Kyoto University,
Graduate School of Economics
Discussion Paper Series

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Discussion Paper No. E-19-004

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July, 2019

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July 2, 2019

Abstract

This study examines a gain/loss asymmetric utility in continuous time in which the investor discounts their utility gain by more than the utility loss. By employing the theory of stochastic differential utility, the model allows a time-variable subjective discount rate. In addition, the model can express various forms of utility functions including a version of the Epstein–Zin utility. Under the model, the optimal consumption/wealth ratio and portfolio weight have different functional forms depending on whether the state variables stay in some region.

Key words: Gain/Loss Asymmetry, Stochastic Differential Utility, Consumption–Investment Problem

JEL Classification: D15, G11, G40

*I thank Masahiko Egami, Katsutoshi Wakai, seminar participants at Kyoto University and Tokyo Keizai university, and conference participants at Nippon Finance Association the 27th annual meeting for their helpful comments. This work was supported by JSPS KAKENHI Grant Numbers 17K03876, 18K12811. All remaining errors are my own.

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1 Introduction

The gain/loss asymmetric preference typically implies that the decision maker values some outcome from an investment over others based on some value as a reference point. This preference includes loss aversion and prospect theory (Kahneman and Tversky (1979)), disappointment aversion (Gul (1991)), and a preference for spread (Wakai (2008)). These preference specifications have been applied to various phenomena in finance, providing rich explanations and predictions. These phenomena include the equity premium puzzle (Benartzi and Thaler (1995)), pricing in initial public offerings (Loughran and Ritter (2002) and Ljungqvist and Wilhelm (2005)), and portfolio choice (Dahlquist et al. (2017)). One of the difficulties associated with gain/loss asymmetric preferences is their mathematical complexity. Compared with a standard expected utility model, the gain/loss asymmetric utility has some type of non-smoothness. This characteristic creates difficulties in mathematical analyses, although this type of model captures notable features of the decision maker's behavior that the standard model cannot.

This paper proposes a continuous-time model of a gain/loss asymmetric utility to provide a tractable form of gain/loss asymmetric preferences. Specifically, I extend the discrete-time gain/loss asymmetric preference suggested by Wakai (2008, 2010) to the stochastic differential utility (SDU) model of Duffie and Epstein (1992b). Wakai (2008) specifies a functional form of the utility function that represents a preference for spread. A preference for spread implies that the decision maker prefers a situation in which bad consumption and good consumption are evenly diversified over time. This is often found in experimental studies such as Loewenstein (1987). Wakai (2010) extends the utility function of Wakai (2008) to a version under uncertainty.

By taking a natural limit of the discrete-time model by Wakai (2008, 2010) with respect to a time interval, I define a gain/loss asymmetric SDU and interpret it as an SDU. This allows me to employ the many mathematical tools available for SDUs and backward stochastic differential equations (BSDEs) such as those of Duffie and Epstein (1992b), El Karoui et al. (1997), and Kraft et al. (2013). Using these mathematical tools, I combine the gain/loss asymmetry and the Epstein–Zin recursive utility of Epstein and Zin (1989) and study the classical Markovian optimal consumption–investment problem of Merton (1969, 1971) under the gain/loss asymmetric SDU. In the optimal consumption–investment problem, I derive a partial differential equation

(PDE) that is satisfied by a value function. This PDE is called the Hamilton–Jacobi–Bellman (HJB) equation, which provides mathematical tractability for various applications.

The HJB equation indicates that optimal consumption and investment under the gain/loss asymmetry are different from that under the standard SDUs. Particularly, when the risk premium of a risky asset is time-varying and uncertain, and when the elasticity of intertemporal substitution (EIS) is high, the gain/loss asymmetric SDU causes non-smooth optimal policies. The sensitivity of the optimal portfolio weight to changes in the risk premium varies depending on the value of the risk premium, whereas it is close to constant in the standard models. Specifically, the sensitivity is high when market conditions are bad, and the sensitivity is low when market conditions are good. The optimal consumption/wealth ratio dips in regions where market conditions are bad, whereas it is smooth in the standard models. Hence, the optimal consumption can decrease rapidly when it can be expected that market conditions will be bad. Interestingly, this result occurs without any jump in a path of underlying state variables. Therefore, the gain/loss asymmetric SDU can endogenize sudden shocks in the consumption path.

The features of the optimal policies in gain/loss asymmetric SDUs can be distinguished from the intertemporal substitution effect. In the high EIS case, the investor easily substitutes current and future consumption, so there is a strong substitution effect. However, under gain/loss asymmetric SDUs, the investor has a strong motivation to evenly diversify good and bad consumption over time, i.e., a preference for spread. For example, when market conditions are bad, the optimal consumption policy in the gain/loss-asymmetric SDU is lower since the market conditions will be bad for a while and since the investor wants to diversify current and future consumption. This effect has the same direction as the income effect, but it works non-smoothly as above.

As mentioned above, the dip in the optimal consumption/wealth ratio implies that the consumption can suddenly change even if an underlying production or endowment process changes only gradually. This property is consistent with recent literature on rare disaster models.[1] In

[1]Rietz (1988) originally proposes a model with rare disasters to address the equity premium puzzle of Mehra and Prescott (1985). After decades, Barro (2006) demonstrates that an asset pricing model with rare disasters can explain the equity premium, by using international data. Recently, Gabaix (2012), Wachter (2013), Farhi and Gabaix (2016), and others show that rare disaster models can replicate various

the rare disaster model, the investor is exposed to sudden shocks in their consumption path. As a result, the consumption path can be discontinuous, and security prices and optimal investment policies take account of these shocks. This framework fits the data and provides plausible explanations about behavior of asset prices. Usually, sudden shocks in rare disaster models are assumed as exogenous jumps in the consumption path, so we can consider that they actually happen in a production process and/or an endowment process which are often out of the asset pricing model. On the other hand, the gain/loss asymmetric SDU allows non-smooth changes in the optimal consumption path without the assumption of exogenous jumps in a production process. Therefore, the gain/loss asymmetric SDU provides a theoretical explanation about a part of the shocks that are usually seen as being exogenous in the literature on rare disaster models, but it can be associated with investor's choices in reality, such as the global financial crisis in the late 2000s.

The gain/loss-asymmetric SDU does not always discount utilities at a constant rate, as well as present-bias models.[2] The discrete-time quasi-hyperbolic discounting model also known as the beta-delta model such as Phelps and Pollak (1968) and Laibson (1997) is one of the most famous implementations of present-bias models. The investor in the quasi-hyperbolic discounting model places more value on a current action than a future action, and his or her short-run discount rate is higher than his or her long-run discount rate. Therefore, it seems that the quasi-hyperbolic discounting model does not discount utilities at a constant rate.

A continuous-time version of the quasi-hyperbolic discounting model is proposed by Harris and Laibson (2013). The authors call this quasi-hyperbolic discounting model an instantaneous gratification model. In this model, the optimal consumption does not satisfy the envelope theorem straightforwardly as well as the gain/loss asymmetric SDU. A main difference between the gain/loss asymmetric SDU and the instantaneous gratification model is how to rationalize the optimal policies. The optimal consumption in the gain/loss asymmetric SDU is determined by a standard procedure: it is a solution to the consumption maximization problem in the HJB equation. On the other hand, the optimal consumption in the instantaneous gratification model is rationalized by a concept of equilibrium in the game theory since the model regards a current

behaviors of asset prices that are often found in the markets.

[2] Ericson and Laibson (2019) is a comprehensive survey of present-bias models.

decision maker and future ones as different persons. In this sense, the instantaneous gratification model is dynamically inconsistent, whereas the gain/loss asymmetric SDU is dynamically consistent.

The features of the gain/loss asymmetric SDU as above are not obvious in the discrete-time gain/loss asymmetric utility model because of its mathematical complexity. Furthermore, the gain/loss asymmetric SDU can be easily applied to existing asset pricing models such as rare disaster models, models with trading constraints, and models under model-uncertainty because it is based on the framework of the SDU. Thus, the mathematical tractability of the continuous-time model helps us to better understand investor behavior under gain/loss asymmetry.

The SDU is introduced by Duffie and Epstein (1992b) and can be used to represent various preferences in continuous time, including the recursive utility of Epstein and Zin (1989). The asset-tpricing implication of using the SDU is provided by Duffie and Epstein (1992a). The representation by the SDU can be used to reinterpret classical problems such as Merton (1969, 1971), Black and Scholes (1973), and Cox et al. (1985). Moreover, the SDU has been applied to various situations and concepts in finance, for example, optimal consumption-investment decisions and asset pricing under model uncertainty (e.g., Chen and Epstein (2002) and Maenhout (2004)), the rare disaster model (e.g., Wachter (2013)), and the cross-section of equity returns (e.g., Ai and Kiku (2013)).

Mathematically, the SDU is equivalent to a solution to a BSDE. This equivalence allows me to employ mathematical techniques of BSDEs[3] to study the SDU. The mathematical properties of the SDU with the Epstein–Zin preference and, in particular, its verification conditions, have been investigated by many economists and mathematicians (e.g., Kraft et al. (2013), Xing (2017), and Kraft et al. (2017)). One difficulty of the Epstein–Zin SDU in terms of mathematics is its non-linearity. In this study, I use the verification techniques of the HJB equation by Kraft et al. (2013).

The remainder of this paper is organized as follows. Section 2 reviews the theory of the recursive gain/loss asymmetric utility by Wakai (2008, 2010) and defines the gain/loss asymmetric SDU. Section 3 discusses the properties of the gain/loss asymmetric SDU in terms of its

[3] El Karoui et al. (1997) provide many useful mathematical tools for BSDEs in the finance literature.

functional behavior and representation. Section 4 considers an optimal consumption-investment problem, such as Merton (1969, 1971), in general settings under the gain/loss asymmetric SDU and gives the HJB equation satisfied by the value function and the optimal policies. Section 5 focuses on two specific optimal consumption-investment problems: a stochastic risk premium case and an independent and identical distribution case. This section also examines the behavior of the value function and the optimal policies in these settings. Section 6 concludes the paper. The Appendixes give proofs of propositions and other notable features of the gain/loss asymmetric SDU omitted in the main text.

2 Gain/Loss Asymmetric Stochastic Differential Utility: From Discrete Time to Continuous Time

In this section, I first review the theory of the recursive gain/loss asymmetric utility in discrete time of Wakai (2008, 2010). Next, I extend the discrete-time utility to a continuous-time utility. Wakai (2008, 2011) suggests a recursive gain/loss asymmetric utility in discrete time without uncertainty to represent a typical person's preference for a spread of utilities over time. Specifically, the author shows that this preference can be expressed as some utility function, U_t, which satisfies the following recursive equation:

$$\begin{aligned} &U_t(\{c_\tau\}_{\tau\geq t}) = \min_{\delta\in[\underline{\delta}_{t+1},\overline{\delta}_{t+1}]} \Big\{(1-\delta)u(c_t) + \delta U_{t+1}(\{c_\tau\}_{\tau\geq t+1})\Big\}, \quad t\leq T-1, \\ &\text{and} \qquad U_T(c_T) = u(c_T), \end{aligned} \tag{2.1}$$

where $\{c_\tau\}_{\tau\geq t}$ is a consumption sequence from time t, and u is an instantaneous utility function. Two sequences, $\{\underline{\delta}_\tau\}_{\tau\geq 1}$ and $\{\overline{\delta}_\tau\}_{\tau\geq 1}$, represent the lower and upper boundaries, respectively, of the subjective discount factor δ, so $0 < \underline{\delta}_t \leq \overline{\delta}_t < 1$ for all t. The difference between the usual discrete-time utility and U_t is the subjective discount factor: in typical cases, the discount factor does not depend on a current and overall utility, but it does so in (2.1). Taking a minimum with respect to the discount factor on the right-hand side of (2.1) is related to a preference for spread.

A preference for spread implies that the decision maker prefers a mixture of several consumption sequences to each sequence on its own. There are two consumption sequences, $c := \{c_\tau\}_{\tau \geq t}$ and $c' := \{c'_\tau\}_{\tau \geq t}$, and the decision maker is indifferent between c and c'. Now, for any given $a \in (0,1)$, let us consider a mixture of c and c' by a denoted as $ac+(1-a)c' = \{ac_\tau+(1-a)c'_\tau\}_{\tau \geq t}$. Then, a preference for spread implies that $ac + (1-a)c'$ is at least as good as c and c' for the decision maker. If c and c' are extreme, for example, $c_\tau = 0$ and $c'_\tau = 1$ when $\tau \leq t^*$, and $c_\tau = 1$ and $c'_\tau = 0$ when $\tau > t^*$, for some given time t^*, then, a mixture of c and c' by $a = 1/2$ is moderate; $ac_\tau + (1-a)c'_\tau = 1/2$, for all τ, and the decision maker who has a preference for spread considers that the moderate option, $ac + (1-a)c'$, is at least as good as the extreme options, c and c'. Wakai (2008) shows that the utility U_t characterized by (2.1) has decision-theoretic axiomatic foundations for a preference for spread as shown above.

Table 1: Survey on choices between consumption sequences in Loewenstein (1987)

	Option	This Weekend	Next Weekend	Two Weekends from Now	Choices
Q1	A	Fancy French	Eat at home	Eat at home	16%
	B	Eat at home	Fancy French	Eat at home	84%
Q2	C	Fancy French	Eat at home	Fancy lobster	57%
	D	Eat at home	Fancy French	Fancy lobster	43%

A preference for spread has been found in people's actual behavior. The survey experiment by Loewenstein (1987) is a typical example and is summarized in Table 1. In this survey experiment, a questioner twice asked respondents which consumption plan they preferred. In Q1, respondents tended to prefer (H, F, H) to (F, H, H) where F stands for "Fancy French" and H stands for "Eat at home." On the other hand, in Q2, respondents tended to prefer (F, H, L) to (H, F, L) where L stands for "Fancy lobster." If a typical respondent prefers F and L to H, the answers to Q1 imply that he or she prefers a choice where good consumption (F) and bad consumption (H) are evenly diversified over time, (i.e., option B) to another option A where good consumption comes first, after which the respondent must accept bad consumption. Q2 implies the same types of preferences. Thus, the respondent has a preference for spread. Here, let us assume a typical respondent has a standard discounted time-additive utility. According to the answers to Q1, $u(F) + \delta_2 u(H) < u(H) + \delta_2 u(F)$ holds, whereas the answers for Q2 imply $u(H) + \delta_2 u(F) < u(F) + \delta_2 u(H)$. Therefore, a typical preference of the respondents cannot be expressed as any standard discounted time-additive utility. In contrast, the utility characterized by (2.1) can support a preference of typical respondents who prefer B to A and C to D.

The recursive equation (2.1) can be expressed as

$$\begin{aligned} U_t(\{c_\tau\}_{\tau\geq t}) = u(c_t) + \underline{\delta}_{t+1} \max\Big\{U_{t+1}(\{c_\tau\}_{\tau\geq t+1}) - u(c_t), 0\Big\} \\ + \overline{\delta}_{t+1} \min\Big\{U_{t+1}(\{c_\tau\}_{\tau\geq t+1}) - u(c_t), 0\Big\}. \end{aligned} \quad (2.2)$$

Recall that $\underline{\delta}_{t+1} \leq \overline{\delta}_{t+1}$. Equation (2.2) indicates that the utility gain, $\max\Big\{U_{t+1}(\{c_\tau\}_{\tau\geq t+1}) - u(c_t), 0\Big\}$ is discounted more than the utility loss, $\min\Big\{U_{t+1}(\{c_\tau\}_{\tau\geq t+1}) - u(c_t), 0\Big\}$. The difference in the discounted factors is a key feature of the gain/loss asymmetry in the model. The discount factor changes depending on whether the future utility, U_{t+1}, exceeds the current utility, $u(c_t)$. When the future utility is larger than the current utility, the decision maker expects better consumption in the future than the current consumption. Thus, the decision maker enjoys the utility gain, but it is greatly discounted. The person may prefer a smoother consumption path over time because it may be less discounted, which implies a preference for spread. Based on the difference between the future utility and current utility, the decision maker judges whether a current state is a utility gain or loss. Therefore, we consider the current utility as a type of reference point in the sense of Kahneman and Tversky (1979).

