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CROWDING IN SCHOOL CHOICE

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Abstract. We consider the problem of matching students to schools when students are able to express preferences over crowding. For example, schools have varying per capita expenditures, average teacher-student ratios, etc. These characteristics of a school are now endogenously determined—matchings with more students to a particular school decrease each of the variables above. We propose a new equilibrium notion, the Rationing Crowding Equilibrium (RCE), that accommodates crowding, no-envy, and respect for priorities. We prove the existence of RCE under mild domain conditions, and establish a Rural Hospitals Theorem and welfare lattice result on the set of RCE. The latter implies the existence of a maximal RCE, and that such RCE are student-optimal. Moreover, the mechanism defined by selection from the maximal RCE correspondence is *strategy-proof*. We also identify an algorithm to find a maximal RCE for a natural subdomain.

Keywords: School choice with crowding; Rationing crowding equilibrium; Student optimality; Strategy-proofness

JEL Classification: C78, D47, D62, I20

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1. Introduction

It is generally accepted among parents that if a school is overcrowded, then the quality of their children's education may suffer. This assumption is supported by the empirical literature. For example, Krueger and Whitmore [2001], Chetty et al. [2011], and Jackson et al. [2016] show that decreasing crowding, as measured by per capita expenditure or teacher-student ratios, etc., has positive effects on measures such as test scores, students' lifetime expected income, and career development. Despite its salience, the school choice and market design literature have largely ignored crowding. This stems largely from the non-existence of the usual equilibrium notions in the presence of externalities and the absence of tractable analytical frameworks to replace them.

This paper proposes a novel framework and equilibrium notion to analyse the *school choice problem with crowding*. We draw on the classical school choice model of Abdulkadiroğlu and Sönmez [2003] wherein each student has a preference over schools and a priority rank at each. The new key feature in our framework is that students' preferences additionally depend on the level of crowding at the school. More specifically, the level is a ratio that divides some aggregate measure of available physical infrastructure, number of teachers, overall financial expenditure, etc. by the total number of students attending [Lewis et al., 2000].[1] We refer to it as a school's *resource ratio*. The students' consumption space is thus composed of school and resource ratio *pairs*. Crucially, this ratio (or any other per capita measure of crowding) is endogenously determined.

Aside from more general student preferences, we allow for a more nuanced interpretation of school capacity. Our motivation is the following: A school may almost always accommodate some extra students by adding seats in the classroom or in a temporary structure. Actual school populations as a result fluctuate year to year based on demographic shifts.[2] Capacity as modelled in the classical model is then implicitly more of a "soft target" as

[1]Other factors include square footage, building "designed" capacity, temporary spaces, number of full-time teachers, teacher aides, substitute teachers, counselors, nurses, mental health specialists, scheduling, financial aid, school day length, school size, subjective feeling, etc.

[2]For example, in Abbotts Creek Elementary School in Wake County, North Carolina, for the 2016-2017 year there were 796 enrolled students, increasing to 870 for the 2019-2020 year, and dropping to 854 in the 2020-2021 year. See "District Facts" at https://www.wcpss.net/domain/100.

opposed to a hard constraint.[3] Our framework explicitly models matchings that are below the capacity of schools, and the subsequent effects on student preferences.

An allocation is thus composed of a matching of students to schools as well as an associated resource ratio at each school. We make two remarks on our modelling choices. First, we define the two components of an allocation independently, as opposed to deriving the resource ratio from the matching. Much like the market clearing condition in classical exchange economies, our equilibrium then imposes a consistency condition between them. Second, we model crowding as a continuous variable. Total resources combines various measures, several of which are continuous e.g. financial expenditure. In practice, schools have large numbers of students. For example, the smallest school in the Wake County Public School System in North Carolina for the 2020-2021 school year had 256 students, and the largest, 2,733 students.[4] A single student's effect on the resource ratio is thus negligible, justifying continuity. If one insists upon the discrete approach, then we remark that our equilibrium notion has trivial discrepancy (i.e. bounded by one seat) between the fraction induced by the matching and the resource ratio at each school (Section 8.2).

We propose a new equilibrium notion, show its existence under mild domain restrictions, and identify interesting structural properties. Our Rationing Crowding Equilibrium (RCE) is a selection from the set of allocations satisfying *fairness*—if student i prefers school s at an allocation, then school s has exhausted its capacity, and all students in that school have higher priority than student i. This implies no-envy in the sense that each student does not prefer any school that is below capacity to their own match, at the respective ratios. We require two further conditions. The allocation must be internally consistent: The resource ratio offered by a school must conform, up to a rounding error, to the number of students actually assigned to the school. This is the key condition accommodating crowding in our approach and separating us from similar notions in the literature.[5] We show that the relative importance of the rounding error decreases monotonically in the size of the problem (Section 8.2). The last condition requires that any empty school is strictly inferior to one's currently matched school.

[3]We still define a "true" hard capacity constraint. For example, many states have laws specifying a minium teacher-student ratio. In 2021 North Carolina requires at least one teacher per 18 students in grades K-3 (G.S. 115C-301 on "Allocation of teachers; class size").

[4]These are Wake STEM Early College High School and Apex Friendship High. We exclude several smaller specialized schools. See "District Facts" and the 2020-2021 Report at https://www.wcpss.net/domain/100.

[5]Without it, we would have a matching with transfers model. See Section 3.2 for details.

Our main results are as follows. Fix a school choice problem with crowding. We show that an RCE exists for this problem so long as it satisfies a mild regularity condition that holds true generically (Theorem 1). We then establish a version of the Rural Hospitals Theorem for our environment, i.e. in each RCE, each school is matched with the same number of students (Theorem 2). We show that the set of RCEs constitute a closed upper semi-lattice under the Pareto dominance partial order (Theorem 3), and so there exists a student-optimal RCE (Proposition 3). From a technical perspective, our proof techniques deviate significantly from those in the literature as crowding was not considered.

A *maximal RCE mechanism* recommends, for each school choice with crowding problem, a maximal RCE (all in this set are welfare-equivalent and student-optimal). We show that these mechanisms are *strategy-proof* (Theorem 4), and we further find an algorithm to calculate them on a restricted domain. It is in general hard to propose an algorithm for equilibrium computation when agents' preferences are not quasi-linear (see the discussion after Theorem 1). This algorithm takes the hybrid format of both the multi-item auction [Demange et al., 1986, Tierney, 2019] and the Deferred-Acceptance algorithm [Gale and Shapley, 1962]. The former adjusts the distribution for unconstrained schools, and the latter adjusts the matching for schools whose capacity constraint has been met.

1.1. **Related literature.** Our model subsumes the classical school choice model of Abdulkadiroğlu and Sönmez [2003] as a special case (Section 3.1). The set of *fair* allocations then coincides with the set of *no justified envy* and *non-wasteful* allocations (the usual combination of properties considered therein) (Proposition 1).

Our RCE are closely related to price-based equilibrium notions in matching with the possibility of monetary transfers; none, however, feature consumption externalities in agents' preferences. Shapley and Shubik [1971] first proved the existence and structural properties of the core and competitive equilibrium allocations for these models. These properties imply that, for each side of the market, there is a unique undominated utility vector it can achieve in the core. Mechanisms that realize one of these vectors are *efficient* and *strategy-proof* [Demange and Gale, 1985] for the side of the market whose utility is maximized. In matching models where prices are not fully flexible, e.g. restricted between price floors and ceilings, Drèze Equilibrium and Rationing Price Equilibrium were proposed as alternative notions [Drèze, 1975, Andersson and Svensson, 2014, Herings, 2018]. Under some mild domain restrictions, there are *constrained efficient* and *group strategy-proof* rules [Andersson and Svensson, 2014]. Our equilibrium concept follows these in spirit, but now allows for consumption externalities (See Section 3.2 for more details); because of this feature, our proof techniques are significantly different.

In some matching models without continuous transfers, the usual fairness properties are naturally expressed with an endogenously determined *cutoff* vector [Balinski and Sönmez, 1999, Sönmez and Ünver, 2010, Azevedo and Leshno, 2016, Dur and Morrill, 2018, Leshno and Lo, 2021]. In the case of school choice, the vector specifies for each school the lowest priority student able to attend. This is thus a competitive approach—cutoffs determine budget sets for agents, agents maximize therein, and markets clear. Again, our approach parallels theirs, but also considers crowding preferences.

Our problem is one of matching with externalities. In general, it is possible that an agent's preference over schools depends on where all the other agents are matched. Sasaki and Toda [1996], Hafalir [2008], Bando [2012], and Bando [2014] propose and study various notions of stability in this environment. Oftentimes there is a natural structure on possible externalities. Dutta and Massó [1997] and Echenique and Yenmez [2007] consider the case where agents have preferences over their peers at the school they are matched to. Narrowing down even further, a large literature considers matching with couples, siblings, or neighbors [Roth, 1984, Aldershof and Carducci, 1996, Roth and Peranson, 1999, Klaus and Klijn, 2005, Kojima et al., 2013, Ashlagi et al., 2014, Dur and Wiseman, 2019, Dur et al., 2021]. Typically, these studies find that externalities eliminate the structures that have been found in the classical literature. A few consider conditions and environments under which stability is possible [Pycia, 2012, Rostek and Yoder, 2020, Pycia and Yenmez, 2021]. Preference over crowding features a specific structure that differs from those considered in this literature, and so our techniques are independent of those.

The technical cornerstone of our discoveries comes from Tierney [2019]. By reinterpreting vectors of crowding ratios as price vectors in a matching with transfers problem, they establish that half of the lattice structure would survive. However, their equilibrium will often not exist in our model, as it presupposes that resource ratios can reach 0. This is unacceptable for school choice, as it implies that a single school can accommodate all the students in the entire system. Our equilibrium is more general. In particular, we allow for priorities to decide seats at schools that have reached their lower bound (as in Andersson and Svensson [2014]), and this feature requires substantial technical innovation.

1.2. **Organization.** In Section 2, we define the school choice with crowding problem. In Section 3, we present Rationing Crowding Equilibrium. Section 4 shows that RCE exist, and Section 5 investigates some properties of them, like the Rural Hospitals Theorem and the welfare lattice. In Section 6, we identify a *fair* and *strategy-proof* mechanism, and in Section 7, we find an algorithm that implements this mechanism on a restricted domain of preferences. In Section 8, we discuss the robustness of our results and some applications beyond school choice. Section 9 concludes with open questions.

2. Model

Let S be the finite set of **schools**, and N be the finite set of **students**. Each school $s \in S$ has resources in the form of teachers, buildings, money, etc. We aggregate this to a single measure and for each school normalize this to one. When a student attends a school, the number of other students and the policy of the school together determine what fraction of those resources, that is, what **resource ratio**, they consume. Formally, each student's consumption space is $[0, 1] \times S$, where the first component is a resource ratio ρ_s and the second is the school s that they attend.[6] This abstraction of the consumption space drives our theoretical innovations. For some applications, it may be essential that the resource ratio be restricted to *simple* fractions of the form $1/k$. This would be the case when crowding really depends *only* on the number of students at a school. Our model is still useful in these cases, since we shall recommend, for each school, a resource ratio that is within one student of being the simple fraction implied by the cohort they admit (see Section 8.2).

Each student $i \in N$ has a complete and transitive **preference relation** R_i over $[0, 1] \times S$. We assume that each is monotonic in the resource ratio: for each $\rho_s, \rho'_s \in [0, 1]$ with $\rho'_s > \rho_s$, and each $s \in S$, $(\rho'_s, s)\ P_i\ (\rho_s, s)$. Let $\mathcal{R}$ be the set of monotonic preference relations, and $\boldsymbol{R} = (R_i)_{i \in N} \in \mathcal{R}^N$ denote a profile of preferences for students.

Each school $s \in S$ has a lower bound b_s on the resource ratio it can provide. This implies a maximal **capacity** b_s^{-1}, and since students are indivisible, it is convenient to assume b_s^{-1} is a natural number. Let $\boldsymbol{b} = (b_s)_{s \in S}$ denote the profile of bounds for schools. We assume that $\sum_{s \in S} b_s^{-1} \geq |N|$. Each school $s \in S$ has a **priority order** $\prec_s$ over the set of students, where $i \prec_s j$ indicates that i has higher priority than j at s. Let $\prec = (\prec_s)_{s \in S}$ denote the profile of priorities for schools. Note that schools' priorities are independent of the resource ratios.

[6] Inclusion of an outside option in our model would not change the results and only add complication to the proofs. We show, however, in Section 3.1, that the richness of our preference space subsumes the outside option in the classical sense. This works by including a school s with unbounded capacity, and restricting preferences so that any admissible school for student i, at any ratio, is preferred to s at any ratio.

A **school choice problem with crowding** or, simply, a problem is a tuple $(S, N, \boldsymbol{R}, \boldsymbol{b}, \prec)$. The canonical school choice problem of Abdulkadiroğlu and Sönmez [2003] is a special case (see Section 3.1 for details). A **distribution** is a vector $\boldsymbol{\rho} \in [0,1]^S$ of resource ratios. A **matching** $\sigma : N \to S$ places each student at a school. Let $\mathcal{M}$ be the set of matchings, and for each $\sigma \in \mathcal{M}$ write $\sigma[s]$ as the set of students matched to s at σ.[7] An **allocation** is a pair $(\boldsymbol{\rho}, \sigma) \in [0,1]^S \times \mathcal{M}$ such that for each $s \in S$,

(1) (Distribution Feasibility) $\rho_s \cdot |\sigma[s]| \leq 1$.
(2) (Respects Capacity) $|\sigma[s]| \leq b_s^{-1}$ and $\rho_s \geq b_s$.

Given a distribution $\boldsymbol{\rho} \in [0,1]^S$ and a school $s \in S$, we also write $(\boldsymbol{\rho}, s)$ to indicate bundle (ρ_s, s). Thus, for an allocation $(\boldsymbol{\rho}, \sigma)$, each $i \in N$ receives $(\boldsymbol{\rho}, \sigma(i))$. We refer to this as ***i*'s component** when the allocation at hand is clear.

A **mechanism** φ recommends, for each problem in a domain $\mathcal{D} \subseteq \mathcal{R}^N$, an allocation. We formalize some senses in which these recommendations might either be good or achievable.

We consider fairness amongst the students. The starting point is the classic condition of Foley [1966], wherein no student prefers another's component in the allocation to their own. However, when a school is at capacity, and its resource ratio is at the lower bound, then possibly an agent who did not get admitted envies one who did. In this case, we require that this envy is *justified*—the admitted student has higher priority. We combine these two ideas formally. Fix a problem $(S, N, \boldsymbol{R}, \boldsymbol{b}, \prec)$. The allocation $(\boldsymbol{\rho}, \sigma)$ is **fair** if $(\boldsymbol{\rho}, s)\ P_i\ (\boldsymbol{\rho}, \sigma(i))$ implies that $\rho_s = b_s$ and, for each $j \in \sigma[s]$, $j \prec_s i$. [8] We repeat this terminology for the associated property of mechanisms.

Fairness: For each problem, the allocation recommended by φ is *fair*.

There may be several *fair* allocations, and we distinguish those most preferred by the agents. Allocation $(\boldsymbol{\rho}, \sigma)$ **Pareto-dominates** $(\boldsymbol{\rho}', \sigma')$ if for each $i \in N$, $(\boldsymbol{\rho}, \sigma(i))\ R_i\ (\boldsymbol{\rho}', \sigma'(i))$, and for some $j \in N$, $(\boldsymbol{\rho}, \sigma(i))\ P_i\ (\boldsymbol{\rho}', \sigma'(i))$. A **student-optimal fair** allocation is *fair* and not Pareto-dominated by any other *fair* allocation.[9]

Student-optimal fairness: For each problem, the allocation recommended by φ is *student-optimal fair*.

[7]Generally, for any function f and any element or set y, we let $f[y]$ denote the pre-image of y under f.

[8]Note that it is not necessarily true that the resource ratio of a school at a *fair* allocation is consistent with the number of matched students to that school.

[9]There may be allocations that are *efficient* (Pareto-undominated by any allocation) but not *fair*. As we are in the context of school choice, we focus on respect for priorities.

Next, we consider the direct revelation incentive compatibility condition.

