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Repeated Games with Recursive Utility: Cournot Duopoly under Gain/Loss Asymmetry

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Abstract

We study the repeated Cournot duopoly with recursive utility where the players discount gains more than losses. First, as in the standard model of discounted utility, we confirm that the optimal punishment equilibrium has a stick-and-carrot structure. Next, we explore its exact form in relation to the role of the asymmetry in discounting. We find that the discount factor used to evaluate losses controls the deterrence of a given punishment, while the discount factor used to evaluate gains influences the enforceability of the penalty. An increase in one of the two discount factors increases the most collusive equilibrium profit unless full collusion is already sustainable. However, the key to collusion is the loss discount factor: regardless of the level of the gain discount factor, full cooperation can be achieved if the loss discount factor is sufficiently high.

Keywords: Cournot duopoly, gain/loss asymmetry, optimal penal code, repeated game, recursive utility, utility smoothing

JEL Classification: C73, D20, D90, L13

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1 Introduction

Repeated games are used to show that mutual cooperation is sustainable in noncooperative environments. A seminal contribution to this field by Abreu (1988) shows that all of the pure strategy subgame perfect equilibrium outcome paths are generated by the reversion strategy profile, where any deviation is penalized by the worst subgame perfect equilibrium called the optimal penal code. Abreu (1986) applies the observation to the repeated Cournot oligopoly. He shows that for the symmetric subgame perfect equilibria, the optimal penal code has a stick-and-carrot structure: a firm deviating from a target outcome path will be penalized severely until it accepts the penalty, and all firms start playing the most cooperative equilibrium path as compensation if the deviator accepts the penalty. This stick-and-carrot optimal penal code can implement an outcome path that may not be sustainable under the conventional Nash reversion strategy profile.

These results, as well as almost all well-known results in repeated games without complete patience, assume that players evaluate a payoff sequence using the discounted utility model. However, experimental studies have reported results that seem to contradict the discounted utility model. One of these anomalies is gain/loss asymmetry, which is a phenomenon whereby a decision maker tends to discount gains more than he discounts losses. This anomaly was first documented by Thaler (1981) and repeatedly confirmed in many subsequent experiments.[1] However, to the best of our knowledge, the economic implications of gain/loss asymmetry have not been investigated in the context of repeated games.

Given the above observations, we analyze the effect of gain/loss asymmetry on the equilibria of repeated games, particularly via the repeated Cournot duopoly.[2] We focus on a version of the preference representation suggested by Wakai (2008), which identifies a notion of intertemporal utility smoothing as a source of gain/loss asymmetry.[3] Formally, we assume that at each time t, the firms evaluate a payoff

[1]See Frederick, Loewenstein, and O'Donoghue (2002) for a survey of experimental studies on gain/loss asymmetry.

[2]Our results can be easily extended to the Cournot oligopoly.

[3]Wakai (2008) shows that his notion of intertemporal utility smoothing is consistent with the experimental result called a *preference for spread*, that is, a subject prefers to spread good and bad outcomes evenly over time.

sequence $U = (u_0, u_1, ...)$ based on the following function:

$$V_t(U) \equiv \min_{\{\delta_{t+\tau}\}_{\tau=1}^{\infty} \in [\underline{\delta}, \overline{\delta}]^{\infty}} \left\{ \sum_{\tau=0}^{\infty} (1 - \delta_{t+\tau+1}) \left(\prod_{\tau'=1}^{\tau} \delta_{t+\tau'} \right) u_{t+\tau} \right\}, \tag{1}$$

where $\overline{\delta}$ and $\underline{\delta}$ are parameters satisfying $0 < \underline{\delta} \leq \overline{\delta} < 1$. The discounted utility model is a special case of evaluating function (1), where $\underline{\delta} = \overline{\delta}$.

To show how gain/loss asymmetry is incorporated into (1), we rewrite it as follows:

$$\begin{aligned} V_t(U) &= \min_{\delta \in [\underline{\delta}, \overline{\delta}]} [(1 - \delta) u_t + \delta V_{t+1}(U)] \\ &= u_t + \underline{\delta} \max \{V_{t+1}(U) - u_t, 0\} + \overline{\delta} \min \{V_{t+1}(U) - u_t, 0\}. \end{aligned} \tag{2}$$

Evaluating function (2) belongs to a class of the recursive utility suggested by Koopmans (1960) that describes history-independent, stationary, and dynamically consistent preferences. In particular, (2) exhibits a key feature called *recursive gain/loss asymmetry*: the difference between future value $V_{t+1}(U)$ and current payoff u_t defines a gain or loss, and gains and losses are discounted by $\underline{\delta}$ and $\overline{\delta}$, respectively. Thus, gains are discounted more than losses. We call $\underline{\delta}$ and $\overline{\delta}$ the gain discount factor and the loss discount factor, respectively, whereas the distance between $\overline{\delta}$ and $\underline{\delta}$ represents a degree of gain/loss asymmetry.

Because gain/loss asymmetry is defined on the recursive preferences, we first isolate the characteristics of the set of pure strategy subgame perfect equilibria induced by the general form of history-independent and stationary recursive utility. We confirm that the implications derived by Abreu (1986, 1988), the existence of the optimal penal code and its stick-and-carrot structure in the Cournot duopoly, extend to the recursive utility. Apart from recursivity that induces dynamically consistent decision, two more assumptions, monotonicity and continuity, are essential to the above results. In particular, the continuity, which is defined on the product topology of the compact payoff space, is crucial because it extends the notion of positive discounting to recursive utility and allows us to establish the one-deviation property.

Having identified the above properties, we focus on the characteristics implied by gain/loss asymmetry via the repeated Cournot duopoly. First, we show that the deterrence of the optimal penal code used to implement the most collusive output is measured by

$$\frac{\overline{\delta}}{1 - \overline{\delta}} \frac{1 - \underline{\delta}}{\underline{\delta}}.$$

This ratio is one for the discounted utility model, but it exceeds one if $\overline{\delta}$ and $\underline{\delta}$ differ. Thus, gain/loss asymmetry increases the level of deterrence because the firms have a stronger incentive to avoid losses than to receive gains so that a less powerful penal code is sufficient to implement a given level of a target payoff sequence.

Second, we examine the role of each of the discount factors separately. We show that an increase in a discount factor, either $\overline{\delta}$ or $\underline{\delta}$, results in a more collusive profit and a stronger penalty, unless either full collusion or the minimax value is already sustainable by an equilibrium. This generalizes the known result for the discounted utility because an increase in either discount factor corresponds to an increase in patience. However, the mechanism that leads to the above result depends on which discount factor increases. The loss discount factor $\overline{\delta}$ controls the deterrence of the optimal penal code because it evaluates the future loss invoked by the current deviation. Thus, as $\overline{\delta}$ increases, the increased fear of suffering from loss allows the same penal code to implement a more collusive path. A higher collusive profit also implements a severer current penalty even when $\underline{\delta}$ remains unchanged. On the other hand, the gain discount factor $\underline{\delta}$ controls the enforceability of the optimal penalty because it evaluates the future gain received as compensation for accepting the current penalty. Thus, as $\underline{\delta}$ increases, the increased anticipation of future gain allows firms to implement a severer current penalty. This severer penalty also strengthens the punishment and therefore implements a more collusive path even when $\overline{\delta}$ remains unchanged.

Third, we derive two Folk theorem–type results and confirm that the loss discount factor $\overline{\delta}$ is the key to collusion. One result is an extension of the standard Folk theorem and states that if the loss discount factor $\overline{\delta}$ is sufficiently high, firms can engage in full cooperation by producing half of the monopoly output, regardless of the level of $\underline{\delta}$. This happens because $\overline{\delta}$ solely determines the deterrence of the Nash reversion strategy profile, which is strong enough to implement the full cooperation if $\overline{\delta}$ is sufficiently high.

The other Folk theorem–type result is a *reverse* Folk theorem, which is new and unique to evaluating function (2). This result states that as $\underline{\delta}$ goes to zero, (i) the optimal penalty converges to the Cournot equilibrium output, and (ii) the best cooperative profit converges to the highest profit implementable by a Nash reversion strategy profile, which is independent of $\underline{\delta}$. This occurs because for each $\overline{\delta}$, $\underline{\delta}$ determines the enforceability of the optimal penalty, which becomes weaker as $\underline{\delta}$ decreases. By showing that the best cooperative profit asymptotically depends only on $\overline{\delta}$, the reverse Folk theorem also emphasizes the key role of the loss discount factor.

Fourth, we investigate how a stronger desire for smoothing allocations over time affects collusion. Such a desire is expressed by increasing $\overline{\delta}$ or by decreasing $\underline{\delta}$, because the firm with evaluating function (1) dislikes the volatility involved in the payoff (or utility) sequence more than the firm with either smaller $\overline{\delta}$ or bigger $\underline{\delta}$. As we have seen, a higher $\overline{\delta}$ leads to more collusive outcomes, and a lower $\underline{\delta}$ leads to less collusive outcomes. Thus, if the former effect outweights the latter effect, a stronger desire results in more collusive outcomes, and *vice versa.* This confirms that the level of the loss discount factor $\overline{\delta}$, not the degree of gain/loss asymmetry, is a key to collusion.

We now review the related literature. As for the models of discounting that exhibit gain/loss asymmetry, evaluating function (2) is closely related to the models suggested by Loewenstein and Prelec (1992) and Shalev (1997). Loewenstein and Prelec (1992) show that if a gain or a loss is defined based on a variation in a utility sequence, gain/loss asymmetry can capture a preference for smoothing a utility distribution over time, which is consistent with several experimental findings contradicting the additive separability assumed in the discounted utility model. Shalev (1997) adopts a related but different notion of utility smoothing suggested by Gilboa (1989) and analyzes a behavior similar to loss aversion (Kahneman and Tversky (1979), Tversky and Kahneman (1991)). However, the aforementioned models describe static choices and, if we apply them to a dynamic setting, they generate dynamically inconsistent decision. On the contrary, evaluating function (2) induces stationary and dynamically consistent decision, which allows us to extend the known results for the discounted utility to the case of gain/loss asymmetry.

This paper contributes to the literature showing that the stick-and-carrot structure of the optimal punishment, first developed by Abreu (1986) in the context of the Cournot oligopoly with homogenous products, extends to other oligopolistic environments. Those extensions include the case of imperfect price information (Abreu, Pearce, and Stacchetti (1986)), Bertrand price competition (Häckner (1996)), and oligopoly with differentiated products (Lambertini and Sasaki (1999)). The idea of stick-and-carrot strategies also applies to tariff-setting trade liberalization (Furusawa (1999)). In contrast, this paper extends to the recursive utility and provides explicit and readily testable formulae for the optimal collusion and punishments under gain/loss asymmetry. We limit our attention to the standard Cournot model with homogenous products, but the main insights easily apply to the frameworks of the aforementioned papers.

Some papers study repeated games without the assumption of discounted utility.

Chade, Prokopovych, and Smith (2008) assume that the players have β–δ preferences, and characterize the Strotz-Pollak equilibria by Peleg and Yaari (1973). Obara and Park (2015) extend the analysis to general discounting functions including the β–δ ones as special cases.[4] We explore the implications of general recursive utility.

In that sense, their companion paper (Obara and Park (2014)) is more closely related to ours because it considers a class of time preferences including recursive utilities as special cases. The authors' results on the structure of Strotz-Pollak equilibria are similar to our results under recursive utility.[5] The difference is that while Obara and Park (2014) examine how the general methodology for the discounted utility model extends to general time preferences possibly without recursive structure (under the Strotz-Pollak equilibrium concept), we focus on the case of recursive gain/loss asymmetry and provide its implications on collusion in detail.

Kochov and Song (2015) employ the recursive utility criterion by Uzawa (1968) and Epstein (1983), where a player's discount factor for the next period depends on the current period payoff. For our criterion, the discount factor also depends on the continuation payoff from the next period on. The difference between the two models is salient when the Uzawa-Epstein type utility has a discount factor that is decreasing in the current period payoff. In this case, Kochov and Song (2015) show that the most efficient equilibrium can involve intertemporal trade so that the players alternately play different action profiles. Our model does not favor such an implication because the players with recursive gain/loss asymmetry dislike volatility brought about by the intertemporal trade under their equilibrium.

The paper proceeds as follows. Section 2 defines the setting of the game and examines the behavior induced by evaluating function (2). Assuming a general form of recursive utility, Section 3 characterizes the optimal penal code and the associated reversion strategy profile as well as the structure of the equilibria of the repeated Cournot duopoly. Section 4 then examines the effect of gain/loss asymmetry on the equilibria of the repeated Cournot duopoly. Section 5 concludes the paper. All proofs are presented in the appendices.

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[4]The β–δ preferences are a class of the discount functions with present bias. Obara and Park (2015) consider both the discounting functions with present bias and the functions with future bias.

[5]The Strotz-Pollak equilibrium concept reduces to the standard equilibrium concept under recursive utility.

2 Model

2.1 General Setting

The stage game, denoted by $G = \left(I, \{A_i\}_{i=1}^{I}, \{u_i\}_{i=1}^{I}\right)$, is an I-player simultaneous move game, where player i's action space A_i is a compact topological space and player i's payoff function $u_i : \prod_{i=1}^{I} A_i \to \mathbb{R}$ is continuous. We define A by $A \equiv \prod_{i=1}^{I} A_i$ and use $q^{(t)} \equiv (q_1^{(t)}, ..., q_I^{(t)}) \in A$ to denote a vector of actions taken at time t by all players, where time t varies over $\mathbb{N} = \{0, 1, 2, ...\}$. We refer to $q^{(t)}$ as an action profile or a time-t action profile if we want to emphasize the time period.

We consider the supergame G^∞ obtained by repeating game G infinitely often. For each t with $t > 0$, let H^{t-1} be defined by $H^{t-1} \equiv A^t$, each element of which, $h^{t-1} \equiv (q^{(0)}, ..., q^{(t-1)})$, is a series of realized actions at all periods before period t. We assume that $H^{-1} \equiv A^0$, which consists of a single element. For each player i, we focus on a pure strategy s_i, that is, a sequence of functions $s_i \equiv \{s_{i,t}\}_{t=0}^{\infty}$, where $s_{i,t} : H^{t-1} \to A_i$ for each t. A strategy profile s is defined by $s \equiv (s_1, ..., s_I)$, and let S be the collection of strategy profiles. We define the time-t action profile $q^{(t)}(s)$ by $q^{(t)}(s) \equiv (q_1^{(t)}(s), ..., q_I^{(t)}(s))$, where $q_i^{(t)}(s)$ is the time-t action taken by player i when the players follow s. A path Q is a sequence of action profiles denoted by $Q \equiv (q^{(0)}, q^{(1)}, ...)$. In particular, $Q(s)$ is the path of the strategy profile s, that is, $Q(s) \equiv (q^{(0)}(s), q^{(1)}(s), ...)$. Moreover, for any path $Q \in (A)^\infty$, let $U_i(Q)$ be the sequence of player i's payoffs $(u_i(q_1^{(0)}, ..., q_I^{(0)}), u_i(q_1^{(1)}, ..., q_I^{(1)}), ...) \in [u_i(A)]^\infty$, where $u_i(A)$ is the range of u_i.

