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Abstract

We extend the domain of social preferences, which depend on not only one's outcomes but also vectors of outcomes of all other agents, from deterministic outcome vectors to lotteries over outcome vectors. First, we axiomatically characterize a class of utility functions which satisfies the expected utility theory and reversal of order as far as these two requirements are consistent with ex post fairness and ex ante fairness. Based on this class, we characterize three classes of utility functions which additionally satisfy ex ante fairness, inequality-aversion, and ex post fairness for probability mixture, respectively. Finally, we characterize our main class of utility functions which satisfies these axioms. Saito (2013) also axiomatizes social preferences on lotteries over the outcome vectors which are an extension of the utility functions proposed by Fehr and Schmidt (1999). We characterize a wider class of utility functions which explains heterogeneous attitudes of how much the decision maker cares about ex ante fairness and ex post fairness toward each agent. Moreover, we provide an additional axiom under which our class of utility functions coincides with Saito's class of utility functions. We reveal that the axiom requires that the attitudes are the same among all the agents.

JEL Codes: D81, D63, D91

Keywords: Social preference, Risk, Fairness

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1 Introduction

People sometimes desire not only to get their payoffs but also to realize fairness. To model these people's tendency, Fehr and Schmidt (1999) and Bolton and Ockenfels (2000) present respective classes of preference relations called social preferences, which depend on not only one's own outcomes but also the vectors of outcomes of all other people. The utility functions which represent the preferences reflect fairness of realized outcome vectors in the sense that given one's outcome, their utility maximizes when all other people have the same outcome as them. However, the functions are defined only on vectors of deterministic outcomes, and we are interested in extending the domain of the functions to lotteries over outcome vectors.

When we extend social preferences to the lotteries, two notions of fairness are revealed: *ex post fairness* and *ex ante fairness*. Ex post fairness means fairness of realized outcome vectors, analyzed by Fehr and Schmidt (1999) and Bolton and Ockenfels (2000). In contrast, ex ante fairness is a concept in which people compare each other's opportunities and dislike the difference between the opportunities.

Saito (2013) provides an example that captures some features of interaction between ex post fairness and ex ante fairness,[1] developed from a classical example of Machina (1989). X and Y are brothers and are huge fans of football. Consider that X has one football ticket. He can use the ticket or can flip a coin to decide who gets to go. Whatever X chooses, they cannot get an equal outcome, but if X chooses to flip a coin, the opportunity to get the ticket is equal. Namely, the expected value of X is equal to that of Y.

To study ex post fairness and ex ante fairness, Fudenberg and Levine (2012) investigate axiomatization of social preferences under risk. They present an axiom of ex post fairness and axioms of ex ante fairness. The axiom of ex post fairness is consistent with the independence axiom, and the axioms of ex ante fairness are inconsistent with the independence axiom. Moreover, they propose a class of utility functions which satisfies both ex ante fairness and ex post fairness. The utility functions are a convex combination between the expected value of a Fehr-Schmidt utility and the Fehr-Schmidt utility

[1] In fact, Saito (2013) does not use the terminology of ex ante fairness or ex post fairness but uses “ equality of opportunity ” or “ equality of outcome. ” We regard equality of opportunity as ex ante fairness and regard equality of outcome as ex post fairness.

of the expected values.

Our goal is to characterize the preferences which satisfy both ex ante fairness and ex post fairness by a class of utility functions while preserving the expected utility theory as far as possible. For that purpose, we assume that the preferences are a weak order and continuous because they are consistent with ex ante fairness and ex post fairness. Based on the results of Fudenberg and Levine (2012), we preserve the independence axiom in each range where a set of the agents whose expected values are less than the decision maker is constant. This is because ex ante fairness is consistent with the independence in each range. Particularly, this is consistent with the above example: X is better off in one outcome vector and is worse off in the other.

In addition, we assume that evaluations of the decision maker are not affected by risk as far as it is consistent with ex ante fairness and ex post fairness. This is because our interest is only fairness-related behaviors under risk, not other behaviors of risk, so we want to focus on these two issues of fairness. More precisely, we can rephrase this as follows. Consider a lottery with two outcome vectors, and assume these two outcome vectors give a disadvantage to the same set of agents. This lottery can be regarded as the compound lottery which consists of two lotteries: the binary lottery whose outcome vectors are the two in the original lottery and the lottery whose outcome vectors are all other vectors in the original lottery.

Next, consider another lottery which is obtained by modifying the compound lottery in the following way. The binary lottery in the compound lottery is replaced with the outcome vector which gives each agent the expected value of the binary lottery. Since the lottery modified in this way and the original lottery give a disadvantage to the same set of agents in expectation, ex ante fairness is not an issue. Furthermore, since the two outcome vectors and the outcome vector of the expected values of the binary lottery give a disadvantage to the same set of agents, ex post fairness is not an issue. Since the modified lottery and the original lottery have the same vectors of the expected values, we assume these two lotteries are indifferent. In other words, we assume risk-neutrality as far as it is consistent with ex ante fairness and ex post fairness. Nevertheless, this kind of axiom is known as “reversal of order” as stated in Epstein et al. (2007) and Seo (2009). Therefore, we use the terminology reversal of order.

Our results are as follows. First, we propose three axioms, which require that the preferences satisfy the expected utility theory and reversal of order unless these two requirements are inconsistent with ex post fairness or ex ante fairness. By these three axioms, we characterize a class of utility functions without axioms about fairness. This class has the potential to express various kinds of fairness by adding proper axioms. We propose three axioms about fairness; ex ante fairness, inequality-aversion, and ex post fairness for probability mixture.[2] Based on the above class, we characterize three respective classes of utility functions, which additionally satisfy each axiom. Finally, we characterize our main class of utility functions, putting all these axioms together. We name this class the PAI model standing for ex post fairness for probability mixture, ex ante fairness, and inequality-aversion.

The PAI model can be described as follows. We assume that a vector of all agents' outcomes is stochastically determined, and we express it by a finite lottery over $\mathbb{R}^n$, where n is a number of agents. Let P be a finite lottery over $\mathbb{R}^n$, P_i be the marginal distribution of the i-th component of P for any agent i, and L be the set of all finite lotteries on $\mathbb{R}^n$. We characterize the following class of utility functions: for $(\alpha_{i,+}, \alpha_{i,-}, \beta_{i,+}, \beta_{i,-})_{i\in N\setminus\{1\}}$ such that $\alpha_{i,-} \geq 0$, $\beta_{i,+} \geq 0$, $\alpha_{i,+}+\beta_{i,+} \geq 0$, $\alpha_{i,-}+\beta_{i,-} \geq 0$, and $\alpha_{i,-} - \alpha_{i,+} = \beta_{i,+} - \beta_{i,-} \geq 0$ for any $i \in N \setminus \{1\}$,

$$u(P) = \sum_{x\in supp(P)} v_{S_P}(x)P(x),$$

where $S_P = \{i \in N \setminus \{1\} | E(P_i) \geq E(P_1)\}$ and

$$\begin{aligned} v_S(x) = x_1 - &\sum_{i\in S} [\alpha_{i,-} \max\{x_i - x_1, 0\} + \beta_{i,-} \max\{x_1 - x_i, 0\}] \\ &- \sum_{i\in N\setminus S} [\alpha_{i,+} \max\{x_i - x_1, 0\} + \beta_{i,+} \max\{x_1 - x_i, 0\}] \end{aligned}$$

for any $S \subset N$.

The PAI model can be interpreted as follows. The decision maker changes her von Neumann-Morgenstern utility functions, depending on whether the expected value of the decision maker is smaller or larger than that of the other agents. Since $\beta_{i,+} \geq 0$, the decision maker (weakly) sympathizes with the agents whose outcome levels are less than hers, and since $\alpha_{i,-} \geq 0$, the decision maker (weakly) envies the agents whose

[2]In this paper, we propose two concepts of ex post fairness, inequality-aversion and ex post fairness for probability mixture, as we will state later.

outcome levels are more than hers. By this condition, a PAI model reduces to a Fehr-Schmidt utility function for deterministic outcomes.

The paper most similar to ours is Saito (2013). Saito (2013) axiomatically characterizes the preferences which are represented by the class of utility functions that are proposed by Fudenberg and Levine (2012), and are a convex combination between the expected value of a Fehr-Schmidt utility and the Fehr-Schmidt utility of the expected values. He calls this class the expected inequality-averse (EIA) model and explains the experimental results that people may want to equalize probabilities of winning a good.

His approach consists of two parts. The first part is devoted to characterizing the class of the Fehr-Schmidt utility functions in deterministic outcomes, which is a different approach from Rohde (2010) that also characterizes the class of the Fehr-Schmidt utility functions. In the second part, he provides two axioms to characterize a convex combination between the expected value of a utility function and the utility function of the expected values. One is a weak independence axiom, and the other is an axiom called "dominance." To understand dominance, consider two lotteries. If one is superior to the other from the viewpoints of both ex ante fairness and ex post fairness, one is preferred over the other. In Saito's characterization, ex ante fairness and ex post fairness are not directly assumed, and it is an open question that when we directly assume ex ante fairness and ex post fairness what kinds of classes of utility functions can be characterized. The PAI model is an answer to the question.

In fact, the EIA model is a special case of the PAI model, and the following example gives an important difference. Consider a trio in an organization, Edgar, Allan, and Poe. Edgar is the boss of Allan and Poe and is wondering who is suitable to give a risky task. If they do their daily task, they can get 1 thousand dollars. If they succeed in the risky task, they can get 2 thousand dollars, but if they fail, they get nothing. The probability of the success and the failure is 1/2 respectively. Let P^A be a lottery which gives 2 or 0 for Allan with probability 1/2 respectively and gives 1 for the two others. Similarly, let P^P be a lottery which gives 2 or 0 for Poe with probability 1/2 respectively and gives 1 for the two others.

In this case, P^A is indifferent to P^P under the EIA model. This is a very natural result, but consider the following additional assumption. The organization is small enough for all the people in it to know each other's tasks. In contrast, outcomes of the tasks are not always known by all. Also, the people in it care equally about Allan and Poe. Allan is talkative and tells the other people the outcome of his task, and Poe is shy and cannot tell the other people the outcome of his task. Then, many people are aware of Allan's outcome and may envy or sympathize with him, but they are relatively unaware of Poe's *unfair* outcome in the sense of ex post fairness. This assumption may make Edgar treat Allan and Poe differently because Edgar may care about the others' evaluation or social images. If he chooses P^A, the others are clearly aware of the *unfair* choice in the sense of ex post fairness.

There is a tendency that people want to be perceived as fair, as Andreoni and Bernheim (2009) show. In other words, when there is an audience, many people want to equalize the outcomes, as Andreoni and Petrie (2004) and Rege and Telle (2004) also point out, which can be understood as one of the "audience effects" in psychology. This leads Edgar to put weight strongly on ex post fairness for Allan, and then Edgar strictly prefers P^P to P^A. The EIA model cannot explain this variety of Edgar's preferences, but the PAI model can.

Moreover, some papers show that people's decisions depend on who is affected by the decisions in risky and social situations. For example, Montinari and Rancan (2018) and Müller and Rau (2019) experimentally find out that people's decisions are influenced by social distance or social context in risky and social situations. Similarly, Long and Krause (2017) point out that decisions of people depend on the age or social proximity of others when they are affected by the decisions through two surveys. In particular, Montinari and Rancan (2018) show that on average, individual behaviors are closer to maximizing the expected value when deciding on behalf of a friend rather than a stranger. Namely, people may place more importance on ex ante fairness when deciding on behalf of a friend. However, the EIA model cannot explain this kind of behavior because how much the decision maker cares about ex ante fairness and ex post fairness toward the other agents does not differ across the agents. The PAI model relaxes this point and may explain various behaviors in the laboratory experiments.