Strategy-proofness: For each problem, each $i \in N$, and each preference relation $R_i' \in \mathcal{R}$ such that $(R_i', \boldsymbol{R}_{-i}) \in \mathcal{D}$,

$$\varphi_i(S, N, \boldsymbol{R}, \boldsymbol{b}, \prec)\ R_i\ \varphi_i(S, N, (R_i', \boldsymbol{R}_{-i}), \boldsymbol{b}, \prec).$$

The reader will note that our definition here is not completely standard. Typically, *strategy-proofness* is only studied for Cartesian product domains, as we wish to allow for each agent to have complete freedom when considering a manipulation report, R_i', to the mechanism. Thus, our notion of *strategy-proofness* is weaker than the standard notion precisely inasmuch as, given some $\boldsymbol{R}_{-i}$, agent i's possible manipulation reports are constrained. However: in what follows, we shall define our mechanism of interest on a domain $\mathcal{D}$ that is open and dense in $\mathcal{R}^N$. That is, our domain almost fills the entire space. Then, by the Kuratowski-Ulam Theorem, for generic $\boldsymbol{R}_{-i} \in \mathcal{R}^{N \setminus i}$, and generic $R_i' \in \mathcal{R}$, the profile $(R_i', \boldsymbol{R}_{-i}) \in \mathcal{D}$.[10] In other words, for a typical profile and a typical manipulation of some agent, the resulting profile will still be in the domain of interest and, therefore, will be an admissible manipulation for the mechanism φ.

3. Solution Concept

Our solution concept is fundamentally Walrasian in spirit. There is a publicly announced vector (a distribution) that induces, for each agent, a menu of options (a list of ratio-school pairs), and agents select their most preferred. Two main points, though, distinguish our concept from price equilibria. If too many students select one school, then rationing occurs as opposed to a price increase. Exogenously given priority information ($\prec$) determines which students are matched. Next, once a distribution is announced, they must consume the object at that distribution quantity (i.e. teacher-student ratio). They cannot "purchase" more or less of the object.

Formally, a **Rationing Crowding Equilibrium (RCE)** is an allocation $(\boldsymbol{\rho}, \sigma)$ that satisfies three conditions:

[10] A set is generic if it is the countable intersection of open-and-dense sets. Endow $\mathcal{R}$ with the topology of closed convergence [Hildenbrand, 2015]. Thus, R is metrizable and hence second-countable. As $\mathcal{D}$ is open and dense, it is both generic (also called comeager) in $\mathcal{R}^N$ and has the Baire property. Given $\boldsymbol{R}_{-i} \in \mathcal{R}^{N \setminus i}$, let $\mathcal{D}_{\boldsymbol{R}_{-i}} = \{R_i' \in \mathcal{R} : (R_i', \boldsymbol{R}_{-i}) \in \mathcal{D}\}$. Then the Kuratowski-Ulam theorem implies that the set $\{\boldsymbol{R}_{-i} \in \mathcal{R}^{N \setminus i} : \mathcal{D}_{\boldsymbol{R}_{-i}}$ is generic in $\mathcal{R}\}$ is generic in $\mathcal{R}^{N \setminus i}$.

(1) (Fairness) $(\boldsymbol{\rho}, \sigma)$ is *fair*.
(2) (Exhaustive Given $\boldsymbol{\rho}$) For each school s with $\sigma[s] \neq \emptyset$,

$$\lfloor \rho_s^{-1} \rfloor = |\sigma[s]|.$$

(3) (Inferior Empty Schools) For each school s with $\sigma[s] = \emptyset$, $\rho_s = 1$, and for each $i \in N$,

$$(\boldsymbol{\rho}, \sigma(i))\ P_i\ (\boldsymbol{\rho}, s).$$

Note that our notion begins by offering each student the menu $\{(\boldsymbol{\rho}, s) : s \in S\}$. If $\rho_s > b_s$, then "demand" for s is just as one expects. Anticipating, however, that ρ_s cannot be reduced if it equals b_s, "demand" is first rationed via priorities in this case. Thus, *fairness* is the analog to consumer maximization. The second condition is the key to adapting our notion to the crowding environment and operationalizes the interpretation of ρ_s as a resource ratio. The amount of resources the school provides to each student is, up to rounding error, the total amount of resource (one) divided by the number of students matched to the school. Together with the definition of an allocation (feasibility), this is the analog of market clearing. The third condition states that each agent finds any empty school strictly worse than their component of the allocation, and is therefore the analog of the requirement that, at equilibrium, unconsumed commodities should be available for free.

The following example illustrates an RCE.

Example 1 (An RCE). Let $S = \{s_1, s_2\}$ and $N = \{1, 2, 3\}$. Agents' preferences are given by the following utility functions:

$$u_1(\boldsymbol{\rho}, s_1) = \frac{3}{22} + \rho_{s_1} \quad \text{and} \quad u_1(\boldsymbol{\rho}, s_2) = \rho_{s_2}$$
$$u_2(\boldsymbol{\rho}, s_1) = \frac{7}{12} + \rho_{s_1} \quad \text{and} \quad u_2(\boldsymbol{\rho}, s_2) = \rho_{s_2}$$
$$u_3(\boldsymbol{\rho}, s_1) = \frac{3}{11} + \rho_{s_1} \quad \text{and} \quad u_3(\boldsymbol{\rho}, s_2) = \rho_{s_2}$$

Each school s has minimum ratio $b_s = \frac{1}{2}$ (and thus capacity $b_s^{-1} = 2$). School s_1 has the priority order $1 \prec_{s_1} 2 \prec_{s_1} 3$. School s_2 has the priority order $3 \prec_{s_2} 1 \prec_{s_2} 2$.

Allocation $(\boldsymbol{\rho}, \sigma) = ((\rho_{s_1}, \rho_{s_2}), (\sigma(1), \sigma(2), \sigma(3))) = (1/2, 7/11, s_1, s_1, s_2)$ is an RCE. *Fairness*: Both agents 1 and 2 find their component at least as good as others'.[11] Agent 3 prefers both 1 and 2's component to her own: $u_3(\boldsymbol{\rho}, \sigma(3)) = 7/11 < 3/11 + 1/2 = u_3(\boldsymbol{\rho}, s_1)$.

[11] $u_1(\boldsymbol{\rho}, \sigma(1)) = \frac{3}{22} + \frac{1}{2} = \frac{7}{11} = u_1(\boldsymbol{\rho}, s_2)$, and $u_2(\boldsymbol{\rho}, \sigma(2)) = \frac{7}{12} + \frac{1}{2} > \frac{7}{11} = u_2(\boldsymbol{\rho}, s_2)$.

Since $\rho_{s_1} = b_{s_1} = 1/2$ and $1 \prec_{s_1} 2 \prec_{s_1} 3$, however, *fairness* is still satisfied. *Exhaustiveness*: We have $\lfloor \rho_{s_1}^{-1} \rfloor = |\sigma[s_1]| = 2$ and $\lfloor \rho_{s_2}^{-1} \rfloor = |\sigma[s_2]| = 1$. *Inferior Empty Schools* is is trivially satisfied, as there is no empty school.

This example is also among the most extreme cases of mismatch between the number of students matched to a school and the reciprocal of the resource ratio. Since there is only one student at s_2, we should hope that $\rho_{s_2} = 1$. Let $\rho'_{s_2} = \rho_{s_2} + \epsilon$ and consider $\boldsymbol{\rho}' = (\rho_{s_1}, \rho'_{s_2})$. For agent 1, $u_1(\boldsymbol{\rho}', \sigma(1)) = \frac{7}{11} < \frac{7}{11} + \epsilon = u_1(\boldsymbol{\rho}', s_2)$. Since $\rho'_{s_2} > b_{s_2} = 1/2$, 1 prefers (ρ', s_2) to their own component, in violation of *fairness*.

In this problem, the entire set of RCEs is

$$\{(\boldsymbol{\rho}, \sigma) : \rho_{s_1} = \frac{1}{2}, \frac{1}{2} < \rho_{s_2} \leq \frac{7}{11}, \sigma(1) = s_1, \sigma(2) = s_1, \sigma(3) = s_2\}$$
$$\cup\{(\boldsymbol{\rho}', \sigma') : \rho'_{s_1} = \frac{1}{2}, \frac{7}{11} \leq \rho'_{s_2} \leq \frac{17}{22}, \sigma'(1) = s_2, \sigma'(2) = s_1, \sigma'(3) = s_1\}.$$

Note that we can have $\rho_{s_2} > 7/11$ when the matching is changed, but that in all cases, school 2 will have only 1 student and $\rho_{s_2} < 1$.

3.1. **Connection to the Standard School Choice Model.** Consider the canonical school choice model of Abdulkadiroglu and Sonmez (2003) . We show how to embed this problem into school choice with crowding, then relate solution concepts across models.

Let schools S, students N, and priorities $\prec$ be defined as before. For each student $i \in N$, let P_i^* be a strict preference relation over the set of schools and $\boldsymbol{P}^* = (P_i^*)_{i \in N}$ be the profile of such preferences. For each school $s \in S$, let $c_s^* \in \mathbb{N}$ be the capacity of school s, and $\boldsymbol{c}^* = (c_s^*)_{s \in S}$ be the capacity profile for S. A **school choice problem** is a tuple $(S, N, \boldsymbol{P}^*, \boldsymbol{c}^*, \prec)$. An allocation is a matching $\sigma : N \to S$ such that for each $s \in S$, $|\sigma[s]| \leq c_s^*$. We recall two central properties in this model. A matching σ satisfies **no justified envy** if for each $i \in N$, there is no $j \in N \setminus i$ and $s \in S$ such that $s \; P_i^* \; \sigma(i)$ and $i \prec_s j$. A matching σ satisfies **non-wastefulness** if for each $i \in N$, and $s \in S$, $s \; P_i^* \; \sigma(i)$ implies $|\sigma[s]| = c_s^*$.

We now construct an associated school choice problem with crowding $(S, N, \boldsymbol{R}, \boldsymbol{b}, \prec)$. Let S, N, and $\prec$ be as in the school choice problem; only $\boldsymbol{R}$ and $\boldsymbol{b}$ need adjustment. For each $i \in N$, let R_i be such that for each $s, s' \in S$, $s \; P_i^* \; s'$ if and only if $(0, s) \; P_i \; (1, s')$. That is, at any distribution level, i prefers s to s' in R_i. For each $s \in S$, let $b_s = \frac{1}{c_s^*}$. Thus, the classical model embeds in ours as a preference restriction. The externality is still present:

students are worse off when they have more classmates. However, on the restricted domain of classical preferences, there is no way for agents to reveal this fact through their choices.

If we wish, we may include a special school, ϕ, in our model with $b_\phi < 1/|N|$. When the canonical school choice model is embedded in ours, this school may function as an outside option in the standard sense, since it can accept all students. This is despite the fact that in our model, we have elected not to consider the outside option.

Proposition 1. *Fix a school choice problem. The following statements are equivalent:*

(1) σ *satisfies* no justified envy *and* non-wastefulness*, and*
(2) $(\boldsymbol{\rho}, \sigma)$ *for some distribution* $\boldsymbol{\rho}$ *is an RCE for the associated school choice problem with crowding.*

Proof. Fix a school choice problem $(S, N, \boldsymbol{P}^*, \boldsymbol{c}^*, \prec)$. Let $(S, N, \boldsymbol{R}, \boldsymbol{b}, \prec)$ be an associated school choice problem with crowding.

Let σ be a matching. Let $\boldsymbol{\rho} \in [0,1]^S$ be such that for each empty school $s \in S$, $\rho_s = 1$ and for each non-empty school $s \in S$, $\rho_s^{-1} = |\sigma[s]|$. Thus $(\boldsymbol{\rho}, \sigma)$ is an allocation and exhaustive.

Matching σ is non-wasteful at empty schools if and only if, for each empty $s \in S$ and each $i \in N$, $\sigma(i)\ R_i^*\ s$. Since R_i^* is strict, this is true if and only if $\sigma(i)\ P_i^*\ s$. Then $(\boldsymbol{\rho}, \sigma(i))\ P_i\ (\boldsymbol{\rho}, s)$. Therefore, *non-wastefulness* for empty schools in $(S, N, \boldsymbol{P}^*, \boldsymbol{c}^*, \prec)$ is equivalent to *inferior empty schools* in $(S, N, \boldsymbol{R}, \boldsymbol{b}, \prec)$.

Now $(\boldsymbol{\rho}, s)\ P_i\ (\boldsymbol{\rho}, \sigma(i))$ if and only if $s\ P_i^*\ \sigma(i)$. So σ is non-wasteful for non-empty schools if and only if $|\sigma[s]| = c_s^* = b_s^{-1}$ if and only if $\rho_s = b_s$. Moreover, σ satisfies no justified envy for $(S, N, \boldsymbol{P}^*, \boldsymbol{c}^*, \prec)$ if and only if, for each $j \in \sigma[s]$, $j \prec_s i$. Conclude that *non-wastefulness* for non-empty schools and *no justified envy* in $(S, N, \boldsymbol{P}^*, \boldsymbol{c}^*, \prec)$ are equivalent to *fairness* in $(S, N, \boldsymbol{R}, \boldsymbol{b}, \prec)$. ■

3.2. **Competitive Foundations of RCE.** Our RCE concept is related to notions of competitive equilibria when prices exhibit rigidities. In the classical exchange problem, a price ceiling may cause demand to outstrip supply, resulting in the failure of market clearing and thus non-existence of equilibria. Drèze [1975] proposed and showed existence for a notion where prices are constrained by ceilings or floors and rationing occurs at such boundaries. In addition to prices, the notion includes a rationing scheme that specifies limits for the net trades of agents. Likewise, in matching models with price controls, Talman and Yang [2008], Andersson and Svensson [2014], and Herings [2018] introduce similar Drèze-style equilibrium concepts.

None of these models consider consumption externalities or, in particular, crowding effects. Our RCE can be seen as the conceptual parallel to their notions, but for the environment where agents have preferences over crowding.

We will define several Drèze-style equilibrium notions for a school choice with crowding problem $(S, N, \boldsymbol{R}, \boldsymbol{b}, \prec)$, then compare them to those above. A rationing scheme in object assignment can be modelled as an indicator variable for each object and agent pair. Unity represents the fact that if the object is rationed, then the agent qualifies for the object and may demand it; otherwise, zero means the agent cannot, in any circumstance, demand the object. For each agent $i \in N$, let $Q_i^s \in \{0, 1\}$ be the ration of object s offered to agent i. Let $Q_i \in \{0, 1\}^S$ be the rations offered to agent i, and $\boldsymbol{Q} = (Q_i)_{i \in N}$ be the rationing scheme. A special case (defined below) is when rationing relies on priorities associated with the objects.

The demand set of agent i at distribution $\boldsymbol{\rho}$ and rationing scheme $\boldsymbol{Q}$ is

$$D_i(\boldsymbol{\rho}, \boldsymbol{Q}) = \{s \in S : Q_i^s = 1, \ \& \, \forall s' \in S \text{ s.t. } Q_i^{s'} = 1 \ (\boldsymbol{\rho}, s) \ R_i \ (\boldsymbol{\rho}, s')\}.$$

Consider a tuple $(\boldsymbol{\rho}, \sigma, \boldsymbol{Q})$ consisting of an allocation and a rationing scheme. Each equilibrium notion is defined by subsets of the conditions below:

(1) Each agent is matched to a school in their demand set.
(2) For each school s with $\sigma[s] \neq \emptyset$, it holds that $\lfloor \rho_s^{-1} \rfloor = |\sigma[s]|$.
(3) For each pair of agent i and school s such that $Q_i^s = 0$, $\rho_s = b_s$.
(4) For each agent $i, j \in N$, and school $s \in S$ such that $Q_i^s = 0$ and $j \in \sigma[s]$, $j \prec_s i$.

A **Crowding Drèze Equilibrium (CDE)** is a tuple $(\boldsymbol{\rho}, \sigma, \boldsymbol{Q})$ that satisfies conditions (1)-(3). Note that $\boldsymbol{Q}$ is a general rationing scheme and does not depend on $\prec$. When Q is consistent with the priority profile $\prec$, then we say that it is a **Priority-Compatible Drèze Equilibrium (PCDE)**. That is, the refinement of CDE by additionally requiring Condition 4 is a PCDE. Our RCE is a further refinement of PCDE by additionally imposing the *inferior empty schools* condition.

Without crowding, these notions coincide with several of those in the literature. Condition 2 is the key difference—the number of students matched to a school is nearly the distribution associated with the school. Without this coupling, we revert back to the interpretation of the distribution as the price where it and quantity demanded are only indirectly related. Formally, without Condition 2, CDE coincides with the notion in Talman and Yang [2008], and if priorities are used, then PCDE, with additional requirement of *constrained efficiency*, coincides with that in Andersson and Svensson [2014]. Finally, notice that when agents' preferences satisfy the standard monotonicity and continuity assumptions, the existence for each of the aforementioned equilibrium notions is guaranteed. This is not true when there is crowding, as Example 2 demonstrates. With mild domain restrictions, however, we can recover existence (Theorem 1).

4. Existence of RCEs

There are profiles in $\mathcal{R}^N$ that do not admit an RCE. We demonstrate one below. Such profiles are rare; they must violate a very mild restriction on preference profiles, one that holds true generically. We first present an exceptional case and then introduce the domain restriction we require.