At each time t, player i evaluates his payoff sequence $U_i(Q)$ by a continuous and strictly monotone function $V_{i,t} : [u_i(A)]^\infty \to \mathbb{R}$, where we adopt the product topology on A^∞ as well as $[u_i(A)]^\infty$. Because A is compact and u_i is continuous on A, it follows from Tychonoff's theorem that A^∞ and $[u_i(A)]^\infty$ are compact. Moreover, the continuity of $V_{i,t}$ implies that the image of $V_{i,t}$ is compact. Given the above properties, we assume that player i's preferences follow the recursive utility defined by

$$V_{i,t}(U_i(Q)) = W_i(u_i(q_1^{(t)}, ..., q_I^{(t)}), V_{i,t+1}(U_i(Q))), \tag{3}$$

where the aggregator function $W_i : u_i(A) \times \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing in both arguments. The recursive utility represents all classes of continuous and dynamically consistent intertemporal preferences, which are history-independent, stationary, and monotone in payoffs.[6]

2.2 Example: The Repeated Cournot Duopoly

When we study the implication of gain/loss asymmetry, we use a particular example, which is the repeated Cournot duopoly having the stage game $G = \left(2, \{A_i\}_{i=1}^2, \{u_i\}_{i=1}^2\right)$ defined as follows. At each period t, the two firms simultaneously decide their supply, $q_1^{(t)}$ and $q_2^{(t)}$, of an identical product, where their production has a constant marginal cost $c > 0$ and $q_1^{(t)}$ and $q_2^{(t)}$ belong to the compact subset A_1 and A_2 of $\mathbb{R}_+$, respectively. We assume that $A_1 = A_2 = [0, M]$, and we will make assumptions on M later. Let $q^{(t)} \equiv (q_1^{(t)}, q_2^{(t)})$ be a combination of time-t products supplied to the market by firms 1 and 2. Firm i's profit is computed based on the price at which the market supply $q_1^{(t)} + q_2^{(t)}$ is equal to the market demand. To derive comparable results to those obtained by much of the applied literature on the Cournot duopoly game, we assume that the market demand for this product is expressed by a linear inverse demand function $p : \mathbb{R}_+ \to \mathbb{R}_+$, that is,

$$p(z) = \begin{cases} a - bz & \text{if } z \leq \dfrac{a}{b}, \\ 0 & \text{if } z > \dfrac{a}{b}, \end{cases}$$

where a and b are positive real numbers satisfying $a > c$ and $[0, \frac{a}{b}] \subset A_1$. Thus, the profit of firm i, when the market supply is $q_1^{(t)} + q_2^{(t)}$, is expressed by the function $u_i(q_1^{(t)}, q_2^{(t)})$, where

$$u_i(q_1^{(t)}, q_2^{(t)}) \equiv p(q_1^{(t)} + q_2^{(t)})q_i^{(t)} - cq_i^{(t)}.$$

To characterize the equilibria below, we follow Abreu (1986) and introduce a few more notations. Let $r^*(q')$ be the best response profit of one firm when the opponent's output is q'. That is,

$$r^*(q') \equiv \max_{q \in A_1} \{p(q + q')q - cq\} = \begin{cases} \dfrac{1}{b}\left(\dfrac{a - c - bq'}{2}\right)^2 & \text{if } q' \leq q^{MC}, \\ 0 & \text{if } q' > q^{MC}, \end{cases}$$

where q^{MC} is the threshold defined by

$$q^{MC} \equiv \frac{a - c}{b}.$$

[6]Given history independence, $V_{i,t}$ depends solely on $(u_i(q^{(t)}), u_i(q^{(t+1)}), ...)$.

If firm j produces more than q^{MC}, the price is forced to be below the marginal cost c. Thus, firm i has no incentive to supply the product in the stage game G. Moreover, we use $r(q)$ to denote the profit of the firm when both firms produce the same level of output q, that is, $r : A_1 \to \mathbb{R}$ that satisfies

$$r(q) \equiv \begin{cases} (a-c-2bq)\,q & \text{if } q \leq \dfrac{a}{2b}, \\ -cq & \text{if } q > \dfrac{a}{2b}. \end{cases}$$

The above definitions also imply that (i) $r^* : A_1 \to \mathbb{R}$ is continuous and decreasing and (ii) $r : A_1 \to \mathbb{R}$ is continuous.

Another notation we use is q^{cn}, which is the Cournot equilibrium quantity defined by the output level that satisfies $r^*(q^{cn}) = r(q^{cn})$. Namely,

$$q^{cn} = \frac{a-c}{3b}.$$

The most collusive output level, denoted by q^m, is defined by

$$q^m \equiv \arg\max_{q \in A_1} r(q) = \frac{a-c}{4b},$$

which is half of the monopoly output. We can then confirm that $r : A_1 \to \mathbb{R}$ is strictly decreasing if $q > q^m$.

Finally, for this example, we assume that the firms have identical recursive utility. Specifically, in (3), we have $V_0 \equiv V_{1,0} = V_{2,0}$ and $W = W_1 = W_2$. We also assume that the largest output level M satisfies

$$W(r(M), V_0(r(q^m), r(q^m), \dots))) < V_0(0, 0, \dots). \tag{4}$$

Note that (4) implies $r(M) < 0$. The above inequality shows that if both firms choose M at some period, they incur a significant loss that cannot be compensated for by sharing monopoly profits in all subsequent periods.

2.3 Recursive Preferences Exhibiting Gain/Loss Asymmetry

We primarily consider a version of the model of utility smoothing as developed by Wakai (2008):

$$\begin{aligned} V_{i,t}(U_i(Q)) &\equiv \min_{\{\delta_{t+\tau}\}_{\tau=1}^{\infty} \in [\underline{\delta_i}, \overline{\delta^i}]^{\infty}} \left\{ \sum_{\tau=0}^{\infty} (1-\delta_{t+\tau+1}) \left(\prod_{\tau'=1}^{\tau} \delta_{t+\tau'} \right) u_i(q^{(t+\tau)}) \right\} \qquad (5) \\ &= \min_{\delta \in [\underline{\delta_i}, \overline{\delta^i}]} [(1-\delta) u_i(q^{(t)}) + \delta V_{i,t+1}(U_i(Q))], \end{aligned}$$

where $\underline{\delta}_i$ and $\overline{\delta}_i$ are parameters satisfying $0 < \underline{\delta}_i \leq \overline{\delta}_i < 1$. Thus, representation (5) leads to a weighted average of a payoff sequence, where the sequence of the weights applied depends on the nature of the payoff sequence.[7] Moreover, representation (5) is a class of the recursive utility (3) with the following aggregator function

$$W_i(u_i(q^{(t)}), V_{i,t+1}(U_i(Q))) = \min_{\delta \in [\underline{\delta_i}, \overline{\delta^i}]} [(1-\delta)u_i(q^{(t)}) + \delta V_{i,t+1}(U_i(Q))]. \tag{6}$$

Following the explanation in the introduction, we will call $\underline{\delta_i}$ and $\overline{\delta_i}$ the gain discount factor and the loss discount factor, respectively, whereas the distance between $\overline{\delta_i}$ and $\underline{\delta_i}$ represents the degree of gain/loss asymmetry.

The gain/loss asymmetry introduced in (5) expresses the desire to lower the volatility involved in a payoff sequence. To see this further, let U denote a sequence of payoffs, and let $\overline{u}$ denote a sequence of a constant payoff u. Then, by following Wakai (2008), we say that firm j is *more time-variability averse* than firm i if for any $\overline{u}$ and any U,

$$V_{i,t}(\overline{u}) \geq V_{i,t}(U) \text{ implies } V_{j,t}(\overline{u}) \geq V_{j,t}(U),$$
$$\text{and the latter is strict if the former is strict.}$$

Two firms agree on the ranking of a constant payoff sequence, whereas any payoff sequence disliked by firm i is disliked by firm j. Moreover, Wakai (2008) shows that this relation is translated into the set inclusion, that is, firm j is more time-variability averse than firm i if and only if

$$[\underline{\delta_j}, \overline{\delta_j}] \supseteq [\underline{\delta_i}, \overline{\delta_i}], \tag{7}$$

where $[\underline{\delta_i}, \overline{\delta_i}]$ and $[\underline{\delta_j}, \overline{\delta_j}]$ represent sets of discount factors for $V_{i,t}$ and $V_{j,t}$, respectively. In this paper, we further extend this notion and say that firm j is *strictly more time-variability averse* than firm i if (i) firm j is more time-variability averse than firm i, and (ii) there exist $\overline{u}$ and U such that

$$V_{i,t}(\overline{u}) = V_{i,t}(U) \text{ but } V_{j,t}(\overline{u}) > V_{j,t}(U).$$

This strict relation is equivalent to (7) with the strict inclusion.

To study the role of discount factors, we further need to introduce a comparative notion that focuses on the degree of patience. For this purpose, let $(u_0, \overline{u_1})$ denote

[7] It is easy to see that $\sum_{\tau=0}^{\infty} (1 - \delta_{t+\tau+1}) \left(\prod_{\tau'=1}^{\tau} \delta_{t+\tau'} \right) = 1$ for any t.

a sequence of payoffs, where u_0 is the payoff of the first period and $\overline{u_1}$ is a constant continuation payoff of u_1. We say that firm j is *more patient* than firm i if for any sequence $(u_0, \overline{u_1})$ satisfying $u_0 \leq u_1$ and any $\overline{u}$,

$$V_{i,t}((u_0, \overline{u_1})) \geq V_{i,t}(\overline{u}) \text{ implies } V_{j,t}((u_0, \overline{u_1})) \geq V_{j,t}(\overline{u}),$$
$$\text{and the latter is strict if the former is strict,}$$

and for any sequence $(u_0, \overline{u_1})$ satisfying $u_0 \geq u_1$ and any $\overline{u}$,

$$V_{i,t}((u_0, \overline{u_1})) \leq V_{i,t}(\overline{u}) \text{ implies } V_{j,t}((u_0, \overline{u_1})) \leq V_{j,t}(\overline{u}),$$
$$\text{and the latter is strict if the former is strict.}$$

Two firms agree on the ranking of a constant payoff sequence, whereas any *increasing* payoff sequence liked by firm i is liked by firm j and any *decreasing* payoff sequence disliked by firm i is disliked by firm j. This relation is translated into the following set ordering, that is, firm j is more patient than firm i if and only if

$$\underline{\delta_i} \leq \underline{\delta_j} \text{ and } \overline{\delta_i} \leq \overline{\delta_j}. \tag{8}$$

We also say that firm j is *strictly more patient* than firm i if (i) firm j is more patient than firm i, and (ii) there exist $\overline{u}$ and $(u_0, \overline{u_1})$ such that

$$V_{i,t}((u_0, \overline{u_1})) = V_{i,t}(\overline{u}) \text{ but } V_{j,t}((u_0, \overline{u_1})) \neq V_{j,t}(\overline{u}).$$

This strict relation is equivalent to (8) with at least one strict inequality. Note that for the discounted utility model, the condition on increasing payoff sequences implies the condition on decreasing payoff sequences, but for evaluating function (5), one does not imply the other.

3 Equilibria under General Recursive Utility

3.1 Penal Code and Reversion Strategy Profile

Let S^* be the set of all pure strategy subgame perfect equilibria of G^∞. We assume that S^* is nonempty.[8] Moreover, for each i, we define $\underline{v}_i$ and $\overline{v}_i$ by

$$\underline{v}_i \equiv \inf \{V_{i,0}(U_i(Q(s))) \,|s \in S^*\}$$

[8]A sufficient condition applied to any W_i is that the stage game G has a pure strategy Nash equilibrium.

and

$$\overline{v}_i \equiv \sup \{V_{i,0}(U_i(Q(s)))\,|s \in S^*\},$$

where $V_{i,0}$ follows (3). The next proposition shows the existence of subgame perfect equilibria $\underline{s}^i$ and $\overline{s}^i$ in S^* under which player i's payoff is $\underline{v}_i$ and $\overline{v}_i$, respectively. This result is an extension of Abreu's (1988) Proposition 2 to the situation of recursive utility (see Appendix A for the proof).

Proposition 1: *Suppose that players evaluate payoff sequences by* (3). *Then, for each* i, *there exist subgame perfect equilibria* $\underline{s}^i$ *and* $\overline{s}^i$ *in* S^* *that satisfy* $V_{i,0}(U_i(Q(\underline{s}^i))) = \underline{v}_i$ *and* $V_{i,0}(U_i(Q(\overline{s}^i))) = \overline{v}_i$, *respectively.*

Next, for an $(I+1)$-tuple of paths $(Q, Q^1, \ldots, Q^I)$, we define a *reversion strategy profile* $s(Q, Q^1, \ldots, Q^I)$ as follows: (i) Q is the initial ongoing path, and players play it until some player deviates unilaterally from it, and (ii) if player j unilaterally deviates from the current ongoing path, Q^j becomes the next ongoing path, and they play it until some player deviates unilaterally from it.[9] If all the players follow this reversion strategy profile, the path becomes Q regardless of whether it is an equilibrium. In Appendix A, we show how to construct $\underline{s}^i$ and $\overline{s}^i$ based on the reversion strategy profiles.

A key result of Abreu (1988) is that for the discounted utility model, any subgame perfect equilibrium path Q is implemented as a subgame perfect equilibrium by the reversion strategy profile $s(Q, Q(\underline{s}^1), \ldots, Q(\underline{s}^I))$. Under the equilibrium, the ongoing path after any player's unilateral deviation is the player's worst equilibrium path. The vector $(\underline{s}^1, ..., \underline{s}^I)$ is called an *optimal penal code*. This result simplifies the analysis of subgame perfect equilibria because we can restrict our attention to the paths that are supported by the optimal penal code. The next proposition shows that the same simplification holds for the recursive utility (see Appendix A for the proof).

Proposition 2: *Suppose that players evaluate payoff sequences by* (3). *Then, for any subgame perfect equilibrium* s^* *in* S^*, *the equilibrium path* $Q(s^*)$ *can be generated as the path of the reversion strategy profile* $s(Q(s^*), Q(\underline{s}^1), ..., Q(\underline{s}^I))$, *where* $s(Q(s^*), Q(\underline{s}^1), ..., Q(\underline{s}^I))$ *is a subgame perfect equilibrium in* S^*.

[9] This reversion strategy profile $s(Q, Q^1, ..., Q^I)$ corresponds to a *simple strategy profile* in Abreu (1988).

Propositions 1 and 2 show that the nonlinearity introduced on the aggregator function W_i does not alter the topological nature of the set of pure strategy subgame perfect equilibria or the effectiveness of the reversion strategy profiles. Therefore, monotonicity, continuity, and recursivity are key properties that derive these results. Note that W_i and u_i need not be identical to W_j and u_j if $i \neq j$.

