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Abstract

We derive a factor pricing model under the economy where the representative agent's preferences follow the smooth model of decision making under ambiguity as proposed by Klibanoff, Marinacci, and Mukerji (2005). A newly derived factor is called an ambiguity factor that captures a component of returns generated by ambiguity aversion.

Keywords: Ambiguity aversion, asset pricing, factor pricing

JEL Classification Numbers: D81, G11, G12

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1 Introduction

Conventionally, finance theory deals with the situation where returns of assets are determined via objectively given probability distributions. This line of research produces empirically testable frameworks for cross-sectional returns, such as the capital asset pricing model (CAPM) suggested by Lintner (1965), Mossin (1966), and Sharpe (1964) and the factor pricing model suggested by Fama and French (1996) and many others. Recently, the theory is extended to the situation where an investor does not know an objective distribution of returns or at least he is unsure about how empirical frequency of data is generated. Such a situation is called ambiguity, which is distinct from the risk the conventional finance theory has been studying.[1]

We aim to extend a factor pricing model to the case where investors face ambiguity. Formally, we approximate the pricing kernel derived under a smooth model of decision making under ambiguity suggested by the Klibanoff, Marinacci, and Mukerji (2005). In their model, the investor believes that there exist a number of possible regimes each of which specifies objective probability of state realization, but he is unsure which regime he faces. The investor behaves as an risk averse expected utility maximizer under each possible regime but expresses an aversion to the ambiguity of a given regime. For this concern, the investor forms a subjective prior over these possible regimes and behaves as to avoid the variability of expected utility computed at each possible regime.

If the investor is ambiguity neutral, Klibanoff et al. (2005) reduces to a standard expected utility, which with rational expectations assumption can produce a standard factor pricing formula. In our approximation, an excess return corresponds to this component is called a factor risk premium. What new in our factor pricing formula is an introduction of an ambiguity factor, which account for the variation of returns due to ambiguity aversion. The expected excess return is then shown to be a summation of two components, factor risk

[1] Knight (1921) is the first person who recognize this distinction.

premium and factor ambiguity premium. We identify the conditions on the representative agent economy that derive this ambiguity-augumented version of a factor pricing formula. A simple canonical example is also provided so that an economy that satisfies such conditions are not empty.

As for the related literature, Ruffino (2014) and Wakai (2015) independently show that the CAPM can be extended to the case where investors face ambiguity. Formally, they adopt the *robust* mean-variance preferences suggested by Maccheroni, Marinacci, and Ruffino (2013), which models an aversion to incomplete knowledge about possible economic regimes the investor faces.[2] The derived relation among equilibrium asset returns, called the *robust* CAPM, differs from the CAPM only in terms of coefficient beta that determines the excess return. This newly derived augumented-beta is shown to be a convex combination of risk beta and ambiguity beta, the former of which is identical to the beta in the standard CAPM. The new component, the ambiguity beta, measure the amount of ambiguity in asset returns priced relative to the ambiguity in market returns.

In this paper, we show that the robust CAPM can be rewritten as a factor model with one risk factor and one ambiguity factor, which is exactly identical to a version of our factor pricing formula. However, the restriction on the factor ambiguity premium is different. In the robust CAPM, the factor ambiguity premium must be less than the factor risk premium, whereas in our model, the factor ambiguity premium can be larger than the factor risk premium.

2 Setting

We consider a single-period portfolio choice problem. Let $(\Omega, \mathcal{F}, P)$ be a probability space. At each state, a single perishable consumption good defined on $\mathbb{R}$ is available. There is a

[2]The robust mean-variance preferences are characterized as the approximation of Klibanoff, Marinacci, and Mukerji's (2005) smooth model of decision making under ambiguity.

single representative agent in this economy, who is endowed with the positive and bounded consumption e measurable with respect to $\mathcal{F}$. There are a finite number $K+1$ of assets that pay a nonnegative amount of consumption good as a dividend. The payoffs of the first K assets are not deterministic, while the $(K+1)$th asset is the risk-free asset that pays one unit of consumption good. We denote by $d=(d_1,...,d_{K+1})$ the vector of assets' dividends, each element of which is a bounded random variable measurable with respect to $\mathcal{F}$. All of assets have net zero supply.

We model ambiguity as follows: The representative agent believes that there are a finite number L of possible regimes in this economy and that he is unsure which regime he faces. Each regime l specifies the probability of state realization, denoted by an absolutely continuous Q_l with respect to P, and investor's belief of possible regimes is expressed by his subjective prior μ defined over L regimes. Let $(L, \mathcal{P}(L), \mu)$ be a probability space that describes the agent belief, where $\mathcal{P}(L)$ is the power set defined on L.

For a random variable x measurable with respect to $\mathcal{F}$, we denote by $E_Q[x]$ a random variable measurable with respect to $\mathcal{P}(L)$, where for each l, $E_Q[x\,|l]$ is the expectation of x under the probability measure Q_l. We also denote by $E_\mu[a]$ the expectation of a random variable a measurable with respect to $\mathcal{P}(L)$ under the probability measure μ. Furthermore, following Maccheroni et. all. (2013), we define the probability measure $\widehat{P}$ on $\mathcal{F}$, called the *reduction* of μ on Ω, by

$$\widehat{P}(A)=\mu(1)Q_1(A)+...+\mu(L)Q_L(A) \text{ for all } A\in\mathcal{F}.$$

Let $E_{\widehat{P}}[x]$ be the expectation of a random variable x measurable with respect to $\mathcal{F}$ under the reduction $\widehat{P}$.

We assume that the representative agent can trade assets without transaction cost and can short and borrow without restrictions. Let c be a feasible consumption, which satisfies the following budget constraints:

$$\theta\cdot q=0, \tag{1}$$

and

$$c = e + \theta \cdot d, \tag{2}$$

where $\theta = (\theta_1, ..., \theta_{K+1}) \in \mathbb{R}^{K+1}$ is the vector of asset holdings and $q = (q_1, ..., q_{K+1}) \in \mathbb{R}_+^{K+1}$ is the vector of assets' prices.

The representative agent's preferences follow a smooth model of decision making under ambiguity as introduced by Klibanoff et al. (2005)

$$V(c) = E_\mu \left[v \left(E_Q \left[u \left(c \right) \right] \right) \right], \tag{3}$$

where both v and u are strictly increasing and strictly concave on the respective domain. The representative agent decides his asset holdings θ so as to maximize the representation (3). Appendix A shows that the equilibrium price q_k satisfies

$$q_k = \frac{E_\mu \left[v' \times E_Q \left[m \times d_k \right] \right]}{E_\mu \left[v' \times E_Q \left[m \right] \right] \times \dfrac{1}{q_{K+1}}}, \tag{4}$$

where v' is $\mathcal{P}(L)$-measurable and m is $\mathcal{F}$-measurable random variables that stand for

$$v' \equiv v' \left(E_Q \left[u \left(e \right) \right] \right) \text{ and } m \equiv u' \left(e \right). \tag{5}$$

We also introduce a few more notations. Let R_k be the gross return of k-th risky asset, and let R_f be the gross return of the risk-free asset. We denote by $Cov_\mu \left[a, b \right]$ the covariance between $\mathcal{P}(L)$-measurable a and b under μ. A variance of $\mathcal{P}(L)$-measurable a under μ, $Var_\mu \left[a \right]$, is similarly defined. Furthermore, for random variables x and y measurable with respect to $\mathcal{F}$, let $Cov_Q \left[x, y \right]$ be a random variable measurable with respect to $\mathcal{P}(L)$, where for each l, $Cov_Q[x, y \,|l\,]$ is the covariance between x and y under Q_l. Later, we also use $Cov_{\widehat{P}} \left[x, y \right]$ to denote the covariance between $\mathcal{F}$-measurable x and y under $\widehat{P}$. A variance of $\mathcal{F}$-measurable x under $\widehat{P}$, $Var_{\widehat{P}} \left[x \right]$, is similarly defined. As commonly used, $\mathbf{1_\Omega}$ denotes the vector of one defined on Ω, whereas $\mathbf{1}_{\boldsymbol{L}}$ denotes the vector of one defined on L.

