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Degree-K subgame perfect Nash equilibria and the folk theorem*

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Abstract

In infinitely repeated n-player games, we introduce a notion of degree-K subgame perfect Nash equilibria, in which any set of players whose size is up to K can coalitionally deviate and can transfer their payoffs within the coalition. If we only assume that players' actions are observable, a coalitional deviation with hidden deviators who play as in the equilibrium cannot be detected by the other players. Hence we consider two models in which the hidden deviators can and cannot be detected, respectively. In the first model, there is an observer who can detect any coalitional deviation and report it to all players. We show an extension of the standard folk theorem; all feasible payoff vectors in which the sum of payoffs within any feasible coalition is strictly larger than the counterpart of the minmax value defined for the coalition arise as a degree-K subgame perfect Nash equilibrium if players are sufficiently patient. In the second model where the hidden deviators cannot be distinguished, we characterize degree-K subgame perfect equilibrium payoff vectors under patience by strategies which punish all players after any deviation. Finally, we adopt a new approach to characterize degree-n subgame perfect Nash equilibrium payoff vectors in the first model, since the punishment in the above folk theorem does not work when the grand coalition is feasible.

Keywords: Folk theorem; Coalition; Perfect monitoring; Subgame perfect Nash equilibrium

JEL Classification: C72; C73; D43

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1 Introduction

The folk theorem which is well known in repeated game theory, states that in an infinitely repeated game with perfect monitoring any feasible and individually rational payoff vector arises as a subgame perfect Nash equilibrium with sufficiently small discounting. And the subgame perfect Nash equilibrium states that no individual could improve his payoff by any deviation in any subgame of an infinitely repeated game.

This paper considers a notion of equilibrium in which coalitional deviations are feasible in infinitely repeated games. More specifically, in the equilibrium no coalition up to some size could improve its payoff by any deviation in any subgame. Futher, this paper analyzes the equilibrium payoff vectors with sufficiently small discounting.

Sometimes players can arrange mutually beneficial deviations in a standard subgame perfect Nash equilibrium, i.e., the equilibrium is unstable if coalitional deviations are allowed. The following 3-player prisoners' dilemma illustrates the

Player 3 chooses C

Player 1 \ Player 2	C	D
C	$3, 3, 3$	$1, 4, 1$
D	$4, 1, 1$	$2, 2, -1$

Player 3 chooses D

Player 1 \ Player 2	C	D
C	$1, 1, 4$	$-1, 2, 2$
D	$2, -1, 2$	$0, 0, 0$

Figure 1: A 3-player prisoners' dilemma

intuition. Clearly, (D, D, D) is the unique Nash equilibrium of this game and each player's equilibrium payoff is 0. However, if both player 1 and player 2 coalitionally deviate and choose C simultaneously, each of them gets payoff 1, i.e, they could mutually benefit from the coalitional deviation. Thus the unique Nash equilibrium becomes unstable as long as the deviations by two players are feasible.

This example further shows that the trivial subgame perfect Nash equilibrium in which all players play the unique Nsah equilibrium (D, D, D) in every period also becomes unstable. This indicates that not all the feasible and individually rational payoff vectors can arise as equilibria.

Hence, we introduce a notion of equilibrium which represents the degree of stability against coalitional deviations in an infinitely repeated n-player game, that is the degree-K subgame perfect Nash equilibrium, where $K \in N$ and $K \leq n$. In a degree-K subgame perfect Nash equilibrium, deviations by all coalitions whose size is up to K are allowed. When coalitions are taken into account, it is natural to assume that players in a coalition can transfer their payoffs. Thus, players base their decisions as to whether to enter into a coalition on the sum of payoffs achieved by the coalition as a whole and not on their individual payoffs. Thus a degree-K subgame perfect Nash equilibrium is defined as a strategy profile such that no feasible coalition could improve its sum of payoffs by any deviation.

Notice that the definition of degree-1 subgame perfect Nash equilibria is the same as the definition of subgame perfect Nash equilibria. The subgame perfect Nash equilibrium has been well studied in the infinitely repeated games with perfect monitoring. Fudenberg and Maskin (1986, 1991) show that the folk theorem holds if $n = 2$ or if a full dimensionality condition is satisfied. And Abreu et al. (1994) find the nonequivalent utility (NEU) condition which is a more general sufficient condition. NEU requires that for no two players, one's stage game payoff function be a positive affine transformation of the other's. Further, Wen (1994) introduces a concept of effective minmax payoffs and shows that any feasible payoff vector can arise as a subgame perfect Nash equilibrium if and only if it Pareto dominates the effective minmax payoff vector.

Therefore, we focus on the case that $K \geq 2$ in this paper. However, a difficulty arises in an infinitely repeated game with perfect monitoring when $K \geq 2$. With perfect monitoring players can only observe the other players' actions, so a coalitional deviation does not reveal the hidden deviators of the coalition who play as in the equilibrium. Thus, players may not distinguish the deviating coalition with perfect monitoring. Consider the above 3-player prisoners' dilemma and suppose that (D, D, D) should be played in some period of a degree-2 subgame perfect Nash equilibrium. Suppose also that the realized action profile is (C, D, D). From player 3' perspective, there is no doubt that player 1 is a deviator but whether the deviating coalition is $\{1\}$ or $\{1, 2\}$ (so player 2 is a hidden deviator) is indistinguishable.

In this paper, we analyze two models, called strengthened perfect monitoring and perfect monitoring, respectively. In a model with strengthened perfect monitoring, we assume that there exists an observer who can detect the deviating coalition and reports it to all players in every period. In this model, a degree-K subgame perfect Nash equilibrium is constructed by an idea that whenever a feasible coalition is reported as deviators all players switch to punish the coalition. In order to enable all players to play those punishments, we extend NEU to the degree-K nonequivalent utility (Degree-K NEU) condition, that is no pair of coalitions whose sizes are up to K have equivalent utility functions, where a coalition's utility function is defined as the sum of its members' utility functions. In a model with perfect monitoring, a degree-K subgame perfect Nash equilibrium is constructed by an idea that all players switch for a specified number of periods to punish all of them after any deviation, since the hidden deviators cannot be distinguished.

Here are our results. A coalition always has the option of playing a best response to the actions chosen by the rest of the players. Thus we define the minmax payoff of a coalition as the smallest best response sum of payoffs of the coalition. We call a payoff vector is degree-K rational if the sum of payoffs within any feasible coalition is strictly larger than its minimax payoff. Clearly, there is no degree-n rational payoff vector, so we focus on the case $n > K \geq 2$. Then we show that any feasible and degree-K rational payoff vector can arise as a degree-K subgame perfect Nash equilibrium with strengthened perfect monitoring if degree-K NEU is satisfied.

In the model with perfect monitoring, a payoff vector is called degree-K

simultaneously punishable if a punishment action profile exists such that the sum of payoffs within any feasible coalition is strictly larger than the best response total payoff of the coalition from the punishment action profile. Then we find that any feasible and degree-K simultaneously punishable payoff vectors can arise as a degree-K subgame perfect Nash equilibrium with perfect monitoring. Note that the result does not require NEU or its extension.

Finally, we study degree-n subgame perfect Nash equilibria. Since any payoff vector is neither degree-n rational nor degree-n simultaneously punishable, we adopt a new approach. Note that degree-n subgame perfect Nash equilibria play only action profiles which maximize the sum of all player's payoffs. Hence, we define a new minmax value of a non-grand coalition as the smallest best response sum of payoffs of the coalition among all action profiles which maximizes the sum of all players' payoffs. We call a payoff vector that maximizes the sum of all players' payoffs degree-n weakly rational if the sum of payoffs within any non-grand coalition is greater than the new minmax value of the coalition defined as above.

Then we find that if the set of feasible and degree-n weakly rational payoff vectors satisfies degree-$(n-1)$ NEU, any payoff vector in the set can arise as a degree-n subgame perfect Nash equilibrium with strengthened perfect monitoring. This result does not extend to perfect monitoring except the case of two players. In 2-player infinitely repeated games, strengthened perfect monitoring and perfect monitoring are equivalent, because there is no hidden deviator when a player finds a deviation.

Our paper isrelated to both the literature on coalitions and the literature on renegotiation-proof equilibria. Within the former literature, Rubinstein (1980) provides a concept of strong perfect equilibria which requires that in a supergame no coalition can make all of its members better off by any deviation. Horniacek (1996) weakens the concept by requiring that no coalition can make at least one of its members better off without making some other member worse off by any deviation. He shows that this sort of equilibrium can be approximated in a discounted supergame. In this paper, we assume that deviators' payoffs are transferable, therefore, no feasible coalition could make a Kaldor-Hicks improvement by any deviation in a degree-K subgame perfect Nash equilibrium. Bernheim et al. (1987) also deal with coalitional deviations but they restrict attention to one-shot games and extensive form games with a finite number of stages. They propose a notion of coalition-proof Nash equilibrium in which no coalition can make a mutually beneficial, self-enforcing joint deviation from it. Different from them, we deal with games with infinite numbers of stages.