Wakai (2010) extends the above recursive gain/loss asymmetric utility to a version under uncertainty. Under uncertainty, the recursive gain/loss asymmetric utility satisfies the following stochastic recursive equation:

$$U_t(\{c_\tau\}_{\tau\geq t}) = \mathrm{E}_t\left[\phi\left(\operatorname*{essinf}_{\delta\in[\underline{\delta}_{t+1},\overline{\delta}_{t+1}]}\Big\{(1-\delta)u(c_t) + \delta\phi^{-1}(U_{t+1}(\{c_\tau\}_{\tau\geq t+1}))\Big\}\right)\right], \quad (2.3)$$

where ϕ is a continuous and strictly increasing function from $\mathbb{R}$ to $\mathbb{R}$, ϕ^{-1} is the functional inverse of ϕ, and E_t is a conditional expectation operator given information up to time t.[4] The

[4]Operators, essinf and esssup, indicate essential infimum and supremum, respectively. The concept of essential infimum (resp. supremum) is one of sophistication of the minimum (resp. maximum) operation in stochastic environments. The formal definitions of essential infimum and supremum are given by standard textbooks of continuous-time asset pricing such as Duffie (2001). Essential infimum allows us to take a minimum of a family of random variables with ignoring their values on negligible sets. In addition, the essential infimum which this paper mainly focuses on exists and it is measureble with respect to a suitable sigma-algebra because a set of mesurable random variables in which all elements

recursive equation (2.3) resembles the recursive characterization of the utility by Epstein and Zin (1989). Let us apply the gain/loss asymmetry by Wakai (2008) to the Epstein–Zin utility heuristically. Then,[5]

$$U_t(\{c_\tau\}_{\tau\geq t}) = \operatorname*{essinf}_{\delta\in[\underline{\delta}_{t+1},\overline{\delta}_{t+1}]} \phi\left((1-\delta)u(c_t) + \delta\phi^{-1}\left(\mathrm{E}_t[U_{t+1}(\{c_\tau\}_{\tau\geq t+1})]\right)\right). \tag{2.4}$$

Therefore, (2.4) takes the expected value of a variable in a different way to equation (2.3): in (2.4), the decision maker takes only the expected value of U_{t+1}, whereas in (2.3), the decision maker takes the expected value given by the entire right–hand side. This difference may be crucial in discrete time; however, surprisingly, the utility under (2.3) has the same continuous analog as the utility under (2.4). This equivalence is discussed in Appendix B.

Now, let us consider a continuous analog of the recursive gain/loss asymmetric utility characterized by (2.3). The original definition of the SDU by Duffie and Epstein (1992b) assumes that a discrete-time utility takes the following form:

$$U_t = W(c_t, m(U_{t+\Delta t}), \Delta t), \tag{2.5}$$

where W is an aggregating function and m is a certainty-equivalent measure of utility. In particular, Duffie and Epstein (1992b) consider the expected-utility-based specification that $m(V) = h^{-1}(\mathrm{E}_t[h(V)])$ for some suitable function h as a certainty-equivalent measure of utility. Duffie and Epstein (1992b) suggest that a drift of an SDU is a derivative of m with respect to Δt at $\Delta t = 0$ minus a second derivative of $M(V_t, V_t)$ with respect to the first argument, where M is a Gateaux derivative of m at the first argument in the direction of the second argument. As shown in (2.3), the discrete-time recursive gain/loss asymmetric utility does not satisfy the functional form (2.5). Thus, the original definition of the SDU cannot be applied to the recursive gain/loss asymmetric utility in a straightforward manner. Therefore, I give a natural definition of the recursive gain/loss asymmetric utility in continuous time, stating that it has the expected

take values on a deterministic closed interval is upward directed. In this paper, I use the essential infimum operation in stochastic environments. However, I also use a usual minimum operation, min, in deterministic equations for simplicity.

[5]The heuristic equation (2.4) is based on an ordinally equivalent form of the original Epstein–Zin utility of Epstein and Zin (1989).

instantaneous growth rate of the discrete-time utility as its drift. The heuristic equation (2.4) satisfies (2.5) with $m(V) = \mathrm{E}_t[V]$, and Appendix B shows that the continuous analog of (2.3) is the same as that of (2.4). Therefore, my definition of the recursive gain/loss asymmetric utility in continuous time can be reinterpreted as the original SDU.

The expected instantaneous growth rate of the recursive gain/loss asymmetric utility is

$$\left.\frac{\mathrm{d}E_t[U_{t+s}]}{\mathrm{d}s}\right|_{s=0} = \lim_{\Delta t \downarrow 0} \frac{E_t[U_{t+\Delta t}] - U_t}{\Delta t}.$$

For notational simplicity, I omit the consumption sequence as an argument of the utility. Moreover, I assume that ϕ is continuously differentiable and that the boundaries of the discount rate are constant over time. To evaluate the expected instantaneous growth rate, I transform the stochastic recursive equation (2.3) to a time-interval-dependent form, as follows:

$$U_t = \mathrm{E}_t\left[\phi\left(\operatorname*{essinf}_{\delta\in[\underline{\delta},\overline{\delta}]}\left\{(1-e^{-\delta\Delta t})u(c_t) + e^{-\delta\Delta t}\phi^{-1}(U_{t+\Delta t})\right\}\right)\right],$$

where $\underline{\delta}$ and $\overline{\delta}$ are the upper and lower boundary of the subjective discount rate, respectively.[6] By the mean value theorem, the differential quotient of $\Delta t \to \mathrm{E}_t[U_{t+\Delta t}]$ at 0 can be expressed as

$$\frac{\mathrm{E}_t[U_{t+\Delta t}] - U_t}{\Delta t} = -\,\mathrm{E}_t\left[\phi'(R_{t+\Delta t})\operatorname*{essinf}_{\delta\in[\underline{\delta},\overline{\delta}]}\left\{\frac{1-e^{-\delta\Delta t}}{\Delta t}(u(C_t) - \phi^{-1}(U_{t+\Delta t}))\right\}\right],$$

where ϕ' is the first derivative of ϕ, $R_{t+\Delta t}$ is a random variable taking values between $\phi^{-1}(U_{t+\Delta t})$ and $\operatorname*{essinf}_{\delta\in[\underline{\delta},\overline{\delta}]}\left\{(1-e^{-\delta\Delta t})u(c_t) + e^{-\delta\Delta t}\phi^{-1}(U_{t+\Delta t})\right\}$. Further, suppose that $\lim_{\Delta t\downarrow 0} U_{t+\Delta t} = U_t$, and that we can exchange the order of the mathematical operations, $\lim, \operatorname{essinf}$ and E_t.[7] Then,

$$\begin{aligned}
\lim_{\Delta t\downarrow 0}\frac{\mathrm{E}_t[U_{t+\Delta t}] - U_t}{\Delta t} &= -\,\mathrm{E}_t\left[\lim_{\Delta t\downarrow 0}\phi'(R_{t+\Delta t})\operatorname*{essinf}_{\delta\in[\underline{\delta},\overline{\delta}]}\left\{\frac{1-e^{-\delta\Delta t}}{\Delta t}(u(c_t) - \phi^{-1}(U_{t+\Delta t}))\right\}\right] \\
&= -\,\mathrm{E}_t\left[\lim_{\Delta t\downarrow 0}\phi'(R_{t+\Delta t})\operatorname*{essinf}_{\delta\in[\underline{\delta},\overline{\delta}]}\left\{\lim_{\Delta t\downarrow 0}\frac{1-e^{-\delta\Delta t}}{\Delta t}(u(c_t) - \phi^{-1}(U_{t+\Delta t}))\right\}\right] \\
&= -\operatorname*{essinf}_{\delta\in[\underline{\delta},\overline{\delta}]}\left\{\delta\phi'(\phi^{-1}(U_t))\Big(u(c_t) - \phi^{-1}(U_t)\Big)\right\},
\end{aligned}$$

[6] Hereafter, I employ a subjective discount rate for discounting utilities over time because it can simplify the notations in continuous-time models.

[7] For example, these assumptions hold when $U_{t+\Delta} = U_t + x_{t+\Delta t}$ where an outcome of $x_{t+\Delta t}$ is $a\sqrt{\Delta t}$ with a constant probability p or $b\sqrt{\Delta t}$ with a probability $1-p$ for constants a and b.

where I use the fact that ϕ' is positive owing to the increasing monotonicity of ϕ in the last equality. Therefore,

$$\frac{\mathrm{d}E_t[U_{t+s}]}{\mathrm{d}s}\Big|_{s=0} = -\operatorname*{essinf}_{\delta\in[\underline{\delta},\overline{\delta}]}\Big\{\delta\phi'(\phi^{-1}(U_t))\Big(u(c_t)-\phi^{-1}(U_t)\Big)\Big\}. \tag{2.6}$$

From (2.6), define a function F as follows:

$$F(c,v;u,\phi,\underline{\delta},\overline{\delta}) := \operatorname*{essinf}_{\delta\in[\underline{\delta},\overline{\delta}]}\Big\{\delta\phi'(\phi^{-1}(v))\Big(u(c)-\phi^{-1}(v)\Big)\Big\}. \tag{2.7}$$

Then, $\mathrm{d}E_t[U_s]/\mathrm{d}s|_{s=t} = -F(c_t,U_t;u,\phi,\underline{\delta},\overline{\delta})$. From the definition, $-F(c_t,U_t;u,\phi,\underline{\delta},\overline{\delta})$ can be regarded as the expected instantaneous growth rate of U_t. Therefore, we can express a continuous analog of the recursive gain/loss asymmetric utility as an SDU as follows:

$$\mathrm{d}U_t = -F(c_t,U_t;u,\phi,\underline{\delta},\overline{\delta})\mathrm{d}t + \mathrm{d}M_t,$$

where $(M_t)_{t\geq 0}$ is some martingale process that represents uncertainty. The functions u and ϕ and the constants $\underline{\delta}$ and $\overline{\delta}$ completely determine F and U_t. In studies on SDUs, the function F is usually called a generator of an SDU. The following is a definition of the gain/loss asymmetric SDU used in this paper.

Definition 1 (The Gain/Loss Asymmetric Stochastic Differential Utility (G/L-A SDU)) *For an instantaneous utility function u, a continuously differentiable and monotone increasing function ϕ, and boundary constants of a subjective discount rate $\underline{\delta}$ and $\overline{\delta}$ with $0<\underline{\delta}\leq\overline{\delta}$, a **gain/loss asymmetric stochastic differential utility (G/L-A SDU)** with quadruplets $(u,\phi,\underline{\delta},\overline{\delta})$ is an SDU with a generator $F(c,v;u,\phi,\underline{\delta},\overline{\delta})$ defined in (2.7).*

3 Characterization of Gain/Loss Asymmetric Stochastic Differential Utility

In this section, I formally characterize the G/L-A SDU using the techniques of BSDEs. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space endowed with a K-dimensional Brownian motion

$\mathbf{B} := (B_t^1, \ldots, B_t^K)_{t\geq 0}$. I denote by $\mathbb{F}^{\mathbf{B}} := (\mathcal{F}_t^{\mathbf{B}})_{t\geq 0}$ an augmentation of the filtration generated by $\mathbf{B}$. Heuristically, $\mathbb{F}^{\mathbf{B}}$ can be regarded as a flow of information.

Let us consider a utility maximization problem during a finite interval $[0, T]$ in which T is a finitely non-negative real number. Let $\mathcal{C}$ be a subset of $\mathbb{R}$, and $C = (C_t)_{t\in[0,T]}$ be a consumption process taking values in $\mathcal{C}$. Additionally, C is $\mathbb{F}^{\mathbf{B}}$-progressively measurable, so we know a value of C_t using the information at time t (i.e., $\mathcal{F}_t^{\mathbf{B}}$).

For any given instantaneous utility u, a continuously differentiable and monotone increasing function ϕ, and boundary constraints $\underline{\delta}$ and $\overline{\delta}$ with $0 < \underline{\delta} \leq \overline{\delta}$, we assume that a decision maker has a G/L-A SDU with $(u, \phi, \underline{\delta}, \overline{\delta})$. Therefore, the decision maker's utility process denoted by $(U_t)_{t\in[0,T]}$ is a solution to the following BSDE:

$$\begin{cases} -\mathrm{d}U_t = F(C_t, U_t; u, \phi, \underline{\delta}, \overline{\delta})\mathrm{d}t - \mathbf{Z}_t^\top \mathrm{d}\mathbf{B}_t, \\ U_T = \overline{u}(C_T), \end{cases} \tag{3.1}$$

where $(\mathbf{Z}_t)_{t\in[0,T]}$ is a K-dimensional $\mathbb{F}^{\mathbf{B}}$-progressively measurable process, and a superscript $\top$ indicates the transpose of a vector or matrix. $\overline{u}$ is a function from $\mathcal{C}$ to $\mathbb{R}$ which represents the bequest utility.

To explore the properties of the G/L-A SDU, I first assume that ϕ is an identity function (i.e., $\phi(v) = v$) and then consider a case of a more complicated ϕ. If $\phi(v) = v$, then BSDE (3.1) can be expressed as

$$\begin{cases} -\mathrm{d}U_t = \underset{\delta\in[\underline{\delta},\overline{\delta}]}{\operatorname{essinf}}\{\delta(u(C_t) - U_t)\}\mathrm{d}t - \mathbf{Z}_t^\top \mathrm{d}\mathbf{B}_t, \\ U_T = \overline{u}(C_T). \end{cases} \tag{3.2}$$

It can be easily seen that

$$\underset{\delta\in[\underline{\delta},\overline{\delta}]}{\operatorname{essinf}}\{\delta(u(C_t) - U_t)\} = \begin{cases} \overline{\delta}(u(C_t) - U_t), & \text{if } U_t > u(C_t), \\ 0, & \text{if } U_t = u(C_t), \\ \underline{\delta}(u(C_t) - U_t), & \text{if } U_t < u(C_t). \end{cases} \tag{3.3}$$

Therefore, the future utility, U_t, is more discounted if it is larger than the current utility, $u(C_t)$. Conversely, the future utility is less discounted if it is smaller than the current utility. This property implies that the utility index has a recursive gain/loss asymmetry.

The generator (3.3) can be rewritten as follows:

$$\operatorname*{essinf}_{\delta\in[\underline{\delta},\overline{\delta}]}\{\delta(u(C_t)-U_t)\} = -\Big(\overline{\delta}\ \mathrm{esssup}\{U_t-u(C_t),0\}+\underline{\delta}\ \mathrm{essinf}\{U_t-u(C_t),0\}\Big).$$

The above representation is consistent with the discrete-time recursive equation given by (2.2), so the G/L-A SDU is a continuous analog of the recursive gain/loss asymmetric utility in discrete time. Let us call a G/L-A SDU with $(u,\phi(v)=v,\underline{\delta},\overline{\delta})$ a *standard G/L-A SDU* with $(u,\underline{\delta},\overline{\delta})$.

Under some regularity conditions, a standard G/L-A SDU can be expressed in an explicit form. To observe this, let us define feasible sets for consumption and discount rate processes. We restrict a space of consumption processes as follows: every consumption process is left continuous on $[0,T)$ and progressively measurable with respect to $\mathbb{F}^{\mathbf{B}}$. It satisfies the following inequalities:

$$\mathrm{E}\left[\int_0^T (u(C_t))^2\mathrm{d}t\right]<\infty, \quad \text{and} \quad \mathrm{E}\left[(\overline{u}(C_T))^2\right]<\infty. \tag{3.4}$$

From the inequalities in (3.4) and the Lipschitz property of the drift term, BSDE (3.2) has a unique solution. I denote by $\mathcal{C}[0,T]$ a set of consumption processes that satisfy the above conditions. Additionally, let us define a set of discount rate processes as follows:

$$\Delta[0,T;\underline{\delta},\overline{\delta}] := \left\{ \delta=(\delta_t)_{t\in[0,T]} \;\middle|\; \begin{array}{c} \delta \text{ is left continuous} \\ \text{and progressively measurable with respect to } \mathbb{F}^{\mathbf{B}}, \\ \text{and } \delta_t\in[\underline{\delta},\overline{\delta}] \text{ for all } t\in[0,T]. \end{array} \right\}.$$

Based on these conditions, we have the following proposition.