3.2 The Repeated Cournot Duopoly

For this example, we focus on a symmetric subgame perfect equilibrium of G^∞. Let Γ be the collection of all symmetric paths, and let $S^*(\Gamma)$ be the collection of all subgame perfect equilibria whose continuation paths on and off the equilibrium are all in Γ. As shown in Lemma A.3 (see Appendix B), Propositions 1 and 2 hold for $S^*(\Gamma)$ replacing S^*. Recall that each firm evaluates a payoff sequence by the same evaluating function V_0 based on the same aggregator function W. Thus, we use $s^v = (s_1^v, s_2^v)$ and $s^w = (s_1^w, s_2^w)$ to denote each player's best and worst symmetric subgame perfect equilibria in $S^*(\Gamma)$, respectively. For any $s^* \in S^*(\Gamma)$, the reversion strategy profile $s(Q(s^*), Q(s^w), Q(s^w))$ is a symmetric subgame perfect equilibrium.

In what follows, a symmetric subgame perfect equilibrium will be simply called an equilibrium. The next proposition shows that the optimal penal code and the best equilibrium satisfy a structure analogous to those of equilibria under the discounted utility model (see Appendix B for the proof).

Proposition 3: *Suppose that firms evaluate payoff sequences by the same form of evaluating function* (3)*.*

(i) The best equilibrium path, denoted by Q^v, is unique, where both firms produce q^v satisfying $q^m \leq q^v \leq q^{cn}$ in each period.

(ii) Let q^w be the largest solution of

$$W(r(q), V_0(U_i(Q^v))) = V_0(r^*(q), r^*(q), \dots), \tag{9}$$

and define the path Q^w by $Q^w = ((q^w, q^w), (q^v, q^v), (q^v, q^v), \dots)$. Then, the reversion strategy profile $s(Q^w, Q^w, Q^w)$ is a worst equilibrium.

(iii) It follows that

$$V_0(U_i(Q^v)) = V_0(r(q^v), r(q^v), \dots) \geq W(r^*(q^v), V_0(U_i(Q^w))), \tag{10}$$

where the inequality is strict only if $q^v = q^m$. Moreover,

$$V_0(U_i(Q^w)) = V_0(r^*(q^w), r^*(q^w), \dots). \tag{11}$$

Proposition 3-(i) and (ii) correspond to Abreu's (1986) Theorem 9 and 13, respectively. The results show that the best equilibrium path is constant, and there exists a worst equilibrium with a stick-and-carrot structure. Namely, under the worst equilibrium path, the firms first produce a penalty output (q^w) and then play the best equilibrium path (Q^v) from the next period onward. Any unilateral deviation lets them restart the path. Again, monotonicity, continuity, and recursivity are key properties to the above result.

4 Implications from Gain/Loss Asymmetry

We study the implications from gain/loss asymmetry via the repeated Cournot duopoly. Because we assume $V_0 = V_{1,0} = V_{2,0}$ and $W = W_1 = W_2$, we define $\overline{\delta} \equiv \overline{\delta_1} = \overline{\delta_2}$ and $\underline{\delta} \equiv \underline{\delta_1} = \underline{\delta_2}$. Moreover, for any output level q, we define $\overleftarrow{q} = (q, q)$. Proposition 3 implies that there uniquely exist q^v and q^w such that (a) any best equilibrium has the path $Q^v \equiv (\overleftarrow{q}^v, \overleftarrow{q}^v, \dots)$ and (b) $Q^w \equiv (\overleftarrow{q}^w, \overleftarrow{q}^v, \overleftarrow{q}^v, \dots)$ is a worst equilibrium path. Because these q^v and q^w can be considered functions of $\underline{\delta}$ and $\overline{\delta}$, we analyze their dependence on $\underline{\delta}$ and $\overline{\delta}$.

Let us adopt Proposition 3 to evaluating function (5). First, the definitions of Q^v and Q^w imply that $V_0(U_i(Q^v)) = V_0\big(r(q^v), r(q^v), \dots\big) = r(q^v)$ and

$$\begin{aligned} V_0(U_i(Q^w)) = W(r(q^w), V_0(U_i(Q^v))) &= \min_{\delta \in [\underline{\delta}, \overline{\delta}]} [(1-\delta) r(q^w) + \delta r(q^v)] \\ &= (1-\underline{\delta}) r(q^w) + \underline{\delta} r(q^v), \end{aligned} \tag{12}$$

where (12) follows because $r(q^v) \geq r(q^w)$. Combining this with (11), we obtain

$$(1-\underline{\delta}) r(q^w) + \underline{\delta} r(q^v) = r^*(q^w). \tag{13}$$

Now, suppose that the full collusion is not sustainable, so that $q^v > q^m$. Then, (10) and (11) imply

$$V_0(U_i(Q^v)) = W(r^*(q^v), V_0(U_i(Q^w))) = \min_{\delta \in [\underline{\delta}, \overline{\delta}]} [(1-\delta) r^*(q^v) + \delta r^*(q^w)],$$

that is,

$$r(q^v) = (1 - \overline{\delta})r^*(q^v) + \overline{\delta}r^*(q^w), \tag{14}$$

where (14) follows because $r^*(q^v) \geq r^*(q^w)$.

It is helpful to understand (13) and (14) in terms of the rates of substitution between the current stage payoff and the continuation value in aggregator function (6). Figure 1 plots two indifference curves for the case of $\overline{\delta} > \underline{\delta}$ in the plane where the horizontal and vertical axes measure the current stage payoff and the future continuation value, respectively. Each indifference curve consists of two lines, whose

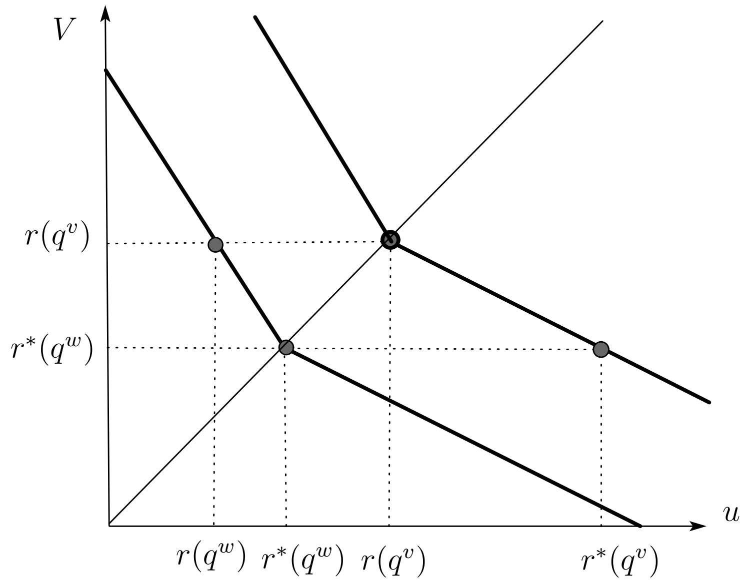

Figure 1: The best and worst equilibria under partial collusion

kink occurs on the 45-degree line. Because the preference exhibits recursive gain/loss asymmetry, the slope is steeper above the 45-degree line than below the line. Thus, the indifference curves are convex toward the origin. Discounted utility is a special case where each indifference curve consists of a single line.

The indifference curve that goes through $(r(q^v), r(q^v))$, which corresponds to the best equilibrium path, also goes through $(r^*(q^v), r^*(q^w))$, which corresponds to the most profitable one deviation from the best equilibrium path. The indifference between the two points is due to (14). The indifference curve that goes through $(r(q^w), r(q^v))$, which corresponds to the worst equilibrium path, also goes through $(r^*(q^w), r^*(q^w))$, which corresponds to the most profitable one deviation from the worst equilibrium path. The indifference between the two points is due to (13).

Figure 1 implicitly assumes that $q^v < q^{cn} < q^w$ (otherwise, we would have $r(q^v) = r^*(q^v)$ or $r(q^w) = r^*(q^w)$). Lemma A.5, proved in Appendix C, verifies that this is the case for any $\underline{\delta} \in (0, 1)$ and $\overline{\delta} \in [\underline{\delta}, 1)$. Therefore, repeated interaction always enables firms to attain some level of collusion via a nontrivial, stick-carrot punishment. Thus, (13) and (14) imply

$$\frac{r(q^v) - r^*(q^w)}{TP(q^v)} = \frac{1 - \overline{\delta}}{\overline{\delta}}, \tag{15}$$

$$\frac{r(q^v) - r^*(q^w)}{TP(q^w)} = \frac{1 - \underline{\delta}}{\underline{\delta}}, \tag{16}$$

where $TP(q) \equiv r^*(q) - r(q)$. $TP(q)$ measures the size of *temptation* at q, or the short-run gain from deviating to a static best response.

In (15), the denominator $TP(q^v) = r^*(q^v) - r(q^v)$ represents the temptation of the firms on the best equilibrium path. The numerator is the difference between the best and worst equilibrium payoffs, which is the size of future rewards for following the equilibrium play. When the firms can collude only partially, they are indifferent between the continuation strategy and the most profitable one deviation. Therefore, (15) defines the rate of substitution of the current payment for the future reward. Here, the rate of substitution is associated with the loss discount factor, because the equilibrium path is constant, whereas a deviator faces a loss in the future (see Figure 1).

Similarly, the denominator of (16) represents the temptation in the initial period of the worst equilibrium. Here, the firms are indifferent between the equilibrium strategy and the most profitable one deviation. Therefore, (16) defines the rate of substitution of the current payment for the future reward. This time, the gain discount factor determines the rate of substitution, because the equilibrium payoff sequence exhibits gain in the future, while a deviator receives the same current and continuation values.

Let us summarize the arguments above.

Proposition 4: *Suppose that firms evaluate payoff sequences by the same evaluating function satisfying* (5). *If* $q^m < q^v$*,*

(i) $\dfrac{r(q^v) - r^*(q^w)}{TP(q^v)} = \dfrac{1 - \overline{\delta}}{\overline{\delta}}$ *and* $\dfrac{r(q^v) - r^*(q^w)}{TP(q^w)} = \dfrac{1 - \underline{\delta}}{\underline{\delta}}$,

(ii) $\dfrac{TP(q^v)}{TP(q^w)} = 1$ *if* $\underline{\delta} = \overline{\delta}$*, and*

(iii) $\dfrac{TP(q^v)}{TP(q^w)} = \dfrac{\overline{\delta}}{1-\overline{\delta}}\dfrac{1-\underline{\delta}}{\underline{\delta}} > 1$ *if* $\underline{\delta} < \overline{\delta}$.

Proposition 4-(i) extends Abreu's (1986) Theorem 15 to evaluating function (5). In Abreu (1986), (15) and (16) are equal to $\dfrac{1-\delta}{\delta}$, which is evaluated by a single discount factor δ. Proposition 4-(iii) states that the ratio of temptations at the best and worst equilibrium paths, which we call the *temptation ratio*, equals the ratio of the rates of substitution of the indifference curves. The latter measures the degree of convexity of indifference curves, that is, the degree of gain/loss asymmetry. By construction, it is more than one and increases as $\underline{\delta}$ decreases or $\overline{\delta}$ increases. This result contrasts to Proposition 4-(ii), which shows that the temptation ratio implied by the discounted utility model is always one. The way the temptation ratio depends on the two discount factors suggests that the role of $\overline{\delta}$ is different from that of $\underline{\delta}$. To see this further, we conduct comparative statics based on the set of discount factors.

Let Δ and Δ' be the sets of discount factors such that $\Delta \equiv [\underline{\delta}, \overline{\delta}]$ and $\Delta' \equiv [\underline{\delta}', \overline{\delta}']$.[10] Let $q^{v,\Delta}$ and $q^{w,\Delta}$ be the output levels specified in Proposition 3 under the evaluating function with the set of discount factors Δ. Specifically, $q^{v,\Delta}$ is the collusive output under the best equilibrium, and $q^{w,\Delta}$ is the penalty output in the initial period of the worst equilibrium. We also define $q^{v,\Delta'}$ and $q^{w,\Delta'}$ similarly. We assume that the largest output level M satisfies (4) under both Δ and Δ'.

Propositions 5 and 6 below show that except a few circumstances, if the firms are strictly more patient, the most collusive output decreases and the penalty output increases. Thus, more collusive outcomes can be sustained via severer penalty. Proposition 6 identifies the case where strictly more patience does not change the most collusive output or the penalty output. We start with Proposition 5 (see Appendix C for the proof).

Proposition 5: *Suppose that for each supergame G^∞, firms evaluate payoff sequences by the same evaluating function satisfying* (5)*, where firms with Δ' are strictly more patient than firms with Δ. If $q^m < q^{v,\Delta} < q^{cn} < q^{w,\Delta} < q^{MC}$, then $q^{v,\Delta'} < q^{v,\Delta}$ and $q^{w,\Delta'} > q^{w,\Delta}$. Namely, $r(q^{v,\Delta'}) > r(q^{v,\Delta})$ and $r^*(q^{w,\Delta'}) < r^*(q^{w,\Delta})$.*

Proposition 5 shows that the best equilibrium payoff increases, and the worst equilibrium payoff decreases as $\overline{\delta}$ or $\underline{\delta}$ increases. Abreu's (1986) original result for the

[10] We allow the case $\underline{\delta} = \overline{\delta}$ and/or $\underline{\delta}' = \overline{\delta}'$ so that we can also compare the results with the discounted utility model.

discounted utility model is a special case of Proposition 5. Note that the condition $q^m < q^{v,\Delta} < q^{cn} < q^{w,\Delta} < q^{MC}$ implies that the best equilibrium does not sustain full collusion, and the worst equilibrium does not attain the minimax value. Proposition 6 deals with the remaining case where full collusion or the minimax value is sustainable.