Within the literature on renegotiation-proofness in repeated games, papers dealing with the stability of subgame perfect Nash equilibria only consider unilateral deviations. Farrell and Maskin (1989) propose the concept of renegotiation-proofness which requires that no continuation payoff be Pareto dominated by another continuation payoff. They conjecture that the renegotiation-proofness requirement generally would greatly reduce the set of subgame perfect Nash equilibrium payoff vectors, especially in a repeated prisoners' dilemma since the standard punishment by the trigger strategies will be excluded by the concept.

However, van Damme (1989) shows that in a repeated prisoners' dilemma requiring renegotiation-proofness does not reduce the set at all by using strategies where a punisher receives reward, i.e., the conjecture is wrong. Pearce (1987) and Benoit and Krishna (1993) extend the concept to infinitely repeated games with imperfect monitoring and finitely repeated games respectively. Unlike them, we deal with the stability of the subgame perfect Nash equilibrium by concerning coalitional deviations. Namely, the question we mentioned at the beginning will still be a question in those papers.

The rest of this paper is organized as follows. In section 2, we give the definition of the degree-K subgame perfect Nash equilibrium and characterize its necessary and sufficient condition. In section 3, we analyze the infinitely repeated games with strengthened perfect monitoring. In section 4, we study the infinitely repeated games with perfect monitoring. In section 5, we analyze degree-n subgame perfect Nash equilibria. Section 6 concludes.

2 Degree-K subgame Nash perfect equilibria

Consider a finite n-player game in normal form $G = \{N, (A_i)_{i\in N}, (u_i)_{i\in N}\}$, where $N = \{1, 2, .., n\}$ denotes a finite set of players, A_i denotes the set of pure actions of player i, and $u_i : A \to R$ is this player's payoff function with $A = \prod_{i\in N} A_i$. Given a set X, let $\Delta(X)$ denote the set of probability distributions over X. Therefore, $S_i = \Delta(A_i)$ denotes the set of mixed actions of player i, and $S = \prod_{i\in N} S_i$ denotes the set of mixed action profiles.[1] The payoff function is extended to mixed actions by taking expectations.

We consider an infinitely repeated game, where the set of players play the stage game G over periods $t = 0, 1, 2, ...$ For $\delta \in (0, 1)$ denote by $G(\delta)$ the infinite repetition of G in which all players discount future payoffs with δ. We assume that mixed actions are observable at the end of each period t. And we also assume that at the end of each period t, all players can observe some other public information from a set which is independent of t. Thus, the set of period $t \geq 0$ histories, H^t with typical element h^t, has the form $H^t = Z^t$ for some set Z, and typical element of Z contains a mixed action profile.[2] Here we define the initial history H^0 to be an arbitrary singleton. The set of all possible histories is $H = \bigcup_{t=0}^{\infty} H^t$.

Let $\sigma_i : H \to S_i$ denote player i's strategy in $G(\delta)$ and let Σ_i denote the set of strategies of player i. Since any history includes all actions in the past periods, there is no loss of generality when we restrict a player's strategy to be a function from H. $\Sigma = \Pi_{i\in N}\Sigma_i$ is the set of strategy profiles, with typical element σ. Player i's discounted payoff from a strategy profile σ is defined by

$$g_i(\sigma) = (1 - \delta) \sum_{t=0}^{\infty} \delta^t u_i(s(t)),$$

[1] We do not take correlated actions into account in this paper.

[2] For instance, standard perfect monitoring with observable mixed actions has $Z = S$.

where $s(t)$ is the realized mixed action profile in period t. For any strategy profile σ, player i's continuation strategy induced by h^t, denoted by $\sigma_i|h^t$, is given by,

$$\sigma_i|h^t(h^\tau) = \sigma_i(h^t h^\tau), \forall h^\tau \in H$$

where $h^t h^\tau$ is the concatenation of the history h^t followed by the history h^τ.

In this paper, we consider the coalitional deviation by multiple players is allowed. Thus let C_K with typical element c be the set of non-empty subsets of N which have K or less elements, where $K \in N$. Then for each $c \in C_K$, we set $Y_c = \Pi_{i\in c} Y_i$, $Y_{-c} = \Pi_{i\in N\setminus c} Y_i$, where $Y \in \{a, s, \sigma, A, S, \Sigma\}$ and set $u_c(\cdot) = \sum_{i\in c} u_i(\cdot)$ and $g_c(\cdot) = \sum_{i\in c} g_i(\cdot)$. In this paper, we regard any notation with subscript $\{i\}$ as the one with subscript i, where $i \in N$. For example we use both a_i and $a_{\{i\}}$. We assume that players' payoffs are transferable in a coalition, i.e., the players of a coalition are only concerned about the sum of payoffs achieved by the coalition. Therefore, we define degree-K Nash equilibria as following.

Definition 1. (Degree-K Nash equilibria) *For $K \in N$, the strategy profile σ is a degree-K Nash equilibrium of $G(\delta)$, if for any $c \in C_K$ and strategy $\sigma'_c \in \Sigma_c$,*

$$g_c(\sigma) \geq g_c(\sigma'_c, \sigma_{-c}).$$

Notice that when $K = 1$, the definition is the same with the definition of Nash equilibrium, i.e., a Nash equilibrium is a degree-1 Nash equilibrium under our definition. Similarly to the Nash equilibrium, this definition imposes no optimality condition in any subgame. Thus, in the following definition, we strengthen the degree-K Nash equilibrium by imposing the sequential rationality requirement that behavior is optimal in all circumstances, not only those that arise in equilibrium but also those that arise out of equilibrium.

Definition 2. (Degree-K subgame perfect Nash equilibria) *For $K \in N$, the strategy profile σ is a degree-K subgame perfect Nash equilibrium of $G(\delta)$ if for any $h^t \in H$, $\sigma|h^t$ is a degree-K Nash equilibrium of $G(\delta)$.*

A subgame perfect Nash equilibrium is a degree-1 subgame perfect Nash equilibrium under our definition. Namely, we extend the subgame perfect Nash equilibrium by requiring coalitional rationalities. Here, the degree-K subgame perfect Nash equilibrium characterizes a stable equilibrium in which all coalitions whose size is up to K are allowed and players in those coalitions can transfer their payoffs. From another point of view, this notion also represents the degree of stability of an equilibrium strategy profile. That is if a strategy profile σ is a degree-K subgame perfect Nash equilibrium but not a degree-$(K+1)$ subgame perfect Nash equilibrium, we may say the degree of stability of σ is K. According to the definition, the following proposition is straightforward.

Proposition 1. *For $K \in N$, a strategy profile σ is a degree-K subgame perfect Nash equilibrium of $G(\delta)$, then for any $K' \leq K$, σ is a degree-K' subgame perfect Nash equilibrium of $G(\delta)$.*

Proof. The proposition holds by Definition 2. □

To confirm that a strategy profile σ is a degree-K subgame perfect Nash equilibrium, we need to check whether an infinite number of strategy profiles are degree-K Nash equilibria. Similarly to the subgame perfect Nash equilibrium, in order to limit the number of alternative strategies that must be checked we introduce a one-shot coalitional deviation principle which is an extension of the one-shot deviation principle. Before stating the one-shot coalitional deviation principle, we first define a one-shot coalitional deviation.

Definition 3. (One-shot coalitional deviation) *For $c \in C_n$, a strategy profile $\sigma'_c \neq \sigma_c$ is a one-shot coalitional deviation for a coalition c from strategy profile σ_c if for any $t \geq 1$ and $h^t \in H^t$*

$$\sigma'_c(h^t) = \sigma_c(h^t).$$

Here, the strategy σ'_c plays identically to σ_c in every period other than period 0. Then the following principle allows us to restrict attention to consider alternative strategies that coalitionally deviate from the equilibrium strategy once and then return to it.

Theorem 1. (One-shot coalitional deviation principle) *A strategy profile σ is a degree-K subgame perfect Nash equilibrium if and only if for any $t \geq 0$, $h^t \in H^t$, $c \in C_K$, and one-shot coalitional deviation $\sigma'_c|h^t$ from strategy profile $\sigma_c|h^t$,*

$$g_c(\sigma|h^t) \geq g_c(\sigma'_c|h^t, \sigma_{-c}|h^t), \tag{1}$$

where $\sigma_{-c} \in \Sigma_{-c}$.

