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Vagueness of Language: Indeterminacy under Two-Dimensional State Uncertainty

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Abstract

We study *indeterminacy of indicative meanings* (disagreements about meanings of messages among players), a kind of language vagueness examined in Blume and Board (2013). They, using a cheap talk model in which the state distribution and the players' language competence were ex-ante uncertain, demonstrated that this vagueness occurs as the equilibrium language. We expand the work of Blume and Board by using a model between an uninformed decision maker and an informed agent in which the state-distribution and the state are both ex-ante uncertain. We show that this two-dimensional uncertainty also leads to indeterminacy of indicative meanings, that is, to a set of conditions in which an agent with different perceptions of state-distribution intentionally uses the same symbol for the different extents of information on the state. Our vagueness can lead to welfare improvement.

Keywords: Information. Language. State-Uncertainty. Vagueness.

JEL Codes: D82, D83, M14.

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1 Introduction

> "*A proposition is vague when there are possible states of things concerning which it is intrinsically uncertain whether, had they been contemplated by the speaker, he would have regarded them as excluded or allowed by the proposition. By intrinsically uncertain we mean not uncertain in consequence of any ignorance of the interpreter, but because the speaker's habits of language were indeterminate*" (Peirce, 1902, pp. 748).[1]

Vagueness of language is a pervasive feature of natural language. There are various interpretations of vagueness of language.[2] Lipman (2009) discussed vagueness related to context-dependent meaning. Blume and Board (2013), hereafter BB, considered a kind of vagueness, *indeterminacy of indicative meanings*, related to language competence.[3] We study indeterminacy of indicative meanings, related to the perception of the state-distribution (or worldview). If a message indicates a set of states rather than the exact state, the language is *imprecise*. If a message can indicate different sets of states, for example, depending on the context, language is *vague*. In addition, if the receivers of the message do not know which set of states the sender of the message indicates, there is *indeterminacy of indicative meanings* (or disagreements about indicative meanings of messages among players).

As an illustration of indeterminacy of indicative meanings, the word "threat" was used inconsistently through the U.S. Federal government. On August 26, 2014 in reaction to ISIS video release on the first beheading of U.S. journalists, Department of Homeland Security Secretary Jeh Johnson said that DHS and the FBI are "unaware of any specific, credible threat to the U.S. homeland" from Islamic State.[4] White House Press Secretary Josh Earnest said that the U.S. has no current plant to raise its own threat level.[5] Secretary of Defense Chuck Hagel said "They are an imminent threat to every interest we have, whether it's in Iraq or anywhere else."[6] Among the U.S.

[1]This reference is contained in the revision in 2012 of Keefe & Smith (1997). Retrieved from https://plato.stanford.edu/entries/vagueness/

[2]There is an increasing number of studies on vagueness of natural language by economists as well as philosophers. See Section 2 for further details.

[3]In BB, the set of messages each player can use and understand is limited and private information.

[4]http://www.washingtontimes.com/news/2014/aug/29/jeh-johnson-no-imminent-threat-against-us-despite-/

[5]http://www.foxnews.com/world/2014/08/29/cameron-talks-tough-on-radical-islam-as-uk-raises-terror-threat/

[6]http://www.bloomberg.com/news/2014-08-21/islamic-state-poses-imminent-threat-to-u-s-

military services, "the word 'aircraft' was understood by the Air Force to include helicopters, but by Army pilots to exclude helicopters" (Snook, 2000, pp. 163).[7]

We attempt to provide a theoretical explanation for this type of vagueness extending cheap talk models developed by Crawford and Sobel (1982), hereafter CS. We assume two-dimensional uncertainty about the state in which the underlying state-distribution and the state is ex-ante uncertain to players.[8] We focus on a static model of communications between an uninformed receiver/decision-maker (R) and an informed sender/agent (S). R chooses an action after taking advice from S. The model predicts that this two-dimensional uncertainty leads the equilibrium language to be vague: S, with different perceptions of the state-distribution (or different views of the entire world), uses the same message for different extents of S's information on the state. For example, there are two uniform distributions with different sizes of support: a small size [0, 1] and a large size [0, 2]. S observes a binary signal on the state-distribution. He also observes a signal on the state, which is the percent rank to the support of the true state distribution in ascending order. S has two types given the first signal: type 1 perceives that the small size (of distribution) is more likely than the large size while type 2 perceives the opposite.[9] Like in CS, there is a conflict of interest between the players: S always prefers a higher action than R given any state. Thus, each type partitions the space for the second signal into finite intervals and sends a different message to R for a different interval. Furthermore, different types use the same message for different extents of information on the state. For example, if type 1 says "small" when his second signal is below 20%, then type 2 also says "small" when his second signal is below 15%, and vice versa. If type 1 says "large" when his second signal is above 20%, then, type 2 also says "large" when his second signal is above 15%, and vice versa. In other words, given the same signal 25%, different types use different words. Furthermore, without knowing S's type, R does not know what S indicates by saying "small."

We also show that vagueness can result in welfare improvement in simplified

hagel-says.html

[7]This example is contained in Crémer, Garicano and Prat (2007).

[8]To our knowledge, little literature on communication games considers two-dimensional uncertainty on the state. However, we believe that our uncertainty setting is realistic in many situations. Knight (1921) said "we live in a world full of contradiction and paradox...the existence of a problem of knowledge depends on the future being different from the past..." (pp. 167).

[9]Uncertainty about the state-distribution in our model is replicated in a model with a single state-distribution and multiple signal channels. However, this alternative model also involves two dimensions of uncertainty. See Section 3 for further details.

discrete models. Consider a referendum on cutting government spending. The government (S) announces the optimal level of spending cut to the public. Given the announcement, the representative voter (R) updates a belief on the optimal level. This voter's updated belief will be implemented. The optimal level is null or small (distribution 1). Or it may be null or large (distribution 2). The government is more biased toward the larger cut than the voter. When the null level is optimal for the voter, the government prefers a small cut but dislikes a medium or large cut. In this example, suppose the distribution is known to everyone. In the former case (distribution 1), the government always recommends a small cut (a babbling equilibrium). But in the latter case (distribution 2), the government differentiates the recommendation depending on the state. Next suppose that the true distribution is uncertain to the public. Given any distribution, the government announces a different recommendation for a different state using vague language and induces a null cut when it is optimal for the voter and a medium cut else. This improvement of information transmission benefits both players.

Last, we examine the effect of heterogenous beliefs on the equilibrium language. Due to psychological, cultural or other factors, the players possess different prior beliefs on the underlying state distribution and update the same information differently. We show that the language will be vague if there are at least either heterogenous prior beliefs or heterogeneous preferences. However, heterogeneity in preferences and heterogeneity in prior beliefs can complement each other in facilitating information transmission. This result shows that heterogeneity can benefit organizations.

The remainder of the paper is organized as follows: Section 2 reviews related literature. Section 3 describes a basic model and the main results. Section 4 uses the uniform-quadratic specification of the CS model. Section 5 extends the model to argue political campaigns and discusses welfare implications of the vagueness. Section 5 also studies a model with heterogenous beliefs. Section 6 concludes the paper. Proofs are in the appendix.

2 Related Literature

Since Lewis (1969) analyzed conventions of languages using coordination games, philosophers and economists have applied game theory to study linguistics. Two of important approaches are "game theoretic pragmatics" and "evolutionary game theory to model cultural language evolution" (De Jaegher, 2003, pp. 406).[10] This study takes the former approach. In particular, as discussed in Section 1, it is closely related to the theory of vagueness of natural language by BB. In BB and our paper, there are two-dimensions of uncertainty: one dimension is on the state (an action type), and the other dimension is on S's type associated with his fundamental ability and invisible from R. The latter dimension of uncertainty causes language for communicating information on the former dimension of uncertainty to be vague. Although BB and we posit two dimensions of uncertainty, BB assumed constraint of players' ability to produce and interpret messages while we assume constraint of players' ability to perceive the underlying distribution of the state. BB studied a common interest game while we consider differences in preferences as well as prior beliefs between the players. Blume and Board (2014) also studied vagueness of language. Unlike this paper, Blume and Board (2014) considered uni-dimensional uncertainty (about the action type) and a communication error, noise. Hence, identical messages by S do not always result in identical interpretations by R; i.e., the language is vague. Due to different preferences among the possible projects, a type of S intentionally selects a message to increase the vagueness in R's side. In their model as well as our work, this intentional vagueness can enhance welfare.

Our work extends the research by Rubinstein (2000) and Lipman (2002, 2009) on the economics of languages. Rubinstein (2000) proposed a model of optimal languages using a different framework from ours. According to Rubinstein, a language is a binary relationship between objects, and the optimal language allows an efficient, precise identification of specific events. Rubinstein also studied the emergence of languages through evolution while we analyze the vagueness of language in a static model and our equilibrium language has a partitional form. Lipman (2002) discussed topics associated with the economics of languages such as the optimal structure of languages; effects of language on choice, debate, and inference rules; and vagueness. Lipman (2009) extended the argument on context-dependent vagueness. He introduced an example where we intentionally use "tall" for different heights for different types of objects: "'tall' for a newborn means about 15 inches, while 'tall' for a professional basketball player means above 6 foot 10" (Lipman, 2009, pp. 1).

De Jaegher (2003) as well as Blume, Board and Kawamura (2007) related a game theoretic rational for vagueness to the idea of Myerson (1991, pages 285-288): noisy

[10] De Jaegher (2003) provided an overview of applications of game theory in linguistics by focusing on these two approaches.

communication channels can mitigate conflict of interest and hence improve information transmission (a carrier pigeon example). De Jaegher (2003) looked at correlated equilibria in coordination games. He argued that vagueness, which is driven by correlation device, can be efficiency enhancing. Blume, Board and Kawamura (2007) extended CS's cheap talk model by including exogenous noises in it. They showed that for any level of conflict of interest, there is some level of noise which enhances welfare as obtained by any communication device.