To understand the result of Proposition 5, we examine the effects of an increase in $\overline{\delta}$ and an increase in $\underline{\delta}$ separately. Because $q^m < q^{v,\Delta}$, (13) and (14) under Δ reduce to

$$(1-\overline{\delta})r^*(q^{v,\Delta}) + \overline{\delta}r^*(q^{w,\Delta}) = r(q^{v,\Delta}), \tag{17}$$

$$(1-\underline{\delta})r(q^{w,\Delta}) + \underline{\delta}r(q^{v,\Delta}) = r^*(q^{w,\Delta}). \tag{18}$$

Suppose the loss discount factor increases to $\overline{\delta}'$ while the gain discount factor remains the same ($\underline{\delta}' = \underline{\delta}$). As shown in Figure 2, the slope of the indifference curves below the 45-degree line flattens. Then, $(r^*(q^{v,\Delta}), r^*(q^{w,\Delta}))$ is located below the indifference curve that goes through $(r(q^{v,\Delta}), r(q^{v,\Delta}))$. That is,

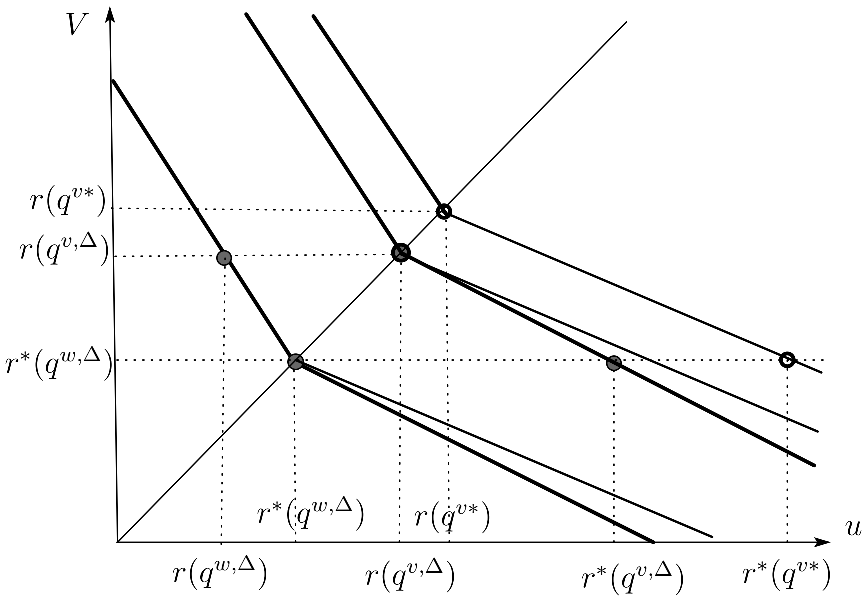

Figure 2: An increase from $\overline{\delta}$ to $\overline{\delta}'$ (less thick line); first-order effects

$$(1-\overline{\delta}')r^*(q^{v,\Delta}) + \overline{\delta}'r^*(q^{w,\Delta}) < r(q^{v,\Delta}). \tag{19}$$

We can also obtain (19) from (17) and $\overline{\delta}' > \overline{\delta}$.

From (18) and (19), we see that both the best and worst equilibria under Δ continue to be equilibria under Δ'.[11] More importantly, as (19) shows, the incentive

[11] Lemma A.6 formally proves this claim.

condition on the best equilibrium path under Δ does not bind under Δ'. The slack is significant because the same punishment can implement a more collusive path. In other words, the *first-order effect* of an increase in the loss discount factor is to strengthen the deterrence of a given level of punishment.

Figure 2 describes the first-order effect. Let q^{v*} be the smallest $q \in [q^m, q^{v,\Delta})$ such that

$$(1 - \overline{\delta}')r^*(q) + \overline{\delta}' r^*(q^{w,\Delta}) \leq r(q). \tag{20}$$

Figure 2 corresponds to the case where (20) holds with equality at $q = q^{v*}$. From this and (18), the reversion strategy profile whose target path is repeated play of (q^{v*}, q^{v*}) and whose punishment path given any firm's unilateral deviation is the worst equilibrium path under Δ is an equilibrium under Δ'. Hence, a more collusive path is implementable under Δ'. If (20) holds at $q = q^m$, this implies $q^{v*} = q^m$ and therefore full collusion is implementable.

Next, suppose the gain discount factor increases to $\underline{\delta}'$ while the loss discount factor remains the same ($\overline{\delta}' = \overline{\delta}$). As shown in Figure 3, the slope of the indifference curves above the 45-degree line flattens. Then, $(r(q^{w,\Delta}), r(q^{v,\Delta}))$ is located above the indifference curve that goes through $(r^*(q^{w,\Delta}), r^*(q^{w,\Delta}))$. That is,

$$(1 - \underline{\delta}')r(q^{w,\Delta}) + \underline{\delta}' r(q^{v,\Delta}) > r^*(q^{w,\Delta}). \tag{21}$$

We can also obtain (21) from (18) and $\underline{\delta}' > \underline{\delta}$.

From (17) and (21), both the best and worst equilibria under Δ are no longer equilibria under Δ'. The point is that an increase in the gain discount factor increases the value of the payoff sequence associated with the punishment path $(\overleftarrow{q}^{w,\Delta}, \overleftarrow{q}^{v,\Delta}, \overleftarrow{q}^{v,\Delta}, \dots)$. Thus, this punishment path no longer deters a deviation when the firms produce the collusive output $q^{v,\Delta}$. On the other hand, because the firms value future compensation more, they are willing to bear severer current penalties for a given level of future compensation. In other words, the *first-order effect* of an increase in $\underline{\delta}$ is to strengthen the penalty enforced by a given level of compensation.

Figure 3 indicates that there exists $q^{w*} > q^{w,\Delta}$ such that

$$(1 - \underline{\delta}')r(q^{w*}) + \underline{\delta}' r(q^{v,\Delta}) = r^*(q^{w*}) \tag{22}$$

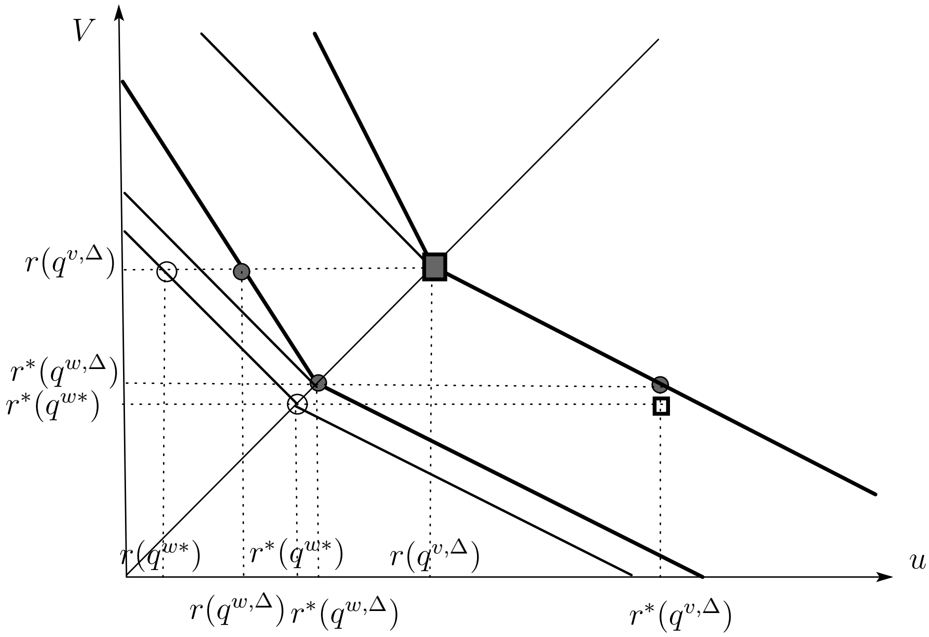

Figure 3: An increase from $\underline{\delta}$ to $\underline{\delta}'$ (less thick line); first-order effects

holds.[12] Because $q^{w*} > q^{w,\Delta}$ implies $r^*(q^{w*}) < r^*(q^{w,\Delta})$, we obtain

$$(1-\overline{\delta})r^*(q^{v,\Delta}) + \overline{\delta}r^*(q^{w*}) < r(q^{v,\Delta}) \tag{23}$$

from (17) (compare "■" and "□" in Figure 3).

Let Q^{w*} be the path such that both firms choose q^{w*} in the initial period and choose $q^{v,\Delta}$ in all subsequent periods. Then, (22) and (23) imply that the reversion strategy profile $s(Q^{w*}, Q^{w*}, Q^{w*})$ is an equilibrium under Δ', where the value of its payoff sequence is $r^*(q^{w*})$.[13] Because $r^*(q)$ is strictly decreasing on $[0, q^{MC}]$ and $q^{w,\Delta} < q^{MC}$, the value of this equilibrium payoff sequence is smaller than the value of the worst equilibrium payoff sequence under Δ. Hence, a severer penalty is implementable under Δ'.

Proposition 5 states that both a more collusive path and a severer penalty are sustainable even when just one of the gain and loss discount factors increases. It is the higher-order effects which account for an increase in q^w when only the loss discount factor increases, and a decrease in q^v when only the gain discount factor

[12] As we will see in the proof of Proposition 5 (Appendix C), this is a consequence of continuity and (4).

[13] Lemma A.6 formally proves this claim.

increases. Suppose only the loss discount factor increases from $\overline{\delta}$ to $\overline{\delta}'$. The first-order effect of this increase verifies that $r(q^{v,\Delta'}) > r(q^{v,\Delta})$. Proposition 3 applied to recursive gain/loss asymmetry implies that $q^{w,\Delta'}$ is the largest q such that

$$r^*(q) = (1-\underline{\delta})r(q) + \underline{\delta}r(q^{v,\Delta'}).$$

Because (18) implies

$$r^*(q^{w,\Delta}) < (1-\underline{\delta})r(q^{w,\Delta}) + \underline{\delta}r(q^{v,\Delta'}),$$

comparing the above equation and inequality proves that $q^{w,\Delta'} > q^{w,\Delta}$.[14]

Next, suppose only the gain discount factor increases from $\underline{\delta}$ to $\underline{\delta}'$. This time, the first-order effect verifies that $r^*(q^{w*})$ is the value of an equilibrium payoff sequence, which is smaller than the value of the worst equilibrium payoff sequence under Δ. From $q^m < q^{v,\Delta}$ and (23), there exists $q \in (q^m, q^{v,\Delta})$ such that

$$(1-\overline{\delta})r^*(q) + \overline{\delta}r^*(q^{w*}) < r(q).$$

It follows from (22) that

$$(1-\underline{\delta}')r(q^{w*}) + \underline{\delta}'r(q) > r^*(q^{w*}).$$

From these, we see that it is an equilibrium path for both firms to choose q in all periods.[15] This proves that $q^{v,\Delta} > q^{v,\Delta'}$.

The next proposition also conducts comparative statics for the case where $q^{v,\Delta} = q^m$ or $q^{w,\Delta} \geq q^{MC}$ (see Appendix D for the proof).

Proposition 6: *Suppose that for each supergame G^∞, firms evaluate payoff sequences by the same evaluating function satisfying (5), where firms with Δ' are strictly more patient than firms with Δ.*

(i) If $q^{v,\Delta} = q^m$, then $q^{v,\Delta'} = q^m$ and $q^{w,\Delta'} \geq q^{w,\Delta}$. The latter inequality is strict if and only if $\underline{\delta}' > \underline{\delta}$.

(ii) If $q^{v,\Delta} > q^m$ and $q^{w,\Delta} \geq q^{MC}$ (and therefore $r^*(q^{w,\Delta}) = 0$), then $q^{w,\Delta'} > q^{w,\Delta}$ (and therefore $r^*(q^{w,\Delta'}) = 0$) and $q^{v,\Delta'} \leq q^{v,\Delta}$. The latter inequality is strict if and only if $\overline{\delta}' > \overline{\delta}$.

[14] Again, this is a consequence of continuity and (4).

[15] Lemma A.6 formally proves this claim.

Proposition 6-(i) addresses the case where full collusion is already sustainable. The proposition first states that more patient firms continue to engage in full collusion. The proposition also shows that the optimal penalty can be strengthened only when the gain discount factor increases. This follows because an increase only in the loss discount factor cannot generate the first-order impact, that is, a reduction in the collusive production level. Hence, a higher-order impact is absent so that the optimal penalty also stays at the same level.[16]

Proposition 6-(ii) examines the case where full collusion is not sustainable but the punishment can be made so severe that the worst equilibrium attains the minimax values of the firms.[17] The proposition first states that more patient firms continue to punish themselves to the minimax values while the level of optimal penalty ($q^{w,\Delta'}$) increases. The proposition also shows that the most collusive output level decreases only when the loss discount factor increases. This follows because an increase only in the gain discount factor cannot generate the first-order impact, that is, a reduction of the value of the worst equilibrium path. Hence, a higher-order impact is absent so that the most collusive output also stays at the same level.

Propositions 5 and 6 are based on a lattice structure of the model. In lattice theory, when firms with Δ' are strictly more patient than firms with Δ, we say that Δ' is greater than Δ according to *induced set ordering* (Topkis (1998)). Therefore, if we regard the "carrot" and "stick" actions, q^v and q^w, as functions of Δ, q^v is decreasing in Δ and q^w is increasing in Δ, according to the induced set ordering.

So far, we have seen the following first-order effects on the levels of q^v and q^w: except the cases investigated in Proposition 6, (i) an increase in $\overline{\delta}$ decreases q^v, and (ii) an increase in $\underline{\delta}$ increases q^w. However, the higher-order effects further decrease q^v and increase q^w. Thus, we next investigate the *aggregate* effect of a change in the discount factor on the levels of q^v and q^w.

First, we introduce a few more notations. Suppose that the firms having the evaluating function with $\Delta \equiv [\underline{\delta}, \overline{\delta}]$ lead to the equilibrium quantity $q^{v,\Delta}$ and $q^{w,\Delta}$, which, to avoid complication, are assumed to satisfy $q^m < q^{v,\Delta} < q^{cn} < q^{w,\Delta} < q^{MC}$. Treating this case as a benchmark, consider two types of strictly more patient firms: one type of firms has the evaluating function with $\Delta^u \equiv [\underline{\delta}, \overline{\delta}^u]$, where $\overline{\delta}^u > \overline{\delta}$, and the other type of firms has the evaluating function with $\Delta^d \equiv [\underline{\delta}^d, \overline{\delta}]$, where $\underline{\delta}^d > \underline{\delta}$. We

[16]This is the only scenario where more patience keeps everything the same.

[17]This is a somewhat exceptional case and never occurs if $a \leq 65c$. If $a > 65c$, however, this occurs under some Δ.

assume that M is so large that (4) holds also for the evaluating functions with Δ^u and Δ^d. Denote the counterparts of $q^{v,\Delta}$ and $q^{w,\Delta}$ for the two evaluating functions by q^{v,Δ^u}, q^{w,Δ^u}, q^{v,Δ^d}, and q^{w,Δ^d}.

Second, to assess the economic impact caused by the change in q^v and q^w, we translate the change in quantity to the change in temptation TP. In particular, we introduce the following version of elasticity defined at the benchmark, that is,

$$\varepsilon_{\Delta^u} \equiv \frac{TP(q^{v,\Delta^u}) - TP(q^{v,\Delta})}{TP(q^{w,\Delta^u}) - TP(q^{w,\Delta})} \frac{TP(q^{w,\Delta})}{TP(q^{v,\Delta})} \text{ and } \varepsilon_{\Delta^d} \equiv \frac{TP(q^{v,\Delta^d}) - TP(q^{v,\Delta})}{TP(q^{w,\Delta^d}) - TP(q^{w,\Delta})} \frac{TP(q^{w,\Delta})}{TP(q^{v,\Delta})},$$

which defines a percent change in $TP(q^v)$ relative to a percent change in $TP(q^w)$. Thus, if $\varepsilon_{\Delta^u} > \varepsilon_{\Delta^d}$ for any Δ^u and Δ^d, we can deduce that an increase in $\overline{\delta}$ generates more impact on the collusive level than an increase in $\underline{\delta}$, and *vice versa*.

