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Efficiency, strategy-proofness, and essentially quasi-linear preferences*

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Abstract

We examine the compatibility of efficiency and strategy-proofness in a package assignment model with transfers where preferences may not be quasi-linear. A preference relation is *essentially quasi-linear* if it is quasi-linear over the set of (consumption) bundles that are at least as desirable as receiving no object and paying nothing, and at which payments are nonnegative. We show that if a domain contains the *essentially quasi-linear domain*, then no mechanism is efficient and strategy-proof. We also show that if there is a mechanism that satisfies efficiency, strategy-proofness, individual rationality, and no subsidy for losers on a domain, the domain must be contained in the essentially quasi-linear domain. Our results demonstrate that the quasi-linearity of preferences is essential for the design of an efficient and strategy-proof mechanism. Our results also have implications for the public goods model with transfers.

Keywords. Strategy-proofness, efficiency, non-quasi-linear preferences, essentially quasi-linear preferences, generalized Vickrey mechanism, maximal domain.

JEL Classification. D44, D71, D61, D82.

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1 Introduction

In many object allocation problems with monetary transfers, agents obtain a package of objects. Examples include spectrum license auctions, treasury bill auctions, and the allocation of emission rights, among others.[1] The primary goal in many object allocation problems is to achieve efficiency. For instance, the Federal Communications Commission (FCC), which has conducted spectrum license auctions intensively over the past three decades, states that "[t]he auction approach is intended to award the licenses to those who will use them most effectively."[2] In many package assignment problems in practice, agents obtain packages at very high prices. As discussed in Section 1.1, high payments may generate income effects, and hence preferences may not be quasi-linear. While there is a substantial literature on object allocation with transfers, many existing studies assume the quasi-linearity of preferences. Without the quasi-linearity assumption, when is it possible to achieve an efficient allocation? Our answer is that quasi-linearity is essential for ensuring the compatibility of efficiency and incentive compatibility.

Formally, we consider a package assignment model with transfers where there are multiple objects and each agent obtains a package of them. Our model is general in the sense that it covers the case where objects are identical, the case where objects are all distinct, and the cases where there are several object types and each object has several copies.[3] A (consumption) bundle consists of a package of objects and a payment.

Each agent has preferences over the consumption bundles. Preferences need not be quasi-linear. A set of preferences is called a *domain*. Many papers in the literature restrict attention to settings in which objects are either all substitutes for each agent or complements for each agent.[4] In contrast, in real-world environments, the

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[1] Bikhchandani and Mamer (1997) and Ausubel (2006) discuss other examples of package assignment problems.

[2] See `https://www.fcc.gov/auctions/about-auctions`.

[3] For example, in the spectrum license auction in the UK in 2018, licenses in the 2.3 and 3.4 GHz spectrum bands were allocated. Each spectrum band was divided into small blocks and a license corresponds to a block in one of the spectrum bands. Blocks in the same spectrum band are considered to be identical. Thus, in this auction, there are two types of objects, each of which has several copies.

[4] Examples include Gul and Stacchetti (1999), Ausubel (2004), Ausubel (2006), Baisa (2020),

substitutability or complementarity of objects often varies across agents, with some agents perceiving objects as substitutes and others perceiving them as complements. For example, in the case of spectrum licenses, incumbents who have already invested in spectrum infrastructure may view spectrum licenses as substitutes, whereas new entrants, who must invest in spectrum infrastructure, may regard them as complements.[5] Moreover, even for a given agent, some objects may be substitutes while others are complements.[6] For this reason, we consider domains that contain the set of all quasi-linear preferences, which allows objects to be substitutes, complements, or mixtures thereof depending on the agent.

A *(direct) mechanism* is a mapping from the set of preference profiles to the set of allocations. *Efficiency* in this paper means that no other allocation makes an agent better off without making any agent worse off or reducing the revenue of the planner. *Strategy-proofness* (dominant strategy incentive compatibility) means that it is a weakly dominant strategy to report her true preferences. For some results, we also impose individual rationality and no subsidy for losers. *Individual rationality* requires that no agent should be worse off than she would be if she had received no object and paid nothing. *No subsidy for losers* means that the payment of an agent who does not receive any object is nonnegative.[7]

In auction contexts, payments are typically assumed to be nonnegative. This assumption is natural when objects are owned by a seller, as payments collected from agents constitute the seller's revenue. However, there are object allocation problems in which objects are not owned by anyone or in which property rights are difficult to specify. Examples include the allocation of emission rights across countries, inherited estates among family members, and fishing rights among fishers. In such environments, monetary transfers do not generate surplus, as there is no natural recipient. Transfers are therefore introduced primarily to achieve efficiency

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among others.

[5] In U.S. spectrum license auctions, regional carriers may treat licenses for different areas as substitutes, while nationwide carriers typically view licenses across regions as complements due to their need for broad geographic coverage.

[6] For example, in fisheries, catch rights for different offshore species can be complements due to shared fixed costs and joint production, while offshore and coastal catch rights may be substitutes.

[7] Some papers consider a stronger condition called *no subsidy*. It requires that the payment of each agent is always nonnegative.

and provide incentives, and from the perspective of agents' welfare it is desirable to keep total payments small. In particular, allowing for negative payments can be natural, as agents who fail to receive objects may need to be compensated by those who do.[8] These considerations highlight the importance of analyzing mechanisms based solely on *efficiency* and *strategy-proofness*.

