tag{40}$$

for all $i \in \mathcal{I}$. Since we arbitrarily choose t, x and $\widetilde{x}$ with $x \leq \widetilde{x}$, the inequality (40) implies that a mapping $x \to v(t,x,i)$ is non-decreasing for all $t \in [0,T]$ and $i \in \mathcal{I}$.

Now, let us consider the following PDE,

$$\begin{aligned}
&\min\{-w_t(t,x,i) - \mathcal{L}^{-\kappa,i}w(t,x,i) - \psi(t,x,i) + \rho(t,i)w(t,x,i),\\
&\qquad w(t,x,i) - \max_{j\in\mathcal{I}\setminus\{i\}}\{w(t,x,j) - c_{i,j}(t,x)\}\} = 0, \quad (t,x,i) \in \overline{\mathcal{K}_T},\\
&w(T,x,i) = g(x,i),
\end{aligned} \tag{41}$$

where

$$\mathcal{L}^{-\kappa,i}f(t,x) = (b(t,x,i) - \kappa_i|\sigma(t,x,i)|)\nabla f(t,x) + \frac{1}{2}(\sigma(t,x,i))^2\frac{\partial^2 f(t,x)}{\partial x^2}.$$

The PDE (41) has a unique continuous viscosity solution. Let $(t,x) \in [0,T) \times \mathbb{R}$ and let $\varphi \in C^{1,2}([0,T)\times\mathbb{R}\times\mathcal{I})$ be a test function such that $v(\cdot,\cdot,i) - \varphi(\cdot,\cdot,i)$ attains a local minimum at (t,x) for all $i \in \mathcal{I}$. Since $y \to v(s,y,j)$ is monotone non-decreasing for all $(s,j) \in [0,T)\times\mathcal{I}$, we have $\nabla\varphi(t,x,i) \geq 0$ for all $i \in \mathcal{I}$. Since v is the viscosity supersolution of the PDE (24) by Proposition 13, we have

$$\begin{aligned}
&\min\{-\varphi_t(t,x,i) - \mathcal{L}^{-\kappa,i}\varphi(t,x,i) - \psi(t,x,i) + \rho(t,i)v(t,x,i),\\
&\qquad v(t,x,i) - \max_{j\in\mathcal{I}\setminus\{i\}}\{v(t,x,j) - c_{i,j}(t,x)\}\}\\
&= \min\{-\varphi_t(t,x,i) - \mathcal{L}^{i}\varphi(t,x,i) - \psi(t,x,i) + \rho(t,i)v(t,x,i) + \kappa|\sigma(t,x,i)\nabla\varphi(t,x,i)|,\\
&\qquad v(t,x,i) - \max_{j\in\mathcal{I}\setminus\{i\}}\{v(t,x,j) - c_{i,j}(t,x)\}\} \geq 0,
\end{aligned}$$

for all $i \in \mathcal{I}$. Hence, v is a viscosity supersolution of the PDE (41). The comparison theorem of the viscosity solutions gives $v \geq w$. Using the similar argument, we also have $v \leq w$. Thus, $v = w$. Since a value function of the optimal switching problem in the right hand side of our desired equality (36) is a unique viscosity solution of the PDE (41), we obtain the equality (36). □

In the infinite horizon case, Proposition 21 also holds under the same conditions as Hypothesis 20. Proposition 21 implies that under the monotone conditions, the optimal switching problem under ambiguity can be regarded as usual optimal switching problems. Thus, we can use existing results in the literature of the optimal switching if the monotone conditions are satisfied. In fact, under the monotone conditions, it is sufficient to solve the PDE (41) instead of the PDE (24) to derive the value function.

The monotone conditions and Proposition 21 are very similar to the results of [6]. [6] study the optimal stopping problem under ambiguity and show that if a payoff function $f(t,x)$ is non-decreasing in x and κ-ignorance is satisfied, then the optimal stopping problem under ambiguity can be regarded as the standard optimal stopping problem in which the drift of X shifts into $b - \kappa|\sigma|$ (Theorem 4.1 in [6]). Our result implies that the optimal switching problem under ambiguity holds the same property as the optimal stopping under ambiguity.

In sections 6.2 and 6.3, we consider two applications of the optimal switching problem under ambiguity in finance. The first application in section 6.2 is a selection of investment funds and it satisfies the monotone conditions. However, the second application (the buy low and sell high problem) in section 6.3 does not satisfy the monotone conditions and it definitely needs negative switching costs.

6.2 Selection of Investment Funds

In this section, we consider an optimal selection of two investment funds under ambiguity in the infinite horizon. Let $d = 1$ and $\mathcal{I} = \{1, 2\}$. Assume that X satisfies the following SDE,

$$\mathrm{d}X_t = b_{\alpha_t} X_t \mathrm{d}t + \sigma_{\alpha_t} X_t \mathrm{d}W_t, \tag{42}$$

where $b_i \in \mathbb{R},\ \sigma_i > 0,\ i = 1, 2$ are constants. The solution to the SDE (42) is

$$X_t^{x,i,\alpha} = x \exp\left\{ \int_0^t \left(b_{\alpha_s} - \frac{1}{2}\sigma_{\alpha_s}^2 \right) \mathrm{d}s + \int_0^t \sigma_{\alpha_s} \mathrm{d}W_s \right\},$$

for all $\alpha \in \mathbb{A}_i[0, \infty)$. Assume that $\phi = 0$ and that ψ is

$$\psi(x) = x^p, \quad x \in [0, \infty),\ 0 < p < 1.$$

The switching costs $c_{1,2}$ and $c_{2,1}$ are constants over x, and they satisfy $c_{1,2} + c_{2,1} > 0$. The constant discount rate ρ, satisfies

$$\rho > p \max_{i \in \mathcal{I}} \left\{ b_i - \frac{1-p}{2}\sigma_i^2 \right\}.$$

The set of multiple priors is

$$\Theta^{x,i} = [-\kappa_i, \kappa_i], \quad \kappa_i \geq 0,$$

for all $x \in \mathbb{R}$ and $i \in \mathcal{I}$. In the above settings, an optimal switching problem of interest is

$$v^\infty(x, i) = \sup_{\alpha \in \mathbb{A}_i[0,\infty)} \inf_{\theta \in \Theta[0,\infty)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} \zeta_t^{\theta,0} (X_t^{x,i,\alpha})^p \mathrm{d}t - \sum_{k=1}^\infty e^{-\rho\tau_k} \zeta_{\tau_k}^{\theta,0} c_{i_{k-1},i_k} \right]. \tag{43}$$

Since the problem (43) satisfies Hypotheses 1, 3, 7 and 14, we can use the results in section 5. Furthermore, the problem (43) also satisfies the monotone conditions (Hypothesis 20).

Without ambiguity (i.e., $\kappa_i = 0$ for all $i \in \mathcal{I}$), the problem (43) is well studied by [20]. We shortly summarize their results as follows.

Proposition 22 (Theorem 4.1 in [20]) *Let*

$$K_i = \frac{1}{\rho - b_i p + \frac{1}{2}\sigma_i^2 p(1-p)}, \tag{44}$$

for all $i \in \mathcal{I}$. Let $i, j \in \mathcal{I},\ i \neq j$.

1. *If $K_i = K_j$, then it is always optimal to switch from regime i to j if the corresponding switching cost is non-positive, and never optimal to switch otherwise.*
2. *If $K_j > K_i$, then the following switching strategies depending on the switching costs are optimal.*
   - (a) *$c_{i,j} \leq 0$: it is always optimal to switch from regime i to j if one first stands in i and it is always optimal not to switch from j to i otherwise.*
   - (b) *$c_{i,j} > 0$:*
     - i. *$c_{j,i} \geq 0$: there exists $\underline{x}_i^* \in [0, \infty)$ such that if one first stands in regime i, then it is optimal to switch from i to j whenever X exceeds $\underline{x}_i^*$. If one first stands in regime j, then it is optimal not to switch from j to i.*

ii. $c_{j,i} < 0$*: there exist* $\underline{x}_i^*, \overline{x}_j^* \in [0, \infty)$ *with* $\overline{x}_j^* < \underline{x}_i^*$ *such that if one first stands in regime* i*, then it is optimal to switch from* i *to* j *whenever* X *exceeds* $\underline{x}_i^*$ *and that if one first stands in regime* j*, then it is optimal to switch from* j *to* i *whenever* X *falls below* $\overline{x}_j^*$*.*

For details of $\underline{x}_i^*$ and $\overline{x}_j^*$ and the functional form of the value function, we refer to [20]. By Proposition 22, the types of the switching strategies are determined by K_i defined in (44) and the switching costs. The most interesting case is Proposition 22.2.(b).ii in which the decision maker continuously switches the regimes.

The problem (43) can be interpreted as an optimal selection of investment funds. An investor chooses a fund to maximize her expected utility with multiple priors. The switching costs are interpreted as costs or benefits in changing funds.

We now assume $K_2 > K_1$ and $c_{1,2} > 0 > c_{2,1}$. Then, heuristically speaking, the fund 2 (regime 2) is more attractive than the fund 1 (regime 1), but one requires the positive switching cost $c_{1,2}$ to switch from the fund 1 to the fund 2. On the other hand, one gets the switching benefit $-c_{2,1}$ when switching from the fund 2 to the fund 1. We can also interpret the fund 2 as a new fund well performing and the fund 1 as an old fund less performing. To obtain customers, the fund 1 begins the campaign that one switching from the fund 2 to the fund 1 obtains the benefit $-c_{2,1}$. Then, the investor has a motivation switching between the fund 1 and 2.

However, in practice, the investor may doubt the good performance of the fund 2 since the fund 2 is new and less experienced. She therefore considers that the fund 2 has a premium of ambiguity. Mathematically, this implies that $\kappa_2 > 0$ and $\kappa_1 = 0$. We now consider the case that $\kappa_2 > 0$ and $\kappa_1 = 0$.

Since the problem (43) satisfies the monotone conditions, we can use the results of [20]. Let

$$K_2^\kappa = \frac{1}{\rho - (b_2 - \kappa_2\sigma_2)p + \frac{1}{2}\sigma_2^2 p(1-p)} > 0.$$

Then, we have

$$K_2^\kappa - K_1 = \frac{pK_2^\kappa K_1}{2}\Big((1-p)(\sigma_1^2 - \sigma_2^2) - 2(b_1 - b_2) - 2\kappa_2\sigma_2\Big).$$

Therefore, the sign of $(1-p)(\sigma_1^2 - \sigma_2^2) - 2(b_1 - b_2) - 2\kappa_2\sigma_2$ determines the type of the switching strategy. On the other hand, we have

$$K_2 - K_1 = \frac{pK_2K_1}{2}\Big((1-p)(\sigma_1^2 - \sigma_2^2) - 2(b_1 - b_2)\Big) > 0.$$

Hence, $(1-p)(\sigma_1^2 - \sigma_2^2) - 2(b_1 - b_2)$ is positive. However, if κ_2 is sufficiently large such that $(1-p)(\sigma_1^2 - \sigma_2^2) - 2(b_1 - b_2) < 2\kappa_2\sigma_2$, then $K_2^\kappa < K_1$. Therefore, the large ambiguity with respect to the fund 2 can change the type of the switching strategy.

To illustrate effects of ambiguity, we conduct a numerical simulation. Let $b_1 = 0.03$, $b_2 = 0.07$, $\sigma_1 = 0.1$, $\sigma_2 = 0.3$, $p = 0.5$, $\rho = 0.03$, $c_{1,2} = 30000$, and $c_{2,1} = -1000$. Then,

$$K_1 = 61.53846\cdots < 160 = K_2.$$

Hence, the investor continuously switches between the fund 1 and 2 without ambiguity. On the other hand, we have

$$\frac{(1-p)(\sigma_1^2 - \sigma_2^2) - 2(b_1 - b_2)}{2\sigma_2} = \frac{1}{15} = 0.0666\cdots.$$

Thus, if $\kappa_2 > 1/15$, then the type of switching strategy changes to that one always chooses the fund 1.

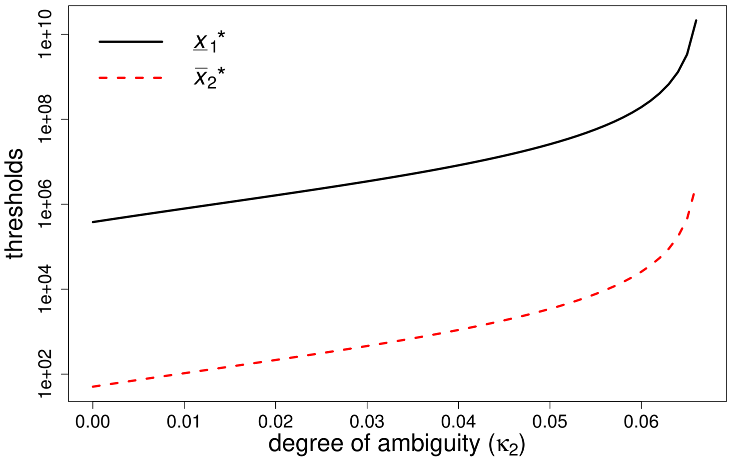

Figure 1: **The Optimal Switching Thresholds in the Selection of Investment Funds.** The vertical axis is a logarithmic scale. $\underline{x}_1^*$ under different κ_2 is plotted in the solid line. $\overline{x}_2^*$ under different κ_2 is plotted in the dashed line.

Figure 1 displays the switching thresholds $\underline{x}_1^*$ and $\overline{x}_2^*$ with different degrees of ambiguity κ_2. If one is investing in the fund 1 at time t and if $X_t \geq \underline{x}_1^*$, then she switches from the fund 1 to the fund 2. On the other hand, if one is investing in the fund 2 at time t and if $X_t \leq \overline{x}_2^*$, then she switches from the fund 2 to the fund 1.