Example 2 (The non-existence of RCE). Let $S = \{s_1, s_2, s_3\}$ and $N = \{1, 2, 3\}$. Agents have the following utility functions:

$$u_1(\boldsymbol{\rho}, s_1) = \rho_{s_1}, \quad u_1(\boldsymbol{\rho}, s_2) = \rho_{s_2}, \quad u_1(\boldsymbol{\rho}, s_3) = -\frac{1}{2} + \rho_{s_3}$$
$$u_2(\boldsymbol{\rho}, s_1) = \rho_{s_1}, \quad u_2(\boldsymbol{\rho}, s_2) = \rho_{s_2}, \quad u_2(\boldsymbol{\rho}, s_3) = -\frac{2}{3} + \rho_{s_3}$$
$$u_3(\boldsymbol{\rho}, s_1) = \rho_{s_1}, \quad u_3(\boldsymbol{\rho}, s_2) = \rho_{s_2}, \quad u_3(\boldsymbol{\rho}, s_3) = -\frac{3}{4} + \rho_{s_3}$$

Each school is allowed to have any priority order, and $b_{s_1} = 1/2$, $b_{s_2} = 1/2$, and $b_{s_3} = 0$.

We show that there is no RCE. By contradiction, suppose that there is. First we claim that no agent is matched with school s_3. If there were, then by *exhaustiveness*, at least one of the other two schools, i.e., s_1 and s_2, should have a ratio greater than $1/2$. Thus the agent matched with s_3 prefers the school with a ratio greater than $1/2$, contradicting *fairness*.

Since $b_{s_1} = b_{s_2} = 1/2$, in the case where s_1 takes two agents, the ratio at s_1 is $1/2$ while s_2 only takes one agent, and by *exhaustiveness*, has a ratio greater than $1/2$. Thus, any agent matched with s_1 prefers s_2, contradicting *fairness*. The same reasoning works for the case where s_2 takes two agents.

We now introduce our domain restriction. Given a preference profile $R \in \mathcal{R}^N$, two schools s_1 and s_{k+1} are **connected by indifference** if there is a distribution $\boldsymbol{\rho}$, a sequence of distinct agents $\{i_1, \ldots, i_k\}$, and a sequence of schools $\{s_1, \ldots, s_{k+1}\}$ such that

(1) $\rho_{s_1} = \frac{1}{n}$ and $\rho_{s_{k+1}} = \frac{1}{m}$ for some $m, n \in \{1, \ldots, |N|\}$ and
(2) $(\boldsymbol{\rho}, s_i) \; I_i \; (\boldsymbol{\rho}, s_{i+1})$ for each student $1 \leq i \leq k$

A preference domain satisfies **no connection by indifference (NCBI)**, if it contains no profile that is connected by indifference. Denote by $\mathcal{D} \subseteq \mathcal{R}^N$ the subdomain of *all* profiles that are not connected by indifference.[12] Call this *the* NCBI domain.

The preference profile presented in Example 2 is not in the NCBI domain as as all three agents are indifferent between $(1/2, s_1)$ and $(1/2, s_2)$. On the other hand, the preference profile presented in Example 1 is not connected by indifference.

The NCBI domain $\mathcal{D}$ is not rectangular, but is open and dense in the full domain.

Theorem 1. *Each profile in the NCBI domain admits an RCE.*

The proof of Theorem 1 is in Appendix A.4. We provide a sketch of the argument here. Start from a *fair* allocation. Thus, we allow for the discrepancy between ρ_s^{-1} and the actual number of matched students to be arbitrarily large. The existence of *fair* allocations is trivial: set the distribution equal to vector $\boldsymbol{b}$ and run the Deferred Acceptance mechanism.

The set of distributions that generate RCE, if non-empty, lie in the upper envelope of the set of distributions that generate a *fair* allocation. So, given that a *fair* allocation always exists, our argument begins with one of these, and seeks to increase the distribution vector while maintaining *fairness*. We do this with some graph theoretic tools.

With a distribution fixed, we study a graph with vertices S and such that an arc represents a student at a given school who finds another school at least as good. A *source set* is a set of vertices such that no arcs enter the set (there may be arcs among vertices in the set, so that no vertex is a source on its own). Lemma 1 tells us that if we can find a set of schools that is a source set in our graph, and for which all schools fail our exhaustiveness condition, then we can perturb upwards the ratios for these schools and arrive at another *fair* allocation.

Thus, our goal is to move students among schools such that we do not violate *fairness* and that we find a source set. This is the only part of the argument that requires NCBI, and is achieved in Lemma 2. In sum, beginning with a *fair* allocation, if it is not an RCE, then we can increase the ratio of some school and find another *fair* allocation. Along the way, we eliminate the problem of empty schools by simply putting students in them; NCBI ensures they will not hinder us in finding a source.

[12]NCBI is similar to the identically named condition in Andersson and Svensson [2014], although the two are applied to different environments. Our condition is stronger than theirs, and the latter is not sufficient to show existence of RCE in our model.

Having established that we can perturb upwards any *fair* allocation that is not an RCE, it remains only to make a limit argument. This is done in Theorem 5 in the Appendix, which actually proves Theorem 1 and Proposition 3 below.

It is known that the exact auction of Demange et al. [1986] and its variants crucially depend on the quasi-linear assumption, and when agents have non-quasi-linear preferences, they are not appropriate methods to show the existence of equilibrium [Zhou and Serizawa, 2021]. Instead, the salary adjustment process of Crawford and Knoer [1981] and Kelso & Crawford [1982], and its variants are frequently used to establish the existence of equilibrium [Ostrovsky, 2008, Herings, 2018, Fleiner et al., 2019]. In general, this method requires that agents always choose their favorite schools among those who have not rejected them yet, and an agent's welfare is independent of the number of tentatively matched agents. This is not true in our model, thus the method fails. Another well-known method is via Scarf's Lemma [Quinzii, 1984]. In this approach, one first shows existence when agents have piece-wise quasi-linear utility functions and then takes the limit, approximating the original continuous utility functions. It is not clear how to establish the existence of RCEs even when agents have piece-wise quasi-linear utility functions, and the main challenges still come from showing *exhaustiveness*.

5. Structural Properties of RCEs

In the classical model, the set of no *justified envy* and *non-wasteful* allocations has a number of remarkable structural properties. Among them, the welfare lattice and the so-called *Rural Hospitals Theorem* stand out as particularly important [Knuth, 1976, Roth and Sotomayor, 1990, Roth, 1986]. On the NCBI domain, we find analogous, sometimes identical properties. RCEs form a welfare lattice, which further implies the classical lattice result via the embedding. We also show that the number of students matched to any school remains constant across all RCE, which establishes a *Rural Hospitals Theorem* for our environment.

Along the way, we find a decomposition result that also has analogues in earlier literature, but was previously unknown. We require, first, a definition. A sequence of distinct agents $(i_1, ..., i_k)$ constitutes a **trading cycle** from a matching σ to a matching τ if $\tau(i_l) = \sigma(i_{l+1})$ for each $1 \leq l \leq k-1$ and $\tau(i_k) = \sigma(i_1)$.

Proposition 2. *Consider two arbitrary RCEs, $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$, for a given preference profile from the NCBI domain.*

(1) *When moving from* $(\boldsymbol{\rho}, \sigma)$ *to* $(\boldsymbol{\gamma}, \tau)$*, if agent* i *experiences a strict welfare-improvement (resp. welfare-reduction), then agents in the same trading cycle as agent* i *experience a non-decreasing welfare change (resp. non-increasing welfare change).*
(2) *If* $\tau(i) \neq \sigma(i)$*, then* i *is involved in a trading cycle from* σ *to* τ*.*

The proof of Proposition 2 is in Appendix A.2. What follows is a sketch of the argument. Schools whose ratio either increases or remains the same cannot take on more students (*exhaustiveness*). Students attending a school whose ratio *increases* must be better off, as its increase causes it to rise above its lower bound and thereby be available for all. Then, using Walrasian-type reasoning, we find that these better-off students must go to a school whose ratio must not have decreased. This of course "closes off" the set of such schools and yields the proposition for them. The proposition for the other agents comes from feasibility, since the schools whose ratios have decreased must take in the rest of the students, and from the properties of RCE—these schools must also have "high enough" ratios by *exhaustiveness*.

The full argument is complicated by several technical details, most difficult among them being: what if $(\boldsymbol{\gamma}, \tau(i)) \; I_i \; (\boldsymbol{\rho}, \sigma(i))$, $\gamma_{\sigma(i)} = \rho_{\sigma(i)} > b_{\sigma(i)}$ and $\gamma_{\tau(i)} = \rho_{\tau(i)} = b_{\tau(i)}$? Student i may displace j at $\tau(i)$ who has $i \prec_{\tau(i)} j$, since i's indifference means that j's presence at $\tau(i)$, under allocation $(\boldsymbol{\sigma}, \rho)$, was not a priority violation. Student j's welfare may drop, and she may envy i at $(\boldsymbol{\gamma}, \tau)$, without priority violation. The problem is that nothing prevents i from being part of a trading path along which some previous agent has increasing welfare and all other previous agents have non-decreasing welfare. For this case we invoke NCBI, and the earlier Walrasian-type reasoning, as we find that such a student i must actually be the end of an indifference chain originating at a school $u \in S$ with $\rho_u = b_u$. Thus, it appears that NCBI is essential for this decomposition to hold.

Note the significance of claim 2 in the proposition. Since the distributions $\boldsymbol{\rho}$ and $\boldsymbol{\gamma}$ are distinct, the equilibrium capacity of a school $s \in S$ may differ from one RCE to another. If $\gamma_s^{-1} > \rho_s^{-1}$, then s might be able to take on more students under $(\boldsymbol{\gamma}, \tau)$, and so may be the endpoint of a trading *path* instead. In fact this is true for *fair* allocations, and so this decomposition does not hold on that larger set of allocations.

Proposition 2 is reminiscent of the classical decomposition property of the marriage market [Roth and Sotomayor, 1990]: Moving from one stable outcome to another, there is a one to one correspondence between agents on the one side who have strictly increased welfare (resp. strictly reduced welfare) and those one the other side whose welfare is strictly reduced (resp. strictly increased). This property holds for competitive equilibrium/core outcomes in the two-sided matching models with transfers as well [Demange and Gale, 1985]. Nevertheless, such a decomposition does not hold in our model. Instead, we provide a generalized decomposition by associating welfare changes with trading cycles of components in an allocation. In our language, the existing decomposition results can be read as follows: if an agent experiences a strict welfare improvement, all the other agents in the same trading cycle will also experience strict welfare improvement. In contrast, Part (1) of Proposition 2 admits the possibility of unchanged welfare in a trading cycle. Besides, we do not have a counterpart of Part (2) of Proposition 2 for the competitive equilibrium in the matching models with transfers [Demange and Gale, 1985].

Proposition 2 is related to but not nested with Lemmas 4 and 5 of Andersson and Svensson [2014]. Like them, our model has divisible and indivisible components, and we also face exogenous upper and lower bounds on the divisible components. However, while they match objects to agents one-to-one, we match schools to students one-to-many, which introduces complications in itself *and* via the imposition of the distribution feasibility constraint. Thus, our matching is entangled with our distribution of the divisible component. Nonetheless, though our models differ in crucial ways, Andersson and Svensson [2014] find a statement analogous to conclusion (1) in the proposition above. They do not provide a full trading cycle decomposition, as we do in statement (2), but their statement *does* hold for stronger hypotheses: the profiles associated with their two equilibria are allowed to differ in a particular way.

We now give our version of the *Rural Hospitals Theorem*, which is a clear corollary of Proposition 2.

Theorem 2. *Fix a profile from the NCBI domain and let $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$ be two RCEs for this profile. Then for each school $s \in S$, the number of students matched to s under σ is equal to the number of students matched to s under τ.*

Example 3 (Illustration of Theorem 2). Consider the set of RCEs in Example 1. There are two RCE matchings. The first one is $\sigma = (\sigma(1), \sigma(2), \sigma(3)) = (s_1, s_1, s_2)$. The second one is $\sigma' = (\sigma'(1), \sigma'(2), \sigma'(3)) = (s_2, s_1, s_1)$.

In either matching, school s_1 is always matched with two students and school s_2 is matched with one student.

Given two distributions, $\boldsymbol{\rho}$ and $\boldsymbol{\gamma}$, let $\boldsymbol{\rho} \vee \boldsymbol{\gamma} \in [0,1]^S$ denote the vector whose s component, for each $s \in S$, is $\max\{\rho_s, \gamma_s\}$.

Theorem 3. *Assume $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$ are RCEs for a preference profile from the NCBI domain. There is a matching μ such that $(\boldsymbol{\rho} \vee \boldsymbol{\gamma}, \mu)$ is an RCE, and for each $i \in N$,*

$$(\boldsymbol{\rho} \vee \boldsymbol{\gamma}, \mu(i)) \; R_i \; \max_{R_i}\{(\boldsymbol{\rho}, \sigma(i)), (\boldsymbol{\gamma}, \tau(i)).\}$$

The full proof of Theorem 3 is in Appendix A.2. Similar to extant results of similar character, a decomposition result, in our case Proposition 2, is the main tool. We simply begin with one of the two RCEs, say $(\boldsymbol{\rho}, \sigma)$, and to arrive at a candidate matching, μ, execute all the welfare-non-decreasing trading cycles from σ to τ. Then we show that $(\boldsymbol{\rho} \vee \boldsymbol{\gamma}, \mu)$ is an RCE. If $(\boldsymbol{\rho} \vee \boldsymbol{\gamma})_s = \gamma_s > \rho_s$, then any student $i \in \sigma[s]$ must have increased welfare, as otherwise they would prefer s at (γ, τ) and the previous inequality gives $\gamma_s > b_s$. Then Proposition 2 and some supporting results in the appendix allow us to conclude that i is part of a cycle among schools whose resource ratio is at least as high under $\boldsymbol{\gamma}$ as it is under $\boldsymbol{\rho}$. This further allows us to use the feasibility of (γ, τ) to conclude the feasibility of $(\boldsymbol{\rho} \vee \boldsymbol{\gamma}, \mu)$. Since all agents are partitioned by Proposition 2, the execution of these cycles will not interfere with each other.

The foregoing argument studied the case when $\boldsymbol{\rho} \neq \boldsymbol{\gamma}$. However, the decomposition holds equally well when $\boldsymbol{\rho} = \boldsymbol{\gamma}$ and so demonstrates that RCE induce a lattice in welfare space.

It is in general not true that the existence of a lattice in distributions implies the existence of a lattice in welfare (see Example 5 below).[13] Here again the NCBI domain seems crucial.

Since we have a lattice in welfare space, a limit argument is sufficient to show the existence of a greatest RCE welfare vector. Several RCEs may induce this vector, all of which have the same distribution. Any RCE that induces this vector is called **maximal**. For Example 1, one such maximal RCE is given by distribution $(1/2, 17/22)$ paired with matching (s_2, s_1, s_1). We formalize the foregoing observations as follows.

Proposition 3. *Given a profile R from the NCBI domain,*

(1) *there is a greatest RCE distribution $\boldsymbol{\rho}^*(R)$,*

[13] The assignment model of Andersson and Svensson [2014] takes the economy described by Example 2.15 in Roth and Sotomayor [1990] as a special case. In that economy, the price can be treated as a fixed price (note the structural similarity between prices and distributions), and there is a price lattice while the welfare lattice fails to hold.

(2) *there is a maximal RCE, with distribution $\boldsymbol{\rho}^*(R)$, and all agents are indifferent between all maximal RCEs, and*
(3) *among all RCEs, only the maximal RCEs satisfy* student-optimal fairness.

The proof of Proposition 3 is in Appendix A.4. It is not true that an RCE compatible with $\boldsymbol{\rho}^*(R)$ always maximizes agents' welfare; maximal RCEs are a subset of the RCEs that are compatible with $\boldsymbol{\rho}^*(R)$.

We conclude this section with two remarks. First, in Section 3.1 we showed that the standard school choice model can be embedded in our model. Recalling that the the embedded, standard model may have an outside option, even though our more general model does not, it follows that the standard Rural Hospitals Theorem of Roth [1986] is a corollary of Theorem 2. Theorem 3 and Proposition 3 mirror the standard welfare lattice results in Roth and Sotomayor [1990], but here we only consider the student side of the market.

Second, even when RCEs exist for profiles outside the NCBI domain, the structural results above may fail to hold.

Example 4 (Lack of structure on the general domain.). The reader will observe that the structures above fail for the same reason they do in the standard school choice model when students' preferences may have indifferences.