We investigate the domains on which *efficiency* and *strategy-proofness* are compatible. It turns out that the class of preferences introduced by Ma et al. (2018) plays an important role. The *relevant consumption set* consists of bundles that are at least as desirable as receiving no object and paying nothing, and at which payments are nonnegative. A preference relation is *essentially quasi-linear* if it is quasi-linear on the relevant consumption set. Bundles in the relevant consumption set correspond to the consumption bundles agents receive in standard auctions used in practice. Accordingly, essential quasi-linearity can be interpreted as meaning that income effects do not arise in standard auction environments.

Our main result depends on the number of agents. For two agents, the essentially quasi-linear domain (the set of essentially quasi-linear preferences) is a *maximal domain* for *efficiency* and *strategy-proofness* (Theorem 1 (ii)). Thus, there is an *efficient* and *strategy-proof* mechanism on the essentially quasi-linear domain, while no such mechanism exists on any larger domain. Moreover, on the essentially quasi-linear domain, *generalized Vickrey mechanisms* (a generalization of the Vickrey mechanism to non-quasi-linear domains) are the only *efficient* and *strategy-proof* mechanisms (Theorem 1 (i)). By contrast, when there are more than two agents, we show that there is no *efficient* and *strategy-proof* mechanism on any domain containing the essentially quasi-linear domain (Theorem 2).

Our results can be further strengthened by imposing *individual rationality* and *no subsidy for losers*. We show that if there is a mechanism on a domain that satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers*, then

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[8] Moreover, *individual rationality* may not always be compelling in real-world allocation problems, as participation is effectively mandatory in some environments. For example, in international allocation problems—such as the division of transboundary resources or internationally negotiated quotas—political considerations or linkages with other agreements may prevent a country from declining the outcome even if it is worse than its outside option. Similarly, in the division of inherited estates, family members may face legal, social, or relational constraints that make withdrawal from the allocation process unrealistic.

the domain must be a subset of the essentially quasi-linear domain (Theorem 3). This result implies that for two agents, the essentially quasi-linear domain is the *unique* maximal domain for the four properties. Thus, the quasi-linearity is essential to achieve an efficient allocation in auction contexts.

Our results indicate that an efficient allocation may fail to be achieved when objects are highly valuable. As we will discuss in Section 1.1, income effects arise when payment levels vary substantially. When objects are highly valuable, the relevant consumption set becomes large, allowing payments to vary widely within that set. As a result, income effects may arise even on the relevant consumption set. Our results therefore suggest that *efficiency* and *strategy-proofness* are incompatible in such environments.

Our results have implications for various domains of interest. For example, we obtain impossibility results if preferences exhibit specific income effects (Corollary 2) or when agents may face borrowing costs (Corollary 3). Our results also have implications for the public goods model with transfers. Specifically, our results imply that in the public goods model with transfers, if there are at least three agents and six alternatives, no mechanism satisfies *efficiency* and *strategy-proofness* on the essentially quasi-linear domain (Corollary 4).

1.1 Related literature

The quasi-linearity has been a fundamental assumption in auction theory. However, agents typically have income effects in many real-life environment, especially those in which payments are high. As Morimoto and Serizawa (2015) argues, "[e]xcessive payments for the auctioned objects may damage bidders' budgets to purchase complements for effective uses of the objects and thus, may influence the benefits from the objects" (Morimoto and Serizawa, 2015, p.447). Another source that makes preferences non-quasi-linear is the existence of distortionary frictions (Saitoh and Serizawa, 2008; Fleiner et al., 2019). An agent may have to borrow money at some interest rate when the payment exceeds her income, which makes preferences non-quasi-linear even if the original one is quasi-linear. These issues are especially relevant when the objects are highly valuable, since the associated payments can vary drastically between substantial and minor amounts.

The existence of income effects changes the results dramatically. When preferences are quasi-linear, a mechanism is *efficient* and *strategy-proof* if and only if it is a *Groves mechanism* (Holmström, 1979). Moreover, a mechanism additionally satisfies *individual rationality* and *no subsidy for losers* if and only if it is a *Vickrey mechanism* (Holmström, 1979; Chew and Serizawa, 2007). However, when preferences can be non-quasi-linear, these results no longer hold. Indeed, Vickrey mechanisms are in general inefficient and do not satisfy *strategy-proofness*. When agents are unit-demand, minimum price Walrasian mechanisms are the only mechanisms that satisfy the above mentioned four properties (Morimoto and Serizawa, 2015; Zhou and Serizawa, 2018; Wakabayashi et al., 2025).[9] In contrast, as we show in this paper, no mechanism satisfies *efficiency* and *strategy-proofness* when agents are multi-demand and the domain contains various non-quasi-linear preferences.[10]

Similar impossibility results to ours have been obtained by Kazumura and Serizawa (2016); Baisa (2020); Malik and Mishra (2021); Shinozaki et al. (2025) However, there are two main differences between our paper and those papers. First, these four papers impose *individual rationality* and a no subsidy condition in addition to *efficiency* and *strategy-proofness*. In contrast, our main results (Theorems 1 and 2) impose only *efficiency* and *strategy-proofness*. As previously discussed, there are many environments where *individual rationality* and *no subsidy* may not be suitable, and their results do not cover such environments.[11]

Second, those four papers and ours focus on different economic environments. Kazumura and Serizawa (2016) and Malik and Mishra (2021) consider specific environments which do not intersect with our model. Baisa (2020) and Shinozaki et al. (2025) consider similar environments to ours. These papers focus on the cases in

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[9] When agents are multi-demand, a Walrasian equilibrium may not exist. Many papers in the literature provide conditions that guarantee the existence of a Walrasian equilibrium. For example, see Kelso and Crawford (1982), Sun and Yang (2006), Teytelboym (2014), Baldwin and Klemperer (2019), Baldwin et al. (2023), and Nguyen and Vohra (2024).