According to Figure 1, in a higher degree of ambiguity κ_2, both of the thresholds $\underline{x}_1^*$ and $\overline{x}_2^*$ are large. This implies that if κ_2 is large, then the investor investing in the fund 1 needs sufficiently large wealth X to switch from the fund 1 to the fund 2. On the other hand, if κ_2 is large, then the investor investing in the fund 2 switches to the fund 1 with smaller wealth than that in small κ_2. Each behavior is well convincing. The large ambiguity makes the fund 2 less attractive, so the investor tends to choose the fund 1.

Remark 23 *If* $\kappa_1 > 0$*, then let*

$$K_1^\kappa = \frac{1}{\rho - (b_1 - \kappa_1\sigma_1)p + \frac{1}{2}\sigma_1^2 p(1-p)} > 0,$$

and

$$K_2^\kappa - K_1^\kappa = \frac{pK_2^\kappa K_1^\kappa}{2}\Big((1-p)(\sigma_1^2 - \sigma_2^2) - 2(b_1 - b_2) - 2(\kappa_2\sigma_2 - \kappa_1\sigma_1)\Big).$$

Hence, in this case, if $(1-p)(\sigma_1^2 - \sigma_2^2) - 2(b_1 - b_2) < 2(\kappa_2\sigma_2 - \kappa_1\sigma_1)$*, then* $K_2^\kappa < K_1^\kappa$*.*

6.3 Buy Low and Sell High

Next, we consider an optimal trading (buy and sell) rule under ambiguity. Without ambiguity, this problem in trading a mean-reverting asset is well studied by [28]. We adopt their settings and consider an optimal trading rule under ambiguity. Let $d = 1$. A trader concerns with trading of a certain asset. A cumulative log return of this asset at time t is denoted by X_t and it satisfies the following SDE.

$$\mathrm{d}X_t = a(b - X_t)\mathrm{d}t + \sigma \mathrm{d}W_t, \tag{45}$$

where $a > 0$, $b \in \mathbb{R}$ and $\sigma > 0$ are constants. Therefore, the asset price at time t is given by $S_t = \exp(X_t)$. We denote the solution to the SDE (45) starting from $X_0 = x$ by X^x.

Furthermore, this asset does not have any dividend and coupon. This implies $\psi = 0$ and $\phi = 0$.

Let $\mathcal{I} = \{1, 2\}$. The regime $i = 1$ means that the trader's position is flat. Hence, she wants to buy the asset at as low a price as possible. The regime $i = 2$ means that the trader's position is long. Hence, she wants to sell the asset at as high a price as possible. If the trader goes from the regime 1 to the regime 2, in other words, if she buys the asset, then the switching cost function is

$$c_{1,2}(x) = e^x(1 + K), \tag{46}$$

where $K \in (0, 1)$ is a constant percentage of slippage or commission per transaction. On the other hand, if the trader goes from the regime 2 to the regime 1, in other words, if she sells the asset, then the cost (benefit) function is

$$c_{2,1}(x) = -e^x(1 - K). \tag{47}$$

The set of multiple priors is

$$\Theta^{x,i} = [-\kappa, \kappa], \quad \kappa \geq 0,$$

for all $x \in \mathbb{R}$ and $i \in \mathcal{I}$. Therefore, we assume κ-ignorance.

The buy low and sell high problem under ambiguity can be interpreted as the following optimal switching problem,

$$v(x, i) = \sup_{\alpha \in \mathbb{A}_i[0,\infty)} \inf_{\theta \in \Theta[0,\infty)} \mathbb{E}\left[-\sum_{k=1}^{\infty} e^{-\rho\tau_k} \zeta^{\theta,0}_{\tau_k} c_{i_{k-1},i_k}(X^x_{\tau_k})\right]. \tag{48}$$

More directly, the problem (48) can be expressed as

$$\begin{aligned}
&v(x, 1) \\
&= \sup_{\alpha \in \mathbb{A}_i[0,\infty)} \inf_{\theta \in \Theta[0,\infty)} \mathbb{E}\left[\sum_{k=1}^{\infty}\left(e^{-\rho\tau_{2k}} \zeta^{\theta,0}_{\tau_{2k}} e^{X^x_{\tau_{2k}}}(1 - K) - e^{-\rho\tau_{2k-1}} \zeta^{\theta,0}_{\tau_{2k-1}} e^{X^x_{\tau_{2k-1}}}(1 + K)\right)\right], \\
&v(x, 2) \\
&= \sup_{\alpha \in \mathbb{A}_i[0,\infty)} \inf_{\theta \in \Theta[0,\infty)} \mathbb{E}\Bigg[e^{-\rho\tau_1} \zeta^{\theta,0}_{\tau_1} e^{X^x_{\tau_1}}(1 - K) \\
&\qquad + \sum_{k=1}^{\infty}\left(e^{-\rho\tau_{2k+1}} \zeta^{\theta,0}_{\tau_{2k+1}} e^{X^x_{\tau_{2k+1}}}(1 - K) - e^{-\rho\tau_{2k}} \zeta^{\theta,0}_{\tau_{2k}} e^{X^x_{\tau_{2k}}}(1 + K)\right)\Bigg].
\end{aligned}$$

The cost/benefit functions (46) and (47) do not satisfy the polynomial growth condition and the strong triangular condition. However, changing variables from X to S, then these functions satisfy the polynomial growth condition. Furthermore, we can easily prove Proposition 5 in the problem (48) (see Lemma 4 in [28] and Remark 17 in this paper). Therefore, we can apply the method in section 5. Note that for sufficiently large constant $C \geq 0$, the following function satisfies the temporary terminal conditions:

$$g(x, i) = -\mathbb{1}_{\{i=1\}} e^x(1 - K) - C.$$

It is easy to show that g satisfies the sufficient condition (31) for sufficiently large C.

According to Proposition 19, the value function v is a viscosity solution of the following system of PDEs.

$$\min\{-\mathcal{L}v(x, 1) + \rho v(x, 1) + \kappa\sigma|\nabla v(x, 1)|, v(x, 1) - v(x, 2) + e^x(1 + K)\} = 0, \tag{49}$$

$$\min\{-\mathcal{L}v(x, 2) + \rho v(x, 2) + \kappa\sigma|\nabla v(x, 2)|, v(x, 2) - v(x, 1) - e^x(1 - K)\} = 0, \tag{50}$$

where

$$\mathcal{L}f(x) = a(b-x)\nabla f(x) + \frac{\sigma^2}{2}\frac{\partial^2 f(x)}{\partial x^2}.$$

Unfortunately, the problem (48) does not satisfy the monotone conditions, therefore we need to solve the system of PDEs (49) and (50). It seems to be difficult to solve this system since it contains the absolute values of the first derivatives of v. However, we can find a continuous solution to the system of PDEs (49) and (50) using the smooth-fit techniques (details of the smooth-fit techniques are in Chapter 5 in [24]).

First, let $\mathcal{C}_1$ be a continuation region of the regime 1 such that

$$\mathcal{C}_1 = (x_1, \infty),$$

for some x_1. Thus, the trader in the flat position buys the asset whenever the asset price falls below e^{x_1}. Also let $\mathcal{C}_2$ be a continuation region of the regime 2 such that

$$\mathcal{C}_2 = (-\infty, x_2),$$

for some x_2. Thus, the trader in the long position sells the asset whenever the asset price exceeds e^{x_2}. Naturally we impose $x_1 \leq x_2$. We assume that

$$\nabla v(x,1) \leq 0,\ \forall x \in \mathcal{C}_1, \quad \text{and} \quad \nabla v(x,2) \geq 0,\ \forall x \in \mathcal{C}_2. \tag{51}$$

By [28], the PDE,

$$-\mathcal{L}V(x,1) + \rho V(x,1) - \kappa\sigma\nabla V(x,1) = 0,$$

on $\mathcal{C}_1$ has a solution such that

$$V(x,1) = C_1\varphi_1(x),$$

where $C_1,\ m = \sqrt{2a}/\sigma$, and $\lambda = \rho/a$ are constants, and

$$\varphi_1(x) = \int_0^\infty t^{\lambda-1}e^{-0.5t^2+m(b+\kappa\sigma/a-x)t}\mathrm{d}t,$$

Similarly, the PDE,

$$-\mathcal{L}V(x,2) + \rho V(x,2) + \kappa\sigma\nabla V(x,2) = 0,$$

on $\mathcal{C}_2$ has a solution such that

$$V(x,2) = C_2\varphi_2(x),$$

where C_2 is a constant and

$$\varphi_2(x) = \int_0^\infty t^{\lambda-1}e^{-0.5t^2-m(b-\kappa\sigma/a-x)t}\mathrm{d}t.$$

Now, let us guess that candidates of the solution to the PDEs (49) and (50) are

$$v(x,1) = \begin{cases} V(x,1), & \text{if } x \in \mathcal{C}_1, \\ V(x,2) - e^x(1+K), & \text{if } x \notin \mathcal{C}_1 \end{cases} \tag{52}$$

$$v(x,2) = \begin{cases} V(x,2), & \text{if } x \in \mathcal{C}_2, \\ V(x,1) + e^x(1-K), & \text{if } x \notin \mathcal{C}_2 \end{cases} \tag{53}$$

Let

$$\varphi_1^*(x) = \int_0^\infty t^{\lambda}e^{-0.5t^2+m(b+\kappa\sigma/a-x)t}\mathrm{d}t, \quad \varphi_2^*(x) = \int_0^\infty t^{\lambda}e^{-0.5t^2-m(b-\kappa\sigma/a-x)t}\mathrm{d}t.$$

Then, $\nabla V(x,1) = -mC_1\varphi_1^*(x)$ and $\nabla V(x,2) = mC_2\varphi_2^*(x)$. Hence by the conditions (51), we need $C_1 \geq 0$ and $C_2 \geq 0$. By the smooth-fit conditions, we need

$$\begin{cases} V(x_1,1) = V(x_1,2) - e^{x_1}(1+K), \\ \nabla V(x_1,1) = \nabla V(x_1,2) - e^{x_1}(1+K), \\ V(x_2,2) = V(x_2,1) + e^{x_2}(1-K), \\ \nabla V(x_2,2) = \nabla V(x_2,1) + e^{x_2}(1-K), \end{cases} \tag{54}$$

$$\begin{cases} v(x,1) \geq v(x,2) - e^{x}(1+K), & \text{on } (x_1,\infty), \\ v(x,2) \geq v(x,1) + e^{x}(1-K), & \text{on } (-\infty,x_2), \end{cases} \tag{55}$$

$$\begin{cases} (-\mathcal{L}+\rho+\kappa\sigma|\nabla|)(V(x,2) - e^{x}(1+K)) \geq 0, & \text{on } (-\infty,x_1), \\ (-\mathcal{L}+\rho+\kappa\sigma|\nabla|)(V(x,1) + e^{x}(1-K)) \geq 0, & \text{on } (x_2,\infty). \end{cases} \tag{56}$$

After simple algebraic computation, the equalities (54) can be expressed as

$$\begin{aligned} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} &= e^{x_1}(1+K) \begin{pmatrix} -\varphi_1(x_1) & \varphi_2(x_1) \\ \varphi_1^*(x_1) & \varphi_2^*(x_1) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1/m \end{pmatrix} \\ &= e^{x_2}(1-K) \begin{pmatrix} -\varphi_1(x_2) & \varphi_2(x_2) \\ \varphi_1^*(x_2) & \varphi_2^*(x_2) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1/m \end{pmatrix} \geq 0. \end{aligned} \tag{57}$$

By the definitions of v, the inequalities (55) are equivalent to

$$V(x,1) \geq V(x,2) - e^{x}(1+K), \quad V(x,2) \geq V(x,1) + e^{x}(1-K), \tag{58}$$

on (x_1,x_2). For the first inequality of (56), we have

$$\begin{aligned} (-\mathcal{L}+\rho+\kappa\sigma|\nabla|)(V(x,2) - e^{x}(1+K)) &= (-\mathcal{L}+\rho+\kappa\sigma|\nabla|)(-e^{x}(1+K)) \\ &= -\Big(\rho - a(b-x) - \frac{\sigma^2}{2} - \kappa\sigma\Big)e^{x}(1+K) \geq 0 \end{aligned}$$

on $(-\infty,x_1)$ since $(-\infty,x_1) \subseteq \mathcal{C}_2$. Thus, the condition expressed by the first inequality is equivalent to

$$x_1 \leq \frac{1}{a}\left(\frac{\sigma^2}{2} + ab + \kappa\sigma - \rho\right). \tag{59}$$

Similarly, the condition expressed by the second inequality of (56) is equivalent to

$$x_2 \geq \frac{1}{a}\left(\frac{\sigma^2}{2} + ab - \kappa\sigma - \rho\right). \tag{60}$$

Finally, we need

$$e^{x_2}(1-K) > e^{x_1}(1+K) \Leftrightarrow x_2 - x_1 > \log(1+K) - \log(1-K). \tag{61}$$

Hence, if C_1, C_2, x_1 and x_2 satisfy the conditions (57) to (61), then the candidates of the solutions (52) and (53) are true viscosity solutions to the system of the PDEs (49) and (50).

To illustrate effects of ambiguity, we conduct a numerical simulation. Let $a = 0.8$, $b = 2$, $\sigma = 0.5$, $\rho = 0.5$, and $K = 0.01$. The values of these parameters are the same as [28]. We compute thresholds (x_1,x_2) with different degrees of ambiguity κ.

Figure 2 displays the thresholds. According to Figure 2, in a larger degree of ambiguity, both of the optimal thresholds become small. The long position trader (that is, the initial regime is 2) considers the worst case that the steady mean of X is smaller than that without ambiguity. Therefore, she sells the asset at a lower price than that without ambiguity.