Let $S = \{s_1, s_2, s_3, s_4\}$ and $N = \{1, 2, 3\}$. Agents have the following preferences:

For each $\boldsymbol{\rho} \in [0,1]^S$,

$$(\boldsymbol{\rho}, s_1)\ I_1\ (\boldsymbol{\rho}, s_2)\ P_1\ (\boldsymbol{\rho}, s_3)\ P_1\ (\boldsymbol{\rho}, s_4)$$
$$(\boldsymbol{\rho}, s_1)\ P_2\ (\boldsymbol{\rho}, s_4)\ P_2\ (\boldsymbol{\rho}, s_3)\ I_2\ (\boldsymbol{\rho}, s_2)$$
$$(\boldsymbol{\rho}, s_2)\ P_3\ (\boldsymbol{\rho}, s_3)\ P_3\ (\boldsymbol{\rho}, s_1)\ I_3\ (\boldsymbol{\rho}, s_4)$$

Each school has unit capacity, i.e., $b_{s_1} = b_{s_2} = b_{s_3} = b_{s_4} = 1$. Schools have the following priority rankings: $1 \prec_{s_1} 2 \prec_{s_1} 3$; $1 \prec_{s_2} 3 \prec_{s_2} 2$; $3 \prec_{s_3} 2 \prec_{s_3} 1$; and $2 \prec_{s_4} 3 \prec_{s_4} 1$.

Let $\boldsymbol{\rho} = (1,1,1,1)$. There are two RCEs compatible with $\boldsymbol{\rho}$, clearly making $\boldsymbol{\rho}^*(R) = \boldsymbol{\rho}$. The first one is $(\boldsymbol{\rho}, \sigma)$ such that $(\sigma(1), \sigma(2), \sigma(3)) = (s_1, s_4, s_2)$. The second one is $(\boldsymbol{\rho}, \tau)$ such that $(\tau(1), \tau(2), \tau(3)) = (s_2, s_1, s_3)$.

It is not hard to see that there is no trading cycle from σ to τ, and so Proposition 2 fails to hold. Also at $(\boldsymbol{\rho}, \sigma)$ school s_3 is empty and at $(\boldsymbol{\rho}, \tau)$ school s_4 is empty school. Therefore, Theorem 2 fails to hold.

Note that in both of the RCEs mentioned, two agents get their favorite possible bundle, and one gets their second favorite. Thus, the only way to improve upon this is with matching $(\mu(1), \mu(2), \mu(3)) = (s_1, s_1, s_2)$ or matching $(\mu'(1), \mu'(2), \mu'(3)) = (s_2, s_1, s_2)$, both of which are infeasible. However, agent 2 prefers $(\boldsymbol{\rho}, \tau(2))$ to $(\boldsymbol{\rho}, \sigma(2))$. Thus Theorem 3 fails to hold. Since there are no maximal RCEs, Proposition 3 fails to hold as well.

Examples 2 and 4 highlight the role of our domain restriction, NCBI. However, it is worth noting that the structural properties hold under a weaker restriction. In particular, we may relax the first condition in definition of connected by indifference, requiring only that $\rho_{s_1} = b_{s_1}$ and $\rho_{s_{k+1}} = b_{s_{k+1}}$. However, this domain restriction is *not* sufficient for our proof of the existence of RCE. See Appendix section A.1 for details.

6. Maximal RCE Mechanisms

By Proposition 3, fixing an environment of schools, students, and lower-bounds, for each profile in the NCBI domain, there is a non-empty set of maximal RCE allocations, between which all students are indifferent. A maximal RCE mechanism is a function that selects, for each profile in the NCBI domain, a maximal RCE allocation; we do not define these mechanisms on the full domain. Thus, all maximal RCE mechanisms are welfare equivalent.

Theorem 4. *Any maximal RCE mechanism is* strategy-proof.

The full proof of Theorem 4 is in Appendix B. For some intuition as to how it works, consider first the properties of $\boldsymbol{\rho}^*(\cdot)$, the greatest RCE distribution. Consider a student i who is not even weakly envied by a student at a different school. That is to say, all students not attending i's school strictly prefer their outcome to i's. Then it better not be feasible to raise the ratio at i's school, because if it were, then we could do so by a very small amount, make i and her classmates happier, and not induce any envy from other students. Thus, for each student i one of the following is true: 1) some student j from another school finds i's outcome at least as good as her own or 2) the ratio at i's school exactly corresponds with the number of students there. For case (2), this means that the ratio at i's school is of the form $1/k$, where k is the number of students at i's school. We say such a school is *completely exhausted*. The above reasoning then implies that, for each student for whom (1) is true, we must be able to find a chain of students $\{j, k, \dots, l\}$ such that j finds i's outcome at least as good as her own, k finds j's outcome at least as good as her own, etc., and l's school is completely exhausted. This is the so-called *connectedness property*, seen in similar environments.

Now let i declare a R_i' such that her preference for her outcome is *strictly* stronger than before. That is, R_i' is a strict Maskin monotonic transform of R_i at her initial outcome. Ignoring complications, if she stays at her original school, then the connectedness property prevents the ratio at her school from rising. If she goes to another school, the connectedness property prevents the ratios of these schools from rising as well, so the only way her new outcome can be R_i better than the original is if she goes to a school that has hit its lower bound. Then we use our decomposition and the connectedness property to find that we cannot displace these students and make them happier, so i induces violation of *fairness* at this new school. In the appendix we show that, if the rule were manipulable, then it would be manipulable via a Maskin monotonic transform, and thus our argument is complete.

7. An Algorithm for Maximal RCE

As discussed after Theorem 1, it is non-trivial to find an algorithm that calculates an RCE in a finite number of steps, in particular, when agents have general preferences as studied here. We can, however, on a restricted domain. The algorithm we introduce uses tools from both the multi-item auction [Demange et al., 1986, Tierney, 2019] and the Deferred-Acceptance algorithm [Gale and Shapley, 1962]. The former helps adjust the distribution for unconstrained schools and the latter helps adjust the matching for schools whose capacity constraint has been met.

Say a preference relation $R \in \mathcal{R}$ is *linear* if there is a vector $v \in \mathbb{R}^S_{++}$ such that the utility function $(r, s) \mapsto rv(s)$ represents R. Note that an equivalent representation is $\log v(s) + \log r$. Thus, we have a structure isomorphic to an auction with quasi-linear preferences. Moreover, if $v(s)$ and r conform to the same discrete grid, then we can be sure that $\boldsymbol{\rho}^*(\boldsymbol{R})$ also conforms to this grid, by the connectedness property. For concreteness, then, fix $\overline{g} \in \mathbb{N}$ and assume valuations and ratios are drawn from the set $\left\{\frac{n}{\overline{g}|N|} : n \in \mathbb{N}\right\}$. Thus, in the additive representation, we have utility functions of the form $\log v(s) + \log k + C$, where $C = -\log \overline{g}\,|N|$. We shall design an algorithm in which the ratio at each school starts at 1 and then decreases. Equivalently, in the quasi-linear representation, we have a procurement auction where the "bid" for each school starts at $\log \overline{g}\,|N|$ and increments as the sequence $\{\log(\overline{g}\,|N| - n) : n \in \mathbb{N}, n \leq \overline{g}\,|N|\}$.

Note that NCBI is now more demanding. In particular, a violation of NCBI is a sequence $\{1,\dots,J+1\}$ of students and schools such that for each j on the sequence, $r^j v_j(j) = r^{j+1}v_j(j+1)$, and such that $r^1 = {}^1\!/_p$ and $r^{J+1} = {}^1\!/_q$, for natural numbers p and q no greater than $|N|$. By recursion on the sequence, we get

$$\frac{p}{q} = \frac{v_J(J+1)}{v_1(1)} \prod_{j=2}^{J} \frac{v_{j-1}(j)}{v_j(j)}.$$

This expression highlights that a violation of NCBI requires the sequence of students and schools to be chosen together; it is not sufficient to find a sequence of valuation ratios that multiply to ${}^p\!/_q$.

Given $R \in \mathcal{R}$ and $S' \subseteq S$ let

$$\mathfrak{D}\left(\boldsymbol{\rho}, S'; R\right) = \{s \in S' : \forall t \in S', (\boldsymbol{\rho}, s)\ R\ (\boldsymbol{\rho}, t)\,.\}$$

Say that R requires S' at $\boldsymbol{\rho}$ if $\mathfrak{D}\left(\boldsymbol{\rho}, S; R\right) \subseteq S'$. Fix a set N of students and a profile $\boldsymbol{R}$ of linear preferences satisfying NCBI. A set $S' \subseteq S$ is *overdemanded* at $\boldsymbol{\rho}$ if the number of agents i for whom R_i requires S' exceeds $\sum_{s\in S'} \lfloor \rho_s^{-1} \rfloor$. Initialize the process with $\boldsymbol{\rho}^0 = (1,\dots,1)$. At each stage n execute the following procedure:

(1) Set $S^* = \{s \in S : \rho_s^n = b_s\}$.
(2) Identify the students N' for whom the set $\mathfrak{D}\left(\boldsymbol{\rho^n}, S; R_i\right) \subseteq S^*$.
(3) Execute deferred acceptance on students N' and schools S^* to arrive at a matching $\sigma : N^* \subseteq N' \to S^*$, where $N' \setminus N^*$ is the set of unmatched students.
(4) For each $i \in N \setminus N^*$, let $D_i^n = \mathfrak{D}\left(\boldsymbol{\rho^n}, S \setminus S^*; R_i\right)$. In words, this is i's demand set from a restricted set of options.
(5) Let $A \subseteq S$ be an overdemanded set from $(D_i^n)_{i \in N\setminus N^*}$ that is minimal, in inclusion, in this property. For each $a \in A$, with $\rho_a^n = {}^p\!/_{\overline{g}|N|}$, set $\rho_a^{n+1} = {}^{p-1}\!/_{\overline{g}|N|}$. All other resource ratios remain fixed.

Proposition 4. *Given a profile $\boldsymbol{R}$ of linear preferences, with rational coefficients and satisfying NCBI, the above algorithm, executed on the grid induced by $\boldsymbol{R}$, ends in a maximal RCE in a finite number of stages.*

The proof of Proposition 4 is in Appendix C.

8. Discussion

8.1. **Dropping Inferior Empty Schools.** Let an allocation be a **weak RCE** if it satisfies all the conditions of RCE except for the *inferior empty schools* condition. There exists a weak RCE in the NCBI domain. A parallel result to Proposition 2 for weak RCEs would decompose changes into trading cycles of positive or negative welfare value, and indifference chains.[14] The Rural Hospitals Theorem (Theorem 2) no longer holds for weak RCEs. Fortunately, all of the other remaining results can be established. The proofs of the above statements are analogous to those in the main text so we omit them.

8.2. **On Crowding As a Continuous Variable.** We use ρ_s to model the amount of resource each student at school s receives. As mentioned above, there are practical situations where ρ_s ought to be of the form $1/k$ for some $k \in \mathbb{N}$. However, the discrepancy between ρ_s and a simple ratio will be small for realistic school sizes and, in particular, it is easy to see that the magnitude of the discrepancy at any RCE is

$$\frac{1}{|\sigma[s]|} - \rho_s = \frac{1}{|\sigma[s]|} - \frac{1}{\rho_s^{-1}} \leq \frac{1}{|\sigma[s]|} - \frac{1}{\lfloor \rho_s^{-1} \rfloor + 1} = \frac{1}{|\sigma[s]|} - \frac{1}{|\sigma[s]| + 1}.$$

Thus, the difference between our modeled ratio and a strictly interpreted resource-to-student ratio is at most the addition of one more student. We think this is a negligible difference for the overwhelming majority of real world applications. Thus by rounding the allocation selected by the maximum RCE mechanism, we can always get a practically implementable allocation that approximately preserves the nice properties of maximal RCEs.

8.3. **Applications beyond School Choice.** The first application is to the problem of **labor markets with financially constrained start-ups**. Consider the labor market where there is a finite set of start-ups F and a finite set of workers W. Each start-up $f \in F$ is subject to some (hard) financial constraints so that expenditure for labor employment $\kappa_f > 0$ is fixed for the modelled time period. Given this, the start-up selects employees following a priority order $\prec_f$. The labor market is protected by a minimum wage $\underline{w}$ such that $0 < \underline{w} \leq \kappa_f$ for each $f \in F$. Each worker $w \in W$ has a complete and transitive preference relation R_w over $\{(t, f) \in \mathbb{R} \times F : \ 0 \leq t \leq \kappa_f\}$ such that for each $f \in F$ and $t, t' \in [0, \kappa_f]$ with $t > t'$, $(t, f) \ P_w \ (t', f)$. The tuple $(F, W, \boldsymbol{R}, \boldsymbol{\kappa}, \prec, \underline{w})$ summarizes primitives of the problem.

We reformulate the problem to apply our results. Let F, W, and $\prec$ be as defined above. For each $f \in F$, let $b_f = \frac{\underline{w}}{\kappa_f}$. Each worker $w \in W$ has a complete and transitive preference

[14] Chains here are defined in the same way as open trading cycles in Andersson and Svensson [2014].

relation R^*_w over $[0,1] \times F$ such that for each $f, f' \in F$ and each $t, t' \in [0,1]$, $(t, f)\ R^*_w\ (t', f')$ if and only if $(t\kappa_f, f)\ R_w\ (t'\kappa'_f, f')$. It is easy to see that all our results hold in the problem $(F, W, \boldsymbol{R}^*, \boldsymbol{b}, \prec)$ and therefore the insights can be carried over to the original problem $(F, W, \boldsymbol{R}, \boldsymbol{\kappa}, \prec, \underline{w})$.

The second application is to the problem of **allocating polluting firms**. Instead of students and schools, consider polluting firms and subnations. Each firm is willing to invest in at one most one subnation. Each subnation s is endowed with an amount of the same environmental resource, and the priority for selecting polluting firms is based on their industrial development policy. Let b_s^{-1} be the maximum number of polluting firms that the subnation is willing to admit. Each firm prefers to locate to a subnation where they are allowed a higher level of pollution. As this is simply a re-interpretation of the model, we are able to directly apply our results for this application.

9. Conclusion

We provide a new framework to model school choice with crowding and establish analogs of several key results in the school choice and matching literature. Many open questions remain. Essentially, each topic considered in the school choice program, e.g. affirmative action, efficiency improvements, lotteries, sibling guarantees, and multi-stage mechanisms, can be re-examined in our framework. Regarding implementation, we show the existence of RCE non-constructively, and so an algorithm for computing RCE on the full domain is also unknown. New classes of questions are also possible, outside of the scope of the classic model. For example, a school may not only have priorities over the students, but also prefer to have a high resource ratio to maintain instructional quality. How should we design mechanisms in this environment? Beyond school choice, crowding has not been considered in general, many-to-many, or matching with contracts. Our techniques may be useful in establishing results if crowding is added.

Appendix A. Existence of RCE and maximal RCE

We proceed in four subsections. In Section A.1 we study the set of *fair* allocations, which contains the set of RCE, and which can be trivially shown to be non-empty on our domain. In Section A.2 we show that RCE, if they exist, induce an upper lattice in welfare space, which then leads to Theorem 3. Proposition 2 is necessary for this argument, so we give its proof here as well. In Section A.3, we uncover some Pareto dominance relations in the set of *fair* allocations ("the *fair* set" for short). These imply that RCE will lie in the welfare-upper-envelope of the *fair* set, and so will exist if the fair set induces a closed set in welfare space. Finally, we conclude in Section A.4 with the (simple) topological arguments that are required for this upper-envelope to exist.

A.1. **Existence of *Fair* Allocations.** A preference domain $\mathcal{D}' \subseteq \mathcal{R}^N$ satisfies **no boundary indifference (NBI)**, if for each pair of schools s and t and each $R \in \mathcal{D}'$, there is no chain of indifference connecting (b_s, s) and (b_t, t). Note that if $\boldsymbol{R}$ satisfies NCBI, then it satisfies NBI.