[10] Essig Aberg and Baisa (2025) measure inefficiency of Vickrey and uniform price mechanisms. Shinozaki and Serizawa (2025) weaken *efficiency* and characterize a mechanism called the bundling unit-demand minimum price Walrasian mechanism by constrained efficiency, *strategy-proofness*, and other auxiliary properties.

[11] Technically, *individual rationality* and *no subsidy (for losers)* imply that the payment of losers must be zero. While those papers exploit this fact to establish their results, we develop a proof strategy to overcome this difficulty.

which preferences are submodular and have positive/negative income effects. This means that objects are all substitutes, and either they are all normal goods or inferior goods. However, as we have discussed, many real-life environments contain both substitutes and complements. Moreover, it is reasonable to consider environments in which some objects are normal while others are inferior.[12] We provide a new notion of income effects that covers such cases and show that our result can be applied to such preferences (Corollary 2).

More technically, it is also worth noting that the domains studied in Baisa (2020) and Shinozaki et al. (2025) are not necessarily subsets of the domains we consider. They require the domain to contain submodular preferences with income effects. While we require the domain to contain the set of quasi-linear preferences, our results can also be applied to domains that do not contain submodular preferences with income effects. Thus, their impossibility results do not imply our result that impose *individual rationality* and *no subsidy for losers* (Theorem 3).

There are papers that study the case where agents have quasi-linear preferences but face hard budget (Che and Gale, 1998, 2000; Pai and Vohra, 2014). Due to the hard budget, the induced preferences are non-quasi-linear. In this setting, there is no mechanism that satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and a no subsidy condition (Dobzinski et al., 2012; Lavi and May, 2012). Preferences in this model are close to quasi-linear preferences with borrowing cost in our model (see Section 6.1). The major difference between preferences in this model and quasi-linear preferences with borrowing cost is that in this model, the budget is hard, i.e., payments cannot exceed the budget. On the other hand, since we allow agents to borrow money, payments can exceed the budget. Moreover, technically, preferences in this model violate continuity whereas we focus only on continuous preferences. Thus, there is no logical relation between the impossibility results by Dobzinski et al. (2012) and Lavi and May (2012) and our results. Moreover, we impose only *efficiency*

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[12] In some cases, spectrum licenses for the current generation and the next generation are auctioned simultaneously (e.g., the UK spectrum license auction in 2018). Because the use of next-generation licenses requires additional fixed investments, the valuation for them typically increases as the payment decreases. By contrast, since the investments for the current-generation licenses are largely sunk, a higher payment raises their relative value. Therefore, next-generation licenses can be viewed as normal goods, while current-generation licenses can be viewed as inferior goods in this environment.

and *strategy-proofness* in our main results, while those papers impose not only these properties but also *individual rationality* and a no subsidy condition.

The essentially quasi-linearity is first introduced in the public goods model with transfers by Ma et al. (2018). They show that if the domain is larger than the essentially quasi-linear domain, *fixed price dictatorships* are the only mechanisms that satisfy *strategy-proofness*, *ontoness*, *individual rationality*, and *no subsidy*, where *ontoness* is a weak efficiency notion that requires each alternative to be selected at some preference profile. Since fixed price dictatorships violate *efficiency*, their result implies that if the domain is larger than the essentially quasi-linear domain, no mechanism satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy*. Compared with this result, our result (Corollary 4) shows a stronger impossibility result in a restricted environment. Indeed, we impose only *efficiency* and *strategy-proofness*, whereas Ma et al. (2018) impose *individual rationality* and *no subsidy* as well. On the other hand, we assume that there are at least three agents and at least six alternatives, while Ma et al. (2018) do not make such an assumption.

Finally, non-quasi-linearity of preferences is recently introduced in various models. Examples include house allocation model with transfers (Andersson and Svensson, 2014; Andersson et al., 2016; Andersson and Svensson, 2016), matching model with transfers (Garratt and Pycia, 2023; Morimoto, 2025), school choice model (Phan et al., 2024), and trading network model (Fleiner et al., 2019; Schlegel, 2022). A general mechanism design model is studied by Kazumura et al. (2020), and they give a necessary and sufficient condition for *strategy-proofness* when preferences can be non-quasi-linear.

2 The model and definitions

There are $n \geq 2$ agents and $m \geq 1$ types of objects. We denote the set of agents by $N \equiv \{1, \ldots, n\}$ and the set of object types by $M \equiv \{1, \ldots, m\}$. Each object $a \in M$ has $\bar{x}_a \in \mathbb{N}$ copies. Thus, $\bar{x} \equiv (\bar{x}_a)_{a \in M}$ is the **social endowment**. If $m = 1$, there are only identical objects. If $\bar{x}_a = 1$ for each $a \in M$, then there are only distinct objects. We assume $\sum_{a \in M} \bar{x}_a > 1$.[13] Denote $\mathbf{0} \equiv (0, 0, \ldots, 0) \in \mathbb{R}^m$. A **package** is a vector

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[13] See Saitoh and Serizawa (2008) and Sakai (2008) for results in the single object case.

$x \equiv (x_a)_{a \in M} \in \mathbb{Z}^m$ such that $\mathbf{0} \leq x \leq \bar{x}$.[14] Let X be the set of packages. Each agent receives a package and pays some amount of money. Thus, the **consumption set** is $X \times \mathbb{R}$, and a typical **(consumption) bundle** for an agent is a pair $z \equiv (x, t) \in X \times \mathbb{R}$, where t is interpreted as the amount paid by the agent.

Each agent i has a complete and transitive preference relation R_i over $X \times \mathbb{R}$. Let P_i and I_i be the strict and indifference relations associated with R_i, respectively. The generic notation for a class of admissible preferences is denoted by $\mathcal{R}$ and we call it a **domain**. The following are standard conditions for a preference relation R_i.