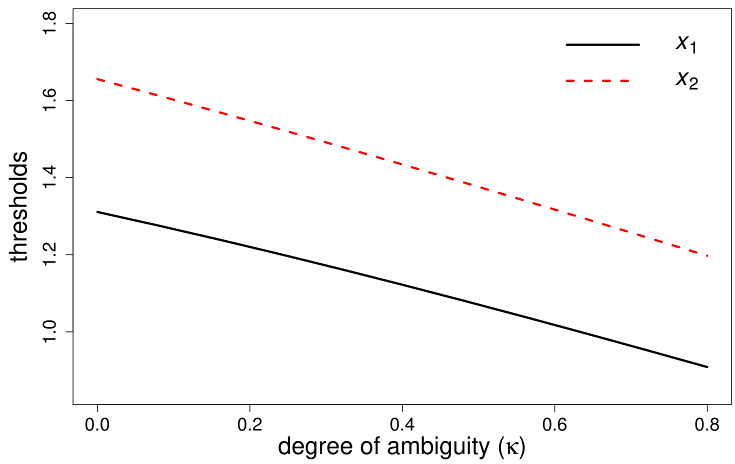

Figure 2: **The Optimal Switching Thresholds in the Buy Low and Sell High Problem.** x_1 under different κ is plotted in the solid line. x_2 under different κ is plotted in the dashed line.

On the other hand, in the flat position case (that is, the initial regime is 1), the trader also buys the asset at a lower price than that without ambiguity. That is because a gain of the trader in the flat position does not realize until she sells the asset. Now, we assume that the trader considers the case when the steady mean of X is larger than that without ambiguity. Then, she can expect a bigger profit in her belief than that in the true probability measure. This is a contradiction since she considers the worst case. Therefore, even if the trader has the flat position, she considers the case that the steady mean of X is smaller than that without ambiguity. Hence, the optimal thresholds of buying the asset under ambiguity is lower than that without ambiguity.

[28] conduct the comparative statics with varying the steady mean of X, i.e., b. Their results are that in a small b, both of the optimal thresholds are also small. These are similar to the results in large ambiguity. However, the results under large ambiguity can not be reproduced by a small b. By the equality (57) with $\kappa = 0$, the optimal thresholds under the steady mean b are equal to the optimal thresholds under the steady mean $\widetilde{b}$ plus $b - \widetilde{b}$ for all $b, \widetilde{b} \in \mathbb{R}$ if the other parameters are the same. Therefore, the optimal thresholds are linear in the steady mean b.

On the other hand, Figure 3 displays equal differences of the optimal thresholds with different degrees of ambiguity. According to Figure 3, the equal differences are not constant, therefore the optimal thresholds are not linear in the degree of ambiguity κ. Our PDEs (49) and (50) cause these non-linearities. The PDEs (49) and (50) can not be expressed as any variational inequality of an optimal switching problem without ambiguity since these do not satisfy the monotone conditions. Indeed, the difference $x_2 - x_1$ without ambiguity is constant over b, whereas $x_2 - x_1$ is small with large κ. Thus, the optimal switching problem under ambiguity can generate this interesting result which can not be reproduced by the problem without ambiguity.

Acknowledgments

The author is grateful to Prof. Masahiko Egami for helpful advices and discussions on this paper.

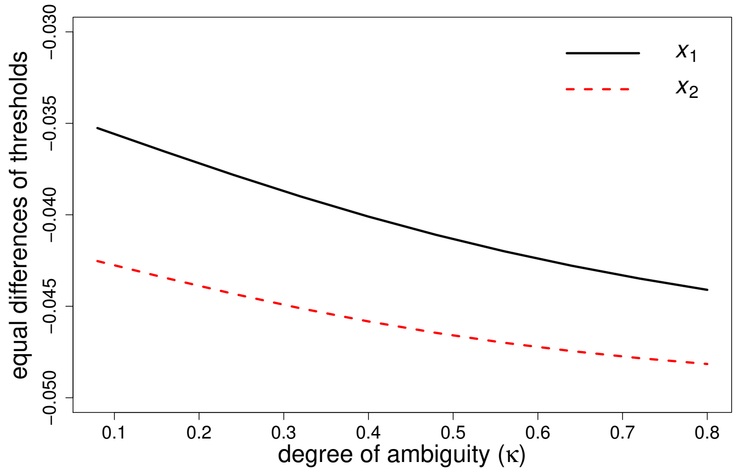

Figure 3: **The Equal Differences of the Optimal Switching Thresholds in the Buy Low and Sell High Problem.** x_1 under different κ is plotted in the solid line. x_2 under different κ is plotted in the dashed line. Each interval of κ is 0.08.

Appendix A The Moment Estimates of X

Proof of Proposition 2. Since $x \to \|x\|^q$ is twice continuously differentiable for all $q \geq 4$, we can apply the Ito's lemma to $\|X_s^{t,x,i,\alpha}\|^q$. Then, for all $s \in [t,T]$, using the quadratic growth condition for b and σ, we have

$$\begin{aligned}\|X_s^{t,x,i,\alpha}\|^q &= \|x\|^q + \int_t^s q\|X_r^{t,x,i,\alpha}\|^{q-2}(X_r^{t,x,i,\alpha})'b(r, X_r^{t,x,i,\alpha}, \alpha_r)\mathrm{d}r \\ &\quad + \frac{1}{2}\int_t^s \Big(q(q-2)\|X_r^{t,x,i,\alpha}\|^{q-4}\|\sigma'(r, X_r^{t,x,i,\alpha}, \alpha_r)X_r^{t,x,i,\alpha}\|^2 \\ &\qquad\qquad q\|X_r^{t,x,i,\alpha}\|^{q-2}\|\sigma(r, X_r^{t,x,i,\alpha}, \alpha_r)\|^2\Big)\mathrm{d}r \\ &\quad + \int_t^s q\|X_r^{t,x,i,\alpha}\|^{q-2}(X_r^{t,x,i,\alpha})'\sigma(r, X_r^{t,x,i,\alpha}, \alpha_r)\mathrm{d}W_r \\ &\leq \|x\|^q + \widehat{C}_q \int_t^s \Big(1 + \|X_r^{t,x,i,\alpha}\|^q\Big)\mathrm{d}r \\ &\quad + q\int_t^s \|X_r^{t,x,i,\alpha}\|^{q-2}(X_r^{t,x,i,\alpha})'\sigma(r, X_r^{t,x,i,\alpha}, \alpha_r)\mathrm{d}W_r,\end{aligned}$$

where $\widehat{C}_q$ is the constant only depending on q and L. The above stochastic integral in the right hand side is a local martingale. Hence, there exists an increasing sequence of stopping times $(\tau_n)_{n\geq 1}$ such that $\tau_n \to \infty$ and

$$\mathbb{E}[\|X_{s\wedge\tau_n}^{t,x,i,\alpha}\|^q] \leq \|x\|^2 + \widehat{C}_q\mathbb{E}\left[\int_t^{s\wedge\tau_n} \Big(1 + \|X_r^{t,x,i,\alpha}\|^q\Big)\mathrm{d}r\right],$$

for all $s \in [t, T]$ and $n \geq 1$, where $a \wedge b = \min\{a, b\}$. By the Fatou lemma, the monotone convergence theorem and the continuity of $X^{t,x,i,\alpha}$, taking a limit, we have

$$\begin{aligned}
1 + \mathbb{E}[\|X_s^{t,x,i,\alpha}\|^q] &\leq 1 + \liminf_{n\to\infty} \mathbb{E}[\|X_{s\wedge\tau_n}^{t,x,i,\alpha}\|^q] \\
&\leq 1 + \|x\|^2 + \widehat{C}_q \mathbb{E}\left[\int_t^s \left(1 + \|X_r^{t,x,i,\alpha}\|^q\right) \mathrm{d}r\right] \\
&= 1 + \|x\|^2 + \widehat{C}_q \int_t^s \mathbb{E}\left[1 + \|X_r^{t,x,i,\alpha}\|^q\right] \mathrm{d}r.
\end{aligned}$$

By the Gronwall lemma, we have

$$\mathbb{E}[\|X_s^{t,x,i,\alpha}\|^q] \leq 1 + \mathbb{E}[\|X_s^{t,x,i,\alpha}\|^q] \leq (1 + \|x\|^q) e^{\widehat{C}_q(s-t)}, \tag{62}$$

for all $0 \leq t \leq s$ and $x \in \mathbb{R}^d$. Similarly, we have

$$\begin{aligned}
\max_{t\leq s\leq T} \|X_s^{t,x,i,\alpha}\|^q \leq \|x\|^q + \widehat{C}_q \int_t^T \left(1 + \|X_r^{t,x,i,\alpha}\|^q\right) \mathrm{d}r& \\
+ q \max_{t\leq s\leq T} \int_t^s \|X_r^{t,x,i,\alpha}\|^{q-2} (X_r^{t,x,i,\alpha})' \sigma(r, X_r^{t,x,i,\alpha}, \alpha_r) \mathrm{d}W_r&.
\end{aligned}$$

By the Burkholder-Davis-Gundy inequality and Jensen inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\max_{t\leq s\leq T} \int_t^s \|X_r^{t,x,i,\alpha}\|^{q-2} (X_r^{t,x,i,\alpha})' \sigma(r, X_r^{t,x,i,\alpha}, \alpha_r) \mathrm{d}W_r\right] \\
&\leq \mathbb{E}\left[\left(\int_t^T \|X_r^{t,x,i,\alpha}\|^{2q-4} \|\sigma'(r, X_r^{t,x,i,\alpha}, \alpha_r) X_r^{t,x,i,\alpha}\|^2 \mathrm{d}r\right)^{1/2}\right] \\
&\leq \mathbb{E}\left[\left(\int_t^T \|X_r^{t,x,i,\alpha}\|^{2q-2} \|\sigma(r, X_r^{t,x,i,\alpha}, \alpha_r)\|^2 \mathrm{d}r\right)^{1/2}\right] \\
&\leq L\mathbb{E}\left[\left(\int_t^T \|X_r^{t,x,i,\alpha}\|^{2q-2} \left(1 + \|X_r^{t,x,i,\alpha}\|^2\right) \mathrm{d}r\right)^{1/2}\right] \\
&\leq \sqrt{2} L \left(\int_t^T \mathbb{E}\left[1 + \|X_r^{t,x,i,\alpha}\|^{2q}\right] \mathrm{d}r\right)^{1/2}.
\end{aligned}$$

Furthermore, using the inequality (62), we have

$$\begin{aligned}
\left(\int_t^T \mathbb{E}\left[1 + \|X_r^{t,x,i,\alpha}\|^{2q}\right] \mathrm{d}r\right)^{1/2} &\leq \left(\int_t^T (1 + \|x\|^{2q}) e^{\widehat{C}_{2q}(r-t)} \mathrm{d}r\right)^{1/2} \\
&\leq \frac{1}{\widehat{C}_{2q}^{1/2}} (1 + \|x\|^q) e^{\widehat{C}_{2q}(T-t)/2}.
\end{aligned}$$

Thus, we obtain

$$\mathbb{E}\left[\max_{t\leq s\leq T} \|X_s^{t,x,i,\alpha}\|^q\right] \leq 1 + \mathbb{E}\left[\max_{t\leq s\leq T} \|X_s^{t,x,i,\alpha}\|^q\right] \leq C_{q,X}(1 + \|x\|^q) e^{C_q(T-t)},$$

where

$$C_{q,X} = \max\left\{1, \widehat{C}_q, \sqrt{\frac{2}{\widehat{C}_q}} qL\right\}, \quad C_q = \frac{\widehat{C}_{2q}}{2}.$$

If $q \in (0,4)$, then by the Jensen inequality, we have

$$\mathbb{E}\left[\max_{t\le s\le T}\|X_s^{t,x,i,\alpha}\|^q\right] = \mathbb{E}\left[\left(\max_{t\le s\le T}\|X_s^{t,x,i,\alpha}\|^4\right)^{q/4}\right] \le \left(\mathbb{E}\left[\max_{t\le s\le T}\|X_s^{t,x,i,\alpha}\|^4\right]\right)^{q/4}$$
$$\le C_{4,X}^{q/4}(1+\|x\|^4)^{q/4}e^{(qC_4/4)(T-t)}$$
$$\le C_{4,X}^{q/4}(1+\|x\|^q)e^{(qC_4/4)(T-t)}.$$

It is easy to show the inequality (3) applying the Ito's lemma to $e^{-\rho s}(1+\|X_s^{t,x,i,\alpha}\|^q)$. □

Appendix B Verification of Y

Proof of Proposition 11. Step.1 Y is at least as large as any objective function. We define a sequence of random variables as follows.

$$X^0 := \eta, \quad X^k := X_{\tau_k}^{\tau_{k-1},X^{k-1},i_{k-1}}, \quad k \ge 1.$$

By the definition, $X^k \in L^{2q}_{\tau_k}(\mathbb{R}^d)$ for all k. Furthermore, for all $k \ge 1$ and $t \in [\tau_{k-1},\tau_k)$, the strong uniqueness of X leads to that

$$X_t^{\nu,\eta,\iota,\alpha} = X_t^{\tau_{k-1},X^{k-1},i_{k-1}}, \tag{63}$$

$\mathbb{P}$-almost surely.

