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Abstract

Based on the smooth model of decision making under ambiguity as proposed by Klibanoff, Marinacci, and Mukerji (2005, 2009), we derive a method that selects assets whose regression constant from the factor regression captures ambiguity premium.

Keywords: Ambiguity aversion, asset pricing, factor pricing

JEL Classification Numbers: D81, G11, G12

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1 Introduction

Recently, based on the smooth model of decision making under ambiguity suggested by the Klibanoff, Marinacci, and Mukerji (2005), Wakai (2018) extended a factor pricing model suggested by Fama and French (1996) to the case where investors face ambiguity. However, to apply his model, econometricians need to estimate variables that are based on the agent's subjective belief.

To overcome this difficulty, in this note, based on a dynamic version of the Klibanoff, Marinacci, and Mukerji (2009), we propose the method in which we can identify the asset whose regression constant from the factor regression captures a premium associated with ambiguity aversion. This method does not require the estimation of variables that are based on the agent's subjective belief, and it can be used to construct a portfolio that captures ambiguity premium.

2 Setting

We consider a two-period portfolio choice problem with time t varying over $\{0, 1\}$. Let $(\Omega, \mathcal{F}, P)$ be a probability space. The filtration $\{\mathcal{F}_t\}$ is given and represents the information structure, where $\mathcal{F}_0$ is trivial and $\mathcal{F}_1$ is $\mathcal{F}$. At each time and state, a single perishable consumption good defined on $\mathbb{R}_+$ is available. There is the single representative agent in this economy, who is endowed with the positive and bounded consumption process $\{e_t\}$ adapted to the filtration $\{\mathcal{F}_t\}$. There are a finite number $K + 1$ of assets that pay a nonnegative amount of time-1 consumption good as a dividend. The payoffs of the first K assets are not deterministic, while the $(K + 1)$th asset is the risk-free asset that pays one unit of time-1 consumption good. All of assets have net zero supply.

We model ambiguity as follows: The representative agent believes that there are a finite number L of possible regimes in this economy and that he is unsure which regime he faces. Each regime l specifies the probability of state realization, denoted by an absolutely continuous Q_l with respect to P, and investor's belief of possible regimes is expressed by his subjective prior μ defined over L regimes. Let $(L, \mathcal{P}(L), \mu)$ be a probability space that describes the agent belief, where $\mathcal{P}(L)$ is the power set defined on L.

For a random variable x measurable with respect to $\mathcal{F}_1$, we denote by $E_Q[x]$ a random variable measurable with respect to $\mathcal{P}(L)$, where for each l, $E_Q[x\,|l]$ is the expectation of x under the probability measure Q_l. We also denote by $E_\mu[a]$ the expectation of a random variable a measurable with respect to $\mathcal{P}(L)$ under the probability measure μ. Furthermore, following Maccheroni et. all. (2013), we define the probability measure $\widehat{P}$ on $\mathcal{F}_1$, called the *reduction* of μ on Ω, by

$$\widehat{P}(A) = \mu(1)Q_1(A) + ... + \mu(L)Q_L(A) \text{ for all } A \in \mathcal{F}_1.$$

Let $E_{\widehat{P}}[x]$ be the expectation of a random variable x measurable with respect to $\mathcal{F}_1$ under the reduction $\widehat{P}$.

We assume that at time 0, the representative agent can trade assets without transaction cost and can short and borrow without restrictions. Let $c = \{c_t\}$ be a feasible consumption process, which satisfies the following budget constraints: At time 0,

$$c_0 + \theta \cdot q_0 = e_0, \tag{1}$$

where $\theta = (\theta^1, ..., \theta^{K+1})$ is the vector of the asset holdings and $q_0 = (q_0^1, ..., q_0^{K+1})$ is the vector of assets' prices, each element of which is a random variable measurable with respect to $\mathcal{F}_0$. At time 1,

$$c_1 = e_1 + \theta \cdot d_1, \tag{2}$$

where $d_1 = (d_1^1, ..., d_1^{K+1})$ is the vector of assets' dividends, each element of which is a bounded random variable measurable with respect to $\mathcal{F}_1$.