Proposition 2 (Explicit Representation of Standard G/L-A SDUs) *For any consumption process $C\in\mathcal{C}[0,T]$, a standard G/L-A SDU with $(u,\underline{\delta},\overline{\delta})$ exists and it can be expressed as*

$$U_t=\operatorname*{essinf}_{\delta\in\Delta[0,T;\underline{\delta},\overline{\delta}]}\mathrm{E}\left[\int_t^T \delta_s e^{-\int_t^s\delta_r\mathrm{d}r}u(C_s)\mathrm{d}s+e^{-\int_t^T\delta_r\mathrm{d}r}\overline{u}(C_T)\;\middle|\;\mathcal{F}_t^{\mathbf{B}}\right], \tag{3.5}$$

for all $t\in[0,T]$.

Proof. See Appendix A. □

Proposition 2 implies that the investors who have a standard G/L-A SDU choose their subjective discount rate, δ, as if their discounted expected utilities are minimized. This explicit representation is similar to a max-min utility which minimizes the discounted expected utility with respect to a probability measure. Here, I compute a definite integral of the discount factors in G/L-A SDUs. A discount factor process of a G/L-A SDU on $[t, T]$ for any $t \in [0, T]$ is $(\delta_s e^{-\int_t^s \delta_r \mathrm{d}r})_{s\in[t,T)}$, and $e^{-\int_t^T \delta_r \mathrm{d}r}$ at time T. Then,

$$\int_t^T \delta_s e^{-\int_t^s \delta_r \mathrm{d}r} \mathrm{d}s + e^{-\int_t^T \delta_r \mathrm{d}r} = -\int_t^T \mathrm{d}e^{-\int_t^s \delta_r \mathrm{d}r} + e^{-\int_t^T \delta_r \mathrm{d}r} = 1 - e^{-\int_t^T \delta_r \mathrm{d}r} + e^{-\int_t^T \delta_r \mathrm{d}r} = 1.$$

Hence, the definite integral of the subjective discount factor process is always standardized to one, so the process can be seen as a weighting function for the instantaneous utilities over time. On the other hand, the probability measure can be seen as a standardized weighting function for the instantaneous utilities over uncertain states, so the max-min utility smooths the utilities over states: a small probability tends to be assigned to a good state where the instantaneous utility takes a larger value, and a high probability tends to be assigned to a bad state. Therefore, the similarity of the two utility representations suggests that the G/L-A SDU smooths the instantaneous utilities over time. A small subjective discount factor tends to be assigned at a time when the current utility is larger than the continuation value so a bad future is expected, i.e., a utility loss. A large subjective discount factor tends to be assigned at a time when the current utility is smaller than the continuation value so a good future is expected, i.e., a utility gain.

As can be seen from the expression (3.5), the G/L-A SDU can be regarded as an extended version of the variational utility by Geoffard (1996). The author proposes a model under a deterministic environment which minimizes an infinite horizon, continuous-time discounted utility with respect to discount factors, that is,

$$U(C) = \min_{\delta\in\Delta} \int_0^\infty f(C_t, B_t, \delta_t) \mathrm{d}t, \tag{3.6}$$

where Δ is a set of admissible paths of rates of time preference, and f is a current felicity function of the current consumption C_t, discount factor B_t, and rate of time preference δ_t.

Compared to (3.6), the G/L-A SDU takes account of uncertainty, so it minimizes an expected discounted utility.

Uncertainty in the G/L-A SDU plays a crucial role in optimal policies. Section 5.2 will show that a current value of wealth has no influence on an optimal choice of rates of time preference when we suppose homotheticity of preference, one of typical assumptions in the literature. As a result, continuous changes in a subjective discount rate do not occur in the I.I.D. case. The deterministic case is also similar. Therefore, we need an extra state variable as another source of randomness to implement continuous changes.

In addition, Proposition 2 leads immediately to the following properties.

Corollary 3

1. *A standard G/L-A SDU is concave on a consumption process C if u and $\overline{u}$ are also concave.*
2. *A standard G/L-A SDU exhibits a homothetic preference if u and $\overline{u}$ are homogeneous with the same degree.*

Here, let us consider a case of a more complicated ϕ. Suppose

$$\phi(v) = \frac{1}{1-\gamma}\Big((1-1/\psi)v\Big)^{\frac{1-\gamma}{1-1/\psi}}, \quad u(c) = \frac{1}{1-1/\psi}c^{1-1/\psi}, \quad \text{and} \quad \overline{u}(c) = \frac{1}{1-\gamma}c^{1-\gamma}, \tag{3.7}$$

where $\gamma > 0,\ \gamma \neq 1$ and $\psi > 0,\ \psi \neq 1$, respectively. It can be easily seen that ϕ is continuously differentiable and strictly increasing. Then, the generator can be expressed as

$$F(c, v; u, \phi, \underline{\delta}, \overline{\delta}) = \min_{\delta \in [\underline{\delta}, \overline{\delta}]} \left\{ \delta \frac{(1-\gamma)v}{1-1/\psi} \left(\frac{c^{1-1/\psi}}{((1-\gamma)v)^{\frac{1-1/\psi}{1-\gamma}}} - 1 \right) \right\}.$$

Therefore, this is a gain/loss asymmetric version of the generator of the Epstein–Zin preference with an EIS ψ and a coefficient of relative risk aversion (RRA) γ. Let us call the G/L-A SDU with $(u, \phi, \underline{\delta}, \overline{\delta})$ defined in (3.7) an *Epstein–Zin G/L-A SDU* with $(\gamma, \psi, \underline{\delta}, \overline{\delta})$.[8] It can be easily

[8] In the unit EIS case ($\psi = 1$), ϕ and u are defined as

$$\phi(v) = \frac{1}{1-\gamma}\exp\{(1-\gamma)v\}, \quad \text{and} \quad u(c) = \log c.$$

seen that an Epstein–Zin G/L-A SDU with $(\gamma, 1/\gamma, \underline{\delta}, \overline{\delta})$ is a standard G/L-A SDU with $(u, \underline{\delta}, \overline{\delta})$ in which $u(c) = c^{1-\gamma}/(1-\gamma)$. This instantaneous utility, u is a constant relative risk aversion (CRRA) utility. Therefore, let us call this Epstein–Zin G/L-A SDU a *CRRA G/L-A SDU* with $(\gamma, \underline{\delta}, \overline{\delta})$.

In the Epstein–Zin G/L-A SDU, the discount rate δ depends on the sign of the following value:

$$\phi^{-1}(U_t) - u(C_t) = u\left((1-\gamma)U_t^{\frac{1}{1-\gamma}}\right) - u(C_t) \Leftrightarrow U_t - \frac{1}{1-\gamma}C_t^{1-\gamma}.$$

If the above value is positive, the decision maker obtains a utility gain that is more discounted. On the other hand, if the above value is negative, the decision maker experiences a utility loss that is less discounted. Furthermore, ψ does not appear directly in the above value. Therefore, the threshold of the utility gain or loss does not depend directly on ψ although ψ could affect the current regime through changes in U_t. Table 2 summarizes the three special cases of G/L-A SDUs, and more detailed properties of G/L-A SDUs are discussed in Appendix C.

Table 2: Three Special Cases of G/L-A SDUs

The G/L-A SDU is defined as $\mathrm{d}U_t = -F(C_t, U_t; u, \phi, \underline{\delta}, \overline{\delta})\mathrm{d}t + \mathrm{d}M_t$, where $F(c, v; u, \phi, \underline{\delta}, \overline{\delta}) = \min_{\delta \in [\underline{\delta}, \overline{\delta}]} \left\{\delta\phi'(\phi^{-1}(v))\left(u(c) - \phi^{-1}(v)\right)\right\}$.

Name	$u(c)$	$\phi(v)$	$\overline{u}(c)$
standard G/L-A SDU with $(u, \underline{\delta}, \overline{\delta})$	arbitrary function	$\phi(v) = v$	arbitrary function
CRRA G/L-A SDU with $(\gamma, \underline{\delta}, \overline{\delta})$	$u(c) = \dfrac{c^{1-\gamma}}{1-\gamma}$	$\phi(v) = v$	$\overline{u}(c) = \dfrac{c^{1-\gamma}}{1-\gamma}$
Epstein-Zin G/L-A SDU with $(\gamma, \psi, \underline{\delta}, \overline{\delta})$	$u(c) = \dfrac{c^{1-1/\psi}}{1-1/\psi}$	$\phi(v) = \dfrac{((1-1/\psi)v)^{\frac{1-\gamma}{1-1/\psi}}}{1-\gamma}$	$\overline{u}(c) = \dfrac{c^{1-\gamma}}{1-\gamma}$

Then, the generator can be expressed as

$$F(c, v) = \min_{\delta \in [\underline{\delta}, \overline{\delta}]} \left\{\delta(1-\gamma)v\left(\log C - \frac{1}{1-\gamma}\log\left((1-\gamma)v\right)\right)\right\}.$$

4 An Optimal Consumption and Investment Problem

This section studies an optimal consumption and investment problem under gain/loss asymmetry. A G/L-A SDU with $(u, \phi, \underline{\delta}, \overline{\delta})$ can be expressed as

$$\begin{cases} -\mathrm{d}U_t = \operatorname*{essinf}_{\delta \in [\underline{\delta}, \overline{\delta}]} \Big\{ \delta \phi'(\phi^{-1}(U_t)) \Big(u(C_t) - \phi^{-1}(U_t) \Big) \Big\} \mathrm{d}t - \mathbf{Z}_t^\top \mathrm{d}\mathbf{B}_t, \\ U_T = \overline{u}(C_T). \end{cases}$$

Hereafter, I consider a standard Merton problem under a G/L-A SDU in a Markovian environment. Following the settings in the previous section, on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbf{B} := (B_t^1, \dots, B_t^K)_{t \geq 0}$ is a K-dimensional Brownian motion, and $\mathbb{F}^{\mathbf{B}} := (\mathcal{F}_t^{\mathbf{B}})_{t \geq 0}$ is an augmentation of the filtration generated by $\mathbf{B}$. In the market, there is one risk-free asset, and there are N risky assets with $N \leq K$. Let $P^0 = (P_t^0)_{t \in [0,T]}$ be a price process of the risk-free asset. Let $\mathbf{P} = (P_t^1, \dots, P_t^N)_{t \in [0,T]}$ be a price vector process of the risky assets. I denote the first N-elements of $\mathbf{B}$ by $\mathbf{B}^N$.

The price processes P^0 and $\mathbf{P}$ satisfy the following system of stochastic differential equations (SDEs).

$$\begin{cases} \mathrm{d}P^0 = r(\mathbf{Y}_t) P_t^0 \mathrm{d}t, \\ \mathrm{d}\mathbf{P}_t = \mathrm{diag}(\mathbf{P}_t) \Big(b(\mathbf{Y}_t)\mathrm{d}t + \sigma(\mathbf{Y}_t) \mathrm{d}\mathbf{B}_t^N \Big), \\ \mathrm{d}\mathbf{Y}_t = b_{\mathbf{Y}}(\mathbf{Y}_t)\mathrm{d}t + \sigma_{\mathbf{Y}}(\mathbf{Y}_t)\mathrm{d}\mathbf{B}_t, \end{cases} \tag{4.1}$$

where $r : \mathbb{R}^M \to \mathbb{R}$, $b : \mathbb{R}^M \to \mathbb{R}^N$, $\sigma : \mathbb{R}^M \to \mathbb{R}^{N \times N}$, $b_{\mathbf{Y}} : \mathbb{R}^M \to \mathbb{R}^M$, and $\sigma_{\mathbf{Y}} : \mathbb{R}^M \to \mathbb{R}^{M \times K}$ are measurable functions, and $\mathbf{Y}$ is a stochastic process taking values in $\mathbb{R}^M$. The process $\mathbf{Y}$ represents state variables. I assume there exists a strong solution to the system of SDEs (4.1).

Let $\alpha = (\alpha_t^1, \dots, \alpha_t^N)_{t \in [0,T]}$ be a portfolio process of the risky assets, and let $W = (W_t)_{t \in [0,T]}$ be a wealth process. For a given portfolio process α and consumption process C, the wealth process W satisfies the following SDE.[9]

$$\begin{aligned}\mathrm{d}W_t = W_t\left(\left(1-\sum_{i=1}^{N}\alpha_t^i\right)\frac{\mathrm{d}P_t^0}{P_t^0}+\sum_{i=1}^{N}\alpha_t^i\frac{\mathrm{d}P_t^i}{P_t^i}\right)-C_t\mathrm{d}t\\ =\Big(W_t(r(\mathbf{Y}_t)+\alpha_t^\top\mu_e(\mathbf{Y}_t))-C_t\Big)\,\mathrm{d}t+W_t\alpha_t^\top\sigma(\mathbf{Y}_t)\mathrm{d}\mathbf{B}_t^N,\end{aligned}$$

where $\mu_e(\mathbf{Y}_t) := b(\mathbf{Y}_t)-r(\mathbf{Y}_t)\mathbf{1}_N$ is an instantaneous excess expected return vector process. The investor has $W_t\alpha_t^i/P_t^i$ shares of the risky asset i at each time t.

Now, let us define the investor's consumption and portfolio in more detail. For simplification of notation, I write $\mathbf{X}=(W,\mathbf{Y})$ and $\mathcal{X}=\mathbb{R}_+\times\mathbb{R}^M$. For any given portfolio α, consumption C, and $\mathbf{x}=(w,\mathbf{y})\in\mathcal{X}$, let us denote by $\mathbf{X}^{t,\mathbf{x};\alpha,C}=(W^{t,(w,\mathbf{y});\alpha,C},\mathbf{Y}^{t,\mathbf{y}})$ the state variable $\mathbf{X}=(W,\mathbf{Y})$ starting at $W_t=w$ and $\mathbf{Y}_t=\mathbf{y}$ and controlled by α and C. For any $t\in[0,T]$ and $\mathbf{x}=(w,\mathbf{y})\in\mathcal{X}$, let $\mathcal{A}(t,\mathbf{x})$ be a set of portfolio processes and consumption processes such that any $(\alpha,C)\in\mathcal{A}(t,\mathbf{x})$ satisfies $C_T=W_T^{t,(w,\mathbf{y});\alpha,C}$,

$$\mathrm{E}\left[\int_t^T(u(C_s))^2\mathrm{d}s\right]<\infty,\quad\text{and}\quad\mathrm{E}\left[(\overline{u}(W_T^{t,(w,\mathbf{y});\alpha,C}))^2\right]<\infty,$$

and (α,C) has a left-continuous path, $\mathbb{P}$-almost surely, and is progressively measurable with respect to $\mathbb{F}^{\mathbf{B}}$. Let us call $\mathcal{A}(t,\mathbf{x})$ a set of admissible controls.

The utility maximization problem for the investor who has a G/L-A SDU, U, with $(u,\phi,\underline{\delta},\overline{\delta})$ is

$$V(t,\mathbf{x}) := \max_{(\alpha,C)\in\mathcal{A}(t,\mathbf{x})}U_t, \tag{4.2}$$

for all $(t,\mathbf{x})\in[0,T]\times\mathcal{X}$. To solve the utility maximization problem (4.2), there are two commonly used methods: the HJB equation approach and the utility gradient approach (e.g., Duffie and Skiadas (1994), Schroder and Skiadas (1999), and El Karoui et al. (2001)). The utility gradient approach requires the differentiability of a generator of an SDU with respect to a utility index, whereas a generator of a G/L-A SDU is not differentiable. Therefore, I use the

[9] I assume that none of the risky assets have a dividend, but this assumption can be relaxed easily in which case the following discussion still holds.