Let $N = \inf\{k \mid \tau_k \ge T\}$ and $\tau_0 = \nu$. By the admissibility of $\alpha = (\tau_k, i_k)_{k\ge 0}$, N is finite $\mathbb{P}$-almost surely. Let $\overline{Z}^{\nu,\eta,\iota,\alpha}$ be a stochastic process such that

$$\overline{Z}_t^{\nu,\eta,\iota,\alpha} = \sum_{k=1}^{N} Z_t^{\tau_{k-1},X^{k-1},i_{k-1}}\mathbb{1}_{[\tau_{k-1},\tau_k)}(t), \quad t \in [0,T], \tag{64}$$

Let D^k be a stochastic process on $[\tau_{k-1},\tau_k]$ such that

$$D_t^k = \exp\left\{-\int_{\tau_{k-1}}^t \rho(s, X_s^{\tau_{k-1},X^{k-1},i_{k-1}}, i_{k-1})\mathrm{d}s\right\}, \quad t \in [\tau_{k-1},\tau_k].$$

By the equality (63), we have

$$D_t^{\nu,\eta,\iota,\alpha} = D_t^1,\ t \in [\tau_0,\tau_1],$$
$$D_t^{\nu,\eta,\iota,\alpha} = D_{\tau_{k-1}}^{\nu,\eta,\iota,\alpha}D_t^k,\ t \in [\tau_{k-1},\tau_k],\ k \ge 2.$$

Then, for any $k \ge 1$, applying the Ito's lemma to $D_t^k Y_t^{\tau_{k-1},X^{k-1},i_{k-1}}$ leads to

$$Y_{\tau_{k-1}}^{\tau_{k-1},X^{k-1},i_{k-1}} \ge D_{\tau_k}^k Y_{\tau_k}^{\tau_{k-1},X^{k-1},i_{k-1}} + \int_{\tau_{k-1}}^{\tau_k} D_s^k\Big(\psi(s, X_s^{\tau_{k-1},X^{k-1},i_{k-1}}, i_{k-1})$$
$$- \varsigma(s, X_s^{\tau_{k-1},X^{k-1},i_{k-1}}, i_{k-1}, Z_s^{\tau_{k-1},X^{k-1},i_{k-1}})\Big)\mathrm{d}s$$
$$- \int_{\tau_{k-1}}^{\tau_k} D_s^k(Z_s^{\tau_{k-1},X^{k-1},i_{k-1}})'\mathrm{d}W_s,$$

where we have used the non-negativity of D^k_t and monotonicity of $K^{\tau_{k-1},X^{k-1},i_{k-1}}_t$. Furthermore, by the pathwise uniqueness of X and Y (see Proposition 10 and (63) and (64)), we have

$$
\begin{aligned}
&Y^{\tau_{k-1},X^{k-1},i_{k-1}}_{\tau_{k-1}}\\
&\quad\geq D^k_{\tau_k} Y^{\tau_k,X^k,i_{k-1}}_{\tau_k} + \int_{\tau_{k-1}}^{\tau_k} D^k_s\Big(\psi(s,X^{\nu,\eta,\iota,\alpha}_s,\alpha_s) - \varsigma(s,X^{\nu,\eta,\iota,\alpha}_s,\alpha_s,\overline{Z}^{\nu,\eta,\iota,\alpha}_s)\Big)\mathrm{d}s\\
&\qquad\qquad - \int_{\tau_{k-1}}^{\tau_k} D^k_s(\overline{Z}^{\nu,\eta,\iota,\alpha}_s)'\mathrm{d}W_s.
\end{aligned}
$$

Since each $Y^{\tau_k,X^k,i_{k-1}}_{\tau_k}$ dominates the lower barrier, we obtain

$$
\begin{aligned}
Y^{\nu,\eta,\iota}_\nu &\geq D^1_{\tau_1} Y^{\tau_1,X^1,i_0}_{\tau_1} + \int_{\tau_0}^{\tau_1} D^1_s\Big(\psi(s,X^{\nu,\eta,\iota,\alpha}_s,\alpha_s) - \varsigma(s,X^{\nu,\eta,\iota,\alpha}_s,\alpha_s,\overline{Z}^{\nu,\eta,\iota,\alpha}_s)\Big)\mathrm{d}s\\
&\quad - \int_{\tau_0}^{\tau_1} D^1_s(\overline{Z}^{\nu,\eta,\iota,\alpha}_s)'\mathrm{d}W_s\\
&\geq D^1_{\tau_1}\Big(Y^{\tau_1,X^1,i_1}_{\tau_1} - c_{i_0,i_1}(\tau_1,X^{\tau_1,X^1,i_0}_{\tau_1})\Big)\\
&\quad + \int_{\tau_0}^{\tau_1} D^1_s\Big(\psi(s,X^{\nu,\eta,\iota,\alpha}_s,\alpha_s) - \varsigma(s,X^{\nu,\eta,\iota,\alpha}_s,\alpha_s,\overline{Z}^{\nu,\eta,\iota,\alpha}_s)\Big)\mathrm{d}s\\
&\quad - \int_{\tau_0}^{\tau_1} D^1_s(\overline{Z}^{\nu,\eta,\iota,\alpha}_s)'\mathrm{d}W_s\\
&= D^1_{\tau_1} Y^{\tau_1,X^1,i_1}_{\tau_1} - D^1_{\tau_1} c_{i_0,i_1}(\tau_1,X^{\nu,\eta,\iota,\alpha}_{\tau_1})\\
&\quad + \int_{\tau_0}^{\tau_1} D^1_s\Big(\psi(s,X^{\nu,\eta,\iota,\alpha}_s,\alpha_s) - \varsigma(s,X^{\nu,\eta,\iota,\alpha}_s,\alpha_s,\overline{Z}^{\nu,\eta,\iota,\alpha}_s)\Big)\mathrm{d}s\\
&\quad - \int_{\tau_0}^{\tau_1} D^1_s(\overline{Z}^{\nu,\eta,\iota,\alpha}_s)'\mathrm{d}W_s\\
&\geq D^1_{\tau_1} Y^{\tau_1,X^1,i_1}_{\tau_2} - D^1_{\tau_1} c_{i_0,i_1}(\tau_1,X^{\nu,\eta,\iota,\alpha}_{\tau_1})\\
&\quad + \int_{\tau_0}^{\tau_1} D^1_s\Big(\psi(s,X^{\nu,\eta,\iota,\alpha}_s,\alpha_s) - \varsigma(s,X^{\nu,\eta,\iota,\alpha}_s,\alpha_s,\overline{Z}^{\nu,\eta,\iota,\alpha}_s)\Big)\mathrm{d}s\\
&\quad + D^1_{\tau_1}\int_{\tau_1}^{\tau_2} D^2_s\Big(\psi(s,X^{\nu,\eta,\iota,\alpha}_s,\alpha_s) - \varsigma(s,X^{\nu,\eta,\iota,\alpha}_s,\alpha_s,\overline{Z}^{\nu,\eta,\iota,\alpha}_s)\Big)\mathrm{d}s\\
&\quad - \int_{\tau_0}^{\tau_1} D^1_s(\overline{Z}^{\nu,\eta,\iota,\alpha}_s)'\mathrm{d}W_s - D^1_{\tau_1}\int_{\tau_1}^{\tau_2} D^2_s(\overline{Z}^{\nu,\eta,\iota,\alpha}_s)'\mathrm{d}W_s\\
&= D^1_{\tau_1} Y^{\tau_2,X^2,i_1}_{\tau_2} - D^1_{\tau_1} c_{i_0,i_1}(\tau_1,X^{\nu,\eta,\iota,\alpha}_{\tau_1})\\
&\quad + \int_{\tau_0}^{\tau_2} D^{\nu,\eta,\iota,\alpha}_s\Big(\psi(s,X^{\nu,\eta,\iota,\alpha}_s,\alpha_s) - \varsigma(s,X^{\nu,\eta,\iota,\alpha}_s,\alpha_s,\overline{Z}^{\nu,\eta,\iota,\alpha}_s)\Big)\mathrm{d}s\\
&\quad - \int_{\tau_0}^{\tau_2} D^{\nu,\eta,\iota,\alpha}_s(\overline{Z}^{\nu,\eta,\iota,\alpha}_s)'\mathrm{d}W_s,
\end{aligned}
$$

where we have used Proposition 10 and (63). By repeating this up to $n \geq 1$, we have

$$
\begin{aligned}
Y^{\nu,\eta,\iota}_\nu &\geq D^{\nu,\eta,\iota,\alpha}_{\tau_n} Y^{\tau_n,X^n,i_{n-1}}_{\tau_n} - \sum_{k=1}^{n-1} D^{\nu,\eta,\iota,\alpha}_{\tau_k} c_{i_{k-1},i_k}(\tau_k,X^{\nu,\eta,\iota,\alpha}_{\tau_k})\\
&\quad + \int_{\tau_0}^{\tau_n} D^{\nu,\eta,\iota,\alpha}_s\Big(\psi(s,X^{\nu,\eta,\iota,\alpha}_s,\alpha_s) - \varsigma(s,X^{\nu,\eta,\iota,\alpha}_s,\alpha_s,\overline{Z}^{\nu,\eta,\iota,\alpha}_s)\Big)\mathrm{d}s\\
&\quad - \int_{\tau_0}^{\tau_n} D^{\nu,\eta,\iota,\alpha}_s(\overline{Z}^{\nu,\eta,\iota,\alpha}_s)'\mathrm{d}W_s,
\end{aligned}
$$

for all n. Since $\tau_n \to T$ $\mathbb{P}$-a.s. and $Y^{\nu,\eta,\iota}$ is continuous, taking a limit, we have

$$Y_\nu^{\nu,\eta,\iota} \geq D_T^{\nu,\eta,\iota,\alpha} g(X_T^{\nu,\eta,\iota,\alpha}, \alpha_T) - \sum_{\nu \leq \tau_k \leq T} D_{\tau_k}^{\nu,\eta,\iota,\alpha} c_{i_{k-1},i_k}(\tau_k, X_{\tau_k}^{\nu,\eta,\iota,\alpha})$$
$$+ \int_\nu^T D_s^{\nu,\eta,\iota,\alpha}\Big(\psi(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s) - \varsigma(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s, \overline{Z}_s^{\nu,\eta,\iota,\alpha})\Big)\mathrm{d}s$$
$$- \int_\nu^T D_s^{\nu,\eta,\iota,\alpha}(\overline{Z}_s^{\nu,\eta,\iota,\alpha})'\mathrm{d}W_s.$$

Similarly to the above, we have

$$D_t^{\nu,\eta,\iota,\alpha} Y_t^{\nu,\eta,\iota} \geq D_T^{\nu,\eta,\iota,\alpha} g(X_T^{\nu,\eta,\iota,\alpha}, \alpha_T) - \sum_{t \leq \tau_k \leq T} D_{\tau_k}^{\nu,\eta,\iota,\alpha} c_{i_{k-1},i_k}(\tau_k, X_{\tau_k}^{\nu,\eta,\iota,\alpha})$$
$$+ \int_t^T D_s^{\nu,\eta,\iota,\alpha}\Big(\psi(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s) - \varsigma(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s, \overline{Z}_s^{\nu,\eta,\iota,\alpha})\Big)\mathrm{d}s$$
$$- \int_t^T D_s^{\nu,\eta,\iota,\alpha}(\overline{Z}_s^{\nu,\eta,\iota,\alpha})'\mathrm{d}W_s,$$

for all $t \in [\nu, T]$. On the other hand, we have

$$D_t^{\nu,\eta,\iota,\alpha} Y_t^{\nu,\eta,\iota,\alpha} = D_T^{\nu,\eta,\iota,\alpha} g(X_T^{\nu,\eta,\iota,\alpha}, \alpha_T) - \sum_{t \leq \tau_k \leq T} D_{\tau_k}^{\nu,\eta,\iota,\alpha} c_{i_{k-1},i_k}(\tau_k, X_{\tau_k}^{\nu,\eta,\iota,\alpha})$$
$$+ \int_t^T D_s^{\nu,\eta,\iota,\alpha}\Big(\psi(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s) - \varsigma(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s, Z_s^{\nu,\eta,\iota,\alpha})\Big)\mathrm{d}s$$
$$- \int_t^T D_s^{\nu,\eta,\iota,\alpha}(Z_s^{\nu,\eta,\iota,\alpha})'\mathrm{d}W_s,$$

for all $t \in [\nu, T]$. Hence, it holds that

$$\begin{aligned}(65)\quad & D_t^{\nu,\eta,\iota,\alpha}\Big(Y_t^{\nu,\eta,\iota} - Y_t^{\nu,\eta,\iota,\alpha}\Big) \\ &\geq - \int_t^T D_s^{\nu,\eta,\iota,\alpha}\Big(\varsigma(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s, \overline{Z}_s^{\nu,\eta,\iota,\alpha}) - \varsigma(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s, Z_s^{\nu,\eta,\iota,\alpha})\Big)\mathrm{d}s \\ &\qquad - \int_t^T D_s^{\nu,\eta,\iota,\alpha}(\overline{Z}_s^{\nu,\eta,\iota,\alpha} - Z_s^{\nu,\eta,\iota,\alpha})'\mathrm{d}W_s \\ &= \int_t^T D_s^{\nu,\eta,\iota,\alpha}\Delta_s'(\overline{Z}_s^{\nu,\eta,\iota,\alpha} - Z_s^{\nu,\eta,\iota,\alpha})\mathrm{d}s - \int_t^T D_s^{\nu,\eta,\iota,\alpha}(\overline{Z}_s^{\nu,\eta,\iota,\alpha} - Z_s^{\nu,\eta,\iota,\alpha})'\mathrm{d}W_s,\end{aligned}$$

where $(\Delta_s)_{\nu \leq s \leq T}$ is a d-dimensional adopted process as follows: Now, we denote by $x_{i,s}$ the ith component of a random vector process $(x_u)_{u \geq 0}$ at time s.
Let $\overline{Z}_s^{\nu,\eta,\iota,\alpha,\, i} = (\overline{Z}_{1,s}^{\nu,\eta,\iota,\alpha}, \ldots, \overline{Z}_{i-1,s}^{\nu,\eta,\iota,\alpha}, \overline{Z}_{i,s}^{\nu,\eta,\iota,\alpha}, Z_{i+1,s}^{\nu,\eta,\iota,\alpha}, \ldots, Z_{d,s}^{\nu,\eta,\iota,\alpha})'$ and
let $Z_s^{\nu,\eta,\iota,\alpha,\, i} = (\overline{Z}_{1,s}^{\nu,\eta,\iota,\alpha}, \ldots, \overline{Z}_{i-1,s}^{\nu,\eta,\iota,\alpha}, Z_{i,s}^{\nu,\eta,\iota,\alpha}, Z_{i+1,s}^{\nu,\eta,\iota,\alpha}, \ldots, Z_{d,s}^{\nu,\eta,\iota,\alpha})'$. $\Delta_{i,s}$ is

$$\Delta_{i,s} = -\frac{\varsigma(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s, \overline{Z}_s^{\nu,\eta,\iota,\alpha,\, i}) - \varsigma(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s, Z_s^{\nu,\eta,\iota,\alpha,\, i})}{\overline{Z}_{i,s}^{\nu,\eta,\iota,\alpha} - Z_{i,s}^{\nu,\eta,\iota,\alpha}},$$

if $\overline{Z}_{i,s}^{\nu,\eta,\iota,\alpha} \neq Z_{i,s}^{\nu,\eta,\iota,\alpha}$ and $\Delta_{i,s} = 0$ otherwise. Then, $(\Delta_s)_{\nu \leq s \leq T}$ is uniformly bounded since $z \to \varsigma(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s, z)$ is uniformly Lipschitz for all s. This implies that the following process,

$$\zeta_s^\Delta = \exp\left\{\int_\nu^s \Delta_u'\mathrm{d}W_u - \frac{1}{2}\int_\nu^s \|\Delta_u\|^2\mathrm{d}u\right\}, \quad s \geq \nu$$

is a martingale. Hence, we can define a new probability measure such that

$$\mathbb{P}_T^{\Delta}(A) := \mathbb{E}[\mathbb{1}_A \zeta_T^{\Delta}], \quad A \in \mathcal{F}_T.$$