The representative agent's preferences follow the *smooth model of decision making under* *ambiguity* as introduced by Klibanoff et al. (2005, 2009)

$$V_0(c) = u(c_0) + \beta v^{-1}\left(E_\mu\left[v\left(E_Q\left[u\left(c_1\right)\right]\right)\right]\right), \tag{3}$$

where both v and u are strictly increasing and strictly concave on the respective domain. The representative agent decides his asset holdings θ so as to maximize the representation (3). Appendix A shows that the equilibrium price q_k satisfies

$$q_k = E_\mu\left[\left(v^{-1}\right)' \times v' \times E_Q\left[m \times d_k\right]\right], \tag{4}$$

where $\left(v^{-1}\right)'$ is constant, v' is $\mathcal{P}(L)$-measurable, and m is $\mathcal{F}_1$-measurable random variables that stand for

$$\left(v^{-1}\right)' \equiv \left(v^{-1}\left(E_\mu[v\left(E_Q\left[u\left(e_1\right)\right]\right)]\right)\right)', \; v' \equiv v'\left(E_Q\left[u\left(e_1\right)\right]\right), \text{ and } m \equiv \frac{\beta u'\left(e_1\right)}{u'(e_0)}. \tag{5}$$

We also introduce a few more notations. Let R_k be the gross return of kth risky asset, and let R_f be the gross return of the risk-free asset. We denote by $Cov_{\widehat{P}}\left[x, y\right]$ the covariance between $\mathcal{F}_1$-measurable x and y under $\widehat{P}$. A variance of $\mathcal{F}_1$-measurable x under $\widehat{P}$, $Var_{\widehat{P}}\left[x\right]$, is similarly defined.

3 Identification of Ambiguity Premium

3.1 Risk Premium

In empirical studies of asset returns, we often assume a factor pricing formula

$$E_P[R_k - R_f] = \sum_{i=1}^{I} \beta_{k,i} E_P[R_{RF_i} - R_f], \tag{6}$$

where a set of gross portfolio returns and the risk-free return, $\{R_{RF_1}, R_{RF_2}, ..., R_{RF_I}, R_f\}$, is assumed to be linearly independent. The model is based on the expected utility by imposing a particular assumption on m, where the factor return R_{RF_i} captures the variation of m that is relevant for asset pricing. Because risk aversion determines the variation of m, the factor risk premium $E_P[R_{RF_i} - R_f]$ represents a risk premium associated with factor returns.

To identify the effect associated with ambiguity aversion, we first want to identify the part of returns associated with the variation of m. We follow the approach used to derive (6) by imposing a spanning condition on m. For this purpose, let $\widehat{q}_k$ be the price of asset k, where the representative agent is assumed to be ambiguity neutral. This implies that

$$\widehat{q}_k = E_{\widehat{P}}\left[m \times d_k\right]. \tag{7}$$

Let $\widehat{m}$ be the projection of m onto the asset span $\left\{\widehat{R}_1, \widehat{R}_2, ..., \widehat{R}_K, \widehat{R}_f\right\}$ under the probability measure $\widehat{P}$, where $\widehat{R}_k$ is the vector of asset k's gross return based on $\widehat{q}_k$. By definition,

$$\widehat{q}_k = E_{\widehat{P}}\left[\widehat{m} \times d_k\right]. \tag{8}$$

Now, for a finite $I < K$, consider a set of gross portfolio returns and the risk-free return $\left\{\widehat{R}_{RF_1}, \widehat{R}_{RF_2}, ..., \widehat{R}_{RF_I}, \widehat{R}_f\right\}$, which is assumed to be linearly independent. We then impose the following.