- **Money monotonicity:** For each $x \in X$ and each pair $t, s \in \mathbb{R}$ with $t < s$, $(x, t) \; P_i \; (x, s)$.

- **Possibility of compensation:** For each $(x, t) \in X \times \mathbb{R}$ and each $y \in X$, there are $s, s' \in \mathbb{R}$ such that $(x, t) \; R_i \; (y, s)$ and $(y, s') \; R_i \; (x, t)$.

- **Continuity:** For each $z \in X \times \mathbb{R}$, the **upper contour set** at z, $UC_i(z) \equiv \{z' \in X \times \mathbb{R} : z' \; R_i \; z\}$, and the **lower contour set** at z, $LC_i(z) \equiv \{z' \in X \times \mathbb{R} : z \; R_i \; z'\}$, are both closed.

- **Object monotonicity:** For each $(x, t) \in X \times \mathbb{R}$ and each $y \in X$ with $y > x$, $(y, t) \; P_i \; (x, t)$.

We denote the set of preferences that satisfy the above four conditions by $\mathcal{R}^O$, and call it the **object monotonic domain**. Throughout the paper, we assume that preferences satisfy the above four conditions. Thus, whenever we take a domain $\mathcal{R}$, it satisfies $\mathcal{R} \subseteq \mathcal{R}^O$.

A standard class of preferences studied in the literature is the class of quasi-linear preferences.

Definition 1. A preference relation R_i is **quasi-linear** if for each pair $(x, t), (y, s) \in X \times \mathbb{R}$ and each $\delta \in \mathbb{R}$, $(x, t) \; I_i \; (y, s)$ implies $(x, t + \delta) \; I_i \; (y, s + \delta)$.

Let $\mathcal{R}^Q$ be the class of quasi-linear preferences and we call it the **quasi-linear domain**. For each $R_i \in \mathcal{R}^Q$, there is a **valuation function** $v_i : X \to \mathbb{R}_+$ such that

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[14] Given a pair of vectors $x, y \in \mathbb{Z}^m$, we write $x \geq y$ to mean $x_a \geq y_a$ for each $a \in M$. Similarly, we write $x > y$ to mean $x_a \geq y_a$ for each $a \in M$ and $x_b > y_b$ for some $b \in M$.

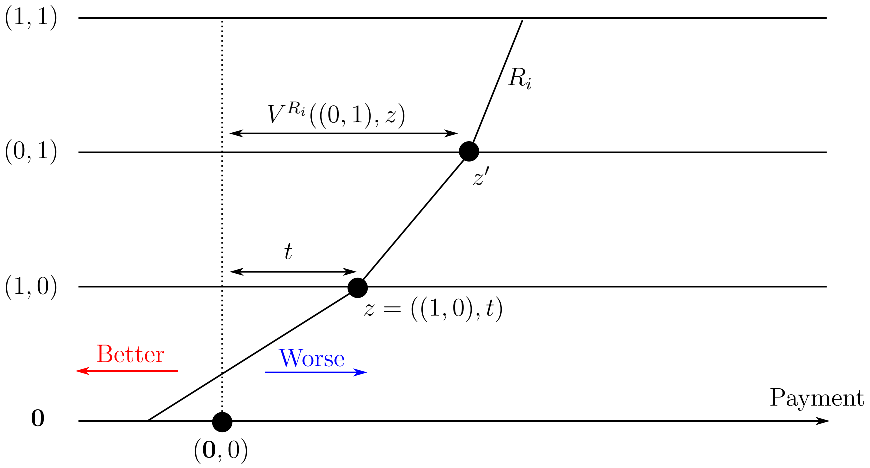

Figure 1: An illustration of the consumption set for $M = \{1, 2\}$ and $\bar{x} = (1, 1)$.

$v_i(\mathbf{0}) = 0$, and for each pair $(x, t), (y, s) \in X \times \mathbb{R}$, $(x, t)\ R_i\ (y, s)$ if and only if $v_i(x) - t \geq v_i(y) - s$.

We now extend the notion of valuation to non-quasi-linear preferences. Given a preference relation R_i, $z \in X \times \mathbb{R}$, and $y \in X$, there is a payment $s \in \mathbb{R}$ such that $z\ I_i\ (y, s)$.[15] We call this payment level the **(compensated) valuation of y at z for R_i**, and denote it by $V^{R_i}(y, z)$. There are two remarks on the notion of valuation that we often use in the rest of the paper.

Remark 1. Given a preference relation R_i and a pair $(x, t), (y, s) \in X \times \mathbb{R}$, we have $(x, t)\ R_i\ (y, s)$ if and only if $V^{R_i}(y, (x, t)) \leq s$.

Remark 2. For each $R_i \in \mathcal{R}^Q$, each $(x, t) \in X \times \mathbb{R}$, and each $y \in X$, $V^{R_i}(y, (x, t)) - t = v_i(y) - v_i(x)$.

Figure 1 is an illustration of the consumption set for $M = \{1, 2\}$ and $\bar{x} = (1, 1)$. In this diagram, each of the four horizontal lines represents the set of real numbers, and each point on the lines represents a payment for the package specified on the left side of the line. The vertical dotted line in this diagram connects the points where the payment is zero. Then, the consumption set in this example consists of these four horizontal lines. For example, the point z corresponds to the consumption bundle

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[15] The existence of such a payment is guaranteed by money monotonicity, possibility of compensation and continuity. For the formal proof of the existence, see Kazumura and Serizawa (2016).

$((1,0),t)$.