Other papers related to ours are as follows. Many papers including Cappelen et al. (2013) and Brock et al. (2013) experimentally confirm that people care about not only ex post fairness but also ex ante fairness. Karni and Safra (2002) and Borah (2020) examine morals or fairness with axiomatic characterizations against the backdrop of

Harsanyi's impartial observer setting (e.g. Harsanyi (1953) and Harsanyi (1955)), while our paper has a different setting. Saito (2015) investigates altruism and impure altruism with axiomatic characterizations. This paper is interested in distinguishing altruism from impure altruism (and distinguishing selfishness from impure selfishness), which is not our goal. Rohde (2010) characterizes the class of the Fehr-Schmidt utility functions in deterministic outcomes, but she does not take risk into account.

This paper is structured as follows. In Section 2, we will propose axioms and characterize several classes of utility functions. In Section 3, we will examine the relationship between our paper and Saito (2013).

2 Representation theorems

We consider a set of n agents $\{1, 2, \ldots, n\}$, and let agent 1 be the decision maker. We will define lotteries over $\mathbb{R}^n$ which represent random vectors of outcomes of n agents.

Definition 1. $P : \mathbb{R}^n \to \mathbb{R}$ is a *lottery* if and only if

1. $P(x) \geq 0$ for all $x \in \mathbb{R}^n$,
2. there exists a finite subset $A \subset \mathbb{R}^n$ such that $P(x) = 0$ for all $x \in A^c$, and
3. $\sum_{x \in supp(P)} P(x) = 1$, where $supp(P) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | P(x_1, \ldots, x_n) > 0\}$.

Let L denote the set of all lotteries. If $(x_1, \ldots, x_n) \in \mathbb{R}^n$ is realized, we assume agent i gets x_i for any $i = 1, \ldots, n$. A lottery is a discrete probability distribution over $\mathbb{R}^n$ with a finite support. Let P_i be the marginal distribution of the i-th component of P for any agent i. We define $\gamma P \oplus (1-\gamma)Q$ for any $P \in L$, any $Q \in L$, and any $\gamma \in [0, 1]$ as $\{\gamma P \oplus (1-\gamma)Q\}(x) = \gamma P(x) + (1-\gamma)Q(x)$ for any $x \in \mathbb{R}^n$. For any $P \in L$ and any function $f : \mathbb{R}^n \to \mathbb{R}$, let $E(P_i)$ be the expected value of P_i for any $i \in N$, and let $E_P(f) = \sum_{x \in suppP} f(x)P(x)$.

We will define expected quasi-comonotonicity, which is explained as follows. Consider that two lotteries are expectedly quasi-comonotonic. Then, if one lottery gives an agent more in expectation compared to the decision maker, the other lottery also gives the agent more in expectation, and vice versa.

Definition 2. Let $P, Q \in L$. P and Q are *expectedly quasi-comonotonic* if and only if there is no $i \in N$ such that $\{E(P_i) - E(P_1)\}\{E(Q_i) - E(Q_1)\} < 0$ holds.

This definition is an extension of the definition of quasi-comonotonicity in Saito (2013). He defines it only on $\mathbb{R}^n$, but we define expected quasi-comonotonicity on the set of the lotteries, L. For each lottery, let a set of the agents whom the lottery gives more compared to the decision maker in expectation be called expectedly richer agents, denoted by $S_P^r = \{i \in N \setminus \{1\} | E(P_i) > E(P_1)\}$. Similarly, let a set of the agents whom the lottery gives equally in expectation be called expectedly equal agents, denoted by $S_P^e = \{i \in N \setminus \{1\} | E(P_i) = E(P_1)\}$, and let a set of the agents whom the lottery gives less in expectation be called expectedly poorer agents, denoted by $S_P^p = \{i \in N \setminus \{1\} | E(P_i) < E(P_1)\}$. By using these notations, the definition of expected quasi-comonotonicity can be rephrased as follows. $P, Q \in L$ are expectedly quasi-comonotonic if and only if $S_P^r \cap S_Q^p = \emptyset$ and $S_P^p \cap S_Q^r = \emptyset$.

Notice that we can classify the lotteries into the ranges where a set of expectedly poorer agents is common. For any range, any probability mixture of two lotteries in the range are in that range, and any two lotteries in the range are expectedly quasi-comonotonic.

Next, we consider a preference relation $\succeq$ on L of the decision maker. We present some axioms of the preference relation which will be used. We sometimes write $(x_1, \ldots, x_n) \in \mathbb{R}^n$ as the lottery whose outcome vector is $(x_1, \ldots, x_n)$ with a probability of 1.

The following axiom requires that $\succeq$ is a weak order and continuous, and they are two of the three axioms of the expected utility theory. In addition, the axiom requires that the better an outcome is, the more preferred the outcome vector which gives all the agents the outcome is.

Axiom 1 (*Rationality*)**.** A preference relation $\succeq$ is a weak order, continuous[3], and monotonic in equal outcome vectors, that is, $(x, \ldots, x) \succ (y, \ldots, y)$ if and only if $x > y$.

[3]This continuity is assumed with respect to probability mixture. Namely, if $P \succ Q$ and $Q \succ R$, there exist some $\eta, \theta \in (0, 1)$ such that

$$\eta P \oplus (1 - \eta) R \succ Q \succ \theta P \oplus (1 - \theta) R.$$

This axiom is almost identical to the “Rationality” axiom in Saito (2013). The difference is as follows. We assume continuity with respect to probability mixture, but Saito (2013) assumes it with respect to not only probability mixture but also outcome mixture.

The expected utility theory assumes the independence axiom, but Fudenberg and Levine (2012) point out that ex ante fairness is not consistent with the independence axiom. Nevertheless, we preserve the independence in each range where any two lotteries are expectedly quasi-comonotonic, as the next axiom requires.

Axiom 2 (*Expectedly quasi-comonotonic independence for probability mixture)***.** If $P, Q, R \in L$ satisfy that any pair among P, Q, and R are expectedly quasi-comonotonic and that $P \succ Q$,

$$\gamma P \oplus (1-\gamma)R \succ \gamma Q \oplus (1-\gamma)R$$

for any $\gamma \in (0,1)$.

Next, we propose the following axiom about reversal of order.

Axiom 3 (*Quasi-comonotonic reversal of order)***.** Suppose $x, y \in \mathbb{R}^n$ are expectedly quasi-comonotonic. For any $P \in L$ and any $\lambda \in [0,1]$,

$$\lambda P \oplus (1-\lambda)\{\gamma x + (1-\gamma)y\} \sim \lambda P \oplus (1-\lambda)\{\gamma x \oplus (1-\gamma)y\}$$

holds for any $\gamma \in (0,1)$.[4]

Since $x, y \in \mathbb{R}^n$ are (expectedly) quasi-comonotonic, any pair among x, y, and $\gamma x + (1-\gamma)y$ are (expectedly) quasi-comonotonic. Additionally, $\lambda P \oplus (1-\lambda)\{\gamma x + (1-\gamma)y\}$ and $\lambda P \oplus (1-\lambda)\{\gamma x \oplus (1-\gamma)y\}$ are also expectedly quasi-comonotonic. Therefore, when comparing these two lotteries, ex ante fairness and ex post fairness are not issues. As stated in the introduction, our interest is fairness-related behaviors under risk but not other behaviors under risk, and these two lotteries have the same vectors of the expected values. To get a parsimonious model, we propose this axiom, which requires that these two lotteries are indifferent. In fact, this axiom is mathematically very similar to the

[4]In fact, our statements we will prove hold even if this axiom is weakened as follows. The indifference holds only when for any $\gamma \in [0,1]$, any pair among $\lambda P \oplus (1-\lambda)x$, $\lambda P \oplus (1-\lambda)y$, and $\lambda P \oplus (1-\lambda)\{\gamma x \oplus (1-\gamma)y\}$ are expectedly quasi-comonotonic. We adopt the original axiom because the additional assumption has little interpretation.

axiom of “ Indifference to Mixture Timing of Constant Acts ” proposed by Ke and Zhang (2020) in the context of ambiguity-aversion. The difference is that they assume reversal of order between any act and a constant act, but we assume it between two (expectedly) quasi-comonotonic outcome vectors.

By these three axioms, we can characterize the following class of utility functions.

Theorem 1. $\succeq$ on L satisfies Axioms 1, 2, and 3 if and only if there exist $(\alpha_{i,+}, \alpha_{i,-}, \beta_{i,+}, \beta_{i,-})_{i\in N\setminus\{1\}}$ such that

1. $\succeq$ is represented by
$$u(P) = \sum_{x\in supp(P)} v_{S_P}(x)P(x),$$
where $S_P = \{i \in N \setminus \{1\}|E(P_i) \geq E(P_1)\}$ and
$$\begin{aligned} v_S(x) = x_1 &- \sum_{i\in S} [\alpha_{i,-} \max\{x_i - x_1, 0\} + \beta_{i,-} \max\{x_1 - x_i, 0\}] \\ &- \sum_{i\in N\setminus S} [\alpha_{i,+} \max\{x_i - x_1, 0\} + \beta_{i,+} \max\{x_1 - x_i, 0\}] \end{aligned}$$
for any $S \subset N$, and

2. $\alpha_{i,-} - \alpha_{i,+} = \beta_{i,+} - \beta_{i,-}$ for any $i \in N \setminus \{1\}$.

Proof. See Appendix A.1.

□

This class of utility functions characterizes the preference relation which satisfies the expected utility theory and reversal of order unless these two requirements are inconsistent with ex ante fairness or ex post fairness. An interpretation of this is that the decision maker evaluates a realizing outcome by v_{S_P}, where S_P is a set of the agents who are not the expectedly poorer agents. In other words, the decision maker changes her von Neumann-Morgenstern utility functions, depending on who the expectedly poorer agents are. This class of utility functions is similar to the class of the expected utility of which von Neumann-Morgenstern utility is a Fehr-Schmidt utility function:
$$E_P(U^{FS}) = \sum_{x\in supp(P)} U^{FS}(x)P(x),$$

where U^{FS}, the utility function proposed by Fehr and Schmidt (1999), is

$$U^{FS}(x) = x_1 - \sum_{i=2}^{n}(\alpha_i \max\{x_i - x_1, 0\} + \beta_i \max\{x_1 - x_i, 0\}),$$

where $\alpha_i \geq 0$ and $\beta_i \geq 0$ for any $i = 1, \dots, n$. The difference between these two classes can be captured by comparison of v_S and U^{FS}.

Both v_S and U^{FS} have two parts, the self-interest part and the part that changes in proportion to the differences between her outcome and those of the others. The coefficients of proportionality depend on each agent. Also, the coefficients of proportionality for each agent also depend on whether the outcome gives more or fewer to the agent compared to the decision maker. α with respective subscripts represents the coefficients when the outcome gives more, and β with respective subscripts represents the coefficients when the outcome gives fewer. Then, if these coefficients are not negative, α means a degree of envy, and β means a degree of sympathy by the definitions of v_S and U^{FS}.

The differences between v_S and U^{FS} are the following two points. First, the coefficients for each agent in v_S depend on whether the agent is expectedly poorer or not, but not in U^{FS}. The coefficients when agent i is expectedly poorer are $\alpha_{i,+}$ and $\beta_{i,+}$, and the coefficients when agent i is not expectedly poorer are $\alpha_{i,-}$ and $\beta_{i,-}$.

Second, the coefficients of proportionality in U^{FS} cannot be negative but can be negative in v_S. In contrast, the parameter condition of v_S is only $\alpha_{i,-} - \alpha_{i,+} = \beta_{i,+} - \beta_{i,-}$. By this condition, this class of utility functions does not depend on whether the coefficients for the expectedly equal agents are $(\alpha_{i,+}, \beta_{i,+})$ or $(\alpha_{i,-}, \beta_{i,-})$. In other words, if $S \subset N$ includes all the expectedly richer agents and does not include any expectedly poorer agent, $u(P) = \sum_{x \in supp(P)} v_S(x)P(x)$ holds, as Lemma 3 in Appendix A1 shows.