Assumption 1: (Spanning Condition on m)

(i) $\widehat{m} = a_0 \widehat{R}_f + \sum_{i=1}^{I} a_i \widehat{R}_{RF_i}$

As shown in Appendix B, Assumption 1 is a well-known condition that leads to the following form of a factor pricing model.

$$E_{\widehat{P}}[\widehat{R}_k - \widehat{R}_f] = \sum_{i=1}^{I} \widehat{\beta}_{k,i} E_{\widehat{P}}[\widehat{R}_{RF_i} - \widehat{R}_f], \tag{9}$$

where $\widehat{\beta}_{k,i}$ is obtained as a coefficient of $\left(\widehat{R}_{RF_i} - \widehat{R}_f\right)$ by the regression of the excess return $\left(\widehat{R}_k - \widehat{R}_f\right)$ on excess factor returns $\left(\widehat{R}_{RF_1} - \widehat{R}_f, ..., \widehat{R}_{RF_I} - \widehat{R}_f\right)$ under $\widehat{P}$.[1]

[1] Theoretically, these coefficients must be identical to those obtained from the regression that includes the constant term, where the coefficient of the constant term turns out to be zero.

3.2 Derivation of Regression Alpha

In this note, we want to identify the effect of ambiguity based on the factor regression. For this purpose, we introduce an assumption that links the agent's belief and the objective probability.

Assumption 2: (Rational Belief)
(i) $\widehat{P} = P$.

This assumption is a version of rational expectation hypothesis adopted to the smooth model of decision making under ambiguity, and it corresponds to a similar assumption adopted to the subjective expected utility model. Given Assumption 2, the agent has the reduction, that is the belief on state realization, which is consistent with the frequency of data. The agent simply does not know how this frequency is generated so that he assumes regimes that seemingly consistent with data. This assumption also contributes to the separation of ambiguity aversion from risk aversion because if the agent is ambiguity neutral, (4) reduces to (7), which is the pricing under the subjective expected utility model with rational expectations.

To identify the effect of ambiguity aversion, we want to introduce a measure γ that relates the price under the ambiguity neutral representative agent and the price under the ambiguity averse representative agent. This measure is defined asset by asset as follows: for each k,

$$\gamma_k \equiv \frac{q_k}{\widehat{q}_k}. \tag{10}$$

For simplicity, we use γ_f instead of γ_{K+1}. If the ambiguity aversion decreases the price of asset, the γ_k is less than one, and vice versa. Thus, in this note, γ_k summarizes the effect of ambiguity aversion on asset k's price, which is used to investigate a premium associated with ambiguity aversion. The relation (10) also implies the following relation

$$R_k = \frac{1}{\gamma_k}\widehat{R}_k. \tag{11}$$

Given (9), the following proposition shows that the regression based on excess factor returns $(R_{RF_1} - R_f, ..., R_{RF_I} - R_f)$ generates a constant term that may capture a premium associated to ambiguity aversion (see Appendix C).

Proposition 1:

Suppose that Assumptions 1 and 2 hold. For each k, the gross return of asset k satisfies the factor pricing formula

$$E_P[R_k - R_f] = \alpha_k + \sum_{i=1}^{I} \beta_{k,i} E_P[R_{RF_i} - R_f], \tag{12}$$

where for each i, $\beta_{k,i}$ is a regression coefficient for $R_{RF_i} - R_f$. Furthermore, α_k satisfies

$$\alpha_k = R_f \left(\sum_{i=1}^{I} \beta_{k,i} - 1 \right) - \frac{\gamma_f}{\gamma_k} R_f \left(\sum_{i=1}^{I} \widehat{\beta}_{k,i} - 1 \right). \tag{13}$$

Moreover, if the representative agent is ambiguity neutral, α_k is zero.

For each i, we call $E_P[R_{RF_i} - R_f]$ the *factor risk premium.* Appendix C also shows that $\widehat{\beta}_{k,i}$ in (9) and $\beta_{k,i}$ in (12) are related by

$$\beta_{k,i} = \frac{\gamma_{RF_i}}{\gamma_k} \widehat{\beta}_{k,i}. \tag{14}$$

3.3 Interpretation of Regression Alpha

Proposition 1 shows that once a factor regression that captures risk premia is correctly specified, the regression constant α_k may captures a component of returns associated with ambiguity aversion. However, a sign of α_k does not necessarily correspond to a sign of the premium associated with ambiguity aversion. For example, assume that γ_f is more than one. Suppose that all of $\gamma_{_{RF_i}}$ and γ_k are similar and they are significantly less than one. Then α_k is negative, even for the case where γ_k is less than all of $\gamma_{_{RF_i}}$. The last assumption implies that asset k is disliked more than any risk factors but such a discount in its price is not captured by α_k.