Proof. Clearly, the necessity is self-evident. Therefore, it suffices to show the contraposition of the sufficiency. Suppose that a profile σ is not a degree-K subgame perfect Nash equilibrium, so there exist a history $h^t \in H^t$, a coalition $c \in C_K$, and a strategy profile σ'_c, such that

$$g_c(\sigma|h^t) < g_c(\sigma'_c|h^t, \sigma_{-c}|h^t).$$

Fix $\varepsilon \in (0, g_c(\sigma'_c|h^t, \sigma_{-c}|h^t) - g_c(\sigma|h^t))$ and let

$$\Delta_c = \max_{a \in A, a' \in A} \{u_c(a') - u_c(a)\}. \tag{2}$$

Then let T be large enough such that

$$(1-\delta)\sum_{s=T}^{\infty} \delta^s \Delta_c < \varepsilon. \tag{3}$$

For all $\tau = 0, 1, ..., T$, we define a strategy $\sigma_c^\tau = (\sigma_i^\tau)_{i \in c} \in \Sigma_c$ as following:

$$\sigma_i^\tau(h^s) = \begin{cases} \sigma'_i|h^t(h^s) & \text{if } s < \tau \\ \sigma_i|h^t(h^s) & \text{if } s \geq \tau \end{cases}$$

for any $i \in c$. Then, by (2) and (3),

$$g_c(\sigma|h^t) < g_c(\sigma_c^T, \sigma_{-c}|h^t),$$

so

$$\sum_{\tau=1}^{T}\{g_c(\sigma_c^\tau, \sigma_{-c}|h^t) - g_c(\sigma_c^{\tau-1}, \sigma_{-c}|h^t)\} = g_c(\sigma_c^T, \sigma_{-c}|h^t) - g_c(\sigma|h^t) > 0.$$

Thus, there must exist $T' \in \{0, 1, ..., T-1\}$ such that

$$g_c(\sigma_c^{T'+1}, \sigma_{-c}|h^t) - g_c(\sigma_c^{T'}, \sigma_{-c}|h^t) > 0.$$

Notice that strategy $\sigma_c^{T'+1}$ agrees with $\sigma_c^{T'}$ over the first T' periods, thus there exists a history $h^{T'}$ such that

$$g_c(\sigma_c^{T'+1}|h^{T'}, \sigma_{-c}|h^t h^{T'}) > g_c(\sigma_c^{T'}|h^{T'}, \sigma_{-c}|h^t h^{T'}). \tag{4}$$

Note that by the definitions of $\sigma_c^{T'+1}$ and $\sigma_c^{T'}$, $\sigma_c^{T'}|h^{T'} = \sigma_c|h^t h^{T'}$ and $\sigma_c^{T'+1}|h^{T'}$ is a one-shot joint deviation from $\sigma_c|h^t h^{T'}$. Therefore, (4) implies that (1) will not hold by a one-shot coalitional deviation. □

So far, we defined the basic equilibrium concepts and showed that it could be examined by the one-shot coalitional deviation principle. As we argued in the introduction, the degree-1 subgame perfect Nash equilibrium is well studied by Fudenberg and Maskin (1986, 1991), Abreu et al. (1994) and Wen (1994). Hence, we focus on the case that $K \geq 2$ in this paper.

However, when $K \geq 2$ a difficulty arises if we assume that all players can only observe the mixed action profile chosen at the end of each period, i.e., the perfect monitoring. Consider a degree-K subgame perfect Nash equilibrium under perfect monitoring, and suppose K' players played differently from the equilibrium at the end of some period, where $K \geq 2$ and $K > K'$. Generally, it is possible that there exist some hidden deviators who play as in the equilibrium but belong to the deviating coalition with the K' players. Obviously, it is crucial for constructing equilibrium strategies whether hidden deviators could be distinguished or not. Thus, in the following two sections, we consider two models in which hidden deviators can be distinguished and cannot be distinguished, respectively.

3 Strengthened perfect monitoring

We first consider a model with strengthened perfect monitoring, in which hidden deviators could be distinguished. We assume that there exists an observer who can detect any deviating coalition $c \in C_n$ and report $r = c$ to all players at the end of each period. Here we assume that mixed actions are observable ex post by the observer. Hence, at the end of each period each player can observe not only the other players' mixed actions but also the observer's report. Therefore, all players are able to know the members of the deviating coalition.

We suppose that if there is no deviator in a period, the observer reports $r = \emptyset$ to all players in that period. So the observer's report r is an element of $\Re = C_n \bigcup \{\emptyset\}$. A history $h^t \in H^t$ includes thus a list of t reports, identifying the reports submitted in periods 0 through $t-1$. Then the set of period $t \geq 0$ histories is given by $H^t = (S \times \Re)^t$, where $(S \times \Re)^t$ with typical element $((s(0), r(0)), (s(1), r(1)), ..., (s(t-1), r(t-1)))$ to be the t-fold product of $(S \times \Re)$ and we define $(S \times \Re)^0$ to be an arbitrary singleton.

Now, we begin to characterize players' payoffs consistent with equilibrium behavior in this model. Let coX be the convex hull of a given set X and $u(\cdot) = (u_1(\cdot), u_2(\cdot), ..., u_n(\cdot))$, so that the set of feasible payoff vectors can be denoted by

$$V = co\{u(a) : a \in A\}.$$

Then we define the minmax payoff for each coalition. For a vector $v \in R^n$, let

$$v_c = \sum_{i \in c} v_i$$

and

$$v_{-c} = \sum_{i \in C_n \setminus c} v_i,$$

where $c \in C_n$.

Note that a coalition $c \in C_n$ always has the option of playing a best response to the actions chosen by the rest of the players. Therefore, by the assumption of transferable payoffs, the sum of payoffs in the stage game for coalition c to guarantee is the smallest best response sum of payoffs among all action profiles, i.e., coalition c's minmax payoff is

$$\underline{v}_c = \min_{s_{-c} \in S_{-c}} \max_{a_c \in A_c} u_c(a_c, s_{-c}).$$

Denote m^c be a minmax profile for coalition c, that is

$$\underline{v}_c = u_c(m^c) = \max_{a_c \in A_c} u_c(a_c, m^c_{-c}).$$

Then for $K \in N$, we call a payoff vector v degree-K rational if for any $c \in C_K$,

$$v_c > \underline{v}_c.$$

We define the set of feasible and degree-K rational payoff vectors by

$$V^K = \{v \in V : v_c > \underline{v}_c \ \forall c \in C_K\}.$$

Intuitively, as K rises V^K shrinks, since it requires more stringent coalitional rationalities, as is shown in the following proposition.

Proposition 2. *$V^K \subseteq V^{K'}$ for any $K' \leq K$ and $V^n = \emptyset$.*

Proof. The former statement holds by the definition of V^K. For the latter statement, if there were $v \in V^n$, then $v_N > \underline{v}_N = \max_{a \in A} u_N(a)$, a contradiction to $v \in V$. □

In the 3-player prisoners' dilemma in the introduction, the minmax profile for player 1 is (D, D, D) and the minmax profile for a coalition of player 1 and player 2 is (C, C, D). Thus $\underline{v}_{\{1\}} = u_1(D, D, D) = 0$ and $\underline{v}_{\{1,2\}} = u_1(C, C, D) + u_2(C, C, D) = 2$. Hence by the symmetry, payoff vectors such as $(1, 1, 1)$ is feasible and degree-1 rational but is not feasible and degree-2 rational. In fact, some inefficient payoff vectors will be eliminated from the set of feasible and degree-K rational payoff vectors as K increases for many infinitely repeated games.

However, there exist some cases in which V^K does not shrink as K rises. Consider the following 3-player Cournot competition game. In the game, player i can decide his production $a_i \in R$ and his payoff is given by $u_i(a) = a_i(Q - a_1 - a_2 - a_3 - b)$, where $i \in \{1, 2, 3\}$ and $Q > b > 0$. Clearly, if player i decides to produce nothing, his payoff is 0. And if he decides to produce $a_i = Q - b$, then other players' payoffs are at most 0. Hence both the minmax payoff for one player and the minmax payoff for a coalition of two players are 0, i.e., $V^1 = V^2 \neq \emptyset$ in an infinitely repeated 3-player Cournot competition game.

The case that there are only two players is well studied by Fudenberg and Maskin (1986) and we state their conclusion by the following theorem.

Theorem 2. *If $n = 2$, then for any $v \in V^1$, there exists $\underline{\delta} \in (0, 1)$ such that for all $\delta \in (\underline{\delta}, 1)$, v is a degree-1 subgame perfect Nash equilibrium payoff vector of $G(\delta)$ with strengthened perfect monitoring.*

Proof. See Fudenberg and Maskin (1986)'s Theorem 1. □

Fudenberg and Maskin (1986) construct an equilibrium strategy by using a joint punishment. That is, all players play a mutual minmax action profile long enough to wipe out any gain of a deviator whenever deviation happens. However, in the case that there are three or more players there may not exist such a convenient punishment which could simultaneously punish every player and minmax any deviator's payoff. Hence we need to change the idea of constructing an equilibrium strategy in the case that there are three or more players. Since the members of a deviating coalition can be distinguished, if a coalition deviates, the coalition can be minmaxed by the other players. To support our idea, we extend Abreu et al. (1994)'s NEU condition to a degree-K nonequivalent utility condition (degree-K NEU condition) which is defined as follows.

Definition 4. *(Degree-K NEU) For $K \in N$, a convex set $X \subset R^n$ satisfies the degree-K NEU condition if for any coalition $c \in C_K$ and $c' \in C_K \backslash c$ there does not exist $\alpha \geq 0$ and $\beta \in R$ such that $x_c = \alpha x_{c'} + \beta$ for all $x \in X$.*

If the feasible payoff vectors set V satisfies degree-1 NEU, then it satisfies NEU.[3] Here, we define the degree-K NEU, because it is a necessary and sufficient

[3]NEU does not imply degree-1 NEU, because NEU allows one player to be indifferent over all actions, i.e., v_i is constant on X for some i.

condition for the following lemma which supports our result.