On the other hand, De Jaegher and Rooij (2011), Franke, Jäger and Rooij (2011), O'Connor (2014), Lambie-Hanson and Parameswaran (2015), Parameswaran and Lambie-Hanson (2016), and Lim and Wu (2017) studied vagueness of language using common interest games. De Jaegher and Rooij (2011) assumed that R imperfectly observes contexts in S's message. They explained the evolution of vagueness using prospect theory. Franke, Jäger and Rooij (2011) showed the evolution of vagueness as a result of cooperative signaling among boundedly rational players (limited memory). O'Connor (2014) considered a model where the states are contiguous. He showed the evolution of vagueness as a result of learning dynamics.

Lambie-Hanson and Parameswaran (2015) introduced R's type of prior belief about the state distribution as well as S's language type. Parameswaran and Lambie-Hanson (2016) introduced R's type of interpreting S's message and considered noises in S's information. Lim and Wu (2017) developed *literal vagueness*, which is a kind of context-dependent vagueness, and showed their vagueness arose as Pareto-optimal equilibrium outcomes in sequential talk models.

Crémer, Garicano and Prat (2007) studied the theory of organizational languages in the presence of bounded rationality. They analyzed the optimal language when the number of available messages is limited due to bounded rationality. They formalized Arrow (1974)'s idea of coding and explored the relationship between the choice of organizational language and the choice of organizational structure while we address organizational languages when the number of messages is endogenously determined. Warglien (2013) discussed two perspectives on organizations and languages. His first perspective focuses on languages in organizations including economic studies by March and Simon (1958) and Arrow (1974). The second perspective is on organizations as language. This paper is related to the first perspective.

There are a number of linguistic and philosophical investigations published on vagueness of natural languages. Williamson (2003) summarized multiple approaches to the issue such as epistemicist view, fuzzy logic, supervaluationism and vagueness as a property of objects. For each specific approach to vagueness in linguistic and philosophical studies, see Williamson (1994) on the epistemicist view, Zadeh (1975) on fuzzy logic, and Fine (1975) as well as Keefe and Smith (1997) on supervaluationism. Our approach is closer to the epistemicist view in the sense that the equilibrium language draws boundaries in S's information but R does not know where the boundaries lie.

Our basic model is an extension of cheap talk models pioneered by CS. In CS, the state distribution is common knowledge, only the state is ex-ante uncertain. A perfectly informed S partitions the state space into finite spaces and reveals the true interval to an uninformed R. That is, the language is imprecise. In our model, the language is vague as well as imprecise.

The idea of considering uncertainty about distributions is related to Morgan and Stocken (2008) and Kawamura (2013). A distribution of populations' preferences is unknown to a policy maker in their papers while the sender's perception of state distribution is unknown to the receiver in our model.

3 Model

3.1 Setup

There are two players, a receiver (R) and a sender (S). R is a decision maker who selects and implements an action. R's action and the state θ decide payoffs for each player. S has more information on the state. Thus, before making a decision, R asks S for advice.

The state-distribution, whose cumulative distribution function (CDF) is denoted by F, is not certain ex-ante. There are n state-distributions, where $n \geq 2$, each of which is a continuous distribution supported on a compact and convex subset of $\mathbb{R}$. CDFs and supports of the distributions are denoted by F_i and $T_i \subset \mathbb{R}$, respectively, for $i \in \{1, 2, ..., n\}$. $F = F_i$ is realized with probability q_i for $i \in \{1, 2, ..., n\}$, where $\sum_{i \in \{1,2,...,n\}} q_i = 1$.

We assume that there are at least two distributions, F_i and F_j, where $i, j \in \{1, 2, ..., n\}$ such that $T_i \cap T_j$ has a positive measure, and:

$$F_i(\theta) \neq F_j(\theta) \text{ almost everywhere over } T_i \cap T_j. \tag{1}$$

F is not directly observed by any player. Instead, S observes a private signal, denoted by $t \in \{1, 2, ..., n\}$. This signal reveals the truth with probability $p \in [0, 1]$ and is a random noise with support $\{1, 2, ..., n\}$ with probability $1 - p$, where the random noise is drawn according to the same distribution as the distribution of state distributions. For example, given $F = F_i$ for $i \in \{1, 2, ..., n\}$, S observes:

$$t = \begin{cases} i & \text{with probability } p + (1-p)\, q_i \\ j & \text{with probability } (1-p)\, q_j \text{ for } j \in \{1, 2, ..., m\} \setminus \{i\} \end{cases} \tag{2}$$

θ, which is drawn according to F, is not directly observed by any player. Instead, S observes a private signal $s \in [0, 1]$ so that:

$$s = F(\theta) \text{ for } \forall F \in \{F_1, F_2, ..., F_n\}. \tag{3}$$

That is, S only observes a rank of the state, which is measured by percent to the support of the true state distribution F in ascending order.

After observing the two signals, S sends a message to R. S's message set is M, which is anything sufficiently large such as $\mathbb{R}$. Thereafter, R selects an action from $\mathbb{R}$.

The timeline is summarized as follows:

1. Nature selects the state distribution $F \in \{F_1, F_2, ..., F_n\}$. Nature also selects the state θ according to F.

2. S observes two signals: a signal $t \in \{1, 2, ..., n\}$ on the state distribution and a signal $s \in [0, 1]$ on the state.

3. S sends a message $m \in M$ to R.

4. R selects an action $a \in \mathbb{R}$.

5. The payoff is realized for every player. The game ends.

The payoff function of player $l \in \{R, S\}$ is denoted by $U^R(a, \theta)$ and $U^S(a, \theta, b)$, respectively. $U^R(a, \theta) \equiv U^S(a, \theta, 0)$ holds for all a and θ, and $U^S(a, \theta, b)$ is twice continuously differentiable. For any θ and b, there is a unique action a so that $U_1^S(a, \theta, b) = 0$. Moreover, $U_{11}^S(a, \theta, b) < 0 < U_{12}^S(a, \theta, b)$ holds for all a and θ, and $0 < U_{13}^S(a, \theta, b)$ holds for all a, b and θ. Functional forms of U^R and U^S are common knowledge.

Let $a^R(\theta) = \{a \in \mathbb{R} : U_1^R(a,\theta) = 0\}$. $a^S(\theta;b)$ is defined similarly. Then, from the definitions of payoff functions, $a^S(\theta;b) = a^R(\theta)$ holds given $b = 0$ for all θ, and $a^S(\theta;b) - a^R(\theta)$ increases with b for any θ and b.

In summary, no player directly observes the state-distribution F or the state θ. S observes two private signals, t (on F) and s (on θ). The other aspects of the game are common knowledge.

The two key parameters are p and b. p measures the level of stochasticity around the problem. p can also measure S's ability to understand the set of potential solutions or the entire world. Then, b measures conflict of interest over the optimal actions between R and S. If $b > 0$, S always prefers a larger action than R.

Comparison of this model with CS and BB with respect to uncertainty—Unlike in CS and BB, there is uncertainty about the state-distribution. In addition, contrary to BB, we do not consider uncertainty nor constraint about S's message set. In BB, a number of messages available to S can be smaller than the number of states, and S's message set is S's private information. In our model, S's message set is large, which is known to everyone.

An alternative model—The uncertainty in this model is replicated in a model with a single distribution and multiple signal channels. However, this alternative model also involves two dimensions of uncertainty. For example, consider an alternative model with a single state-distribution and multiple signal channels in which the CDF of the state-distribution is given by $\widetilde{F}(\theta) = \sum_{k \in \{1,2,...,n\}} q_k F_k(\theta)$. S observes two signals t and s. There are n channels generating the second signal s such that given the state θ, s goes through the i-th channel with probability $\frac{q_i F_i(\theta)}{\sum_{k \in \{1,2,...,n\}} q_k F_k(\theta)}$. Given that the true signal channel is the i-th channel, the first signal t is determined so that $t = i$ with probability $p + (1-p) q_i$, and $t = j$ with probability $(1-p) q_j$ for $j \in \{1,2,...,n\} \setminus \{i\}$. The second signal s is determined so that $s = F_i(\theta)$. Hence, R is not certain about what S observes on the state (signal s) and how S interprets it (S's information on the signal channel).

3.2 Definitions

The solution concept is *Perfect Bayesian equilibrium* (PBE). In PBE, a strategy for S associates his information with a message m and is optimal for S given R's reaction to his message. R updates his belief on the true state using Bayes' rule. A strategy for R associates each message m with an action a and is optimal for R given his updated belief.

In addition, the following definitions are used in the entire paper.

Types—S is called *Type t* if he observes signal $t \in \{1, 2, ..., n\}$ on the state distribution. S's type represents how he perceives the state distribution (or his worldview). For $\forall t \in \{1, 2, ..., n\}$, Type t infers $F = F_t$ with probability $p + (1 - p) q_t$ and $F = F_j$ with probability $(1 - p) q_j$ for $j \in \{1, 2, ..., n\} \setminus \{t\}$.

Partition Equilibrium—We define "*a partition equilibrium with* N_1-N_2-...-N_n *intervals*" to be a PBE in which Type t partitions his signal space into N_t intervals and sends a different message given signals belonging to a different interval of his information partition for each $t \in \{1, 2, ..., n\}$. Let σ_t^* specify an equilibrium strategy of Type t, a mapping from s to m. Let $\sigma^* = \{\sigma_1^*, \sigma_2^*, ...\sigma_n^*\}$ be strategies of all types. Let ρ^* specify an equilibrium strategy of R, a mapping from m to a.

Outcome Equivalent—We do not distinguish equilibria which involve the same partitions but different messages and hence are outcome equivalent to one another.

In addition, we assume that S uses *one message per interval* (i.e. S uses the same m given signals belonging to the same interval of the partition). This assumption implies that, unless there is full revelation, there remain *off-equilibrium messages*. Hence, we assume that given off-equilibrium message, R infers nothing additionally to common prior.

We also suppose that both types commonly use some message only if they strictly prefer the message to any other messages. In cheap talk models, m is a mere symbol because it does not reveal any information by itself. Hence, the result remains unchanged if S randomizes his messages so that there are no off-equilibrium messages as in CS.

Indicative Meanings and Vagueness—Let $F(\theta|m, t)$ denote a CDF conditional on message m sent by S and S's type t. Then, we interpret that $F(\theta|m, t)$ represents type t's *indicative meaning* by sending m.

Suppose that both types t and t' send some message m on the equilibrium path, and the extensions of the message is different for different types such that $F(\cdot|m, t) \neq F(\cdot|m, t')$, then, we say the language is *vague*.