Proposition 5. *A* fair *allocation exists on any NBI domain.*

Proof. Consider $\boldsymbol{\rho} \in [0,1]^S$ such that $\rho_s = b_s$. Then we have a standard school-choice problem where the capacity of school $s \in S$ is b_s^{-1}. NBI implies that student preferences are strict, so the set of stable matchings is non-empty (Roth and Sotomayor [1990]). Let σ be a stable matching for this problem. Clearly, it satisfies no-blocking, in the sense of our model. Thus, $(\boldsymbol{\rho}, \sigma)$ is a *fair* allocation. ■

A.2. **The upper-lattice property.** Given two allocations, $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$, construct the labeled, directed *transfer graph* T on vertices S so that $s \xrightarrow{i} t \in T$ if $\sigma(i) = s$, $\tau(i) = t$, and $s \neq t$. Define the sets

$$S^+ = \{s \in S : \gamma_s > \rho_s\} \qquad N^+ = \{i \in N : (\boldsymbol{\gamma}, \tau(i))\ P_i\ (\boldsymbol{\rho}, \sigma(i))\}$$

$$S^= = \{s \in S : \gamma_s = \rho_s > b_s\} \qquad N^= = \{i \in N : (\boldsymbol{\gamma}, \tau(i))\ I_i\ (\boldsymbol{\rho}, \sigma(i))\}$$

$$S^* = \{s \in S : \gamma_s = \rho_s = b_s\}.$$

We denote by $s \rightsquigarrow t \subseteq T(\sigma, \tau)$ a simple path in the transfer graph, which is to say, a path with no repeated arcs. Note that since our graph may contain several arcs, with the same orientation, between a given pair of vertices, there may be many distinct paths from s to t, even on the same ordered list of vertices. We distinguish between different paths either by decoration, so that $s \rightsquigarrow' t \neq s \rightsquigarrow t$, or superscript index, so that $s \rightsquigarrow^m t \neq s \rightsquigarrow^n t$ when $n \neq m$. A path is positive if it contains an N^+-labeled arc. A positive path is totally-positive if it contains only labels from $N^+ \cup N^=$.

The in-degree of a set of vertices $V \subseteq S$ is the number of edges $s \to t \in T$ with $s \notin V$ and $t \in V$. Symmetrically, the out-degree is the number of such edges where $s \in V$ and $t \notin V$.

Given the language just introduced, we rephrase Proposition 2.

Proposition 6. *Let $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$ be two RCEs for a profile $\boldsymbol{R}$ satisfying NBI. Let T be the transfer graph from $(\boldsymbol{\rho}, \sigma)$ to $(\boldsymbol{\gamma}, \tau)$, and let the sets $S^+, S^=, S^*, N^+, N^=$ be defined as above. Then any path $s \rightsquigarrow t \subseteq T$ touching a S^+ school is positive. Any positive path is a totally-positive cycle, confined to $\overline{S} = S^+ \cup S^= \cup S^*$. Moreover, these cycles can be constructed so that they are all mutually disjoint.*

Proof. We first establish several claims.

Claim 1. *For each $s \in \overline{S}$ with $\sigma[s] \neq \emptyset$, the out-degree of s in T is at least as large as its in-degree.*

Proof of claim. For each $s \in \overline{S}$, with $\sigma^{-1}[s]$ non-empty,

$$\lfloor \gamma_s^{-1} \rfloor \leq \lfloor \rho_s^{-1} \rfloor = |\sigma^{-1}[s]|,$$

and so there cannot be more students at s under $(\boldsymbol{\gamma}, \tau)$ then under $(\boldsymbol{\rho}, \sigma)$, and so for any arc entering s, there must be at least one exiting. ◇

Claim 2. *If $s \xrightarrow{i} t \in T$ has $s \in \overline{S}$ and $i \in N^+ \cup N^=$, then $t \in \overline{S}$.*

Proof of claim. If $t \notin \overline{S}$, then $\gamma_t < \rho_t$, and furthermore, $\rho_t > b_t$. Since $(\boldsymbol{\rho}, \sigma)$ is an RCE and $\sigma(i) = s$, $(\boldsymbol{\rho}, s)\ R_i\ (\boldsymbol{\rho}, t)$. Since, $\gamma_t < \rho_t$, $(\boldsymbol{\rho}, s)\ P_i\ (\boldsymbol{\gamma}, t)$, and since $\tau(i) = t$, $i \notin N^+ \cup N^=$. ◇

Claim 3. *$\sigma[S^+] \subseteq N^+$ and $\tau(N^+) \subseteq S^+ \cup S^*$.*

Proof of claim. For $s \in S^+$, $\gamma_s > b_s$. Therefore, for each $i \in N$, $(\boldsymbol{\gamma}, \tau(i))\ R_i\ (\boldsymbol{\gamma}, s)$. In particular, for $i \in \sigma[s]$, preference monotonicity gives

$$(\boldsymbol{\gamma}, \tau(i))\ R_i\ (\boldsymbol{\gamma}, s)\ P_i\ (\boldsymbol{\rho}, s) = (\boldsymbol{\rho}, \sigma(i)).$$

Thus, $i \in N^+$.

Let $i \in N^+$. If $\tau(i) \notin S^+$, then by preference monotonicity

$$(\boldsymbol{\rho}, \tau(i))\ R_i\ (\boldsymbol{\gamma}, \tau(i))\ P_i\ (\boldsymbol{\rho}, \sigma(i)),$$

and so

$$b_{\tau(i)} = \rho_{\tau(i)} \geq \gamma_{\tau(i)} \geq b_{\tau(i)},$$

yielding $\tau(i) \in S^*$. ◇

Claim 4. *Consider a path in T of the following form:*

$$t \xrightarrow{i} u \xrightarrow{j} v$$

with $i \in N^+$ and $j \notin N^+$. Then $j \in N^=$, $u \in S^*$, $v \in S^+ \cup S^=$, and $\sigma[v] \neq \emptyset$.

Proof of claim. By Claim 3, $u \in S^+$ implies $j \in N^+$, and so $\gamma_u \leq \rho_u$. Since $i \in N^+$,

$$(\boldsymbol{\rho}, u) \; R_i \; (\boldsymbol{\gamma}, u) = (\boldsymbol{\gamma}, \tau(i)) \; P_i \; (\boldsymbol{\rho}, \sigma(i)) \, .$$

It follows that $u \in S^*$ and, since $\sigma(j) = u$, $j \prec_u i$. Thus if j is made worse off going to $(\boldsymbol{\gamma}, \tau)$, we have

$$(\boldsymbol{\gamma}, u) = (\boldsymbol{\rho}, u) = (\boldsymbol{\rho}, \sigma(j)) \; P_j \; (\boldsymbol{\gamma}, \tau(j)) \, ,$$

implying, since $\tau(i) = u$, that $i \prec_u j$. In sum, we have $j \prec_u i$ and $i \prec_u j$, a contradiction. Therefore, $j \in N^=$. Since $u \in S^*$, if $\gamma_v = b_v$, then j is indifferent between (b_u, u) and (b_v, v), contradicting NBI. Moreover, if $\rho_v > \gamma_v$, then $\rho_v > b_v$ and

$$(\boldsymbol{\rho}, v) \; P_j \; (\boldsymbol{\gamma}, v) = (\boldsymbol{\gamma}, \tau(j)) \; I_j \; (\boldsymbol{\rho}, \sigma(j)) \, ,$$

contradicting that $(\boldsymbol{\rho}, \sigma)$ is an RCE. Conclude that $v \in S^+ \cup S^=$. Finally, if $\sigma[v] = \emptyset$, then

$$(\boldsymbol{\rho}, u) = (\boldsymbol{\rho}, \sigma(j)) \; P_j \; (\boldsymbol{\rho}, v) = (1, v) \; R_j \; (\boldsymbol{\gamma}, v) = (\boldsymbol{\gamma}, \tau(j)) \, ,$$

where the strict relation is by *inferior empty schools*, contradicting that $j \in N^=$. ◇

Let $s \xrightarrow{1} u \in T$ have $1 \in N^+$. We shall extend this to a path $s \rightsquigarrow t$. By Claim 3, $u \in \overline{S}$. If it were the case that $\sigma[u] = \emptyset$, then since $(\boldsymbol{\rho}, \sigma)$ is an RCE, *inferior empty schools* implies,

$$(\boldsymbol{\rho}, s) \; P_1 \; (\boldsymbol{\rho}, u) = (1, u) = (\boldsymbol{\gamma}, u) \, ,$$

contradicting that $1 \in N^+$. By Claim 1, there is $u \xrightarrow{2} v \in T$. If $2 \in N^+$, we could then start with this edge instead. Continuing inductively, let j be the first agent on the path who is not in N^+ (if such an agent does not exist, the argument yields a totally postive cycle, as desired). We have that $s \rightsquigarrow t$ decomposes to

$$s \xrightarrow{1} u \rightsquigarrow v \rightarrow \sigma(j) \xrightarrow{j} w,$$

where $u \rightsquigarrow v$, if it exists, is labeled by N^+ agents. By Claim 4, $j \in N^=$, $\sigma(j) \in S^*$, $w \in S^+ \cup S^=$, and $\sigma[w] \neq \emptyset$. If $w \in S^+$, then the path is extended by an arc $w \xrightarrow{k} w'$ with $k \in N^+$ (Claims 1 and 3). Our argument has returned to its starting point; our goal is simply to show that a path initiated by a positive arc cannot have a terminal arc, remains within $\overline{S}$, and is labeled only by $N^+ \cup N^=$ agents. Thus, we proceed constructively, and when we arrive at an arc with an N^+ label, call this the *escape condition* of our proof.

Assume, therefore, that $w \in S^=$. Claim 1 implies there is $w \overset{k}{\to} w' \in T$. Note that $k \in N^+ \cup N^=$, as otherwise,

$$(\boldsymbol{\gamma}, w) = (\boldsymbol{\rho}, w) = (\boldsymbol{\rho}, \sigma(k))\ P_k\ (\boldsymbol{\gamma}, \tau(k)),$$

violating that $(\boldsymbol{\gamma}, \tau)$ is an RCE. If $k \in N^+$, we have encountered the escape condition again, so assume $k \in N^=$. By Claim 2, $w' \in \overline{S}$. If $w' \in S^+$, then $\rho_{w'} < 1$ so $\sigma[w'] \neq \emptyset$, and there must be an outgoing N^+ arc from w' (Claims 1 and 3); again we have the escape condition. Thus, to continue the argument, assume $w' \in S^* \cup S^=$. Now if $w' \in S^*$, we have

$$\sigma(j) \overset{j}{\to} w \overset{k}{\to} w'$$

with $\sigma(j), w' \in S^*$, and $j, k \in N^=$. This is an indifference chain connecting two S^* schools, contradicting NBI. Conclude that $w' \in S^=$, so

$$(\boldsymbol{\rho}, w)\ I_k\ (\boldsymbol{\gamma}, w') = (\boldsymbol{\rho}, w'),$$

where the indifference is because $k \in N^=$. Since $(\boldsymbol{\rho}, \sigma)$ is an RCE, by *inferior empty schools*, $\sigma[w'] \neq \emptyset$. We can then repeat the foregoing arguments and continue the path. In particular, w' must have an outgoing arc $w' \overset{k'}{\to} w''$, and $k' \in N^+ \cup N^=$. If $k' \in N^+$, we get the escape condition, and if $k' \in N^=$, we again conclude that $w'' \in S^=$ and $\sigma[w''] \neq \emptyset$.

Conclude that we can further decompose $s \rightsquigarrow t$ to

$$s \overset{1}{\to} u \rightsquigarrow v \to \sigma(j) \overset{j}{\to} w \rightsquigarrow x \to t$$

where

(1) $u \rightsquigarrow v$, is labeled by N^+ agents, and is contained in $\overline{S}$ by Claim 3,
(2) $j \in N^=$ and $\sigma(j) \in S^*$,
(3) $w \rightsquigarrow x$, is within $S^=$ and labeled by $N^=$ agents,
(4) $t \in S^+$.

These segments need not all exist. If the last segment exists, then we are back where we started and repeat the argument. In any case, we have shown that any N^+ labeled arc $s \to u$ induces a path that is always labeled by $N^+ \cup N^=$ agents, is always within $\overline{S}$ (except possibly for the very first vertex, s), and can always be extended. It follows that we can find a cyclic sub-path, not necessarily including s. However, by deleting the cycle from T (viewing the graph as a set of labeled edges), we preserve the vertex degree inequality of Claim 1, and none of the other claims are affected. Thus, we may repeat the argument. Eventually, we must find a cycle involving s, implying $s \in \overline{S}$.

Note, finally, that if $s \to t \in T$ and $t \in S^+$, then since $\rho_t < \gamma_t \leq 1$, $\sigma[t] \neq \emptyset$, and by Claim 1, there is $t \xrightarrow{i} u \in T$, and $i \in N^+$. Invoking the argument above, we find that any path that touches a S^+ school is a totally positive cycle. ■

Now we apply Proposition 6 to the NCBI domain and complete the proofs of Proposition 2 and Theorem 3.

Proof of Proposition 2. Note that Proposition 6 can be applied in reverse, from $(\boldsymbol{\gamma}, \tau)$ to $(\boldsymbol{\rho}, \sigma)$ to get the first claim of this proposition. Thus, to complete the decomposition, it remains to show that there are no paths in T that are not cycles. We have already shown this for signed paths, so suppose $s \rightsquigarrow t \subseteq T$ is labeled only by $N^=$ agents. Then each vertex on the path belongs to $S^* \cup S^=$. If $u \xrightarrow{i} v \in s \rightsquigarrow t$ has $\sigma[v] = \emptyset$, then by inferior empty schools, $(\boldsymbol{\rho}, u)\ P_i\ (1, v) = (\boldsymbol{\gamma}, v)$, contradicting that $i \in N^=$. If $\tau[u] = \emptyset$ then reverse the argument. Clearly only s or t could be empty in one of the two RCEs, and one of the two arguments just made applies to each, so no vertex touched by the path is empty at either RCE. Then, since $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$ are both exhaustive, for each vertex u touched by the path, $\lfloor \rho_u^{-1} \rfloor = |\sigma[u]|$ and $\lfloor \gamma_u^{-1} \rfloor = |\tau[u]|$. Since $u \in S^* \cup S^=$, $\rho_u = \gamma_u$ and so $|\sigma[u]| = |\tau[u]|$. Thus, $s \rightsquigarrow t$ can be extended to $u \to s \rightsquigarrow t \to v$. If $u \rightsquigarrow v$ is labeled only by $N^=$ agents, then we may repeat the argument. Since there are finitely many vertices, we can eventually extend to a cycle, either by exhaustion of the argument in this paragraph, or by extending to a signed path. ■

Proof of Theorem 3. Let T be the transfer graph from $(\boldsymbol{\rho}, \sigma)$ to $(\boldsymbol{\gamma}, \tau)$. Let μ be the matching that results from executing all the positive paths in T on σ. That is, if i labels an arc on a positive path, then $\mu(i) = \tau(i)$, and otherwise $\mu(i) = \sigma(i)$. Let $\boldsymbol{\zeta} = \boldsymbol{\rho} \vee \boldsymbol{\gamma}$. We show that $(\boldsymbol{\zeta}, \mu)$ is an RCE. Since positive paths are totally positive cycles, the number of students at each school is unchanged from σ to μ, so $(\boldsymbol{\zeta}, \mu)$ satisfies *exhaustiveness*. To check that $(\boldsymbol{\zeta}, \mu)$ is an allocation, it is sufficient to check the schools whose distribution has increased. That is, pick $s \in S^+$, where S^+ is defined as above. Since $|\tau^{-1}[s]| = |\sigma^{-1}[s]|$, and $\zeta_s = \gamma_s$, distribution feasibility at school s then follows from the distribution feasibility of $(\boldsymbol{\gamma}, \tau)$.