We propose this class as the broadest one in this paper, and this class has the potential to explain various fairness-related behaviors by adding proper axioms. We will propose the following three axioms, which express kinds of fairness. The first axiom requires ex ante fairness.

Axiom 4 (*Ex ante fairness*)**.** If $P, Q \in L$ satisfy $P \sim Q$, then for any $\gamma \in (0, 1)$,

$$\gamma P \oplus (1 - \gamma)Q \succeq P \sim Q.$$

The meanings of Axiom 4 are as follows. There are two indifferent lotteries, P and Q. Under Axioms 1 and 2, if P and Q are expectedly quasi-comonotonic, $\gamma P \oplus (1-\gamma)Q \sim P \sim Q$ holds. Then, this axiom is effective only when P and Q are not expectedly quasi-comonotonic, so assume that. In this case, there exists an agent such that compared to the agent, one lottery gives more to the decision maker in expectation, but the other lottery gives fewer to the decision maker in expectation. Axiom 4 requires that the decision maker (weakly) wants to select the two lotteries stochastically. This axiom comes from Machina's example which is the original version of Saito's example in the introduction. In addition, this axiom is mathematically the same as "Uncertainty Aversion" as stated by Gilboa and Schmeidler (1989).

The second axiom expresses inequality-aversion. Let $(x,(y)_{-i})$ denote a vector of $\mathbb{R}^n$ which gives x only to agent i and gives y to all other agents.

Axiom 5 (*Inequality-aversion*)**.** For any $i \in N \setminus \{1\}$, $(0,\dots,0) \succeq (1,(0)_{-i})$ and $(0,\dots,0) \succeq (-1,(0)_{-i})$ hold.

This axiom means that the decision maker (weakly) envies the agents whose outcome levels are more than hers, and the decision maker (weakly) sympathizes with the agents whose outcome levels are fewer than hers, as is a property of the Fehr-Schmidt utility functions. This axiom is identical to the axiom of inequality-aversion, which is also assumed by Saito (2013). Notice that this axiom expresses a part of the concept of ex post fairness in the sense that equality of outcomes is (weakly) preferred. Additionally, this axiom does not treat a lottery essentially.

See the third axiom, which expresses ex post fairness under risk.

Axiom 6 (*Ex post fairness for probability mixture*)**.** If $P,Q \in L$ satisfy $supp(P) = \{(0,\dots,0),(1,\dots,1)\}$, $supp(Q) = \{(0,(1)_{-i}),(1,(0)_{-i})\}$, and $P(0,\dots,0) = Q(1,(0)_{-i}) = P(1,\dots,1) = Q(0,(1)_{-i}) = 1/2$ for any $i \neq 1$, $P \succeq Q$ holds.

This axiom is adequate to call ex post fairness in the sense that equality of outcomes is (weakly) preferred even if an expected value of the decision maker is the same as those of the other agents. This axiom is an extension of the axiom of ex post fairness proposed by Fudenberg and Levine (2012). They propose their ex post fairness only when $n = 2$, and we extend it to the general number. Both inequality-aversion and ex post fairness for probability mixture are a part of the concept of ex post fairness, but inequality-aversion does not treat lotteries. To supplement inequality-aversion, we need this axiom, which treats lotteries.

We can show the following three propositions corresponding to the above three axioms respectively, based on Theorem 1. By the first proposition, we characterize the preferences which satisfy ex ante fairness in addition to Axioms 1, 2, and 3 by a class of utility functions.

Proposition 1. $\succeq$ on L satisfies Axioms 1, 2, 3, and 4 if and only if there exist $(\alpha_{i,+}, \alpha_{i,-}, \beta_{i,+}, \beta_{i,-})_{i \in N \setminus \{1\}}$ such that

1. $\succeq$ is represented by
$$u(P) = \sum_{x \in supp(P)} v_{S_P}(x) P(x),$$
where $S_P = \{i \in N \setminus \{1\} | E(P_i) \geq E(P_1)\}$ and
$$\begin{aligned} v_S(x) = x_1 &- \sum_{i \in S} [\alpha_{i,-} \max\{x_i - x_1, 0\} + \beta_{i,-} \max\{x_1 - x_i, 0\}] \\ &- \sum_{i \in N \setminus S} [\alpha_{i,+} \max\{x_i - x_1, 0\} + \beta_{i,+} \max\{x_1 - x_i, 0\}] \end{aligned}$$
for any $S \subset N$, and

2. $\alpha_{i,-} - \alpha_{i,+} = \beta_{i,+} - \beta_{i,-} \geq 0$ for any $i \in N \setminus \{1\}$.

Proof. See Appendix A.2.

□

The differences between Theorem 1 and this proposition are the parameter conditions, $\alpha_{i,-} - \alpha_{i,+} \geq 0$ and $\beta_{i,+} - \beta_{i,-} \geq 0$. $\alpha_{i,-} - \alpha_{i,+} \geq 0$ requires that if a lottery gives fewer to agent i than to the decision maker in expectation, she exhibits a forgiving attitude toward agent i even though the decision maker's outcome is less than that of agent i. Similarly, $\beta_{i,+} - \beta_{i,-} \geq 0$ requires that if a lottery gives more to agent i than to the decision maker in expectation, she feels less sympathy even when the decision maker's outcome is larger than that of agent i.

By the following second proposition, we characterize the preferences which satisfy inequality-aversion in addition to Axioms 1, 2, and 3 by a class of utility functions.

Proposition 2. $\succeq$ on L satisfies Axioms 1, 2, 3, and 5 if and only if there exist $(\alpha_{i,+}, \alpha_{i,-}, \beta_{i,+}, \beta_{i,-})_{i \in N \setminus \{1\}}$ such that

1. $\succeq$ is represented by

$$u(P) = \sum_{x \in supp(P)} v_{S_P}(x) P(x),$$

where $S_P = \{i \in N \setminus \{1\} | E(P_i) \geq E(P_1)\}$ and

$$\begin{aligned} v_S(x) = x_1 &- \sum_{i \in S} [\alpha_{i,-} \max\{x_i - x_1, 0\} + \beta_{i,-} \max\{x_1 - x_i, 0\}] \\ &- \sum_{i \in N \setminus S} [\alpha_{i,+} \max\{x_i - x_1, 0\} + \beta_{i,+} \max\{x_1 - x_i, 0\}] \end{aligned}$$

for any $S \subset N$, and

2. $\alpha_{i,-} \geq 0$, $\beta_{i,+} \geq 0$, and $\alpha_{i,-} - \alpha_{i,+} = \beta_{i,+} - \beta_{i,-}$ for any $i \in N \setminus \{1\}$.

Proof. ($\Rightarrow$) By the proof of Theorem 1, we only have to prove $\beta_{i,+} \geq 0$ and $\alpha_{i,-} \geq 0$ for any $i \in N \setminus \{1\}$. Fix $i \neq 1$ arbitrarily. By Axiom 5,

$$\begin{aligned} (0, \ldots, 0) \succeq (1, (0)_{-i}) \Leftrightarrow\ & u(0, \ldots, 0) \geq u(1, (0)_{-i}) \\ \Leftrightarrow\ & 0 \geq -\alpha_{i,-} \\ \Leftrightarrow\ & \alpha_{i,-} \geq 0 \end{aligned}$$

holds. Similarly to the above, we can show $\beta_{i,+} \geq 0$.
($\Leftarrow$) We only have to check Axiom 5 by the proof of Theorem 1, and it is easy to make sure that the utility functions satisfy Axiom 5 in the same way as the above discussion.

□

The differences between Theorem 1 and this proposition are the parameter conditions, $\alpha_{i,-} \geq 0$ and $\beta_{i,+} \geq 0$. By these parameter conditions, these utility functions reduce into a Fehr-Schmidt utility function on $\mathbb{R}^n$, and the parameter conditions correspond to the parameter conditions of the Fehr-Schmidt utility functions, $\alpha_i \geq 0$ and $\beta_i \geq 0$. Consequently, $\alpha_{i,-}$ expresses a degree of envy, and $\beta_{i,+}$ expresses a degree of sympathy in deterministic situations.

By the following third proposition, we characterize the preferences which satisfy ex post fairness for probability mixture in addition to Axioms 1, 2, and 3 by a class of utility functions.

Proposition 3. $\succeq$ on L satisfies Axioms 1, 2, 3, and 6 if and only if there exist $(\alpha_{i,+}, \alpha_{i,-}, \beta_{i,+}, \beta_{i,-})_{i \in N \setminus \{1\}}$ such that

1. $\succeq$ is represented by

$$u(P) = \sum_{x \in supp(P)} v_{S_P}(x) P(x),$$

where $S_P = \{i \in N \setminus \{1\} | E(P_i) \geq E(P_1)\}$ and

$$\begin{aligned} v_S(x) = x_1 &- \sum_{i \in S} [\alpha_{i,-} \max\{x_i - x_1, 0\} + \beta_{i,-} \max\{x_1 - x_i, 0\}] \\ &- \sum_{i \in N \setminus S} [\alpha_{i,+} \max\{x_i - x_1, 0\} + \beta_{i,+} \max\{x_1 - x_i, 0\}] \end{aligned}$$

for any $S \subset N$, and

2. $\alpha_{i,+} + \beta_{i,+} \geq 0$, $\alpha_{i,-} + \beta_{i,-} \geq 0$, and $\alpha_{i,-} - \alpha_{i,+} = \beta_{i,+} - \beta_{i,-}$ for any $i \in N \setminus \{1\}$.

Proof. ($\Rightarrow$) By the proof of Theorem 1, we only have to prove $\alpha_{i,+} + \beta_{i,+} \geq 0$ and $\alpha_{i,-} + \beta_{i,-} \geq 0$ for any $i \in N \setminus \{1\}$. Fix $i \neq 1$ arbitrarily. Assume that $P, Q \in L$ satisfy $supp(P) = \{(0, \ldots, 0), (1, \ldots, 1)\}$, $supp(Q) = \{(0, (1)_{-i}), (1, (0)_{-i})\}$, and $P(0, \ldots, 0) = Q(1, (0)_{-i}) = P(1, \ldots, 1) = Q(0, (1)_{-i}) = 1/2$. Then,

$$\begin{aligned} P \succeq Q &\Leftrightarrow u(P) \geq u(Q) \\ &\Leftrightarrow P(0, \ldots, 0) v_{N \setminus \{1\}}(0, \ldots, 0) + P(1, \ldots, 1) v_{N \setminus \{1\}}(1, \ldots, 1) \\ &\quad \geq Q(1, (0)_{-i}) v_{N \setminus \{1\}}(1, (0)_{-i}) + Q(0, (1)_{-i}) v_{N \setminus \{1\}}(0, (1)_{-i}) \\ &\Leftrightarrow \frac{1}{2}\{v_{N \setminus \{1\}}(0, \ldots, 0) - v_{N \setminus \{1\}}(1, (0)_{-i})\} \\ &\quad + \frac{1}{2}\{v_{N \setminus \{1\}}(1, \ldots, 1) - v_{N \setminus \{1\}}(0, (1)_{-i})\} \geq 0 \\ &\Leftrightarrow \alpha_{i,-} + \beta_{i,-} \geq 0 \end{aligned}$$

holds by Axiom 6. Since $\alpha_{i,-} - \alpha_{i,+} = \beta_{i,+} - \beta_{i,-}$, $\alpha_{i,-} + \beta_{i,-} \geq 0$ and $\alpha_{i,+} + \beta_{i,+} \geq 0$ for all $i \in N \setminus \{1\}$.
($\Leftarrow$) We must only check Axiom 6 by the proof of Theorem 1, and it is easy to make sure that the utility functions satisfy Axiom 6 in the same way as the above discussion.