Indeterminacy of Indicative Meanings—Let $F(\theta|m)$ denote a conditional CDF (without information on t). We say that there is *indeterminacy of indicative meanings* if $F(\cdot|m) \neq F(\cdot|m, t)$ for any t who sends m on the equilibrium path (i.e., R cannot determine the indicative meaning of m without knowing S's type).

Language—We call the set of messages and indicative meanings of the messages *language*.

Let $\Delta x_i = x_i - x_{i-1}$ for any sequences x_1, x_2, ..., x_{i-1} and x_i from real numbers.

3.3 Equilibria

Our main questions include: How does the two-dimensional state-uncertainty affect the equilibrium language? To answer this question, we compare models under different dimensions of state uncertainty. In both cases, only S observes a private signal s on θ.

Under *one-dimensional state-uncertainty*, every player has perfect information on the state distribution F. (I.e., $p = 1$ and type signal t is public information.)

Under *two-dimensional state-uncertainty*, S has private information on F (and hence θ). (I.e., $p \in [0, 1]$ and type signal t is S's private information.)

Hence, under one-dimensional state-uncertainty, the game is comprised of two subgames so that there is one subgame per type of S. Hence, it is a simple extension of CS. Under two-dimensional state-uncertainty, there is only one subgame because S's type is not observable to R.

Lemma 1 *Fix $b \in (0, \infty)$. Under one-dimensional state-uncertainty (i.e., t is public information), every PBE is a partition equilibrium. Further, every equilibrium is outcome equivalent to an equilibrium in which different types of S use different messages on the equilibrium path.*

Proof. See Appendix A.1. ∎

With one dimension of uncertainty, S's type is observable to R. Hence, different types do not need to use the same message, and hence the equilibrium language is not vague.

Lemma 2 *Fix $b \in (0, \infty)$ and $p \in [0, 1]$. Under two-dimensional state-uncertainty (i.e., t is S's private information), every PBE is a partition equilibrium.*

Proof. See Appendix A.1. ∎

As under one-dimensional state-uncertainty, under two-dimensional state-uncertainty, equilibrium language has a partitional form when there is conflict of interest (i.e., $b \neq 0$). However, different types should use the same message as follows.

Proposition 1 *Fix* $b \in (0, \infty)$ *and* $p \in [0, 1]$. *Consider any equilibrium under two-dimensional state-uncertainty (i.e., t is S's private information). Consider a set of types, denoted by* $\overline{N}$, *such that* $\bigcap_{i \in \overline{N}} T_j$ *has a positive measure and condition (1) holds for every pair of* $i, j \in \overline{N}$, *where* $i \neq j$. *Then, all types belonging to* $\overline{N}$ *should commonly use at least one message. Further, if* $p \in (0, 1)$, *there is indeterminacy of indicative meanings of a message which is sent by all types belonging to* $\overline{N}$ *on equilibrium path.*

Proof. See Appendix A.1. ■

With two dimensions of state-uncertainty, each type has incentives to pretend to be the other types. As a result, the extensions of their language is vague—different types commonly use some messages (words) for different extensions. Hence, the multiple dimensional state-uncertainty causes the language to be vague.

As discussed in Sections 1 and 2, BB also showed that the same message is used for different extensions by different types. However, their type depends on S's language competence (i.e., S's message set).

As BB argued, it can be natural in some situations that R can select an outside option, which yields state-dependent payoffs to both players and whose payoffs are ex-ante known to every one. In BB, the game with this option is called *a default game*. According to Chiba and Leong (2013), equilibria still have partitional forms in the leading case of CS model with an outside option in R's choice set. With the default option, our main result remains unchanged.

4 Uniform-Quadratic Case

From now on, we focus on a uniform-quadratic case, introduced in CS. Its tractability allows us to examine what our vague language looks like.

We consider two state-distributions, i.e., $n = 2$. Hence, there are two types for S. There are two uniform distributions, F_1 and F_2, whose supports are $T_1 = [0, 1]$ and $T_2 = [0, L]$, respectively, where $L > 1$. The true state-distribution F is F_1 with probability $q_1 = \frac{1}{2}$ and F_2 with the remaining probability. Hence, Type 1 infers that F is F_1 with probability $\frac{1+p}{2}$ and F_2 with probability $\frac{1-p}{2}$, and vice versa for the inference by Type 2.

The payoff function of each player is $U^R(a, \theta) = -(\theta - a)^2$ and $U^S(a, \theta, b) = -(\theta + b - a)^2$ where $b > 0$. The model specification in this section satisfies assumptions required to obtain the main results in Section 3.

Figure 1 shows the outcome given $L = 2$ (i.e., $T_1 = [0,1]$ and $T_2 = [0,2]$), $b = \frac{1}{4}$ and $p = \frac{1}{2}$ under two-dimensional state uncertainty. There is a partition equilibrium in which both S types must commonly use two messages. For each type, the boundary point in his signal space is $x_1 = \frac{7}{5}y_1$ and $y_1 = \frac{\sqrt{3201}-45}{84} \approx \frac{1}{7}$. Figure 2 shows the outcome given $L = 2$ and $b = \frac{1}{4}$ under one-dimensional state uncertainty. Type 1 uses one message at most, and Type 2 uses two messages at most. Both S types do not need to use a common message.

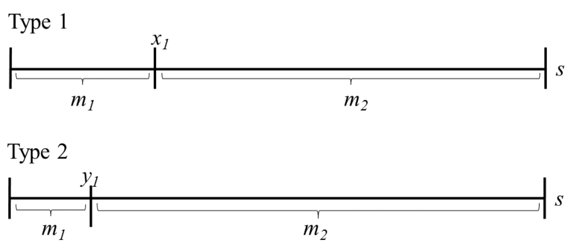

Figure 1:
Two-dimensional state uncertainty (*b*=1/4, *p*=1/2)
Types 1 and 2 should commonly use symbols m_1 and m_2. y_1= (5/7) x_1 and $x_1 \approx 1/7$.

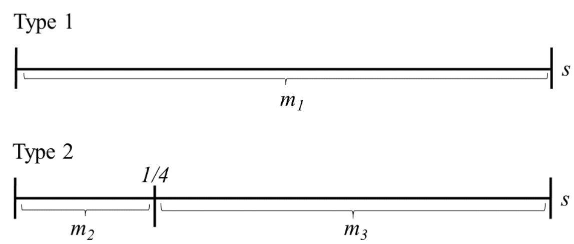

Figure 2: One-dimensional state uncertainty (*b*=1/4)
Both types do not need to use common symbols. Type 1 uses only one symbol. Type 2 uses two symbols.

Remark 1 *Fix $p \in [0,1]$ and $b \in (0,\infty)$. In a partition equilibrium with N_1-N_2 intervals, both types of S commonly use messages, m_1, m_2, ..., m_{N_1}. There is indeterminacy of indicative meanings at least for m_{N_1}. Moreover, if $p \in (0,1]$, there is indeterminacy of indicative meanings for all m_1, m_2, ..., m_{N_1}, and, for each of the common messages, information provided by Type 2 second order stochastically dominates (SOSD) information provided by Type 1.*

Type 1 perceives that the smaller size (of state-distribution) is more likely than the larger size, and vice versa for the perception of Type 2. Nevertheless, Type 1 provides information with the same mean but larger variances than Type 2. We also show in Appendix A.2 that Type 1 uses each common message for the wider range of states than Type 2.

To understand intuitions of this result, we briefly describe the equilibrium outcome after defining notations.

Here are notations to be used in this section. $x(N_1) \equiv (x_0, ..., x_{N_1})$ denotes Type 1's partition in a partition equilibrium with N_1-N_2 intervals where $0 = x_0 < x_1 <$

... $< x_{N_1} = 1$, and (x_{t-1}, x_t) is called the t-th interval of $x(N_1)$. Type 2's partition, denoted $y(N_2) \equiv (y_0, ..., y_{N_2})$, is defined similarly. Thus, S's equilibrium strategy $\sigma^* = \{\sigma_1^*, \sigma_2^*\}$ is characterized by $x(N_1)$ and $y(N_2)$.

$N_2(b,p)$ denotes the maximum number of messages used by Type 2 in an equilibrium. $N_1(b,p)$ denotes a number of messages used by Type 1 in equilibrium given Type 2 uses $N_2(b,p)$ messages.

In a partition equilibrium with N_1-N_2 intervals, there is a unique N_1 given N_2, and $N_1 \leq N_2$. The relationship between $x(N_1)$ and $y(N_2)$ is given by:

$$x_i = \begin{cases} \frac{L(1+p)+1-p}{1+p+L(1-p)} y_i & \text{for } i \in \{1, ..., N_1 - 1\} \\ 1 & \text{for } i = N_1. \end{cases} \tag{4}$$

Type 2's partition $y(N_2)$ satisfies:

$$\Delta y_i = \begin{cases} \Delta y_{i-1} + \frac{8b}{L(1+p)+1-p} & \text{for } i \in \{2, ..., N_2\} \setminus \{N_1, N_1+1\} \\ \Delta y_{i-1} + \frac{8b}{L(1+p)+1-p} + \frac{4}{L(1+p)+1-p} G_{b,p}(N_1) & \text{for } i \in \{N_1, N_1+1\}. \end{cases} \tag{5}$$

where:

$$G_{b,p}(N_1) = \frac{\Pr\left[m=m_{N_1}|t=1\right]\left(E\left[\theta|m=m_{N_1}, t=2\right] - E[\theta|m=m_N, t=1]\right)}{\Pr\left[m=m_{N_1}|t=1\right] + \Pr\left[m=m_{N_1}|t=2\right]}. \tag{6}$$

Types 1 and 2 send $m = m_i$ given $s \in (x_{i-1}, x_i)$ (if this interval exists) and $s \in (y_{i-1}, y_i)$, respectively, for $i \in \{1, ..., N_2\}$, and $m_i \neq m_{i'}$ for any $i \neq i'$.

There may be multiple equilibria with different N_2 (and hence N_1). As observed in CS, both players are better off as N_2 increases. It is reasonable to focus on a partition equilibrium with $N_1(b,p)$-$N_2(b,p)$ intervals. See Appendix A.2 for further details and proofs of the remark in this section.