The following proposition shows that the temptation changes mainly via the channel of the first-order effect.

Proposition 7: *Consider three supergames where the firms evaluate payoff sequences by the same evaluating function satisfying (5) with Δ, Δ^u, or Δ^d defined as above. Suppose $q^m < q^{v,\Delta} < q^{cn} < q^{w,\Delta} < q^{MC}$, $q^m < q^{v,\Delta^u}$, and $q^m < q^{v,\Delta^d}$. Then, we have*

(i) $\dfrac{TP(q^{v,\Delta^u})}{TP(q^{w,\Delta^u})} > \dfrac{TP(q^{v,\Delta})}{TP(q^{w,\Delta})} > \dfrac{TP(q^{v,\Delta^d})}{TP(q^{w,\Delta^d})}$,

(ii) $TP(q^{v,\Delta^u}) > TP(q^{v,\Delta})$, $TP(q^{v,\Delta^d}) > TP(q^{v,\Delta})$, $TP(q^{w,\Delta^u}) > TP(q^{w,\Delta})$, $TP(q^{w,\Delta^d}) > TP(q^{w,\Delta})$, *and*

(iii) $\varepsilon_{\Delta^u} > 1 > \varepsilon_{\Delta^d}$.

Proposition 7-(i) is a direct consequence of Proposition 4-(iii), which states that the temptation ratio is higher for firms with Δ^u and lower for firms with Δ^d than firms with Δ. Proposition 5 and (16) also imply Proposition 7-(ii), that is, both $TP(q^{v,\Delta})$ and $TP(q^{w,\Delta})$ increase as $\underline{\delta}$ or $\overline{\delta}$ increases.[18] As for Proposition 7-(iii), observe that

$$\frac{TP(q^{v,\Delta^u})}{TP(q^{w,\Delta^u})} = \frac{TP(q^{v,\Delta}) + \{TP(q^{v,\Delta^u}) - TP(q^{v,\Delta})\}}{TP(q^{w,\Delta}) + \{TP(q^{w,\Delta^u}) - TP(q^{w,\Delta})\}},$$

[18]Due to the assumptions on the stage-game payoffs, $TP(q^{v,\Delta})$ increases as $q^{v,\Delta}$ decreases. (16) is also rewritten as $TP(q^{w,\Delta}) = \left(r(q^{v,\Delta}) - r^*(q^{w,\Delta})\right) \frac{\overline{\delta}}{1-\underline{\delta}}$, which increases as $\underline{\delta}$ or $\overline{\delta}$ increases. Note that we invoke (16) because if $a > 6c$, $TP(q)$ is not increasing on $[q^{cn}, q^{MC}]$, even weakly.

and a similar equation holds for $\dfrac{TP(q^{v,\Delta^d})}{TP(q^{w,\Delta^d})}$. Given Propositions 7-(i) and (ii), the above equality implies that

$$\frac{TP(q^{v,\Delta^u}) - TP(q^{v,\Delta})}{TP(q^{w,\Delta^u}) - TP(q^{w,\Delta})} > \frac{TP(q^{v,\Delta})}{TP(q^{w,\Delta})} > \frac{TP(q^{v,\Delta^d}) - TP(q^{v,\Delta})}{TP(q^{w,\Delta^d}) - TP(q^{w,\Delta})},$$

which is equivalent to (iii). Thus, an increase in $\overline{\delta}$ influences q^v more than q^w, whereas an increase in $\underline{\delta}$ influences q^w more than q^v.

The different roles of the discount factors become more evident when we explore folk-theorem type results by taking a limit of $\overline{\delta}$ or $\underline{\delta}$. For this analysis, we fix $\overline{\delta} \in (0, 1)$ and consider any evaluating function (5) whose loss discount factor is $\overline{\delta}$. We assume that M satisfies (4) for any such evaluating function, which is equivalent to assuming that (4) holds when $\underline{\delta} = \overline{\delta}$. As a benchmark, we also consider the discounted utility model with $\overline{\delta}$, and let $q^{v,N}$ be the most collusive output level that can be sustained by *the Nash reversion strategy profile* under this benchmark model. In other words, $q^{v,N}$ maximizes $r(q)$ under the constraint that the reversion strategy profile $s((\overleftarrow{q}, \overleftarrow{q}, \dots), (\overleftarrow{q}^{cn}, \overleftarrow{q}^{cn}, \dots), (\overleftarrow{q}^{cn}, \overleftarrow{q}^{cn}, \dots))$ is an equilibrium. The following proposition examines the possible outcomes of collusion in all evaluating functions whose loss discount factor is given (see Appendix E for the proof).

Proposition 8: *Fix $\overline{\delta} \in (0, 1)$ and M as above, and consider all supergames where the firms evaluate payoff sequences by the same evaluating function satisfying* (5) *whose loss discount factor is $\overline{\delta}$. There exists $\delta^* \in (0, 1)$, which is independent of $\overline{\delta}$, M, and the choice of $\underline{\delta}$, such that:*

(i) if $\overline{\delta} \geq \delta^*$, then $q^v = q^{v,N} = q^m$ for any $\underline{\delta} \leq \overline{\delta}$, and

(ii) if $\overline{\delta} < \delta^*$, then $q^{v,N} > q^m$. Moreover, when $\underline{\delta}$ approaches to zero, (a) q^w approaches to q^{cn}, and (b) q^v approaches to $q^{v,N}$.

Proposition 8-(i) corresponds to the folk theorem of evaluating function (5), whereas Proposition 8-(ii) is regarded as the *reverse* folk theorem, which is new and unique to evaluating function (5). To understand the above results, observe that if the loss discount factor is $\overline{\delta}$, the most collusive output q the Nash reversion strategy profile can implement is exactly $q^{v,N}$ because the incentive constraint is

$$(1 - \overline{\delta})r^*(q) + \overline{\delta}r(q^{cn}) \leq r(q). \tag{24}$$

This is identical to the incentive constraint for the discounted utility model with $\overline{\delta}$. Because the Nash reversion strategy profile is in $S^*(\Gamma)$, this implies that $q^{v,N} \geq q^v$. Then, Proposition 8-(i) follows immediately because, under the discounted utility model with $\overline{\delta}$, the Nash reversion strategy profile implements q^m for any sufficiently large $\overline{\delta}$. Thus, regardless of $\underline{\delta}$, full collusion is achieved for any sufficiently large $\overline{\delta}$.

We also provide a more intuitive explanation via Figure 4. On the indifference curve that goes through $(r(q^m), r(q^m))$, the firms' payoff is $r(q^m)$. The vertical line that goes through $(r^*(q^m), 0)$ corresponds to the paths where a firm chooses the static best response against the most collusive output q^m in the initial period. By observing the two lines, the one deviation from (q^m, q^m) is beneficial only when the intersection of the two lines is located below $(r^*(q^m), r(q^{cn}))$ (as in the case of the dotted line). Thus, the most collusive output sequence is attainable via the Nash reversion strategy profile when the slope of the indifference curve below the 45-degree line flattens (toward the direction of the solid line), that is, when the loss discount factor $\overline{\delta}$ increases.

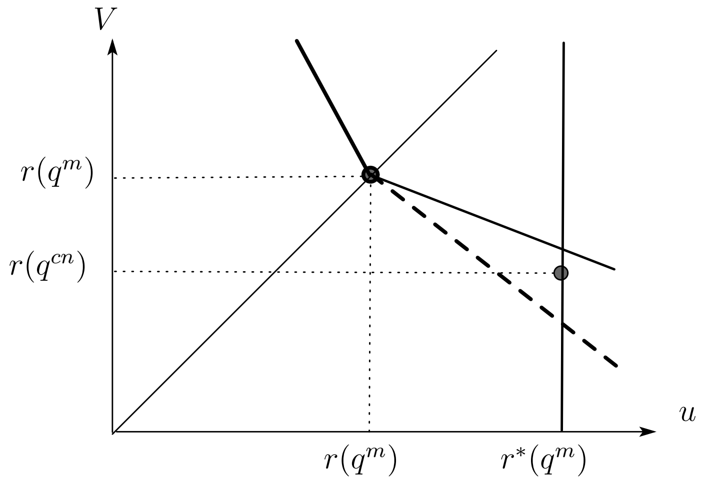

Figure 4: Sustainability of full collusion

On the other hand, Proposition 8-(ii) shows that as $\underline{\delta}$ approaches to zero, the best equilibrium path converges to the best equilibrium path implementable by the Nash reversion strategy profile if $q^{v,N} > q^m$. To see this result, consider the constraint (13) for the worst equilibrium:

$$(1-\underline{\delta})r(q^w) + \underline{\delta}r(q^v) = r^*(q^w).$$

If the firms barely evaluate future compensation, or if $\underline{\delta} \to 0$, the above equation converges to $r(q^w) = r^*(q^w)$, which is equivalent to $q^w = q^{cn}$.

Next, consider the constraint (14) for the best equilibrium:

$$(1-\overline{\delta})r^*(q^v) + \overline{\delta}r^*(q^w) = r(q^v).$$

This equation shows that q^v converges to $q^{v,N}$ if q^w converges to q^{cn}, where q^v always stays below $q^{v,N}$ because the Nash reversion strategy profile is in $S^*(\Gamma)$. This result contrasts with the implications from the discounted utility model, where a reduction in the level of the discount factor makes both the best and the worst equilibrium paths converge to the repeated play of the Cournot Nash equilibrium.

We also provide a graphical explanation via Figure 5. The indifference curve that goes through $(r^*(q^w), r^*(q^w))$ and $(r(q^w), r(q^v))$ represents the constraint (13) satisfied at the worst equilibrium s^w. However, once $\underline{\delta}$ decreases, the indifference curve that goes through $(r^*(q^w), r^*(q^w))$ becomes steeper. This implies that s^w is no longer an equilibrium because the value of the deviation which always plays a static best response against the other firm's output, $(r^*(q^w), r^*(q^w))$, is higher than the value of s^w indicated by $(r(q^w), r(q^v))$. As Propositions 5 and 6 show, the worst equilibrium shifts toward the direction of a less severe penalty, but even this penalty does not work if $\underline{\delta}$ decreases further. In the limit, the indifference curve that represents constraint (13) will converge to the indifference curve going through $(r(q^{cn}), r(q^{cn}))$, which is parallel to the vertical axis above the 45-degree line. At this limit, q^{cn} becomes the optimal penalty, and $q^{v,N}$ becomes the most cooperative output. Thus, q^w converges to q^{cn}, and q^v converges to $q^{v,N}$.

Finally, in this section, we investigate how the level of time-variability aversion affects the best and worst equilibria. As before, we fix $\Delta \equiv [\underline{\delta}, \overline{\delta}]$ and $\Delta' \equiv [\underline{\delta}', \overline{\delta}']$ so that (4) is satisfied for both Δ and Δ'. First, we examine the case that leads to clear comparative statics results (see Appendix F for the proof).

Proposition 9: *Let* $\Delta \equiv [\underline{\delta}, \overline{\delta}]$ *and* $\Delta' \equiv [\underline{\delta}', \overline{\delta}']$ *where* $\Delta \subset \Delta'$. *Consider the two supergames where the firms evaluate payoff sequences by the same evaluating function satisfying* (5) *with* Δ *or* Δ'. *Suppose* $q^{v,\Delta} > q^m$.

(i) If $q^{w,\Delta'} = q^{w,\Delta}$, *then* $q^{v,\Delta'} < q^{v,\Delta}$.

(ii) If $q^{v,\Delta'} = q^{v,\Delta}$, *then* $q^{w,\Delta'} < q^{w,\Delta}$.

To understand Proposition 9, recall that the loss discount factor affects the level of collusion more strongly than the gain discount factor, and the gain discount factor

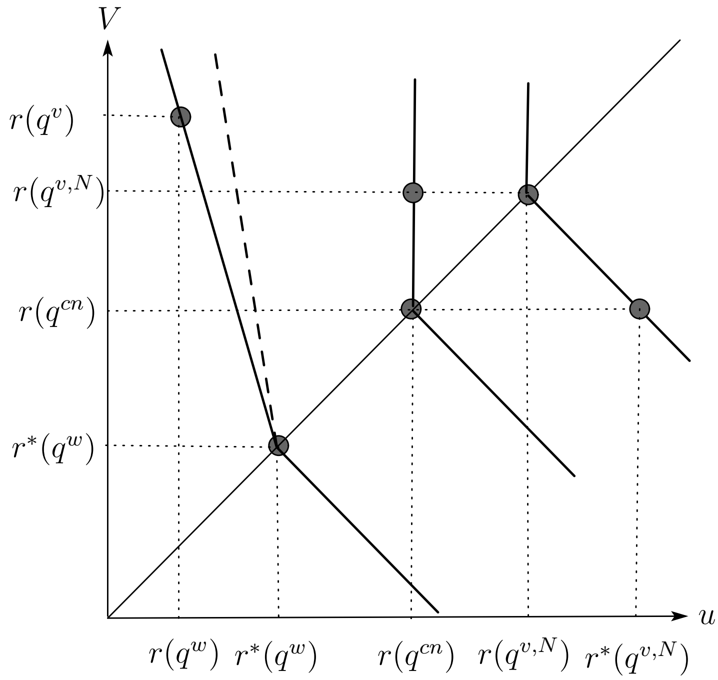

Figure 5: A reverse folk theorem

affects the level of penalty more strongly than the loss discount factor. If $q^{w,\Delta'} = q^{w,\Delta}$ (part (i)), the loss discount factor increases to a greater extent than the gain discount factor decreases with the change from Δ to Δ'. Hence, the change facilitates stronger collusion even under the same level of punishment. If $q^{v,\Delta'} = q^{v,\Delta} > q^m$ (part (ii)), the gain discount factor decreases to a greater extent than the loss discount factor increases with the change from Δ to Δ'. Hence, the change allows a less severe punishment to implement the same level of collusion.

Proposition 9-(i) shows the case where the strictly more time-variability aversion can facilitate stronger collusion, but the same conclusion does not always hold. Our last proposition confirms this point, which also covers the case where Proposition 5 does not apply (see Appendix G for the proof).

Proposition 10: *Let* $\Delta \equiv [\underline{\delta}, \overline{\delta}]$ *and* $\Delta' \equiv [\underline{\delta}', \overline{\delta}']$ *where* $\Delta \subset \Delta'$. *Suppose that for each supergame* G^∞, *firms evaluate payoff sequences by the same evaluating function satisfying* (5) *with* Δ *or* Δ'. *Suppose further that* $q^{v,\Delta} > q^m$. *Then, there exists* $\overline{\delta}^* \in (\overline{\delta}, 1)$, *which depends solely on* Δ, *such that*

(i) if $\overline{\delta}' \in [\overline{\delta}, \overline{\delta}^*)$, *there exists* $\underline{\delta}^* \in (0, \underline{\delta})$ *such that* $q^{v,\Delta'} > q^{v,\Delta}$ *if* $\underline{\delta}' < \underline{\delta}^*$, *and*

(ii) if $\overline{\delta}' \geq \overline{\delta}^*$, $q^{v,\Delta'} < q^{v,\Delta}$.