This non-monotonic relation between α_k and ambiguity premium is caused by the fact that factor returns and associated coefficients are also affected by ambiguity aversion. Thus, we want to identify the case where a positive α_k implies a positive premium associated with ambiguity aversion and a negative α_k implies that a negative premium associated with ambiguity aversion. The following summarizes such cases.

Proposition 2:

Suppose that Assumptions 1 and 2 hold. For each k, consider (12). Suppose that γ_f is more than γ_k. Then,

1. *If $\sum_{i=1}^{I} \beta_{k,i} > 1$, a positive α_k captures a positive premium associated with ambiguity aversion.*
2. *If $\sum_{i=1}^{I} \beta_{k,i} < 1$, a negative α_k captures a negative premium associated with ambiguity aversion*

The key feature of Proposition 2 is that we can identify a premium associated with ambiguity aversion without directly estimating γ_k. For the first case, (13) implies that α_k is positive if and only if

$$\left(\sum_{i=1}^{I} \beta_{k,i} - 1 \right) > \frac{\gamma_f}{\gamma_k} \left(\sum_{i=1}^{I} \widehat{\beta}_{k,i} - 1 \right). \tag{15}$$

Because the assumptions in Proposition 2 leads to $\frac{\gamma_f}{\gamma_k} > 1$, $\sum_{i=1}^{I} \beta_{k,i} > 1$ and (15) imply

$$\sum_{i=1}^{I} \beta_{k,i} > \sum_{i=1}^{I} \widehat{\beta}_{k,i}. \tag{16}$$

Given (14), (16) shows that on a risk-adjusted average, asset k's price decreases more than those of risk factors. Thus, α_k captures the ambiguity effect net of those embedded in factor returns, which is the left-over premium caused by ambiguity aversion.

Similarly, for the second case, (13) implies that α_k is negative if and only if

$$\left(\sum_{i=1}^{I}\beta_{k,i}-1\right)<\frac{\gamma_f}{\gamma_k}\left(\sum_{i=1}^{I}\widehat{\beta}_{k,i}-1\right). \tag{17}$$

Because the assumptions in Proposition 2 leads to $\frac{\gamma_f}{\gamma_k}>1$, $\sum_{i=1}^{I}\beta_{k,i}<1$ and (17) imply

$$\sum_{i=1}^{I}\beta_{k,i}<\sum_{i=1}^{I}\widehat{\beta}_{k,i}. \tag{18}$$

Given (14), (18) shows that on a risk-adjusted average, asset k's price decreases less than those of risk factors. Thus, α_k captures the ambiguity effect net of those embedded in factor returns, which is the reduction in premium in returns due to net hedging demand by the ambiguity averse agent.

The validity of assumptions is a key for Proposition 2. For the risk-free asset, most of simulation studies confirms that $\gamma_f>1$ (For example, see Ju and Miao (2012)). Also, a risky asset whose dividends are positively correlated with an aggregate endowment should subject to ambiguity. Thus, $\gamma_k<1$ and $\gamma_{RF_i}<1$ should hold in general.

Appendix A: The Derivation of Equation (4)

The investor maximizes (3) with the constraints (1) and (2), where the constraint (2) is automatically satisfied. The first order condition with respect to c_0 leads to

$$u'(c_0)=\lambda, \tag{19}$$

where λ is the Lagrange multiplier corresponding to the constraint (1). Similarly, the first order condition with respect to θ_k leads to

$$\beta\left(v^{-1}\right)'\times E_\mu\left[v'\times E_Q\left[u'\left(c_1\right)d_k\right]\right]=\lambda q_k. \tag{20}$$

Thus, (19) and (20) imply that at the equilibrium,

$$q_k = E_\mu \left[\left(v^{-1}\right)' \times v' \times E_Q \left[m \times d_k \right] \right], \tag{21}$$

which is (4), where

$$\left(v^{-1}\right)' \equiv \left(v^{-1}\left(E_\mu[v\left(E_Q\left[u\left(e_1\right)\right]\right)]\right)\right)', \; v' \equiv v'\left(E_Q\left[u\left(e_1\right)\right]\right), \text{ and } m \equiv \frac{\beta u'\left(e_1\right)}{u'(e_0)}.$$