Lemma 1. *For $K \in N$, if a convex set $X \subset R^n$ satisfies degree-K NEU if and only if there exists a set $\{x^c\}_{c\in C_K} \subseteq X$ such that for any $c \in C_K$ and $c' \in C_K \backslash c$,*

$$x_c^c < x_c^{c'}.$$

Proof. We prove this lemma by Abreu et al. (1994)'s approach. For the sufficiency, we first show that for each pair of coalitions c and $c' \neq c$ there exist $x^{cc'} \in X$ and $x^{c'c} \in X$ such that $x_c^{cc'} < x_{c'}^{cc'}$ and $x_{c'}^{c'c} > x_c^{c'c}$. Let $X_{cc'} = \{(x_c, x_{c'}) : x \in X\}$. Then one of the following two conditions holds, since the stage game G satisfies degree-K NEU.
(i) $dim X_{cc'} = 2$;
(ii) $dim X_{cc'} = 1$ and $X_{cc'}$ is a line with negative slope.
No matter which condition holds there exist $\chi^c \in X_{cc'}$ and $\chi^{c'} \in X_{cc'}$ such that $\chi_c^c < \chi_c^{c'}$ and $\chi_{c'}^{c'} < \chi_{c'}^c$. By the definition of $X_{cc'}$, there exist $x^{cc'} \in X$ and $x^{c'c} \in X$ such that $(x_c^{cc'}, x_{c'}^{cc'}) = \chi^c$ and $(x_c^{c'c}, x_{c'}^{c'c}) = \chi^{c'}$, respectively. Further, $x^{cc'}$ and $x^{c'c}$ satisfy $x_c^{cc'} < x_c^{c'c}$ and $x_{c'}^{c'c} < x_{c'}^{cc'}$.

Now we show the existence of the set $\{x^c\}_{c\in C_K}$. By the above proof, for each pair of coalitions $c' \in C_K$ and $c'' \in C_K \backslash c'$ there exist $x^{c'c''} \in V$ and $x^{c''c'} \in V$ such that $x_{c'}^{c'c''} < x_{c'}^{c''c'}$ and $x_{c''}^{c''c'} < x_{c''}^{c'c''}$. For each coalition $c \in C_K$, order the $|C_K| \times |C_K - 1|$ payoff vectors $x^{c'c''}$ in increasing size (break ties arbitrarily) from the point of view of coalition c, and assign these ordered vectors strictly decreasing weights $\theta_\iota > 0$, $\iota = 1, 2, ..., |C_K| \times |C_K - 1|$, summing to one. Let x^c be the resulting convex combination of the $|C_K| \times |C_K - 1|$ payoff vectors $x^{c'c''}$. By the sequence inequality, $x_c^c < x_c^{c'}$ for any $c' \in C_K \backslash c$.

For the necessity, suppose that X does not satisfy degree-K NEU, i.e., for some coalition $c \in C_K$ and $c' \in C_K \backslash c$ there exist $\alpha \geq 0$ and $\beta \in R$ such that $x_c = \alpha x_{c'} + \beta$ for all $x \in X$. If $\alpha = 0$, then $x_c = \beta$ for all $x \in X$. If $\alpha > 0$, then $x_c = \alpha x_{c'} + \beta$ for all $x \in X$, i.e., x_c and $x_{c'}$ are positively correlated. Obviously, no matter which condition holds there does not exist such a set $\{x^c\}_{c\in C_K} \subseteq X$. □

Before stating our result, let us introduce two more lemmas which support our result.

Lemma 2. *For any $\alpha > \beta \geq 0$, there exists $\bar{\delta} \in (0, 1)$ such that for any $\delta \in (\bar{\delta}, 1)$, there exists a natural number l such that*

$$\alpha > \frac{\delta^l}{1 - \delta^l} > \beta. \tag{5}$$

Proof. If $\beta = 0$, (5) holds for for any natural number $l > \log_\delta \frac{\alpha}{\alpha+1}$. If $\beta > 0$, for any $\delta \in (\frac{(\alpha+1)\beta}{\alpha(\beta+1)}, 1)$,

$$\log_\delta \frac{\beta}{\beta + 1} - \log_\delta \frac{\alpha}{\alpha + 1} > 1.$$

Notice that $\log_\delta \frac{\alpha}{\alpha+1} > 0$, so there exists a natural number l such that

$$\log_\delta \frac{\beta}{\beta+1} > l > \log_\delta \frac{\alpha}{\alpha+1}.$$

Then (5) holds, since $\delta \in (0,1)$. □

Lemma 3. *For all $\varepsilon > 0$ and all $A' \subseteq A$, there exist $\underline{\delta} \in (0,1)$ such that for all $\delta \in (\underline{\delta}, 1)$ and all $v \in co\{u(a) : a \in A'\}$, there exists a sequence of action profiles whose discounted average payoffs are v and whose continuation payoffs at any time t are within ε of v.*

Proof. See Mailath and Samuelson (2006)'s lemma 3.7.2. □

Now we can show that all payoff vectors in V^K arise as a degree-K subgame perfect Nash equilibrium with strengthened perfect monitoring if V^K satisfies degree-K NEU and if the players are sufficiently patient. The idea behind the proof of the following theorem is simple. If a coalition deviates, the coalition is minmaxed by the other players long enough to wipe out any gain from its deviation. To induce the other players minmax the deviating coalition, the equilibrium gives a "reward" after minmaxing periods. The degree-K NEU ensures that the continuation strategies with the reward can be constructed.

Theorem 3. *In the case $n \geq 3$ and $K < n$, if V^K satisfies degree-K NEU, then for any $v \in V^K$, there exists $\underline{\delta} \in (0,1)$ such that for all $\delta \in (\underline{\delta}, 1)$, v is a degree-K subgame perfect Nash equilibrium payoff vector of $G(\delta)$ with strengthened perfect monitoring.*

Proof. Fix any $v \in V^K$. By Lemma 1, there exists a set $\{x^c\}_{c \in C_K} \subseteq V^K$ such that for any $c \in C_K$ and $c' \in C_K \backslash \{c\}$

$$x_c^c < x_c^{c'}.$$

For each $c \in C_K$, fix a payoff vector $y^c \in V$ which satisfies $y_c^c = \min_{a \in A} u_c(a)$. Now for each $c \in C_K$, consider this payoff vector

$$(1-\eta)v + \eta\zeta x^c + \eta(1-\zeta)y^c$$

where $\eta \in (0,1)$ and $\zeta \in (0,1)$. Then for any $c \in C_K$ and $c' \in C_K \backslash \{c\}$,

$$(1-\eta)v_c + \eta\zeta x_c^c + \eta(1-\zeta)y_c^c < (1-\eta)v_c + \eta\zeta x_c^{c'} + \eta(1-\zeta)y_c^{c'},$$

due to $y_c^c = \min_{a \in A} u_c(a)$ and $x_c^c < x_c^{c'}$. Because $v \in V^K$ and $x^c \in V^K$, so for all $c \in C_K$ and $c' \in C_K$, $v_{c'} > \underline{v}_{c'}$ and $x_{c'}^c > \underline{v}_{c'}$. Hence there exists $\bar{\eta} > 0$ such that for any $\eta \in (0, \bar{\eta})$

$$(1-\eta)v_{c'} + \eta\zeta x_{c'}^c + \eta(1-\zeta)y_{c'}^c > \underline{v}_{c'},$$

for any $c \in C_K$ and $c' \in C_K$, i.e.,

$$(1-\eta)v + \eta\zeta x^c + \eta(1-\zeta)y^c \in V^K$$

for all $c \in C_K$. By $y_c^c = \min_{a\in A} u_c(a) \le \underline{v}_c < v_c$, there must exist $\bar{\zeta} > 0$ such that for any $\zeta \in (0, \bar{\zeta})$ and any $\eta \in (0, 1)$,

$$(1-\eta)v_c + \eta(\zeta x_c^c + (1-\zeta)y_c^c) < v_c,$$

for all $c \in C_K$.

Fix $\eta \in (0, \bar{\eta})$ and $\zeta \in (0, \bar{\zeta})$ and let $v^c = (1-\eta)v + \eta\zeta x^c + \eta(1-\zeta)y^c$ for each $c \in C_K$. Then

$$v^c \in V^K, \tag{6}$$

$$v_c^c < v_c, \tag{7}$$

and for any $c' \in C_K \backslash \{c\}$

$$v_c^c < v_c^{c'}. \tag{8}$$

Due to (6) and (8), there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$,

$$\min_{c\in C_K} \frac{v_c^c - \underline{v}_c - |c|\varepsilon}{|c|\varepsilon} > \max_{c\in C_K, c'\in C_K\backslash\{c\}} \{\frac{v_c^c - u_c(m^{c'}) - |c|\varepsilon}{v_c^{c'} - v_c^c + |c|\varepsilon}, 0\}.$$

Further, fix $\varepsilon \in (0, \bar{\varepsilon})$. By Lemma 2, there exists $\delta_1 > 0$ for every $\delta \in [\delta_1, 1)$, there exists a natural number l such that

$$\min_{c\in C_K} \frac{v_c^c - \underline{v}_c - |c|\varepsilon}{|c|\varepsilon} > \frac{\delta^l}{1-\delta^l} > \max_{c\in C_K, c'\in C_K\backslash\{c\}} \frac{v_c^c - u_c(m^{c'}) - |c|\varepsilon}{v_c^{c'} - v_c^c + |c|\varepsilon}.$$