Let N' be the set of agents on a totally positive cycle. By Proposition 6, totally positive cycles are confined to $\overline{S}$, so each $i \in N'$ gets $(\boldsymbol{\zeta}, \mu(i)) = (\boldsymbol{\gamma}, \tau(i))$. The total-positivity of these paths also yields

$$\forall i \in N', (\boldsymbol{\zeta}, \mu(i)) = (\boldsymbol{\gamma}, \tau(i))\ R_i\ (\boldsymbol{\rho}, \sigma(i))\,. \tag{1}$$

Each i not on a totally positive cycle gets $\mu(i) = \sigma(i)$. Let $\sigma(i) = s$. There are two cases, $s \in S^+$ or not. If $s \in S^+$, then Proposition 6 implies that i cannot label any arc in T, as any such arc is then part of a totally positive cycle. Thus, $\tau(i) = \sigma(i)$ and we have $(\boldsymbol{\zeta}, \mu(i)) = (\boldsymbol{\gamma}, \tau(i))$. Then by preference monotonicity we have

$$\forall i \in \sigma[S^+] \setminus N', (\boldsymbol{\zeta}, \mu(i)) = (\boldsymbol{\gamma}, \tau(i))\ P_i\ (\boldsymbol{\rho}, \sigma(i)) \tag{2}$$

If $s \notin S^+$, then $\zeta_s = \rho_s$ and so $(\boldsymbol{\zeta}, \mu(i)) = (\boldsymbol{\rho}, \sigma(i))$. If $i \in N^+$, then since $\gamma_s \leq \rho_s$, it must be that $\tau(i) \neq \sigma(i) = s$. Then i would be on a totally positive cycle. Therefore, $i \notin N^+$ and again we conclude

$$\forall i \in \sigma[S \setminus S^+] \setminus N', (\boldsymbol{\zeta}, \mu(i)) = (\boldsymbol{\rho}, \sigma(i))\ R_i\ (\boldsymbol{\gamma}, \tau(i))\,. \tag{3}$$

In all cases, we have found that, at $(\boldsymbol{\zeta}, \mu)$, agents are consuming either their bundle under $(\boldsymbol{\rho}, \sigma)$ or their bundle under $(\boldsymbol{\gamma}, \tau)$. Moreover, since

$$\left(\sigma[S^+] \setminus N'\right) \cup \left(\sigma[S \setminus S^+] \setminus N'\right) = \left(\sigma[S^+] \cup \sigma[S \setminus S^+]\right) \setminus N' = N \setminus N',$$

lines 1, 2, and 3 yield

$$\forall i \in N, (\boldsymbol{\zeta}, \mu(i))\ R_i\ \max_{R_i}\{(\boldsymbol{\rho}, \sigma(i))\,, (\boldsymbol{\gamma}, \tau(i))\} \tag{4}$$

Suppose $(\boldsymbol{\zeta}, s)\ P_i\ (\boldsymbol{\zeta}, \mu(i))$, which by line 4 implies

$$(\boldsymbol{\zeta}, s)\ P_i\ \max_{R_i}\{(\boldsymbol{\rho}, \sigma(i))\,, (\boldsymbol{\gamma}, \tau(i))\}.$$

Assume there is $j \in \mu[s]$, so $(\boldsymbol{\zeta}, \mu(j)) = (\boldsymbol{\zeta}, s)$. Then since $(\boldsymbol{\zeta}, \mu(j)) \in \{(\boldsymbol{\rho}, \sigma(j))\,, (\boldsymbol{\gamma}, \tau(j))\}$, plugging the appropriate case into the previous line yields$j \prec_s i$.

Assume, therefore, that s is empty under μ. Then it is empty under σ, and so $\zeta_s = \rho_s = 1$. Thus, again invoking line 4,

$$(1, s) = (\boldsymbol{\rho}, s)\ P_i\ (\boldsymbol{\zeta}, \mu(i))\ R_i\ (\boldsymbol{\rho}, \sigma(i))\,,$$

contradicting that $(\boldsymbol{\rho}, \sigma)$ is an RCE. Therefore, $(\boldsymbol{\zeta}, \mu)$ satisfies *fairness*.

Finally, by Theorem 2, the set of empty schools remains the same in $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$, and so also in $(\boldsymbol{\zeta}, \mu)$. Thus, by line 4, $(\boldsymbol{\zeta}, \mu)$ satisfies *inferior empty schools*. ■

A.3. **Domination lemmas.** Given an allocation $(\boldsymbol{\rho}, \sigma)$, let $s \xrightarrow{i} t \in \Gamma$ if $\sigma(i) = s \neq t$, and $(\boldsymbol{\rho}, t)\ R_i\ (\boldsymbol{\rho}, s)$. We say that Γ is the *weak envy* graph of $(\boldsymbol{\rho}, \sigma)$.

Recall that a *source set* in a directed graph is a set of vertices that no edge enters. Formally, it is a set $S' \subseteq S$ such that if $s \to t \in \Gamma$ and $s \notin S'$, then $t \notin S'$.

Say a school $s \in S$ is *totally exhausted* at $(\boldsymbol{\rho}, \sigma)$ if $|\sigma^{-1}[s]|\rho_s = 1$.

Lemma 1. *Let $(\boldsymbol{\rho}, \sigma)$ be a* fair *allocation with weak-envy graph Γ. Suppose $S' \subseteq S \setminus S^*$, not empty, is a source set in Γ and that no school in S' is totally exhausted. Then there is an RCE $(\boldsymbol{\gamma}, \tau)$, Pareto-dominating $(\boldsymbol{\rho}, \sigma)$ and with $\gamma \gneq \rho$.*

Proof. Let $N' = \sigma^{-1}[S']$. For each $s \in S'$, let $n_s = |\sigma^{-1}[s]|$. We shall construct an assignment market isomorphic to the problem we currently face when restricted to N' and S'. To aid comparison of our current model with the assignment market we employ, we use the terms *stability* and *blocking*. For our model, clearly s and i block $(\boldsymbol{\rho}, \sigma)$ if $\sigma(i) \neq s$ and either $\sigma[s] = \emptyset$ or i has (justified) envy at s. An allocation is stable if there are no blocks. Let $\mathfrak{S}$ be a set of $\sum_{s \in S'} n_s$ elements. Let $f : \mathfrak{S} \to S'$ have $|f[s]| = n_s$. We view $\mathfrak{S}$ as the set of copies of the elements of S'.

Each $\mathfrak{s} \in \mathfrak{S}$ consumes a point $(l, i) \in \mathbb{R} \times N'$ and has simple preferences represented by utility function $W_{\mathfrak{s}}(l, i) = l$; copies of schools care only about resources. Each copy has an outside option denoted $\underline{w}_{\mathfrak{s}}$, so that $\mathfrak{s}$ will withdraw from the matching (now an option) before accepting a bundle giving utility less than $\underline{w}_{\mathfrak{s}}$. With abuse of notation, we retain the same notation for the students. Each $i \in N'$ consumes a point $(r, \mathfrak{s}) \in \mathbb{R} \times \mathfrak{S}$ and has preferences so that

$$(r, \mathfrak{s})\ R_i\ (r', \mathfrak{s}') \iff (r, f(\mathfrak{s}))\ R_i\ (r', f(\mathfrak{s}')).$$

Let U_i be a continuous utility function representation of R_i. Assume that the outside option utility for agents is $-\infty$. When an agent and a school match, one unit of divisible resource is produced, independent of their identities. We now have a one-to-one assignment market, matching the sets N' and $\mathfrak{S}$ together, and each matched pair having a unit of divisible resource to divide. Demange and Gale [1985] show that, in this model, there is a unique agent-optimal stable utility profile $(\boldsymbol{u}, \boldsymbol{w})$, with at least one and possibly several matchings that yield these utilities. Moreover, there is at least one $\mathfrak{s} \in \mathfrak{S}$ with $w_{\mathfrak{s}} = \underline{w}_{\mathfrak{s}}$.

Let $\hat{\sigma} : N' \to \mathfrak{S}$ be a bijection such that each $\hat{\sigma}(i) \in f[\sigma(i)]$. The RCE $(\boldsymbol{\rho}, \sigma)$ induces the following allocation on the constructed assignment market: Each $i \in N'$ gets $(\rho_{f(\mathfrak{s})}, \mathfrak{s})$ and $\mathfrak{s} \in \mathfrak{S}$ gets $(1 - \rho_{f(\mathfrak{s})}, \hat{\sigma}^{-1}(\mathfrak{s}))$.[15]

We show in this paragraph that, so long as $\underline{w}_{\mathfrak{s}} \leq 1 - \rho_{f(\mathfrak{s})}$, the allocation $(\boldsymbol{\rho}, \hat{\sigma})$ in the constructed assignment market is stable. Clearly, no individual rationality constraints are violated. Since $(\rho_{f(\mathfrak{s})}, f(\mathfrak{s}))\ R_i\ (\rho_{f(\mathfrak{s}')}, f(\mathfrak{s}'))$ when $f(\mathfrak{s}) = \sigma(i)$, i and $\mathfrak{s}'$ could only form a blocking pair by giving i at least $\rho_{f(\mathfrak{s}')}$, leaving only $1 - \rho_{f(\mathfrak{s}')}$ for $\mathfrak{s}'$.

Fix $\varepsilon > 0$ and set each $\underline{w}_{\mathfrak{s}} = 1 - \rho_{f(\mathfrak{s})} - \varepsilon$. Let $(\boldsymbol{u}, \boldsymbol{w})$ be the agent-optimal utility profile for this problem, and let $\hat{\mu}$ be a matching that supports it. Since $(\boldsymbol{\rho}, \hat{\sigma})$ is stable, Demange and Gale [1985] also show that, for each $\mathfrak{s} \in \mathfrak{S}$, $w_{\mathfrak{s}} \leq 1 - \rho_{f(\mathfrak{s})}$.

Suppose there are $\mathfrak{s}, \mathfrak{s}' \in f[s]$ with $w_{\mathfrak{s}} < w_{\mathfrak{s}'}$. Let $j = \hat{\mu}^{-1}(\mathfrak{s}')$. By feasibility, j is getting no more than $r_j = 1 - w_{\mathfrak{s}'}$ units of resource at $\mathfrak{s}'$. By monotonicity of j's preferences, and since she cannot distinguish $\mathfrak{s}$ and $\mathfrak{s}'$, $U_j(r_j, \mathfrak{s}) \geq u_j$. However,

$$W_{\mathfrak{s}}(1 - r_j, j) = 1 - r_j = w_{\mathfrak{s}'} > w_{\mathfrak{s}},$$

and so j and $\mathfrak{s}$ form a blocking pair. Conclude then that copies of the same school all get the same level of utility.

For each $s \in S'$, let $\gamma_s = 1 - w_{f[s]}$, where this latter is an abuse of notation but is well-defined by our previous observation. For each $s \notin S'$, let $\gamma_s = \rho_s$. For $s \in S'$ we have

$$\gamma_s = 1 - w_{f[s]} \geq 1 - (1 - \rho_s) = \rho_s,$$

and since $w_{\mathfrak{s}} = \underline{w}_{\mathfrak{s}}$ for some copy of some school, the above inequality is strict for at least one $s \in S'$. Next, define μ so that, for each $s \in S'$, $\mu[s] = \{i \in N : \hat{\mu}(i) \in f[s]\}$, and for $s \notin S'$, $\mu[s] = \sigma[s]$.

Recall that for each $s \in S'$, $\rho_s > b_s$, so there is a block of $(\boldsymbol{\gamma}, \mu)$ involving $s \in S'$ if and only if there is $i \in N$ with $(\boldsymbol{\gamma}, s)\ P_i\ (\boldsymbol{\gamma}, \mu(i))$. Since S' is a source set in Γ, for ε small enough, it remains a source set in the weak-envy graph for $(\boldsymbol{\gamma}, \mu)$. Thus, there is no block with agents outside N'. Since $(\boldsymbol{u}, \boldsymbol{w})$ is stable, for each $i \in N'$ and each $s \in S'$, $(\boldsymbol{\gamma}, \mu(i))\ R_i\ (\boldsymbol{\gamma}, s)$. Thus the only remaining possible block is between $i \in N'$ and $s \notin S'$. Suppose such a block exists. Then

$$(\boldsymbol{\rho}, s) = (\boldsymbol{\gamma}, s)\ P_i\ (\boldsymbol{\gamma}, \mu(i))\ R_i\ (\boldsymbol{\rho}, \sigma(i)),$$

[15]We use $f^{-1}(x)$ to denote the *unique* inverse of a bijection and $f[x]$ to denote the set-valued pre-image.

where the last relation is because the stable match in the one-to-one problem Pareto dominates $(\boldsymbol{\rho}, \sigma)$ for the N' agents. Thus, i would also like to block with s at $(\boldsymbol{\rho}, \sigma)$. However, since $\gamma_s = \rho_s$ and $\mu[s] = \sigma[s]$, if school s is party to the block at $(\boldsymbol{\gamma}, \mu)$ then it is at $(\boldsymbol{\rho}, \sigma)$ as well, contradicting that the latter is a *fair* allocation.

It remains to check that $(\boldsymbol{\gamma}, \mu)$ is feasible, which requires only checking feasibility for the S' schools. Since each $s \in S'$ has $n_s\rho_s < 1$, for ε small enough,

$$n_s\gamma_s = n_s(1 - w_{f[s]}) \leq n_s(1 - \underline{w}_{f[s]}) = n_s(\rho_s + \varepsilon) < 1,$$

as desired. ■

Let Γ be the weak-envy graph of allocation $(\boldsymbol{\rho}, \sigma)$, and let $t \rightsquigarrow s \subseteq \Gamma$. Construct τ so that for each $i \in N$ with $u \xrightarrow{i} v \in t \rightsquigarrow s$, $\tau(i) = v$, and otherwise $\tau(i) = \sigma(i)$. We allow for $s = t$, so that the path may be a cycle. If $(\boldsymbol{\rho}, \tau)$ is an allocation, then we say that $t \rightsquigarrow s$ is *feasible*. We say that τ is the matching that results from *executing* the path on matching σ.

Given allocation $(\boldsymbol{\rho}, \sigma)$ with weak envy graph Γ, the set of *vertices upstream of* s is

$$U_s = \{t \in S : \exists t \rightsquigarrow s \subseteq \Gamma\}.$$

Lemma 2. *Given profile $\boldsymbol{R}$ from the NCBI domain, let $(\boldsymbol{\rho}, \sigma)$ be a* fair *allocation that is not an RCE. Then there is another* fair *allocation for $\boldsymbol{R}$ that Pareto dominates $(\boldsymbol{\rho}, \sigma)$.*

Proof. Observe that if the set of upstream vertices U_s is empty for some $s \in S$, and if ρ_s is less than 1, then we can set $\gamma_s = \rho_s + \epsilon$ and all else equal, and $(\boldsymbol{\gamma}, \sigma)$ is a *fair* allocation if ϵ is small enough. If $\sigma[s]$ is non-empty, we are done.

As usual, let $S^* = \{s \in S : \rho_s = b_s\}$. Execute the following procedure as many times as possible, starting with $\sigma_0 = \sigma$: choose $s \in S \setminus S^*$ with $|\sigma_m^{-1}[s]| \leq \lfloor \rho_s^{-1} \rfloor - 1$. Letting Γ_m be the weak-envy graph of $(\boldsymbol{\rho}, \sigma_m)$, find a (minimal) path $t \rightsquigarrow s \subseteq \Gamma_m$ with $t \in S^*$. That is, by taking sub-paths, $t \rightsquigarrow s$ touches S^* only at t. Execute the path to arrive at a new allocation $(\boldsymbol{\rho}, \sigma_{m+1})$. Observe that $(\boldsymbol{\rho}, \sigma_{m+1})$ is a *fair* allocation, as no agent has entered a S^*-school, so no violations of *fairness* can be introduced. Of course, we now have a failure of exhaustiveness at t, if not before. Nonetheless, by the definition of the weak-envy graph, $(\boldsymbol{\rho}, \sigma_{m+1})$ Pareto weakly dominates $(\boldsymbol{\rho}, \sigma_m)$.[16] We have therefore proven the following claim:

[16]Allocation $(\boldsymbol{\rho}, \sigma)$ Pareto weakly dominates $(\boldsymbol{\rho}', \sigma')$ if for each $i \in N$, $(\boldsymbol{\rho}, \sigma(i)) \ R_i \ (\boldsymbol{\rho}', \sigma'(i))$.

Claim. *Let Γ be the weak-envy graph of a* fair *allocation $(\boldsymbol{\rho}, \sigma)$. Assume $t \rightsquigarrow s \subseteq \Gamma$ touches the set of constrained vertices, S^*, only at t. Then the allocation $(\boldsymbol{\rho}, \tau)$ that results from executing $t \rightsquigarrow s$ on $(\boldsymbol{\rho}, \sigma)$ is a* fair *allocation that Pareto weakly dominates the original.*

If any $(\boldsymbol{\rho}, \sigma_{m+1})$ Pareto dominates $(\boldsymbol{\rho}, \sigma_m)$, we are done. Thus, we may assume $(\boldsymbol{\rho}, \sigma_{m+1})$ is welfare equivalent to $(\boldsymbol{\rho}, \sigma)$. This process can be repeated at most finitely many times. Let $(\boldsymbol{\rho}, \mu)$ be the result and Γ_μ the associated weak-envy graph.

Case 1: There is $s \in S \setminus S^*$ with $|\mu^{-1}[s]| \leq \lfloor \rho_s^{-1} \rfloor - 1$.

Our procedure above moves agents *out* of S^* vertices along chains of weak-envy (actually, chains of indifference). Thus, since the procedure was executed to exhaustion, there are no S^* vertices in the set U_s of upstream vertices of s in graph Γ_μ.

By definition, U_s is a source in Γ_μ. If no school in U_s is totally exhausted under $(\boldsymbol{\rho}, \mu)$ then we may invoke Lemma 1 to arrive at our desired conclusion. Suppose, then, that there is $t \in U_s$ that is totally exhausted. Since $\{s\} \cup U_s \subseteq S \setminus S^*$, and since $(\boldsymbol{\rho}, \mu)$ is an RCE, all the arcs between these vertices in Γ_μ represent indifferences. Suppose there is $t' \in U_s$, $t' \neq t$, that is also totally exhausted. Then there are two chains of indifference, $t \rightsquigarrow s$ and $t' \rightsquigarrow' s$, in Γ_μ. The concatenation of these, $t \rightsquigarrow s \leftsquigarrow' t'$, represents a chain of indifference connecting t and t'. This violates NCBI as both of these vertices are totally exhausted and so $\rho_t^{-1}, \rho_{t'}^{-1} \in \mathbb{N}$. Therefore, t is the *only* member of U_s that is totally exhausted.