One way to describe a preference relation in the diagram is to draw "indifference curves." A typical indifference curve is illustrated in Figure 1. The indifference curve passes through z and z'. This means that z and z' are indifferent for a preference relation R_i. To specify which preference relation an indifference curve belongs to, we sometimes write the notation for the preference relation next to the indifference curve as in Figure 1. By money monotonicity, bundles to the left (resp. right) of an indifference curve are better (resp. worse) than the bundles on the indifference curve. The valuation of a package at a bundle corresponds to the payment at the point that is indifferent to the bundle. Thus, for example, the valuation of $(0,1)$ at z is equal to the payment at z'.

A **package allocation** is an n-tuple $(x_i)_{i\in N} \in X^n$ such that $\sum_{i\in N} x_i \leq \bar{x}$. We denote the set of package allocations by A. A **(feasible) allocation** is an n-tuple $((x_i,t_i))_{i\in N} \in (X \times \mathbb{R})^n$ such that $(x_i)_{i\in N} \in A$. We denote the set of allocations by Z. A **preference profile** is an n-tuple $R \equiv (R_1,\ldots,R_n) \in \mathcal{R}^n$. Given $R \in \mathcal{R}^n$ and $i,j \in N$, let $R_{-i} \equiv (R_k)_{k\in N\setminus\{i\}}$ and $R_{-i,j} \equiv (R_k)_{k\in N\setminus\{i,j\}}$.

A **mechanism** on $\mathcal{R}^n$ is a function $f : \mathcal{R}^n \to Z$. Given a mechanism f and $R \in \mathcal{R}^n$, we denote the bundle assigned to agent i by $f_i(R)$ and we write $f_i(R) \equiv (x_i^f(R), t_i^f(R))$, where $x_i^f(R)$ is the package that agent i receives and $t_i^f(R)$ is her payment.

We introduce four properties of mechanisms. An allocation $((x_i,t_i))_{i\in N} \in Z$ is **(Pareto) efficient** for $R \in \mathcal{R}^n$ if there is no allocation $((y_i,s_i))_{i\in N} \in Z$ such that (i) for each $i \in N$, $(y_i,s_i)\ R_i\ (x_i,t_i)$, (ii) for some $j \in N$, $(y_j,s_j)\ P_j\ (x_j,t_j)$, and (iii) $\sum_{i\in N} s_i \geq \sum_{i\in N} t_i$.[16] Thus, if an allocation is efficient, it is impossible to make an agent better off without harming other agents or reducing the revenue of the planner. One interpretation of the efficiency notion is that it incorporates the planner's preferences, assuming that she is interested only in total revenue. Under this interpretation, the notion coincides with standard Pareto efficiency.[17]

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[16] There are other ways to define efficiency. We employ one of the weakest notions of efficiency. For a comparison with other efficiency notions, see Kazumura and Serizawa (2016).

[17] An alternative interpretation is in terms of reallocation among agents. Given an allocation, if there exists a reallocation of objects and transfers among agents, without introducing money from outside the economy, that makes at least one agent strictly better off without making any agent

By object monotonicity, if an allocation $((x_i, t_i))_{i \in N} \in Z$ is efficient for $R \in \mathcal{R}^n$, then no object remains unassigned, i.e., $\sum_{i \in N} x_i = \bar{x}$.

Remark 3. An allocation $((x_i, t_i))_{i \in N} \in Z$ is efficient for $R \in \mathcal{R}^n$ if and only if there is no allocation $((y_i, s_i))_{i \in N} \in Z$ such that (i) for each $i \in N$, $(y_i, s_i)\ I_i\ (x_i, t_i)$, and (ii) $\sum_{i \in N} s_i > \sum_{i \in N} t_i$.[18]

The efficiency of a mechanism requires that for each preference profile, an efficient allocation should be selected.

Efficiency: For each $R \in \mathcal{R}^n$, $f(R)$ is efficient for R.

The next property requires that no agent should benefit from misrepresenting her preferences.

Strategy-proofness: For each $R \in \mathcal{R}^n$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $f_i(R)\ R_i\ f_i(R'_i, R_{-i})$.

The following property requires that an agent should be never assigned a bundle that makes her worse off than she would be if she had received no object and paid nothing.

Individual rationality: For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R)\ R_i\ (\mathbf{0}, 0)$.

The last property requires that the payment of losers (the agents who receive no object) should be nonnegative.

No subsidy for losers: For each $R \in \mathcal{R}^n$ and each $i \in N$, if $x^f_i(R) = \mathbf{0}$, $t^f_i(R) \geq 0$.

3 Generalized Vickrey mechanisms

Groves and Vickrey mechanisms are defined on the quasi-linear domain, because they are defined by means of valuation functions. In this section, we define these mechanisms and extend Vickrey mechanisms to larger domains.

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worse off, then the original allocation would be considered undesirable. Our efficiency is equivalent to the non-existence of such reallocation. In this sense, our efficiency is also appropriate even when there is no natural recipient of payments.

[18] This result follows from money monotonicity.

Given $i \in N$, $R_{-i} \in \mathcal{R}^{n-1}$, and $x \in X$, let

$$\sigma_i(R_{-i}; x) \equiv \max\{\sum_{j \in N \setminus \{i\}} V^{R_j}(x_j, (\mathbf{0}, 0)) : (x_j)_{j \in N} \in A, x_i = x\}.$$

Thus, $\sigma_i(R_{-i}; x)$ is the maximum sum of valuations at $(\mathbf{0}, 0)$ that the agents other than agent i can achieve when agent i obtains x. Now we define Groves and Vickrey mechanisms on the quasi-linear domain.