Then for any $c \in C_K$ and any $c' \in C_K \backslash \{c\}$,

$$(1-\delta^l)u_c(m^{c'}) + \delta^l v_c^{c'} > v_c^c - |c|\varepsilon > (1-\delta^l)\underline{v}_c + \delta^l v_c^c. \tag{9}$$

Hence there exists $\delta_2 \in [\delta_1, 1)$ such that for every $\delta \in [\delta_2, 1)$

$$v_c^c - |c|\varepsilon > (1-\delta)\max_{a\in A} u_c(a) + \delta\{(1-\delta^l)\underline{v}_c + \delta^l v_c^c\}. \tag{10}$$

By Lemma 3, there exists $\underline{\delta} \in [\delta_2, 1)$ such that for every $\delta \in [\underline{\delta}, 1)$, there exist a sequence of action profiles $\pi^{\emptyset} = \{s^{\emptyset}(t)\}_{t=0}^{\infty}$ and $|C_K|$ sequences of action profiles $\pi^c = \{s^c(t)\}_{t=0}^{\infty}$ $(c \in C_K)$ which satisfy the following equalities and inequalities:

$$(1-\delta)\sum_{t=0}^{\infty} \delta^t u(s^{\emptyset}(t)) = v \tag{11}$$

$$(1-\delta)\sum_{t=0}^{\infty} \delta^t u_i(s^{\emptyset}(T+t)) > v_i - \varepsilon \quad \forall i \in N \quad \forall T \ge 1 \tag{12}$$

$$s^c(t) = m^c \quad \forall t < l$$

$$(1-\delta)\sum_{t=0}^{\infty} \delta^t u(s^c(l+t)) = v^c \tag{13}$$

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t u_i(s^c(T+t)) > v_i^c - \varepsilon \quad \forall i \in N \quad \forall T > l. \tag{14}$$

Consider the following strategy profile σ^*:

(A) All players play following $\pi^{\emptyset}$ in each period until $r \in C_K$ is reported;[4]

(B) Whenever $c \in C_K$ is reported, then all players start to play following π^c from the next period until another $r' \in C_K$ is reported.[5]

Now we use the one-shot coalitional deviation principle to show that σ^* is a degree-K subgame perfect Nash equilibrium strategy profile. Suppose a coalition $c \in C_K$ chooses a one-shot coalitional deviation in period $t \geq 0$. Notice that for any history $h^t \in H^t$, if all players play following $\sigma^*|h^t$, they should play $s^{\emptyset}(t)$ or $s^{c'}(t')$ in period t, where $c' \in C_K$ and $t' \geq 0$.

First, if they should play $s^{\emptyset}(t)$ or $s^{c'}(t')$ in period t, where $t' \geq l$, then the coalition's sum of continuation payoffs by a one-shot coalitional deviation is less than

$$v_c^c - |c|\varepsilon < g_c(\sigma^*|h^t),$$

due to (7), (8) and (10)-(14). Second, if they should play $s^{c'}(t')$ in period t, where $t' < l$ and $c' \neq c$, then the coalition's sum of continuation payoffs by a one-shot coalitional deviation is less than

$$v_c^c - |c|\varepsilon,$$

due to (10). But if $u_c(m^{c'}) < v_c^{c'}$, we have

$$g_c(\sigma^*|h^t) = (1-\delta^{l-t'})u_c(m^{c'}) + \delta^{l-t'}v_c^{c'} \geq (1-\delta^l)u_c(m^{c'}) + \delta^l v_c^{c'} > v_c^c - |c|\varepsilon$$

due to (9), and if $u_c(m^{c'}) \geq v_c^{c'}$, wa also have

$$g_c(\sigma^*|h^t) = (1-\delta^{l-t'})u_c(m^{c'}) + \delta^{l-t'}v_c^{c'} \geq v_c^{c'} > v_c^c - |c|\varepsilon,$$

due to (8). Third, if they should play $s^c(t')$ in period t, where $t' < l$, then the coalition's sum of continuation payoffs by a one-shot coalitional deviation is at most

$$(1-\delta)u_c(m^c)+\delta\{(1-\delta^l)u_c(m^c)+\delta^l v_c^c\} < (1-\delta^{l-t'})u_c(m^c)+\delta^{l-t'}v_c^c = g_c(\sigma^*|h^t),$$

due to (6).

Therefore, if all players play following the strategy profile σ^*, for any history $h^t \in H^t$ no feasible coalition could benefit from a one-shot coalitional deviation, i.e., σ^* is a degree-K subgame perfect Nash equilibrium strategy profile. Therefore any $v \in V^K$ is a degree-K subgame perfect Nash equilibrium payoff vector of $G(\delta)$ with strengthened perfect monitoring. □

[4] If $r \in C_n \backslash C_K$ is reported, we suppose that all players ignore the report and continue to play following $\pi^{\emptyset}$.

[5] As in the former footnote, we suppose that any report $r' \in C_n \backslash C_K$ is ignored.

Compared to Fudenberg and Maskin (1986)'s full dimensionality condition, degree-K NEU is a weaker sufficient condition. In fact, there are many games which satisfy degree-K NEU but do not satisfy the full dimensionality condition, for example constant-sum games. In other words, Theorem 3 applies to some infinitely repeated constant-sum games.

Consider the following 5-player constant-sum game. In the game, each player chooses two of the rest players and give each of them 1 dollar. Each player's payoff is the revenue of himself at the end of the game. If a player receives nothing, then his payoff is -2 and his worst payoff in the game is also -2. Therefore, a player's minmax payoff is -2 and

$$V^1 = \{(v_1, v_2, v_3, v_4, v_5) : v_1 + v_2 + v_3 + v_4 + v_5 = 0, v_i > -2 \forall i\}$$

by the symmetry of the game. It is easy to see that V^1 satisfies degree-1 NEU, so all payoff vectors in V^1 arise as degree-1 subgame perfect Nash equilibrium with strengthened perfect monitoring.

A 2-player coalition can give one dollar each other, but if they receive nothing from the other players, the coalition's sum of payoffs is -2. In fact, the other players can give nothing to the coalition. Therefore, any 2-player coalition's minmax payoff is -2 and

$$V^2 = \{(v_1, v_2, v_3, v_4, v_5) : v_1+v_2+v_3+v_4+v_5 = 0, v_i > -2 \forall i, v_i+v_j > -2 \forall j \neq i\}$$

by the symmetry of the game. Note that V^2 is non-empty, because $(0, 0, 0, 0, 0) \in V^2$. Therefore all payoff vectors in V^1 arise as degree-1 subgame perfect Nash equilibrium with strengthened perfect monitoring, since V^2 satisfies degree-2 NEU.

A 3-player coalition can give nothing to the other players, since any member of the coalition can give the other two members one dollar. And the coalition must receive two dollars from the other players, so any 3-player coalition's minmax payoff is 2 and V^3 is empty.

4 Perfect monitoring

In this section, we consider the perfect monitoring, i.e., each player can only observe the other players' mixed actions. Hence, the hidden deviators who play as in the equilibrium cannot be distinguished. Then a history $h^t \in H^t$ is thus a list of t mixed action profiles, identifying the mixed actions played in periods 0 through $t-1$. And the set of period $t \geq 0$ histories is given by $H^t = S^t$, where S^t with typical element $(s(0), s(1), ..., s(t-1))$ to be the t-fold product of S and we define S^0 to be an arbitrary singleton.

In a model with strengthened perfect monitoring, the observer reveals all members of the deviating coalition, including those who play as in the equilibrium. However, in a model with perfect monitoring, a coalitional deviation does not reveal the members of the coalition who play as in the equilibrium. This is because players can only observe the other players' mixed actions with perfect monitoring. Notice that a deviator who plays against the equilibrium forms the deviating coalition if only coalitions whose size is 1 are feasible. Therefore, the degree-1 subgame perfect Nash equilibrium payoff vectors sets are the same both with strengthened perfect monitoring and with perfect monitoring.

Theorem 4. *If $n = 2$ and $V^1 \neq \emptyset$ or if V^1 satisfies degree-1 NEU, then for any $v \in V^1$, there exists $\underline{\delta} \in (0,1)$ such that for all $\delta \in (\underline{\delta}, 1)$, v is a degree-1 subgame perfect Nash equilibrium payoff vector of $G(\delta)$ both with strengthened perfect monitoring and with perfect monitoring.*

Proof. This theorem holds by Theorems 2 and 3. □

Now we focus on the case that $n > K \geq 2$ since $V^n = \emptyset$. As we argued, a coalitional deviation does not reveal the members of the coalition who play as in the equilibrium. So in some period of a degree-K subgame perfect Nash equilibrium, if the number of players who play differently from the equilibrium is less than K, any feasible coalition which contains those players may be the deviating coalition.

Recall the equilibrium strategy which is constructed by Fudenberg and Maskin (1986) in Theorem 2. In the equilibrium strategy, all players will play a mutual minmax action profile long enough to wipe out any gain of a deviator whenever deviation happens. Here, we apply the idea to solve the above problem, i.e., whenever deviation happens, all players switch to play an action profile s long enough to punish everyone, where $s \in S$. However, there may not exist an action profile s which can simultaneously punish every player with his minmax value when $n \geq 3$.