Furthermore, by the Girsanov theorem, the following process,

$$W_t^{\Delta} := \int_{\nu}^{t} \Delta_s \mathrm{d}s - W_t, \quad t \in [\nu, T],$$

is a d-dimensional Brownian motion under $\mathbb{P}_T^{\Delta}$. We denote by $\mathbb{E}_T^{\Delta}$ an expectation operator under $\mathbb{P}_T^{\Delta}$. Since $\overline{Z}^{\nu,\eta,\iota,\alpha}$ and $Z^{\nu,\eta,\iota,\alpha}$ are in $\mathbb{H}_d^2[\nu,T]$, it holds that

$$\mathbb{E}_T^{\Delta}\left[\int_{\nu}^{T} (D_s^{\nu,\eta,\iota,\alpha})^2 \|\overline{Z}_s^{\nu,\eta,\iota,\alpha} - Z_s^{\nu,\eta,\iota,\alpha}\|^2 \mathrm{d}s\right] < \infty.$$

This implies that the stochastic integral

$$\int_{\nu}^{u} D_s^{\nu,\eta,\iota,\alpha}(\overline{Z}_s^{\nu,\eta,\iota,\alpha} - Z_s^{\nu,\eta,\iota,\alpha})' \mathrm{d}W_s^{\Delta}, \quad u \in [\nu, T],$$

is a martingale under $\mathbb{P}_T^{\Delta}$. Hence, taking conditional expectation of the inequality (65) under the probability measure $\mathbb{P}_T^{\Delta}$ given by $\mathcal{F}_t$, we obtain

$$Y_t^{\nu,\eta,\iota} - Y_t^{\nu,\eta,\iota,\alpha} \geq 0,$$

$\mathbb{P}$-almost surely for all $t \in [\nu, T]$.

Step.2 Optimality of Y. We first prove the admissibility of α^*. Let $\overline{Z}_s^{\nu,\eta,\iota,\alpha^*}$ be a stochastic process defined as (64). Then, by the definition α^*, $K_s^{\tau_{k-1},X^*_{\tau_{k-1}},i^*_{k-1}} = 0$ for all $k \geq 1$ and $s \in [\tau^*_{k-1}, \tau^*_k]$. Furthermore, it holds that

$$Y_{\tau_k^*}^{\tau^*_{k-1},X^*_{\tau^*_{k-1}},i^*_{k-1}} = Y_{\tau_k^*}^{\tau^*_{k-1},X^*_{\tau^*_{k-1}},i^*_{k}} - c_{i^*_{k-1},i^*_k}(\tau^*_k, X^*_{\tau^*_k}),$$

for all $k \geq 1$. Hence, the following equality holds.

$$\begin{aligned} D_t^{\nu,\eta,\iota,\alpha^*} Y_t^{\nu,\eta,\iota} &= D_{t\vee\tau_n^*}^{\nu,\eta,\iota,\alpha^*} Y_{t\vee\tau_n^*}^{\tau_n^*,X^*_{\tau_n^*},i^*_{n-1}} - \sum_{k=1}^{n} D_{\tau_k^*}^{\nu,\eta,\iota,\alpha^*} c_{i^*_{k-1},i^*_k}(\tau^*_k, X^*_{\tau^*_k}) \mathbb{1}_{[\nu,\tau^*_k]}(t) \\ &\quad + \int_t^{t\vee\tau_n^*} D_s^{\nu,\eta,\iota,\alpha^*}\Big(\psi(s, X_s^*, \alpha_s^*) - \varsigma(s, X_s^*, \alpha_s, \overline{Z}_s^{\nu,\eta,\iota,\alpha^*})\Big)\mathrm{d}s \\ &\quad - \int_t^{t\vee\tau_n^*} D_s^{\nu,\eta,\iota,\alpha^*}(\overline{Z}_s^{\nu,\eta,\iota,\alpha^*})' \mathrm{d}W_s, \end{aligned} \tag{66}$$

for all $n \geq 1$, where $a \vee b = \max\{a, b\}$. Let $N^* = \inf\{k \mid \tau_k^* \geq T\}$ and $B = \{N^* = +\infty\}$. Suppose that $\mathbb{P}(B) > 0$. Then, as $\mathcal{I}$ is a finite set, there exists a finite loop $i_0, i_1, \ldots, i_m, i_0$, $i_0 \in \mathcal{I}, i_0 \neq i_1$ such that

$$Y_{\tau^*_{k_{q+l}}}^{\nu,\eta,i_{l-1}} = Y_{\tau^*_{k_{q+l}}}^{\nu,\eta,i_l} - c_{i_{l-1},i_l}(\tau^*_{k_{q+l}}, X^*_{\tau^*_{k_{q+l}}}) \text{ on } B,$$

for all $l = 1, \ldots, m+1$, $q \geq 0$ and $i_{m+1} = i_0$, where $(\tau^*_{k_q})_{q\geq 1}$ is a subsequence of $(\tau^*_k)_{k\geq 0}$. Let $\overline{\tau} = \lim_{q\to\infty} \tau^*_{k_q}$. Then $\overline{\tau} < T$ on B and

$$Y_{\overline{\tau}}^{\nu,\eta,i_{l-1}} = Y_{\overline{\tau}}^{\nu,\eta,i_l} - c_{i_{l-1},i_l}(\overline{\tau}, X^*_{\overline{\tau}}) \text{ on } B,$$

for all $l = 1, \dots, m+1$. This implies that

$$\sum_{l=1}^{m+1} c_{i_{l-1},i_l}(\overline{\tau}, X^*_{\overline{\tau}}) = 0 \text{ on } B,$$

which is contradiction to Hypothesis 4.3. Therefore, $\mathbb{P}(B) = 0$ and N^* is finite $\mathbb{P}$-almost surely. Hence, taking the limit of (66), we have

$$\begin{aligned} D_t^{\nu,\eta,\iota,\alpha^*} Y_t^{\nu,\eta,\iota} &= D_T^{\nu,\eta,\iota,\alpha^*} g(X^*_T, \alpha^*_T) - \sum_{t \le \tau^*_k \le T} D_{\tau^*_k}^{\nu,\eta,\iota,\alpha^*} c_{i^*_{k-1},i^*_k}(\tau^*_k, X^*_{\tau^*_k}) \\ &\quad + \int_t^T D_s^{\nu,\eta,\iota,\alpha^*} \Big(\psi(s, X^*_s, \alpha^*_s) - \varsigma(s, X^*_s, \alpha_s, \overline{Z}_s^{\nu,\eta,\iota,\alpha^*})\Big) \mathrm{d}s \\ &\quad - \int_t^T D_s^{\nu,\eta,\iota,\alpha^*} (\overline{Z}_s^{\nu,\eta,\iota,\alpha^*})' \mathrm{d}W_s. \end{aligned} \tag{67}$$

Since $(Y^{\nu,\eta,\iota}, \overline{Z}^{\nu,\eta,\iota,\alpha^*}) \in \mathbb{S}^2[\nu,T] \times \mathbb{H}^2_d[\nu,T]$ and since Hypotheses 1, 3, 4 and 7 are satisfied, $\sum_{\nu \le \tau^*_k \le T} c_{i^*_{k-1},i^*_k}(\tau^*_k, X^*_{\tau^*_k})$ is quadratic integrable under $\mathbb{P}$. Hence, α^* is admissible.

We consider the solution of the BSDE (16) at $(\nu,\eta,\iota,\alpha^*)$,
denoted by $(Y^{\nu,\eta,\iota,\alpha^*}, Z^{\nu,\eta,\iota,\alpha^*})$. Then, combining (67) and $(Y^{\nu,\eta,\iota,\alpha^*}, Z^{\nu,\eta,\iota,\alpha^*})$, we obtain that

$$\begin{aligned} &D_t^{\nu,\eta,\iota,\alpha^*} \Big(Y_t^{\nu,\eta,\iota} - Y_t^{\nu,\eta,\iota,\alpha^*}\Big) \\ &\quad = \int_t^T D_s^{\nu,\eta,\iota,\alpha^*} \Delta'_s (\overline{Z}_s^{\nu,\eta,\iota,\alpha^*} - Z_s^{\nu,\eta,\iota,\alpha^*}) \mathrm{d}s - \int_t^T D_s^{\nu,\eta,\iota,\alpha^*} (\overline{Z}_s^{\nu,\eta,\iota,\alpha^*} - Z_s^{\nu,\eta,\iota,\alpha^*})' \mathrm{d}W_s, \end{aligned}$$

where $(\Delta_s)_{0 \le s \le T}$ is the stochastic process defined in Step.1. As well as Step.1, we conclude that

$$Y_t^{\nu,\eta,\iota} = Y_t^{\nu,\eta,\iota,\alpha^*},$$

$\mathbb{P}$-almost surely for all $t \in [\nu, T]$. □

Appendix C Verification in the Infinite Horizon

Proof of Proposition 18. Step.1 Monotonicity of $\widehat{Y}$. Fix an arbitrary $0 \le T \le \widetilde{T}$, $\nu \in \mathcal{T}_0^T$ and $\eta \in L^{2q}_\nu(\mathbb{R}^d)$. Let $(\widehat{Y}^{T,\nu,\eta,i,n}, \widehat{Z}^{T,\nu,\eta,i,n}, \widehat{K}^{T,\nu,\eta,i,n})_{n \ge 0}$ be the Picard's iterations of $(\widehat{Y}^{T,\nu,\eta,i}, \widehat{Z}^{T,\nu,\eta,i}, \widehat{K}^{T,\nu,\eta,i})$ constructed in Theorem 8.
Also let $(\widehat{Y}^{\widetilde{T},\nu,\eta,i,n}, \widehat{Z}^{\widetilde{T},\nu,\eta,i,n}, \widehat{K}^{\widetilde{T},\nu,\eta,i,n})_{n \ge 0}$ be the Picard's iterations of
$(\widehat{Y}^{\widetilde{T},\nu,\eta,i}, \widehat{Z}^{\widetilde{T},\nu,\eta,i}, \widehat{K}^{\widetilde{T},\nu,\eta,i})$ constructed in Theorem 8. Then, by the non-negative reward condition, temporary terminal condition and Proposition 2.2 in [12], we have

$$\begin{aligned} e^{-\rho T} \widehat{Y}_T^{\widetilde{T},\nu,\eta,i,0} &= \inf_{\theta \in \Theta[T,\widetilde{T}]} \mathbb{E}\Big[e^{-\rho \widetilde{T}} \zeta_{\widetilde{T}}^{\theta,T} g(X_{\widetilde{T}}^{\nu,\eta,i}, i) \\ &\qquad + \int_T^{\widetilde{T}} e^{-\rho t} \zeta_t^{\theta,T} \Big(\psi(X_t^{\nu,\eta,i}, i) - \theta'_t \phi(X_t^{\nu,\eta,i}, i)\Big) \mathrm{d}t \;\Big|\; \mathcal{F}_T \Big] \\ &\ge \inf_{\theta \in \Theta[T,\widetilde{T}]} \mathbb{E}\Big[e^{-\rho \widetilde{T}} \zeta_{\widetilde{T}}^{\theta,T} g(X_{\widetilde{T}}^{\nu,\eta,i}, i) \;\Big|\; \mathcal{F}_T \Big] \ge e^{-\rho T} g(X_T^{\nu,\eta,i}, i). \end{aligned}$$

Hence, $\widehat{Y}_T^{\widetilde{T},\nu,\eta,i,0} \geq g(X_T^{\nu,\eta,i},i)$ for all $i \in \mathcal{I}$. On the other hand, for all $i \in \mathcal{I}$, $(\widehat{Y}_t^{\widetilde{T},\nu,\eta,i,0}, \widehat{Z}_t^{\widetilde{T},\nu,\eta,i,0})$ is the solution to the following BSDE on $[\nu,T]$,

$$-\mathrm{d}y_t = \Big(\psi(X_t^{\nu,\eta,i},i) - \rho y_t - \varsigma(X_t^{\nu,\eta,i},i,z_t)\Big)\mathrm{d}t - z_t'\mathrm{d}W_t,$$
$$y_T = \widehat{Y}_T^{\widetilde{T},\nu,\eta,i,0}, \quad (y,z) \in \mathbb{S}^2[\nu,T] \times \mathbb{H}_d^2[\nu,T].$$

By the comparison theorem, $\widehat{Y}_t^{\widetilde{T},\nu,\eta,i,0} \geq \widehat{Y}_t^{T,\nu,\eta,i,0}$ for all $t \in [\nu,T]$ and $i \in \mathcal{I}$. Similarly, by the non-negative reward condition, temporary terminal condition and Proposition 7.1 in [11], we have

$$\widehat{Y}_T^{\widetilde{T},\nu,\eta,i,n} \geq g(X_T^{\nu,\eta,i},i),$$

for all $n \geq 1$. Hence, recursively applying the comparison theorem, we obtain that $\widehat{Y}_t^{\widetilde{T},\nu,\eta,i,n} \geq \widehat{Y}_t^{T,\nu,\eta,i,n}$ for all $t \in [\nu,T]$, $i \in \mathcal{I}$ and $n \geq 1$. Taking a limit, we also have $\widehat{Y}_t^{\widetilde{T},\nu,\eta,i} \geq \widehat{Y}_t^{T,\nu,\eta,i}$ for all $t \in [\nu,T]$ and $i \in \mathcal{I}$.