5 Extensions

We have considered the uniform-quadratic case in CS so that our work is comparable to the current literature such as BB. However, the uniform-quadratic case requires heavy computations. Hence, we will use a further simplified model (with discrete states) to examine welfare implications of our vagueness. We will also discuss whether S reveals distribution type if he can. Finally, we will examine the effect of heterogenous beliefs on the equilibrium language.

5.1 Political Campaign (Discrete Case)

We simplify a uniform-quadratic model in Section 4. Specifically, we consider two discrete-distributions. Given $F = F_1$, the state is $\theta = 0$ or $\theta = 1$, which are equally likely. Given $F = F_2$, the state is $\theta = 0$ and $\theta = L$, where $L > 1$, which are equally likely. We use this simplified model to examine welfare implications of vagueness.

We think that this model can help understand vagueness of languages in political campaigns. "The basis of a good campaign strategy is, first, to recognize the reality of the campaign—or broader context in which the campaign is being run—and second, to set an appropriate goal that, if achieved, will win the election." (Sides, Shaw, Grossmann, and Lipsitz, chapter 5)

Consider a referendum on a policy proposal such as a spending cut. The government is S, and a representative voter is R. The true state tells whether to implement a spending cut $(a > 0)$ or not $(a = 0)$ and how large the spending cut should be. Given the government's announcement, the voter has a view of the optimal level, which will be implemented. Given $F = F_1$, the optimal spending cut level is null or small. Given $F = F_2$, it is null or relatively large.

Remark 2 *Fix $1 < L < 2$ and $b \in \left(\frac{1}{2}, \frac{1+L}{4}\right)$. Then, there is indeterminacy of indicative meanings in the equilibrium language under two-dimensional state uncertainty. Furthermore, this vagueness improves welfare.*

We look at R's ex-ante expected payoff because it is well known that S's ex-ante expected payoff increases if R's payoff is increases.

Fix $b \in \left(\frac{1}{2}, \frac{1+L}{4}\right)$. Under two-dimensional state uncertainty, the both distribution types commonly use two messages. If type 1 sends $m = \textit{Null}$ and $m = \textit{Large}$ given $s = 0$ and $s = 1$ respectively, then, so does type 2, and vice versa. Given $s = 1$, type 1 infers $\theta = 1$ with probability $\frac{1+p}{2}$ and $\theta = L$ with the remaining probability while type 2 infers $\theta = 1$ with probability $\frac{1-p}{2}$ and $\theta = L$ with the remaining probability. Given $s = 0$, both types infer $\theta = 0$ with probability 1. As a result, R will choose $a = 0$ and $a = \frac{1+L}{2}$ in response to $m = \textit{Null}$ and $m = \textit{Large}$ respectively. See Figure 3. R's ex-ante expected payoff is $-\frac{(1-L)^2}{8}$.

On the other hand, under one-dimensional state uncertainty, if $F = F_1$ (i.e., $\theta = 0$ or $\theta = 1$) or S has type 1, the players do not agree with the optimal choice. S always wants to induce the highest inducible action. There is only a babbling equilibrium, and $a = \frac{1}{2}$ is chosen in any state. If $F = F_2$ (i.e., $\theta = 0$ or $\theta = L$) or S has type 2, the players can agree with the optimal choice. Even for S, $a = L$ is too large in state $\theta = 0$. Hence S fully reveals the state. $a = \theta$ is chosen in every state. See Figure 4. Hence, under one-dimensional uncertainty, R's ex-ante expected payoff is $-\frac{1}{8}$.

Therefore, two-dimensional uncertainty causes vagueness (specifically $m = Large$ in this case), but it incentivizes both types of S to separate his messages depending on the signal. As a result, both players' payoff increases on average.

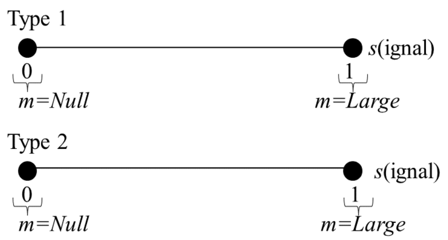

Figure 3: Simplified Model. Two-dimensional state uncertainty given $b = (3+L)/8$. Both types commonly use two messages *Null* and *Large* to induce actions 0 and (1+L)/2, respectively.

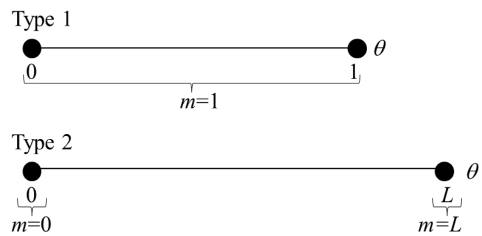

Figure 4: Simplified Model. One-dimensional state uncertainty given $b = (3+L)/8$
Type 1 does not reveal the state and always induces action ½ while type 2 reveals the state and action 0 or L.

If $b \notin \left(\frac{1}{2}, \frac{1+L}{4}\right)$, the vagueness does not improve welfare. For $b < \frac{1}{2}$, S has closer preferences to R and hence fully reveals information under one-dimensional state uncertainty. But under two dimensional uncertainty, both distribution types commonly use two messages. Thus, the additional dimension of uncertainty causes welfare loss. For $b > \frac{1+L}{4}$, S always wants to induce the highest inducible action. In this case, two-dimensional state uncertainty is worse.

What if there is one more communication stage after S observes a signal t (on F) but before he observes a signal s (on θ)?

Remark 3 *Fix $1 < L < 2$ and $b \in \left(\frac{1}{2}, \frac{1+L}{4}\right)$. Then, in the first communication stage, both distribution types (t=1,2) pool. In the second communication stage, different distribution types commonly use the same messages.*

If type information is revealed, type 2 is better-off. But type 1 is worse-off because the second stage communication cannot be informative. Thus, type 1 pretends to be type 2. Indeterminacy of indicative meanings remain. See Appendix A.3 for further details and proofs of remarks in this section.

5.2 Agree to Disagree (Different Prior Beliefs)

The basic model shows that the language is vague (indeterminacy of indicative meanings) if there is heterogeneity in preferences. The section introduces heterogenous prior beliefs into the basic model. Due to psychological, cultural or other factors, the players are endowed with different prior beliefs and update the same information differently. Tversky and Kahnemen (1974) and Aumann (1976) initiated arguments on the effect of psychological, cultural and other factors on prior beliefs of players. Since then, a number of examples in the literature such as Che and Kartik (2009) and Van den Steen (2010) modelled communication games with different prior beliefs. According to Che and Kartik (2009), "Although game-theoretic models often assume a common prior, referred to as the Harsanyi doctrine, there is a significant and growing literature that analyzes games with heterogenous priors" (pp. 817).

We will show the following. First, language is vague if there is heterogeneity in at least either prior beliefs or preferences. Second, heterogeneity in preferences and heterogeneity in prior beliefs can complement each other in facilitating information transmission (non-monotonicity).

We use the uniform-quadratic specification in Section 4 given $L = 2$. Additionally, we assume the difference that both players have *different prior beliefs* on the distribution. Each player $l \in \{R, S\}$ believes $F = F_1$ with probability q_l and $F = F_2$ with probability $1 - q_l$, where $q_l \in (0, 1)$. They know their own prior and the other player's prior.

Remark 4 *Fix $b = 0$ and $p \in (0, 1]$. Then, any PBE is a partition equilibrium in which both types of S commonly use at least one message for different extensions. The indicative meanings of the commonly used message are indeterminate.*

The last analysis examines organizational issues. Two dimensional state uncertainty causes two dimensional heterogeneity between the players. How do these two dimensions of heterogeneity affect the equilibrium language?

Again we assume different preferences ($b > 0$). For simplicity, we suppose that S has no information on the state distribution ($p = 0$). Thus, there is one type of S, and the only type partitions the signal space. Let $x(N) \equiv (x_0, ..., x_N)$ denote a partition of the signal space $[0, 1]$ with N intervals, where $0 = x_0 < ... < x_N = 1$. Let $N^{DP}(b, q^R, q^S)$ denote the maximum number of intervals for some b, q_R and q_S given $p = 0$.

Remark 5 *Fix $p = 0$. Then, $N^{DP}\left(b, q^R, q^S\right)$ is not monotonic with $|q_R - q_S|$ given b. Similarly, $N^{DP}\left(b, q^R, q^S\right)$ is not monotonic with b given $|q_R - q_S|$.*

In a partition equilibrium with N intervals, the partition should satisfy:

$$\Delta x_i = \Delta x_{i-1} + \tfrac{4}{2-q_R} b + \tfrac{4(q_R - q_S)}{2-q_R} x_i \quad \text{for } i \in \{2, ..., N\}.$$

Unless $q_R = q_S$, there is non-monotonicity as shown in Figures 5 and 6. Fixing $|q_R - q_S| \neq 0$, $N^{DP}\left(b, q^R, q^S\right)$ is not monotonic with $|b|$. Fixing $|b|$, $N^{DP}\left(b, q^R, q^S\right)$ is not monotonic with $|q_R - q_S|$.

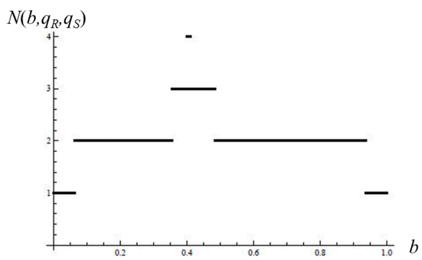

Figure 5: The maximum number of intervals $N(b,q_R,q_S)$ and the difference in preferences b given q_R=1/4 and q_S=3/4.

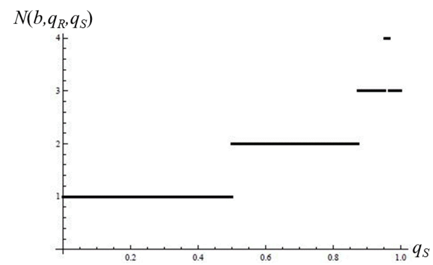

Figure 6: The maximum number of intervals $N(b,q_R,q_S)$ and S's prior belief q_S for distribution F_1 given b=3/8 and q_R=1/2.