Here, we first characterize the degree-K subgame perfect Nash equilibrium payoffs which can be achieved by using a given $s \in S$ as simultaneous punishment. Since a coalition $c \in C_n$ always has the option of playing a best response to s_{-c}, its sum of payoffs is at least

$$\underline{v}_c^s = \max_{a_c \in A_c} u_c(a_c, s_{-c}).$$

Therefore a payoff vector v which can be achieved by s should satisfy

$$v_c > \underline{v}_c^s$$

for any $c \in C_K$ and we call the payoff vector v degree-K simultaneously punishable with respect to s.

Then for a given mixed action profile $s \in S$, the set of feasible and degree-K simultaneously punishable payoff vectors with respect to s can be defined by

$$V_s^K = \{v \in V : v_c > \underline{v}_c^s \forall c \in C_K\},$$

and the set of feasible and degree-K simultaneously punishable payoff vectors is

$$\bigcup_{s \in S} V_s^K = \bigcup_{s \in S} \{v \in V : v_c > \underline{v}_c^s \forall c \in C_K\},$$

where $K \in N$. Intuitively, $\bigcup_{s\in S} V_s^K$ shrinks as K rises, since it requires more stringent coalitional rationalities.

Notice that not all payoff vectors in V^K can be achieved by the idea, since not all feasible coalitions could be simultaneously and extremely punished. However, if there exists a mixed action profile s which simultaneously minmax all coalitions whose size is up to K, then $V_s^K = V^K$, where $K \in N$. Recall the 3-player Cournot competition game and consider the action profile $s = (Q-b, Q-b.Q-b)$. If a player decides to produce $Q - b$, the best response of any other player is producing nothing, i.e., $\underline{v}_c^s = 0$ for any $c \in C_2$. Then we have $V_s^K = V^K$ for $K = 1, 2$.

Proposition 3. *For any $K > K'$, $\bigcup_{s\in S} V_s^K \subseteq \bigcup_{s\in S} V_s^{K'}$. $\bigcup_{s\in S} V_s^K \subseteq V^K$ for all $K \in N$ and $\bigcup_{s\in S} V_s^n = V^n = \emptyset$.*

Proof. $\bigcup_{s\in S} V_s^K \subseteq \bigcup_{s\in S} V_s^{K'}$ holds by the definition of $\bigcup_{s\in S} V_s^K$. For a given $s \in S$, we have

$$\begin{aligned} v_c &> \underline{v}_c^s \\ &= \max_{a_c\in A_c} u_c(a_c, s_{-c}) \\ &\geq \min_{s_{-c}\in S_{-c}} \max_{a_c\in A_c} u_c(a_c, s_{-c}) \\ &= \underline{v}_c, \end{aligned}$$

for any $v \in V_s^K$, so $v \in V^K$, i.e., $V_s^K \subseteq V^K$. Therefore $\bigcup_{s\in S} V_s^K \subseteq V^K$ for any $K \in N$. Since $V^n = \emptyset$, $\bigcup_{s\in S} V_s^n = V^n = \emptyset$. □

In fact, all payoff vectors in $\bigcup_{s\in S} V_s^K$ can arise as degree-K subgame perfect Nash equilibria with perfect monitoring by the idea of simultaneous punishment. And we show this result without degree-K NEU.

Theorem 5. *In the case $n > K \geq 2$, for any $v \in \bigcup_{s\in S} V_s^K$ there exists $\underline{\delta} \in (0, 1)$ such that for all $\delta \in (\underline{\delta}, 1)$, v is a degree-K subgame perfect Nash equilibrium payoff vector of $G(\delta)$ with perfect monitoring.*

Proof. Suppose $v \in V_s^K$, where $s \in S$. By the definition of V_s^K and the definition of $\underline{v}_c^s$, we have

$$v_c > \underline{v}_c^s = \max_{a_c\in A_c} u_c(a_c, s_{-c}) \geq u_c(s) \tag{15}$$

for any $c \in C_K$. Due to $v_i > u_i(s)$ for any $i \in N$, there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$

$$\min_{i\in N}\{\frac{v_i - u_i(s) - \varepsilon}{\varepsilon}\} > \max_{c\in C_K}\{\frac{\underline{v}_c^s - u_c(s)}{v_c - \underline{v}_c^s}\} \geq 0.$$

Further, fix $\varepsilon \in (0, \bar{\varepsilon})$, by Lemma 2, there exists $\delta_1 \in (0, 1)$ such that for every $\delta \in [\delta_1, 1)$, there exists a natural number l such that

$$\min_{i\in N}\{\frac{v_i - u_i(s) - \varepsilon}{\varepsilon}\} > \frac{\delta^l}{1-\delta^l} > \max_{c\in C_K}\{\frac{\underline{v}_c^s - u_c(s)}{v_c - \underline{v}_c^s}\}.$$

Then for any $c \in C_K$,

$$(1-\delta^l)u_c(s) + \delta^l v_c > \underline{v}_c^s \tag{16}$$

and for any $i \in N$

$$v_i - \varepsilon > (1-\delta^l)u_i(s) + \delta^l v_i.$$

Hence there exists $\delta_2 \in [\delta_1, 1)$ such that for any $\delta \in [\delta_2, 1)$ and any $c \in C_k$,

$$v_c - |c|\varepsilon > (1-\delta)\max_{a\in A} u_c(a) + \delta\{(1-\delta^l)u_c(s) + \delta^l v_c\}. \tag{17}$$

By Lemma 3, there exists $\underline{\delta} \in (\delta_2, 1)$ such that for all $\delta \in [\underline{\delta}, 1)$, there exist 2 sequences of action profiles $\widehat{\pi}^z = \{s^z(t)\}_{t=0}^{\infty}$ $(z = 0, 1)$ which satisfy the following equalities and inequalities.

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t u(s^0(t)) = v \tag{18}$$

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t u_i(s^0(T+t)) > v_i - \varepsilon \quad \forall i \in N \quad \forall T \geq 1 \tag{19}$$

$$s^1(t) = s \quad \forall t < l$$
$$s^1(t) = s^0(t-l) \quad \forall t \geq l \tag{20}$$

Now consider the following strategy profile σ^*:

(A) All players play following $\widehat{\pi}^0$ in each period if all players continued playing following $\widehat{\pi}^0$.

(B) If there are at most K players who play differently from $\widehat{\pi}^z$, then all players start to play following $\widehat{\pi}^1$ in each period if all players continued playing following $\widehat{\pi}^1$, where $z = 0, 1$.[6]

Finally, we show that σ^* is a degree-K subgame perfect Nash equilibrium strategy profile by using the one-shot coalitional deviation principle. Suppose a coalition $c \in C_K$ chooses a one-shot coalitional deviation in period $t \geq 0$. Notice that for any history $h^t \in H^t$, if all players play following $\sigma^*|h^t$, they should play $s^0(t)$ or $s^1(t')$, where $t' \geq 0$, in period t.

On the one hand, if they should play $s^0(t)$ or $s^1(t')$ in period t, where $t' \geq l$, then the coalition's sum of continuation payoffs by a one-shot coalitional deviation is at most

$$v_c - |c|\varepsilon < g_c(\sigma^*|h^t),$$

due to (17)-(20). On the other hand, if they should play $s^1(t')$ in period t, where $t' < l$, then the coalition's sum of continuation payoffs by a one-shot coalitional deviation is at most

$$\begin{aligned}(1-\delta)\max_{a_c\in A_c} u_c(a_c, s_{-c}) + \delta\{(1-\delta^l)u_c(s) + \delta^l v_c\} &< (1-\delta^l)u_c(s) + \delta^l v_c \\ &\leq (1-\delta^{l-t'})u_c(s) + \delta^{l-t'} v_c \\ &= g_c(\sigma^*|h^t)\end{aligned}$$

[6] Similarly to the former equilibrium construction, we suppose that all players ignore any coalitional deviation in which there are more than K players who play differently from $\widehat{\pi}^1$ and continue to play $\widehat{\pi}^1$.

due to (15) and (16).

Therefore, if all players play following the strategy profile σ^*, for any history $h^t \in H^t$, no feasible coalition can benefit from a one-shot coalitional deviation, i.e., σ^* is a degree-K subgame perfect Nash equilibrium strategy profile. Therefore any $v \in \bigcup_{s\in S} V_s^K$ can arise as a degree-K subgame perfect Nash equilibrium with perfect monitoring. □

Although we show the above theorem without degree-K NEU, one may wonder if the set of feasible and degree-K simultaneously punishable payoff vectors might be empty in any infinitely repeated games which do not satisfy the degree-K NEU condition. Consider the following 3-player game from Fudenberg and Maskin (1986).