Proposition 10-(i) shows that strictly more time-variability averse firms may only achieve less collusive outcomes, if the loss discount factor does not increase much *and* if the gain discount factor decreases to a large extent. This is the case where strictly more time-variability aversion prevents collusion. In contrast, Proposition 10-(ii) shows that strictly more time-variability aversion enhances collusion if the loss discount factor increases to a certain level regardless of the decrease in the gain discount factor. The result is reminiscent of the folk theorem in Proposition 8-(i). Namely, a given level of collusion is always sustainable if the firms' loss discount factor is sufficiently large. This reinforces our point that the loss discount factor $\overline{\delta}$ is a key to collusion.

5 Summary

This paper studied the repeated Cournot duopoly under the recursive utility that exhibits gain/loss asymmetry. First, we found that the key results obtained for the discounted utility are extended to the general recursive utility: The reversion strategy profile that has the optimal penal code as the punishment can implement any of the equilibrium paths, where the optimal penal code has the stick-and-carrot structure in the Cournot duopoly setting. Second, we investigated the effect of the gain/loss asymmetry on the best and worst equilibria in the repeated Cournot duopoly. We demonstrated that an increase in the loss discount factor $\overline{\delta}$ strengthens the deterrence of the reversion strategy profile, whereas an increase in the gain discount factor $\underline{\delta}$ strengthens the enforceability of the optimal penalty. Thus, higher patience leads to stronger cooperation and stronger punishment because it requires an increase in either $\overline{\delta}$ or $\underline{\delta}$, but a stronger desire for smoothing allocations over time does not necessarily lead to stronger cooperation because it requires either an increase in $\overline{\delta}$ or a decrease in $\underline{\delta}$. We also introduced the reverse Folk theorem, which examines the behavior when the gain discount factor $\underline{\delta}$ approaches to zero. This theorem confirmed that the loss discount factor $\overline{\delta}$ is the key to implementing cooperative production because it defines a lower bound of the cooperative profit via the Nash reversion strategy profile.

6 Acknowledgment

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Appendix A: Proofs of Propositions 1 and 2

For the proofs of Propositions 1 and 2, we introduce additional notations. Let $q_{-i}^{(t)} \equiv (q_j^{(t)})_{j \neq i}$: this is the time-$t$ action profile by all players except player i. We also denote by (s_i, s_{-i}) a strategy profile consisting of s_i and s_{-i}, where $s_{-i} \equiv (s_j)_{j \neq i}$. Moreover, for a given strategy profile s and a given history h^{t-1}, let $Q(s; h^{t-1}) = (q^{(0)}(s; h^{t-1}), q^{(1)}(s; h^{t-1}), \dots)$ be a path such that (a) its play at time $0, 1, \dots, t-1$ coincides with h^{t-1}, and (b) the players follow s from time t onward. Formally,

(a) $(q^{(0)}(s; h^{t-1}), ..., q^{(t-1)}(s; h^{t-1})) = h^{t-1}$, and

(b) $q^{(\tau)}(s; h^{t-1}) = (s_{i,\tau}(q^{(0)}(s; h^{t-1}), \dots, q^{(\tau-1)}(s; h^{t-1})))_{i=1}^{I}$ for all $\tau \geq t$.

Note that $Q(s) = Q(s; h^{-1})$. For any $t > 0$, $Q(s) = Q(s; h^{t-1})$ holds if and only if $h^{t-1} = (q^{(0)}(s), ..., q^{(t-1)}(s))$.

Next, we introduce the following property.

Definition (one-deviation property): *A strategy profile s is said to satisfy the one-deviation property if for all t and all h^{t-1}, no player can increase her utility by changing her current action given the opponents' strategies and the rest of her own strategy.*

The following lemma is the key to prove Propositions 1 and 2.

Lemma A.1: *A strategy profile s is a subgame perfect equilibrium if and only if it satisfies the one-deviation property.*

Proof. The necessity of the one-deviation property follows from the definition of the subgame perfect equilibrium.

For the sufficiency, assume that a strategy profile s satisfies the one-deviation property. Suppose, by way of contradiction, that s is not a subgame perfect equilibrium. Then, there exist time t, history h^{t-1}, and a player i with a strategy s'_i such that

$$V_{i,t}(U_i(Q(s'_i, s_{-i}; h^{t-1}))) > V_{i,t}(U_i(Q(s; h^{t-1}))). \tag{25}$$

Given such s'_i, we consider a sequence of histories $\{h^\tau\}_{\tau=t}^{\infty}$ such that for all $\tau \geq t$,

$$h^\tau \equiv (q^{(0)}(s'_i, s_{-i}; h^{t-1}), \dots , q^{(\tau)}(s'_i, s_{-i}; h^{t-1})).$$

Then, it follows from the product topology adopted to $(A)^\infty$ that $Q(s; h^\tau)$ converges to $Q(s'_i, s_{-i}; h^{t-1})$ as τ goes to infinity.

Note that $Q(s; h^t)$ is a path induced by player i's one deviation from $Q(s; h^{t-1})$. Under the path, player i switches to s'_i at time t given h^{t-1}, and then switches back to s_i from time $t + 1$ onward against s_{-i}. Therefore, by the one-deviation property,

$$V_{i,t}(U_i(Q(s; h^{t-1}))) \geq V_{i,t}(U_i(Q(s; h^t))).$$

Similarly, for any $\tau > t$, $Q(s; h^\tau)$ is a path induced by player i's one deviation from $Q(s; h^{\tau-1})$. Namely, player i switches to s'_i at time τ given $h^{\tau-1}$, and then switches back to s_i from time $\tau + 1$ onward against s_{-i}. Therefore, by the one-deviation property,

$$V_{i,\tau}(U_i(Q(s; h^{\tau-1}))) \geq V_{i,\tau}(U_i(Q(s; h^\tau))).$$

By repeated application of the recursive relation (3) and the strict monotonicity of W_i in the second argument, it follows that

$$V_{i,t}(U_i(Q(s; h^{\tau-1}))) \geq V_{i,t}(U_i(Q(s; h^\tau)))$$

for any $\tau \geq t$. It follows from iterating this relation that

$$V_{i,t}(U_i(Q(s; h^{t-1}))) \geq V_{i,t}(U_i(Q(s; h^\tau)))$$

for any $\tau \geq t$. Because $Q(s; h^\tau)$ converges to $Q(s'_i, s_{-i}; h^{t-1})$ as $\tau \to \infty$, the continuity of $V_{i,t}$ implies

$$V_{i,t}(U_i(Q(s; h^{t-1}))) \geq V_{i,t}(U_i(Q(s'_i, s_{-i}; h^{t-1}))),$$

which contradicts (25). ■

For the argument below, we define paths $Q^{i,*}$ and $Q^{i,\#}$ for each i as follows: Let $\{Q^{i,k}\}_{k=1}^{\infty}$ be a sequence of subgame perfect equilibrium paths such that $\lim_{k\to\infty} V_{i,0}(U_i(Q^{i,k})) = \underline{v}_i$. Because A^∞ is compact, it has a subsequence converging to $Q^{i,*} = (q^{i,*(0)}, q^{i,*(1)}, \dots)$ such that $V_{i,0}(U_i(Q^{i,*})) = \underline{v}_i$. Similarly, let $\{Q'^{i,k}\}_{k=1}^{\infty}$ be a sequence of subgame perfect equilibrium paths such that $\lim_{k\to\infty} V_{i,0}(U_i(Q'^{i,k})) = \overline{v}_i$. Because A^∞ is compact, it has a subsequence converging to $Q^{i,\#}$, where $V_{i,0}(U_i(Q^{i,\#})) = \overline{v}_i$.

The proofs of Propositions 1 and 2 follow directly from Abreu (1988) with a minor modification to accommodate the change in the evaluating function. In particular, the proofs presented in Theorems 5.5 and 5.6 of Fudenberg and Tirole (1991) immediately extend to the recursive utility. The essence of their proofs is summarized by the following Lemma.

Lemma A.2: *Let $Q^{0,*} = (q^{0,*(0)}, q^{0,*(1)}, \dots)$ be a path that is the limit of a sequence of subgame perfect equilibrium paths. Then, the reversion strategy profile $s^* \equiv s(Q^{0,*}, Q^{1,*}, \dots, Q^{I,*})$ is a subgame perfect equilibrium.*

Proof. Suppose, by way of contradiction, s^* is not a subgame perfect equilibrium. From Lemma A.1, some player j has a profitable one deviation at some h^{t-1}, where he chooses q_j' at time t. By the construction of s^*, there exists $i \in \{0, \dots, I\}$ and $\tau \geq 0$ such that

$$(q^{(t)}(s^*; h^{t-1}), q^{(t+1)}(s^*; h^{t-1}), \dots) = (q^{i,*(\tau)}, q^{i,*(\tau+1)}, \dots).$$

By this and history independence, it follows that

$$W_j(u_j(q_j', q_{-j}^{i,*(\tau)}), \underline{v}_j) > V_{j,\tau}(U_j(Q^{i,*})).$$

Because $V_{j,\tau}$, u_j, and W_j are continuous and $Q^{i,*}$ is the limit of a sequence of subgame perfect equilibrium paths, there exists a subgame perfect equilibrium path $\hat{Q} = (\hat{q}^{(0)}, \hat{q}^{(1)}, \dots)$ such that

$$W_j(u_j(q_j', \hat{q}_{-j}^{(\tau)}), \underline{v}_j) > V_{j,\tau}(U_j(\hat{Q})). \tag{26}$$

Because $\hat{Q}$ is a subgame perfect equilibrium path, there exists another subgame perfect equilibrium path $\tilde{Q}$ such that

$$W_j(u_j(q_j', \hat{q}_{-j}^{(\tau)}), V_{j,0}(U_j(\tilde{Q}))) \leq V_{j,\tau}(U_j(\hat{Q})) \tag{27}$$

holds.[19] By monotonicity of W_j, (26) and (27) imply $V_{j,0}(U_j(\tilde{Q})) < \underline{v}_j$. This is a contradiction against the definition of $\underline{v}_j$. ■

The proofs of Propositions 1 and 2:

For Proposition 1, the existence of $\underline{s}^i$ and $\overline{s}_i$ is a straightforward consequence of Lemma A.2 by setting $Q^{0,*} = Q^{i,*}$ and $Q^{0,*} = Q^{i,\#}$, respectively.

As for Proposition 2, fix $s^* \in S^*$. Clearly, $Q(s^*)$ is the limit of the sequence of subgame perfect equilibrium paths, $(Q(s^*), Q(s^*), \dots)$. Because $Q(\underline{s}^i) = Q^{i,*}$ for each i, Lemma A.2 immediately proves that the reversion strategy profile $s(Q(s^*), Q(\underline{s}^1), ..., Q(\underline{s}^I))$ is a subgame perfect equilibrium. ■

Appendix B: Proof of Proposition 3

In what follows, for any output level q, we write $\overleftarrow{q} = (q, q)$.

We first show the following lemma.

Lemma A.3: *Suppose that firms evaluate payoff sequences by the same evaluating function satisfying* (3)*. Then, Propositions 1 and 2 hold for* $S^*(\Gamma)$ *replacing* S^**.*

Proof. It suffices to show that Γ is compact and $S^*(\Gamma)$ is nonempty. For the first claim, Γ is compact because it is a closed subset of compact A^∞. For the second claim, the strategy profile where the firms produce the Cournot equilibrium quantity $\overleftarrow{q}^{cn}$ at any history is a subgame perfect equilibrium that induces a symmetric path both on and off the equilibrium. Thus, $S^*(\Gamma)$ is nonempty. ■

Given that s^v and s^w are best and worst equilibria, respectively, the next lemma follows immediately from the definition of the stage game.

Lemma A.4: *Suppose that firms evaluate payoff sequences by the same evaluating function satisfying* (3)*. Then,*

$$V_0(r(q^{cn}), r(q^{cn}), \dots) \leq V_0(U_i(Q(s^v))) \leq V_0(r(q^m), r(q^m), \dots), \text{ and}$$
$$V_0(0, 0, \dots) \leq V_0(U_i(Q(s^w))) \leq V_0(r(q^{cn}), r(q^{cn}), \dots).$$

[19] By the stationarity of the action space and history-independent recursive preferences, the set of subgame perfect equilibrium paths and the set of continuation paths of a subgame perfect equilibrium coincide.

The proof of (i):

Let s^w be given, and consider the following system of inequalities.

$$V_0(r(q), r(q), \dots) \geq W(r^*(q), V_0(U_i(Q(s^w)))), \quad q^m \leq q \leq q^{cn}. \tag{28}$$

Because $r^*(q^{cn}) = r(q^{cn})$, Lemma A.4 implies that $q = q^{cn}$ is a solution of the system. By continuity, the smallest solution of (28) exists, which we denote by q^v.

If $q^v = q^m$, (28) implies that the reversion strategy profile $s((\overleftarrow{q}^m, \overleftarrow{q}^m, \dots), Q(s^w), Q(s^w))$ is an equilibrium. Any path in Γ other than $(\overleftarrow{q}^m, \overleftarrow{q}^m, \dots)$ gives each firm a smaller payoff than $V_0(r(q^m), r(q^m), \dots)$. Hence, the claim holds.

If $q^v > q^m$, (28) implies that the reversion strategy profile $s((\overleftarrow{q}^v, \overleftarrow{q}^v, \dots), Q(s^w), Q(s^w))$ is an equilibrium. Next, fix $Q = (q^{(0)}, q^{(1)}, \dots) \in \Gamma$ such that $V_0(U_i(Q)) \geq V_0(r(q^v), r(q^v), \dots)$ and $(q_1^{(0)}, q_1^{(1)}, \dots) \neq (q^v, q^v, \dots)$. For each t, define $\hat{q}_1^{(t)} \equiv \max\{q_1^{(t)}, q^m\}$ and also define $\hat{q}_1 = \inf_t \hat{q}_1^{(t)}$. Because of symmetry and $q^v > q^m$, $q_1^{(t)} < q^v$ for some t. Therefore, $\hat{q}_1^{(t)} \in [q^m, q^v)$. This implies $\hat{q}_1 \in [q^m, q^v)$, and therefore $\hat{q}_1$ is not a solution of (28). Because $r(\hat{q}_1) \geq r(\hat{q}_1^{(t)}) \geq r(q_1^{(t)})$ for any t, this implies that

$$V_t(U_i(Q)) \leq V_0(r(\hat{q}_1), r(\hat{q}_1), \dots) < W(r^*(\hat{q}_1), V_0(U_i(Q(s^w))))$$

for any t. Because $r^*(q_1^{(t)}) \geq r^*(\hat{q}_1^{(t)})$ for any t, continuity implies

$$V_t(U_i(Q)) < W(r^*(q_1^{(t)}), V_0(U_i(Q(s^w))))$$

for some t. Therefore, the reversion strategy profile $s(Q, Q(s^w), Q(s^w))$ is not an equilibrium. From Lemma A.3, Q is not an equilibrium path. Hence, no equilibrium improves the payoff of the path $(\overleftarrow{q}^v, \overleftarrow{q}^v, \dots)$, which establishes the claim. ■

The proof of (ii):

Recall that q^v is the smallest solution of (28). From part (i), we have $Q^v = (\overleftarrow{q}^v, \overleftarrow{q}^v, \dots)$. The proof is divided into the following three steps.