Given the above notations, as shown in Appendix A, (4) is rewritten as follows:

$$E_{\widehat{P}}[R_k - R_f] = -\frac{\left\{ E_\mu \left[v' \times Cov_Q \left[m, R_k \right] \right] + Cov_\mu \left[v' \times E_Q \left[m \right], E_Q \left[R_k \right] \right] \right\}}{E_\mu \left[v' \times E_Q \left[m \right] \right]}. \tag{6}$$

3 Factor Pricing under Ambiguity

3.1 Motivation

In empirical studies of asset returns, we often assume a factor pricing formula

$$E_P[R_k - R_f] = \sum_{i=1}^{I} \beta_{k,i} E_P[R_{RF_i} - R_f], \tag{7}$$

where a set of gross portfolio returns and the risk-free return, $\{R_{RF_1}, R_{RF_2}, ..., R_{RF_I}, R_f\}$, is assumed to be linearly independent. The model is based on the expected utility by imposing a particular assumption on m, where the factor return R_{RF_i} captures the variation of m that is relevant for asset pricing. Because risk aversion determines the variation of m, the factor risk premium $E_P[R_{RF_i} - R_f]$ represents a risk premium associated with factor returns.

In this paper, we intend to derive a similar decomposition of returns by introducing a premium associated with ambiguity aversion. Intuitively, we want to have

$$E_P[R_k - R_f] = \text{(factor ambiguity premium)} + \text{(factor risk premium)},$$

where the latter has a form similar to that shown in (7). Because risk aversion and ambiguity aversion interwind nonlinearly, it is not always possible to decompose returns exactly as above. We later provide a canonical example in which the above decomposition holds, where the part of returns associated with the variation of m is regarded as to risk premium and the part of the returns associated with the variation of v' is regarded as to ambiguity premium.

3.2 Risk Premium

For asset returns based on (4), we first want to identify the part of returns associated with the variation of m. We follow the approach used for the expected utility model and derive a version of a factor pricing formula by imposing a spanning condition on m. For this purpose, notice that there exist a prior $\widetilde{\mu}$ defined on $(L, \mathcal{P}(L))$ so that

$$q_k = \frac{E_{\widetilde{\mu}}\left[E_Q\left[m \times d_k\right]\right]}{E_{\widetilde{\mu}}\left[E_Q\left[m\right]\right] \times R_f},$$

where $\widetilde{\mu}(l) \equiv \dfrac{v' \times \mu(l)}{E_\mu\left[v'\right]}$. Then an argument similar to Gollier (2011) shows that there exists a prior $\widetilde{P}$ defined on $(\Omega, \mathcal{F})$ such that (4) is rewritten as

$$q_k = \frac{E_{\widetilde{P}}\left[m \times d_k\right]}{E_{\widetilde{P}}\left[m\right] \times R_f}, \tag{8}$$

where

$$\widetilde{P}(A) = \widetilde{\mu}(1)Q_1(A) + ... + \widetilde{\mu}(L)Q_L(A) \text{ for all } A \in \mathcal{F}.$$

Equation (8) is a common pricing formula derived under the expected utility with a subjective prior $\widetilde{P}$.

Now, for a finite $I \leq K$, consider a set of gross portfolio returns and the risk-free return $\{R_{RF_1}, R_{RF_2}, ..., R_{RF_I}, R_f\}$, which is assumed to be linearly independent. We then impose the following.

Assumption 1: (Spanning Condition on m)

(i) $m = a_0 R_f + \sum_{i=1}^{I} a_i R_{RF_i} + \epsilon$, where $a_i \neq 0$ for any $i \in \{1, ..., I\}$.

(ii) $E_{\widetilde{P}}\left[\epsilon \times R_k\right] = 0$ for all k.

Assumption 1 is a well-known condition that leads to the following form of a factor pricing model.

$$E_{\widetilde{P}}[R_k - R_f] = \sum_{i=1}^{I} \widetilde{\beta}_{k,i} E_{\widetilde{P}}[R_{RF_i} - R_f], \tag{9}$$

where $\widetilde{\beta}_{k,i}$ is obtained as a coefficient of $(R_{RF_i} - R_f)$ by the regression of the excess return $(R_k - R_f)$ on excess factor returns $(R_{RF_1} - R_f, ..., R_{RF_I} - R_f)$ under $\widetilde{P}$.[3]

[3] Theoretically, these coefficients must be identical to those obtained from the regression that includes the constant term, where the coefficient of the constant term turns out to be zero.

Equation (9) states that if we *use* $\widetilde{P}$, the excess returns should be written as a factor pricing model where the factor return R_{RF_i} captures the variation of m that is relevant for asset pricing. However, the empirical factor pricing formula (7) is based on the objective probability P. Thus, in the next subsection, we aim to identify the condition under which a bias of the subjective prior $\widetilde{P}$ relative to the objective probability P tells us an ambiguity related premium.

3.3 Alpha from Ambiguity Aversion

In this subsection, we want to identify the effect of ambiguity. For this purpose, we introduce an assumption that links the agent's belief and the objective probability.

Assumption 2: (Rational Belief)
(i) $\widehat{P} = P$.

This assumption is a version of rational expectation hypothesis adopted to the smooth model of decision making under ambiguity, and it corresponds to a similar assumption adopted to the subjective expected utility model. Given Assumption 2, the agent has the reduction, that is the belief on state realization, which is consistent with the frequency of data. The agent simply does not know how this frequency is generated so that he assumes regimes that seemingly consistent with data. This assumption also contributes to the separation of the ambiguity aversion from risk aversion because if the agent is ambiguity neutral, (4) reduces to the pricing under the subjective expected utility model with rational expectations.

The following proposition shows that the regression constant captures a premium related to ambiguity aversion (see Appendix B).

Proposition 1:
Suppose that Assumptions 1 and 2 hold. For each k, the gross return of asset k satisfies the

factor pricing formula

$$E_P[R_k - R_f] = \alpha_k + \sum_{i=1}^{I} \beta_{k,i} E_P[R_{RF_i} - R_f], \tag{10}$$

or

$$R_k - R_f = \alpha_k + \sum_{i=1}^{I} \beta_{k,i} \left(R_{RF_i} - R_f\right) + \varepsilon_k, \tag{11}$$

where for each i, $\beta_{k,i}$ *is a regression coefficient for* $R_{RF_i} - R_f$. *Furthermore,* $E_P\left[\varepsilon_k\right] = 0$ *and* α_k *satisfies*

$$\alpha_k = -\frac{E_\mu\left[v' \times E_Q\left[m \times \varepsilon_k\right]\right]}{E_\mu\left[v' \times E_Q\left[m\right]\right]}. \tag{12}$$

Moreover, if the representative agent is ambiguity neutral, α_k *is zero.*

For each i, we call $E_P[R_{RF_i} - R_f]$ the *factor risk premium.* Note that $\beta_{k,i}$ in (10), which is obtained as a coefficient of $(R_{RF_i} - R_f)$ by the regression of the excess return $(R_k - R_f)$ on a set of returns $(\mathbf{1}_{\mathbf{\Omega}}, R_{RF_1} - R_f, ..., R_{RF_I} - R_f)$ under P, need not be identical to the regression coefficient $\widetilde{\beta}_{k,i}$ shown in (9). Also note that we cannot eliminate the effect of ambiguity aversion from the factor returns in (10) because ambiguity aversion influences the levels of $\{q_1, ..., q_K\}$.