Step.2 n-step dominated. Since $T \to \widehat{Y}_t^{T,\nu,\eta,i,n}$ is increasing by Step.1 and since $n \to \widehat{Y}_t^{T,\nu,\eta,i,n}$ is also increasing, we can exchange the orders of taking the limits such that

$$\lim_{T\to\infty} \widehat{Y}_t^{T,\nu,\eta,i} = \lim_{T\to\infty}\lim_{n\to\infty} \widehat{Y}_t^{T,\nu,\eta,i,n} = \lim_{n\to\infty}\lim_{T\to\infty} \widehat{Y}_t^{T,\nu,\eta,i,n} = \lim_{n\to\infty} \widehat{Y}_t^{\infty,\nu,\eta,i,n},$$

where

$$\widehat{Y}_t^{\infty,\nu,\eta,i,n} = \lim_{T\to\infty} \widehat{Y}_t^{T,\nu,\eta,i,n}, \quad n \geq 1.$$

By Proposition 2.2 in [12] and the comparison theorem, it holds that

$$e^{-\rho\nu}\widehat{Y}_\nu^{T,\nu,\eta,i,0} = \inf_{\theta\in\Theta[\nu,T]} \mathbb{E}\bigg[\zeta_T^{\theta,\nu} e^{-\rho T} g(X_T^{\nu,\eta,i},i) + \int_\nu^T e^{-\rho t}\zeta_t^{\theta,\nu}\Big(\psi(X_t^{\nu,\eta,i},i) - \theta_t'\phi(X_t^{\nu,\eta,i},i)\Big)\mathrm{d}t \;\Big|\; \mathcal{F}_\nu\bigg], \tag{68}$$

for all $T \geq \nu$. Now, we choose an arbitrary $\theta \in \Theta[\nu,\infty)$. Then, by the equality (68) and the temporary terminal condition, we have

$$e^{-\rho\nu}\widehat{Y}_\nu^{T,\nu,\eta,i,0} \leq \mathbb{E}\left[\int_\nu^T e^{-\rho t}\zeta_t^{\theta,\nu}\Big(\psi(X_t^{\nu,\eta,i},i) - \theta_t'\phi(X_t^{\nu,\eta,i},i)\Big)\mathrm{d}t \;\Big|\; \mathcal{F}_\nu\right],$$

for all $T \geq \nu$. By the Lebesgue dominated convergence theorem, we have

$$e^{-\rho\nu}\widehat{Y}_\nu^{\infty,\nu,\eta,i,0} \leq \mathbb{E}\left[\int_\nu^\infty e^{-\rho t}\zeta_t^{\theta,\nu}\Big(\psi(X_t^{\nu,\eta,i},i) - \theta_t'\phi(X_t^{\nu,\eta,i},i)\Big)\mathrm{d}t \;\Big|\; \mathcal{F}_\nu\right].$$

Since θ is arbitrary, we obtain that

$$e^{-\rho\nu}\widehat{Y}_\nu^{\infty,\nu,\eta,i,0} \leq \inf_{\theta\in\Theta[\nu,\infty)} \mathbb{E}\left[\int_\nu^\infty e^{-\rho t}\zeta_t^{\theta,\nu}\Big(\psi(X_t^{\nu,\eta,i},i) - \theta_t'\phi(X_t^{\nu,\eta,i},i)\Big)\mathrm{d}t \;\Big|\; \mathcal{F}_\nu\right]. \tag{69}$$

Now, we assume that for some $n \geq 1$,

$$e^{-\rho\widetilde{\tau}}\widehat{Y}_{\widetilde{\tau}}^{\infty,\widetilde{\tau},\widetilde{\eta},j,n-1} \leq \sup_{\alpha\in\mathbb{A}_{j,n-1}[\widetilde{\tau},\infty)} \inf_{\theta\in\Theta[\nu,\infty)} \mathbb{E}\bigg[\int_{\widetilde{\tau}}^\infty e^{-\rho t}\zeta_t^{\theta,\widetilde{\tau}}\Big(\psi(X_t^{\widetilde{\tau},\widetilde{\eta},j,\alpha},\alpha_t) - \theta_t'\phi(X_t^{\widetilde{\tau},\widetilde{\eta},j,\alpha},\alpha_t)\Big)\mathrm{d}t - \sum_{k=1}^{n-1} e^{-\rho\tau_k}\zeta_{\tau_k}^{\theta,\widetilde{\tau}} c_{i_{k-1},i_k}(X_{\tau_k}^{\widetilde{\tau},\widetilde{\eta},j,\alpha}) \;\Big|\; \mathcal{F}_{\widetilde{\tau}}\bigg],$$

where $\widetilde{\tau} \in \mathcal{T}_\nu$ and $\widetilde{\eta} \in L^{2q}_{\widetilde{\tau}}(\mathbb{R}^d)$, and $\alpha \in \mathbb{A}_{j,n-1}[\widetilde{\tau}, \infty)$ is a set of the admissible controls on $[\widetilde{\tau}, \infty)$ changing the regimes at most $n-1$ times. On the other hand, by Proposition 7.1 in [11] and the uniqueness of $\widehat{Y}$, it holds that

$$(70) \qquad e^{-\rho\nu}\widehat{Y}^{T,\nu,\eta,i,n}_\nu = \sup_{\widetilde{\tau}\in\mathcal{T}^T_\tau} \inf_{\theta\in\Theta[\nu,T]} \mathbb{E}\Big[e^{-\rho T}\zeta^{\theta,\nu}_T g(X^{\nu,\eta,i}_T, i)\mathbb{1}_{\{\widetilde{\tau}=T\}} + e^{-\rho\widetilde{\tau}}\zeta^{\theta,\nu}_{\widetilde{\tau}} \max_{j\in\mathcal{I}\setminus\{i\}} \left\{ \widehat{Y}^{T,\widetilde{\tau},X^{\nu,\eta,i}_{\widetilde{\tau}},j,n-1}_{\widetilde{\tau}} - c_{i,j}(X^{\nu,\eta,i}_{\widetilde{\tau}}) \right\} \mathbb{1}_{\{\widetilde{\tau}<T\}} + \int_\nu^{\widetilde{\tau}} e^{-\rho t}\zeta^{\theta,\nu}_t \Big(\psi(X^{\nu,\eta,i}_t, i) - \theta'_t\phi(X^{\nu,\eta,i}_t, i)\Big)\mathrm{d}t \;\Big|\; \mathcal{F}_\nu \Big].$$

Let τ^* be an optimal stopping time of the maximization problem in the right hand side of (70). Then, by Proposition 2.3 in [11], we have

$$\tau^* = \inf\left\{ t\in[\nu,T] \mid \widehat{Y}^{T,t,X^{\nu,\eta,i}_t,i,n}_t = \max_{j\in\mathcal{I}\setminus\{i\}} \left\{ \widehat{Y}^{T,t,X^{\nu,\eta,i}_t,j,n-1}_t - c_{i,j}(X^{\nu,\eta,i}_t) \right\} \right\}.$$

Hence,

$$\begin{aligned} &e^{-\rho T}\zeta^{\theta,\nu}_T g(X^{\nu,\eta,i}_T, i)\mathbb{1}_{\{\tau^*=T\}} + e^{-\rho\tau^*}\zeta^{\theta,\nu}_{\tau^*} \max_{j\in\mathcal{I}\setminus\{i\}} \left\{ \widehat{Y}^{T,\tau^*,X^{\nu,\eta,i}_{\tau^*},j,n-1}_{\tau^*} - c_{i,j}(X^{\nu,\eta,i}_{\tau^*}) \right\} \mathbb{1}_{\{\tau^*<T\}} \\ &= e^{-\rho\tau^*}\zeta^{\theta,\nu}_{\tau^*} \Big(\widehat{Y}^{T,\tau^*,X^{\nu,\eta,i}_{\tau^*},j^*,n-1}_{\tau^*} - c_{i,j^*}(X^{\nu,\eta,i}_{\tau^*})\mathbb{1}_{\{\tau^*<T\}} \Big), \end{aligned}$$

where j^* satisfies

$$\widehat{Y}^{T,\tau^*,X^{\nu,\eta,i}_{\tau^*},i,n}_{\tau^*} = \widehat{Y}^{T,\tau^*,X^{\nu,\eta,i}_{\tau^*},j^*,n-1}_{\tau^*} - c_{i,j^*}(X^{\nu,\eta,i}_{\tau^*}),$$

if $\tau^* < T$, and $j^* = i$ otherwise. By the monotonicity of $\widehat{Y}$, we have

$$\widehat{Y}^{T,\tau^*,X^{\nu,\eta,i}_{\tau^*},j^*,n-1}_{\tau^*} \le \widehat{Y}^{\infty,\tau^*,X^{\nu,\eta,i}_{\tau^*},j^*,n-1}_{\tau^*}.$$

Hence, we obtain

$$\begin{aligned} e^{-\rho\nu}\widehat{Y}^{T,\nu,\eta,i,n}_\nu &\le \inf_{\theta\in\Theta[0,T]} \mathbb{E}\Bigg[\int_\nu^{\tau^*} e^{-\rho t}\zeta^{\theta,\nu}_t \Big(\psi(X^{\nu,\eta,i}_t, i) - \theta'_t\phi(X^{\nu,\eta,i}_t, i)\Big)\mathrm{d}t \\ &\qquad + e^{-\rho\tau^*}\zeta^{\theta,\nu}_{\tau^*}\widehat{Y}^{\infty,\tau^*,X^{\nu,\eta,i}_{\tau^*},j^*,n-1}_{\tau^*} - e^{-\rho\tau^*}\zeta^{\theta,\nu}_{\tau^*} c_{i,j^*}(X^{\nu,\eta,i}_{\tau^*})\mathbb{1}_{\{\tau^*<T\}} \;\Big|\; \mathcal{F}_\nu \Bigg] \\ &\le \inf_{\theta\in\Theta[\nu,\tau^*)} \mathbb{E}\Bigg[\int_\nu^{\tau^*} e^{-\rho t}\zeta^{\theta,\nu}_t \Big(\psi(X^{\nu,\eta,i}_t, i) - \theta'_t\phi(X^{\nu,\eta,i}_t, i)\Big)\mathrm{d}t \\ &\qquad - e^{-\rho\tau^*}\zeta^{\theta,\nu}_{\tau^*} c_{i,j^*}(X^{\nu,\eta,i}_{\tau^*})\mathbb{1}_{\{\tau^*<T\}} \\ &\qquad + \zeta^{\theta,\nu}_{\tau^*} \sup_{\alpha\in\mathbb{A}_{j^*,n-1}[\tau^*,\infty)} \inf_{\theta\in\Theta[\tau^*,\infty)} \mathbb{E}\Bigg[\int_{\tau^*}^\infty e^{-\rho t}\zeta^{\theta,\tau^*}_t \Big(\psi(X^{\tau^*,X^{\nu,\eta,i}_{\tau^*},j^*,\alpha}_t, \alpha_t) \\ &\qquad\qquad - \theta'_t\phi(X^{\tau^*,X^{\nu,\eta,i}_{\tau^*},j^*,j,\alpha}_t, \alpha_t)\Big)\mathrm{d}t \\ &\qquad\qquad - \sum_{k=1}^{n-1} e^{-\rho\tau_k}\zeta^{\theta,\tau^*}_{\tau_k} c_{i_{k-1},i_k}(X^{\tau^*,X^{\nu,\eta,i}_{\tau^*},j^*,\alpha}_{\tau_k}) \;\Big|\; \mathcal{F}_{\tau^*}\Bigg] \;\Big|\; \mathcal{F}_\nu\Bigg] \\ &\le \sup_{\alpha\in\mathbb{A}_{i,n}[\nu,\infty)} \inf_{\theta\in\Theta[\nu,\infty)} \mathbb{E}\Bigg[\int_\nu^\infty e^{-\rho t}\zeta^{\theta,\nu}_t \Big(\psi(X^{\nu,\eta,i,\alpha}_t, \alpha_t) - \theta'_t\phi(X^{\nu,\eta,i,\alpha}_t, \alpha_t)\Big)\mathrm{d}t \\ &\qquad - \sum_{k=1}^{n} e^{-\rho\tau_k}\zeta^{\theta,\nu}_{\tau_k} c_{i_{k-1},i_k}(X^{\nu,\eta,i,\alpha}_{\tau_k}) \;\Big|\; \mathcal{F}_\nu\Bigg], \end{aligned}$$

where we have used the uniqueness of the strong solution of X. Taking a limit, we have

(71)
$$e^{-\rho\nu}\widehat{Y}_\nu^{\infty,\nu,\eta,i,n} \leq \sup_{\alpha\in\mathbb{A}_{i,n}[\nu,\infty)} \inf_{\theta\in\Theta[\nu,\infty)} \mathbb{E}\left[\int_\nu^\infty e^{-\rho t}\zeta_t^{\theta,\nu}\Big(\psi(X_t^{\nu,\eta,i,\alpha},\alpha_t)-\theta_t'\phi(X_t^{\nu,\eta,i,\alpha},\alpha_t)\Big)\mathrm{d}t \right.$$
$$\left. -\sum_{k=1}^n e^{-\rho\tau_k}\zeta_{\tau_k}^{\theta,\nu} c_{i_{k-1},i_k}(X_{\tau_k}^{\nu,\eta,i,\alpha}) \;\Big|\; \mathcal{F}_\nu\right].$$

By the inequalities (69) and (71), we can prove that the inequality (71) holds for all $n \geq 1$ using the induction method. Since $\mathbb{A}_{i,n}[t,\infty) \subseteq \mathbb{A}_i[t,\infty)$ for all $n \geq 1$, the inequality (71) leads to

$$\lim_{T\to\infty}\widehat{Y}_t^{T,t,x,i} = \lim_{n\to\infty}\widehat{Y}_t^{\infty,t,x,i,n} \leq v^\infty(x,i), \tag{72}$$

for all $(t,x,i)\in[0,\infty)\times\mathbb{R}^d\times\mathcal{I}$. By the monotonicity of $\widehat{Y}$ and the inequality (72), we have

$$\widehat{Y}_t^{T,t,x,i} \leq v^\infty(x,i), \tag{73}$$

for all $(t,T,x,i)\in[0,\infty)^2\times\mathbb{R}^d\times\mathcal{I}$.