(Step 1) q^w, the largest solution of (9), is well-defined.

By $r(q^{cn}) = r^*(q^{cn})$ and Lemma A.4, the left-hand side of (9) is not smaller than its right-hand side at $q = q^{cn}$. Moreover, it follows from Lemma A.4, (4), and $r^*(M) \geq 0$ that

$$\begin{aligned} W(r(M), V_0(U_i(Q^v))) &\leq W(r(M), V_0(r(q^m), r(q^m), \dots)) \\ &< V_0(0, 0, \dots) \\ &\leq V_0(r^*(M), r^*(M), \dots). \end{aligned}$$

Hence, the left-hand side of (9) is smaller than its right-hand side at $q = M$. From these and continuity, q^w is well-defined and satisfies $q^w \geq q^{cn}$. Note also that the above argument implies

$$W(r(q), V_0(U_i(Q^v))) < V_0(r^*(q), r^*(q), \dots) \quad \text{for all } q > q^w. \tag{29}$$

□

(Step 2) If $Q = (q^{(0)}, q^{(1)}, \dots)$ is a worst equilibrium path, $V_0(U_i(Q)) \geq V_0(r^*(q^w), r^*(q^w), \dots)$.

Because Q is a worst equilibrium path, the reversion strategy profile $s(Q, Q, Q)$ is an equilibrium. Consider a firm's deviation where it chooses a static best response against the opponent's action at every history. Because any deviation makes the opponent restart Q in the next period, the payoff from this deviation is $V_0(r^*(q_1^{(0)}), r^*(q_1^{(0)}), \dots)$.

Because this deviation is not profitable,

$$\begin{aligned} V_0(r^*(q_1^{(0)}), r^*(q_1^{(0)}), \dots) &\leq V_0(U_i(Q)) \\ &= W(r(q_1^{(0)}), V_1(U_i(Q))) \\ &\leq W(r(q_1^{(0)}), V_0(U_i(Q^v))) \end{aligned} \tag{30}$$

holds, where the last inequality follows because Q^v is the best equilibrium path. From this and (29), we obtain $q_1^{(0)} \leq q^w$. Because r^* is decreasing, it follows from (30) that $V_0(r^*(q^w), r^*(q^w), \dots) \leq V_0(U_i(Q))$, as desired. □

(Step 3) If we define $Q^w \equiv (\overleftarrow{q}^w, \overleftarrow{q}^v, \overleftarrow{q}^v, \dots)$, the reversion strategy profile $s(Q^w, Q^w, Q^w)$ is a worst equilibrium.

From the definition of q^w,

$$\begin{aligned} V_0(U_i(Q^w)) = W(r(q^w), V_0(U_i(Q^v))) &= V_0(r^*(q^w), r^*(q^w), \dots) \\ &= W(r^*(q^w), V_0(r^*(q^w), r^*(q^w), \dots)) \\ &= W(r^*(q^w), V_0(U_i(Q^w))). \end{aligned} \tag{31}$$

Thus, any one deviation at the initial period is not profitable. Next, consider one deviations while Q^v is played. The continuation payoff after any nontrivial one deviation is (31), which is, by Step 2, not greater than the worst equilibrium payoff. Because Q^v is an equilibrium path, no one deviation while it is played is profitable. This proves that $s(Q^w, Q^w, Q^w)$ is an equilibrium. From Step 2 and (31), this is worst.
□

The proof of (iii):

The equality in (10) is an immediate consequence of part (i). The inequality in (10) follows because q^v is the smallest solution of (28), and it must hold with equality if $q^v > q^m$. The result in part (ii) implies (11). ■

Appendix C: Proof of Proposition 5

We first derive the following lemma.

Lemma A.5: *Suppose that firms evaluate payoff sequences by the same evaluating function satisfying* (5)*. Then,* q^v *and* q^w *defined in Proposition 3 satisfy* $q^m \leq q^v < q^{cn} < q^w$.

Proof. Let the preference represented by (5) be given, and define a function f by

$$f(q) = r(q) - (1 - \overline{\delta})r^*(q) - \overline{\delta}r(q^{cn}). \tag{32}$$

Because the assumptions on the stage game payoffs imply $r'(q^{cn}) = (r^*)'(q^{cn}) < 0$, it follows that $f'(q^{cn}) < 0$. Because $f(q^{cn}) = 0$, there exists $q' < q^{cn}$ such that

$$r(q') > (1 - \overline{\delta})r^*(q') + \overline{\delta}r(q^{cn}). \tag{33}$$

This implies that the reversion strategy profile $s((\overleftarrow{q}', \overleftarrow{q}', \dots), (\overleftarrow{q}^{cn}, \overleftarrow{q}^{cn}, \dots), (\overleftarrow{q}^{cn}, \overleftarrow{q}^{cn}, \dots))$ is an equilibrium. Therefore, $q^v \leq q' < q^{cn}$ follows. It also follows that $q^m \leq q^v$ from Proposition 3-(i). From (13), $q^v < q^{cn}$ implies $q^w \neq q^{cn}$. Because we have seen that $q^w \geq q^{cn}$ in the proof of Proposition 3-(ii), this proves that $q^w > q^{cn}$. ■

The proof of Proposition 5 is based on the lemma shown below, for which we need a few more notations introduced by Abreu (1986). For any $\underline{x} \geq q^m$ and any $\overline{x} \geq \underline{x}$, we define the following two paths:

$$Q(\underline{x}) \equiv ((\underline{x}, \underline{x}), (\underline{x}, \underline{x}), \dots), \quad Q(\overline{x}, \underline{x}) \equiv ((\overline{x}, \overline{x}), (\underline{x}, \underline{x}), (\underline{x}, \underline{x}), \dots).$$

Next, we define the following two reversion strategy profiles:

$$s(\overline{x}, \underline{x}) \equiv (Q(\overline{x}, \underline{x}), Q(\overline{x}, \underline{x}), Q(\overline{x}, \underline{x})), \quad s^*(\overline{x}, \underline{x}) \equiv (Q(\underline{x}), Q(\overline{x}, \underline{x}), Q(\overline{x}, \underline{x})).$$

Lemma A.6: *Suppose that firms evaluate payoff sequences by the same evaluating function satisfying* (5). *Then, for any* $\underline{x} \geq q^m$ *and any* $\overline{x} \geq \underline{x}$, *the reversion strategy profiles* $s(\overline{x}, \underline{x})$ *and* $s^*(\overline{x}, \underline{x})$ *are equilibria if and only if the following two conditions hold.*

(i) $(1-\underline{\delta})r(\overline{x}) + \underline{\delta}r(\underline{x}) \geq r^*(\overline{x})$ *and*

(ii) $r(\underline{x}) \geq (1-\overline{\delta})r^*(\underline{x}) + \overline{\delta}\{(1-\underline{\delta})r(\overline{x}) + \underline{\delta}r(\underline{x})\}$.

Proof. Fix $\underline{x} \geq q^m$ and $\overline{x} \geq \underline{x}$. If either $s(\overline{x}, \underline{x})$ or $s^*(\overline{x}, \underline{x})$ is played, the continuation path given any history is either $Q(\overline{x}, \underline{x})$ or $Q(\underline{x})$. Further, any one deviation at any history makes $Q(\overline{x}, \underline{x})$ the continuation path from the next period onward. Therefore, a necessary and sufficient condition for both $s(\overline{x}, \underline{x})$ and $s^*(\overline{x}, \underline{x})$ to be equilibria is

$$V_0(U_i(Q(\overline{x}, \underline{x}))) \geq W(r^*(\overline{x}), V_0(U_i(Q(\overline{x}, \underline{x})))), \quad (34)$$

$$V_0(U_i(Q(\underline{x}))) \geq W(r^*(\underline{x}), V_0(U_i(Q(\overline{x}, \underline{x})))). \quad (35)$$

It follows from $\overline{x} \geq \underline{x} \geq q^m$ that $r(\overline{x}) \leq r(\underline{x})$. Hence, (34) is equivalent to

$$(1-\underline{\delta})r(\overline{x}) + \underline{\delta}r(\underline{x}) \geq \min_{\delta \in [\underline{\delta}, \overline{\delta}]} \Big[(1-\delta)r^*(\overline{x}) + \delta\big\{(1-\underline{\delta})r(\overline{x}) + \underline{\delta}r(\underline{x})\big\}\Big],$$

which is further equivalent to condition (i). It also follows from $r^*(\underline{x}) \geq r(\underline{x}) \geq r(\overline{x})$ that $r^*(\underline{x}) \geq (1-\underline{\delta})r(\overline{x}) + \underline{\delta}r(\underline{x})$. Hence, (35) is equivalent to

$$r(\underline{x}) \geq (1-\overline{\delta})r^*(\underline{x}) + \overline{\delta}\{(1-\underline{\delta})r(\overline{x}) + \underline{\delta}r(\underline{x})\},$$

which is exactly condition (ii). ■

The proof of Proposition 5:

Fix $\Delta = [\underline{\delta}, \overline{\delta}]$ such that $q^m < q^{v,\Delta} < q^{cn} < q^{w,\Delta} < q^{MC}$. Fix also $\Delta' = [\underline{\delta}', \overline{\delta}']$ such that $\Delta' \neq \Delta$, $\underline{\delta} \leq \underline{\delta}'$, and $\overline{\delta} \leq \overline{\delta}'$.

It follows from (4) and $q^{v,\Delta} > q^m$ that

$$r^*(M) \geq 0 > (1-\underline{\delta}')r(M) + \underline{\delta}'r(q^m) > (1-\underline{\delta}')r(M) + \underline{\delta}'r(q^{v,\Delta}).$$

Because $r(q^{v,\Delta}) > r(q^{w,\Delta})$ due to $q^m < q^{v,\Delta} < q^{w,\Delta}$, it follows from (13) and $\underline{\delta}' \geq \underline{\delta}$ that

$$r^*(q^{w,\Delta}) = (1-\underline{\delta})r(q^{w,\Delta}) + \underline{\delta}r(q^{v,\Delta}) \leq (1-\underline{\delta}')r(q^{w,\Delta}) + \underline{\delta}'r(q^{v,\Delta}),$$

where the inequality is strict if $\underline{\delta}' > \underline{\delta}$. From these and continuity, there exists $q^{w*} \geq q^{w,\Delta}$ such that

$$r^*(q^{w*}) = (1 - \underline{\delta}')r(q^{w*}) + \underline{\delta}' r(q^{v,\Delta}). \tag{36}$$

Moreover, $q^{w*} > q^{w,\Delta}$ if $\underline{\delta}' > \underline{\delta}$. For a later purpose, we note that this conclusion is valid even if $q^{v,\Delta} = q^m$ or $q^{w,\Delta} \geq q^{MC}$.

We claim that

$$(1 - \overline{\delta}')r^*(q^{v,\Delta}) + \overline{\delta}' \left\{(1 - \underline{\delta}')r(q^{w*}) + \underline{\delta}' r(q^{v,\Delta})\right\} < r(q^{v,\Delta}). \tag{37}$$

To see that, note that (14) can be rewritten as follows:

$$r(q^{v,\Delta}) = (1 - \overline{\delta})r^*(q^{v,\Delta}) + \overline{\delta} r^*(q^{w,\Delta}) \geq (1 - \overline{\delta}')r^*(q^{v,\Delta}) + \overline{\delta}' r^*(q^{w*}), \tag{38}$$

where the inequality follows because $\overline{\delta}' \geq \overline{\delta}$, $q^{v,\Delta} < q^{w,\Delta} \leq q^{w*}$, and r^* is decreasing.

Because $\Delta' \neq \Delta$, either $\overline{\delta}' > \overline{\delta}$ or $\underline{\delta}' > \underline{\delta}$ holds. Suppose $\overline{\delta}' > \overline{\delta}$. Then, because $q^{v,\Delta} < q^{w,\Delta}$ and $q^{v,\Delta} < q^{MC}$ imply $r^*(q^{v,\Delta}) > r^*(q^{w,\Delta})$, the inequality in (38) is strict. Next, suppose $\underline{\delta}' > \underline{\delta}$. Then, $q^{w*} > q^{w,\Delta}$. This implies $r^*(q^{w,\Delta}) > r^*(q^{w*})$ because $q^{w,\Delta} < q^{MC}$. Hence, the inequality in (38) is strict. Consequently, (38) holds with strict inequality in either case, and substituting (36) establishes (37).

From (37) and continuity, there exists $\underline{x} \in (q^m, q^{v,\Delta})$ such that

$$(1 - \overline{\delta}')r^*(\underline{x}) + \overline{\delta}' \{(1 - \underline{\delta}')r(q^{w*}) + \underline{\delta}' r(\underline{x})\} < r(\underline{x}).$$

Because (36) implies

$$r^*(q^{w*}) < (1 - \underline{\delta}')r(q^{w*}) + \underline{\delta}' r(\underline{x}),$$

there exists $\overline{x} > q^{w*}$ such that

$$(1 - \overline{\delta}')r^*(\underline{x}) + \overline{\delta}' \{(1 - \underline{\delta}')r(\overline{x}) + \underline{\delta}' r(\underline{x})\} < r(\underline{x}), \tag{39}$$

$$r^*(\overline{x}) < (1 - \underline{\delta}')r(\overline{x}) + \underline{\delta}' r(\underline{x}). \tag{40}$$

Applying Lemma A.6 to (39) and (40), we see that $Q(\underline{x})$ is an equilibrium path under Δ' where the value of its payoff sequence is $r(\underline{x}) > r(q^{v,\Delta})$. Because the value of the best equilibrium payoff sequence under Δ' may be greater, we conclude that $q^{v,\Delta'} < q^{v,\Delta}$.