Execute $t \rightsquigarrow s$ on $(\boldsymbol{\rho}, \mu)$ to arrive at a *fair* allocation $(\boldsymbol{\rho}, \tau)$ with associated weak-envy graph Γ_τ. As above, if we have found a Pareto improvement, we are done, so we may assume it is welfare equivalent to $(\boldsymbol{\rho}, \mu)$. Our next task is to show that $\{s\} \cup U_s$ is a source in Γ_τ, recalling that U_s is the set of upstream vertices in Γ_μ.

Let $u \xrightarrow{i} v \in \Gamma_\tau$ have $u \notin U_s$. If $\tau(i) = \mu(i)$ then clearly $u \xrightarrow{i} v \in \Gamma_\mu$, and since U_s is a source set in Γ_μ, $v \notin U_s$. If $\tau(i) \neq \mu(i)$, then i labels some arc on the path $t \rightsquigarrow s \subseteq \Gamma_\mu$ we just executed. Stated formally, there is $u' \xrightarrow{i} u \in t \rightsquigarrow s$, $\mu(i) = u'$, and $\tau(i) = u$. By construction, the only school not in U_s that is touched by this path is s, so in fact $u' \xrightarrow{i} u$ is the last arc of the path, and so $u = s$. Thus, we have shown that if $u \xrightarrow{i} v \in \Gamma_\tau$ has $u \notin U_s$ but $v \in U_s$, then $u = s$; the only arcs in Γ_τ (if there are any at all) that enter U_s are those coming from s.

Suppose there is a path $w \rightsquigarrow s \subseteq \Gamma_\tau$ that is not a path in Γ_μ. By taking sub-paths, assume we have the shortest such path, so that

$$w \rightsquigarrow s = w \xrightarrow{k} w' \rightsquigarrow s,$$

with $w' \rightsquigarrow s \subseteq \Gamma_\mu$. This latter inclusion, however, implies that $w' \in U_s$, along with all the other vertices touched by $w' \rightsquigarrow s$, and so by the previous paragraph, $w = s$. Conclude that the only paths to s in Γ_τ that are not in Γ_μ are of the form $s \to t \rightsquigarrow s$, where $t \rightsquigarrow s \subseteq \Gamma_\mu$. It follows that $\{s\} \cup U_s$ is a source set in Γ_τ.

Recall that our original path $t \rightsquigarrow s \subseteq \Gamma_\mu$ represented only indifferences. Since t is totally exhausted at $(\boldsymbol{\rho}, \mu)$, $\rho_t^{-1} \in \mathbb{N}$. By NCBI, it follows that ρ_s^{-1} is *not* an integer, implying that $\rho_s^{-1} > \lfloor \rho_s^{-1} \rfloor$. This further implies that s remains not totally exhausted if another student is added to it, and so is not totally exhausted at $(\boldsymbol{\rho}, \tau)$. The schools in the middle of the path have not changed the number of students they admit from μ to τ, so they remain not totally exhausted. Clearly,

$$|\tau^{-1}[t]| = |\mu^{-1}[t]| - 1 = \lfloor \rho_t^{-1} \rfloor - 1,$$

so t is not totally exhausted at $(\boldsymbol{\rho}, \tau)$. Since t was the *only* totally exhausted site in U_s under $(\boldsymbol{\rho}, \mu)$, we now have that $\{s\} \cup U_s$ is a source set in Γ_τ with no exhausted schools, and we therefore invoke Lemma 1.

Case 2: Each $s \in S$ with $|\mu^{-1}[s]| \leq \lfloor \rho_s^{-1} \rfloor - 1$ has $s \in S^*$, so $\rho_s^{-1} = b_s^{-1} \in \mathbb{N}$.

Assume $N' = \{j \in N : (\boldsymbol{\rho}, s)\ P_j\ (\boldsymbol{\rho}, \mu(j))\}$ is non-empty, and let $j = \min_{\prec_s} N'$. Define matching τ so that $\tau(j) = s$ and otherwise $\tau(i) = \mu(i)$. Then $(\boldsymbol{\rho}, \tau)$ is clearly a *fair* allocation that Pareto dominates $(\boldsymbol{\rho}, \mu)$, and therefore $(\boldsymbol{\rho}, \sigma)$. We proceed, therefore, under the assumption that each arc $t \xrightarrow{i} s \in \Gamma_\mu$ represents indifference.

If there is $t \in U_s$ with $\rho_t^{-1} \in \mathbb{N}$, then by taking sub-paths, assume $t \rightsquigarrow s$ is a minimal path starting from such a t. That is, for every $s' \in S$ touched by the path except t and s, $\rho_{s'} > b_{s'}$. Decompose $t \rightsquigarrow s$ as $t \to u \rightsquigarrow v \to s$. Then $u \rightsquigarrow v$ touches no S^* vertices and so, since $(\boldsymbol{\rho}, \mu)$ is a *fair* allocation, the edge $t \to u$ and all edges in $u \rightsquigarrow v$ represent indifference. We showed in the previous paragraph that $v \to s$ represents indifference. Thus, since $\rho_t^{-1} \in \mathbb{N}$, this path is a contradiction to NCBI. Conclude that U_s contains neither a totally exhausted vertex, nor a S^* vertex, and so we invoke Lemma 1. ■

A.4. **Topological argument to complete the proof.**

Theorem 5. *Given $\boldsymbol{R} \in \mathcal{R}^N$ satisfying NCBI, let $\mathcal{E}$ be the set of RCE for $\boldsymbol{R}$. Then*

(1) *$\mathcal{E}$ is not empty,*
(2) *$\mathcal{E}$ induces a closed upper-lattice in welfare space, and*
(3) *the set of distributions supporting the elements of $\mathcal{E}$ has a $\leq$-greatest element, $\boldsymbol{\rho}^*(\boldsymbol{R})$, which itself supports the welfare-greatest elements of $\mathcal{E}$.*

Proof. For each $i \in N$, let u_i be a continuous utility function representation for R_i. Fixing a matching σ, the function $\boldsymbol{\rho} \in [0,1]^S \overset{U}{\mapsto} (u_i(\boldsymbol{\rho}, \sigma))_{i \in N}$ is continuous. Closed subsets of $[0,1]^S$ are compact and so map to compact sets under this function. The set $\mathcal{D}^\sigma \subseteq [0,1]^S$ of distributions $\boldsymbol{\rho}$ such that $(\boldsymbol{\rho}, \sigma)$ is a *fair* allocation is closed: To see this, recall simply that a violation of *fairness* requires strict preference, and no new strict preference can be introduced in the limit of a sequence of distributions of *fair* allocations. Let $\mathcal{D} = \cup_\sigma \mathcal{D}^\sigma$. Since there are only finitely many possible matchings, $\mathcal{D}$ is compact.

Let $\mathcal{U} = U(\mathcal{D})$, which is compact. Let $\boldsymbol{u} \in \mathcal{U}$ be $\leq$-maximal. By Lemma 2, there is an RCE that induces $\boldsymbol{u}$. Thus, the $\leq$-upper envelope of $\mathcal{U}$ corresponds to RCE. By Theorem 3, the $\leq$-upper envelope of $\mathcal{U}$ is a lattice. Therefore, $\mathcal{U}$ has a $\leq$-greatest element. ■

Proof of Theorem 1. It follows directly from Part (1) of Theorem 5. ■

Proof of Proposition 3. By Lemma 2 and Theorem 5, it follows that the correspondence of welfare-greatest RCE, i.e., the maximal RCE, on the NCBI domain is non-empty, essentially single-valued, and satisfies *student-optimal fairness*. ■

Appendix B. Proof of Theorem 4: *Strategy-proofness*

First, we establish the following lemma, which is an immediate consequence of lemmas in Section A.3, but highlights a structural feature that will be important in the proof of *strategy-proofness* below.

Lemma 3. *Assume that $\boldsymbol{R} \in \mathcal{R}^N$ satisfies NCBI. Suppose $(\boldsymbol{\rho}, \sigma)$ is a* fair *allocation for $\boldsymbol{R}$ at which either $s \in S$ is not totally exhausted, or $s \in S^*$. Let U_s be the set of vertices upstream of s under the weak-envy graph of $(\boldsymbol{\rho}, \sigma)$. If U_s contains no totally exhausted schools, there is another* fair *allocation that Pareto dominates $(\boldsymbol{\rho}, \sigma)$.*

Proof. Observe that if U_s is empty, then we can set $\gamma_s = \rho_s + \epsilon$ and all else equal, and $(\boldsymbol{\gamma}, \sigma)$ is an RCE if ϵ is small enough. Thus, we may assume that U_s is non-empty for all $s \in S$.

By Case 2 of Lemma 2, we may assume each S^* school is totally exhausted. Thus if some $s \in S$ has U_s containing no totally exhausted schools, then it contains no S^* schools either. That is, $U_s \subseteq S \setminus S^*$, and is non-empty. We now invoke Lemma 1 to get the desired result. ■

Recall that preference relation R' is a Maskin Monotonic transform of preference relation R at bundle (x, m) if $(y, t)\ R'\ (x, m)$ implies that $(y, t)\ R\ (x, m)$. Let $\mathcal{T}(R, (x, m))$ be the set of Maskin monotonic transforms of R at (x, m). It is obvious that the correspondence of RCE is Maskin monotonic, which is to say that if $(\boldsymbol{\rho}, \sigma)$ is a RCE for $\boldsymbol{R}$, and $\boldsymbol{R}'$ has, for each $i \in N$, $R'_i \in \mathcal{T}(R_i, (\boldsymbol{\rho}, \sigma(i)))$, then $(\boldsymbol{\rho}, \sigma)$ is a RCE for $\boldsymbol{R}'$. We first uncover some structural properties of φ with respect to Maskin Monotoic Transforms.

An undirected graph is a *tree* if there is exactly one path in the graph between any pair of vertices. In particular, a tree is simple—there is at most one edge between any pair of vertices. With abuse of terminology, we shall call a directed graph a *tree* if its underlying undirected graph is a tree *and* there is a special vertex r, called the *root*, from which all paths emerge. That is, for all non-root vertices s, there is a path $r \rightsquigarrow s$ in the graph. Finally, a directed graph is a *forest* if it is comprised of disjoint directed trees, having no edge or vertex in common. Let Γ be the weak-envy graph for allocation $\varphi(\boldsymbol{R})$ with preferences $\boldsymbol{R}$. By Lemma 3, we can find a subgraph $\Gamma' \subseteq \Gamma$ that is a directed forest and such that each totally exhausted $s \in S$ with $\boldsymbol{\rho}^*_s(\boldsymbol{R}) > b_s$ is a root vertex. Call such Γ' a *minimal forest* for $\boldsymbol{R}$. The following observations imply that, for generic profiles, the minimal forest is unique. In any case, we have the following lemma.

Lemma 4. *Let $\boldsymbol{R}$ and $\boldsymbol{R}'$ be members of $\mathcal{D}$ with the following properies: 1) for each $i \in N$, $R'_i \in \mathcal{T}(R_i, \varphi_i(\boldsymbol{R}))$ and 2) the weak-envy graph for $\boldsymbol{R}'$ at $\varphi(\boldsymbol{R})$ contains a minimal forest for $\boldsymbol{R}$ at $\varphi(\boldsymbol{R})$. Then $\boldsymbol{\rho}^*(\boldsymbol{R}') = \boldsymbol{\rho}^*(\boldsymbol{R})$.*

Proof. Let Γ' be a minimal forest for $\boldsymbol{R}$. First consider $\boldsymbol{R}'' \in \mathcal{D}$ such that, for each $i \in N$, $R''_i \in \mathcal{T}(R_i, \varphi_i(\boldsymbol{R}))$ and such that the weak-envy graph of $\boldsymbol{R}''$ at $\varphi(\boldsymbol{R})$ is *precisely* Γ'. By Maskin monotonicity, $\varphi(\boldsymbol{R})$ is an RCE for $\boldsymbol{R}''$, so $\boldsymbol{\rho}^*(\boldsymbol{R}'') \geq \boldsymbol{\rho}^*(\boldsymbol{R})$. By the lattice property, $\boldsymbol{R}''$ welfare can only increase from $\varphi(\boldsymbol{R})$ to $\varphi(\boldsymbol{R}'')$. By Theorem 2, the change in school-assignment betweeen these two consists entirely of trading cycles. By Proposition 2, all trading cycles between these two allocations must be welfare non-negative, and therefore must be cycles in Γ'. However, Γ' has no cycles, and therefore the matching under $\varphi(\boldsymbol{R}')$, say σ, is the same as that under $\varphi(\boldsymbol{R})$. Now if $\boldsymbol{\rho}^*_s(\boldsymbol{R}') > \boldsymbol{\rho}^*_s(\boldsymbol{R})$, then clearly s is not totally exhausted at $\varphi(\boldsymbol{R})$. Thus there is a path $t \rightsquigarrow s \subseteq \Gamma'$. In particular, there is $u \xrightarrow{i} s \in \Gamma'$. However, we then have $\boldsymbol{\rho}^*_s(\boldsymbol{R}'') > b_s$ and

$$(\boldsymbol{\rho^*}_s(\boldsymbol{R}''), s)\ P''_i\ (\boldsymbol{\rho^*}_s(\boldsymbol{R}), s)\ R''_i\ (\boldsymbol{\rho^*}_s(\boldsymbol{R}), u)\,,$$

implying, since $\varphi(\boldsymbol{R}'')$ is an RCE, that $\boldsymbol{\rho}^*_u(\boldsymbol{R}'') > \boldsymbol{\rho}^*_u(\boldsymbol{R})$. It follows that u is not totally exhausted at $\varphi(\boldsymbol{R})$ and so we may repeat the argument. In fact, we may repeat the argument all the way up the path $t \rightsquigarrow s$ to vertex t, getting a contradiction to feasibility since $|\sigma[t]|\, \boldsymbol{\rho}^*_t(\boldsymbol{R}) = 1$. We conclude, therefore, that $\boldsymbol{\rho}^*(\boldsymbol{R}'') = \boldsymbol{\rho}^*(\boldsymbol{R})$.

Now let $\boldsymbol{R}' \in \mathcal{D}$ have, for each $i \in N$, $R'_i \in \mathcal{T}(R_i, \varphi(\boldsymbol{R}))$ and that Γ' is a subgraph of the weak-envy graph of $\boldsymbol{R}'$ at $\varphi(\boldsymbol{R})$. As above, $\boldsymbol{\rho}^*(\boldsymbol{R}') \geq \boldsymbol{\rho}^*(\boldsymbol{R})$. However, note that we may choose $\boldsymbol{R}''$ above so that, for each $i \in N$, $R''_i \in \mathcal{T}(R'_i, \varphi(\boldsymbol{R}))$. Thus, $\boldsymbol{\rho}^*(\boldsymbol{R}) = \boldsymbol{\rho}^*(\boldsymbol{R}'') \geq \boldsymbol{\rho}^*(\boldsymbol{R}')$ and so $\boldsymbol{\rho}^*(\boldsymbol{R}') = \boldsymbol{\rho}^*(\boldsymbol{R})$. ■

An immediate corollary of the previous lemma is the following observation, which seems important enough to be labeled a Theorem.

Theorem 6 (The Locality Theorem)**.** *Let $\boldsymbol{R}' \in \mathcal{D}$ be a profile such that, for each $i \in N$, $(\boldsymbol{\rho}^*(\boldsymbol{R}), s)\ I_i\ \varphi_i(\boldsymbol{R})$ implies $(\boldsymbol{\rho}^*(\boldsymbol{R}), s)\ I'_i\ \varphi_i(\boldsymbol{R})$. Then $\boldsymbol{\rho}^*(\boldsymbol{R}') = \boldsymbol{\rho}^*(\boldsymbol{R})$.*

This result makes it easier to transfer insight across preference domains, because one can make nearly arbitrary changes in the preference profile and induce no change in the maximal distribution so long as the key set of indifference sets are preserved. In particular, it shows that studying $\boldsymbol{\rho}^*$ with restricted domains, such as linear or quasilinear domains, is sufficient.

We are now prepared to prove the incentive compatibility of φ.