Definition 2. A mechanism f on $(\mathcal{R}^Q)^n$ is a **Groves mechanism** if for each $R \in (\mathcal{R}^Q)^n$,

$$(x_i^f(R))_{i \in N} \in \text{argmax}_{(x_i)_{i \in N} \in A} \sum_{i \in N} v_i(x_i),$$

and for each $i \in N$, there is $h_i : (\mathcal{R}^Q)^{n-1} \to \mathbb{R}$ such that $t_i^f(R) = h_i(R_{-i}) - \sigma_i(R_{-i}; x_i^f(R))$. A mechanism f on $(\mathcal{R}^Q)^n$ is a **Vickrey mechanism** if it is a Groves mechanism and for each $i \in N$, $h_i(\cdot) = \sigma_i(\cdot; \mathbf{0})$.

We now generalize Vickrey mechanisms to larger domains. We do so by means of compensated valuations.

Definition 3. Let $\mathcal{R}$ be an arbitrary domain. A mechanism f on $\mathcal{R}^n$ is a **generalized Vickrey mechanism** if for each $R \in \mathcal{R}^n$,

$$(x_i^f(R))_{i \in N} \in \text{argmax}_{(x_i)_{i \in N} \in A} \sum_{i \in N} V^{R_i}(x_i, (\mathbf{0}, 0)),$$

and for each $i \in N$, $t_i^f(R) = \sigma_i(R_{-i}; \mathbf{0}) - \sigma_i(R_{-i}; x_i^f(R))$.

A generalized Vickrey mechanism is defined in the same manner as a Vickrey mechanism except that compensated valuations at $(\mathbf{0}, 0)$ are used instead of valuation functions.

4 Essentially quasi-linear preferences

In this section, we define a class of non-quasi-linear preferences. Given a preference relation R_i, the **relevant consumption set** for R_i is defined as

$$X(R_i) \equiv \{(x, t) \in X \times \mathbb{R} : (x, t)\ R_i\ (\mathbf{0}, 0) \text{ and } t \geq 0\}.$$

In words, the relevant consumption set consists of *individually rational* bundles

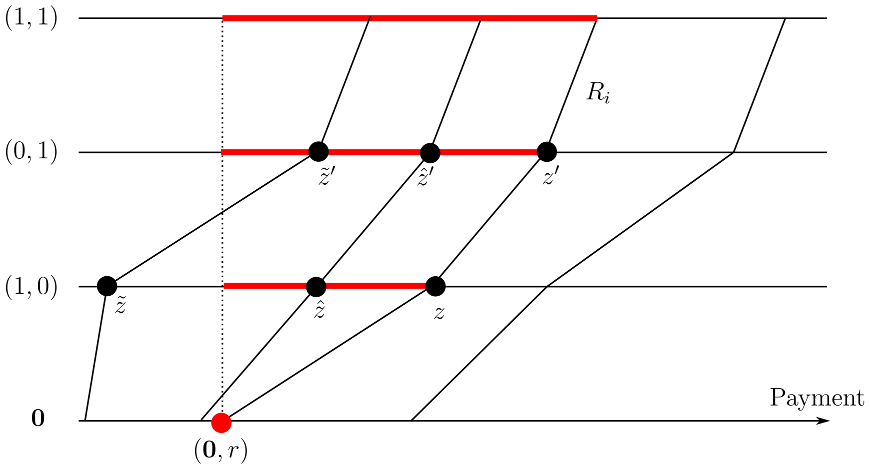

Figure 2: An illustration of the relevant consumption set and an essentially quasi-linear preference relation.

with a nonnegative payment. Figure 2 illustrates the relevant consumption set for a preference relation R_i when $\bar{x} = (1,1)$. The relevant consumption set for R_i consists of the bundle $(\mathbf{0}, 0)$ and the bundles on the three thick red lines.

There are auctions that are widely used in practice or studied in the literature, such as the first or second price auction, the Vickrey auction, the simultaneous ascending or descending auction, core-selecting auctions. Here, we refer to such auctions as *standard auctions*. In such auctions, *individual rationality* is typically satisfied. Moreover, it is unlikely that a bidder can obtain a positive monetary transfer. Thus, the relevant consumption set can be interpreted as the set of bundles that arise in standard auctions.

Definition 4. A preference relation R_i is **essentially quasi-linear** if for each $(x,t) \in X(R_i)$ and each $y \in X$ with $V^{R_i}(y,(x,t)) \geq 0$,

$$V^{R_i}(y,(x,t)) - t = V^{R_i}(y,(\mathbf{0},0)) - V^{R_i}(x,(\mathbf{0},0)).$$

Remark 4. A preference relation R_i is essentially quasi-linear if and only if for each $(x,t) \in X(R_i)$, $V^{R_i}(\bar{x},(x,t)) - t = V^{R_i}(\bar{x},(\mathbf{0},0)) - V^{R_i}(x,(\mathbf{0},0))$.

Let $\mathcal{R}^E$ be the class of essentially quasi-linear preferences, and call it the **essentially quasi-linear domain**.

The essential quasi-linearity requires that the preference relation be quasi-linear when the consumption set is restricted to the relevant consumption set. Thus, essentially quasi-linear preferences exhibit no income effects within the relevant consumption set. Since the relevant consumption set corresponds to bundles that an agent may receive in standard auctions, the essential quasi-linearity requires that the agent has no income effect as long as she participates in a standard auction.