Step.3 Convergence. To prove the opposite inequality of (72), we use the ϵ-optimal argument such as Corollary 2.1 in [1]. Fix any $(t,x,i)\in[0,\infty)\times\mathbb{R}^d\times\mathcal{I}$. Let $J^T(t,x,i,\alpha)$ be an objective function in the finite horizon $[0,T]$. Then, by the time-homogeneity, we have

$$Y_t^{T,t,x,i} = Y_0^{T-t,0,x,i} \geq J^{T-t}(0,x,i,\alpha),$$

for all $0\leq t\leq T$, $x\in\mathbb{R}^d$, $i\in\mathcal{I}$ and $\alpha\in\mathbb{A}_i[0,T-t]$. Now, we fix an arbitrary $t\geq 0$ and $x\in\mathbb{R}^d$. For any $\epsilon>0$, we choose a control $\alpha^\epsilon=(\tau_k^\epsilon,i_k^\epsilon)_{k\geq 0}\in\mathbb{A}_i[0,\infty)$ such that

$$J(x,i,\alpha^\epsilon)\geq v^\infty(x,i)-\epsilon.$$

For all $T\geq t$, define

$$\alpha_s^{\epsilon,T-t} := \alpha_s, \quad s\in[0,T-t].$$

Then, $\alpha^{\epsilon,T-t}\in\mathbb{A}_i[0,T-t]$ for all $T\geq t$. For all $T\geq t$, let

$$\theta^{T-t} := \arg\inf_{\theta\in\Theta[0,T-t]}\mathbb{E}\Bigg[\int_0^{T-t} e^{-\rho s}\zeta_s^{\theta,0}\Big(\psi(X_s^{0,x,i,\alpha^\epsilon},\alpha_s^\epsilon)-\theta_s'\phi(X_s^{0,x,i,\alpha^\epsilon},\alpha_s^\epsilon)\Big)\mathrm{d}s$$
$$-\sum_{k=1}^\infty e^{-\rho\tau_k^\epsilon}\zeta_{\tau_k^\epsilon}^{\theta,0}c_{i_{k-1}^\epsilon,i_k^\epsilon}(X_{\tau_k^\epsilon}^{0,x,i,\alpha^\epsilon})\mathbb{1}_{\{\tau_k<T-t\}} + e^{-\rho(T-t)}\zeta_{T-t}^{\theta,0}g(X_{T-t}^{0,x,i,\alpha^\epsilon},\alpha_{T-t}^\epsilon)\Bigg].$$

Also let

$$\theta_s^{\infty,T-t} := \begin{cases}\theta_s^{T-t}, & \text{if } s<T-t,\\ 0, & \text{otherwise,}\end{cases}$$

for all $T\geq t$. It is easy to check $\theta^{\infty,T-t}\in\Theta[0,\infty)$. Then, we have

$$\begin{aligned}
&J(x,i,\alpha^\epsilon)\\
&\leq \mathbb{E}\Bigg[\int_0^\infty e^{-\rho s}\zeta_s^{\theta^{\infty,T-t},0}\Big(\psi(X_s^{0,x,i,\alpha^\epsilon},\alpha_s^\epsilon)-(\theta_s^{\infty,T-t})'\phi(X_s^{0,x,i,\alpha^\epsilon},\alpha_s^\epsilon)\Big)\mathrm{d}s\\
&\qquad -\sum_{k=1}^\infty e^{-\rho\tau_k^\epsilon}\zeta_{\tau_k^\epsilon}^{\theta^{\infty,T-t},0}c_{i_{k-1}^\epsilon,i_k^\epsilon}(X_{\tau_k^\epsilon}^{0,x,i,\alpha^\epsilon})\Bigg]\\
&= J^{T-t}(0,x,i,\alpha^{\epsilon,T-t}) + \mathbb{E}\Bigg[\zeta_{T-t}^{\theta^{\infty,T-t},0}\int_{T-t}^\infty e^{-\rho s}\psi(X_s^{0,x,i,\alpha^\epsilon},\alpha_s^\epsilon)\mathrm{d}s\\
&\qquad -\zeta_{T-t}^{\theta^{\infty,T-t},0}\sum_{k=1}^\infty e^{-\rho\tau_k^\epsilon}c_{i_{k-1}^\epsilon,i_k^\epsilon}(X_{\tau_k^\epsilon}^{0,x,i,\alpha^\epsilon})\mathbb{1}_{\{\tau_k^\epsilon>T-t\}} - e^{-\rho(T-t)}\zeta_{T-t}^{\theta^{\infty,T-t},0}g(X_{T-t}^{0,x,i,\alpha^\epsilon},\alpha_{T-t}^\epsilon)\Bigg],
\end{aligned}$$

for all $T \geq t$. By the polynomial growth condition and the strong triangular condition, we have

$$\mathbb{E}\left[\int_{T-t}^{\infty} e^{-\rho s}\psi(X_s^{0,x,i,\alpha^\epsilon}, \alpha_s^\epsilon)\mathrm{d}s - \sum_{k=1}^{\infty} e^{-\rho\tau_k^\epsilon} c_{i_{k-1}^\epsilon, i_k^\epsilon}(X_{\tau_k^\epsilon}^{0,x,i,\alpha^\epsilon})\mathbb{1}_{\{\tau_k^\epsilon > T-t\}} \,\Big|\, \mathcal{F}_{T-t}\right]$$
$$\leq C_1(1 + \|X_{T-t}^{0,x,i,\alpha^\epsilon}\|^q)e^{-\rho(T-t)},$$

for all $T \geq t$, where C_1 is a positive constant not depending on T, t and x. Thus, we have

$$\mathbb{E}\Bigg[\zeta_{T-t}^{\theta^\infty,T-t,0}\int_{T-t}^{\infty} e^{-\rho s}\psi(X_s^{0,x,i,\alpha^\epsilon}, \alpha_s^\epsilon)\mathrm{d}s$$
$$-\zeta_{T-t}^{\theta^\infty,T-t,0}\sum_{k=1}^{\infty} e^{-\rho\tau_k^\epsilon} c_{i_{k-1}^\epsilon, i_k^\epsilon}(X_{\tau_k^\epsilon}^{0,x,i,\alpha^\epsilon})\mathbb{1}_{\{\tau_k^\epsilon > T-t\}} - e^{-\rho(T-t)}\zeta_{T-t}^{\theta^\infty,T-t,0} g(X_{T-t}^{0,x,i,\alpha^\epsilon}, \alpha_{T-t}^\epsilon)\Bigg]$$
$$\leq C_2\mathbb{E}\left[\zeta_{T-t}^{\theta^\infty,T-t,0}\left(1 + \|X_{T-t}^{0,x,i,\alpha^\epsilon}\|^q\right)e^{-\rho(T-t)}\right]$$
$$\leq C_3(1 + \|x\|^q)e^{-c_\infty(T-t)},$$

for all $T \geq t$, where C_2, C_3 and c_∞ are positive constants not depending on T, t and x. This implies that for sufficiently large $\widetilde{T}$, it holds that

$$J(x,i,\alpha^\epsilon) \leq J^{T-t}(0,x,i,\alpha^{\epsilon,T-t}) + C_3(1 + \|x\|^q)e^{-c_\infty(T-t)} \leq J^{T-t}(0,x,i,\alpha^{\epsilon,T-t}) + \epsilon, \tag{74}$$

for all $T \geq \widetilde{T}$. Hence, we have

$$\liminf_{T\to\infty} Y_t^{T,t,x,i} \geq \liminf_{T\to\infty} J^{T-t}(0,x,i,\alpha^{\epsilon,T-t}) \geq J(x,i,\alpha^\epsilon) - \epsilon \geq v^\infty(x,i) - 2\epsilon.$$

Since ϵ is arbitrarily chosen, we obtain

$$\liminf_{T\to\infty} \widehat{Y}_t^{T,t,x,i} \geq v^\infty(x,i), \tag{75}$$

for all $(t,x,i) \in [0,\infty) \times \mathbb{R}^d \times \mathcal{I}$. Thus, we obtain the desired equality (34). For all $i \in \mathcal{I}$, the convergence of (75) is locally uniform with respect to t and x by the inequalities (73) and (74). Furthermore, $Y_t^{T,t,x,i}$ is continuous in t and x for all $T \geq 0$ and $i \in \mathcal{I}$. Therefore, $v^\infty(x,i)$ is continuous in x for all $i \in \mathcal{I}$. □

Appendix D A Viscosity Solution in the Infinite Horizon

Proof of Proposition 19. Let $v^T(t,x,i) = \widehat{Y}_t^{T,t,x,i}$ for $0 \leq t \leq T$, $x \in \mathbb{R}^d$ and $i \in \mathcal{I}$. By the definition, we have

$$v^T(t,x,i) \geq \max_{j\in\mathcal{I}\setminus\{i\}}\{v^T(t,x,j) - c_{i,j}(x)\},$$

for all $0 \leq t \leq T$, $x \in \mathbb{R}^d$ and $i \in \mathcal{I}$. Hence, taking a limit, we have

$$v^\infty(x,i) \geq \max_{j\in\mathcal{I}\setminus\{i\}}\{v^\infty(x,j) - c_{i,j}(x)\},$$

for all $(x,i) \in \mathbb{R}^d \times \mathcal{I}$.

Furthermore, we can show the followings.

Lemma 24 *For all $T > 0$, $x \in \mathbb{R}^d$ and $i \in \mathcal{I}$, $v^T(\cdot, x, i)$ is non-increasing. Furthermore, there exists a positive constant C such that*

$$|v^T(t,x,i) - v^T(s,x,i)| \le C(1 + \|x\|^q), \tag{76}$$

for all $0 \le s \le t \le T$, $x \in \mathbb{R}^d$ and $i \in \mathcal{I}$.

We will show Lemma 24 after the proof of Proposition 19.

Let $C^2(\mathbb{R}^d)$ be a set of twice continuously differentiable functions from $\mathbb{R}^d$ onto $\mathbb{R}$. Let $B(x) = \{y \in \mathbb{R}^d \mid \|y - x\| \le 1\}$ be a unit ball on $\mathbb{R}^d$ centered on x. Now, let us show the viscosity solution property of v^∞.

Step.1 Viscosity subsolution. We arbitrarily choose $\varphi \in C^2(\mathbb{R}^d)$ and $\overline{x} \in \mathbb{R}^d$ such that $\max\{v^\infty(\cdot, i) - \varphi\} = v^\infty(\overline{x}, i) - \varphi(\overline{x}) = 0$. Let

$$\widehat{\varphi}(x) := \varphi(x) + \|x - \overline{x}\|^4.$$

Let $(t_k, x_k) \in [0, k] \times B(\overline{x})$ for all $k = 1, 2, 3, \dots$ such that

$$\max\{v^k(\cdot,\cdot,i) - \widehat{\varphi}\} = v^k(t_k, x_k, i) - \widehat{\varphi}(x_k).$$

Since $v^k(\cdot, x, i)$ is non-increasing for all $x \in \mathbb{R}^d$ by Lemma 24, we have $t_k = 0$ for all k. We choose a subsequence of $(x_k)_{k\ge1}$ which converges to some $x_0 \in \mathbb{R}^d$. For convenience, we also denote this subsequence by $(x_k)_{k\ge1}$. Then, since $(x_k)_{k\ge1} \subseteq B(\overline{x})$, the Dini theorem leads to

$$\lim_{k\to\infty} v^k(0, x_k, i) = v^\infty(x_0, i).$$

Thus, we have

$$\begin{aligned}
0 &\le v^\infty(\overline{x}, i) - \varphi(\overline{x}) - (v^\infty(x_0, i) - \varphi(x_0)) \\
&\le \lim_{k\to\infty} \Big(v^k(0, \overline{x}, i) - \widehat{\varphi}(\overline{x}) - (v^k(0, x_k, i) - \widehat{\varphi}(x_k)) - \|x_k - \overline{x}\|^4 \Big) \\
&\le \lim_{k\to\infty} \Big(- \|x_k - \overline{x}\|^4 \Big) = -\|x_0 - \overline{x}\|^4.
\end{aligned}$$

Hence, $x_0 = \overline{x}$.