HJB equation approach. The HJB equation for (4.2) with respect to $(t, \mathbf{x} = (w, \mathbf{y}))$ is

$$-\frac{\partial v}{\partial t} - \max_{(\alpha,C)} \left\{ \mathcal{L}^{\alpha,C} v + \min_{\delta \in [\underline{\delta}, \overline{\delta}]} \left\{ \delta \phi'(\phi^{-1}(v)) \Big(u(C) - \phi^{-1}(v) \Big) \right\} \right\} = 0, \tag{4.3}$$

where

$$\begin{aligned}
\mathcal{L}^{\alpha,C} v &= \Big(w \Big(r(\mathbf{y}) + \alpha^\top \mu_e(\mathbf{y}) \Big) - C \Big) v_w + \frac{1}{2} w^2 \alpha^\top \Sigma(\mathbf{y}) \alpha v_{ww} \\
&\qquad + w \alpha^\top \Sigma_{\mathbf{PY}}(\mathbf{y}) v_{w\mathbf{y}} + (b_{\mathbf{Y}}(\mathbf{y}))^\top v_{\mathbf{y}} + \frac{1}{2} \mathrm{tr} \left\{ \Sigma_{\mathbf{Y}}(\mathbf{y}) v_{\mathbf{yy}} \right\}, \\
\Sigma(\mathbf{y}) &= \sigma(\mathbf{y})(\sigma(\mathbf{y}))^\top, \quad \Sigma_{\mathbf{PY}}(\mathbf{y}) = \sigma(\mathbf{y})(\sigma_{\mathbf{Y}}^N(\mathbf{y}))^\top, \quad \Sigma_{\mathbf{Y}}(\mathbf{y}) = \sigma_{\mathbf{Y}}(\mathbf{y})(\sigma_{\mathbf{Y}}(\mathbf{y}))^\top, \\
v_w &= \frac{\partial v}{\partial w}, \quad v_{\mathbf{y}} = \frac{\partial v}{\partial \mathbf{y}}, \quad v_{ww} = \frac{\partial v}{\partial w \partial w}, \quad v_{\mathbf{yy}} = \frac{\partial v}{\partial \mathbf{y} \partial \mathbf{y}^\top}, \quad v_{w\mathbf{y}} = \frac{\partial v}{\partial w \partial \mathbf{y}},
\end{aligned}$$

and $\sigma_{\mathbf{Y}}^N(\mathbf{y})$ is the sub-matrix of the first N columns of $\sigma_{\mathbf{Y}}(\mathbf{y})$. The terminal condition is

$$v(T, \mathbf{x} = (w, \mathbf{y})) = \overline{u}(w).$$

The HJB equation (4.3) contains a minimization problem for δ. This type of HJB equation is usually called the Hamilton–Jacobi–Bellman–Isaacs (HJBI) equation. HJBI equations may not have classical solutions, but the HJB equation (4.3) is elliptic, so we can consider a viscosity solution: a larger class of solutions to PDEs. However, it is mathematically hard work to rigorously prove the existence of a viscosity solution in a general setting, which would be beyond the scope of this paper. Furthermore, we can discuss the interesting economic features of G/L-A SDUs by assuming the HJB equation (4.3) has a classical solution. Therefore, I assume that the HJB equation (4.3) has a classical solution.

We need to prove a solution to the HJB equation (4.3) is a value function defined in (4.2). To the best of my knowledge, no one has proved yet whether a solution to the HJB equation in the Epstein–Zin utility is *always* the associated value function. However, under some parameter constraints, we can easily prove that a solution to the HJB equation (4.3) is the value function, by using the result of Kraft et al. (2013). The details is in Appendix D.

In the HJB equation (4.3), the maximization problem over α and C can be separated as follows:

$$
\begin{aligned}
&-\frac{\partial v}{\partial t}-\max_{(\alpha,C)}\left\{\mathcal{L}^{\alpha,C}v+\min_{\delta\in[\underline{\delta},\overline{\delta}]}\left\{\delta\phi'(\phi^{-1}(v))\Big(u(C)-\phi^{-1}(v)\Big)\right\}\right\}\\
&=-\frac{\partial v}{\partial t}-r(\mathbf{y})wv_w-(b_{\mathbf{Y}}(\mathbf{y}))^{\top}v_{\mathbf{y}}-\frac{1}{2}\mathrm{tr}\left\{\Sigma_{\mathbf{Y}}(\mathbf{y})v_{\mathbf{yy}}\right\}\\
&\quad-\max_{\alpha}\left\{w(\mu_e(\mathbf{y})v_w+\Sigma_{\mathbf{PY}}(\mathbf{y})v_{w\mathbf{y}})^{\top}\alpha+\frac{1}{2}w^2v_{ww}\alpha^{\top}\Sigma(\mathbf{y})\alpha\right\}\\
&\quad-\max_{C}\left\{\min_{\delta\in[\underline{\delta},\overline{\delta}]}\left\{\delta\phi'(\phi^{-1}(v))\Big(u(C)-\phi^{-1}(v)\Big)\right\}-Cv_w\right\}.
\end{aligned}
$$

The optimization problem over α is the same form as the standard model. However, the optimization problem over C is clearly different. The solution to the optimization problem over C is different functional form depending on the value function v.

To compare the optimal consumption of the G/L-A SDU with those of the standard model, let us look the optimal condition of consumption in the standard model. In the standard model in which a discount rate is a constant δ, the optimal consumption C^* satisfies the envelope condition as follows:

$$
\delta u'(C^*)=\frac{v_w}{\phi'(\phi^{-1}(v))}=\frac{\mathrm{d}\phi^{-1}(v)}{\mathrm{d}w},
$$

where u' and ϕ' are the first derivatives of u and ϕ, respectively. This envelope condition states that the marginal value of the optimal consumption, $\delta u'(C^*)$, is equal to the marginal value of wealth, $\mathrm{d}\phi^{-1}(v)/\mathrm{d}w$. However, the optimal consumption in the G/L-A SDU does not always satisfy the envelope theorem due to the non-differentiability of the generator. Under some standard assumptions, we can characterize the optimal consumption in the G/L-A SDU by a new condition.

Proposition 4 *Suppose that $v_w \geq 0$ and u is increasing and concave. Then, the optimal consumption policy in the G/L-A SDU, denoted by C^*, satisfies*

$$
\begin{aligned}
\underline{\delta}u'(C^*)&=\frac{\mathrm{d}\phi^{-1}(v)}{\mathrm{d}w}, \quad \text{if } u(C(\underline{\delta},v))-\phi^{-1}(v)>0,\\
\overline{\delta}u'(C^*)&=\frac{\mathrm{d}\phi^{-1}(v)}{\mathrm{d}w}, \quad \text{if } u(C(\overline{\delta},v))-\phi^{-1}(v)<0,\\
u(C^*)&=\phi^{-1}(v), \quad \text{otherwise},
\end{aligned}
$$

where a function $(\delta, v) \to C(\delta, v)$ *is a solution to the following equation for* C*,*

$$\delta u'(C) = \frac{\mathrm{d}\phi^{-1}(v)}{\mathrm{d}w}.$$

Proof. See Appendix A. $\square$

Proposition 4 tells us that there are *three* situations which the investor who chooses the optimal consumption can experience. First two are a utility loss and gain as aforementioned. In these situations, the investor's marginal utility of consumption is equal to the marginal value of the wealth, so the envelope condition holds. The last one is a neutral situation that the investor does not experience both of a utility gain and loss. In this situation, the investor cannot equate the marginal value of current consumption to the marginal value of wealth due to the gain/loss asymmetry. To observe this, let us consider a simple example: if the investor experiences a utility gain, he or she has a motivation to increase his or her consumption because the marginal value of current consumption is larger than the marginal value of wealth. However, in the last situation, the investor enters the region of a utility loss before the marginal value of current consumption becomes equal to the marginal value of wealth. The marginal value of consumption in the utility loss region is always smaller than the marginal value of wealth, so the investor chooses his or her consumption at the boundary of the utility loss and gain. As a result, the envelope condition does not hold, and the investor chooses his or her consumption as its value measured by utility, $u(C)$, is equal to the adjusted continuation value, $\phi^{-1}(v)$. The property that the envelope condition does not hold in some region is an important difference of the G/L-A SDU from other standard SDUs.

Hereafter, I focus on the HJB equation of the Epstein-Zin G/L-A SDU with $(\gamma, \psi, \underline{\delta}, \overline{\delta})$. In addition, I suppose $\psi \neq 1$ and $\gamma \neq 1$. Let us assume that a solution to the HJB equation (4.3) has the following form:

$$v(t, \mathbf{x} = (w, \mathbf{y})) = \frac{w^{1-\gamma}}{1-\gamma} \exp\{(1-\gamma) g(t, \mathbf{y})\} \tag{4.4}$$

where $g(t, \mathbf{y})$ is some continuously differentiable function. Then,

$$\phi'(\phi^{-1}(v))\Big(u(C) - \phi^{-1}(v)\Big) = w^{1-\gamma} \exp\{(1-\gamma)g(t, \mathbf{y})\} \frac{(\beta/\exp\{g(t, \mathbf{y})\})^{1-1/\psi} - 1}{1 - 1/\psi},$$

where $\beta = C/w$. Hence,

$$\begin{aligned}
&\max_C \left\{ \min_{\delta \in [\underline{\delta}, \overline{\delta}]} \left\{ \delta\phi'(\phi^{-1}(v))\Big(u(C) - \phi^{-1}(v)\Big) \right\} - Cv_w \right\} \\
&= w^{1-\gamma} \exp\{(1-\gamma)g(t, \mathbf{y})\} \max_\beta \left\{ \min_{\delta \in [\underline{\delta}, \overline{\delta}]} \left\{ \delta \frac{(\beta/\exp\{g(t, \mathbf{y})\})^{1-1/\psi} - 1}{1 - 1/\psi} \right\} - \beta \right\}.
\end{aligned}$$

Solving the above optimization problem, we have

$$\begin{aligned}
a(g(t, \mathbf{y})) &:= \max_\beta \left\{ \min_{\delta \in [\underline{\delta}, \overline{\delta}]} \left\{ \delta \frac{(\beta/\exp\{g(t, \mathbf{y})\})^{1-1/\psi} - 1}{1 - 1/\psi} \right\} - \beta \right\} \\
&= \begin{cases} \dfrac{1}{1-1/\psi} \left(\dfrac{\underline{\delta}^\psi}{\psi} \exp\{(1-\psi)g(t, \mathbf{y})\} - \underline{\delta} \right), & \text{if } g(t, \mathbf{y}) < \log \underline{\delta}, \\ -\exp\{g(t, \mathbf{y})\}, & \text{if } \log \underline{\delta} \leq g(t, \mathbf{y}) \leq \log \overline{\delta}, \\ \dfrac{1}{1-1/\psi} \left(\dfrac{\overline{\delta}^\psi}{\psi} \exp\{(1-\psi)g(t, \mathbf{y})\} - \overline{\delta} \right), & \text{if } g(t, \mathbf{y}) > \log \overline{\delta}, \end{cases}
\end{aligned}$$

and the optimal consumption/wealth ratio (C^*/w) is

$$\beta^*(t, \mathbf{y}) := \frac{C^*}{w} = \begin{cases} \underline{\delta}^\psi \exp\{(1-\psi)g(t, \mathbf{y})\}, & \text{if } g(t, \mathbf{y}) < \log \underline{\delta}, \\ \exp\{g(t, \mathbf{y})\}, & \text{if } \log \underline{\delta} \leq g(t, \mathbf{y}) \leq \log \overline{\delta}, \\ \overline{\delta}^\psi \exp\{(1-\psi)g(t, \mathbf{y})\}, & \text{if } g(t, \mathbf{y}) > \log \overline{\delta}. \end{cases}$$

Obviously, a function $g \to a(g)$ is continuously differentiable and monotone decreasing. On the other hand, when we regard the optimal consumption/wealth ratio, β^*, as a function of g, a function $g \to \beta^*$ is *not* continuously differentiable, though it is continuous. This implies that the optimal consumption/wealth ratio can change non-smoothly even though the function g, i.e., the value function changes smoothly. These non-smooth changes are more clear when $\psi > 1$, that is the substitution effect is larger than the income effect. Then, a slope of the optimal consumption/wealth ratio jumps at the boundaries of the three situations. For example, if g changes from below $\log \underline{\delta}$ to above $\log \underline{\delta}$, the slope of β^* jumps from positive one to negative one. Therefore, the level of β^* can change non-smoothly on this boundary.[10]

For the optimization problem with respect to the portfolio, we have

$$
\begin{aligned}
&\max_{\alpha}\left\{w(\mu_e(\mathbf{y})v_w+\Sigma_{\mathbf{PY}}(\mathbf{y})v_{w\mathbf{y}})^\top\alpha+\frac{1}{2}w^2v_{ww}\alpha^\top\Sigma(\mathbf{y})\alpha\right\}\\
&=w^{1-\gamma}\exp\{(1-\gamma)g(t,\mathbf{y})\}\max_{\alpha}\left\{\Big(\mu_e(\mathbf{y})+(1-\gamma)\Sigma_{\mathbf{PY}}(\mathbf{y})g_{\mathbf{y}}(t,\mathbf{y})\Big)^\top\alpha-\frac{\gamma}{2}\alpha^\top\Sigma(\mathbf{y})\alpha\right\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Q(g_{\mathbf{y}}(t,\mathbf{y}),\mathbf{y}) &:= \max_{\alpha}\left\{\Big(\mu_e(\mathbf{y})+(1-\gamma)\Sigma_{\mathbf{PY}}(\mathbf{y})g_{\mathbf{y}}(t,\mathbf{y})\Big)^\top\alpha-\frac{\gamma}{2}\alpha^\top\Sigma(\mathbf{y})\alpha\right\}\\
&=\frac{1}{2\gamma}\Big(\mu_e(\mathbf{y})+(1-\gamma)\Sigma_{\mathbf{PY}}(\mathbf{y})g_{\mathbf{y}}(t,\mathbf{y})\Big)^\top\Sigma^{-1}(\mathbf{y})\Big(\mu_e(\mathbf{y})+(1-\gamma)\Sigma_{\mathbf{PY}}(\mathbf{y})g_{\mathbf{y}}(t,\mathbf{y})\Big),
\end{aligned}
$$

[10] In the unit EIS case, I guess that the value function also takes the functional form (4.4). Then,

$$
\phi'(\phi^{-1}(v))\Big(u(C)-\phi^{-1}(v)\Big)=w^{1-\gamma}\exp\{(1-\gamma)g(t,\mathbf{y})\}\Big(\log\beta-g(t,\mathbf{y})\Big),
$$

where $\beta=C/w$. Hence,

$$
\begin{aligned}
&\max_{C}\left\{\min_{\delta\in[\underline{\delta},\overline{\delta}]}\left\{\delta\phi'(\phi^{-1}(v))\Big(u(C)-\phi^{-1}(v)\Big)\right\}-Cv_w\right\}\\
&=w^{1-\gamma}\exp\{(1-\gamma)g(t,\mathbf{y})\}\max_{\beta}\left\{\min_{\delta\in[\underline{\delta},\overline{\delta}]}\left\{\delta\Big(\log\beta-g(t,\mathbf{y})\Big)\right\}-\beta\right\}.
\end{aligned}
$$

Solving the above optimization problem, we have

$$
\begin{aligned}
a(g(t,\mathbf{y})) &:= \max_{\beta}\left\{\min_{\delta\in[\underline{\delta},\overline{\delta}]}\left\{\delta\Big(\log\beta-g(t,\mathbf{y})\Big)\right\}-\beta\right\}\\
&=\begin{cases}\underline{\delta}\Big(\log\underline{\delta}-g(t,\mathbf{y})\Big)-\underline{\delta}, & \text{if } g(t,\mathbf{y})<\log\underline{\delta},\\ -\exp\{g(t,\mathbf{y})\}, & \text{if } \log\underline{\delta}\le g(t,\mathbf{y})\le\log\overline{\delta},\\ \overline{\delta}\Big(\log\overline{\delta}-g(t,\mathbf{y})\Big)-\overline{\delta}, & \text{if } g(t,\mathbf{y})>\log\overline{\delta},\end{cases}
\end{aligned}
$$

and the optimal consumption/wealth ratio (C^*/w) is

$$
\beta^*(t,\mathbf{y}):=\frac{C^*}{w}=\begin{cases}\underline{\delta}, & \text{if } g(t,\mathbf{y})<\log\underline{\delta},\\ \exp\{g(t,\mathbf{y})\}, & \text{if } \log\underline{\delta}\le g(t,\mathbf{y})\le\log\overline{\delta},\\ \overline{\delta}, & \text{if } g(t,\mathbf{y})>\log\overline{\delta}.\end{cases}
$$

As well as the general case when $\psi\neq 1$, a function $g\to a(g)$ is continuously differentiable and monotone decreasing. A function $g\to\beta^*$ is not continuously differentiable but continuous. The rest is the same as the general case.

and the optimal portfolio weight is

$$\alpha^*(t,\mathbf{y}) := \frac{1}{\gamma}\Sigma^{-1}(\mathbf{y})\Big(\mu_e(\mathbf{y}) + (1-\gamma)\Sigma_{\mathbf{PY}}(\mathbf{y})g_{\mathbf{y}}(t,\mathbf{y})\Big).$$

As in the classical results, such as Merton (1969, 1971, 1973), the optimal portfolio weight is separated into a myopic term, $\Sigma^{-1}\mu_e/\gamma$, and the hedging demand, $(1-\gamma)\Sigma^{-1}\Sigma_{\mathbf{PY}}g_{\mathbf{y}}/\gamma$. This shows that the gain/loss asymmetry affects the optimal portfolio choice only through the hedging demand. In some cases, there is no hedging demand: for example, (1) the absence of an extra state variable, $\mathbf{Y}$; (2) an uncorrelated extra state variable (i.e., $\Sigma_{\mathbf{PY}} = 0$); or (3) log utility (i.e., $\gamma = 1$). In these instances, the optimal portfolio weight is the same as the symmetric case, which is the myopic term. Note that, even if the investor with a G/L-A SDU is faced with situations (2) and (3), the gain/loss asymmetry still affects the investor's consumption choice.