Proposition 1 shows that the regression constant α_k captures a premium related to ambiguity: If the agent is ambiguity neutral (that is, v is linear), α_k is zero because $E_\mu\left[E_Q\left[m \times \varepsilon_k\right]\right] = E_P\left[m \times \varepsilon_k\right] = 0$. Indeed, under Assumption 2, we can show that for a regression on any set of portfolio returns, α_k needs to satisfy (12), which represents $\dfrac{-1}{q_{K+1}}$ times a *price of the regression error.* If this idiosyncratic variation is beneficial to hedge risk or ambiguity, it requires the negative premium, and vice versa. In fact, even for an ambiguity neutral agent, α_k can be nonzero for some regression. What Assumption 1 introduces is the condition under which α_k represents a premium contingent on the existence of ambiguity aversion, once a factor regression is correctly specified.

Note that equation (10) with a similar interpretation of α_k holds under Gilboa and Schmeidler's (1989) multiple-priors model with an effectively selected prior $\widetilde{P}$. The advantage of Klibanoff et al. (2005) is that we can introduce Assumption 2, which enables us to derive more elaborate interpretation of the regression constant α_k, as shown in the next section.

For the following study, we also want to emphasize the relation between the regression coefficient α_k and the residual ε_k.

Corollary 1:

Suppose that Assumptions 1 and 2 hold. Assume that Ω *is finite, where the number of states is less than or equal to* $K+1$*, that is, the asset market is complete. Moreover, let* $I = |\Omega| - 1$*. Then, generically, the gross return of asset* k *satisfies the factor pricing formula* (10) *with* $\alpha_k = 0$*, so that in terms of the factor pricing formula, the economy with an ambiguity averse agent is observationally equivalent to that with an ambiguity neutral agent.*[4]

Corollary 1 states that if the asset market is complete and the factor returns completely span any payoff vector, we cannot identify the effect of ambiguity via the factor pricing formula. This result shows that to identify the effect of ambiguity, we need either an incomplete asset market or a degenerate factor structure under which the number of factors is less than $|\Omega| - 1$.

3.4 Ambiguity Premium

As Appendix C shows, α_k can be decomposed into two parts, one is for a premium mainly related to the ambiguity aversion, and the other is a premium generated by the interaction between ambiguity aversion and risk aversion. However, there is no clear condition that identifies the latter effect. Thus, we want to investigate a condition that makes α_k, the joint effect of pure ambiguity and an interaction, have a factor pricing formula. We can at least show that the following condition works.

Assumption 3:

[4] From (8), $q_k = \frac{E_P[\widetilde{m} \times d_k]}{E_P[\widetilde{m}] \times R_f}$, where $\widetilde{m} = m\frac{d\widetilde{P}}{dP}$. Then $\widetilde{m}$ is generically spanned by I factors, so that (8) holds under P. Moreover, for this case, $\varepsilon_k = 0$.

For each k, $Cov_Q[m, \varepsilon_k \,|l] = 0$ for all l.

Assumption 3 states that the regression residual does not generate factor risk premium at each regime l. This shuts a part of the connection between risk and ambiguity. Then, by using well known statistical relation, we can rewrite α_k as follows

$$\alpha_k = -\frac{E_\mu \left[v' \times E_Q \left[m\right] \times E_Q \left[\varepsilon_k\right]\right]}{E_\mu \left[v' \times E_Q \left[m\right]\right]}. \tag{13}$$

In this formula, the conditional connection between m and ε_k shown in $E_Q \left[m \times \varepsilon_k\right]$ is broken down to $E_Q \left[m\right] \times E_Q \left[\varepsilon_k\right]$. Thus, by treating $v' \times E_Q \left[m\right]$ as a ambiguity pricing kernel defined on $(L, \mathcal{P}(L))$, we can price a random variable $E_Q \left[\varepsilon_k\right]$. For this part, we adopt a condition similar to Assumption 1.

Assumption 4: (Spanning Condition on $v' \times E_Q \left[m\right]$)

(i) There exist $J \leq K$ portfolios, each of whose returns is denoted by R_{A_j}, such that $\left\{E_Q \left[R_{A_1}\right], ..., E_Q \left[R_{A_J}\right], E_Q \left[R_f\right]\right\}$, becomes a set of linearly independent vectors define on the state space L.

(ii) $v' \times E_Q \left[m\right] = \widetilde{a}_0 E_Q \left[R_f\right] + \sum_{j=1}^{J} \widetilde{a}_j E_Q \left[R_{A_j}\right] + \widetilde{\epsilon}$, where $\widetilde{a}_j \neq 0$ for any $j \in \{1, ..., J\}$.

(iii) $E_\mu \left[\widetilde{\epsilon} \times E_Q \left[\varepsilon_k\right]\right] = 0$ for all k.

For the next proposition, we define the two more notations. Let k be given, where $1 \leq k \leq K$. For each j satisfying $1 \leq j \leq J$,

$$\beta^E_{k,j} \equiv \frac{Cov_\mu \left[E_Q \left[R_k\right], E_Q \left[R_{A_j}\right]\right]}{Var_\mu \left[E_Q \left[R_{A_j}\right]\right]}, \tag{14}$$

and for each i and j satisfying $1 \leq i \leq I$ and $1 \leq j \leq J$,

$$\beta^E_{RF_i,j} \equiv \frac{Cov_\mu \left[E_Q \left[R_{RF_i}\right], E_Q \left[R_{A_j}\right]\right]}{Var_\mu \left[E_Q \left[R_{A_j}\right]\right]}. \tag{15}$$

By definition, $\beta^E_{k,j}$ is a coefficient of $E_Q \left[R_{A_j}\right]$ obtained by a regression of $E_Q \left[R_k\right]$ on $\left\{\mathbf{1_L}, E_Q \left[R_{A_j}\right]\right\}$ under $(L, \mathcal{P}(L), \mu)$. Similarly, $\beta^E_{RF_i,j}$ is a coefficient of $E_Q \left[R_{A_j}\right]$ obtained by a regression of

$E_Q\left[R_{RF_i}\right]$ on $\left\{\mathbf{1}_{\boldsymbol{L}}, E_Q\left[R_{A_j}\right]\right\}$ under $(L, \mathcal{P}(L), \mu)$. Then the following proposition shows that α_k has a particular structure (see Appendix D).

Proposition 2:

Suppose that Assumptions 1 to 4 hold. There exists a set of real numbers $\{\gamma_1, ..., \gamma_J\}$ *such that for each* k, *the gross return of asset* k *follows the factor pricing formula* (10), *where* α_k *satisfies*

$$\alpha_k = \sum_{j=1}^{J} \gamma_j Var_\mu\left[E_Q\left[R_{A_j}\right]\right] \left\{\beta^E_{k,j} - \sum_{i=1}^{I} \beta_{k,i} \times \beta^E_{RF_i,j}\right\}. \tag{16}$$

Proposition 2 allows us to define, at least in a theoretical ground, a portfolio that captures a ambiguity premium. Such a portfolio is called a *ambiguity factor*, which satisfies the following assumption.

Assumption 5: (Ambiguity Factors)

For each j, there exists a portfolio AF_j such that

(i) for each i, $\beta_{AF_j,i} = 0$,

(ii) for each j' satisfying $j' \neq j$, $\beta^E_{AF_j,j'} = 0$, and

(iii) $\beta^E_{AF_j,j} = 1$.

Assumption 5, together with Proposition 2, leads to the following.

Corollary 2:

Suppose that Assumptions 1 to 5 hold. For each k, α_k *in* (16) *satisfies*

$$\alpha_k = \sum_{j=1}^{J} \left\{\beta^E_{k,j} - \sum_{i=1}^{I} \beta_{k,i} \times \beta^E_{RF_i,j}\right\} E_P[R_{AF_j} - R_f], \tag{17}$$

where for each j,

$$E_P[R_{AF_j} - R_f] = \gamma_j Var_\mu\left[E_Q\left[R_{A_j}\right]\right]. \tag{18}$$

Note that as (18) shows, for each $j \leq J$, the factor ambiguity premium $E_P[R_{AF_j} - R_f]$ is negative if γ_j is negative. Also, in equation (17), $\beta^E_{k,j}$ represents the exposure to the factor ambiguity premium from R_k itself. However, adjustment $-\sum_{i=1}^{I} \beta_{k,i} \times \beta^E_{RF_i,j}$ is necessary because α_k is based on ε_k.