Recall that Proposition 3-(ii) implies in this particular environment that $q^{w,\Delta'}$ is the greatest q such that

$$r^*(q) = (1-\underline{\delta}')r(q) + \underline{\delta}' r(q^{v,\Delta'}).$$

It follows from (36) and $r(q^{v,\Delta'}) > r(q^{v,\Delta})$ that

$$r^*(q^{w*}) < (1-\underline{\delta}')r(q^{w*}) + \underline{\delta}' r(q^{v,\Delta'}),$$

and it follows from (4) that

$$r^*(M) > (1-\underline{\delta}')r(M) + \underline{\delta}' r(q^{v,\Delta'}).$$

From these and continuity, we obtain $q^{w,\Delta'} > q^{w*} \geq q^{w,\Delta}$, which completes the proof.
■

Appendix D: Proof of Proposition 6

The proof follows that of Proposition 5 with a minor modification. Let us fix $\Delta = [\underline{\delta}, \overline{\delta}]$ and $\Delta' = [\underline{\delta}', \overline{\delta}']$ such that $\Delta' \neq \Delta$, $\underline{\delta} \leq \underline{\delta}'$, and $\overline{\delta} \leq \overline{\delta}'$. As before, $q^{w*} \geq q^{w,\Delta}$ satisfying (36) exists, and the inequality is strict if $\underline{\delta}' > \underline{\delta}$. Moreover, (38) also holds but not necessarily with strict inequality.[20] Substituting (36) into (38) yields

$$r(q^{v,\Delta}) \geq (1-\overline{\delta}')r^*(q^{v,\Delta}) + \overline{\delta}' \left\{(1-\underline{\delta}')r(q^{w*}) + \underline{\delta}' r(q^{v,\Delta})\right\}. \tag{41}$$

Applying Lemma A.6 to (36) and (41), we see that $Q(q^{v,\Delta})$ is an equilibrium path under Δ'.

The proof of (i):

Because $q^{v,\Delta} = q^m$ in this case, we have seen that $Q(q^m)$ is an equilibrium path under Δ'. Hence, $q^{v,\Delta'} = q^m = q^{v,\Delta}$.

Suppose $\underline{\delta}' = \underline{\delta}$. Then, because $q^{v,\Delta'} = q^{v,\Delta}$, Proposition 3-(ii) implies that $q^{w,\Delta}$ and $q^{w,\Delta'}$ are the largest solution of the same equation. Thus, it follows that $q^{w,\Delta'} = q^{w,\Delta}$, as desired. Next, suppose $\underline{\delta}' > \underline{\delta}$. Then, $q^{w*} > q^{w,\Delta}$ holds. Again, Proposition 3-(ii) implies that $q^{w,\Delta'} \geq q^{w*} > q^{w,\Delta}$, as desired. ■

[20] In the proof of Proposition 5, we derived (38) from (14), which may not hold if $q^{v,\Delta} = q^m$. In the case of $q^{v,\Delta} = q^m$, we have $r(q^{v,\Delta}) \geq (1-\overline{\delta})r^*(q^{v,\Delta}) + \overline{\delta} r^*(q^{w,\Delta})$, from which we obtain (38).

The proof of (ii):

Suppose $\overline{\delta}' > \overline{\delta}$. The proof of Proposition 5 for the case of $\overline{\delta}' > \overline{\delta}$ does not depend on $q^{w,\Delta} < q^{MC}$. Hence, the same proof verifies that $q^{v,\Delta'} < q^{v,\Delta}$ and $q^{w,\Delta'} > q^{w,\Delta}$.

Next, suppose $\overline{\delta}' = \overline{\delta}$, and therefore $\underline{\delta}' > \underline{\delta}$. We have seen that $Q(q^{v,\Delta})$ is an equilibrium path under Δ' so that $q^{v,\Delta'} \leq q^{v,\Delta}$. This, $\underline{\delta}' > \underline{\delta}$, and (13) for Δ imply

$$r^*(q^{w,\Delta}) = (1 - \underline{\delta})r(q^{w,\Delta}) + \underline{\delta}r(q^{v,\Delta}) < (1 - \underline{\delta}')r(q^{w,\Delta}) + \underline{\delta}'r(q^{v,\Delta'}).$$

From the definition of $q^{w,\Delta'}$ and the routine argument based on (4) and continuity, it follows that $q^{w,\Delta'} > q^{w,\Delta}$.

Because $q^{w,\Delta'} > q^{w,\Delta} \geq q^{MC}$ in this case, $r^*(q^{w,\Delta'}) = r^*(q^{w,\Delta}) = 0$. That is, the value of the worst equilibrium payoff sequence is zero under both Δ and Δ'. Therefore, any symmetric and constant path $(\overleftarrow{q}, \overleftarrow{q}, \dots)$ is an equilibrium path under Δ' if and only if $r(q) \geq (1 - \overline{\delta}')r^*(q)$. Because $\overline{\delta}' = \overline{\delta}$, this holds if and only if the same path is an equilibrium path under Δ. Because the best equilibrium path is constant (Proposition 3-(i)), this proves that $q^{v,\Delta'} = q^{v,\Delta}$. ■

Appendix E: Proof of Proposition 8

The proof of (i):

Let

$$\delta^* = \frac{r^*(q^m) - r(q^m)}{r^*(q^m) - r(q^{cn})},$$

which does not depend on $\overline{\delta}$, M, or the choice of $\underline{\delta}$. Because $r^*(q^m) > r(q^m) > r(q^{cn})$, we have $\delta^* \in (0, 1)$. Suppose $\overline{\delta} \geq \delta^*$ and fix any evaluating function whose loss discount factor is $\overline{\delta}$. The Nash reversion strategy profile

$$s((\overleftarrow{q}^m, \overleftarrow{q}^m, \dots), (\overleftarrow{q}^{cn}, \overleftarrow{q}^{cn}, \dots), (\overleftarrow{q}^{cn}, \overleftarrow{q}^{cn}, \dots))$$

is an equilibrium because $\overline{\delta} \geq \delta^*$ implies

$$(1 - \overline{\delta})r^*(q^m) + \overline{\delta}r(q^{cn}) \leq r(q^m). \tag{42}$$

Hence, $q^v = q^m$ and $q^{v,N} = q^m$ hold. ■

The proof of (ii):

Suppose $\overline{\delta} < \delta^*$. Because (42) does not hold, we have $q^{v,N} > q^m$. For any $\underline{\delta} \leq \overline{\delta}$, define $q^{v,\underline{\delta}}$ and $q^{w,\underline{\delta}}$ as the counterparts of q^v and q^w in Proposition 3 for the evaluating function with $[\underline{\delta}, \overline{\delta}]$.

We first show that $q^{w,\underline{\delta}}$ converges to q^{cn} as $\underline{\delta}$ goes to zero. (13) implies

$$(1 - \underline{\delta})TP(q^{w,\underline{\delta}}) = \underline{\delta}\{r(q^{v,\underline{\delta}}) - r^*(q^{w,\underline{\delta}})\} \tag{43}$$

for any $\underline{\delta}$. Because $0 < r(q^{v,\underline{\delta}}) - r^*(q^{w,\underline{\delta}}) \leq r(q^m)$, the right-hand side of (43) converges to zero as $\underline{\delta} \to 0$. Hence, $TP(q^{w,\underline{\delta}})$ converges to zero. Because $TP(q)$ is continuous and is zero only at $q = q^{cn}$, this proves that $q^{w,\underline{\delta}}$ converges to q^{cn}.

It remains to show that $q^{v,\underline{\delta}}$ converges to $q^{v,N}$ as $\underline{\delta}$ goes to zero. By definition, $q^{v,N}$ is the smallest $q \in [q^m, q^{cn}]$ such that

$$(1 - \overline{\delta})r^*(q) + \overline{\delta}r(q^{cn}) = (1 - \overline{\delta})r^*(q) + \overline{\delta}r^*(q^{cn}) \leq r(q). \tag{44}$$

For any $\underline{\delta}$, $q^{v,\underline{\delta}}$ is the smallest $q \in [q^m, q^{cn}]$ such that

$$(1 - \overline{\delta})r^*(q) + \overline{\delta}r^*(q^{w,\underline{\delta}}) \leq r(q). \tag{45}$$

Because $r^*(q^{w,\underline{\delta}}) \leq r^*(q^{cn})$ for any $\underline{\delta}$, it follows that $q^{v,\underline{\delta}} \leq q^{v,N}$ for any $\underline{\delta}$. Moreover, for any small $\varepsilon > 0$, $q = q^{v,N} - \varepsilon$ does not satisfy (44). Because $r^*(q^{w,\underline{\delta}}) \to r^*(q^{cn})$ as $\underline{\delta} \to 0$, $q = q^{v,N} - \varepsilon$ does not satisfy (45) for all small $\underline{\delta}$. Hence, $q^{v,N} - \varepsilon < q^{v,\underline{\delta}} \leq q^{v,N}$ for all small $\underline{\delta}$. Because $\varepsilon > 0$ is arbitrary, this implies that $q^{v,\underline{\delta}} \to q^{v,N}$ as $\underline{\delta} \to 0$. ■

Appendix F: Proof of Proposition 9

The proof of (i):

Suppose $q^{v,\Delta'} \geq q^{v,\Delta}$. Because $q^{v,\Delta'} \geq q^{v,\Delta} > q^m$, Proposition 4-(iii) implies

$$\frac{TP(q^{v,\Delta'})}{TP(q^{w,\Delta'})} = \frac{\overline{\delta}'}{1 - \overline{\delta}'} \cdot \frac{1 - \underline{\delta}'}{\underline{\delta}'} > \frac{\overline{\delta}}{1 - \overline{\delta}} \cdot \frac{1 - \underline{\delta}}{\underline{\delta}} = \frac{TP(q^{v,\Delta})}{TP(q^{w,\Delta})},$$

where the inequality follows from $\Delta \subset \Delta'$. Because $q^{w,\Delta'} = q^{w,\Delta}$, we have $TP(q^{v,\Delta'}) > TP(q^{v,\Delta})$. However, this implies $q^{v,\Delta'} < q^{v,\Delta}$, a contradiction. ■

The proof of (ii):

Suppose $\overline{\delta}' > \overline{\delta}$. Because $q^{v,\Delta'} = q^{v,\Delta} > q^m$, (15) holds under Δ and Δ'. Therefore,

$$\frac{r(q^{v,\Delta}) - r^*(q^{w,\Delta})}{TP(q^{v,\Delta})} = \frac{1 - \overline{\delta}}{\overline{\delta}} > \frac{1 - \overline{\delta}'}{\overline{\delta}'} = \frac{r(q^{v,\Delta'}) - r^*(q^{w,\Delta'})}{TP(q^{v,\Delta'})}.$$

Because $q^{v,\Delta'} = q^{v,\Delta}$, we have $r^*(q^{w,\Delta'}) > r^*(q^{w,\Delta})$. Hence, $q^{w,\Delta'} < q^{w,\Delta}$. Next, suppose $\overline{\delta}' = \overline{\delta}$. Then, we must have $\underline{\delta}' < \underline{\delta}$. Because $q^{v,\Delta'} = q^{v,\Delta} > q^m$, either Proposition 5 or 6-(ii) applies. In either case, $q^{w,\Delta'} < q^{w,\Delta}$. ■

Appendix G: Proof of Proposition 10

Define

$$\overline{\delta}^* \equiv \frac{r^*(q^{v,\Delta}) - r(q^{v,\Delta})}{r^*(q^{v,\Delta}) - r(q^{cn})} < 1, \tag{46}$$

which depends solely on Δ. To show $\overline{\delta}^* > \overline{\delta}$, suppose, by way of contradiction, that $\overline{\delta}^* \leq \overline{\delta}$. Then, it follows from (46) and Lemma A.5 that

$$r(q^{v,\Delta}) \geq (1-\overline{\delta})r^*(q^{v,\Delta}) + \overline{\delta}r(q^{cn}) > (1-\overline{\delta})r^*(q^{v,\Delta}) + \overline{\delta}r(q^{w,\Delta}).$$

Because $q^{v,\Delta} > q^m$, this contradicts (17).

The proof of (i):

Suppose $\overline{\delta}' \in [\overline{\delta}, \overline{\delta}^*)$. Given (46), it follows from the continuity of r^* that

$$r(q^{v,\Delta}) < (1-\overline{\delta}')r^*(q^{v,\Delta}) + \overline{\delta}'r^*(\hat{q}^w) \tag{47}$$

for some $\hat{q}^w > q^{cn}$, where by (17), $\hat{q}^w < q^{w,\Delta}$. Choose $\underline{\delta}^* > 0$ so that for any $q \geq \hat{q}^w$,

$$r^*(q) > (1-\underline{\delta}^*)r(q) + \underline{\delta}^* r(q^m). \tag{48}$$

Note that $\underline{\delta}^*$ exists because any $q \geq \hat{q}^w$ satisfies $q > q^{cn}$ and therefore $r^*(q) > r(q)$. Moreover, (18) implies

$$r^*(q^{w,\Delta}) = (1-\underline{\delta})r(q^{w,\Delta}) + \underline{\delta}r(q^{v,\Delta}) < (1-\underline{\delta})r(q^{w,\Delta}) + \underline{\delta}r(q^m).$$

Because (48) must be satisfied at $q = q^{w,\Delta}$, $\underline{\delta}^* < \underline{\delta}$.

Suppose $\underline{\delta}' < \underline{\delta}^*$. From (48),

$$r^*(q) > (1-\underline{\delta}')r(q) + \underline{\delta}'r(q^m) \geq (1-\underline{\delta}')r(q) + \underline{\delta}'r(q^{v,\Delta'})$$

for any $q \geq \hat{q}^w$. This implies $q^{w,\Delta'} < \hat{q}^w$ from Proposition 3-(ii). Proposition 3-(iii) and this imply

$$r(q^{v,\Delta'}) \geq (1-\overline{\delta}')r^*(q^{v,\Delta'}) + \overline{\delta}'r^*(q^{w,\Delta'}) \geq (1-\overline{\delta}')r^*(q^{v,\Delta'}) + \overline{\delta}'r^*(\hat{q}^w). \tag{49}$$

Define a function h by

$$h(q) = r(q) - (1 - \overline{\delta}')r^*(q) - \overline{\delta}' r^*(\hat{q}^w).$$

By (47) and (49), $h(q^{v,\Delta}) < 0$ and $h(q^{v,\Delta'}) \geq 0$. Note also that $h(q^{cn}) > 0$ because $r(q^{cn}) = r^*(q^{cn}) > r^*(\hat{q}^w)$. Because h is concave on $[0, q^{cn}]$, these imply $q^{v,\Delta} < q^{v,\Delta'}$, as desired. ■

The proof of (ii):

Suppose $\overline{\delta}' \geq \overline{\delta}^*$. From the definition of $\overline{\delta}^*$,

$$r(q^{v,\Delta}) \geq (1 - \overline{\delta}')r^*(q^{v,\Delta}) + \overline{\delta}' r(q^{cn}) > (1 - \overline{\delta}')r^*(q^{v,\Delta}) + \overline{\delta}' r^*(q^{w,\Delta'}),$$

where the inequality follows from Lemma A.5. Because $q^{v,\Delta} > q^m$, there exists $q' \in (q^m, q^{v,\Delta})$ such that

$$r(q') \geq (1 - \overline{\delta}')r^*(q') + \overline{\delta}' r^*(q^{w,\Delta'}).$$

This implies that $(\overleftarrow{q'}, \overleftarrow{q'}, \dots)$ is an equilibrium path under Δ', which proves that $q^{v,\Delta'} < q^{v,\Delta}$. ■

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