Player 3 chooses Yes

Player 1 \ Player 2	Yes	No
Yes	$1,1,1$	$0,0,0$
No	$0,0,0$	$0,0,0$

Player 3 chooses No

Player 1 \ Player 2	Yes	No
Yes	$0,0,0$	$0,0,0$
No	$0,0,0$	$1,1,1$

Figure 2: 3-player game

In the game, each player faces the same binary question. Each of them will be rewarded 1 dollar if their answers are the same, and get nothing if their answers are not the same. Note that the three players have the same payoffs, so this game does not satisfy degree-K NEU, where $K = 1, 2, 3$. Consider an action profile s in which all players choose the two answers with equal probability. It is easy to know that s is a Nash equilibrium action profile, so $V_s^1 = \{(b, b, b) : b \in (0.25, 1]\}$. For any $s' \in S\backslash s$, there must exist two players and an answer such that the probability for both of them to choose the answer is larger than 0.25. Then the best response of the the remaining player is to choose the answer with probability 1. Because the three players have the same payoffs, $V_{s'}^1 \subset V_s^1$, i.e.,

$$\bigcup_{s\in S} V_s^1 = V_s^1 = \{(b, b, b) : b \in (0.25, 1]\}.$$

For the action profile s and any coalition c containing two players, the best response of the coalition is to take the same action. So $V_s^2 = \{(b, b, b) : b \in (0.5, 1]\}$. For any $s' \in S\backslash s$, there must exist one player who chooses some answer with probability greater than 0.5. Then the best response of the other two players is to choose that answer with probability 1. Because the three players have the same payoffs, $V_{s'}^2 \subset V_s^2$, i.e.,

$$\bigcup_{s\in S} V_s^2 = V_s^2 = \{(b, b, b) : b \in (0.5, 1]\}.$$

This example shows that failure of degree-K NEU is consistent with nonemptiness of $\bigcup_{s\in S} V_s^K$ for $K < n$.

5 Degree-n subgame perfect Nash equilibria

So far, we have characterized the degree-K subgame perfect Nash equilibrium payoff vectors under strengthened perfect monitoring and perfect monitoring, where $K < n$. The approach does not apply to the degree-n subgame perfect NAsh equilibrium because $\bigcup_{s\in S} V_s^n = V^n = \emptyset$. However, it does not mean that there does not exist any degree-n subgame perfect Nash equilibrium payoff vector. Notice that any degree-n subgame perfect Nash equilibrium continuation payoff at any history maximizes the sum of all players' payoffs. Hence, in this section, we provide a new approach to characterize the conditions for equilibrium payoff vectors.

At first, we consider an infinitely repeated game with strengthened perfect monitoring. Let

$$\bar{S} = \{s \in S | u_N(s) = \max_{a\in A} u_N(a)\}$$

and

$$W = co\{u(s) : s \in \bar{S}\}.$$

Note that in the stage game the maximum sum of all players' payoffs is $\max_{a\in A} u_N(a)$, i.e, W is the set of payoff vectors which maximizes the sum of payoffs of the grand coalition. Therefore, at any history of a degree-n subgame perfect Nash equilibrium, the prescribed action profile must belong to $\bar{S}$. Because a coalition $c \in C_{n-1}$ always has the option of playing a best response to the actions chosen by the rest of the players, in a degree-n subgame perfect Nash equilibrium the coalition c's smallest sum of payoffs consistent with optimization is

$$\underline{w}_c = \min_{s_{-c}\in\bar{S}_{-c}} \max_{a_c\in A_c} u_c(a_c, s_{-c}).$$

Define $\bar{m}^c \in \bar{S}$ as a profile which satisfies

$$\underline{w}_c = u_c(\bar{m}_c^c, \bar{m}_{-c}^c) = \max_{a_c\in A_c} u_c(a_c, \bar{m}_{-c}^c),$$

where $c \in C_{n-1}$. Then the set of feasible and degree-n weakly rational payoff vectors can be defined by

$$W^n = \{v \in W : v_c > \underline{w}_c \forall c \in C_{n-1}\}.$$

As we argued, a degree-n subgame perfect Nash equilibrium specifies an action profile in $\bar{S}$ at any history. Then the grand coalition will not deviate in any period. Therefore, we effectively consider a degree-$(n-1)$ subgame perfect Nash equilibrium strategy in which any action profile at any history belongs to $\bar{S}$. Then we can show that if W^n satisfies degree-$(n-1)$ NEU, then any $v \in W^n$ can arise as a degree-n subgame perfect Nash equilibrium with strengthened perfect monitoring.

Theorem 6. *If W^n satisfies degree-$(n-1)$ NEU, then for any $v \in W^n$, there exists $\underline{\delta} \in (0,1)$ such that for all $\delta \in (\underline{\delta}, 1)$, v is a degree-n subgame perfect Nash equilibrium payoff vector of $G(\delta)$ with strengthened perfect monitoring.*

Proof. Fix any $v \in W^n$. By Lemma 1, there exists a set $\{x^c\}_{c \in C_{n-1}} \subseteq W^n$ such that for any $c \in C_{n-1}$ and $c' \in C_{n-1} \backslash \{c\}$

$$x_c^c < x_c^{c'},$$

since W^n satisfies degree-$(n-1)$ NEU. For each $c \in C_{n-1}$, fix a payoff vector $y^c \in W^n$ which satisfies $y_c^c = \min_{s \in \bar{S}} u_c(s)$. Now for each $c \in C_{n-1}$, consider this payoff vector

$$(1-\eta)v + \eta\zeta x^c + \eta(1-\zeta)y^c$$

where $\eta \in (0,1)$ and $\zeta \in (0,1)$. Then for any $c \in C_{n-1}$ and $c' \in C_{n-1} \backslash \{c\}$,

$$(1-\eta)v_c + \eta\zeta x_c^c + \eta(1-\zeta)y_c^c < (1-\eta)v_c + \eta\zeta x_c^{c'} + \eta(1-\zeta)y_c^{c'},$$

due to $y_c^c = \min_{s \in \bar{S}} u_c(s)$ and $x_c^c < x_c^{c'}$. So for all $c \in C_{n-1}$ and $c' \in C_{n-1}$, $v_{c'} > \underline{w}_{c'}$ and $x_{c'}^c > \underline{w}_{c'}$, since $v \in W^n$ and $x^c \in W^n$. Hence there exists $\bar{\eta} > 0$ such that for any $\eta \in (0, \bar{\eta})$

$$(1-\eta)v_{c'} + \eta\zeta x_{c'}^c + \eta(1-\zeta)y_{c'}^c > \underline{w}_{c'},$$

for all $c \in C_{n-1}$ and $c' \in C_{n-1}$, i.e., $(1-\eta)v + \eta\zeta x^c + \eta(1-\zeta)y^c \in W^n$ for all $c \in C_{n-1}$. By $y_c^c = \min_{s \in \bar{S}} u_c(s) \leq \underline{w}_c < v_c$, there must exist $\bar{\zeta} > 0$ such that for any $\zeta \in (0, \bar{\zeta})$,

$$(1-\eta)v_c + \eta(\zeta x_c^c + (1-\zeta)y_c^c) < v_c,$$

for all $c \in C_{n-1}$.

Therefore, fix $\eta \in (0, \bar{\eta})$ and $\zeta \in (0, \bar{\zeta})$ and let $v^c = (1-\eta)v + \eta\zeta x^c + \eta(1-\zeta)y^c$ for each $c \in C_{n-1}$. Then $v^c \in W^n$ and

$$v_c^c < v_c \tag{21}$$

and for any $c' \in C_{n-1} \backslash \{c\}$

$$v_c^c < v_c^{c'}. \tag{22}$$

For any $c \in C_{n-1}$, since $v^c \in W^n$, we have

$$v_c^c > \underline{w}_c = u_c(\bar{m}^c). \tag{23}$$

Then there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$,

$$\min_{c \in C_{n-1}} \frac{v_c^c - u_c(\bar{m}^c) - |c|\varepsilon}{|c|\varepsilon} > \max_{c \in C_{n-1}, c' \in C_{n-1} \backslash \{c\}} \{ \frac{v_c^c - u_c(\bar{m}^{c'}) - |c|\varepsilon}{v_c^{c'} - v_c^c + |c|\varepsilon}, 0 \},$$

due to (23) and (24). Further, fix $\varepsilon \in (0, \bar{\varepsilon})$, by Lemma 2, there exists $\delta_1 > 0$ for every $\delta \in [\delta_1, 1)$, there exists a natural number l such that

$$\min_{c \in C_{n-1}} \frac{v_c^c - u_c(\bar{m}^c) - |c|\varepsilon}{|c|\varepsilon} > \frac{\delta^l}{1 - \delta^l}$$

and

$$\frac{\delta^l}{1-\delta^l} > \max_{c\in C_{n-1}, c'\in C_{n-1}\setminus\{c\}} \frac{v_c^c - u_c(\bar{m}^{c'}) - |c|\varepsilon}{v_c^{c'} - v_c^c + |c|\varepsilon}.$$

Then for any $c \in C_{n-1}$ and any $c' \in C_{n-1}\setminus\{c\}$,

$$v_c^c - |c|\varepsilon > (1-\delta^l)u_c(\bar{m}^c) + \delta^l v_c^c$$

and

$$(1-\delta^l)u_c(\bar{m}^{c'}) + \delta^l v_c^{c'} > v_c^c - |c|\varepsilon. \tag{24}$$

Hence there exists $\delta_2 \in [\delta_1, 1)$ such that for every $\delta \in [\delta_2, 1)$

$$v_c^c - |c|\varepsilon > (1-\delta)\max_{a\in A} u_c(a) + \delta\{(1-\delta^l)u_c(\bar{m}^c) + \delta^l v_c^c\} \tag{25}$$

for any $c \in C_{n-1}$.