□

The differences between Theorem 1 and this proposition are the parameter conditions, $\alpha_{i,+} + \beta_{i,+} \geq 0$ and $\alpha_{i,-} + \beta_{i,-} \geq 0$. To understand the parameter conditions of $\alpha_{i,+} + \beta_{i,+} \geq 0$ and $\alpha_{i,-} + \beta_{i,-} \geq 0$, consider the following example. There are two agents called agent 1 and agent 2, and agent 1 is the decision maker.

Consider $\frac{1}{2}(1,-1) \oplus \frac{1}{2}(-1,1)$. The vector of the expected values of this lottery is $(0,0)$. If we assume Axioms 1, 2, and 3 but not Axiom 6, there can be the preference, $\frac{1}{2}(1,-1) \oplus \frac{1}{2}(-1,1) \succ (0,0)$. This is because

$$u\left(\frac{1}{2}(1,-1) \oplus \frac{1}{2}(-1,1)\right) = \frac{1}{2}(1-2\beta_{2,-}) + \frac{1}{2}(-1-2\alpha_{2,-}) = -(\alpha_{2,-}+\beta_{2,-}),$$

and $-(\alpha_{2,-}+\beta_{2,-})$ can be positive if we do not assume Axiom 6. (Notice that $u(0,0) = 0$.) The outcomes for agent 1 of the lottery $\frac{1}{2}(1,-1) \oplus \frac{1}{2}(-1,1)$ never equal to those for agent 2, but the outcome for agent 1 of the lottery $(0,0)$ always equals to that for agent 2. Both of the vectors of the expected values of these two lotteries are $(0,0)$. Therefore, if agent 1 prefers $\frac{1}{2}(1,-1) \oplus \frac{1}{2}(-1,1)$ to $(0,0)$, agent 1 prefers tossing a coin. However, $(0,0)$ is preferred from the viewpoint of ex post fairness because the outcome for agent 1 are the same as that for agent 2. To avoid kinds of preferences which satisfy $\frac{1}{2}(1,-1) \oplus \frac{1}{2}(-1,1) \succ (0,0)$, Axiom 6 is required.

Notice that both Axiom 5 and Axiom 6 express the tendency of people to dislike unequal outcomes, but the axiom of inequality-aversion does not deal with the lotteries essentially. The above issue is raised only when we take the lotteries into account. That is why we need Axiom 6, which supplements Axiom 5.

By putting these propositions together, we can show the following theorem as our main theorem. We name the following class of utility functions the PAI model standing for three axioms, ex post fairness for probability mixture, ex ante fairness, and inequality-aversion.

Theorem 2. $\succeq$ on L satisfies Axioms 1, 2, 3, 4, 5, and 6 if and only if there exist $(\alpha_{i,+}, \alpha_{i,-}, \beta_{i,+}, \beta_{i,-})_{i \in N \setminus \{1\}}$ such that

1. $\succeq$ is represented by

$$u(P) = \sum_{x \in supp(P)} v_{S_P}(x) P(x),$$

where $S_P = \{i \in N \setminus \{1\} | E(P_i) \geq E(P_1)\}$ and

$$\begin{aligned} v_S(x) = x_1 &- \sum_{i \in S} [\alpha_{i,-} \max\{x_i - x_1, 0\} + \beta_{i,-} \max\{x_1 - x_i, 0\}] \\ &- \sum_{i \in N \setminus S} [\alpha_{i,+} \max\{x_i - x_1, 0\} + \beta_{i,+} \max\{x_1 - x_i, 0\}] \end{aligned}$$

for any $S \subset N$, and

2. $\alpha_{i,-} \geq 0$, $\beta_{i,+} \geq 0$, $\alpha_{i,+} + \beta_{i,+} \geq 0$, $\alpha_{i,-} + \beta_{i,-} \geq 0$, and $\alpha_{i,-} - \alpha_{i,+} = \beta_{i,+} - \beta_{i,-} \geq 0$ for any $i \in N \setminus \{1\}$.

Proof. We can show this theorem by the proofs of Theorem 1 and Propositions 1, 2, and 3.

□

The PAI model satisfies the properties of the classes in Theorem 1 and Propositions 1, 2, and 3. In particular, the PAI model satisfies inequality-aversion, ex post fairness for probability mixture, and ex ante fairness, and depending on who the expectedly poorer agents are, the decision maker's von Neumann-Morgenstern utility functions are changed.

The following proposition implies that the PAI model can be rewritten as a minimization by Axioms 1, 2, 3, and 4.

Proposition 4. $\succeq$ on L satisfies Axioms 1, 2, 3, and 4 if and only if there exist $(\alpha_{i,+}, \alpha_{i,-}, \beta_{i,+}, \beta_{i,-})_{i \in N \setminus \{1\}}$ such that

1. $\succeq$ is represented by

$$u(P) = \min_{S \subset N \setminus \{1\}} \sum_{x \in supp(P)} v_S(x) P(x),$$

where

$$\begin{aligned} v_S(x) = x_1 &- \sum_{i \in S} [\alpha_{i,-} \max\{x_i - x_1, 0\} + \beta_{i,-} \max\{x_1 - x_i, 0\}] \\ &- \sum_{i \in N \setminus S} [\alpha_{i,+} \max\{x_i - x_1, 0\} + \beta_{i,+} \max\{x_1 - x_i, 0\}] \end{aligned}$$

for any $S \subset N$, and

2. $\alpha_{i,-} - \alpha_{i,+} = \beta_{i,+} - \beta_{i,-} \geq 0$ for any $i \in N \setminus \{1\}$.

Proof. By Proposition 1, we only have to show

$$\sum_{x \in supp(P)} v_{S_P}(x) P(x) \leq \sum_{x \in supp(P)} v_T(x) P(x) \text{ for any } T \subset N \setminus \{1\},$$

where $S_P = \{i \in N \setminus \{1\} | E(P_i) \geq E(P_1)\}$. By the inequality of (A2) in the proof of Proposition 1, we can show the above inequality. (Let $T = S_{\gamma P \oplus (1-\gamma) Q}$.)

□

For any $x \in \mathbb{R}^n$, $S = \{i \in N \setminus \{1\} \mid x_i \geq x_1\}$ minimizes $v_S(x)$ because $\alpha_{i,-} - \alpha_{i,+} \geq 0$ and $\beta_{i,+} - \beta_{i,-} \geq 0$. Intuitively, this proposition extends this result to the lotteries. Notice that by construction, the PAI model can also be rewritten as this description with the proper parameter conditions.

3 The relationship with Saito (2013)

Saito (2013) also examines ex ante fairness and ex post fairness and characterizes the following class of utility functions, which is called an expected inequality-averse model $(\alpha, \beta, \delta) \in \mathbb{R}_+^{n-1} \times \mathbb{R}_+^{n-1} \times [0, 1]$ (an EIA model in short). For any lottery $P \in L$,

$$V(P) = \delta U^{FS}(E(P_1), \ldots, E(P_n)) + (1 - \delta)E_P(U^{FS}),$$

where U^{FS} is a Fehr-Schmidt utility function:

$$U^{FS}(x) = x_1 - \sum_{i=2}^{n}(\alpha_i \max\{x_i - x_1, 0\} + \beta_i \max\{x_1 - x_i, 0\}),$$

where $\alpha_i \geq 0$ and $\beta_i \geq 0$ for all $i \in N$. The term $U^{FS}(E(P_1), \ldots, E(P_n))$ satisfies ex ante fairness, and the term $E_P(U^{FS})$ satisfies ex post fairness. Then, δ is a degree of how much the decision maker cares about ex ante fairness relative to ex post fairness. Obviously, the EIA model is a general case of the class of the Fehr-Schmidt utility functions. In fact, this model is a special case of the PAI model, as the following proposition shows.

Proposition 5. For any EIA model, there exists a PAI model such that the utility function of the EIA model coincides with that of the PAI model.

Proof. See Appendix A.3.

□

By the proof of this theorem,

$$\begin{cases} \alpha_{i,-} = \alpha_i \\ \beta_{i,-} = -\delta\alpha_i + (1 - \delta)\beta_i \\ \alpha_{i,+} = -\delta\beta_i + (1 - \delta)\alpha_i \\ \beta_{i,+} = \beta_i \end{cases}$$

for all $i \neq 1$. Then, $\alpha_{i,+}$ is a convex combination of α_i and $-\beta_i$. Similarly, $\beta_{i,-}$ is a convex combination of $-\alpha_i$ and β_i. By this theorem, the EIA model can be also interpreted as stated in Section 2. Namely, the decision maker changes her von Neumann-Morgenstern utility functions, depending on who the expectedly poorer agents are.

We will consider the axiomatic differences between the PAI model and the EIA model. The axiomatic differences can be exposed by the following axiom.

Axiom 7. For any $i \in N \setminus \{1\}$, let $Q^i \in L$ satisfy $supp(Q^i) = \{(0,(1)_{-i}),(1,(0)_{-i})\}$ and $1/2 = Q^i(0,(1)_{-i}) = Q^i(1,(0)_{-i})$. There exists $\delta \in [0,1]$ such that for any $i \in N \setminus \{1\}$,

$$Q^i \sim \delta(E(Q^i_1),\dots,E(Q^i_n)) \oplus (1-\delta)(E_{Q^i}(e),\dots,E_{Q^i}(e)),$$

where $e : \mathbb{R}^n \to \mathbb{R}$ satisfies $x \sim (e(x),\dots,e(x))$[5].

Under Axioms 1, 2, 3, 4, and 6, we can show that $(E(Q^i_1),\dots,E(Q^i_n)) \succeq Q^i \succeq (E_{Q^i}(e),\dots,E_{Q^i}(e))$ always holds for any $i \in N \setminus \{1\}$. This is because $\alpha_{i,-}+\beta_{i,-} \geq 0$ and $\beta_{i,+}-\beta_{i,-} \geq 0$. Then, there exists $\delta_i \in [0,1]$ such that $Q^i \sim \delta_i(E(Q^i_1),\dots,E(Q^i_n)) \oplus (1-\delta_i)(E_{Q^i}(e),\dots,E_{Q^i}(e))$. This axiom states that there is a common parameter δ among Q^i for any $i \in N \setminus \{1\}$. Therefore, this axiom does not restrict preferences when $n = 2$.

Assuming this axiom in addition to Axioms 1, 2, 3, 4, 5, and 6, we get the same class of utility functions as Saito (2013). We first characterize the preferences which satisfy these axioms by a class of utility functions. We call the class of utility functions the PAI+ model.

Proposition 6. $\succeq$ on L satisfies Axioms 1, 2, 3, 4, 5, 6, and 7 if and only if there exist $(\alpha_{i,+},\alpha_{i,-},\beta_{i,+},\beta_{i,-})_{i \in N\setminus\{1\}}$ such that

1. $\succeq$ is represented by

$$u(P) = \sum_{x \in supp(P)} v_{S_P}(x)P(x),$$

[5] $e(x)$ is well-defined in both Saito's utility functions and ours.

where $S_P = \{i \in N \setminus \{1\} | E(P_i) \geq E(P_1)\}$ and

$$v_S(x) = x_1 - \sum_{i \in S} [\alpha_{i,-} \max\{x_i - x_1, 0\} + \beta_{i,-} \max\{x_1 - x_i, 0\}]$$
$$- \sum_{i \in N \setminus S} [\alpha_{i,+} \max\{x_i - x_1, 0\} + \beta_{i,+} \max\{x_1 - x_i, 0\}],$$

for any $S \subset N$, and

2. $\alpha_{i,+} + \beta_{i,+} \geq 0$, $\beta_{i,+} \geq 0$, $\alpha_{i,-} + \beta_{i,-} \geq 0$, $\alpha_{i,-} \geq 0$, $\alpha_{i,-} - \alpha_{i,+} = \beta_{i,+} - \beta_{i,-} \geq 0$, and $(\alpha_{j,-} + \beta_{j,+})(\alpha_{i,-} - \alpha_{i,+}) = (\alpha_{i,-} + \beta_{i,+})(\alpha_{j,-} - \alpha_{j,+})$ for any $i, j \in N \setminus \{1\}$.