Finally, the HJB equation (4.3) can be reduced to the following PDE for $g(t,\mathbf{y})$:

$$\begin{aligned} &g_t(t,\mathbf{y}) + a(g(t,\mathbf{y})) + Q(g_{\mathbf{y}}(t,\mathbf{y}),\mathbf{y}) + r(\mathbf{y}) \\ &\qquad + b_{\mathbf{Y}}^\top(\mathbf{y})g_{\mathbf{y}}(t,\mathbf{y}) + \frac{1}{2}\mathrm{tr}\Big\{\Sigma_{\mathbf{Y}}(\mathbf{y})\Big((1-\gamma)g_{\mathbf{y}}(t,\mathbf{y})g_{\mathbf{y}}^\top(t,\mathbf{y}) + g_{\mathbf{yy}}(t,\mathbf{y})\Big)\Big\} = 0, \end{aligned} \tag{4.5}$$

with the terminal condition $g(T,\mathbf{y}) = 0$ for all $\mathbf{y} \in \mathbb{R}^M$.

5 The Effects of Gain/Loss Asymmetry on the Optimal Consumption and Investment

This section specifies the asset price dynamics and examines the effects of the gain/loss asymmetry on the optimal consumption and investment. I will consider two cases: a stochastic risk premium case and an independent and identical distribution (I.I.D.) case. Throughout this section, I assume an Epstein–Zin G/L-A SDU with $(\gamma, \psi, \underline{\delta}, \overline{\delta})$. Furthermore, I assume there exists a classical solution to the HJB equation (4.3).

5.1 Stochastic Risk Premium

I now introduce an extra state variable. Suppose $N = M = 1, K = 2$ and let us denote by P^0 a risk-free asset price and by P^1 a risky asset price. The dynamics of P^0 and P^1 are as follows:

$$\begin{cases} \mathrm{d}P_t^0 = rP_t^0\mathrm{d}t, \\ \mathrm{d}P_t^1 = P_t^1\Big((r + Y_t)\mathrm{d}t + \sigma\mathrm{d}B_t\Big), \\ \mathrm{d}Y_t = \kappa(\overline{y} - Y_t)\mathrm{d}t + \sigma_y\Big(\rho\mathrm{d}B_t + \sqrt{1-\rho^2}\mathrm{d}\widetilde{B}_t\Big), \end{cases} \tag{5.1}$$

where $r > 0$, $\sigma > 0$, $\kappa > 0$, $\overline{y} \in \mathbb{R}$, $\sigma_y > 0$ and $\rho \in [-1, 1]$ are constants, and B and $\widetilde{B}$ are mutually independent one-dimensional standard Brownian motions. r is a constant risk-free rate, σ is a volatility of return on P^1, and the state variable, Y, is the instantaneous risk premium on P^1. Moreover, ρ is a correlation coefficient between P^1 and Y. This model is a complete market model if $\rho = 1$ or -1, and it is an incomplete market model otherwise.

The asset dynamics equation (5.1) assumes a time-varying stochastic risk premium. The optimal consumption and investment with a stochastic risk premium under a CRRA agent has been studied by many scholars such as Kim and Omberg (1996) and Wachter (2002). The Epstein–Zin preference agent's problem (without gain/loss asymmetry) has also been studied by Kraft et al. (2013) and Xing (2017) among others.

The distribution parameter functions can be expressed as

$$r(y) = r,\ \mu_e(y) = y,\ \sigma(y) = \sigma,\ b_{\mathbf{Y}}(y) = \kappa(\overline{y} - y),\ \text{and}\ \sigma_{\mathbf{Y}}(y) = \sigma_y\begin{pmatrix} \rho & \sqrt{1-\rho^2} \end{pmatrix}.$$

Therefore, the reduced HJB equation (4.5) can be rewritten as

$$g_t(t,y) + a(g(t,y)) + r + \frac{y^2}{2\gamma\sigma^2} + \Big(\kappa\overline{y} + \Big(\frac{1-\gamma}{\gamma}\frac{\rho\sigma_y}{\sigma} - \kappa\Big)y\Big)g_y(t,y) \\ + \frac{\sigma_y^2}{2}\Big((1-\gamma)\frac{(1-\gamma)\rho^2 + \gamma}{\gamma}g_y^2(t,y) + g_{yy}(t,y)\Big) = 0. \tag{5.2}$$

The reduced HJB function (5.2) does not have an explicit solution, so I compute a numerical solution using the finite difference (FD) method.

Campbell et al. (2004) examine an optimal consumption and investment with a (gain/loss symmetric) Epstein–Zin preference under the asset price dynamics in (5.1). The authors compute distribution parameter values based on the work of Campbell and Viceira (1999). I use the parameter values in Campbell et al. (2004) to compute a numerical solution in my model.

For the preference parameters, let us consider three cases: a CRRA G/L-A SDU (Case 1) and two Epstein–Zin G/L-A SDUs (Case 2 and 3). In Case 1, the EIS is the reciprocal of the RRA; that is, $\psi = 1/\gamma = 0.1$. In Case 2 and 3, I assume that the investor prefers early resolution of uncertainty: $\gamma = 10$ and $\psi = 1.1$. The boundaries of the subjective discount rates are identical in Case 1 and 2: $\underline{\delta} = 0.05$ and $\overline{\delta} = 0.12$. The upper boundary of the subjective discount rate in Case 3 is the same as the other two cases: $\overline{\delta} = 0.12$, but the lower boundary is smaller: $\underline{\delta} = 0.02$. The time to maturity, T, is 100, so I evaluate a long-run value of g. Table 3 shows the parameters.

Table 3: **Parameter Values in Stochastic Risk Premium**.

Distribution Parameters (by Campbell et al. (2004))

r	σ	κ	$\overline{y}$	σ_y	ρ
0.0082	0.0790	0.0439	0.0213	0.0057	-0.9626

Preference Parameters

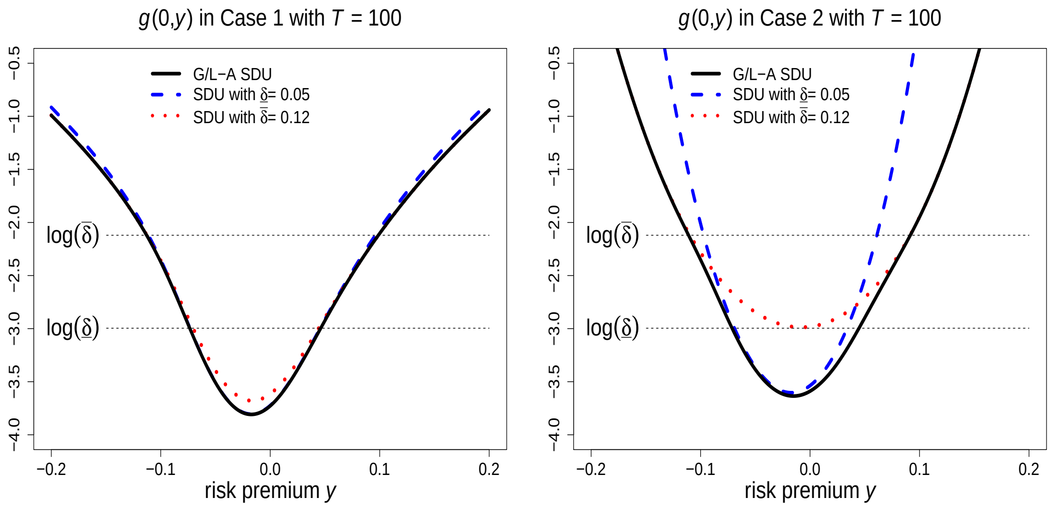
γ	ψ	$\underline{\delta}$	$\overline{\delta}$	T (Time to maturity)
10.0	Case 1: $0.1 = 1/\gamma$	Case 1 and 2: 0.05	0.12	100
	Case 2 and 3: 1.1	Case 3: 0.02		

In the FD method, I set the step size of the time variable to $T/50000 = 0.002$ with a maximum and minimum of y as 0.2 and -0.2, respectively. The mesh size of y is $0.4/1000 = 0.0004$, and the boundary values of g for y are linearly extrapolated.

Figure 1 shows the relationships between y and $g(0, y)$ in Case 1 and 2. In Case 1 (i.e., the CRRA case), the shape of g of the G/L-A SDU on the left figure resembles those of symmetric SDUs. In contrast, the shape of g of the G/L-A SDU in Case 2 (i.e., the Epstein–Zin case) differs from those of symmetric SDUs. If $g < \log \underline{\delta}$, the G/L-A SDU's g is close to the symmetric g with $\underline{\delta}$, whereas it is close to the symmetric g with $\overline{\delta}$ if $g > \log \overline{\delta}$.

In Case 1, the value of ψ is 0.1, which means the investor has a small EIS. This implies that the investor has strong motivation to pursue consumption smoothing, so the intertemporal consumption choice does not matter. Therefore, the subjective discount rate, which determines

Figure 1: g



the relative importance of current and future consumption, is also less important. Accordingly, the value function is not sensitive to the subjective discount rate, and the gain/loss asymmetry has a limited effect on the investor's utility.

In contrast to Case 1, the investor in Case 2 has an Epstein–Zin type utility, which allows both the RRA and the EIS to be high. A high EIS ($\psi = 1.1$) indicates that the investor easily substitutes a decrease (resp. an increase) in his or her current consumption to an increase (resp. a decrease) in future consumption. Therefore, the intertemporal consumption choice of the investor in Case 2 is more important than in Case 1. As a result, the value function in Case 2 is more sensitive to the subjective discount rate than the value function in Case 1, and the gain/loss asymmetry has a larger impact on the investor's utility. This implies that the gain/loss asymmetry is crucial when the EIS is high.

In addition, Figure 1 shows that the utility gain or loss coincides with the market condition. The utility loss, in which g is less than $\log \underline{\delta}$, occurs when the risk premium, y, stays in a region around zero. Since there does not exist any short-sell constraint, the expected return on investment of the risk asset is low when y is close to 0. Accordingly, the investor expects a bad future, and the future utility is less than the current utility; that is, a utility loss occurs. On the other hand, the utility gain, in which g is greater than $\log \overline{\delta}$, occurs when the risk premium stays in a region far from zero. The investor expects to enjoy successful investments in this region and, thus, increases their investment in risky assets including short selling. Therefore, the future utility is larger than the current utility, and a utility gain occurs.

Figure 2: **Optimal Policies α^* and β^* in Case 1 (CRRA case $\psi = 1/\gamma$)**

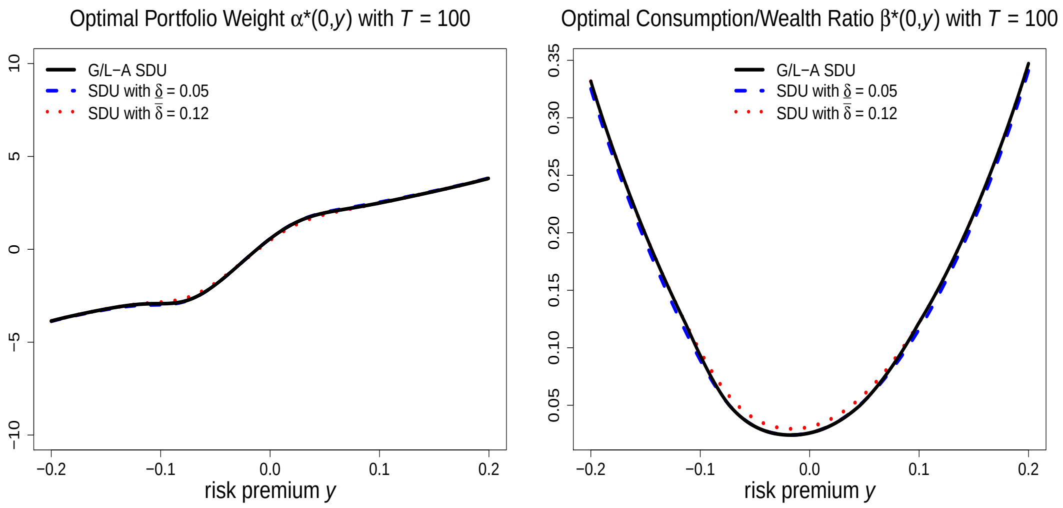


Figure 2 shows the optimal portfolio weight and consumption/wealth ratio in Case 1. As argued above, the optimal policies under gain/loss asymmetry resemble those in the symmetric cases.

Figure 3: **Optimal Policies α^* and β^* in Case 2 (Epstein-Zin case $\psi \neq 1/\gamma$)**

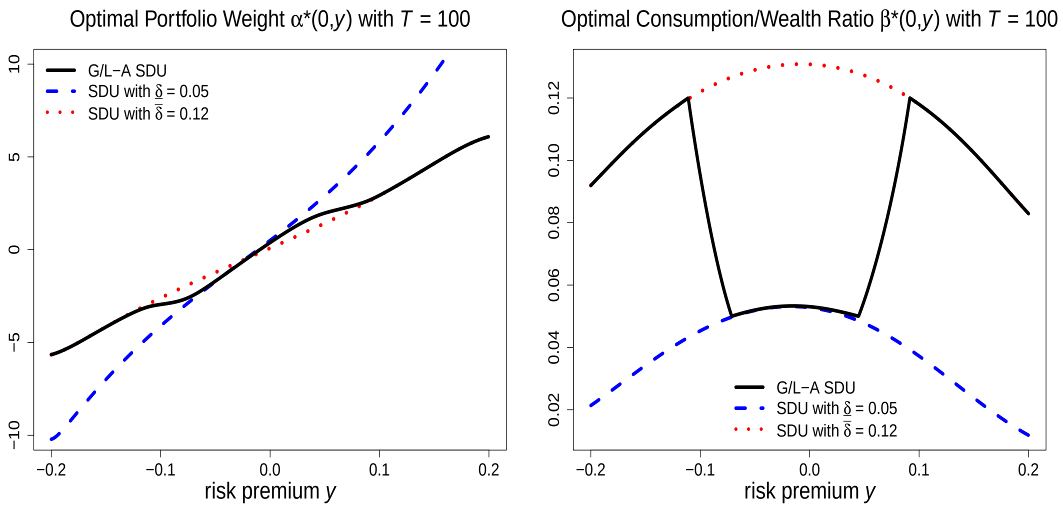


Figure 3 shows the optimal portfolio weight and consumption/wealth ratio in Case 2. In contrast to Figure 2, the optimal policies under gain/loss asymmetry differ from those in the symmetric cases. In the gain/loss asymmetric case, the sensitivity of the optimal portfolio weight to the risk premium, y, changes depending on the value of y, whereas the sensitivity is close to a constant everywhere in the symmetric cases. In the utility loss region, in which $g < \log \underline{\delta}$, the sensitivity is high. In the utility gain region, in which $g > \log \overline{\delta}$, the sensitivity is low. In the utility loss region, the investor's subjective discount rate is low, $\underline{\delta}$, so the future utility is more valuable than the current utility. As a result, the investor significantly increases their (long or short) position of risky assets to increase the future utility even if the risk premium increases or decreases by a small amount. However, in the utility gain region, the relative importance of the future utility decreases because it is more discounted in the utility gain region than it is in the loss region. Therefore, the sensitivity of the investor's position to changes in the risk premium is lower in the gain region than it is in the loss region.