The more messages the language involves, the better off both players are. Hence, the last lemma implies that heterogeneity can be beneficial to organizations. See Appendix A.4 for further details and proofs of remarks in this section.

6 Conclusion

This paper provides a potential theoretical explanation for the observation that natural language involves a kind of vagueness, indeterminacy of indicative meanings. We have shown that there can be indeterminacy of indicative meanings when there is two-dimensional state uncertainty (the state distribution as well as the state itself is ex-ante uncertain to players). An informed agent communicates with an uninformed decision maker using a message for different indicative meanings depending on how the agent perceives the entire world (the state distribution). Under some condition, this type of vagueness incentivizes the agent to reveal more information and hence leads to welfare improvement.

Our work suggests several avenues for future research. The first avenue is investigation of sufficient or/and necessary conditions for the optimality of our vagueness in static models. The second is studying evolution of the optimal language and examining whether vagueness of language remains over time. The third is conducting experimental tests for this model.

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Appendix A

A.1 Proofs for Section 3

A.1.1 Definitions

Define a^l for $l \in \{R, S\}$:

$$a^S(\theta; b) := \underset{a\in\mathbb{R}}{\arg\max} U^S(a, \theta, b) \text{ for } \forall\theta, \forall b$$

a^R is given similarly. From assumptions, a^l is well defined.

Define $\overline{U^l}$ for $l \in \{R, S\}$:

$$\begin{aligned}\overline{U^S}(a, s', b; i) &= E\left[U^S(a, \theta, b) \,|s = s', t = i\right] \\ &= (p + (1-p) q_i) U^S\left(a, F_i^-(s'), b\right) + \sum_{j\in\{1,2,\ldots,n\}\setminus\{i\}} (1-p) q_j U^S\left(a, F_j^-(s'), b\right).\end{aligned}$$

$\overline{U^R}$ is expressed similarly. $\overline{U^l}$ is a convex combination of U^l and hence preserves similar properties to those of U^l such as U^l is continuously twice differentiable, $\overline{U^R}(a, s; i) = \overline{U^S}(a, s, 0; i)$ for $\forall a, \forall s$; $\overline{U^S_{11}} < 0 < \overline{U^S_{12}}$ for $\forall a, \forall s$; and $0 < \overline{U^S_{13}}$ for $\forall a, \forall b$.

Thus, there also well defined A^l for $l \in \{R, S\}$:

$$A^S(s, i, b) := \underset{a\in\mathbb{R}}{\arg\max} \overline{U^S}(a, s, b; i)$$

Then, it follows from $0 < \overline{U^S_{13}}$ that $A^S(s, i, b) = A^R(s, i)$ if $b = 0$ for $\forall s, \forall i$ and $A^S(s, i, b) - A^R(s, i)$ increases with b for any b and θ. A^l is bounded below and above by $A^l(0, i)$ and $A^l(1, i)$, respectively because the support $\cup_{j\in\{1,2,\ldots,n\}} T_j$ is bounded.

A.1.2 Proof for Lemma 1

Under one-dimensional state uncertainty, there are n subgames, one game per type of S. Lemma 1 and Theorem 1 in CS directly applies to each subgame since S perfectly knows θ and payoff functions satisfy required conditions. Thus, each type uses finite messages and induces finite actions. Last, even if different types commonly use some message, R infers differently for different types. Thus, the claim holds.

A.1.3 Proof for Lemma 2

It suffices to show the next three claims: every PBE is a partition equilibrium; at least two types of S uses at least one message; when multiple types use a common message, they use it for different extensions.

Claim 1 *Every PBE is a partition equilibrium for any $b \in (0, \infty)$ and $p \in (0, 1)$.*

Proof. Follow Lemma 1 and Theorem 1 in CS. Thus, if the set of actions induced in equilibrium is finite, the equilibrium should have a partitional form because $\overline{U_{12}^S} > 0$. Thus. it suffices to show that if multiple actions are induced by one type, for every two distinct actions a and a' induced by one type, there is $\epsilon > 0$ such that $|a' - a| \geq \epsilon$.

Consider two cases: no message is commonly used by multiple types; second, at least one message is used by at least two types.

In the first case, the analysis is equivalent to the proof of Lemma 1 because a message reveals the type and payoff functions $\overline{U^l}$ satisfied required conditions for Lemma 1 and Theorem 1.

In the second case, suppose a is induced by Type 1 and Type 2. Suppose $a' > a$ is also induced by both types. Let a be induced by Type 1 given s_1 and Type 2 given s_2, respectively. Let a' be induced by Type 1 given s_1' and Type 2 given s_2', respectively. Hence, by weakly revealed preferences and continuity there exists a unique $\overline{s_i}$ such that $\overline{U^S}\left(a, \overline{s_i}, b; i\right) = \overline{U^S}\left(a', \overline{s_i}, b; i\right)$. Thus:

$$a < A^S\left(\overline{s_i}, i, b\right) < a' \text{ for } i \in \{1, 2\}.$$

Type k, where $k \in \{1, 2\}$, does not induce a for any $s > \overline{s_i}$ nor a' for any $s > \overline{s_i}$. This and $\overline{U_{12}^R} > 0$ implies that:

$$a \leq \gamma A^R\left(\overline{s_1}, 1\right) + (1 - \gamma) A^R\left(\overline{s_2}, 2\right) \leq a' \text{ for some } \gamma \in (0, 1).$$

Thus, there is an $\epsilon > 0$ such that $a' - a \geq \epsilon$.

Last, suppose a is induced by Type 1 and Type 2. Suppose $a' > a$ is induced only by Type 1. For Type 1, there exists a unique $\overline{s_1}$ such that $\overline{U^S}\left(a, \overline{s_1}, b; 1\right) = \overline{U^S}\left(a', \overline{s_1}, b; 1\right)$, but for Type 2, $\overline{U^S}\left(a, s, b; 2\right) > \overline{U^S}\left(a', s, b; 2\right)$ holds for any s. Thus:

$$a < A^S\left(\overline{s_1}, 1, b\right) < a',$$
$$a < A^S\left(1, 2, b\right) < a'.$$

Both types induce a, but type 1 does not induce it for any $s > \overline{s_i}$. Only Type 1 induces a' and he does not induce it for $s < \overline{s_1}$:

$$\begin{aligned} &A^R\left(\overline{s_1}, 1\right) \leq a', \\ &a \leq \gamma A^R\left(\overline{s_1}, 1\right) + (1-\gamma) A^R(1,2) \text{ for some } \gamma \in (0,1). \end{aligned}$$

Thus, there is an $\epsilon > 0$ such that $a' - a \geq \epsilon$. ■

A.1.4 Proof for Proposition 1

Fix $b \in (0, \infty)$ and $p \in [0,1]$. Consider any equilibrium under two-dimensional state-uncertainty (i.e., t is S's private information).

Consider a set of types $\overline{N}$ such that $\bigcap_{i \in \overline{N}} T_i$ has a positive measure and condition (1) (i.e., $F_i(\theta) \neq F_j(\theta)$ almost everywhere over $T_i \cap T_j$) holds for every pair of $i, j \in \overline{N}$, where $i \neq j$.

Let F_i^- represent an inverse function of F_i supported on $[0,1]$.

The next claim suffices to show that *all types belonging to $\overline{N}$ should commonly use at least one message.*

Claim 2 *For any $i \in \overline{N}$, an action a' induced by Type i given $s = s_i$ (and hence in its neighborhood) should be induced by Type j given some $s' \in \left[\min\left\{s_j, s_i\right\}, \max\left\{s_j, s_i\right\}\right]$ (and hence in its neighborhood) for every $j \in \overline{N}$.*

This claim implies that all types belonging to $\overline{N}$ should induce at least one same action. Hence, all types belonging to $\overline{N}$ should commonly use at least one message.

Proof. Take $\theta' \in \bigcap_{i \in \overline{N}} T_i$ such that $F_i(\theta') \neq F_j(\theta')$ for every pair of $i, j \in \overline{N}$, where $i \neq j$. any $i \neq j$. Let $s_i = F_i(\theta')$.

Without loss of generality, consider $1, 2 \in \overline{N}$ and $s_1 > s_2$. Then:

$$\begin{aligned} & F_1^-(s_2) < F_1^-(s_1) = \theta' = F_2^-(s_2) < F_2^-(s_1) \\ \Leftrightarrow \quad & a^S\left(F_1^-(s_2)\right) < a^S\left(\theta'\right) < a^S\left(F_2^-(s_1)\right). \end{aligned}$$

Besides, Type 1 (i.e., S observing $t = 1$) infers:

$$F = \begin{cases} F_1 & \text{with probability } p + (1-p)\, q_1 \\ F_2 & \text{with probability } (1-p)\, q_2 \\ F_j & \text{where } j \in N \backslash \{1,2\} \quad \text{otherwise} \end{cases},$$

and Type 2 (i.e., S observing $t = 2$) infers:

$$F = \begin{cases} F_1 & \text{with probability } (1-p)\, q_1 \\ F_2 & \text{with probability } p + (1-p)\, q_2 \\ F_j & \text{where } j \in N \backslash \{1,2\} \quad \text{otherwise} \end{cases}.$$

Hence:

$$\begin{aligned}
\overline{U^S}(a, s_1, b; 1) &= (p + (1-p)\, q_1)\, U^S(a, \theta', b) + (1-p)\, q_2 U^S\left(a, F_2^-(s_1), b\right) \\
&+ \sum_{\forall j \in N \backslash \{1,2\}} (1-p)\, q_j U^S\left(a, F_j^-(s_1), b\right), \\
\overline{U^S}(a, s_1, b; 2) &= (1-p)\, q_1 U^S(a, \theta', b) + (p + (1-p)\, q_2)\, U^S\left(a, F_2^-(s_1), b\right) \\
&+ \sum_{\forall j \in N \backslash \{1,2\}} (1-p)\, q_j U^S\left(a, F_j^-(s_1), b\right), \\
\overline{U^S}(a, s_2, b; 1) &= (p + (1-p)\, q_1)\, U^S\left(a, F_1^-(s_2), b\right) + (1-p)\, q_2 U^S(a, \theta', b) \\
&+ \sum_{\forall j \in N \backslash \{1,2\}} (1-p)\, q_j U^S\left(a, F_j^-(s_2), b\right), \\
\overline{U^S}(a, s_2, b; 2) &= (1-p)\, q_1 U^S\left(a, F_1^-(s_2), b\right) + (p + (1-p)\, q_2)\, U^S(a, \theta', b) \\
&+ \sum_{\forall j \in N \backslash \{1,2\}} (1-p)\, q_j U^S\left(a, F_j^-(s_2), b\right).
\end{aligned}$$