Corollary 2 allows us to rewrite (10) as follows

$$
\begin{aligned}
E_P[R_k - R_f] =& \sum_{j=1}^{J} \left\{ \beta^E_{k,j} - \sum_{i=1}^{I} \beta_{k,i} \times \beta^E_{i,j} \right\} E_P[R_{AF_j} - R_f] \\
&+ \sum_{i=1}^{I} \beta_{k,i} E_P[R_{RF_i} - R_f].
\end{aligned} \tag{19}
$$

Also, we can rewrite (19) as follows

$$
\begin{aligned}
E_P[R_k - R_f] =& \sum_{j=1}^{J} \beta^E_{k,j} E_P[R_{AF_j} - R_f] \\
&+ \sum_{i=1}^{I} \beta_{k,i} \left\{ E_P[R_{RF_i} - R_f] - \sum_{j=1}^{J} \beta^E_{i,j} E_P[R_{AF_j} - R_f] \right\}.
\end{aligned} \tag{20}
$$

The first line of (20) represents *gross* factor ambiguity premiums associate with R_k. In the second line, for each i,

$$
E_P[R_{RF_i} - R_f] - \sum_{j=1}^{J} \beta^E_{RF_i,j} E_P[R_{AF_j} - R_f]
$$

is the *ambiguity-adjusted* factor risk premium or *net* factor risk premium.

The key difference of the factor ambiguity premium from the factor risk premium is that the coefficient for the former effect, either $\beta^E_{k,j} - \sum_{i=1}^{I} \beta_{k,i} \times \beta^E_{RF_i,j}$ or $\beta^E_{k,j}$, is not obtained as a coefficient from the regression based on (19) or (20). This follows because the factor ambiguity premium is based on the variations on $(L, \mathcal{P}(L), \mu)$, while the regression is based on the variations on $(\Omega, \mathcal{F}, P)$. Thus, to identify the factor ambiguity premium, we need to identify a coefficient and an associated factor portfolio separately.

3.5 Relation between $\widetilde{\beta}_{k,i}$ and $\beta_{k,i}$

Equation (9) is based on $\widetilde{P}$, whereas equation (10) is based on P. Thus, the sensitivity to factor risk premium $\widetilde{\beta}_{k,i}$ is different from $\beta_{k,i}$. To focus on the effect of ambiguity restrictively to the expectation of asset returns, it is informative to know a condition in which $\widetilde{\beta}_{k,i}$ is identical to $\beta_{k,i}$.

Now, define $\widetilde{\varepsilon}_k$ by

$$\widetilde{\varepsilon}_k \equiv R_k - R_f - \left\{ \sum_{i=1}^{I} \widetilde{\beta}_{k,i} \left(R_{RF_i} - R_f \right) \right\},$$

where (9) implies that $E_{\widetilde{P}}\left[\widetilde{\varepsilon}_k\right] = 0$. Consider the following condition.

Assumption 6:

For each k, $Cov_P[m, \widetilde{\varepsilon}_k] = 0$.

Assumption 6 implies $E_P[m \times \{\widetilde{\varepsilon}_k - E_P\left[\widetilde{\varepsilon}_k\right]\}] = 0$, that is,

$$\alpha_k = E_P\left[\widetilde{\varepsilon}_k\right] \text{ and } \varepsilon_k = \widetilde{\varepsilon}_k - E_P\left[\widetilde{\varepsilon}_k\right].$$

Because factor returns are linearly independent, this shows that

$$\widetilde{\beta}_{k,i} = \beta_{k,i}.$$

Note that if Assumption 6 holds, ambiguity premium α_k captures a bias embedded in $\widetilde{P}$.

3.6 Canonical Example

In this subsection, we provide a canonical example that satisfies Assumptions 1, 3, and 4. First, let $\mathcal{F}^e$ and $\mathcal{F}^r$ be σ-algebras defined on Ω^e and Ω^r, respectively. Let $\Omega = \Omega^e \times \Omega^r$, where $(\Omega, \mathcal{F}, P)$ is a probability space with a factor σ-algebra $\mathcal{F}$. For each k, the dividend d_k consists of the two parts, that is,

$$d_k = d_k^e + d_k^r,$$

where d_k^e is measurable only with respect to $(\Omega^e, \mathcal{F}^e)$ and d_k^r is measurable only with respect to $(\Omega^r, \mathcal{F}^r)$. Moreover, the endowment e is measurable only with respect to $(\Omega^e, \mathcal{F}^e)$. Because of this construction, we call Ω^e the endowment space and Ω^r the idiosyncratic space. We assume that the endowment space Ω^e is complete, that is, Ω^e is finite with S states, where there exists a set of lineally independent S assets whose dividends are measurable only with respect to $(\Omega^e, \mathcal{F}^e)$. For convenience, let the first $S-1$ risky assets be such linearly independent assets without idiosyncratic dividends, which complete the endowment span together with the risk-free asset. This construction guarantees Assumption 1, where the first $S-1$ risky assets become I risk factors. Furthermore, Corollary 1 is not applied because a factor structure is degenerate, that is, there is no risk factor that spans the idiosyncratic space.

Second, at each regime l, for each $s, s' \in \Omega^e$, $Q_l(.|s) = Q_l(.|s')$, that is, the conditional probability is identical for all $s \in \Omega^e$. This means that the idiosyncratic dividend d_k^r is independently distributed from d_k^e. By construction,

$$R_k = R_k^e + R_k^r = (R_k^e + E_P[R_k^r]) + (R_k^r - E_P[R_k^r]),$$

where

$$R_k^e \equiv \frac{d_k^e}{q_k} \text{ and } R_k^e \equiv \frac{d_k^r}{q_k}.$$

Furthermore, $(R_k^e + E_P[R_k^r])$ is spanned by $\{R_1, ..., R_{S-1}, R_f\}$, and because the conditional probability is identical for all $s \in \Omega^e$, $E_P[R_k^r\,|s] = E_P[R_k^r]$ and $E_P\left[(R_k^r - E_P[R_k^r])\,|s\right] = E_P\left[(R_k^r - E_P[R_k^r])\right] = 0$ for all $s \in \Omega^e$. Thus, $(R_k^r - E_P[R_k^r])$ is orthogonal to $\{R_1, ..., R_{S-1}, R_f\}$, that is,

$$\varepsilon_k = R_k^r - E_P[R_k^r].$$

This implies Assumption 3 because m is measurable only with respect to $(\Omega^e, \mathcal{F}^e)$ and at each regime l, for each $s, s' \in \Omega^e$, $Q_l(.|s) = Q_l(.|s')$.

Third, let L satisfy $L \leq S = |\Omega^e|$. Then, generically, $\{E_Q\left[R_1\right], ..., E_Q\left[R_{S-1}\right], R_f\}$ spans all variations defined on $(L, \mathcal{P}(L), \mu)$. Thus, Assumption 4 is generically satisfied with $\widetilde{\epsilon} = 0$.

Fourth, the above argument implies also that Assumption 6 holds, that is, $\widetilde{\beta}_{k,i} = \beta_{k,i}$ for all k and all i.

4 Relation to the CAPM and Fama and French (1996)

For an application of Klibanoff et al. (2005), we often consider the case of $J = 2$. For example, Ju and Miao (2012) assumes that the agent perceives the two regimes, one of the regimes is a booming regime and the other is a recession regime. They show by a simulation study that Klibanoff et al.'s (2005) model is consistent with many of regularities observed in financial data. Moreover, Gallant, Jahan-Parvar, and Liu (2015) estimate Ju and Miao's (2012) model via the Bayesian method and show that financial data implies the high degree of ambiguity aversion.