The right figure in Figure 3 shows that the optimal consumption/wealth ratio under the gain/loss asymmetry has a dip in the utility loss region. In the standard theory of the symmetric case, the investor with a high EIS easily substitutes their current consumption for future consumption. When $\psi > 1$, this substitution effect dominates the income effect. Therefore, the optimal consumption/wealth ratio is inversely proportional to the market condition. Considering that the utility loss or gain coincides with the market condition, this dip cannot happen in the symmetric case.

The dip can be explained by changes in the subjective discount rate. In the utility loss region, the investor is satisfied with low consumption because the utility is less discounted. On the other hand, the investor needs large consumption in the utility gain region because of the large subjective discount rate. This third effect decreases the investor's strong motivation to substitute as a reaction to the market condition, and the dip occurs in the consumption/wealth ratio. Note that the third effect by the gain/loss asymmetry occurs even if $\psi < 1$, but its magnitude is small.

The third effect can be interpreted as a preference for spread. The intertemporal substitution of consumption means that the investor increases current consumption as compensation for a decrease in future consumption and vice versa. Therefore, the investor with a strong motivation for an intertemporal substitution of consumption chooses a consumption plan whereby more (resp. less) is consumed currently and less (resp. more) is consumed in the future. On the other hand, a preference for spread means that the investor prefers a consumption plan in which good and bad consumption are evenly diversified over time. As a result, although a bad state in the market can be expected, the investor under the G/L-A SDU does not increase their current consumption but, instead, decreases it in order to diversify the intertemporal consumption stream over time. In this sense, the G/L-A SDU expresses a preference other than habit formation. Habit formation implies that the investor smooths current consumption compared with past consumption (or a consumption level known by current information), but a preference for spread means that the investor diversifies current and future consumption.

Figure 4: **Value Function g and Optimal Policies α^* and β^* in Case 3 (Less Discounting in a Utility Loss $\underline{\delta} = 0.02$)**

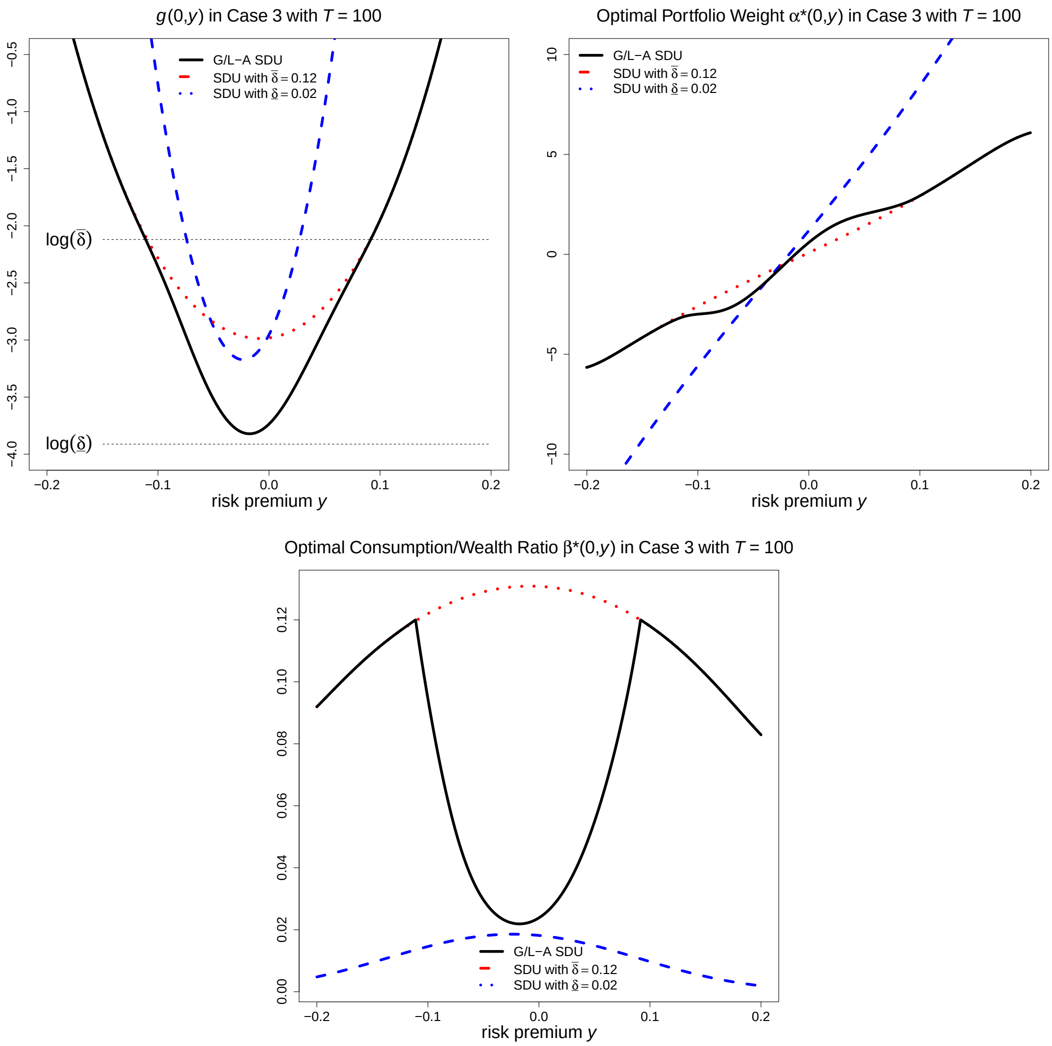


Figure 4 shows graphs of g and optimal policies α^* and β^* in Case 3, in which a utility loss is less discounted ($\underline{\delta} = 0.02$). The graph of g in Figure 4 indicates that g is always larger than $\log \underline{\delta}$, in other words, the investor who maximizes his or her utility does not experience a utility loss in Case 3. This implies that, in Case 3, the marginal value of current consumption in a utility loss region is always strictly smaller than the marginal value of wealth. Therefore, a consumption plan which attains a utility loss is not worth choosing for the investor in Case 3.

Although the investor in Case 3 does not experience a utility loss, his or her optimal policies and value function are different from those in the symmetric case. The optimal portfolio weight is obviously non-linear at risk premium, and the optimal consumption/wealth ratio has a dip. In Case 3, the investor will experience a utility loss when he or she continues increasing his or her consumption. As aforementioned, a large consumption in a utility loss is costly, so the investor chooses a consumption in the boundary between the two regions. On the boundary, a preference for spread affects intertemporal consumption choice more strongly than the substitution effect, so the portfolio weight becomes non-linear and the dip occurs in the consumption/wealth ratio.

Although the non-smooth change in the optimal policies is a key feature of the G/L-A SDU, other preferences may also replicate endogenously. One candidate is model uncertainty; specifically, a max-min expected utility. Chen and Epstein (2002) extend the discrete-time max-min expected utility to the SDU. Some of these max-min SDUs also have a non-smooth term in their HJB equations, for example, the κ-ignorance model in Chen and Epstein (2002). However, non-smoothness appears in the term of the first derivative of the value function.[11] Accordingly, it is difficult to investigate the existence of the dip owing to mathematical complexity although many studies have found that model uncertainty has other rich implications. In this sense, the G/L-A SDU exhibits moderate non-smoothness.

The other candidate is a model of multiple equilibria, such as the sunspot model. A multiple equilibria model necessarily involves an equilibrium selection problem; for example, which equilibrium occurs? On the other hand, the non-smooth change in the G/L-A SDU is caused by the characterization of its preference. Therefore, the G/L-A SDU can avoid the equilibrium selection problem.

Finally, the dip occurs in the utility loss region, which means that the investor decreases

[11]For example, the HJB equation in the κ-ignorance model is

$$-\frac{\partial v}{\partial t} - \max_{(\alpha,C)} \left\{ \mathcal{L}^{\alpha,C} v + \delta\phi'(\phi^{-1}(v))\Big(u(C) - \phi^{-1}(v)\Big) + \kappa|v_w| \right\} = 0,$$

where κ is a non-negative constant.

consumption non-smoothly when the market conditions become bad. Therefore, the G/L-A SDU has a similar consumption path to that of some consumption-based asset pricing models assuming non-smooth decreases in consumption such as in the rare disaster models. Although any sample path of the risk premium, Y, does not have a jump, the optimal consumption path in the G/L-A SDU has non-smooth changes similar to a jump in consumption because of the preference, not because of an assumption of the market model. Therefore, the G/L-A SDU provides a foundation for non-smooth changes in a consumption path from decision theory.

5.2 The I.I.D. Case

Let us consider the case where the risky asset prices follow I.I.D. dynamics. This implies that the risky asset price vector satisfies the following SDE:

$$\mathrm{d}\mathbf{P}_t = \mathrm{diag}(\mathbf{P}_t)\Big(b\mathrm{d}t + \sigma\mathrm{d}\mathbf{B}_t^N\Big)$$

where $b \in \mathbb{R}^N$ and $\sigma \in \mathbb{R}^{N\times N}$ are constants. Additionally, I assume that the risk-free rate, r, is also a constant. Therefore, there is no state variable that affects the asset price dynamics as well as the investor's consumption and investment choice. As a result, the value function does not depend on $\mathbf{y}$.

The reduced HJB equation (4.5) can be expressed as

$$a(g(t)) + \frac{1}{2\gamma}\mu_e^\top \Sigma^{-1}\mu_e + r = -g_t(t), \tag{5.3}$$

with the terminal condition, $g(T) = 0$. μ_e is the excess expected return vector, $\mu_e = b - r\mathbf{1}$, and Σ is a dispersion matrix, $\Sigma = \sigma\sigma^\top$. The optimal portfolio weight vector is constant: $\alpha^* = \Sigma^{-1}\mu_e/\gamma$.

The reduced HJB equation (5.3) has an explicit solution which can be represented as a combination of monotone transformations of three logistic functions. However, I only consider a steady state of g as $T \to \infty$ without derivation of the solution because the steady state has sufficient information in order to study behavior of the value function and optimal policies. The function, $g \to a(g)$, is monotone decreasing and continuous, and the left-hand-side of (5.3) does not contain the time variable, t, without the argument of g. Therefore, there exists a steady value of g as $T \to \infty$ if the left-hand-side of (5.3) can take zero.[12] To confirm this, let us consider the limiting behavior of $a(g)$ as g tends to positive infinity or negative infinity. To compute the limiting behavior, we need to consider two cases depending on the value of ψ, as follows:

$$\lim_{g\to\infty} a(g) = \begin{cases} -\dfrac{\overline{\delta}}{1-1/\psi} < 0, & \text{if } \psi > 1, \\ -\infty, & \text{if } \psi < 1, \end{cases} \qquad \lim_{g\to-\infty} a(g) = \begin{cases} \infty, & \text{if } \psi > 1, \\ -\dfrac{\underline{\delta}}{1-1/\psi} > 0, & \text{if } \psi < 1. \end{cases}$$

Hence, if $\psi < 1$, there exists a steady value of g because the right-hand-side of (5.3) can take a zero due to the monotonicity. On the other hand, if $\psi > 1$, the existence of a steady value of g depends on the values of the parameters; that is, if the inequality

$$-\frac{\overline{\delta}}{1-1/\psi} + \frac{1}{2\gamma}\mu_e^\top \Sigma^{-1}\mu_e + r < 0 \tag{5.4}$$

holds, then there exists a steady value of g. The following proposition summarizes the result of the steady value.

Proposition 5 *Suppose that $\psi < 1$ or $\psi > 1$ and that the inequality (5.4) holds. Then, there*

[12]It can be easily seen that g converges to a steady state as $T \to \infty$ if the left-hand side of (5.3) can take zero. Let us define $\tau = T - t$, and write $\widehat{g}(\tau) = g(T-t)$. Then, $\widehat{g}_\tau = -g_t$, and we can express the HJB equation as follows:

$$a(\widehat{g}(\tau)) + \frac{1}{2\gamma}\mu_e^\top \Sigma^{-1}\mu_e + r = \widehat{g}_\tau(\tau),$$

with $\widehat{g}(0) = 0$. If $\widehat{g}_\tau(0) < 0$, then $\widehat{g}$ decreases as τ increases. Therefore, due to the monotone decreasing property of a, $a(\widehat{g}) + \frac{1}{2\gamma}\mu_e^\top \Sigma^{-1}\mu_e + r$ increases when τ increases. As a result, when $a(\widehat{g}) + \frac{1}{2\gamma}\mu_e^\top \Sigma^{-1}\mu_e + r = 0$, $\widehat{g}$ reaches the steady state. In the same manner, we can prove the case when $\widehat{g}_\tau(0) > 0$.

exists a steady value of g as $T \to \infty$ such that

$$\lim_{T\to\infty} g(t) = \begin{cases} \dfrac{1}{1-\psi}\left(\log\left(\psi\underline{\delta} + (1-\psi)\Big(\dfrac{1}{2\gamma}\mu_e^\top\Sigma^{-1}\mu_e + r\Big)\right) - \psi\log\underline{\delta}\right), \\ \qquad\qquad \text{if } \dfrac{1}{2\gamma}\mu_e^\top\Sigma^{-1}\mu_e + r \le \underline{\delta}, \\ \log\Big(\dfrac{1}{2\gamma}\mu_e^\top\Sigma^{-1}\mu_e + r\Big), \\ \qquad\qquad \text{if } \underline{\delta} \le \dfrac{1}{2\gamma}\mu_e^\top\Sigma^{-1}\mu_e + r \le \overline{\delta}, \\ \dfrac{1}{1-\psi}\left(\log\left(\psi\overline{\delta} + (1-\psi)\Big(\dfrac{1}{2\gamma}\mu_e^\top\Sigma^{-1}\mu_e + r\Big)\right) - \psi\log\overline{\delta}\right), \\ \qquad\qquad \text{if } \dfrac{1}{2\gamma}\mu_e^\top\Sigma^{-1}\mu_e + r \ge \overline{\delta}. \end{cases}$$

The steady optimal consumption/wealth ratio is

$$\lim_{T\to\infty} \beta^*(t) = \begin{cases} \psi\underline{\delta} + (1-\psi)\Big(\dfrac{1}{2\gamma}\mu_e^\top\Sigma^{-1}\mu_e + r\Big), & \text{if } \dfrac{1}{2\gamma}\mu_e^\top\Sigma^{-1}\mu_e + r \le \underline{\delta}, \\ \dfrac{1}{2\gamma}\mu_e^\top\Sigma^{-1}\mu_e + r, & \text{if } \underline{\delta} \le \dfrac{1}{2\gamma}\mu_e^\top\Sigma^{-1}\mu_e + r \le \overline{\delta}, \\ \psi\overline{\delta} + (1-\psi)\Big(\dfrac{1}{2\gamma}\mu_e^\top\Sigma^{-1}\mu_e + r\Big), & \text{if } \dfrac{1}{2\gamma}\mu_e^\top\Sigma^{-1}\mu_e + r \ge \overline{\delta}. \end{cases}$$

Proposition 5 implies that a change in the subjective discount rate cannot happen in the I.I.D case until the time variable t becomes sufficiently close to the maturity T. Therefore, the investor keeps the utility gain or loss in the long term. The reason for this is the homotheticity of the Epstein–Zin G/L-A SDU. In the I.I.D. case, there are two variables as arguments of the value function: wealth w and time t. However, because of the homotheticity, the value function is expressed as $(w\exp\{g(t)\})^{1-\gamma}/(1-\gamma)$. On the other hand, the current utility is expressed as $(w\beta)^{1-\gamma}/(1-\gamma)$, and the consumption/wealth ratio, β, is a control variable. Hence, the investor can control a current status of utility, e.g., a utility gain or loss, independently of a value of the wealth variable. Therefore, the wealth variable has no influence on whether the optimal consumption plan causes a utility gain or loss. In other words, the extra state variable, $\mathbf{Y}$, which affects the asset price dynamics, causes an effective change in the subjective discount rate.