Then,

$$A^S(s_2, b; 1) < A^S(s_2, b; 2) < A^S(s_1, b; 1) < A^S(s_1, b; 2)$$

due to $U_{11}^S < 0$, $F_j^-(s_2) < F_j^-(s_1)$, $F_1^-(s_2) < \theta' < F_2^-(s_1)$ because:

$$\overline{U_1^S}(a, s_2, b; 1) > \overline{U^S}_1(a, s_2, b; 2) > \overline{U^S}_1(a, s_1, b; 1) > \overline{U^S}_1(a, s_1, b; 2) \text{ for any } a.$$

Also, for any $a > a'$

$$\begin{aligned} & \overline{U^S}(a, s_2, b; 1) - \overline{U^S}\left(a', s_2, b; 1\right) < \overline{U^S}(a, s_2, b; 2) - \overline{U^S}\left(a', s_2, b; 2\right) \\ < \ & \overline{U^S}(a, s_1, b; 1) - \overline{U^S}\left(a', s_1, b; 1\right) < \overline{U^S}(a, s_1, b; 2) - \overline{U^S}\left(a', s_1, b; 2\right) \end{aligned}$$

due to $U_{12}^S > 0$, $F_j^-(s_2) < F_j^-(s_1)$, $F_1^-(s_2) < \theta' < F_2^-(s_1)$.

Take an action a' induced by Type 1 given $s = s_1$ (and hence in its neighborhood). Then, a' should be induced by Type 2 given some $s' \in [s_2, s_1]$ (and hence in its neighborhood) as follows.

If $a' \in \left(A^S(s_2, b; 2), A^S(s_1, b; 2)\right)$, by continuity, there must be $s' \in (s_2, s_1)$ such that $A^S(s', b; 2) = a'$. The action a' should be induced by Type 2 given s in the neighborhood of s'.

Consider $a' \leq A^S(s_2, b; 2)$. If some action $\widetilde{a} > a'$ is inducible on the equilibrium path, due to the revealed preference:

$$0 \geq \overline{U^S}(\widetilde{a}, s_1, b; 1) - \overline{U^S}\left(a', s_1, b; 1\right),$$

and

$$\overline{U^S}(\widetilde{a}, s_1, b; 1) - \overline{U^S}\left(a', s_1, b; 1\right) > \overline{U^S}(\widetilde{a}, s_2, b; 2) - \overline{U^S}\left(a', s_2, b; 2\right).$$

In this case, because Type 1 given s_1 weakly prefers a' to $\widetilde{a}$, Type 2 given $s = s_2$ strictly prefers a' to $\widetilde{a}$. If a' is highest inducible action, Type 2 given $s = s_2$ strictly prefers a' to any action $a < a'$ because $\overline{U^S}(a, s_2, b; 2)$ is increasing with a for $a < a'$. The action a' should be induced by Type 2 given s in the neighborhood of s_2.

Consider $a' \geq A^S(s_1, b; 2)$. If some action $\widetilde{a} < a'$ is inducible on the equilibrium path, due to the revealed preference:

$$0 \leq \overline{U^S}\left(a', s_1, b; 2\right) - \overline{U^S}(\widetilde{a}, s_1, b; 2),$$

and:

$$\overline{U^S}\left(a', s_1, b; 1\right) - \overline{U^S}(\widetilde{a}, s_1, b; 1) < \overline{U^S}\left(a', s_1, b; 2\right) - \overline{U^S}(\widetilde{a}, s_1, b; 2).$$

In this case, because Type 1 given s_1 weakly prefers a' to $\widetilde{a}$, Type 2 given $s = s_1$ strictly prefers a' to $\widetilde{a}$. If a' is the lowest inducible action, Type 2 given $s = s_1$ strictly prefers a' to any action $a > a'$ because $\overline{U^S}(a, s_1, b; 2)$ is decreasing with a for $a > a'$. The action a' should be induced by Type 2 given s in the neighborhood of s_1.

Similarly, an action induced by Type 2 given $s = s_2$ is also induced by Type 1. (Take a' induced by Type 2 given $s = s_2$. If $a' \in \left(A^S(s_2, b; 1), A^S(s_1, b; 1)\right)$, a' should be induced by Type 1 given s in the neighborhood of s' where $A^S(s', b; 1) = a'$. If $a' \leq A^S(s_2, b; 1)$, a' should be induced by Type 1 given s in the neighborhood of s_2. If $a' \geq A^S(s_1, b; 1)$, a' should be induced by Type 1 given s in the neighborhood of s_1.)

The above proof applies to any types $i, j \in \overline{N}$. ■

The next claim suffices to show that *if* $p \in (0, 1)$, *all types belonging to* $\overline{N}$ *should commonly use at least one message for different extensions.*

Claim 3 *Fix* $p \in (0, 1)$. *When two types commonly use some message and at least one type use multiple messages, then, the two types use the common message for different extensions.*

Proof. If three actions $a < a' < a''$ are induced by Type 1, and a and a'' are induced by Type 2 as well, then, a' should be induced by Type 2.

It suffices to consider two cases: first, two actions are induced by Type 1 and Type 2; second, only one action is induced by both types.

In the fist case, let a and a' be induced by Type 1 and Type 2, where $a < a'$. Type 1 and Type 2 are indifferent between the two actions at $s = x$ and $s = y$, respectively. Let m be used to induce a. The underlying states given which each sends m includes the neighborhoods of $\theta = F_1^-(x)$ and $\theta = F_2^-(x)$ for Type 1 and the neighborhoods of $\theta = F_1^-(y)$ and $\theta = F_2^-(y)$ for Type 2. Thus, it suffices to show $x \neq y$. This inequality holds because S's indifference conditions require:

$$\begin{aligned} & \overline{U^S}(a, x, b; 1) = \overline{U^S}\left(a', x, b; 1\right) \\ \Leftrightarrow \quad & \frac{p + (1-p) q_1}{(1-p) q_2} = \frac{U^S\left(a', F_2^-(x), b\right) - U^S\left(a, F_2^-(x), b\right)}{U^S\left(a, F_1^-(x), b\right) - U^S\left(a', F_1^-(x), b\right)} \end{aligned}$$

and:

$$\begin{aligned} & \overline{U^S}(a, y, b; 2) = \overline{U^S}\left(a', y, b; 2\right) \\ \Leftrightarrow \quad & \frac{(1-p) q_1}{p + (1-p) q_2} = \frac{U^S\left(a', F_2^-(y), b\right) - U^S\left(a, F_2^-(y), b\right)}{U^S\left(a, F_1^-(y), b\right) - U^S\left(a', F_1^-(y), b\right)}, \end{aligned}$$

but $\frac{p+(1-p)q_1}{(1-p)q_2} \neq \frac{(1-p)q_1}{p+(1-p)q_2}$.

In the second case, suppose Type 1 induces a for $s \in (x, x')$ and a' for $s \in (x', x'')$, where $a < a'$; Type 2 induces a for $s \in (y, y')$ and a'' for $s \in (y', y'')$, where

$a < a'' \neq a'$. Single crossing properties imply $x' \neq y'$. Suppose not, $x' = y'$. Then, $\min\{a', a''\} > \max\{A^S(x', 1, b), A^S(x', 2, b)\}$. $\overline{U^S}(a, x', b; 1)$ and $\overline{U^S}(a', y, b; 2)$ both decreases with a over $a \geq \max\{A^S(x', 1, b), A^S(x', 2, b)\}$, and Type 1's choice implies that a' is highest inducible action over $a \geq \max\{A^S(x', 1, b), A^S(x', 2, b)\}$, i.e., $a' \geq a''$. Type 2's choice implies $a' \leq a''$. There is a contradiction. ■

If $p = 0$ or $p = 1$, $x = y$ should hold, i.e., all types belonging to $\overline{N}$ commonly use at least one message for the same extension.

A.2 Section 4 (the Uniform-Quadratic Case)

We present results to support arguments in Section 4, including Remark 1.

A.2.1 One-dimensional state uncertainty

We additionally defines the following notations for the subgame given Type t, where $t \in \{1, 2\}$, in a partition equilibrium with N_1-N_2 intervals.

Lemma 3 *Fix $b \in (0, \infty)$. Consider a partition equilibrium with N_1-N_2 intervals. In a subgame given Type t, where $t \in \{1, 2\}$:*

(1) Type t's partition satisfies $\Delta\theta_i = \Delta\theta_{i-1} + 4b$ for $i \in \{2, ..., N_t\}$.
(2) Type t sends $m = m_i$ given $\theta \in (\theta_{i-1}, \theta_i)$ so that $m_i \neq m_{i'}$ for any $i' \neq i$.
(3) R selects $a = a_i$ given $m = m_i$ such that $a_i = \frac{\theta_{i-1}+\theta_i}{2}$ for $i \in \{1, ..., N_t\}$.

Proof. For each subgame, the outcome is defined by S's indifference conditions (S's ICs) and R's best responses (R's BRs).