Proof. See Appendix A.4.

□

The PAI+ model is the same as the EIA model. See the following proposition.

Proposition 7. For any PAI+ model, there exists an EIA model such that the utility function of the EIA model coincides with that of the PAI+ model.

Proof. Assume

$$\begin{cases} \alpha_i = \alpha_{i,-} \\ \beta_i = \beta_{i,+} \end{cases}$$

for any $i \in N \setminus \{1\}$. In addition, if we can pick any $i \in N \setminus \{1\}$ such that $\alpha_{i,-} + \beta_{i,+} \neq 0$, assume that

$$\delta = \frac{\alpha_{i,-} - \alpha_{i,+}}{\alpha_{i,-} + \beta_{i,+}}.$$

Notice that $\delta \in [0, 1]$ because $\alpha_{i,-} - \alpha_{i,+} \geq 0$ and $\alpha_{i,+} + \beta_{i,+} \geq 0$. Since $(\alpha_{j,-} + \beta_{j,+})(\alpha_{i,-} - \alpha_{i,+}) = (\alpha_{i,-} + \beta_{i,+})(\alpha_{j,-} - \alpha_{j,+})$ for any $i, j \in N \setminus \{1\}$, $\beta_{j,-} = -\delta\alpha_j + (1 - \delta)\beta_j$ and $\alpha_{j,+} = -\delta\beta_j + (1 - \delta)\alpha_j$ hold for any $j \in N$. Therefore, we can prove this proposition by the proof of Proposition 5.

If $\alpha_{i,-} + \beta_{i,+} = 0$ for any $i \in N \setminus \{1\}$, $\alpha_{i,-} = 0$ and $\beta_{i,+} = 0$ because $\alpha_{i,-} \geq 0$ and $\beta_{i,+} \geq 0$. Then, $\alpha_{i,+} = 0$ and $\beta_{i,-} = 0$ because $\alpha_{i,+} + \beta_{i,+} \geq 0$, $\alpha_{i,-} + \beta_{i,-} \geq 0$, and $\alpha_{i,-} - \alpha_{i,+} = \beta_{i,+} - \beta_{i,-} \geq 0$. In this case, the PAI+ model is the same as the EIA model when $\alpha_i = 0$ and $\beta_i = 0$. Notice that the EIA model does not depend on δ in this case.

□

By Propositions 5, 6, and 7, the PAI+ model coincides with the EIA model. When $n = 2$, Axiom 7 does not restrict preferences, and the PAI+ model coincides with the EIA model without Axiom 7. In other words, when $n = 2$, the PAI model coincides with the EIA model.

By the proof of Proposition 7, since $\delta = \frac{\alpha_{i,-} - \alpha_{i,+}}{\alpha_{i,-} + \beta_{i,+}}$, $\delta \in [0, 1]$ does not hold without Axioms 4 and 6. Therefore, even if we assume Axioms 1, 2, 3, and 7, we cannot characterize a meaningful class of utility functions differently from Propositions 1, 2, and 3.

To describe the differences between the PAI model and the EIA model intuitively, we will revisit the example we introduce in the introduction. There are three people, Edgar, Allan, and Poe. The outcome $(x, y, z) \in \mathbb{R}^3$ means that Edgar gets x, Allan gets y, and Poe gets z. Let us consider Edgar's preferences. Assume that $(1, 2, 1) \sim (1, 1, 2)$ and $(1, 0, 1) \sim (1, 1, 0)$. In this case, $P^A = \frac{1}{2}(1, 2, 1) \oplus \frac{1}{2}(1, 0, 1)$ is indifferent to $P^P = \frac{1}{2}(1, 1, 2) \oplus \frac{1}{2}(1, 1, 0)$ for Edgar under any EIA model because $E_{P^A}(U^{FS}) = E_{P^P}(U^{FS})$ and $U^{FS}(E(P_1^A), \ldots, E(P_n^A)) = U^{FS}(E(P_1^P), \ldots, E(P_n^P))$ hold. However, under the PAI model, Edgar can prefer P^P to P^A. Assume $\beta_{2,-} > \beta_{3,-}$. Since $\alpha_{2,-} = \alpha_{3,-}$ by $(1, 2, 1) \sim (1, 1, 2)$,

$$
\begin{aligned}
& P^P \succ P^A \\
\Leftrightarrow\ & \frac{1}{2}(1 - \alpha_{3,-}) + \frac{1}{2}(1 - \beta_{3,-}) > \frac{1}{2}(1 - \alpha_{2,-}) + \frac{1}{2}(1 - \beta_{2,-}) \\
\Leftrightarrow\ & \beta_{2,-} > \beta_{3,-}.
\end{aligned}
$$

Therefore, $P^P \succ P^A$ holds when $\alpha_{2,-} = \alpha_{3,-}$ and $\beta_{2,-} > \beta_{3,-}$. Therefore, the EIA model cannot explain Edgar's preferences but the PAI model can.

A Appendix

A.1 The proof of Theorem 1

($\Rightarrow$) For all $S \subset N \setminus \{1\}$, define $L_S = \{P \in L | E(P_i) \geq E(P_1) \text{ for all } i \in S\} \cap \{P \in L | E(P_i) \leq E(P_1) \text{ for all } i \in N \setminus (S \cup \{1\})\}$. Notice that for any $S \subset N \setminus \{1\}$, $\gamma P \oplus (1 - \gamma)Q$ is in L_S for any $P \in L_S$, any $Q \in L_S$, and any $\gamma \in [0, 1]$. In addition, any $P \in L_S$ and any $Q \in L_S$ are expectedly quasi-comonotonic for any $S \subset N \setminus \{1\}$. Besides, $L = \cup_{S \subset N \setminus \{1\}} L_S$.

By Axioms 1 and 2, the preference relation $\succeq$ is a weak order, continuous, and independent in L_S for any $S \subset N \setminus \{1\}$. Then, for any $S \subset N \setminus \{1\}$, there exists a von Neumann-Morgenstern utility function $v_S : \mathbb{R}^n \to \mathbb{R}$ such that $u_S : L_S \to \mathbb{R}$ represents $\succeq$ in L_S, where

$$u_S(P) = \sum_{x \in supp(P)} v_S(x)P(x).$$

In addition, we can assume that $v_S(1, \ldots, 1) = 1$ and $v_S(-1, \ldots, -1) = -1$ because positive affine transformation of v_S preserves the preference relation. Thus, $u_S(1, \ldots, 1) = 1$ and $u_S(-1, \ldots, -1) = -1$ hold.

We will prove (i) for any $S \subset N \setminus \{1\}$ and any $\gamma \in (0, 1)$, if $x, y \in \mathbb{R}^n$ are expectedly quasi-comonotonic,

$$v_S(\gamma x + (1 - \gamma)y) = \gamma v_S(x) + (1 - \gamma)v_S(y) \tag{A1}$$

holds, and (ii) for any $S \subset N \setminus \{1\}$, any $x \in \mathbb{R}^n$, and any $a \in [0, \infty)$, $v_S(ax) = av_S(x)$.

To prove (i), we will show the next lemma.

Lemma 1. For any $k \in \mathbb{Z}_{>0}$, any $P^1, \ldots, P^k \in L$, any $\lambda \in (0, 1)$, and any $T \subset N \setminus \{1\}$, there exists $P \in L_T$ such that $\lambda P \oplus (1 - \lambda)P^j \in L_T$ for any $j = 1, \ldots, k$.

Proof. Select $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ so that $y_i > y_1 + \max_{j=1,\ldots,k} \frac{(1-\lambda)(E(P_1^j)-E(P_i^j))}{\lambda}$ for any $i \in T$ and $y_i < y_1 + \min_{j=1,\ldots,k} \frac{(1-\lambda)(E(P_1^j)-E(P_i^j))}{\lambda}$ for any $i \in N \setminus (T \cup \{1\})$. Then, for any $j = 1, \ldots, k$,

$$\begin{aligned} & y_i > y_1 + \frac{(1-\lambda)(E(P_1^j) - E(P_i^j))}{\lambda} \\ \Leftrightarrow\ & \lambda(y_i - y_1) + (1-\lambda)(E(P_i^j) - E(P_1^j)) > 0 \\ \Leftrightarrow\ & E((\lambda y \oplus (1-\lambda)P^j)_i) > E((\lambda y \oplus (1-\lambda)P^j)_1) \end{aligned}$$

for any $i \in T$. Similarly, $E((\lambda y \oplus (1 - \lambda)P^j)_i) < E((\lambda y \oplus (1 - \lambda)P^j)_1)$ for any $i \in N \setminus (T \cup \{1\})$. Therefore, $\lambda y \oplus (1 - \lambda)P^j \in L_T$ holds for any $j = 1, \ldots, k$.

$\square$

Fix $S \subset N \setminus \{1\}$ and $\gamma \in (0, 1)$ arbitrarily, and fix any $x, y \in \mathbb{R}^n$ such that $x, y \in \mathbb{R}^n$ are expectedly quasi-comonotonic. By Lemma 1, there exist $P \in L_S$ and $\lambda \in (0, 1)$ such that $\lambda P \oplus (1 - \lambda)\{\gamma x + (1 - \gamma)y\} \in L_S$ and $\lambda P \oplus (1 - \lambda)\{\gamma x \oplus (1 - \gamma)y\} \in L_S$. By Axiom 3,

$$u_S(\lambda P \oplus (1 - \lambda)\{\gamma x + (1 - \gamma)y\}) = u_S(\lambda P \oplus (1 - \lambda)\{\gamma x \oplus (1 - \gamma)y\}).$$

Since

$$u_S(\lambda P \oplus (1-\lambda)\{\gamma x + (1-\gamma)y\}) = \lambda \sum_{x' \in supp(P)} v_S(x')P(x') + (1-\lambda)v_S(\gamma x + (1-\gamma)y)$$

and

$$\begin{aligned} & u_S(\lambda P \oplus (1-\lambda)\{\gamma x \oplus (1-\gamma)y\}) \\ = & \lambda \sum_{x' \in supp(P)} v_S(x')P(x') + (1-\lambda)\gamma v_S(x) + (1-\lambda)(1-\gamma)v_S(y) \end{aligned}$$

hold, $v_S(\gamma x + (1-\gamma)y) = \gamma v_S(x) + (1-\gamma)v_S(y)$. Therefore, we proved (i).

By (A1), $u_S(0,\dots,0) = 0$ because $u_S(1,\dots,1) = 1$ and $u_S(-1,\dots,-1) = -1$ for any $S \subset N \setminus \{1\}$. Therefore, (A1) shows that for any $x \in \mathbb{R}^n$ and any $a \in [0,1]$,

$$u_S(ax + (1-a)(0,\dots,0)) = au_S(x) + 0 = au_S(x)$$

for any $S \subset N \setminus \{1\}$ because $(0,\dots,0)$ is in L_T for any $T \subset N \setminus \{1\}$. Similarly, for any $x \in \mathbb{R}^n$ and any $a \in (1,\infty)$, $x = \frac{1}{a}(ax) + (1-\frac{1}{a})(0,\dots,0)$ holds, and then $u_S(x) = \frac{1}{a}u_S(ax)$ for any $S \subset N \setminus \{1\}$. Hence, $v_S(ax) = av_S(x)$ holds for any $S \subset N \setminus \{1\}$. Therefore, we proved (ii).