Now, by Proposition 13, for all $k \ge 1$, we have

$$\begin{aligned}
0 &\ge -\frac{\partial \widehat{\varphi}(x_k)}{\partial t} - \mathcal{L}^i \widehat{\varphi}(x_k) - \psi(x_k, i) + \rho v^k(0, x_k, i) + \varsigma(x_k, i, \sigma'(x_k, i)\nabla\widehat{\varphi}(x_k)) \\
&= -\mathcal{L}^i \widehat{\varphi}(x_k) - \psi(x_k, i) + \rho v^k(0, x_k, i) + \varsigma(x_k, i, \sigma'(x_k, i)\nabla\widehat{\varphi}(x_k)).
\end{aligned}$$

Hence, by the Dini theorem, taking a limit of the above inequality, we have

$$0 \ge -\mathcal{L}^i \varphi(\overline{x}) - \psi(\overline{x}, i) + \rho v^\infty(\overline{x}, i) + \varsigma(\overline{x}, i, \sigma'(\overline{x}, i)\nabla\varphi(\overline{x})).$$

This implies that v^∞ is a viscosity subsolution of the PDE (35).

Step.2 Viscosity supersolution. We arbitrarily choose $\varphi \in C^2(\mathbb{R}^d)$ and $\underline{x} \in \mathbb{R}^d$ such that $\min\{v^\infty(\cdot, i) - \varphi\} = v^\infty(\underline{x}, i) - \varphi(\underline{x}) = 0$. For $m = 1, 2, 3, \dots,$ let

$$\varphi_m(t, x) := \varphi(x) - \|x - \underline{x}\|^4 - \frac{t}{m}$$

Now, fix an arbitrary m temporarily. Let $(t_k, x_k) \in [0, k] \times B(\underline{x})$ for all $k = 1, 2, 3, \dots$ such that

$$\min\{v^k(\cdot,\cdot,i) - \varphi_m\} = v^k(t_k, x_k, i) - \varphi_m(t_k, x_k).$$

For any $k \geq 1,\ t \in [0, k]$ and $x \in B(\underline{x})$, by Lemma 24, we have

$$\begin{aligned} v^k(0,x,i) - \varphi_m(0,x) - (v^k(t,x,i) - \varphi_m(t,x)) &\leq -\frac{t}{m} + C\Big(1 + \|x\|^q\Big) \\ &\leq -\frac{t}{m} + C\Big(1 + \max_{y \in B(\underline{x})} \|y\|^q\Big). \end{aligned}$$

We now suppose that

$$t > mC\Big(1 + \max_{y \in B(\underline{x})} \|y\|^q\Big). \tag{77}$$

Then,

$$\begin{aligned} v^k(0,x,i) - \varphi_m(0,x) - (v^k(t,x,i) - \varphi_m(t,x)) &\leq -\frac{t}{m} + C\Big(1 + \max_{y \in B(\underline{x})} \|y\|^q\Big) \\ &< 0, \end{aligned}$$

for all t satisfying the inequality (77). This implies that for sufficient large $\widetilde{k}$, all t_k with $k \geq \widetilde{k}$ are in the following compact subset.

$$\left[0, mC\Big(1 + \max_{y \in B(\underline{x})} \|y\|^q\Big)\right].$$

Now, we choose a subsequence of $(t_k, x_k)_{k \geq \widetilde{k}}$ converging some (t_0, x_0). We also write this subsequence as $(t_k, x_k)_{k \geq 1}$ for convenience. Then, by the Dini theorem, we have

$$\lim_{k \to \infty} v^k(t_k, x_k, i) = v^\infty(x_0, i).$$

Hence, we have

$$\begin{aligned} 0 &\leq v^\infty(x_0,i) - \varphi(x_0) - (v^\infty(\underline{x},i) - \varphi(\underline{x})) \\ &\leq \lim_{k \to \infty} \Big(v^k(t_k,x_k,i) - \varphi_m(t_k,x_k) - (v^k(t_k,\underline{x},i) - \varphi_m(t_k,\underline{x})) - \|x_k - \underline{x}\|^4\Big) \\ &\leq \lim_{k \to \infty} \Big(-\|x_k - \underline{x}\|^4\Big) = -\|x_0 - \underline{x}\|^4, \end{aligned}$$

so $x_0 = \underline{x}$.

Now, by Proposition 13, we have

$$\begin{aligned} 0 &\leq -\frac{\partial \varphi_m(t_k,x_k)}{\partial t} - \mathcal{L}^i \varphi_m(t_k,x_k) - \psi(x_k,i) + \rho v^k(t_k,x_k,i) + \varsigma(x_k,i,\sigma'(x_k,i)\nabla\varphi_m(t_k,x_k)) \\ &= \frac{1}{m} - \mathcal{L}^i \varphi_m(t_k,x_k) - \psi(x_k,i) + \rho v^k(t_k,x_k,i) + \varsigma(x_k,i,\sigma'(x_k,i)\nabla\varphi_m(t_k,x_k)), \end{aligned}$$

for all $k \geq 1$. Thus, by the Dini theorem, taking a limit with respect to k, we have

$$0 \leq \frac{1}{m} - \mathcal{L}^i \varphi(\underline{x}) - \psi(\underline{x},i) + \rho v^\infty(\underline{x},i) + \varsigma(\underline{x},i,\sigma'(\underline{x},i)\nabla\varphi(\underline{x})).$$

Since m is arbitrarily chosen, tending m to infinity, we have

$$0 \leq -\mathcal{L}^i \varphi(\underline{x}) - \psi(\underline{x},i) + \rho v^\infty(\underline{x},i) + \varsigma(\underline{x},i,\sigma'(\underline{x},i)\nabla\varphi(\underline{x})).$$

This implies that v^∞ is a viscosity supersolution of the PDE (35). □

Proof of Lemma 24. For all $0 \le h \le t \le T,\ x \in \mathbb{R}^d$ and $i \in \mathcal{I}$, we have

$$
\begin{aligned}
v^T(t,x,i) = \widehat{Y}_t^{T,t,x,i} &= \widehat{Y}_{t-h}^{T-h,t-h,x,i} \quad \text{(time-homogeneous Markov property)} \\
&\le \widehat{Y}_{t-h}^{T,t-h,x,i} \quad \text{(monotonicity of } \widehat{Y}\text{)} \\
&= v^T(t-h,x,i).
\end{aligned}
$$

Hence, $v^T(\cdot,x,i)$ is non-increasing for all $T>0,\ x \in \mathbb{R}^d$ and $i \in \mathcal{I}$.

Now, we prove the inequality (76). Since $t \to v^T(t,x,i)$ is non-increasing for all T,x and i, it suffices to derive an upper boundary of $v^T(0,x,i) - v^T(T,x,i)$. Then, by the polynomial growth conditions for ϕ,ψ,g, and c and Propositions 2 and 5, it is easy to show that

$$
\begin{aligned}
&v^T(0,x,i) - v^T(T,x,i) = \widehat{Y}_0^{T,0,x,i} - g(x,i) \\
&\le \sup_{\alpha\in\mathbb{A}_i[0,T]} \inf_{\theta\in\Theta[0,T]} \mathbb{E}\left[\int_0^T \zeta_t^{\theta,0} e^{-\rho t}\Big(\psi(X_t^{0,x,i,\alpha},\alpha_t) - \theta_t'\phi(X_t^{0,x,i,\alpha},\alpha_t)\Big)\mathrm{d}t \right.\\
&\qquad \left. +\zeta_T^{\theta,0} e^{-\rho T} g(X_T^{0,x,i,\alpha},i) - \sum_{0\le\tau_k\le T} \zeta_{\tau_k}^{\theta,0} e^{-\rho\tau_k} c_{i_{k-1},i_k}(X_{\tau_k}^{0,x,i,\alpha})\right] - g(x,i) \\
&\le \sup_{\alpha\in\mathbb{A}_i[0,T]} \mathbb{E}\Big[e^{-\rho T} g(X_T^{0,x,i,\alpha},i) - g(x,i) \\
&\qquad \left. + \int_0^T e^{-\rho t}\psi(X_t^{0,x,i,\alpha},\alpha_t)\mathrm{d}t - \sum_{0\le\tau_k\le T} e^{-\rho\tau_k} c_{i_{k-1},i_k}(X_{\tau_k}^{0,x,i,\alpha})\right] \\
&\le C(1+\|x\|^q),
\end{aligned}
$$

where C is a positive constant not depending on T and x. □

References

[1] E. Bayraktar and M. Egami, *On the one-dimensional optimal switching problem*, Mathematics of Operations Research, 35 (2010), pp. 140–159, URL `http://dx.doi.org/10.1287/moor.1090.0432`.

[2] J.-M. Bismut, *Contrôle des systèms linéares quadratiques: Applications de l'intégrale stochastique*, Séminaire de Probabilités XII, Lecture Notes in Mathematics, 649 (1978), pp. 180–264.

[3] K. A. Brekke and B. Øksendal, *Optimal switching in an economic activity under uncertainty*, SIAM Journal on Control and Optimization, 32 (1994), pp. 1021–1036, URL `http://dx.doi.org/10.1137/S0363012992229835`.

[4] M. J. Brennan and E. S. Schwartz, *Evaluating natural resource investments*, Journal of Business, 58 (1985), pp. 135–157, URL `http://www.jstor.org/stable/2352967`.

[5] Z. Chen and L. Epstein, *Ambiguity, risk, and asset returns in continuous time*, Econometrica, 70 (2002), pp. 1403–1443, URL `http://dx.doi.org/10.1111/1468-0262.00337`.

[6] X. Cheng and F. Riedel, *Optimal stopping under ambiguity in continuous time*, Mathematics and Financial Economics, 7 (2013), pp. 29–68, URL `http://dx.doi.org/10.1007/s11579-012-0081-6`.

[7] M. G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bulletin of the American Mathematical Society, 27 (1992), pp. 1–67, URL http://dx.doi.org/10.1090/S0273-0979-1992-00266-5.

[8] A. Dixit, *Entry and exit decisions under uncertainty*, Journal of Political Economy, 97 (1989), pp. 620–638, URL http://www.jstor.org/stable/1830458.

[9] B. Djehiche, S. Hamadène, and A. Popier, *A finite horizon optimal multiple switching problem*, SIAM Journal on Control and Optimization, 48 (2009), pp. 2751–2770, URL http://dx.doi.org/10.1137/070697641.

[10] B. El Asri, *Optimal multi-modes switching problem in infinite horizon*, Stochastics and Dynamics, 10 (2010), pp. 231–261, URL http://dx.doi.org/10.1142/S0219493710002930.

[11] N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng, and M. C. Quenez, *Reflected solutions of backward SDE's, and related obstacle problems for PDE's*, The Annals of Probability, 25 (1997), pp. 702–737, URL http://dx.doi.org/10.1214/aop/1024404416.

[12] N. El Karoui, S. Peng, and M. C. Quenez, *Backward stochastic differential equations in finance*, Mathematical Finance, 7 (1997), pp. 1–71, URL http://dx.doi.org/10.1111/1467-9965.00022.

[13] D. Ellsberg, *Risk, ambiguity, and the Savage axioms*, Quarterly Journal of Economics, 75 (1961), pp. 643–669, URL http://dx.doi.org/10.2307/1884324.

[14] I. Gilboa and D. Schmeidler, *Maxmin expected utility with non-unique prior*, Journal of Mathematical Economics, 18 (1989), pp. 141–153, URL http://dx.doi.org/10.1016/0304-4068(89)90018-9.

[15] S. Hamadène and M.-A. Morlais, *Viscosity solutions of systems of PDEs with interconnected obstacles and switching problem*, Applied Mathematics and Optimization, 67 (2013), pp. 163–196, URL http://dx.doi.org/10.1007/s00245-012-9184-y.

[16] S. Hamadène and J. Zhang, *Switching problem and related system of reflected backward SDEs*, Stochastic Processes and their Applications, 120 (2010), pp. 403–426, URL http://dx.doi.org/10.1016/j.spa.2010.01.003.

[17] Y. Hu and S. Tang, *Multi-dimensional BSDE with oblique reflection and optimal switching*, Probability Theory and Related Fields, 147 (2010), pp. 89–121, URL http://dx.doi.org/10.1007/s00440-009-0202-1.

[18] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer-Verlag New York, 2 ed., 1991.

[19] N. V. Krylov, *Controlled Diffusion Processes*, Springer-Verlag Berlin Heidelberg, 1980.

[20] V. Ly Vath and H. Pham, *Explicit solution to an optimal switching problem in the two-regime case*, SIAM Journal on Control and Optimization, 46 (2007), pp. 395–426, URL http://dx.doi.org/10.1137/050638783.

[21] M.-M. Ngo and H. Pham, *Optimal switching for the pairs trading rule: A viscosity solutions approach*, Journal of Mathematical Analysis and Applications, 441 (2016), pp. 403–425, URL http://dx.doi.org/10.1016/j.jmaa.2016.03.060.

[22] E. Pardoux and S. Peng, *Adapted solution of a backward stochastic differential equation*, Systems and Control Letters, 14 (1990), pp. 55–61, URL http://dx.doi.org/10.1016/0167-6911(90)90082-6.

[23] S. Peng, *Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyer's type*, Probability Theory and Related Fields, 113 (1999), pp. 473–499, URL http://dx.doi.org/10.1007/s004400050214.

[24] H. Pham, *Continuous-Time Stochastic Control and Optimization with Financial Applications*, Springer-Verlag Berlin Heidelberg, 2009.

[25] F. Riedel, *Optimal stopping with multiple priors*, Econometrica, 77 (2009), pp. 857–908, URL http://dx.doi.org/10.3982/ECTA7594.

[26] R. Rouge and N. El Karoui, *Pricing via utility maximization and entropy*, Mathematical Finance, 10 (2000), pp. 259–276, URL http://dx.doi.org/10.1111/1467-9965.00093.

[27] D. Schmeidler, *Subjective probability and expected utility without additivity*, Econometrica, 57 (1989), pp. 571–587, URL http://dx.doi.org/10.2307/1911053.

[28] H. Zhang and Q. Zhang, *Trading a mean-reverting asset: Buy low and sell high*, Automatica, 44 (2008), pp. 1511–1518, URL http://dx.doi.org/10.1016/j.automatica.2007.11.003.