At any boundary point $\theta = \theta_i$ for $i \in \{1, ..., N_i - 1\}$, any type of S should be indifferent between two actions, $a = a_i$ and $a = a_{i+1}$:

$$\begin{aligned} &-(\theta_i + b - a_i)^2 = -(a_{i+1} - \theta_i - b)^2 \\ &\Leftrightarrow a_{i+1} + a_i = 2\theta_i + 2b \end{aligned} \qquad \text{(S's ICs)}$$

Given $m = m_i$ for $i \in \{1, ..., N_t\}$, R updates his belief on the state using Bayes' rule and selects the optimal action a=a_i:

$$a_i = E[\theta | \theta \in (\theta_{i-1}, \theta_i)] = \tfrac{\theta_{i-1}+\theta_i}{2}. \qquad \text{(R's BRs)}$$

Hence, regardless of S's type, S's ICs and R's BRs define the partition such that $\Delta\theta_i = \Delta\theta_{i-1} + 4b$ for $i \in \{2, ..., N_t\}$. ■

Lemma 4 *Fix $b \in (0, \infty)$. Consider a subgame given Type t, where $t \in \{1, 2\}$. Then, there is a positive integer $N_t^{CS}(b)$ such that there is an equilibrium where type t partitions the state space into N intervals for $N \in \{1, 2, ..., N_t^{CS}(b)\}$. Further, $N_t^{CS}(b)$ decreases with b.*

Proof. The previous lemma implies $\Delta\theta_i = \theta_1 + 4(i-1)b$ for $i \in \{1, ..., N\}$ for any type. Hence, the existence of an equilibrium with N intervals requires $\theta_1 > 0$ and:

$$\theta_N = \sum_{i=1}^{N} \Delta\theta_i = \quad N\theta_1 + 2N(N-1)b = i.$$

For any $b > 0$, $N_t^{CS}(b)$ is well defined such that $N_1^{CS}(b) = n$ *for* $b \in [b_{n+1}, b_n)$ *for* $n \geq 1$ *and* $N_2^{CS}(b) = N$ *for* $b \in [Lb_{N+1}, Lb_N)$ *for* $N \geq 1$. ■

A.2.2 Two-dimensional state uncertainty

Lemma 5 *Fix $p \in (0, 1)$ and $b \in (0, \infty)$. Any PBE is a partition equilibrium. Further, in a partition equilibrium with N_1-N_2 intervals:*

(1) There is a unique N_1 for N_2, and $N_1 \leq N_2$.
(2) $G_{b,p}(N_1) = \frac{\Delta x_{N_1}}{\Delta x_{N_1} + \Delta y_{N_1}} \left(\frac{(L(1+p)+1-p)(y_{N_1-1}+y_{N_1})}{4} - \frac{(1+p+L(1-p))(x_{N_1-1}+x_{N_1})}{4} \right) \geq 0.$
(3) Type 1's partition $x(N_1)$ satisfies:

$$\begin{aligned} \Delta x_i &= \Delta x_{i-1} + \tfrac{8b}{1+p+L(1-p)} \quad \textit{for } i \in \{2, ..., N_1 - 1\}. \\ \Delta x_i &\leq \Delta x_{i-1} + \tfrac{8b}{1+p+L(1-p)} \quad \textit{for } i = N_1. \end{aligned}$$

(4) Type 2's partition $y(N_2)$ satisfies:

$$\Delta y_i = \begin{cases} \Delta y_{i-1} + \frac{8b}{L(1+p)+1-p} & \textit{for } i \in \{2, ..., N_2\} \setminus \{N_1, N_1+1\}, \\ \Delta y_{i-1} + \frac{8b}{L(1+p)+1-p} + \frac{4}{L(1+p)+1-p} G_{b,p}(N_1) & \textit{for } i \in \{N_1, N_1+1\}. \end{cases}$$

(5) Relationship between partitions of both types is:

$$x_i = \begin{cases} \frac{L(1+p)+1-p}{1+p+L(1-p)} y_i & \textit{for } i \in \{1, ..., N_1 - 1\}, \\ 1 \left(\leq \frac{L(1+p)+1-p}{1+p+L(1-p)} y_{N_1} \right) & \textit{for } i = N_1. \end{cases}$$

(6) Both types send $m = m_i$ given $s \in (x_{i-1}, x_i)$ (if this interval exists) and $s \in (y_{i-1}, y_i)$, respectively, for $i \in \{1, ..., N_2\}$, where $m_i \neq m_j$ for any $j \neq i$.

(7) R selects $a = a_i$ given $m = m_i$ such that:

$$a_i = \begin{cases} \frac{(L(1+p)+1-p)(y_{i-1}+y_i)}{4} & \text{for } i \in \{1,...,N_2\} \setminus \{N_1\}, \\ \frac{(L(1+p)+1-p)(y_{i-1}+y_i)}{4} - G_{b,p}(N_1) & \text{for } i = N_1, \end{cases}$$

where $a_1, ..., a_{N_1}$ is also described as follows:

$$a_i = \begin{cases} \frac{(1+p+L(1-p))(x_{i-1}+x_i)}{4} & \text{for } i \in \{1,...,N_2\} \setminus \{N_1\}, \\ \frac{(1+p+L(1-p))(x_{i-1}+x_i)}{4} + \frac{\Delta y_{N_1}}{\Delta x_{N_1}} G_{b,p}(N_2) & \text{for } i = N_1. \end{cases}$$

Proof. First, we show claims, "on the equilibrium path, every action induced by Type 1 should be also induced by Type 2" and "on the equilibrium path, every action induced by Type 2 given $s \in \left[0, \frac{1+p+L(1-p)}{L(1+p)+1-p}\right]$ should be also induced by Type 1." We use the observation that given some signal s, Type 1's ideal action is $a = \frac{1+p+L(1-p)}{2}s + b$ while Type 2' ideal action given s is $a = \frac{L(1+p)+1-p}{2}s + b$.

To define a partition, we consider indifference conditions of each type of S (S's ICs) and R's best responses (R's BRs).

Suppose a_{i+1} and a_i are induced in the i-th and $(i+1)$-th intervals, respectively, by Type 1. At the boundary point $s = x_i$, Type 1's ICs require:

$$a_{i+1} + a_i = (1 + p + L(1-p)) x_i + 2b. \qquad \text{(S's ICs (t=1))}$$

$1 + p + L(1-p)$Similarly, suppose a_{i+1} and a_i are induced in the i-th and $(i+1)$-th intervals, respectively, by Type 2. At the boundary point $s = y_i$, Type 2's ICs require:

$$a_{i+1} + a_i = (L(1+p) + 1 - p) y_i + 2b. \qquad \text{(S's ICs (t=2))}$$

Thus, if a_i and a_{i+1} are induced by both types of S, then: $x_i = \frac{L(1+p)+1-p}{1+p+L(1-p)} y_i$. Type 1's ideal action given signal $s = x_i$ is equivalent to Type 2's ideal action given signal $s = y_i$ for $i \in \{1, 2, ..., N_1\text{-}1\}$.

R's BRs require that given $m = m_i$ for any $i \in \{1, 2, ..., N_1\}$ (i.e., both types use m_i):

$$a_i = \frac{\Pr[m=m_i|t=1]E[\theta|m_i,t=1] + \Pr[m=m_i|t=2]E[\theta|m_i,t=2]}{\Pr[m=m_i|t=1] + \Pr[m=m_i|t=2]} \qquad \text{(R's BRs)}$$

given $m = m_i$ for $i \in \{N_1+1, ..., N_2\}$ (i.e., only Type 2 use m_i) is

$$a_i = \quad E\left[\theta | m_{N_1}, t=2\right], \qquad \text{(R's BRs)}$$

where $E\left[\theta | m_i, t=1\right] = \frac{(1+p+L(1-p))(x_{i-1}+x_i)}{4}$ and $E\left[\theta | m_{N_1}, t=2\right] = \frac{(L(1+p)+1-p)(y_{i-1}+y_i)}{4}$. S's ICs imply $E\left[\theta | m_{N_1}, t=1\right] = E\left[\theta | m_{N_1}, t=2\right]$ for $i \in \{1, ..., N_1-1\}$ while $E\left[\theta | m_{N_1}, t=1\right] \leq E\left[\theta | m_{N_1}, t=2\right]$. Thus, a_i is well defined as claimed. ■

Lemma 6 *Fix $p \in (0,1)$ and $b \in (0,\infty)$. Consider m_t in a partition equilibrium with N_1-N_2 intervals, where $t \in \{1,2,...,N_1-1\}$. Information provided by Type 2 second order stochastically dominates (SOSD) information provided by Type 1.*

Proof. Type 1 uses m_i given $s \in (x_{i-1}, x_i)$ while Type 2 uses m_i given signal $s \in (y_{i-1}, y_i)$. The two conditional distributions have the same mean

$$\frac{(L(1+p)+1-p)(y_{i-1}+y_i)}{4} = \frac{(1+p+L(1-p))(x_{i-1}+x_i)}{4},$$

but different supports; and the former has larger variances than the latter as follows.

Type	Support	Variances
t=1	$(x_{i-1}, x_i) \cup (Lx_{i-1}, Lx_i)$	$\frac{\left(1+p+L^2(1-p)\right)\left(x_{i-1}^2+x_{i-1}x_i+x_i^2\right)}{6} - \frac{(1+p+L(1-p))(x_{i-1}+x_i)^2}{4^2}$
t=2	$(y_{i-1}, y_i) \cup (Ly_{i-1}, Ly_i)$	$\frac{\left(L^2(1+p)+1-p\right)\left(y_{i-1}^2+y_{i-1}y_i+y_i^2\right)}{6} - \frac{(L(1+p)+1-p)^2(y_{i-1}+y_i)^2}{4^2}$

■

A.3 Proofs for Section 5.1 (Discrete Model)

A.3.1 Proof for Remark 2

We find sufficient conditions which satisfy the next three claims.

Claim 1. Under two-dimensional uncertainty, there exists an equilibrium in which every type uses different messages for different signals, as shown in Figure 3.

Claim 2. Under two-dimensional uncertainty, there is no equilibrium in which different types use different messages for $s = 1$. (Language is vague.)

Claim 3. Under one-dimensional uncertainty, there only exists a babbling equilibrium given type 1 while there is a separation equilibrium given type 2, as shown in Figure 4.

Claim 1 holds if $b < \frac{1+L}{4}$ as follows. Under two-dimensional uncertainty, in the supposed equilibrium, each type induces $a = 0$ given $s = 0$ and $a = \frac{1+L}{2}$ given $s = 1$. Thus, this equilibrium exists if both types prefer $a = 0$ to $a = 1$ given $s = 0$. We obtain the condition.

Claim 2 holds if $b > \frac{p(L-1)}{2}$ as follows. Under two-dimensional uncertainty, if different types use different messages for $s = 1$, type 1 induces $a = 0$ given $s = 0$ and $a = \frac{1+p+(1-p)L}{2}$ given $s = 1$ while type 2 induces $a = 0$ given $s = 0$ and $a = \frac{(1+p)L+1-p}{2}$ given $s = 1$. However, if $b > \frac{p(L-1)}{2}$, type 1 does not most prefer $a = \frac{1+p+(1-p)L}{2}$ given $s = 1$ among all inducible actions.