6 Concluding Remarks

This study examines the gain/loss asymmetric SDU (G/L-A SDU) by extending the recursive preference representation in discrete time of Wakai (2008, 2010). The investor who has the G/L-A SDU discounts the utility gain, in which the current utility is smaller than the future utility, by more than the utility loss for which the current utility is larger than the future utility. By employing the theory of the SDU, the G/L-A SDU can express various preferences including a version of the Epstein–Zin preference. Furthermore, the G/L-A SDU has different properties to those of the standard SDU. In particular, when the risk premium of the risky asset is time-varying and uncertain and, when the EIS is high, this difference is clearer. The sensitivity of the optimal portfolio weight to changes in the risk premium varies depending on the value of the risk premium. The optimal consumption/wealth ratio has a dip, which occurs when the market conditions are bad; that is, the risk premium is close to zero.

There are many possibilities in terms of extending the G/L-A SDU. A particularly interesting extension is a partial equilibrium asset pricing model. In the rare disaster models, sudden consumption shocks affect the asset price dynamics in equilibrium. These shocks can be regarded as shocks in production processes through equilibrium in a good market, so they are exogenous in the model. However, the G/L-A SDU can generate non-smooth changes, even though the market return does not have a jump. Therefore, the consumption shocks in the rare disaster models may be reinterpreted in terms of the dip in the G/L-A SDU. Additionally, the G/L-A SDU exhibits time-varying sensitivity of the optimal portfolio weight to changes in the stochastic risk premium. Taking into account feedback in market equilibrium, this may cause non-smooth changes in the risk premium as in state change models such as the Markov regime-switching model of Hamilton (1989). Thus, the G/L-A SDU may provide an explanation for these state change models from the viewpoint of economic theory. This study, therefore, provides a tractable tool to examine the above phenomena in finance from the perspective of an investor's gain/loss asymmetric behavior.

A Proofs

Proof of Proposition 2. We fix an arbitrary $C \in \mathcal{C}[0,T]$. Note that U is left continuous and progressively measurable by the left-continuity and progressive measurability of C. We define by δ^* a discount rate process as follows:

$$\widehat{\delta}_t = \overline{\delta}\mathbb{1}_{\{U_t - u(C_t) \geq 0\}} + \underline{\delta}\mathbb{1}_{\{U_t - u(C_t) < 0\}},$$
$$\delta^*_t = \begin{cases} \lim\limits_{s \uparrow t} \widehat{\delta}_s, & \text{if } t \in (0,T], \\ \widehat{\delta}_0, & \text{if } t = 0. \end{cases}$$

Obviously, δ^* is in $\Delta[0,T;\underline{\delta},\overline{\delta}]$. Now, we consider a new BSDE with respect to $(\widehat{U}_t, \widehat{\mathbf{Z}}_t)_{t\in[0,T]}$ as follows:

$$\begin{cases} -\mathrm{d}\widehat{U}_t = \delta^*_t(u(C_t) - U_t)\mathrm{d}t - \widehat{\mathbf{Z}}_t^\top \mathrm{d}\mathbf{B}_t, \\ \widehat{U}_T = \overline{u}(C_T). \end{cases}$$

Since the above BSDE meets the standard conditions, there exists a unique solution. It can be easily seen that

$$\operatorname*{essinf}_{\delta\in[\underline{\delta},\overline{\delta}]} \{\delta(u(C_t) - U_t)\} = \delta^*_t(u(C_t) - U_t), \quad \mathrm{d}\mathbb{P}\otimes\mathrm{d}t\text{-a.e.}$$

Therefore, by the comparison theorem, we have $U_t = \widehat{U}_t$ $\mathbb{P}$-a.s. for all $t \in [0,T]$. Hence,

$$\operatorname*{essinf}_{\delta\in[\underline{\delta},\overline{\delta}]} \left\{\delta(u(C_t) - \widehat{U}_t)\right\} = \delta^*_t(u(C_t) - U_t) = \delta^*_t(u(C_t) - \widehat{U}_t), \quad \mathrm{d}\mathbb{P}\otimes\mathrm{d}t\text{-a.e.} \tag{A.1}$$

This implies that $\widehat{U}$ is also a solution to the following BSDE:

$$\begin{cases} -\mathrm{d}\widehat{U}_t = \delta^*_t(u(C_t) - \widehat{U}_t)\mathrm{d}t - \widehat{\mathbf{Z}}_t^\top \mathrm{d}\mathbf{B}_t, \\ \widehat{U}_T = \overline{u}(C_T). \end{cases} \tag{A.2}$$

On the other hand, for any $\delta \in \Delta[0,T;\underline{\delta},\overline{\delta}]$ and $C \in \mathcal{C}[0,T]$, I consider the following BSDE with respect to $(U_t^\delta, \mathbf{Z}_t^\delta)$:

$$\begin{cases} -\mathrm{d}U_t^\delta = \delta_t(u(C_t) - U_t^\delta)\mathrm{d}t - (\mathbf{Z}_t^\delta)^\top \mathrm{d}\mathbf{B}_t, \\ U_T^\delta = \overline{u}(C_T). \end{cases} \tag{A.3}$$

Since the generator of the BSDE (A.3) is linear in U^δ and any $\delta \in \Delta[0,T;\underline{\delta},\overline{\delta}]$ is uniformly bounded, from Proposition 2.2 in El Karoui et al. (1997), its solution can be expressed as

$$U_t^\delta = \mathrm{E}\left[\int_t^T \delta_s e^{-\int_t^s \delta_r \mathrm{d}r} u(C_s)\mathrm{d}s + e^{-\int_t^T \delta_r \mathrm{d}r}\overline{u}(C_T) \,\middle|\, \mathcal{F}_t^{\mathbf{B}}\right], \quad \mathbb{P}\text{-a.s.},$$

for all $t \in [0,T]$.

Since $\delta^* \in \Delta[0,T;\underline{\delta},\overline{\delta}]$ and using the equality (A.1) and BSDEs (A.2) and (A.3), from Proposition 3.1 in El Karoui et al. (1997), we obtain

$$U_t = \widehat{U}_t = \operatorname*{essinf}_{\delta\in\Delta[0,T;\underline{\delta},\overline{\delta}]} U_t^\delta, \quad \mathbb{P}\text{-a.s.},$$

for all $t \in [0,T]$. Hence, we conclude that

$$U_t = \operatorname*{essinf}_{\delta\in\Delta[0,T;\underline{\delta},\overline{\delta}]} \mathrm{E}\left[\int_t^T \delta_s e^{-\int_t^s \delta_r \mathrm{d}r} u(C_s)\mathrm{d}s + e^{-\int_t^T \delta_r \mathrm{d}r}\overline{u}(C_T) \,\middle|\, \mathcal{F}_t^{\mathbf{B}}\right],$$

for all $t \in [0,T]$. □

Proof of Proposition 4. Since $v_w \geq 0$ and u is concave, $C(\delta, v)$ is an increasing function of δ for any fixed v. This implies that there does not exist v that satisfies both of $u(C(\underline{\delta}, v)) - \phi^{-1}(v) > 0$ and $u(C(\overline{\delta}, v)) - \phi^{-1}(v) < 0$.

If $u(C(\underline{\delta}, v)) - \phi^{-1}(v) > 0$, then the investor experiences a utility loss at $C(\underline{\delta}, v)$, so the discount rate is $\underline{\delta}$. This implies that the optimal consumption is $C^* = C(\underline{\delta}, v)$ in the region in which a consumption C satisfies $u(C) - \phi^{-1}(v) > 0$. In the region in which a consumption satisfies $u(C) - \phi^{-1}(v) \leq 0$, the discount rate is $\overline{\delta}$. Hence, for any C being $u(C) - \phi^{-1}(v) < 0$, the first-order derivative of the objective function of the consumption problem is

$$\begin{aligned}
\overline{\delta}\phi'(\phi^{-1}(v))u'(C) - v_w &= \phi'(\phi^{-1}(v))\left(\overline{\delta}u'(C) - \frac{\mathrm{d}\phi^{-1}(v)}{\mathrm{d}w}\right) \\
&\geq \phi'(\phi^{-1}(v))\left(\underline{\delta}u'(C) - \frac{\mathrm{d}\phi^{-1}(v)}{\mathrm{d}w}\right) \\
&\geq \phi'(\phi^{-1}(v))\left(\underline{\delta}u'(C(\underline{\delta}, v)) - \frac{\mathrm{d}\phi^{-1}(v)}{\mathrm{d}w}\right) = 0,
\end{aligned}$$

in which I used the positivity and decreasing monotonicity of u'. This inequality implies that the optimal consumption on $u(C) - \phi^{-1}(v) \leq 0$ satisfies $u(C^*) = \phi^{-1}(v)$, i.e., $C^* = u^{-1}(\phi^{-1}(v))$. However, since the objective function of the consumption problem is continuous with respect to C, the value of the objective function at $C = C(\underline{\delta}, v)$ is not smaller than at $C = u^{-1}(\phi^{-1}(v))$. Therefore, if $u(C(\underline{\delta}, v)) - \phi^{-1}(v) > 0$, then the optimal consumption is $C(\underline{\delta}, v)$. In the same manner, we can prove that the optimal consumption is $C(\overline{\delta}, v)$ if $u(C(\overline{\delta}, v)) - \phi^{-1}(v) < 0$.

We now consider the case when $u(C(\underline{\delta}, v)) - \phi^{-1}(v) < 0$ and $u(C(\overline{\delta}, v)) - \phi^{-1}(v) > 0$. Since u is increasing and $u(C(\overline{\delta}, v)) - \phi^{-1}(v) \geq 0$, $C(\overline{\delta}, v)$ is not smaller than any C being $u(C) - \phi^{-1}(v) < 0$. The first-order derivative of the objective function of the consumption problem in the region where $u(C) - \phi^{-1}(v) < 0$ is

$$\begin{aligned}
\overline{\delta}\phi'(\phi^{-1}(v))u'(C) - v_w &= \phi'(\phi^{-1}(v))\left(\overline{\delta}u'(C) - \frac{\mathrm{d}\phi^{-1}(v)}{\mathrm{d}w}\right) \\
&\geq \phi'(\phi^{-1}(v))\left(\overline{\delta}u'(C(\overline{\delta}, v)) - \frac{\mathrm{d}\phi^{-1}(v)}{\mathrm{d}w}\right) = 0,
\end{aligned}$$

in which I have used the decreasing monotonicity of u' and $C \leq C(\overline{\delta}, v)$ in this region. Similarly, in the region where $u(C) - \phi^{-1}(v) > 0$, the first-order derivative of the objective function is

$$\underline{\delta}\phi'(\phi^{-1}(v))u'(C) - v_w \leq 0.$$

Therefore, by the continuity of the objective function, the optimal consumption satisfies $u(C^*) = \phi^{-1}(v)$ in this case. $\square$

B Equivalence in Continuous Analogs of Different Discrete-Time Equations

In this section, I discuss the equivalence in continuous analogs of (2.3) and (2.4). The stochastic recursive equation (2.4) is shown again:

$$U_t(\{c_\tau\}_{\tau\geq t}) = \operatorname*{essinf}_{\delta\in[\underline{\delta}_{t+1},\overline{\delta}_{t+1}]} \phi\left((1-\delta)u(c_t)+\delta\phi^{-1}\Big(\mathrm{E}_t[U_{t+1}(\{c_\tau\}_{\tau\geq t+1})]\Big)\right).$$

Here, I suppose that the boundary processes, $\{\underline{\delta}_\tau\}_{\tau\geq 1}$ and $\{\overline{\delta}_\tau\}_{\tau\geq 1}$, are constant over time, and omit the consumption sequences. Equation (2.4) can be transformed to the following time-interval-dependent form:

$$U_t = \phi\left(\operatorname*{essinf}_{\delta\in[\underline{\delta},\overline{\delta}]}\left\{(1-e^{-\delta\Delta t})u(c_t)+e^{-\delta\Delta t}\phi^{-1}\Big(\mathrm{E}_t[U_{t+\Delta t}]\Big)\right\}\right),$$

where I have used the increasing monotonicity of ϕ. Then, the differential quotient is

$$\frac{\mathrm{E}_t[U_{t+\Delta t}]-U_t}{\Delta t} = -\phi'(R_{t+\Delta t})\operatorname*{essinf}_{\delta\in[\underline{\delta},\overline{\delta}]}\left\{\frac{1-e^{-\delta\Delta t}}{\Delta t}\Big(u(c_t)-\phi^{-1}\Big(\mathrm{E}_t[U_{t+\Delta t}]\Big)\Big)\right\},$$

where $R_{t+\Delta t}$ is a random variable taking values between $\phi^{-1}\Big(\mathrm{E}_t[U_{t+\Delta t}]\Big)$ and $\operatorname*{essinf}_{\delta\in[\underline{\delta},\overline{\delta}]}\left\{(1-e^{-\delta\Delta t})u(c_t)+e^{-\delta\Delta t}\phi^{-1}\Big(\mathrm{E}_t[U_{t+\Delta t}]\Big)\right\}$, and I have used the mean value theorem. I further suppose that $\lim_{\Delta t\downarrow 0}\mathrm{E}_t[U_{t+\Delta t}]=U_t$ and we can exchange the order of essinf and lim. Then, we have

$$\lim_{\Delta t\downarrow 0}\frac{\mathrm{E}_t[U_{t+\Delta t}]-U_t}{\Delta t} = -\operatorname*{essinf}_{\delta\in[\underline{\delta},\overline{\delta}]}\Big\{\delta\phi'(\phi^{-1}(U_t))\Big(u(c_t)-\phi^{-1}(U_t)\Big)\Big\},$$

where I have used the fact that ϕ' is positive because of the increasing monotonicity of ϕ. Therefore, the two equations (2.3) and (2.4) have the same instantaneous growth rate of the utility. This implies that they also have the same stochastic differential utility representation.