Define $\alpha_{i,S} = -v_S((0)_{-i}, 1)$ and $\beta_{i,S} = -v_S((0)_{-i}, -1)$ for any $i \in N$ and any $S \subset N \setminus \{1\}$. By (i) and (ii), for any $x, y \in \mathbb{R}^n$ where x and y are expectedly quasi-comonotonic, $v_S(x+y) = v_S(x) + v_S(y)$ for any $S \subset N \setminus \{1\}$. Since

$$x = (x_1,\dots,x_1) + \sum_{i=2}^{n}((0)_{-i}, x_i - x_1)$$

holds,

$$\begin{aligned} v_S(x) &= v_S(x_1,\dots,x_1) + v_S(\sum_{i=2}^{n}((0)_{-i}, x_i - x_1)) \\ &= v_S(x_1,\dots,x_1) + \sum_{i=2}^{n} v_S((0)_{-i}, x_i - x_1) \end{aligned}$$

for any $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ and any $S \subset N \setminus \{1\}$. Since $(x_1,\dots,x_1)$ and $\sum_{i=2}^{n}((0)_{-i}, x_i - x_1)$ are expectedly quasi-comonotonic, the first equality holds, and since $((0)_{-j}, x_j - x_1)$ and $\sum_{i=j+1}^{n}((0)_{-i}, x_i - x_1)$ are expectedly quasi-comonotonic for any $j = 2,\dots,n-1$, the second equality holds. Therefore,

$$v_S(x) = x_1 - \sum_{i=2}^{n} \alpha_{i,S} \max\{x_i - x_1, 0\} - \sum_{i=2}^{n} \beta_{i,S} \max\{x_1 - x_i, 0\}.$$

Define $u : L \to \mathbb{R}$ as

$$u(P) = u_{S_P}(P)$$

for any $P \in L$, where $S_P = \{i \in N \setminus \{1\} \mid E(P_i) \geq E(P_1)\}$. We will show that $P \succ Q$ if and only if $u(P) > u(Q)$. To prove that, we will first show the following lemma.

Lemma 2. For any $P \in L$, there exists $x_P \in \mathbb{R}$ such that $u_S(P) = u_S(x_P, \dots, x_P) = x_P$ for any $S \subset N$ which satisfies $P \in L_S$.

Proof. Let $E = \{(x, \dots, x) \in \mathbb{R}^n | x \in \mathbb{R}\}$. For any $x \in \mathbb{R}$ and any $S \subset N$, $u_S(x, \dots, x) = x$ holds because (ii), $u_S(1, \dots, 1) = 1$, and $u_S(-1, \dots, -1) = -1$. Then, $u_S(E) = \mathbb{R}$ for any $S \subset N$. Thus, for any $P \in L$ and any $S \subset N$ which satisfies $P \in L_S$, there exists $x_P \in \mathbb{R}$ such that $u_S(P) = u_S(x_P, \dots, x_P) = x_P$. Since $(x_P, \dots, x_P) \in L_S$ for any $S \subset N$ and $P \sim (x_P, \dots, x_P)$ by the definition of u_S, for any $P \in L$, there exists $x_P \in \mathbb{R}$ such that $u_S(P) = u_S(x_P, \dots, x_P) = x_P$ for any $S \subset N$ which satisfies $P \in L_S$. Then, we complete the proof.

□

Consider $P, Q \in L$. By Lemma 2, there exists $x_P \in \mathbb{R}$ such that $u_{S_P}(P) = u_{S_P}(x_P, \dots, x_P) = x_P = u_{S_Q}(x_P, \dots, x_P)$ holds. Since $(x_P, \dots, x_P) \in L_{S_P} \cap L_{S_Q}$, $P \sim (x_P, \dots, x_P) \succ Q$ if and only if $u(P) = u(x_P, \dots, x_P) > u(Q)$ by the definition of u. Therefore, we proved that u satisfies $P \succ Q$ if and only if $u(P) > u(Q)$.

Before we will show the condition of the parameters, we will show that $\alpha_{i,S}$ depends only on whether $i \in S$ or not and that $\beta_{i,S}$ also depends only on whether $i \in S$ or not. Fix $i \in N \setminus \{1\}$ arbitrarily. By Lemma 2, there exists $x_{(1,(0)_{-i})} \in \mathbb{R}$ such that for any $S, T \subset N \setminus \{1\}$ where $(1, (0)_{-i}) \in L_S \cap L_T$,

$$u_S(1, (0)_{-i}) = 0 - \alpha_{i,S} = x_{(1,(0)_{-i})} = u_T(1, (0)_{-i}) = 0 - \alpha_{i,T}.$$

Thus, $\alpha_{i,S} = \alpha_{i,T}$. Therefore, for any $S \subset N \setminus \{1\}$ which includes i, $\alpha_{i,S}$ can be redefined by one number, $\alpha_{i,-}$.

Similarly, there exists $x_{(-1,(0)_{-i})} \in \mathbb{R}$ such that for any $S, T \subset N \setminus \{1\}$ where $(-1, (0)_{-i}) \in L_S \cap L_T$,

$$u_S(-1, (0)_{-i}) = 0 - \beta_{i,S} = x_{(-1,(0)_{-i})} = u_T(-1, (0)_{-i}) = 0 - \beta_{i,T}$$

by Lemma 2. Thus, $\beta_{i,S} = \beta_{i,T}$. Therefore, for any $S \subset N \setminus \{1\}$ which does not include i, $\beta_{i,S}$ can be redefined by one number, $\beta_{i,+}$.

Let $\check{P} = \frac{1}{2}(1,(0)_{-i}) \oplus \frac{1}{2}(-2,(0)_{-i})$. There exists $x_{\check{P}} \in \mathbb{R}$ such that for any $S, T \subset N \setminus \{1\}$ where $\check{P} \in L_S \cap L_T$,

$$u_S(\check{P}) = \frac{1}{2}(-\alpha_{i,S}) + \frac{1}{2}(-2\beta_{i,S}) = x_{\check{P}} = u_T(\check{P}) = \frac{1}{2}(-\alpha_{i,T}) + \frac{1}{2}(-2\beta_{i,T})$$

by Lemma 2. Thus, since $\beta_{i,S} = \beta_{i,T}$, $\alpha_{i,S} = \alpha_{i,T}$ holds. Therefore, for any $S \subset N \setminus \{1\}$ which does not include i, $\alpha_{i,S}$ can be redefined by one number, $\alpha_{i,+}$.

Let $\hat{P} = \frac{1}{2}(-1,(0)_{-i}) \oplus \frac{1}{2}(2,(0)_{-i})$. There exists $x_{\hat{P}} \in \mathbb{R}$ such that for any $S, T \subset N \setminus \{1\}$ where $\hat{P} \in L_S \cap L_T$,

$$u_S(\hat{P}) = \frac{1}{2}(-\beta_{i,S}) + \frac{1}{2}(-2\alpha_{i,S}) = x_{\hat{P}} = u_T(\hat{P}) = \frac{1}{2}(-\beta_{i,T}) + \frac{1}{2}(-2\alpha_{i,T})$$

by Lemma 2. Thus, since $\alpha_{i,S} = \alpha_{i,T}$, $\beta_{i,S} = \beta_{i,T}$. Therefore, for any $S \subset N \setminus \{1\}$ which includes i, $\beta_{i,S}$ can be redefined by one number, $\beta_{i,-}$.

Finally, we will show the condition of the parameters. Fix $i \in N \setminus \{1\}$ arbitrarily. Consider the lottery $\bar{P} = \frac{1}{2}(1,(0)_{-i}) \oplus \frac{1}{2}(-1,(0)_{-i})$, which is in L_S for any $S \subset N \setminus \{1\}$. When we pick $S, T \subset N$ where $i \notin S$ and $i \in T$,

$$u_S(\bar{P}) = -\frac{1}{2}\alpha_{i,+} - \frac{1}{2}\beta_{i,+}$$

and

$$u_T(\bar{P}) = -\frac{1}{2}\alpha_{i,-} - \frac{1}{2}\beta_{i,-}$$

hold. By Lemma 2, there exists $x_{\bar{P}}$ such that $u_S(\bar{P}) = x_{\bar{P}} = u_T(\bar{P})$, and then,

$$\alpha_{i,+} + \beta_{i,+} = \alpha_{i,-} + \beta_{i,-}.$$

Therefore, $\alpha_{i,-} - \alpha_{i,+} + \beta_{i,-} - \beta_{i,+} = 0$ holds for all $i \in N \setminus \{1\}$.

($\Leftarrow$)

To prove that u satisfies the axioms, we will first show the following lemma. For any $P \in L$, let $u_S(P) = \sum_{x \in supp(P)} v_S(x)P(x)$.

Lemma 3. For any $P \in L$, if $S \subset N$ satisfies $S_P^r \subset S$ and $S \cap S_P^p = \emptyset$, $u(P) = u_S(P)$ holds. In particular, for any $P \in L$, $u(P) = u_S(P)$, where S satisfies $P \in L_S$.

Proof. Notice that $S_P = S_P^r \cup S_P^e$. In addition, for any $S \subset N$, if $S_P^r \subset S$ and $S \cap S_P^p = \emptyset$, then $S \setminus S_P^r \subset S_P^e$. First, we will show that

$$u_S(P) = u_{S\cup\{i^e\}}(P)$$

for any $i^e \in S_P^e$ and any $S \subset N$ where $S_P^r \subset S$ and $S \cap S_P^p = \emptyset$.

If $i^e \in S$, it is obvious, and then, assume $i^e \notin S$.

$$\begin{aligned}
u_S(P) - u_{S\cup\{i^e\}}(P) =& \sum_{x \in supp(P)} (v_S(x) - v_{S\cup\{i^e\}})P(x) \\
=& \sum_{x \in supp(P)} [(\alpha_{i^e,-} - \alpha_{i^e,+}) \max\{x_{i^e} - x_1, 0\} \\
&+ (\beta_{i^e,-} - \beta_{i^e,+}) \max\{x_1 - x_{i^e}, 0\}]P(x) \\
=& \sum_{x \in supp(P)} [(\alpha_{i^e,-} - \alpha_{i^e,+})(\max\{x_{i^e} - x_1, 0\} - \max\{x_1 - x_{i^e}, 0\})]P(x) \\
=&(\alpha_{i^e,-} - \alpha_{i^e,+}) \sum_{x \in supp(P)} (x_{i^e} - x_1)P(x) = 0
\end{aligned}$$

holds because the last equality comes from $i^e \in S_P^e$. We can repeat this manipulation, and then, we complete the proof.

□

First, we will check Axiom 1. We can easily show that $\succsim$ is a weak order and monotonic in equal outcome vectors. We will show continuity. Let $P, Q \in L$ satisfy $P \succ Q$. Define $f : [0,1] \to \mathbb{R}$ as follows:

$$f(\alpha) = u(\alpha P \oplus (1-\alpha)Q).$$

We only have to show f is continuous. For any $S \subset N$, let $A_S = \{\alpha \in [0,1] \mid \alpha P \oplus (1-\alpha)Q \in L_S\}$. Notice that for any $S \subset N$, A_S is a closed interval by the definition of A_S and L_S. Additionally, $\bigcup_{S\subset N} A_S = [0,1]$. For any $S \subset N$, f is continuous at α for any $\alpha \in A_S$ by Lemma 3 and the definition of u_S. Then, f is continuous.