■

Appendix B: The Derivation of Equation (9)

Equation (8) implies that for the risk-free asset,

$$\widehat{q}_f = \frac{1}{\widehat{R}_f} = E_{\widehat{P}}\left[\widehat{m}\right]. \tag{22}$$

Furthermore, (8) is rewritten as

$$1 = E_{\widehat{P}}\left[m \times \widehat{R}_k \right]. \tag{23}$$

By applying the standard statistical relation, (23) leads to

$$1 = Cov_{\widehat{P}}\left[m, \widehat{R}_k \right] + E_{\widehat{P}}\left[\widehat{m}\right] E_{\widehat{P}}\left[\widehat{R}_k \right]. \tag{24}$$

Thus, (22) and (24) imply

$$E_{\widehat{P}}[\widehat{R}_k - \widehat{R}_f] = -\widehat{R}_f Cov_{\widehat{P}}\left[\widehat{m}, \widehat{R}_k \right]. \tag{25}$$

By Assumption 1, for each i,

$$E_{\widehat{P}}[\widehat{R}_{RF_i} - \widehat{R}_f] = \sum_{j=1}^{I} - \widehat{R}_f a_j Cov_{\widehat{P}}\left[\widehat{R}_{RF_j}, \widehat{R}_{RF_i} \right]. \tag{26}$$

Let $\widehat{A}$ be the $I \times 1$ matrix having the ith element of $E_{\widehat{P}}[\widehat{R}_{RF_i} - \widehat{R}_f]$, let $\widehat{V}$ be the $I \times I$ matrix having the (i,j)th element of $Cov_{\widehat{P}}\left[\widehat{R}_{RF_i}, \widehat{R}_{RF_j}\right]$, and let $\widehat{B}$ be the $I \times 1$ matrix having the ith element of $-\widehat{R}_f a_i$. Then (26) is written in the matrix form of

$$\widehat{A} = \widehat{V}\widehat{B}.$$

By assumption, V is invertible, so that

$$\widehat{B} = (\widehat{V})^{-1}\widehat{A}. \tag{27}$$

Moreover, (25) implies that for each k,

$$E_{\widehat{P}}[\widehat{R}_k - \widehat{R}_f] = \sum_{i=1}^{I} - \widehat{R}_f a_i Cov_{\widehat{P}}\left[\widehat{R}_{RF_i}, \widehat{R}_k\right]. \tag{28}$$

Let $\widehat{C}_k$ be the $I \times 1$ matrix having the ith element of $Cov_{\widehat{P}}\left[\widehat{R}_{RF_i}, \widehat{R}_k\right]$. Because $\widehat{V}$ is symmetric, given (27), (28) implies that

$$E_{\widehat{P}}[\widehat{R}_k - \widehat{R}_f] = \widehat{B}^T\widehat{C}_k = (\widehat{A})^T(\widehat{V})^{-1}\widehat{C}_k = (\widehat{A})^T\widehat{\beta}, \tag{29}$$

where $\widehat{\beta} = (\widehat{V})^{-1}\widehat{C}_k$, the ith element of which is denoted by $\widehat{\beta}_{k,i}$. Then (29) is rewritten as

$$E_{\widehat{P}}[\widehat{R}_k - \widehat{R}_f] = \sum_{i=1}^{I}\widehat{\beta}_{k,i}E_{\widehat{P}}[\widehat{R}_{RF_i} - \widehat{R}_f], \tag{30}$$

which is (9). Note that $\widehat{\beta}_{k,i}$ is the coefficient of $\left(\widehat{R}_{RF_i} - \widehat{R}_f\right)$ by the regression of the excess return $\left(\widehat{R}_k - \widehat{R}_f\right)$ on excess factor returns $\left(\widehat{R}_{RF_1} - \widehat{R}_f, ..., \widehat{R}_{RF_I} - \widehat{R}_f\right)$ under $\widehat{P}$. ■