To derive a clear intuition about our model, we follow the above authors and focus on the case of $J = 2$. The intuition gained here can be easily applied to the case of $J > 2$. When $J = 2$, (19) becomes

$$E_P[R_k - R_f] = \left\{ \beta^E_{k,1} - \sum_{i=1}^{I} \beta_{k,i} \times \beta^E_{i,1} \right\} E_P[R_{AF_1} - R_f] + \sum_{i=1}^{I} \beta_{k,i} E_P[R_{RF_i} - R_f]. \tag{21}$$

Thus, a single ambiguity factor R_{AF_1} captures the factor ambiguity premium. For an expositional purpose, we use R_{AF} and β^E_k in place of R_{AF_1} and $\beta^E_{k,1}$ in the following subsection.

4.1 Ambiguity-augumented CAPM

In this section, we derive an ambiguity-augumented version of the CAPM. Let R_M be a gross return of a market portfolio whose dividend is equal to e (equivalently, is proportional to e). We assume that the market portfolio is tradable. Without loss of generality, let the first risky asset be the market portfolio.

It is well known that the CAPM suggested by Lintner (1965), Mossin (1966), and Sharpe

(1964) can be derived in the standard expected utility framework if Assumption 1 holds with

$$m = a_0 R_f + a_1 R_M + \epsilon \text{ satisfying } a_1 < 0.$$

This condition is satisfied if $|\Omega^e| = 2$ in the canonical example of Section 3.6 or u is an appropriate form of a quadratic function. Then (10) becomes

$$E_P[R_k - R_f] = \alpha_k + \beta_k E_P[R_M - R_f],$$

where β_k is defined by

$$\beta_k \equiv \frac{Cov_P\left[R_M, R_k\right]}{Var_P\left[R_M\right]}. \tag{22}$$

This β_k is identical to a regression coefficient, and we refer to it as *risk beta.* The CAPM also implies that $\alpha_k = 0$ for all k because it does not assume the presence of ambiguity. Moreover, (8) and risk aversion implies that to hold e as an equilibrium position, $E_P[R_M - R_f] > 0$.

Given the assumption of $J = 2$, it is no longer necessary for α_k to be 0. Here, we assume that $E_Q\left[R_f\right]$ and $E_Q\left[R_M\right]$ can span the state space L (it is generically possible), that is, $R_{A_1} = R_M$. Furthermore, to guarantee Assumption 3, the state space Ω and the assets' dividend satisfy the condition stated in the canonical example of Section 3.6, where either (a) $|\Omega^e| = 2$ or (b) u is an appropriate form of a quadratic function and all of d_k^e is spanned by $\{\mathbf{1}_{\boldsymbol{\Omega}}, e\}$.[5] Then

$$E_P[R_k - R_f] = \left\{\beta_k^E - \beta_k\right\} E_P[R_{AF} - R_f] + \beta_k E_P[R_M - R_f], \tag{23}$$

where R_{AF} is an ambiguity factor return identified by Assumption 5. The β_k^E is defined by (14), that is,

$$\beta_k^E \equiv \frac{Cov_\mu\left[E_Q\left[R_M\right], E_Q\left[R_k\right]\right]}{Var_\mu\left[E_Q\left[R_M\right]\right]},$$

which is a coefficient of $E_Q\left[R_M - R_f\right]$ derived from the regression of the excess return $E_Q\left[R_k - R_f\right]$ on $\{\mathbf{1}_{\boldsymbol{L}}, E_Q\left[R_M - R_f\right]\}$ under μ. Thus, we refer to β_k^E as *ambiguity beta.*

[5] Either case satisfies Assumption 6.

We first investigate a sign of the factor ambiguity premium $E_P[R_{AF} - R_f]$. Let $l = 1$ be a booming regime and let $l = 2$ be a recession regime, where $E_Q\left[u\left(e\right)|l = 1\right] > E_Q\left[u\left(e\right)|l = 2\right]$. This implies that $v'\left(E_Q\left[u\left(e\right)|l = 1\right]\right) < v'\left(E_Q\left[u\left(e\right)|l = 2\right]\right)$. On the other hand, the assumption of the model does not provide an enough information to induce $E_Q\left[m\,|l = 1\right] < E_Q\left[m\,|l = 2\right]$ or $E_Q\left[R_M\,|l = 1\right] > E_Q\left[R_M\,|l = 2\right]$. However, because R_M is perfectly correlated with e, it is likely that $E_Q\left[m\,|l = 1\right] < E_Q\left[m\,|l = 2\right]$ and $E_Q\left[R_M\,|l = 1\right] > E_Q\left[R_M\,|l = 2\right]$. If so, as Appendix D show, $\gamma_1 > 0$. Thus, $E_Q\left[R_M\right]$ does not provides a hedge against ambiguity so that an associated factor ambiguity premium $E_P[R_{AF} - R_f]$ must be positive. Moreover, the factor ambiguity premium $E_P[R_{AF} - R_f]$ tends to increase as the agent becomes more ambiguity averse. This happens because γ_1 tends to increase as the volatility of v' increases.

We want to emphasize a few important aspects of (23). First, the interpretation of risk beta β_k is analogous to that for the CAPM beta, that is, only the risk contributing to market volatility is priced. Thus, a positively contributed asset earns a positive excess return because it bears market risk, whereas a negatively contributed asset earns a negative excess return because it provides a hedge against market risk. Second, as for the interpretation of β_k^E, $Var_\mu\left[E_Q[R_M]\right]$ measures market ambiguity, and $Cov_\mu\left[E_Q[R_k], E_Q[R_M]\right]$ measures the contribution of asset k's expected return to market ambiguity. Thus, β_k^E defines the compensation scheme for ambiguity, which is analogous to that for market risk. Third, the coefficient of the ambiguity factor R_{AF} is not identical to the coefficient obtained by the multivariable regression of $R_k - R_f$ on $\{\mathbf{1_\Omega}, R_{AF} - R_f, R_M - R_f\}$ under P. It is a difference between ambiguity beta and risk beta. Therefore, only the net ambiguity exposure is compensated by the ambiguity premium.

To emphasize the effect of ambiguity aversion, (23) can be rewritten as

$$E_P[R_k - R_f] = \beta_k^E E_P[R_{AF} - R_f] + \beta_k \left\{E_P[R_M - R_f] - E_P[R_{AF} - R_f]\right\}.$$

The first term indicates the *gross* factor ambiguity premium associated with R_k, whereas the second term indicates the *net* factor risk premium associated with R_k. Because $\beta_M = \beta_M^E = 1$, the latter is an incremental risk premium embedded in $E_P[R_M - R_f]$ after subtracting the ambiguity effect. To judge the relative importance of risk aversion to ambiguity aversion, it is informative to know the sign of the net factor risk premium. However, the theory does not provide enough information to conclude whether $E_P[R_M - R_f]$ is larger than $E_P[R_{AF} - R_f]$.

4.2 Ambiguity-augumented Fama-French Three Factor Model

Many researchers have proposed factor models, but for expositional purpose, we focus on the model proposed by Fama and French (1996), referred to as the Fama-French three factor model(**changed on 9/21/17**). The three factor returns consist of a return of a broad stock market index portfolio, denoted by R_M, a return of a portfolio composed of firms with small BP ratios minus firms with large BP ratios, denoted by R_{HL}, a return of a portfolio composed of firms with small capitalization minus firms with large capitalization, denoted by R_{SB}. Each of risk factor returns have been shown to have a positive excess returns. For the following study, we assume that $R_{RF_1} = R_M$ and $R_{A_1} = R_M$. As shown in the above, because R_M is strongly correlated with e, the factor ambiguity premium $E_P[R_{AF} - R_f]$ is likely to be positive.