We will check Axiom 2. Consider $P, Q, R \in L$ where any pair among P, Q, and R are expectedly quasi-comonotonic. By Lemma 3, for any $\gamma \in (0,1)$,

$$\begin{aligned}
u(\gamma P \oplus (1-\gamma)R) =& \sum_{x \in supp(\gamma P \oplus (1-\gamma)R)} v_{S^r_{\gamma P \oplus (1-\gamma)R}}(x)\{\gamma P \oplus (1-\gamma)R\}(x) \\
=& \gamma \sum_{x \in supp(P)} v_{S^r_{\gamma P \oplus (1-\gamma)R}}(x)P(x) + (1-\gamma) \sum_{x \in supp(R)} v_{S^r_{\gamma P \oplus (1-\gamma)R}}(x)R(x)
\end{aligned}$$

holds. Similarly, for any $\gamma \in (0,1)$,

$$
\begin{aligned}
u(\gamma Q \oplus (1-\gamma)R) &= \sum_{x \in supp(\gamma Q \oplus (1-\gamma)R)} v_{S^r_{\gamma Q \oplus (1-\gamma)R}}(x)\{\gamma Q \oplus (1-\gamma)R\}(x) \\
&= \gamma \sum_{x \in supp(Q)} v_{S^r_{\gamma Q \oplus (1-\gamma)R}}(x)Q(x) + (1-\gamma) \sum_{x \in supp(R)} v_{S^r_{\gamma Q \oplus (1-\gamma)R}}(x)R(x)
\end{aligned}
$$

holds.

Remind that $P', Q' \in L$ are expectedly quasi-comonotonic if and only if $S^r_{P'} \cap S^p_{Q'} = \emptyset$ and $S^p_{P'} \cap S^r_{Q'} = \emptyset$. Since P, R and Q, R are expectedly quasi-comonotonic, $S^r_P \subset S^r_{\gamma P \oplus (1-\gamma)R}$ and $S^r_{\gamma P \oplus (1-\gamma)R} \cap S^p_P = \emptyset$. Then, $u_{S^r_{\gamma P \oplus (1-\gamma)R}}(P) = u_{S^r_P}(P)$ holds by Lemma 3. Similarly, since $S^r_Q \subset S^r_{\gamma Q \oplus (1-\gamma)R}$ and $S^r_{\gamma Q \oplus (1-\gamma)R} \cap S^p_Q = \emptyset$, $u_{S^r_{\gamma Q \oplus (1-\gamma)R}}(Q) = u_{S^r_Q}(Q)$ holds. Additionally, since $S^r_R \subset S^r_{\gamma P \oplus (1-\gamma)R}$ and $S^r_{\gamma P \oplus (1-\gamma)R} \cap S^p_R = \emptyset$, $u_{S^r_{\gamma P \oplus (1-\gamma)R}}(R) = u_{S^r_R}(R)$ holds, and since $S^r_R \subset S^r_{\gamma Q \oplus (1-\gamma)R}$ and $S^r_{\gamma Q \oplus (1-\gamma)R} \cap S^p_R = \emptyset$, $u_{S^r_{\gamma Q \oplus (1-\gamma)R}}(R) = u_{S^r_R}(R)$ holds. Therefore,

$$
u(\gamma P \oplus (1-\gamma)R) - u(\gamma Q \oplus (1-\gamma)R) = \gamma \left\{ \sum_{x \in supp(P)} v_{S^r_P}(x)P(x) - \sum_{x \in supp(Q)} v_{S^r_Q}(x)Q(x) \right\}
$$

holds. Since $\sum_{x \in supp(P)} v_{S^r_P}(x)P(x) = u(P)$ and $\sum_{x \in supp(P)} v_{S^r_Q}(x)Q(x) = u(Q)$ by Lemma 3,

$$
u(\gamma P \oplus (1-\gamma)R) - u(\gamma Q \oplus (1-\gamma)R) > 0
$$

holds because $P \succ Q$. As a result, we proved u satisfies Axiom 2.

Next, we will show u satisfies Axiom 3. Suppose $x, y \in \mathbb{R}^n$ are expectedly quasi-comonotonic. Fix $P \in L$ and $\lambda \in [0,1]$ arbitrarily. Let $\acute{P} = \lambda P \oplus (1-\lambda)\{\gamma x + (1-\gamma)y\}$ and $\grave{P} = \lambda P \oplus (1-\lambda)\{\gamma x \oplus (1-\gamma)y\}$. Then, $S_{\acute{P}} = S_{\grave{P}}$ holds because $(E(\acute{P}_1), \dots, E(\acute{P}_n)) = (E(\grave{P}_1), \dots, E(\grave{P}_n))$. By (i), since x and y are expectedly quasi-comonotonic, $v_{S_{\acute{P}}}(\gamma x + (1-\gamma)y) = \gamma v_{S_{\acute{P}}}(x) + (1-\gamma)v_{S_{\acute{P}}}(y)$. Therefore,

$$
\begin{aligned}
u(\acute{P}) &= \lambda \sum_{x' \in supp(P)} v_{S_{\acute{P}}}(x')P(x') + (1-\lambda)v_{S_{\acute{P}}}(\gamma x + (1-\gamma)y) \\
&= \lambda \sum_{x' \in supp(P)} v_{S_{\acute{P}}}(x')P(x') + (1-\lambda)\{\gamma v_{S_{\acute{P}}}(x) + (1-\gamma)v_{S_{\acute{P}}}(y)\} \\
&= \sum_{x' \in supp(\grave{P})} v_{S_{\grave{P}}}(x')\grave{P}(x') = u(\grave{P}).
\end{aligned}
$$

As a result, u satisfies Axiom 3.

A.2 Proof of Proposition 1

($\Rightarrow$) By the proof of Theorem 1, we only have to prove $\beta_{i,+} - \beta_{i,-} \geq 0$ (which is equivalent to $\alpha_{i,-} - \alpha_{i,+} \geq 0$) for any $i \in N \setminus \{1\}$. Fix $i \in N \setminus \{1\}$ arbitrarily. For any $\alpha_{i,-}$ and any $\beta_{i,+}$, there exist $a, b \in \mathbb{R}$ such that $a - \alpha_{i,-} = b - \beta_{i,+}$. Since $u(a+1, (a)_{-i}) = a - \alpha_{i,-}$ and $u(b-1, (b)_{-i}) = b - \beta_{i,+}$, $(a+1, (a)_{-i}) \sim (b-1, (b)_{-i})$. Then,

$$a - \alpha_{i,-} = b - \beta_{i,+} = \frac{1}{2}(a - \alpha_{i,-}) + \frac{1}{2}(b - \beta_{i,+}).$$

Since $\frac{1}{2}(a+1, (a)_{-i}) \oplus \frac{1}{2}(b-1, (b)_{-i}) \in L_{N\setminus\{1\}}$, Lemma 3, and Axiom 4 hold,

$$\begin{aligned}
&\frac{1}{2}(a+1, (a)_{-i}) \oplus \frac{1}{2}(b-1, (b)_{-i}) \succeq (a+1, (a)_{-i}) \\
\Leftrightarrow\ &\frac{1}{2}(a - \alpha_{i,-}) + \frac{1}{2}(b - \beta_{i,-}) \geq \frac{1}{2}(a - \alpha_{i,-}) + \frac{1}{2}(b - \beta_{i,+}) \\
\Leftrightarrow\ &\beta_{i,+} - \beta_{i,-} \geq 0,
\end{aligned}$$

which is what we want to show.

($\Leftarrow$) We only have to check Axiom 4 by the proof of Theorem 1. Consider $P, Q \in L$ where $P \sim Q$. We only have to prove that for any $\gamma \in (0, 1)$,

$$u(\gamma P \oplus (1-\gamma)Q) \geq \gamma u(P) + (1-\gamma)u(Q)$$

because $\gamma u(P) + (1-\gamma)u(Q) = u(P) = u(Q)$ for any $\gamma \in (0, 1)$. We will prove

$$\begin{aligned}
&u(\gamma P \oplus (1-\gamma)Q) - \gamma u(P) - (1-\gamma)u(Q) \\
&= \sum_{x \in supp(\gamma P \oplus (1-\gamma)Q)} [\gamma P(x)\{v_{S_{\gamma P \oplus (1-\gamma)Q}}(x) - v_{S_P}(x)\} + (1-\gamma)Q(x)\{v_{S_{\gamma P \oplus (1-\gamma)Q}}(x) - v_{S_Q}(x)\}] \\
&= \gamma \sum_{x \in supp(P)} P(x)\{v_{S_{\gamma P \oplus (1-\gamma)Q}}(x) - v_{S_P}(x)\} \\
&\quad + (1-\gamma) \sum_{x \in supp(Q)} Q(x)\{v_{S_{\gamma P \oplus (1-\gamma)Q}}(x) - v_{S_Q}(x)\} \geq 0
\end{aligned}$$

for any $\gamma \in (0, 1)$.

Since $\alpha_{i,-} - \alpha_{i,+} = \beta_{i,+} - \beta_{i,-} \geq 0$,

$$\begin{aligned}
v_{S_{\gamma P \oplus (1-\gamma) Q}}(x) - v_{S_P}(x) = & \sum_{i \in S_P \setminus S_{\gamma P \oplus (1-\gamma) Q}} [(\alpha_{i,-} - \alpha_{i,+}) \max\{x_i - x_1, 0\} \\
& + (\beta_{i,-} - \beta_{i,+}) \max\{x_1 - x_i, 0\}] \\
& + \sum_{i \in S_{\gamma P \oplus (1-\gamma) Q} \setminus S_P} [(\alpha_{i,+} - \alpha_{i,-}) \max\{x_i - x_1, 0\} \\
& + (\beta_{i,+} - \beta_{i,-}) \max\{x_1 - x_i, 0\}] \\
= & \sum_{i \in S_P \setminus S_{\gamma P \oplus (1-\gamma) Q}} (\alpha_{i,-} - \alpha_{i,+})(\max\{x_i - x_1, 0\} - \max\{x_1 - x_i, 0\}) \\
& - \sum_{i \in S_{\gamma P \oplus (1-\gamma) Q} \setminus S_P} (\alpha_{i,-} - \alpha_{i,+})(\max\{x_i - x_1, 0\} - \max\{x_1 - x_i, 0\}) \\
= & \sum_{i \in S_P \setminus S_{\gamma P \oplus (1-\gamma) Q}} (\alpha_{i,-} - \alpha_{i,+})(x_i - x_1) \\
& - \sum_{i \in S_{\gamma P \oplus (1-\gamma) Q} \setminus S_P} (\alpha_{i,-} - \alpha_{i,+})(x_i - x_1)
\end{aligned}$$

for any $x \in \mathbb{R}^n$. Then,

$$\begin{aligned}
\sum_{x \in supp(P)} P(x)\{v_{S_{\gamma P \oplus (1-\gamma) Q}}(x) - v_{S_P}(x)\} = & \sum_{x \in supp(P)} P(x) \Bigg[\sum_{i \in S_P \setminus S_{\gamma P \oplus (1-\gamma) Q}} (\alpha_{i,-} - \alpha_{i,+})(x_i - x_1) \\
& - \sum_{i \in S_{\gamma P \oplus (1-\gamma) Q} \setminus S_P} (\alpha_{i,-} - \alpha_{i,+})(x_i - x_1) \Bigg] \\
= & \sum_{i \in S_P \setminus S_{\gamma P \oplus (1-\gamma) Q}} (\alpha_{i,-} - \alpha_{i,+}) \sum_{x \in supp(P)} P(x)(x_i - x_1) \\
& - \sum_{i \in S_{\gamma P \oplus (1-\gamma) Q} \setminus S_P} (\alpha_{i,-} - \alpha_{i,+}) \sum_{x \in supp(P)} P(x)(x_i - x_1) \\
\geq & \, 0 \qquad \text{(A2)}
\end{aligned}$$

holds. The last inequality comes from the definition of S_P. Similarly,

$$\sum_{x \in supp(Q)} Q(x)\{v_{S_{\gamma P \oplus (1-\gamma) Q}}(x) - v_{S_Q}(x)\} \geq 0$$

holds. Therefore,

$$\gamma \sum_{x \in supp(P)} P(x)\{v_{S_{\gamma P \oplus (1-\gamma) Q}}(x) - v_{S_P}(x)\} + (1-\gamma) \sum_{x \in supp(Q)} Q(x)\{v_{S_{\gamma P \oplus (1-\gamma) Q}}(x) - v_{S_Q}(x)\} \geq 0$$

for any $\gamma \in (0, 1)$. As a result, we proved u satisfies Axiom 4.