C Properties of G/L-A SDU

In this section, I explore characteristics of G/L-A SDU as preferences. First, the individual G/L-A SDUs have different mean-reverting speeds by their gain/loss asymmetries. To observe this clearly, consider a G/L-A SDU under a constant consumption process. Let $U^{u,\phi,\underline{\delta},\overline{\delta}}(C)$ be a G/L-A SDU with $(u, \phi, \underline{\delta}, \overline{\delta})$ under $C \in \mathcal{C}[0,T]$ as a consumption process. For any given constant $\overline{c} \in \mathcal{C}$, let $\overline{C}^{\overline{c}} \in \mathcal{C}[0,T]$ be a consumption process with $\overline{C}^{\overline{c}}_t = \overline{c}$ for all $t \in [0,T]$. Let us call $\overline{C}^{\overline{c}}$ a constant consumption process with $\overline{c}$. Here, for any $(u, \phi, \underline{\delta}, \overline{\delta})$, $U^{u,\phi,\underline{\delta},\overline{\delta}}(\overline{C}^{\overline{c}})$ does not have any random component because the consumption process $\overline{C}^{\overline{c}}$ is not random. Therefore, the SDU equation of $U^{u,\phi,\underline{\delta},\overline{\delta}}(\overline{C}^{\overline{c}})$ is represented as

$$
\begin{aligned}
-\mathrm{d}U^{u,\phi,\underline{\delta},\overline{\delta}}_t(\overline{C}^{\overline{c}}) &= F(\overline{C}^{\overline{c}}_t, U^{u,\phi,\underline{\delta},\overline{\delta}}_t(\overline{C}^{\overline{c}}); u, \phi, \underline{\delta}, \overline{\delta})\mathrm{d}t \\
&= \underset{\delta\in[\underline{\delta},\overline{\delta}]}{\operatorname{essinf}} \{\delta\phi'(\phi^{-1}(U^{u,\phi,\underline{\delta},\overline{\delta}}_t(\overline{C}^{\overline{c}})))\Big(u(\overline{c}) - \phi^{-1}(U^{u,\phi,\underline{\delta},\overline{\delta}}_t(\overline{C}^{\overline{c}}))\Big)\}\mathrm{d}t,
\end{aligned}
$$

and $U^{u,\phi,\underline{\delta},\overline{\delta}}_T(\overline{C}^{\overline{c}}) = \overline{u}(\overline{C}^{\overline{c}}_T) = \overline{u}(\overline{c})$. This solution is obvious, which is

$$
U^{u,\phi,\underline{\delta},\overline{\delta}}_t(\overline{C}^{\overline{c}}) = \begin{cases}
\phi\Big((1-e^{-\overline{\delta}(T-t)})u(\overline{c}) + e^{-\overline{\delta}(T-t)}\phi^{-1}(\overline{u}(\overline{c}))\Big), & \text{if } \phi(u(\overline{c})) < \overline{u}(\overline{c}), \\
u(\overline{c}), & \text{if } \phi(u(\overline{c})) = \overline{u}(\overline{c}), \\
\phi\Big((1-e^{-\underline{\delta}(T-t)})u(\overline{c}) + e^{-\underline{\delta}(T-t)}\phi^{-1}(\overline{u}(\overline{c}))\Big), & \text{if } \phi(u(\overline{c})) > \overline{u}(\overline{c}),
\end{cases}
$$

for all $t \in [0,T]$. If $\phi(u(\overline{c})) < \overline{u}(\overline{c})$, then the current utility is greatly discounted, which means it converges to $\overline{u}(\overline{c})$ quickly. On the other hand, the current utility is less discounted when $\phi(u(\overline{c})) > \overline{u}(\overline{c})$, in which case it converges to $\overline{u}(\overline{c})$ slowly.

Second, the gain/loss asymmetry affects the risk attitude. Let us consider G/L-A SDUs with different degrees of gain/loss asymmetry. Following Duffie and Epstein (1992b), let us define the comparative risk-averse property as follows.

Definition 6 (Comparative Risk Aversion) *A preference represented by a G/L-A SDU $U^{u^*,\phi^*,\underline{\delta}^*,\overline{\delta}^*}$ is more risk averse than a preference represented by another G/L-A SDU $U^{u,\phi,\underline{\delta},\overline{\delta}}$ if for any constant consumption process $\overline{C}^{\overline{c}}$ and consumption process $C \in \mathcal{C}[0,T]$, $U^{u,\phi,\underline{\delta},\overline{\delta}}_t(\overline{C}^{\overline{c}}) \geq U^{u,\phi,\underline{\delta},\overline{\delta}}_t(C)$ implies $U^{u^*,\phi^*,\underline{\delta}^*,\overline{\delta}^*}_t(\overline{C}^{\overline{c}}) \geq U^{u^*,\phi^*,\underline{\delta}^*,\overline{\delta}^*}_t(C)$.*

Based on the definition of comparative risk aversion, I implicitly assume that G/L-A SDUs with an identical constant consumption process take the same value for them to be comparable.[13] In Proposition 7 and Corollary 8, I assume two utilities take the same value under an identical constant consumption process. The following proposition shows that large gain/loss asymmetry strengthens the degree of risk aversion in the sense of comparative risk aversion.

Proposition 7 *For any given u and ϕ, $U^{u,\phi,\underline{\delta}^*,\overline{\delta}^*}$ is more risk averse than $U^{u,\phi,\underline{\delta},\overline{\delta}}$ if $[\underline{\delta},\overline{\delta}] \subseteq [\underline{\delta}^*,\overline{\delta}^*]$ and $F(c,v;u,\phi,\underline{\delta},\overline{\delta})$ is uniformly Lipschitz with respect to v.*

The Lipschitz condition for F is needed when applying the comparison theorem for BSDEs. The comparison theorem for BSDEs states that an SDU is no less than another SDU, almost surely, if the generator and bequest utility of the former are also almost surely no less than the generator and bequest utility of the latter (see Theorem 2.2 in El Karoui et al. (1997)). If the comparison theorem holds without the Lipschitz condition, it can be omitted.

Proof of Proposition 7. Let $\overline{C}^{\overline{c}}$ be a constant consumption process with $\overline{c} \in \mathcal{C}$, and let C be an arbitrary consumption process. Furthermore, suppose $U_t^{u,\phi,\underline{\delta},\overline{\delta}}(C) \leq U_t^{u,\phi,\underline{\delta},\overline{\delta}}(\overline{C}^{\overline{c}})$, for all $t \in [0,T]$. The condition $[\underline{\delta},\overline{\delta}] \subseteq [\underline{\delta}^*,\overline{\delta}^*]$ implies that the generator of $U^{u,\phi,\underline{\delta},\overline{\delta}}$ is, almost surely, no less than that of $U^{u,\phi,\underline{\delta}^*,\overline{\delta}^*}$. Therefore, the comparison theorem for BSDEs gives $U_t^{u,\phi,\underline{\delta},\overline{\delta}}(C) \geq U_t^{u,\phi,\underline{\delta}^*,\overline{\delta}^*}(C)$ for all $t \in [0,T]$. Thus,

$$U_t^{u,\phi,\underline{\delta}^*,\overline{\delta}^*}(\overline{C}^{\overline{c}}) = U_t^{u,\phi,\underline{\delta},\overline{\delta}}(\overline{C}^{\overline{c}}) \geq U_t^{u,\phi,\underline{\delta},\overline{\delta}}(C) \geq U_t^{u,\phi,\underline{\delta}^*,\overline{\delta}^*}(C).$$

Hence, we obtain the desired result. □

In general, the Epstein–Zin G/L-A SDU does not satisfy the Lipschitz condition for their generators. This non-Lipschitz property of the Epstein–Zin type of SDUs is their key feature, but it is mathematically complex.

The condition for the comparative risk aversion of Duffie and Epstein (1992b) can also be applied to G/L-A SDUs. The following corollary briefly summarizes the comparative risk aversion results for changes in ϕ and u.

Corollary 8

[13] Duffie and Epstein (1992b) and Chen and Epstein (2002) also refer to this implicit assumption.

1. *Suppose that two generators, F and $\widehat{F}$, and a continuously twice-differentiable, monotone increasing, and concave function, h, satisfy*

$$\widehat{F}(c, h(v); u, \phi, \underline{\delta}, \overline{\delta}) = h'(v) F(c, v; u, \phi, \underline{\delta}, \overline{\delta}),$$

   for all c and v. Then, a G/L-A SDU with generator $\widehat{F}$ is more risk averse than a G/L-A SDU with generator F.

2. *For any given constants $\gamma, \gamma^*, \psi, \underline{\delta}$, and $\overline{\delta}$, an Epstein–Zin G/L-A SDU with $(\gamma^*, \psi, \underline{\delta}, \overline{\delta})$ is more risk averse than another Epstein–Zin G/L-A SDU with $(\gamma, \psi, \underline{\delta}, \overline{\delta})$ if $\gamma^* \geq \gamma$.*

3. *For any given constants $\gamma, \gamma^*, \underline{\delta}$, and $\overline{\delta}$, a CRRA G/L-A SDU with $(\gamma^*, \underline{\delta}, \overline{\delta})$ is more risk averse than another CRRA G/L-A SDU with $(\gamma, \underline{\delta}, \overline{\delta})$ if $\gamma^* \geq \gamma$.*

Finally, under uncertainty, the gain/loss asymmetry strengthens absolute risk aversion. Suppose u is a linear function and ϕ is an identity function. Let C be an arbitrary consumption process, and define $EC := (\mathrm{E}[C_t])_{t\in[0,T]}$. Then, in the absence of gain/loss asymmetry (i.e., $\underline{\delta} = \overline{\delta}$), we have

$$U_t(C) = U_t(EC),$$

for all $t \in [0, T]$. Thus, the decision maker is risk neutral in this case. However, if $\underline{\delta} < \overline{\delta}$,

$$U_t(C) < U_t(EC),$$

for all $t \in [0, T]$ if $C_t \neq EC_t$ in a positive mass of $\mathrm{d}\mathbb{P} \otimes \mathrm{d}t$. In this case, the decision maker is risk averse. Although the instant utility u is linear, a G/L-A SDU exhibits a preference for risk aversion.

D Verification of the G/L-A SDU

In a general setting, including the Epstein–Zin G/L-A SDU, it is not obvious that a solution to the HJB equation (4.3) is the value function because the Lipschitz property of a generator does not necessarily hold. However, Kraft et al. (2013) show that the solution is the value function under some condition, which they call condition L. Their result can be applied to the HJB equation (4.3) as well.

Proposition 9 (Verification Theorem) *Suppose the following.*

1. *There exists a classical solution to the HJB equation (4.3) denoted by v;*
2. ***Condition L**: There exists a positive constant k such that*

$$\phi'(\phi^{-1}(v'))\Big(u(c)-\phi^{-1}(v')\Big)-\phi'(\phi^{-1}(v''))\Big(u(c)-\phi^{-1}(v'')\Big)\le k(v'-v''),$$

   for any $v', v''\in\mathbb{R}$ with $v'\ge v''$ and $c\in\mathcal{C}$;

3. *For any $(t,\mathbf{x})\in[0,T)\times\mathcal{X}$ and $(\alpha,C)\in\mathcal{A}(t,\mathbf{x})$, the local martingale*

$$M_s^{t,\mathbf{x};\alpha,C}:=\int_t^s\left(\frac{\partial v(r,X_r^{t,\mathbf{x};\alpha,C})}{\partial\mathbf{x}}\right)^\top\begin{pmatrix}W_r^{t,\mathbf{x};\alpha,C}\alpha_r^\top\sigma(\mathbf{Y}_r^{t,\mathbf{x}})\mathrm{d}\mathbf{B}_r^N\\ \sigma_{\mathbf{Y}}(\mathbf{Y}_r^{t,\mathbf{x}})\mathrm{d}\mathbf{B}_r\end{pmatrix},\quad s\in[t,T],$$

   is a true martingale.

Then, $v\ge V$ for all $(t,\mathbf{x})\in[0,T]\times\mathcal{X}$. Furthermore, in addition to the above conditions, there exists an admissible feedback control [14] *(α^*,C^*) such that*

$$\begin{aligned}&\mathcal{L}^{\alpha^*(t,\mathbf{x}),C^*(t,\mathbf{x})}v(t,\mathbf{x})+\min_{\delta\in[\underline{\delta},\bar{\delta}]}\Big\{\delta\phi'(\phi^{-1}(v(t,\mathbf{x})))\Big(u(C^*(t,\mathbf{x}))-\phi^{-1}(v(t,\mathbf{x}))\Big)\Big\}\\&\qquad=\max_{(\alpha,C)}\left\{\mathcal{L}^{\alpha,C}v(t,\mathbf{x})+\min_{\delta\in[\underline{\delta},\bar{\delta}]}\Big\{\delta\phi'(\phi^{-1}(v(t,\mathbf{x})))\Big(u(C)-\phi^{-1}(v(t,\mathbf{x}))\Big)\Big\}\right\},\end{aligned}$$

for all $(t,\mathbf{x})\in[0,T)\times\mathcal{X}$. Then, $v=V$ for all $(t,\mathbf{x})\in[0,T]\times\mathcal{X}$.

Proof of Proposition 9. If F also satisfies condition L, Theorem 3.1 in Kraft et al. (2013) can be applied to the G/L-A SDU, in which case the verification is complete. For any $v',v''\in\mathbb{R}$

[14] By Kraft et al. (2013), a function $(\alpha,C):[0,T]\times\mathcal{X}\to\mathbb{R}^n\times\mathcal{C}$ is an admissible feedback control if, for any $\mathbf{x}=(w,\mathbf{y})\in\mathbb{R}_+\times\mathbb{R}^M$, the following system of SDEs of the state variables,

$$\begin{cases}\mathrm{d}W_t=\big(W_t(r(\mathbf{Y}_t)+\alpha(t,\mathbf{X}_t)^\top\mu_e(\mathbf{Y}_t))-C(t,\mathbf{X}_t)\big)\,\mathrm{d}t+W_t\alpha(t,\mathbf{X}_t)^\top\sigma(\mathbf{Y}_t)\mathrm{d}\mathbf{B}_t^N,\quad W_0=w\\ \mathrm{d}\mathbf{Y}_t=b_{\mathbf{Y}}(\mathbf{Y}_t)\mathrm{d}t+\sigma_{\mathbf{Y}}(\mathbf{Y}_t)\mathrm{d}\mathbf{B}_t,\quad \mathbf{Y}_0=\mathbf{y}\end{cases}$$

has a unique solution $\mathbf{X}^{\mathbf{x}}=(W^{w,\mathbf{y}},\mathbf{Y}^{\mathbf{y}})$ such that $(\alpha(t,\mathbf{X}_t^{\mathbf{x}}),C(t,\mathbf{X}_t^{\mathbf{x}}))_{t\in[0,T]}\in\mathcal{A}(0,\mathbf{x})$.

with $v' \geq v''$ and $c \in \mathcal{C}$, we have

$$\begin{aligned}
&F(c,v') - F(c,v'') \\
&= \min_{\delta\in[\underline{\delta},\overline{\delta}]} \left\{\delta\phi'(\phi^{-1}(v'))\Big(u(c) - \phi^{-1}(v')\Big)\right\} - \min_{\delta\in[\underline{\delta},\overline{\delta}]} \left\{\delta\phi'(\phi^{-1}(v''))\Big(u(c) - \phi^{-1}(v'')\Big)\right\} \\
&\leq \max_{\delta\in[\underline{\delta},\overline{\delta}]} \left\{\delta\Big(\phi'(\phi^{-1}(v'))\Big(u(c) - \phi^{-1}(v')\Big) - \phi'(\phi^{-1}(v''))\Big(u(c) - \phi^{-1}(v'')\Big)\Big)\right\} \\
&\leq k \max_{\delta\in[\underline{\delta},\overline{\delta}]} \left\{\delta(v' - v'')\right\} = k\overline{\delta}(v' - v'').
\end{aligned}$$

Therefore, F also satisfies condition L if a function, $(c,v) \to \phi'(\phi^{-1}(v))\Big(u(c) - \phi^{-1}(v)\Big)$ satisfies the condition. $\square$

Condition L is also a sufficient condition of the existence of gain/loss-asymmetric SDUs (See Pardoux (1999)). It is easily seen that the standard G/L-A SDU satisfies condition L. Here, let us consider the Epstein–Zin G/L-A SDU, in which the functional form of $\phi'(\phi^{-1}(v))(u(c) - \phi^{-1}(v))$ is

$$\phi'(\phi^{-1}(v))\Big(u(c) - \phi^{-1}(v)\Big) = \frac{(1-\gamma)v}{1-1/\psi}\left(\frac{c^{1-1/\psi}}{((1-\gamma)v)^{\frac{1-1/\psi}{1-\gamma}}} - 1\right),$$

where γ is the coefficient of relative risk aversion, and ψ is the elasticity of intertemporal substitution. Then, by Proposition 3.2 in Kraft et al. (2013), condition L holds in the following four cases:

1. $\gamma > 1$ and $\psi > 1$ (early resolution of uncertainty),
2. $\gamma > 1$ and $\psi < 1$ with $\gamma\psi \leq 1$ (late resolution of uncertainty),
3. $\gamma < 1$ and $\psi < 1$ (late resolution of uncertainty),
4. $\gamma < 1$ and $\psi > 1$ with $\gamma\psi \geq 1$ (early resolution of uncertainty).

Many empirical studies show that the coefficient of RRA, γ, is more than one. Therefore, cases 1 and 2 are more interesting economically. As discussed in Kraft et al. (2013), the empirical values of the EIS given in the literature are mixed. Case 1 coincides with the long-run risk model of Bansal and Yaron (2004), in which the authors empirically identify the EIS as being more than one. On the other hand, many traditional studies find that the EIS is less than one, corresponding to case 2. The EIS is also an important parameter in the Epstein–Zin G/L-A SDU as I demonstrate in an optimal consumption and investment plan in this paper.

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