We assume that Assumption 1 is satisfied with the three factors defined above. Moreover, to guarantee Assumption 3, the state space Ω and the assets' dividend satisfy the condition stated in the canonical example of Section 3.6, where either (a) $|\Omega^e| = 4$ or (b) all of d_k^e is spanned by $\{\mathbf{1}_{\boldsymbol{\Omega}}, R_M, R_{HL}, R_{SB}\}$.[6] If we take the factor risk premium as a basis, (21) becomes

$$E_P[R_k - R_f] = \left\{\beta_k^E - \beta_{k,1} - \beta_{k,2} \times \beta_{HL}^E - \beta_{k,3} \times \beta_{SB}^E\right\} E_P[R_{AF} - R_f] \\ + \beta_{k,1} E_P[R_M - R_f] + \beta_{k,2} E_P[R_{HL} - R_f] + \beta_{k,3} E_P[R_{SB} - R_f].$$

[6]Either case satisfies Assumption 6.

where $\beta_{HL}^E = \beta_{HL,1}^E$ and $\beta_{SB}^E = \beta_{SB,1}^E$. The coefficient of the ambiguity factor is adjusted to capture the factor ambiguity premium net of the risk effect. On the other hand, if we take the factor ambiguity premium as a basis, (21) can be rewritten as

$$\begin{aligned} E_P[R_k - R_f] = \beta_k^E E_P[R_{AF} - R_f] & \\ + \beta_{k,1} \{E_P[R_M - R_f] - E_P[R_{AF} - R_f]\} & \\ + \beta_{k,2} \left\{E_P[R_{HL} - R_f] - \beta_{HL}^E E_P[R_{AF} - R_f]\right\} & \\ + \beta_{k,3} \left\{E_P[R_{SB} - R_f] - \beta_{SB}^E E_P[R_{AF} - R_f]\right\}. & \end{aligned}$$

Here, to compute each of the net factor risk premium, we need to subtract the return associated with the ambiguity effect, which must reflect the correlation with the ambiguity factor, that is, β_{HL}^E or β_{SB}^E. Again, to judge the relative importance of risk aversion to ambiguity aversion, it is informative to know the sign of the net factor risk premium. However, the theory does not provide enough information.

5 Relation to Maccheroni et al. (2013)

We have elicited a part of asset returns associated with ambiguity aversion by adopting a linear approximation to a pricing kernel. In this section, we want to investigate a relation of our results to asset pricing implications derived from another type of approximation of Klibanoff et al. (2005) proposed by Maccheroni et al. (2013).

First, we assume that the asset markets are complete, where Ω is finite and q_s is the price of the Arrow-Debreu asset for state s.[7] Let W be the value of the endowment defined by

$$W \equiv \sum_{s=1}^{|\Omega|} e_s \times q_s,$$

where W is understood as a function of an endowment and state prices. Let $\mathbf{w}$ denote the K-dimensional vector of portfolio weights on the K risky assets. Then, the return of the

[7] Alternatively, we can weaken this assumption by making e tradable.

portfolio $R_{\mathbf{w}}$ is given by

$$R_{\mathbf{w}} = R_f + \mathbf{w} \cdot (\mathbf{R} - R_f \mathbf{1}),$$

where $\mathbf{R}$ is the K-dimensional vector of risky asset returns and $\mathbf{1}$ is the K-dimensional unit vector. Let Σ_P be the variance-covariance matrix of R_k under P, and let Σ_μ be the variance-covariance matrix of $E_Q[R_k]$ under μ.

Given Assumption 2, we now consider the representative agent whose preferences follow the *robust mean-variance preferences* as introduced by Maccheroni et al. (2013), which is shown to be a local approximation of Klibanoff et al. (2005). Formally, the representative agent chooses a portfolio weight $\mathbf{w}$ based on the following function:

$$E_P[R_{\mathbf{w}}] - \frac{\lambda}{2} Var_P[R_{\mathbf{w}}] - \frac{\theta}{2} Var_\mu [E_Q[R_{\mathbf{w}}]],$$

which is equivalent to

$$R_f + \mathbf{w} \cdot E_P[\mathbf{R} - r_f \mathbf{1}] - \frac{\lambda}{2} \mathbf{w}^T \Sigma_P \mathbf{w} - \frac{\theta}{2} \mathbf{w}^T \mathbf{\Sigma}_\mu \mathbf{w}, \tag{24}$$

where $E_P[\mathbf{R} - R_f \mathbf{1}]$ is the K-dimensional vector of expected return for $\mathbf{R} - R_f \mathbf{1}$. We assume that both λ and θ are positive. By the Arrow–Pratt analysis, the first assumption roughly implies that investors are risk averse. Maccheroni et al. (2013) also shows that the second assumption roughly implies that investors are ambiguity averse in the way defined by Klibanoff et al. (2005).

To derive an equilibrium relation based on (24), we also introduce the following:

Assumption 7:
(i) Σ_P is positive definite.
(ii) $\mathbf{1}^T [\lambda \Sigma_P + \theta \Sigma_\mu]^{-1} E_P[\mathbf{R} - R_f \mathbf{1}] > 0$.

Condition (i) guarantees that $[\lambda \Sigma_P + \theta \Sigma_\mu]$ is symmetric and positive definite. Condition (ii) corresponds to that usually assumed in mean-variance analysis in finance.

The vector of optimal portfolio weights $\mathbf{w}^*$ is that which maximizes (24). The first-order condition implies that $\mathbf{w}^*$ satisfies

$$[\lambda \Sigma_P + \theta \Sigma_\mu] \, \mathbf{w}^* = E_P[\mathbf{R} - R_f \mathbf{1}]. \tag{25}$$

Compared with the mean-variance preferences (that is, $\theta = 0$), ambiguity aversion additionally introduces the term $\theta \Sigma_\mu$ on the left-hand side. By solving (25), we obtain

$$\mathbf{w}^* \equiv \left[\lambda \Sigma_{\overline{Q}} + \theta \Sigma_\mu\right]^{-1} E_P[\mathbf{R} - R_f \mathbf{1}]. \tag{26}$$

Thus, the agent's optimal portfolio is a linear combination of the risk-free asset and the portfolio of risky assets, each of whose weights is defined by

$$\mathbf{w}_k^M \equiv \frac{\mathbf{w}_k^*}{(\mathbf{1} \cdot \mathbf{w}^*)},$$

where the denominator is positive given Assumption 7-(ii).

At the equilibrium, the demand for assets is equal to the supply of assets. Thus, $\mathbf{w}^M = (\mathbf{w}_1^M, ..., \mathbf{w}_K^M)$ becomes the portfolio weights in the market portfolio, which is the asset that pays e_s at each state s, and the demand for the risk-free asset is zero, that is, $\mathbf{1} \cdot \mathbf{w}^* = 1$. Let R_M be the return of the market portfolio defined by $R_M \equiv \mathbf{w}^M \cdot \mathbf{R}$. Then, Appendix E shows that the equilibrium relationship between the return of the risky asset k and the return of the market portfolio satisfies the following formula called *robust CAPM*

$$E_P[R_k - R_f] = \phi_k E_P[R_M - R_f], \tag{27}$$

where

$$\phi_k \equiv \frac{Cov_P(R_k, R_M) + \frac{\theta}{\lambda} Cov_\mu(E_Q[R_k], E_Q[R_M])}{Var_P(R_M) + \frac{\theta}{\lambda} Var_\mu(E_Q[R_M])}. \tag{28}$$

The CAPM is a special case of (28), where the agent is ambiguity neutral by having $\theta = 0$.[8]

[8] In a multi-agents' framework, Raffino (2014) and Wakai (2015) independently derives (27) by imposing some form of homogeneity in preference parameters.

Given β_k and β_k^E defined in Section 4.1, we can also rewrite (28) as follows:

$$\phi_k = \beta_k \left(\frac{Var_P(R_M)}{Var_P(R_M) + \frac{\theta}{\lambda} Var_\mu(E_Q[R_M])} \right) + \beta_k^E \left(\frac{\frac{\theta}{\lambda} Var_\mu(E_Q[R_M])}{Var_P(R_M) + \frac{\theta}{\lambda} Var_\mu(E_Q[R_M])} \right).$$

Thus, the more ambiguity averse investors become (that is, the larger θ is), the more the ambiguity beta dominates ϕ_k, and vice versa.