Proof of Theorem 4. Let $\boldsymbol{R}' = (R'_i, R_{-i}) \in \mathcal{D}$. Suppose $\varphi_i(\boldsymbol{R}')\ P_i\ \varphi_i(\boldsymbol{R})$. Let

$$R''_i \in \mathcal{T}(R'_i, \varphi_i(\boldsymbol{R}')) \cap \mathcal{T}(R_i, \varphi_i(\boldsymbol{R}))$$

have the following properties. For each $s \in S$, if $(\boldsymbol{\rho}^*(\boldsymbol{R}'), s) \neq \varphi_i(\boldsymbol{R}')$ then $\varphi_i(R')\ P''_i\ (1, s)$. Also, let R''_i have the same indifference set through $\varphi_i(\boldsymbol{R})$ as R_i does. Note that this assumption implies $\varphi_i(\boldsymbol{R}')\ P''\ \varphi_i(\boldsymbol{R})$. Let $\boldsymbol{R}'' = (R''_i, \boldsymbol{R}_i)$. By the Locality Theorem, $\boldsymbol{\rho}^*(\boldsymbol{R}'') = \boldsymbol{\rho}^*(\boldsymbol{R})$. By Maskin monotonicity, $\varphi(\boldsymbol{R}')$ is an RCE for R''. Therefore $\varphi_i(R''_i, \boldsymbol{R}_{-i})\ R''_i\ \varphi_i(\boldsymbol{R}')\ P''_i\ \varphi_i(\boldsymbol{R})$. It follows that $\varphi_i(R''_i, \boldsymbol{R}_{-i})\ P_i\ \varphi_i(\boldsymbol{R})$. Therefore, if i can manipulate φ at $\boldsymbol{R}$, then i can manipulate via a preference such as R''_i. Without loss of generality, we assume henceforth that $\boldsymbol{R}' = \boldsymbol{R}''$.

We are considering two allocations, $\varphi(\boldsymbol{R})$ and $\varphi(\boldsymbol{R}')$ with the same distribution vector $\boldsymbol{\rho} = \boldsymbol{\rho}^*(\boldsymbol{R}) = \boldsymbol{\rho}^*(\boldsymbol{R}')$. We shall construct a classical school choice problem from these and derive a contradiction to the *strategy-proofness* of the student-optimal stable rule (Roth and Sotomayor [1990]) in this context.

The set of classical schools is denoted S. As usual, let $S^* = \{s \in S : \boldsymbol{\rho}_s = b_s\}$ and call these (crowded) schools *constrained*. We collapse all the unconstrained schools into one classical school, $\grave{\mathsf{s}}$. School priorities in the classical model will be denoted $\lhd$. For $s \in S^*$, which maps to $\mathsf{s} \in \mathsf{S}$, set $\lhd_{\mathsf{s}} = \prec_s$. Set $k \lhd_{\grave{\mathsf{s}}} j$ if $k \in \tau[S \setminus S^*]$ and $j \in \tau[S^*]$. We shall not need to further specify $\lhd_{\grave{\mathsf{s}}}$.

Next we break ties in student preferences. We begin with an intermediate step, deciding that $\grave{\mathsf{s}}$ shall inherit the rank of the highest ranked unconstrained school. That is, let $s \in S \setminus S^*$ have, for each $t \in S \setminus S^*$, $(\boldsymbol{\rho}, s)\ R_j\ (\boldsymbol{\rho}, t)$. Then, for $\mathsf{u} \in \mathsf{S}$, $\mathsf{u}\ \mathsf{R}_j\ \grave{\mathsf{s}}$ only if $(\boldsymbol{\rho}, u)\ R_j\ (\boldsymbol{\rho}, s)$. With this step, we have defined the weak preference R_j on S. It remains to break ties on this relation. Note that by NCBI, R_j is in fact strict when restricted to $\mathsf{S} \setminus \{\grave{\mathsf{s}}\}$, so there is at most one non-singleton indifference class, and it has the form $\{\mathsf{t}, \grave{\mathsf{s}}\}$. Before completing our tie-breaking specification, let us first make the following observation:

Claim 5. *Let $(\boldsymbol{\rho}, s)\ I_j\ (\boldsymbol{\rho}, \sigma(j))$ or $(\boldsymbol{\rho}, s)\ I'_j\ (\boldsymbol{\rho}, \tau(j))$. Then $s \in S \setminus S^*$.*

Proof of claim. Recall Lemma 3. First, if $\sigma(j)$ is exhausted at $(\boldsymbol{\rho}, \sigma)$, then the claim follows directly from NCBI. Otherwise, there is $t \rightsquigarrow \sigma(j) \subseteq \Gamma$, where Γ is the weak-envy graph of $\boldsymbol{R}$ at $(\boldsymbol{\rho}, \sigma)$, with t totally exhausted. By taking subpaths we may find assume this is a shortest (by length) path with this property. Thus, at most one school touched by the path is in S^*, and it must be t, as otherwise we could shorten the path further. Therefore, the path must consist entirely of indifferences, and so $t \rightsquigarrow \sigma(j) \to s$, with $s \in S^*$, contradicts NCBI.

Note that the symmetric proof holds for $(\boldsymbol{\rho}, s)\ I'_j\ (\boldsymbol{\rho}, \tau(j))$. ◇

We now break the tie in the indifference set $\{\mathsf{t}, \grave{\mathsf{s}}\}$. Here are the rules:

(1) If $\sigma(j)$ maps to $\grave{\mathsf{s}}$, then $\grave{\mathsf{s}}\ \mathsf{P}_j\ \mathsf{t}$.
(2) Otherwise $\mathsf{t}\ \mathsf{P}_j\ \grave{\mathsf{s}}$.

We now show that $\bar{\sigma}$ is stable for the classical school-choice problem with preferences $\mathbf{R}$. Suppose $\mathsf{s}\ \mathsf{P}_j\ \bar{\sigma}(j)$. There is $s \in S$ such that $(\boldsymbol{\rho}, s)\ R_j\ (\boldsymbol{\rho}, \sigma(j))$. If this relation is strict, then $s \in S^*$, since $(\boldsymbol{\rho}, \sigma)$ is an RCE. This further implies that $\lhd_{\mathsf{s}} = \prec_s$ and that, for each $k \in \sigma[s]$, $k \prec_s j$. If the relation is an indifference, then by the claim, $\mathsf{s} = \grave{\mathsf{s}}$. However, $\grave{\mathsf{s}}\ \mathsf{P}_j\ \bar{\sigma}(j)$ could only have happened via Rule (1), which could only happen if $\bar{\sigma}(j) = \grave{\mathsf{s}}$, a contradiction. In sum, all envy is justified by the priorities.

We now show that τ is stable for $\mathbf{R}'$. Observe that, by construction, P'_i top ranks $\bar{\tau}(i)$, so we may restrict attention to $j \neq i$. In this case, $\mathsf{R}'_j = \mathsf{R}_j$. Suppose $\mathsf{s}\ \mathsf{P}_j\ \bar{\tau}(j)$. Given the argument of the previous paragraph, it is clear we can skip to the case that $(\boldsymbol{\rho}, s)\ I_j\ (\boldsymbol{\rho}, \tau(j))$.

The claim then implies that $\mathsf{s} = \grave{\mathsf{s}}$ and so $\bar{\tau}(j) \neq \grave{\mathsf{s}}$ and $j \in \tau[S^*]$. Thus for each $k \in \bar{\tau}[\grave{\mathsf{s}}]$, $k \in \tau[S \setminus S^*]$, and so $k \lhd_{\grave{\mathsf{s}}} j$. Again, all envy is justified by the priorities.

Now we claim that $\bar{\sigma}$ is the student optimal stable match for $\mathbf{R}$. Suppose that $\bar{\mu}$ is a stable match that weakly dominates $\bar{\sigma}$. By our tie-breaking construction, if $(\boldsymbol{\rho}, \sigma(j))\ R_j\ (\boldsymbol{\rho}, t)$, then $\bar{\sigma}(j)\ \mathsf{P}_j\ \mathsf{t}$. In particular, since $(\boldsymbol{\rho}, \sigma)$ is an RCE for $\boldsymbol{R}$, either $\bar{\sigma}(j) = \grave{\mathsf{s}}$ or $\bar{\sigma}(j)\ \mathsf{P}_j\ \grave{\mathsf{s}}$. Thus, going from $\bar{\sigma}$ to $\bar{\mu}$ cannot involve moving students into $\grave{\mathsf{s}}$ who are not already there. Then by feasibility, no students can move out of $\grave{\mathsf{s}}$. Thus, $\bar{\mu}[\grave{\mathsf{s}}] = \bar{\sigma}[\grave{\mathsf{s}}]$. In other words, $\bar{\mu}$ is a reassignment of the agents at constrained schools. Let μ be a matching in the crowded school model that coincides with $\bar{\mu}$ on S^* and with σ otherwise. Suppose there is $j \in N$ with $\bar{\mu}(j)\ \mathsf{P}_j\ \bar{\sigma}(j)$. Then $j \in \sigma[S^*]$ and $\mu(j) \in S^*$. Since $(\boldsymbol{\rho}, \sigma)$ is maximal for $\boldsymbol{R}$, there is $k \in N$ with $k \prec_{\mu(j)} j$ and $(\boldsymbol{\rho}, \mu(j))\ P_k\ (\boldsymbol{\rho}, \mu(k))$. By construction, $k \lhd_{\bar{\mu}(j)} j$. If $\mu(k) \in S^*$, then $\bar{\mu}(k) \neq \grave{\mathsf{s}}$ and $\bar{\mu}$ is blocked in the classical model, as preferences over S^* map directly to preferences over $\mathsf{S} \setminus \grave{\mathsf{s}}$. Thus, $\mu(k) = \sigma(k) \in S \setminus S^*$. Since $(\boldsymbol{\rho}, \sigma)$ is an RCE, for each $t \in S \setminus S^*$, $(\boldsymbol{\rho}, \sigma(k))\ R_k\ (\boldsymbol{\rho}, t)$, so recalling that $\grave{\mathsf{s}}$ inherits the rank of the highest unconstrained school, we have $\bar{\mu}(j)\ \mathsf{P}_k\ \bar{\mu}(k) = \grave{\mathsf{s}}$. Again we conclude that $\bar{\mu}$ is blocked in the classical model.

Now observe that $\bar{\tau}(i)$ is the top-ranked school for R'_i, so $\bar{\mu}(i) = \bar{\tau}(i)$ for any stable $\bar{\mu}$ that dominates $\bar{\tau}$ for $\mathbf{R}'$. By assumption, $(\boldsymbol{\rho}, \tau(i))\ P_i\ (\boldsymbol{\rho}, \sigma(i))$, so by construction $\bar{\tau}(i)\ \mathsf{P}_i\ \bar{\sigma}(i)$, contradicting that i is not able to manipulate the student optimal stable rule. ■

Appendix C. Proof of Proposition 4: The Algorithm

First, observe that the procedure ends in finitely many stages. By construction, the ratio at each school s will be no lower than b_s, and so $\boldsymbol{\rho}^n \geq \boldsymbol{b}$ at every stage. If indeed the algorithm reaches $\boldsymbol{\rho}^n = \boldsymbol{b}$, then it terminates, with the final stage being exactly the Deferred Acceptance algorithm. NCBI guarantees that this results in a maximal RCE.

As $\boldsymbol{R} \in \mathcal{D}$, $\boldsymbol{\rho}^*(\boldsymbol{R})$ is defined. Clearly, $\boldsymbol{\rho}^0 \geq \boldsymbol{\rho}^*(\boldsymbol{R})$. We claim that $\boldsymbol{\rho}^n \geq \boldsymbol{\rho}^*(\boldsymbol{R})$ for each n. To simplify notation, let $\boldsymbol{\rho}^* = \boldsymbol{\rho}^*(\boldsymbol{R})$. Assume that $\boldsymbol{\rho} = \boldsymbol{\rho}^n \geq \boldsymbol{\rho}^*$. We shall show that $\boldsymbol{\rho}^{n+1} \geq \boldsymbol{\rho}^*$ and then induction completes the argument. If $\boldsymbol{\rho} > \boldsymbol{\rho}^*$, then there is nothing to show. Similarly, if $\rho_s = \rho^*_s$ implies that $\rho_s = b_s$, then by definition of the algorithm, $\rho^{n+1}_s \geq \rho_s$. Therefore, let $S' = \{s \in S : \rho_s = \rho^*_s > b_s\}$ be non-empty. Thus at maximal RCE $(\boldsymbol{\rho}^*, \tau)$ for $\boldsymbol{R}$, for each $s \in S'$, each i has $(\boldsymbol{\rho}^*, \tau(i))\ R_i\ (\boldsymbol{\rho}^*, s)$.

Note that, by the definition of a maximal RCE, τ is precisely the outcome of doing steps (1)-(4) in the algorithm for distribution $\boldsymbol{\rho}^*$, and then matching the remaining students to a school they demand. This will be possible as there are no overdemanded sets of schools. Thus, each $s \in S'$ is trivially not a member of any overdemanded set.

Let $S^+ = \{t \in S : \rho_t > \rho^*_t\}$. We make an *ad hoc* modification of distribution feasibility. Let τ' be the matching that results from steps (1)-(4) of the algorithm, at distribution $\boldsymbol{\rho}^*$, but ignoring the capacity constraint on the schools in S^+. That is, any student who applies to any $s \in S^+ \cap S^*$ at any point in Deferred Acceptance is immediately assigned to s. By the resource monotonicity of Deferred Acceptance rule (Ehlers and Klaus [2016]), each $i \in N$ has $(\boldsymbol{\rho}^*, \tau'(i))\ R_i\ (\boldsymbol{\rho}^*, \tau(i))$.

Now by preference monotonicity, for each $i \in N$, $(\boldsymbol{\rho}, \tau'(i))\ R_i\ (\boldsymbol{\rho}^*, \tau'(i))$. The allocation $(\boldsymbol{\rho}, \tau')$ is not necessarily *fair*. However, the only violations of *fairness* are of the form $(\boldsymbol{\rho}, t)\ P_i\ (\boldsymbol{\rho}, \tau'(i))$ for $t \in S^+$, as $(\boldsymbol{\rho}^*, \tau')$ is *fair*. Construct τ'' so that $\tau''(i) \in \mathfrak{D}\,(\boldsymbol{\rho}, S^+; R_i)$ if there is $t \in S^+$ with $(\boldsymbol{\rho}, t)\ R_i\ (\boldsymbol{\rho}, \tau'(i))$, and otherwise $\tau''(i) = \tau'(i)$. Thus, for each $i \in N$, $(\boldsymbol{\rho}, \tau''(i))\ R_i\ (\boldsymbol{\rho}, \tau'(i))$, and $(\boldsymbol{\rho}, \tau'')$ is *fair*. Clearly, $(\boldsymbol{\rho}, \tau'')$ is not an allocation; hence we refer to it as a *tentative* allocation. Let N' be the set of step (2) for ρ, which is of course the set N' calculated in stage n. As $(\boldsymbol{\rho}, \tau'')$ is *fair* for N, it is *fair* when restricted to N'. NCBI implies that the *fair* set for the restricted problem of N' and constrained schools, $\{t \in S : \rho_t = b_t\}$, is a welfare lattice; let σ be the *student-optimal fair* allocation for this problem. If $i \in N'$ is unmatched by σ, or if $i \notin N'$, then set $\sigma(i) = \tau''(i)$. Then $(\boldsymbol{\rho}, \sigma)$ is *fair*. Moreover, for each $i \in N$, $(\boldsymbol{\rho}, \sigma(i))\ R_i\ (\boldsymbol{\rho}, \tau''(i))$. Together with our earlier deductions, we have that $(\boldsymbol{\rho}, \sigma(i))\ R_i\ (\boldsymbol{\rho}^*, \tau(i))$. By construction, σ, restricted to the constrained schools, is the result of steps (1) through (3) of the algorithm in stage n. Moreover, on the rest of the schools, the tentative matching σ is compatible with demands calculated in step (4).

Let $s \in S'$. We showed in the second paragraph above that, for each $i \in N$, $(\boldsymbol{\rho}^*, \tau(i))\ R_i\ (\boldsymbol{\rho}^*, s)$. As $\rho_s = \rho^*_s$, we have just deduced then that $(\boldsymbol{\rho}, \sigma(i))\ R_i\ (\boldsymbol{\rho}, s)$. It follows that since s is not a member of an overdemanded set at $\boldsymbol{\rho}^*$, then it is not at $\boldsymbol{\rho}$. Thus s will not be incremented and so $\rho^{n+1}_s = \rho_s \geq \rho^*_s$ as desired.

Clearly the algorithm terminates only when it has arrived at an RCE. Thus, it must arrive at an RCE for distribution $\boldsymbol{\rho}^*(\boldsymbol{R})$. However, among the students at schools at their lower bounds, we have found the *student-optimal fair* outcome. By the standard Rural Hospitals Theorem of, e.g., Roth [1986], any student not matched to one of these and would *prefer* to be, cannot be matched in a *fair* way. Thus, the algorithm terminates at a maximal RCE.

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