A.3 Proof of Proposition 5

Assume

$$\begin{cases}\alpha_{i,-} = \alpha_i \\ \beta_{i,-} = -\delta\alpha_i + (1-\delta)\beta_i \\ \alpha_{i,+} = -\delta\beta_i + (1-\delta)\alpha_i \\ \beta_{i,+} = \beta_i.\end{cases}$$

Notice that $\alpha_{i,-} \geq 0$, $\beta_{i,+} \geq 0$, $\alpha_{i,+} + \beta_{i,+} \geq 0$, $\alpha_{i,-} + \beta_{i,-} \geq 0$, and $\alpha_{i,-} - \alpha_{i,+} = \beta_{i,+} - \beta_{i,-} \geq 0$ hold for any $i \in N \setminus \{1\}$ because $\alpha_i \geq 0$, $\beta_i \geq 0$, and $\delta \in [0,1]$.

Consider $P \in L$. Then,

$$\begin{aligned}
U^{FS}(E(P_1),\ldots,E(P_n)) &= \sum_{x\in supp(P)} x_1 P(x) - \sum_{i\in S_P} \alpha_i \left\{ \sum_{x\in supp(P)} x_i P(x) - \sum_{x\in supp(P)} x_1 P(x) \right\} \\
&\quad - \sum_{i\in N\setminus(S_P\cup\{1\})} \beta_i \left\{ \sum_{x\in supp(P)} x_1 P(x) - \sum_{x\in supp(P)} x_i P(x) \right\} \\
&= \sum_{x\in supp(P)} \left\{ \left(1 + \sum_{i\in S_P} \alpha_i - \sum_{i\in N\setminus(S_P\cup\{1\})} \beta_i \right) x_1 P(x) - \sum_{i\in S_P} \alpha_i x_i P(x) \right. \\
&\quad \left. + \sum_{i\in N\setminus(S_P\cup\{1\})} \beta_i x_i P(x) \right\} \\
&= \sum_{x\in supp(P)} P(x) \left\{ x_1 - \sum_{i\in S_P} \alpha_i (x_i - x_1) - \sum_{i\in N\setminus(S_P\cup\{1\})} \beta_i (x_1 - x_i) \right\} \\
&= \sum_{x\in supp(P)} P(x) \left\{ x_1 - \sum_{i\in S_P} \alpha_i \max\{x_i - x_1, 0\} + \sum_{i\in S_P} \alpha_i \max\{x_1 - x_i, 0\} \right. \\
&\quad \left. - \sum_{i\in N\setminus(S_P\cup\{1\})} \beta_i \max\{x_1 - x_i, 0\} + \sum_{i\in N\setminus(S_P\cup\{1\})} \beta_i \max\{x_i - x_1, 0\} \right\}
\end{aligned}$$

holds because $x_i - x_1 = \max\{x_i - x_1, 0\} - \max\{x_1 - x_i, 0\}$. In addition,

$$\begin{aligned}
E_P(U^{FS}) &= \sum_{x \in supp(P)} U^{FS}(x)P(x) \\
&= \sum_{x \in supp(P)} P(x) \left\{ x_1 - \sum_{i \in N} \alpha_i \max\{x_i - x_1, 0\} - \sum_{i \in N} \beta_i \max\{x_1 - x_i, 0\} \right\} \\
&= \sum_{x \in supp(P)} P(x) \left\{ x_1 - \sum_{i \in S_P} \alpha_i \max\{x_i - x_1, 0\} - \sum_{i \in N \backslash (S_P \cup \{1\})} \alpha_i \max\{x_i - x_1, 0\} \right. \\
&\quad \left. - \sum_{i \in S_P} \beta_i \max\{x_1 - x_i, 0\} - \sum_{i \in N \backslash (S_P \cup \{1\})} \beta_i \max\{x_1 - x_i, 0\} \right\}
\end{aligned}$$

holds. Therefore,

$$\begin{aligned}
V(P) &= \sum_{x \in supp(P)} P(x) \left\{ x_1 - \sum_{i \in S_P} \alpha_i \max\{x_i - x_1, 0\} \right. \\
&\quad - \sum_{i \in N \backslash (S_P \cup \{1\})} \{-\delta\beta_i + (1-\delta)\alpha_i\} \max\{x_i - x_1, 0\} \\
&\quad \left. - \sum_{i \in S_P} \{-\delta\alpha_i + (1-\delta)\beta_i\} \max\{x_1 - x_i, 0\} - \sum_{i \in N \backslash (S_P \cup \{1\})} \beta_i \max\{x_1 - x_i, 0\} \right\} \\
&= \sum_{x \in supp(P)} v_{S_P}(x)P(x)
\end{aligned}$$

holds, and we proved the theorem.

A.4 Proof of Proposition 6

($\Rightarrow$) By Theorem 2, we only have to show $(\alpha_{j,-} + \beta_{j,+})(\alpha_{i,-} - \alpha_{i,+}) = (\alpha_{i,-} + \beta_{i,+})(\alpha_{j,-} - \alpha_{j,+})$ for any $i, j \in N \backslash \{1\}$. Fix $i, j \in N \backslash \{1\}$ arbitrarily. $(E(Q^i_1), \ldots, E(Q^i_n)) = (\frac{1}{2}, \ldots, \frac{1}{2})$ and $(E_{Q^i}(e), \ldots, E_{Q^i}(e)) = \left(\frac{1-\alpha_{i,-}-\beta_{i,+}}{2}, \ldots, \frac{1-\alpha_{i,-}-\beta_{i,+}}{2}\right)$ hold. Then, since there exists $\delta \in [0,1]$ such that $Q^i \sim \delta(E(Q^i_1), \ldots, E(Q^i_n)) \oplus (1-\delta)(E_{Q^i}(e), \ldots, E_{Q^i}(e))$ by Axiom 7,

$$\begin{aligned}
& u(Q^i) = u(\delta(E(Q^i_1), \ldots, E(Q^i_n)) \oplus (1-\delta)(E_{Q^i}(e), \ldots, E_{Q^i}(e)) \\
\Leftrightarrow\ & \frac{1 - \beta_{i,-} - \alpha_{i,-}}{2} = \frac{\delta}{2} + \frac{1-\delta}{2}(1 - \beta_{i,+} - \alpha_{i,-}) \\
\Leftrightarrow\ & \delta(\beta_{i,+} + \alpha_{i,-}) = \alpha_{i,-} - \alpha_{i,+}
\end{aligned}$$

holds. The first equivalence comes from $u(0, (1)_{-i}) = 1 - \beta_{i,+}$ and $u(1, (0)_{-i}) = -\alpha_{i,-}$. By Axiom 7, δ satisfies $\delta(\beta_{j,+} + \alpha_{j,-}) = \alpha_{j,-} - \alpha_{j,+}$.

If $\beta_{i,+} + \alpha_{i,-} = 0$, $\alpha_{i,-} - \alpha_{i,+} = 0$. Then, $(\alpha_{j,-} + \beta_{j,+})(\alpha_{i,-} - \alpha_{i,+}) = (\alpha_{i,-} + \beta_{i,+})(\alpha_{j,-} - \alpha_{j,+})$. Similarly, if $\beta_{j,+} + \alpha_{j,-} = 0$, $(\alpha_{j,-} + \beta_{j,+})(\alpha_{i,-} - \alpha_{i,+}) = (\alpha_{i,-} + \beta_{i,+})(\alpha_{j,-} - \alpha_{j,+})$.

If $\beta_{i,+} + \alpha_{i,-} > 0$ and $\beta_{j,+} + \alpha_{j,-} > 0$, $(\alpha_{j,-} + \beta_{j,+})(\alpha_{i,-} - \alpha_{i,+}) = (\alpha_{i,-} + \beta_{i,+})(\alpha_{j,-} - \alpha_{j,+})$ holds because $\delta = \frac{\alpha_{i,-}-\alpha_{i,+}}{\alpha_{i,-}+\beta_{i,+}} = \frac{\alpha_{j,-}-\alpha_{j,+}}{\alpha_{j,-}+\beta_{j,+}}$.

($\Leftarrow$) By Theorem 2, we only have to show Axiom 7. Since for any $i \in N \setminus \{1\}$,

$$u(Q^i) = \frac{1 - \beta_{i,-} - \alpha_{i,-}}{2}$$

and

$$\begin{aligned}
&u(\delta(E(Q_1^i), \dots, E(Q_n^i)) \oplus (1-\delta)(E_{Q^i}(e), \dots, E_{Q^i}(e)) \\
&= u\left(\delta\left(\frac{1}{2}, \dots, \frac{1}{2}\right) \oplus (1-\delta)\left(\frac{1 - \alpha_{i,-} - \beta_{i,+}}{2}, \dots, \frac{1 - \alpha_{i,-} - \beta_{i,+}}{2}\right)\right) \\
&= \frac{\delta}{2} + \frac{1-\delta}{2}(1 - \beta_{i,+} - \alpha_{i,-})
\end{aligned}$$

for any $\delta \in [0, 1]$,

$$u(Q^i) = u(\delta(E(Q_1^i), \dots, E(Q_n^i)) \oplus (1-\delta)(E_{Q^i}(e), \dots, E_{Q^i}(e))$$

holds if and only if

$$\delta(\alpha_{i,-} + \beta_{i,+}) = \alpha_{i,-} - \alpha_{i,+}. \tag{A3}$$

Then, we only have to show that there exists $\delta \in [0, 1]$ such that (A3) holds for any $i \in N \setminus \{1\}$.

Fix $i \in N \setminus \{1\}$ arbitrarily. First, assume $0 \geq \alpha_{i,-} + \beta_{i,+}$. Since $\alpha_{i,-} + \beta_{i,-} \geq 0$,

$$\beta_{i,-} - \beta_{i,+} \geq \alpha_{i,-} + \beta_{i,-} \geq 0.$$

Since $\beta_{i,+} - \beta_{i,-} \geq 0$, $\beta_{i,+} - \beta_{i,-} = 0$ and $\alpha_{i,-} + \beta_{i,-} = 0$. Then, $\alpha_{i,-} - \alpha_{i,+} = 0$ and $\alpha_{i,-} + \beta_{i,+} = 0$ because $\beta_{i,+} - \beta_{i,-} = \alpha_{i,-} - \alpha_{i,+}$. Therefore, if $0 \geq \alpha_{i,-} + \beta_{i,+}$, (A3) holds for any $\delta \in [0, 1]$.

Assume $\alpha_{i,-} + \beta_{i,+} > 0$. Let $\delta = \frac{\alpha_{i,-}-\alpha_{i,+}}{\alpha_{i,-}+\beta_{i,+}}$. Since $\alpha_{i,-} - \alpha_{i,+} \geq 0$ and $\alpha_{i,+} + \beta_{i,+} \geq 0$,

$$0 \leq \frac{\alpha_{i,-} - \alpha_{i,+}}{\alpha_{i,-} + \beta_{i,+}} \leq 1.$$

By the above discussion, if there exists $j \in N \setminus \{1\}$ such that $0 \geq \alpha_{j,-} + \beta_{j,+}$, $\delta(\alpha_{j,-} + \beta_{j,+}) = \alpha_{j,-} - \alpha_{j,+}$ holds. In addition, if there exists $j \in N \setminus \{1\}$ such that $\alpha_{j,-} + \beta_{j,+} >$

0, $\delta = \frac{\alpha_{j,-} - \alpha_{j,+}}{\alpha_{j,-} + \beta_{j,+}}$ holds because $(\alpha_{j,-} + \beta_{j,+})(\alpha_{i,-} - \alpha_{i,+}) = (\alpha_{i,-} + \beta_{i,+})(\alpha_{j,-} - \alpha_{j,+})$. Therefore, there exists $\delta \in [0, 1]$ such that (A3) holds for any $i \in N \setminus \{1\}$, and we complete the proof.

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