Next, a regression analysis leads to

$$E_P[R_k - R_f] = \alpha_k + \beta_k E_P[R_M - R_f],$$

where β_k is equal to the risk beta defined in (22). Thus, α_k must satisfy

$$\alpha_k = (\phi_k - \beta_k) E_P[R_M - R_f], \tag{29}$$

where ϕ_k is defined in (28). Equation (29) is further rewritten as follows:

$$\alpha_k = \left(\beta_k^E - \beta_k\right) \left(\frac{\frac{\theta}{\lambda} Var_\mu(E_Q[R_M])}{Var_P(R_M) + \frac{\theta}{\lambda} Var_\mu(E_Q[R_M])} \right) E_P[R_M - R_f].$$

Now, let AF be an ambiguity factor portfolio such that $\beta_{AF} = 0$ and $\beta_{AF}^E = 1$. Then,

$$E_P[R_{AF} - R_f] = \left(\frac{\frac{\theta}{\lambda} Var_\mu(E_Q[R_M])}{Var_P(R_M) + \frac{\theta}{\lambda} Var_\mu(E_Q[R_M])} \right) E_P[R_M - R_f] < E_P[R_M - R_f]. \tag{30}$$

Thus, (27) is rewritten as

$$E_P[R_k - R_f] = \left(\beta_k^E - \beta_k\right) E_P[R_{AF} - R_f] + \beta_k E_P[R_M - R_f], \tag{31}$$

where the first term represents the factor ambiguity premium. Equation (31) is also rewritten as follows:

$$\begin{aligned} E_P[R_k - R_f] &= \beta_k^E E_P[R_{SF} - R_f] + \beta_k \left(E_P[R_M - R_f] - E_P[R_{SF} - R_f]\right) \qquad (32) \\ &= \beta_k^E E_P[R_{SF} - R_f] + \beta_k \left(\frac{Var_P(r_M)}{Var_P(r_M) + \frac{\theta}{\lambda} Var_\mu(E_Q[r_M])} \right) E_P[R_M - R_f], \end{aligned}$$

where in (32), $(E_P[R_M - R_f] - E_P[R_{SF} - R_f])$ represents the ambiguity-adjusted factor risk premium.

Now, we want to compare the robust CAPM (that is, (31)) with the ambiguity-augumented CAPM (that is, (23)), where for the latter model, u is assumed to be quadratic and the state space and dividends satisfy the conditions stated in Section 4.1. First, the robust CAMP holds even for the case of $L \neq 2$, but the ambiguity-augumented CAPM does not hold in general if $L \neq 2$. Thus, for the robust mean-variance preferences, the equilibrium forces the asset returns to have the factor ambiguity premium represented only by the first term of (31). Second, in (31), the factor ambiguity premium is less than the factor risk premium because of (30), whereas in (23), the factor ambiguity premium may be larger than the factor risk premium.

Appendix A: The Derivation of Equations (4) and (6)

The investor maximizes (3) with the constraints (1) and (2), where the constraint (2) is automatically satisfied. By the first order condition with respect to θ_k leads to

$$E_\mu \left[v' \left(E_Q \left[u \left(e + \theta \cdot d\right)\right]\right) \times E_Q \left[u' \left(e + \theta \cdot d\right) \times d_k\right]\right] = \lambda q_k, \tag{33}$$

where λ is the Lagrange multiplier corresponding to the constraint (1). In particular, at the equilibrium, (33) implies that the equilibrium price q_k must satisfy

$$E_\mu \left[v' \left(E_Q \left[u \left(e\right)\right]\right) \times E_Q \left[u' \left(e\right) \times d_k\right]\right] = E_\mu \left[v' \times E_Q \left[m \times d_k\right]\right] = \lambda q_k. \tag{34}$$

For the risk-free asset,

$$E_\mu \left[v' \left(E_Q \left[u \left(e\right)\right]\right) \times E_Q \left[u' \left(e\right)\right]\right] = E_\mu \left[v' \times E_Q \left[m\right]\right] = \lambda q_{K+1}. \tag{35}$$

Thus, (34) and (35) imply that at the equilibrium,

$$q_k = \frac{E_\mu \left[v' \times E_Q \left[m \times d_k\right]\right]}{E_\mu \left[v' \times E_Q \left[m\right]\right] \times \dfrac{1}{q_{K+1}}}, \tag{36}$$

which is (4).

By applying the standard statistical relation, (36) leads to

$$1 = \frac{\left\{E_\mu \left[v' \times Cov_Q \left[m, R_k\right]\right] + E_\mu \left[v' \times E_Q \left[m\right] E_Q \left[R_k\right]\right]\right\}}{E_\mu \left[v' \times E_Q \left[m\right]\right] \times R_f}.$$

The second term of the numerator in the right-hand side is rewritten as

$$E_\mu \left[v' \times E_Q \left[m\right] E_Q \left[R_k\right]\right] = Cov_\mu \left[v' \times E_Q \left[m\right], E_Q \left[R_k\right]\right] + E_\mu \left[v' \times E_Q \left[m\right]\right] E_\mu \left[E_Q \left[R_k\right]\right].$$

Thus, the above two equations imply

$$E_{\widehat{P}}[R_k - R_f] = -\frac{\left\{E_\mu \left[v' \times Cov_Q \left[m, R_k\right]\right] + Cov_\mu \left[v' \times E_Q \left[m\right], E_Q \left[R_k\right]\right]\right\}}{E_\mu \left[v' \times E_Q \left[m\right]\right]}. \tag{37}$$

which is (6). ■

Appendix B: The Proof of Proposition 1

Given (37), by Assumptions 1 and 2, for each portfolio return R_{RF_i}

$$\begin{aligned} & E_P[R_{RF_i} - R_f] \\ = & -\frac{\{E_\mu[v' \times Cov_Q[m, R_{RF_i}]] + Cov_\mu[v' \times E_Q[m], E_Q[R_{RF_i}]]\}}{E_\mu[v' \times E_Q[m]]}. \end{aligned} \tag{38}$$

Now, we define the following regression

$$E_P[R_k - R_f] = \alpha_k + \sum_{i=1}^{I} \beta_{k,i} E_P[R_{RF_i} - R_f], \tag{39}$$

or

$$R_k - R_f = \alpha_k + \sum_{i=1}^{I} \beta_{k,i}(R_{RF_i} - R_f) + \varepsilon_k, \tag{40}$$

where $E_P[\varepsilon_k] = 0$. By applying (40) to (37), Assumptions 1 and 2 and (38) imply that

$$E_P[R_k - R_f] = \widehat{\alpha}_k + \sum_{i=1}^{I} \beta_{k,i} E_P[R_{RF_i} - R_f], \tag{41}$$

where

$$\widehat{\alpha}_k = -\frac{\{E_\mu[v' \times Cov_Q[m, \varepsilon_k]] + Cov_\mu[v' \times E_Q[m], E_Q[\varepsilon_k]]\}}{E_\mu[v' \times E_Q[m]]}.$$

By comparing (39) and (41),

$$\alpha_k = \widehat{\alpha}_k = -\frac{\{E_\mu[v' \times Cov_Q[m, \varepsilon_k]] + Cov_\mu[v' \times E_Q[m], E_Q[\varepsilon_k]]\}}{E_\mu[v' \times E_Q[m]]}. \tag{42}$$

The first term of the numerator of the right-hand side becomes

$$E_\mu[v' \times Cov_Q[m, \varepsilon_k]] = E_\mu[v' \times E_Q[m \times \varepsilon_k]] - E_\mu[v' \times E_Q[m] E_Q[\varepsilon_k]],$$

and the second term of the numerator of the right-hand side becomes

$$Cov_\mu[v' \times E_Q[m], E_Q[\varepsilon_k]] = E_\mu[v' \times E_Q[m] E_Q[\varepsilon_k]] - E_\mu[v' \times E_Q[m]] E_\mu[E_Q[\varepsilon_k]],$$

where the last term is zero because $E_\mu[E_Q[\varepsilon_k]] = 0$. Given the above two equations, (42) is rewritten as

$$\alpha_k = -\frac{E_\mu[v' \times E_Q[m \times \varepsilon_k]]}{E_\mu[v' \times E_Q[m]]},$$

which completes the proof. ■

Appendix C:

Consider the situation where the agent is risk neutral and ambiguity neutral. Then, an expected return of any asset is equal to the risk-free rate. Treating this case as a benchmark, consider the situation where the agent is risk neutral. Then equation (10) becomes

$$E_P[R_k - R_f] = \alpha_k,$$

where α_k satisfies equation (12) and it is zero if the agent is ambiguity neutral. Risk neutrality also implies that

$$\alpha_k = -\frac{E_\mu\left[v' \times E_Q\left[m \times \varepsilon_k\right]\right]}{E_\mu\left[v' \times E_Q\left[m\right]\right]} = -\frac{E_\mu\left[v' \times E_P\left[m\right] \times E_Q\left[\varepsilon_k\right]\right]}{E_\mu\left[v' \times E_Q\left[m\right]\right]} \tag{43}$$

because m is constant. The key here is that the variation of m is absent because of $E_P[m]$. If we replace $E_P[m]$ in (43) with $E_Q[m]$, it generates an interaction between risk aversion and ambiguity aversion because $E_Q[m]$ becomes nonconstant as soon as the agent becomes risk averse. Thus, by using $E_P[m]$, we can isolate the effect of ambiguity aversion that is independent of risk aversion.

Following the above argument, we decompose the regression constant α_k into two parts as follows

$$\alpha_k = A_k + B_k,$$

where

$$A_k \equiv -\frac{E_\mu\left[v' \times E_P\left[m\right] \times E_Q\left[\varepsilon_k\right]\right]}{E_\mu\left[v' \times E_Q\left[m\right]\right]} \tag{44}$$

and

$$B_k \equiv -\frac{E_\mu\left[v' \times \left\{E_Q\left[m \times \varepsilon_k\right] - E_P\left[m\right] \times E_Q\left[\varepsilon_k\right]\right\}\right]}{E_\mu\left[v' \times E_Q\left[m\right]\right]}. \tag{45}$$

By comparing (44) and (45), we call A_k a *pure ambiguity premium*, which is a portion of returns remained after the effect of risk aversion is subtracted. Given this interpretation of A_k, B_k represents an *interaction effect* between risk aversion and ambiguity aversion. It is easy to see that both A_k and B_k are zero if the agent is ambiguity neutral. Note that technically speaking, risk aversion determines $E_P[m]$ so that we cannot isolate the exact effect of ambiguity aversion from the effect of risk aversion.

Appendix D: The Proof of Proposition 2

By Assumption 4, (13) is rewritten as

$$\alpha_k = E_\mu\left[\left\{\gamma_0 E_Q\left[R_f\right] + \sum_{j=1}^{J}\gamma_j E_Q\left[R_{A_j}\right]\right\} E_Q\left[\varepsilon_k\right]\right], \tag{46}$$

where $\gamma_0 \equiv -\dfrac{\widetilde{a}_0}{E_\mu\left[v' \times E_Q[m]\right]}$ and $\gamma_j \equiv -\dfrac{\widetilde{a}_j}{E_\mu\left[v' \times E_Q[m]\right]}$ for each j.

We further evaluate (46) by

$$\begin{aligned}
\alpha_k &= E_\mu\left[\left\{\gamma_0 E_Q\left[R_f\right] + \sum_{j=1}^{J}\gamma_j E_Q\left[R_{A_j}\right]\right\} E_Q\left[\varepsilon_k\right]\right] \\
&= \sum_{j=1}^{J}\gamma_j E_\mu\left[E_Q\left[R_{A_j}\right] \times E_Q\left[\varepsilon_k\right]\right] \\
&= \sum_{j=1}^{J}\gamma_j \left\{Cov_\mu\left[E_Q\left[R_{A_j}\right], E_Q\left[\varepsilon_k\right]\right] + E_\mu\left[E_Q\left[R_{A_j}\right]\right] E_\mu\left[E_Q\left[\varepsilon_k\right]\right]\right\} \\
&= \sum_{j=1}^{J}\gamma_j Cov_\mu\left[E_Q\left[R_{A_j}\right], E_Q\left[\varepsilon_k\right]\right].
\end{aligned}$$

By definition of ε_k, for each j,

$$\begin{aligned}
&\gamma_j Cov_\mu\left[E_Q\left[R_{A_j}\right], E_Q\left[\varepsilon_k\right]\right] \\
=&\gamma_j Cov_\mu\left[E_Q\left[R_{A_j}\right], E_Q\left[R_k - R_f - \alpha_k - \sum_{i=1}^{I}\beta_{k,i}\left(R_{RF_i} - R_f\right)\right]\right] \\
=&\gamma_j\left\{Cov_\mu\left[E_Q\left[R_{A_j}\right], E_Q\left[R_k\right]\right] - \sum_{i=1}^{I}\beta_{k,i} Cov_\mu\left[E_Q\left[R_{A_j}\right], E_Q\left[R_{RF_i}\right]\right]\right\} \\
=&\gamma_j Var_\mu\left[E_Q\left[R_{A_j}\right]\right]\left\{\beta^E_{k,j} - \sum_{i=1}^{I}\beta_{k,i} \times \beta^E_{RF_i,j}\right\}.
\end{aligned}$$

Thus,

$$\alpha_k = \sum_{j=1}^{J} \gamma_j Var_\mu \left[E_Q \left[R_{A_j} \right] \right] \left\{ \beta_{k,j}^E - \sum_{i=1}^{I} \beta_{k,i} \times \beta_{RF_i,j}^E \right\}, \tag{47}$$

as desired. ■

Appendix E:

Multiplying $(\mathbf{w}^M)^T$ to (26) leads to

$$(\mathbf{w}^M)^T \left[\lambda \Sigma_P + \theta \Sigma_\mu \right] \mathbf{w}^* = E_P[R_M - R_f],$$

which is equivalent to

$$(\mathbf{1} \cdot \mathbf{w}^*)(\mathbf{w}^M)^T \left[\lambda \Sigma_P + \theta \Sigma_\mu \right] \mathbf{w}^M = E_P[R_M - R_f]. \tag{48}$$

(48) is positive because $[\lambda \Sigma_P + \theta \Sigma_\mu]$ is positive definite and $(\mathbf{1} \cdot \mathbf{w}^*) > 0$.

Similarly, multiplying $\mathbf{w}(k)^T$ to (26) leads to

$$\mathbf{w}(k)^T \left[\lambda \Sigma_P + \theta \Sigma_\mu \right] \mathbf{w}^* = E_P[R_k - R_f], \tag{49}$$

where $\mathbf{w}(k)$ is a K-dimensional vector of asset weights that assigns one for asset k and zero for others. Then, (49) is rewritten as

$$(\mathbf{1} \cdot \mathbf{w}^*)\mathbf{w}(k)^T \left[\lambda \Sigma_P + \theta \Sigma_\mu \right] \mathbf{w}^M = E_P[R_k - R_f]. \tag{50}$$

Dividing (50) by (48) produces the following formula

$$E_P[R_k - R_f] = \phi_k E_P[R_M - R_f],$$

where

$$\phi_k \equiv \frac{\mathbf{w}(k)^T \left[\lambda \Sigma_P + \theta \Sigma_\mu \right] \mathbf{w}^M}{(\mathbf{w}^M)^T \left[\lambda \Sigma_P + \theta \Sigma_\mu \right] \mathbf{w}^M} = \frac{Cov_P(R_k, R_M) + \dfrac{\theta}{\lambda} Cov_\mu(E_Q[R_k], E_Q[R_M])}{Var_P(R_M) + \dfrac{\theta}{\lambda} Var_\mu(E_Q[R_M])},$$

as desired. ■

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