From Lemma 3, there exists $\underline{\delta} \in [\delta_2, 1)$ such that for every $\delta \in [\underline{\delta}, 1)$, there exist a sequence of action profiles $\bar{\pi}^{\emptyset} = \{s^{\emptyset}(t)\}_{t=0}^{\infty}$ and $|C_{n-1}|$ sequences of action profiles $\bar{\pi}^c = \{s^c(t)\}_{t=0}^{\infty}$ $(c \in C_{n-1})$ which satisfy the following equalities and inequalities:

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t u(s^{\emptyset}(t)) = v \tag{26}$$

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t u_i(s^{\emptyset}(T+t)) > v_i - \varepsilon \quad \forall i \in N \quad \forall T \geq 1 \tag{27}$$

$$s^c(t) = \bar{m}^c \quad \forall t < l$$

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t u(s^c(l+t)) = v^c \tag{28}$$

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t u_i(s^c(T+t)) > v_i^c - \varepsilon \quad \forall i \in N \quad \forall T > l. \tag{29}$$

Consider the following strategy profile σ^*:

(A) All players play following $\bar{\pi}^{\emptyset}$ in each period until $r \in C_{n-1}$ is reported;[7]

(B) Whenever if $c \in C_{n-1}$ is reported, then all players start to play following π^c from the next period until another $r' \in C_{n-1}$ is reported.[8]

Now we show that σ^* is a degree-n subgame perfect Nash equilibrium strategy profile by using the one-shot coalitional deviation principle. Note that for any history $h^t \in H^t$, if all players play following $\sigma^*|h^t$, then

$$g_N(\sigma^*|h^t) = \max_{a\in A} u_N(a),$$

i.e., the grand coalition cannot benefit from any one-shot coalitional deviation. Suppose a coalition $c \in C_{n-1}$ chooses a one-shot coalitional deviation in period

[7] If $r = c_n$ is reported, we suppose that all players ignore the report and continue to play following $\bar{\pi}^{\emptyset}$.

[8] As in the former footnote, we suppose that the report $r = c_n$ is ignored.

$t \geq 0$. Notice that for any history $h^t \in H^t$, if all players play following $\sigma^*|h^t$, in period t they should play $s^{\emptyset}(t)$ or $s^{c'}(t')$, where $c' \in C_{n-1}$ and $t' \geq 0$.

First, if they should play $s^{\emptyset}(t)$ or $s^{c'}(t')$ in period t, where $t' \geq l$, then the coalition's sum of continuation payoffs by a one-shot coalitional deviation is less than

$$v_c^c - |c|\varepsilon < g_c(\sigma^*|h^t),$$

due to (21), (22) and (26)-(29). Second, if they should play $s^{c'}(t')$ in period t, where $t' < l$ and $c' \neq c$, then the coalition's sum of continuation payoffs by a one-shot coalitional deviation is less than

$$v_c^c - |c|\varepsilon,$$

due to (25). But if $u_c(\bar{m}^{c'}) < v_c^{c'}$, we have

$$g_c(\sigma^*|h^t) = (1-\delta^{l-t'})u_c(\bar{m}^{c'}) + \delta^{l-t'}v_c^{c'} \geq (1-\delta^l)u_c(\bar{m}^{c'}) + \delta^l v_c^{c'} > v_c^c - |c|\varepsilon$$

due to (24), and if $u_c(\bar{m}^{c'}) \geq v_c^{c'}$, we also have

$$g_c(\sigma^*|h^t) = (1-\delta^{l-t'})u_c(\bar{m}^{c'}) + \delta^{l-t'}v_c^{c'} \geq v_c^{c'} > v_c^c - |c|\varepsilon$$

due to (22). Third, if they should play $s^c(t')$ in period t, where $t' < l$, then the coalition's sum of continuation payoffs by a one-shot coalitional deviation is at most

$$\begin{aligned}(1-\delta)\underline{w}_c + \delta\{(1-\delta^l)u_c(\bar{m}^c) + \delta^l v_c^c\} &< (1-\delta^l)u_c(\bar{m}^c) + \delta^l v_c^c \\ &\leq (1-\delta^{l-t'})u_c(\bar{m}^c) + \delta^{l-t'}v_c^c \\ &= g_c(\sigma^*|h^t),\end{aligned}$$

due to (23).

Therefore, if all players play following the strategy profile σ^*, for any history $h^t \in H^t$ no coalition could benefit from a one-shot coalitional deviation, i.e., σ^* is a degree-n subgame perfect Nash equilibrium strategy profile. Therefore any $v \in W^n$ is a degree-n subgame perfect Nash equilibrium payoff vector of $G(\delta)$ with strengthened perfect monitoring. □

Now consider an infinitely repeated game with perfect monitoring. Unfortunately, since players cannot distinguish hidden deviators, it is difficult to find a degree-n subgame perfect Nash equilibrium in this model expect the case $n = 2$. In that case, perfect monitoring is equivalent to strengthened perfect monitoring, and we have the following results.

Theorem 7. *If $n = 2$ and W^2 satisfies degree-1 NEU, then for any $v \in W^2$, there exists $\underline{\delta} \in (0,1)$ such that for all $\delta \in (\underline{\delta}, 1)$, v is a degree-2 subgame perfect Nash equilibrium payoff vector of $G(\delta)$ with perfect monitoring.*

Proof. It holds by Theorem 6. □

		Player 2	
		C	D
Player 1	C	$0, 3$	$1, 1$
	D	b_1, b_2	$3, 0$

Figure 3: An example of non-empty W^2

Here, we give an example where the set W^n is non-empty. Consider the 2-player game in Figure 3, where $b_1 + b_2 < 3$, $b_1 > 0$ and $b_2 > 0$.

Clearly, $\bar{S} = \{(C, C), (D, D)\}$ and $W = \{(v_1, v_2) \in R^2_+ : v_1 + v_2 = 3\}$. For player 1, the action C is dominated by the action D, and for player 2 the action D is dominated by the action C, since $b_1 > 0$ and $b_2 > 0$. Therefore,

$$W^2 = \{(v_1, v_2) : v_1 + v_2 = 3, v_i > b_i \forall i \in \{1, 2\}\}.$$

6 Conclusion

We introduced a notion of equilibrium, that is the degree-K subgame perfect Nash equilibrium which is an extension of subgame perfect Nash equilibrium. In the equilibrium, the deviations by any coalition whose size is up to K are allowed, and in a coalition we assumed that players' payoffs are transferable. We showed that whether a strategy profile is a degree-K subgame perfect Nash equilibrium or not can be confirmed by the one-shot coalitional deviation principle which is an extension of one-shot deviation principle. We focused on the case that $K \geq 2$, since the degree-1 subgame perfect Nash equilibrium is well studied by previous researches.

When $K \geq 2$, there may exist hidden deviators, so we studied two models where the hidden deviators can and cannot be distinguished, respectively. We first studied an infinitely repeated n-player game with strengthened perfect monitoring. In this model, we assumed that there is an observer who is able to detect a deviating coalition and reports it to all players. We constructed a degree-K subgame perfect Nash equilibrium by the idea that whenever a feasible coalition is reported by the observer, the coalition is minmaxed by the other players long enough to wipe out any gain from its deviation. Then if the set of feasible and degree-K rational payoff vectors satisfies degree-K NEU, we show that any payoff vectors in that set can arise as a degree-K subgame perfect Nash equilibrium with strengthened perfect monitoring.

Then, we analyzed an infinitely repeated n-player game with perfect monitoring. Because a deviation does not reveal all members of the coalition, we constructed a degree-K subgame perfect Nash equilibrium by a simple idea that whenever any deviation by a feasible coalition happens, all players switch to play a joint punishment path to punish everyone. And we showed any payoff vector which is feasible and degree-K simultaneously punishable can arise as a degree-K subgame perfect Nash equilibrium with perfect monitoring.

Finally, we studied degree-n subgame perfect Nash equilibria by providing a new approach to characterize the candidates of the equilibrium payoff vectors. We found that the action profile at any history of any equilibrium must maximize the grand coalition's sum of payoffs, to prevent a deviation by the grand coalition. And we showed that all payoff vectors in the set of feasible and degree-n weakly rational payoff vectors can arise as a degree-n subgame perfect Nash equilibrium with strengthened perfect monitoring, if degree-$(n-1)$ NEU is satisfied. We do not have a counterpart of this result for perfect monitoring, except for the case $n = 2$, where strengthened perfect monitoring and perfect monitoring are equivalent.

It would be interesting to extend the analysis developed here by assuming players' payoffs are not transferable in a coalition. Then players base their decisions as to whether to enter into a coalition on their individual payoffs as coalition members. Intuitively, in this case, the hidden deviators can be ignored since if the deviator who can be observed in a coalition is punished, the coalition will not deviate from a desired path. Namely, it may be that more payoff vectors can arise as equilibria even with perfect monitoring.

Recall the 3-player game in Figure 2. Fudenberg and Maskin (1986) use this game to show that the folk theorem will fail if no restriction is imposed on the set of feasible and degree-1 rational payoff vectors. They showed that each player's payoff must be at least 0.25 in any equilibrium, while each player's minmax payoff is 0. Notice that the payoff vector set is exactly the set of feasible and degree-1 simultaneously punishable payoff vectors and we do not impose any restriction on the set. Thus another interesting extension is to analyze the relationship between the set of feasible and degree-K rational payoff vectors and the set of feasible and degree-K simultaneously punishable payoff vectors. We leave the challenging work for future research.

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