Quasi-linear preferences are, of course, essentially quasi-linear. Thus, $\mathcal{R}^Q \subseteq \mathcal{R}^E$. An essentially quasi-linear preference relation is illustrated in Figure 2. Since this preference relation is quasi-linear in the relevant consumption set, indifference curves are parallel *within* the relevant consumption set. However, essential quasi-linearity does not require indifference curves to be parallel if one of them passes through a bundle outside of the relevant consumption set. Thus, for instance, the indifference curve between z and z' in Figure 2 must be parallel to that between $\hat{z}$ and $\hat{z}'$. In contrast, it need not be parallel to the indifference curve between $\tilde{z}$ and $\tilde{z}'$, since $\tilde{z}$ is not in the relevant consumption set.

5 Main results

It is well known that on the quasi-linear domain, Groves mechanisms are the only mechanisms that are *efficient* and *strategy-proof*, and that Vickrey mechanisms are the only mechanisms that additionally satisfy *individual rationality* and *no subsidy for losers*.

Fact 1 (Holmström (1979); Chew and Serizawa (2007))**.** *Groves mechanisms are the only mechanisms that satisfy efficiency and strategy-proofness on $(\mathcal{R}^Q)^n$. Vickrey mechanisms are the only mechanisms that satisfy efficiency, strategy-proofness, individual rationality, and no subsidy for losers.*

We investigate the possibility of designing an *efficient* and *strategy-proof* mechanism when agents may have non-quasi-linear preferences. More specifically, we study how far the domain can be expanded beyond the quasi-linear domain while guaranteeing the existence of an *efficient* and *strategy-proof* mechanism. Our results depend on the number of agents.

Before stating our results, we define the notion of a maximal domain.

Definition 5. A domain $\mathcal{R}$ is a **maximal domain** for a list of properties if

(i) there is a mechanism on $\mathcal{R}^n$ that satisfies the properties, and
(ii) for each $\mathcal{R}' \supsetneq \mathcal{R}$, no mechanism on $(\mathcal{R}')^n$ satisfies the properties.

Remark 5. A maximal domain for a list of properties may not be unique.

The following theorem shows that in the two-agent case, generalized Vickrey mechanisms are the only mechanisms that satisfy *efficiency* and *strategy-proofness* on the essentially quasi-linear domain, and moreover, that this domain is maximal for *efficiency* and *strategy-proofness*.

Theorem 1. *Let $n = 2$.*
(i) On $(\mathcal{R}^E)^2$, a mechanism satisfies efficiency and strategy-proofness if and only if it is a generalized Vickrey mechanism.
(ii) $\mathcal{R}^E$ is a maximal domain for efficiency and strategy-proofness.

Notice that Theorem 1 (i) is not a straightforward extension of Fact 1. First, Theorem 1 (i) applies only to the two-agent case, whereas Fact 1 holds for an arbitrary number of agents. Indeed, the proof of Theorem 1 (i) relies on the two-agent assumption. Second, generalized Vickrey mechanisms are characterized using only *efficiency* and *strategy-proofness*. In contrast, on the quasi-linear domain, the class of *efficient* and *strategy-proof* mechanisms is the class of Groves mechanisms, which is much larger than the class of (generalized) Vickrey mechanisms.

Theorem 1 (ii) implies that no mechanism satisfies *efficiency* and *strategy-proofness* on any domain larger than the essentially quasi-linear domain. However, Theorem 1 (ii) does not rule out the possibility that there exist maximal domains for *efficiency* and *strategy-proofness* other than the essentially quasi-linear domain. As we see in Section 5.1, we can pin down the class of maximal domains once *individual rationality* and *no subsidy for losers* are imposed.

In contrast to the case of $n = 2$, when $n \geq 3$, *efficiency* and *strategy-proofness* are incompatible even on the essentially quasi-linear domain.

Theorem 2. *Let $n \geq 3$. No mechanism on $(\mathcal{R}^E)^n$ satisfies efficiency and strategy-proofness.*

As in Theorem 1 (ii), Theorem 2 implies that no mechanism satisfies *efficiency* and *strategy-proofness* on any domain larger than the essentially quasi-linear domain.

Moreover, Theorem 2 implies that a maximal domain for *efficiency* and *strategy-proofness* lies between the quasi-linear domain and the essentially quasi-linear domain.[19]

5.1 Individual rationality and no subsidy for losers

We have shown that preferences should be essentially quasi-linear for the existence of an *efficient* and *strategy-proof* mechanism. However, there exist domains that are neither subsets or supersets of the essentially quasi-linear domain, and on which an *efficient* and *strategy-proof* mechanism may exist. By imposing *individual rationality* and *no subsidy for losers*, however, the results in the previous section can be strengthened.

Before stating the main results of this subsection, we present a technical result that we use to derive them.

Proposition 1. *Let* $\mathcal{R}$ *be such that* $\mathcal{R}^Q \subseteq \mathcal{R} \nsubseteq \mathcal{R}^E$. *Then, there is no efficient and strategy-proof mechanism on* $\mathcal{R}^n$ *that coincides with a (generalized) Vickrey mechanism on* $(\mathcal{R}^Q)^n$.

By Fact 1 and Proposition 1, we obtain the following result.

Theorem 3. *Let* $\mathcal{R}$ *be such that* $\mathcal{R} \supseteq \mathcal{R}^Q$.
(i) Let $n = 2$. $\mathcal{R}$ *is a maximal domain for efficiency, strategy-proofness, individual rationality and no subsidy for losers if and only if* $\mathcal{R} = \mathcal{R}^E$.
(ii) Let $n \geq 3$. *If there is a mechanism on* $\mathcal{R}^n$ *that satisfies efficiency, strategy-proofness, individual rationality and no subsidy for losers, then* $\mathcal{R} \subsetneq \mathcal{R}^E$.