Claim 3 holds if $b \in \left(\frac{1}{2}, \frac{L}{2}\right]$ as follows. Under one-dimensional uncertainty, each type of S knows the state (because $s = \theta$) and induces $a = \theta$ by sending $m = s$ in a separating equilibrium if it exists. We can show that a separating equilibrium exists for a type if this type wants to induce low action $a = 0$ for $\theta = 0$. The condition is $b \leq \frac{1}{2}$ for type 1 and $b \leq \frac{L}{2}$ for type 2.

Therefore, if $1 < L < 2$, then, $b \in \left(\frac{1}{2}, \frac{1+L}{4}\right)$ suffices for the three claims.

Then, fix $b \in \left(\frac{1}{2}, \frac{1+L}{4}\right)$. We look at the most informative equilibrium. Under two-dimensional state uncertainty, both types send $m = Null$ given $s = 0$ and $m = Large$ given $s = 1$, as shown in Figure 3. R's ex-ante expected payoff is:

$$\underbrace{\frac{1}{2}\left(-\frac{(1-L)^2}{8}\right)}_{F=F_1} + \underbrace{\frac{1}{2}\left(-\frac{(1-L)^2}{8}\right)}_{F=F_2} = -\frac{(1-L)^2}{8},$$

where:

$$\begin{cases} \frac{1}{2}\underbrace{\left(-(0-0)^2\right)}_{\theta=0} + \frac{1}{2}\underbrace{\left(-\left(1-\frac{1+L}{2}\right)^2\right)}_{\theta=1} = -\frac{(1-L)^2}{8} & \text{if } F = F_1, \\ \frac{1}{2}\underbrace{\left(-(0-0)^2\right)}_{\theta=0} + \frac{1}{2}\underbrace{\left(-\left(L-\frac{1+L}{2}\right)^2\right)}_{\theta=L} = -\frac{(1-L)^2}{8} & \text{if } F = F_2. \end{cases}$$

Under one-dimensional uncertainty, we consider a babbling equilibrium given type 1 and a separation equilibrium given type 2, as shown in Figure 4. R's ex-ante expected payoff is:

$$\underbrace{\tfrac{1}{2}\left(-\frac{1}{4}\right)}_{F=F_1}+\underbrace{\tfrac{1}{2}(0)}_{F=F_2}=-\tfrac{1}{8},$$

where:

$$\begin{cases}\tfrac{1}{2}\underbrace{\left(-\left(0-\frac{1}{2}\right)^2\right)}_{\theta=0}+\tfrac{1}{2}\underbrace{\left(-\left(1-\frac{1}{2}\right)^2\right)}_{\theta=1}=-\tfrac{1}{4} & \text{if } F=F_1,\\ \tfrac{1}{2}\underbrace{\left(-(0-0)^2\right)}_{\theta=0}+\tfrac{1}{2}\underbrace{\left(-(L-L)^2\right)}_{\theta=L}=0 & \text{if } F=F_2.\end{cases}$$

Hence, R is better-off under two-dimensional state uncertainty.

Fix $b<\frac{1}{2}$. Under one-dimensional state uncertainty, there is a separation equilibrium given each type. R's ex-ante expected payoff is:

$$\underbrace{\tfrac{1}{2}(0)}_{F=F_1}+\underbrace{\tfrac{1}{2}(0)}_{F=F_2}=0.$$

Hence, R is worse-off under two-dimensional state uncertainty.

Fix $b>\frac{1+L}{4}$. Under two-dimensional state uncertainty, there only exists a babbling equilibrium in which only $a=\frac{1+L}{4}$ is induced by both types. R's ex-ante expected payoff is:

$$\underbrace{\tfrac{1}{2}\left(-\frac{L^2-2L+5}{16}\right)}_{F=F_1}+\underbrace{\tfrac{1}{2}\left(-\frac{5L^2-2L+1}{16}\right)}_{F=F_2}=-\tfrac{3L^2-2L+3}{8}\left(<-\tfrac{L^2-2L+3}{16}\right).$$

Under one-dimensional state uncertainty, there only exists a babbling equilibrium in which only $a=\frac{1}{2}$ ($a=\frac{L}{2}$) is induced given type 1 (type 2). R's ex-ante expected payoff is:

$$\underbrace{\tfrac{1}{2}\left(-\frac{1}{4}\right)}_{F=F_1}+\underbrace{\tfrac{1}{2}\left(-\frac{(1-L)^2}{8}\right)}_{F=F_2}=-\tfrac{L^2-2L+3}{16}.$$

Hence, R is worse-off under two-dimensional state uncertainty.

A.3.1 Proof for Remark 3

Let $a_0 = 0$, $a_1 = \frac{1+p+(1-p)L}{2}$ and $a_2 = \frac{(1+p)L+1-p}{2}$ (i.e., $a_0 < a_1 < a_2$). If the type is revealed in the first stage, type 1 can induce a_0 and a_1 while type 2 can induce a_0 and a_2, respectively, in the second stage. But type 1 knows he will prefer a_2 to a_1 given $s = 1$. Hence, type 1 pretends to be type 2.

A.4 Proofs for Section 5.2 (Different Prior Beliefs)

A.4.1 Proof for Remark 4

The proof and the outcome is similar to those for a model in section 4. We only show differences in equilibrium outcome from the basic model.

Type 2's partition $y\left(N_2\right)$ satisfies:

$$\Delta y_i = \begin{cases} \Delta y_{i-1} + 4\frac{(q_R-q_S)(1-p)}{2-q_R(1-p)} y_i & \textit{for } i \in \{2,...,N_2\} \setminus \{N_1, N_1+1\}, \\ \Delta y_{i-1} + 4\frac{(q_R-q_S)(1-p)}{2-q_R(1-p)} y_i + \frac{2G_{b,p}(N_1)}{(1+(1-p)(1-q_R)+p)} & \textit{for } i \in \{N_1, N_1+1\}. \end{cases}$$

Relationship between partitions of both types is:

$$x_i = \begin{cases} \frac{1+(1-p)(1-q_S)+p}{1+(1-p)(1-q_S)} y_i & \textit{for } i \in \{N_1, N_1+1\}, \\ 1 \left(\leq \frac{1+(1-p)(1-q_S)+p}{1+(1-p)(1-q_S)} y_{N_1} \right) & \textit{for } i = N_1. \end{cases}$$

R selects $a = a_i$ given $m = m_i$ such that:

$$a_i = \begin{cases} \frac{(1+(1-p)(1-q_R)+p)(y_{i-1}+y_i)}{2} & \textit{for } i \in \{1,...,N_2\} \setminus \{N_1\}, \\ \frac{(1+(1-p)(1-q_R)+p)(y_{i-1}+y_i)}{2} - G_{b,p}\left(N_1\right) & \textit{for } i = N_1. \end{cases}$$

Due to the difference in prior beliefs $|q_R - q_S|$, any type of S cannot credibly separate messages for signals. Then, as in the basic model, the communication should have a partitional form. The players use a smaller number of messages as $|q_R - q_S|$ is larger. Moreover, in a partition equilibrium N_1-N_2 intervals, both types of S commonly use messages $m_1, m_2, ..., m_{N_1}$. Type 1's ideal action given signal s=x_i is equivalent to Type 2's ideal action given $s = y_i$, which is $a = \left(1 + \left(1-p\right)\left(1-q_S\right) + p\right) y_i + b$, for $i \in \{1,...,N_1 - 1\}$.

A.4.2 Proof for Remark 5

In *a partition equilibrium with N intervals, the partition satisfies:*

$$\Delta x_i = \Delta x_{i-1} + \frac{4}{2-q_R} b + \frac{4(q_R - q_S)}{2-q_R} x_i \quad \textit{for } i \in \{2, ..., N\}.$$

S send $m = m_i$ *given* $s \in (x_{i-1}, x_i)$ *for* $i \in \{1, ..., N\}$ *so that* $m_i \neq m_{i'}$ *for any* $i' \neq i$. *Then, R selects* $a = a_i$ given $m = m_i$ *such that* $a_i = \frac{(2-q_R)(x_{i-1}+x_i)}{4}$ *for* $i \in \{1, ..., N\}$.

Therefore, there observed non-monotonic relationships unless $q_R = q_S$ (common prior belief). For example, fixing $q_R = \frac{1}{4}$ and $q_S = \frac{3}{4}$, then, larger $|b|$ can result in smaller or larger $N^{DP}\left(b, q^R, q^S\right)$:

$$N^{DP}\left(b, q^R, q^S\right) = \begin{cases} 4 & \text{for } b \in \left(\frac{3375}{8416}, \frac{223}{544}\right), \\ 3 & \text{for } b \in \left(\frac{17}{48}, \frac{225}{464}\right) \setminus \left(\frac{3375}{8416}, \frac{223}{544}\right), \\ 2 & \text{for } b \in \left(\frac{11}{16}, \frac{15}{16}\right) \setminus \left(\frac{17}{48}, \frac{225}{464}\right), \\ 1 & \text{otherwise.} \end{cases}$$

For another example, fixing $b = \frac{3}{8}$ and $q_R = \frac{1}{2}$, then, larger $|q_R - q_S|$ can result in smaller or larger $N^{DP}\left(b, q^R, q^S\right)$:

$$N^{DP}\left(b, q^R, q^S\right) = \begin{cases} 4 & \text{for } q^S \in \left(\frac{29-3\sqrt{21}}{16}, \frac{4+(476-12\sqrt{1545}])^{1/3}+(476+12\sqrt{1545}])^{1/3}}{16}\right), \\ 3 & \text{for } q^S \in \left(\frac{7}{8}, 1\right) \setminus \left(\frac{29-3\sqrt{21}}{16}, \frac{4+(476-12\sqrt{1545}])^{1/3}+(476+12\sqrt{1545}])^{1/3}}{16}\right), \\ 2 & \text{for } b \in \left(\frac{1}{2}, \frac{7}{8}\right), \\ 1 & \text{for } b \in \left(0, \frac{1}{2}\right). \end{cases}$$