Appendix C: The Proof of Proposition 1

Given (11), the variance-covariance matrix V of R_{RF_i} is

$$\Lambda\widehat{V}\Lambda, \tag{31}$$

where $i$the diagonal element of Λ is $\dfrac{1}{\gamma_{RF_i}}$ and an off diagonal element of Λ is zero. Also, the covariance vector between R_k and R_{RF_i} becomes

$$\frac{1}{\gamma_k}\Lambda\widehat{C}_k. \tag{32}$$

Let $\beta_{k,i}$ be the coefficient of $(R_{RF_i} - R_f)$ by the regression of the excess return $(R_k - R_f)$ on excess factor returns $(R_{RF_1} - R_f, ..., R_{RF_I} - R_f)$ under $\widehat{P}$. By the standard argument,

$$\begin{aligned}\beta &= \left(\Lambda \widehat{V} \Lambda\right)^{-1} \left(\frac{1}{\gamma_k} \Lambda \widehat{C}_k\right) \\ &= \frac{1}{\gamma_k} \left(\Lambda \widehat{V} \Lambda\right)^{-1} \left(\Lambda \widehat{C}_k\right) \\ &= \frac{1}{\gamma_k} (\Lambda)^{-1} (\widehat{V})^{-1} (\Lambda)^{-1} \left(\Lambda \widehat{C}_k\right) \\ &= \frac{1}{\gamma_k} (\Lambda)^{-1} (\widehat{V})^{-1} \widehat{C}_k = \frac{1}{\gamma_k} (\Lambda)^{-1} \widehat{\beta},\end{aligned} \tag{33}$$

where the ith element of β is $\widehat{\beta}_{k,i}$. Then (33) implies that

$$\beta_{k,i} = \frac{\gamma_{RF_i}}{\gamma_k} \widehat{\beta}_{k,i}. \tag{34}$$

Now,

$$E_{\widehat{P}}[\widehat{R}_k - \widehat{R}_f] = \gamma_k E_{\widehat{P}}[R_k - R_f] - R_f (\gamma_f - \gamma_k) \tag{35}$$

and

$$\begin{aligned}\sum_{i=1}^{I} \widehat{\beta}_{k,i} E_{\widehat{P}}[\widehat{R}_{RF_i} - \widehat{R}_f] &= \sum_{i=1}^{I} \left\{ \gamma_{RF_i} \widehat{\beta}_{k,i} E_{\widehat{P}}[R_{RF_i} - R_f] - \widehat{\beta}_{k,i} R_f \left(\gamma_f - \gamma_{RF_i}\right) \right\} \\ &= \gamma_k \sum_{i=1}^{I} \left\{ \beta_{k,i} E_{\widehat{P}}[R_{RF_i} - R_f] - \frac{1}{\gamma_k} \widehat{\beta}_{k,i} R_f \left(\gamma_f - \gamma_{RF_i}\right) \right\}.\end{aligned} \tag{36}$$

Then by (35) and (36), (30) becomes

$$\gamma_k E_{\widehat{P}}[R_k - R_f] - R_f (\gamma_f - \gamma_k) = \gamma_k \sum_{i=1}^{I} \left\{ \beta_{k,i} E_{\widehat{P}}[R_{RF_i} - R_f] - \frac{1}{\gamma_k} \widehat{\beta}_{k,i} R_f \left(\gamma_f - \gamma_{RF_i}\right) \right\},$$

which is

$$E_{\widehat{P}}[R_k - R_f] = \alpha_k + \sum_{i=1}^{I} \beta_{k,i} E_{\widehat{P}}[R_{RF_i} - R_f], \tag{37}$$

where

$$\begin{aligned}\alpha_k &= \frac{1}{\gamma_k} R_f (\gamma_f - \gamma_k) - \frac{1}{\gamma_k} \sum_{i=1}^{I} \widehat{\beta}_{k,i} R_f \left(\gamma_f - \gamma_{RF_i}\right) \\ &= R_f \left(\sum_{i=1}^{I} \beta_{k,i} - 1\right) - \frac{\gamma_f}{\gamma_k} R_f \left(\sum_{i=1}^{I} \widehat{\beta}_{k,i} - 1\right),\end{aligned}$$

which is (13). Note that (37) defines α_k as the regression constant because $\beta_{k,i}$ is the regression coefficient.■

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