Theorem 3 states that the domain must be a subset of the essentially quasi-linear domain for the existence of a mechanism that satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers*. Moreover, Theorem 3 (i) states that in the two-agent case, the essentially quasi-linear domain is the *unique*

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[19] An *efficient* and *strategy-proof* mechanism exists on a domain that is close to the essentially quasi-linear domain. In addition to the requirement of the essential quasi-linearity, suppose that a preference relation R_i satisfies the following condition: For each $(x,t) \in X(R_i)$ and each $y \in X$ with $V^{R_i}(y,(x,t)) < 0$, $t - V^{R_i}(y,(x,t)) \geq V^{R_i}(x,(\mathbf{0},0)) - V^{R_i}(y,(\mathbf{0},0))$. If $\mathcal{R}$ is the set of preferences that satisfy this condition, then generalized Vickrey mechanisms are *efficient* and *strategy-proof* on $\mathcal{R}^n$.

maximal domain for the four properties among the domains that contain the quasi-linear domain.

6 Applications

6.1 Implications for various domains

Our results are useful to verify whether there is a mechanism that satisfies *efficiency* and *strategy-proofness* (and *individual rationality* and *no subsidy for losers*) on various domains of interest. In this section, we show that our results can be applied to several domains studied in the literature.

The object monotonic domain contains the essentially quasi-linear domain as well as preferences that are not essentially quasi-linear. Thus, no *efficient* and *strategy-proof* mechanism exists on the object monotonic domain.

Corollary 1. *No mechanism on* $(\mathcal{R}^O)^n$ *satisfies efficiency and strategy-proofness.*

Some papers such as Saitoh and Serizawa (2008), Baisa (2020), and Shinozaki et al. (2025) consider preferences with income effects.

Definition 6. A preference relation R_i exhibits **positive income effects** (resp. **negative income effects**) for $x \in X$ if for each pair $t, s \in \mathbb{R}$, $(x, t)\ I_i\ (\mathbf{0}, s)$ implies that, for each $\delta \in \mathbb{R}_{++}$, $(x, t-\delta)\ P_i\ (\mathbf{0}, s-\delta)$ (resp. $(\mathbf{0}, s-\delta)\ P_i\ (x, t-\delta)$).

Although income is not modeled explicitly, zero payment can be interpreted as the endowed income. Thus, a decrease in payment by $\delta > 0$ corresponds to an increase in income by δ. Then, $(x, t-\delta)\ P_i\ (\mathbf{0}, s-\delta)$ (resp. $(\mathbf{0}, s-\delta)\ P_i\ (x, t-\delta)$) implies that x becomes more preferred (resp. less preferred) when the income is increased by δ. In other words, x is a normal (resp. inferior) package for R_i.

The notion of income effects here is more general than that in other papers such as Saitoh and Serizawa (2008), Baisa (2020), and Shinozaki et al. (2025). They consider preferences that exhibit either positive income effects for every package or negative income effects for every package. In contrast, we allow preferences to exhibit positive income effects for some packages and negative income effects for other packages.

It is clear that if a preference relation exhibits positive or negative income effects for some $x \in X \setminus \{\mathbf{0}\}$, then it is not essentially quasi-linear. Thus, by Theorem 3, no mechanism satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* if the domain contains even a single preference relation that exhibits positive or negative income effects for some package.

Corollary 2. *Let R_i be a preference relation that exhibits positive or negative income effects for some $x \in X \setminus \{\mathbf{0}\}$. Let $\mathcal{R}$ be such that $\mathcal{R}^Q \cup \{R_i\} \subseteq \mathcal{R}$. Then no mechanism on $\mathcal{R}^n$ satisfies efficiency, strategy-proofness, individual rationality, and no subsidy for losers.*

Another class of preferences studied in the literature is the class of quasi-linear preferences with borrowing cost (Saitoh and Serizawa, 2008; Fleiner et al., 2019). Here, income is modeled explicitly. Suppose that each agent must borrow money at some interest rate when the payment for a package exceeds her income. For each $i \in N$, let $w_i \in \mathbb{R}_+$ and $r_i \in \mathbb{R}_+$ be agent i's income level and the interest rate that she faces, respectively. When the payment for a package is $t \in \mathbb{R}$, agent i's actual cost is given by a function $c(\cdot; w_i, r_i) : \mathbb{R} \to \mathbb{R}$ defined as

$$c(t; w_i, r_i) = \begin{cases} t & \text{if } t \leq w_i, \\ w_i + (1 + r_i)(t - w_i) & \text{if } t > w_i. \end{cases}$$

That is, if the payment does not exceed her income, the agent pays the amount directly; if it exceeds her income, she borrows the difference at the interest rate r_i. We call this function the **borrowing cost function**.

Definition 7. A preference relation R_i is **quasi-linear with borrowing cost** if there exist a valuation function $v_i : X \to \mathbb{R}_+$ with $v_i(\mathbf{0}) = 0$, an income level $w_i \in \mathbb{R}_+ \cup \{+\infty\}$, and an interest rate $r_i \in \mathbb{R}_+$ such that for each pair $(x, t), (y, s) \in X \times \mathbb{R}$, $(x, t)\ R_i\ (y, s)$ if and only if $v_i(x) - c(t; w_i, r_i) \geq v_i(y) - c(s; w_i, r_i)$.

Let $\mathcal{R}^{Q,B}$ be the class of quasi-linear preferences with borrowing cost. We assume that each agent's income level and interest rate are her private information. An extreme case is $w_i = +\infty$, in which case the agent never needs to borrow and her preference relation is quasi-linear. Thus, $\mathcal{R}^Q \subseteq \mathcal{R}^{Q,B}$. Moreover, there exist preference relations in $\mathcal{R}^{Q,B}$ that are not essentially quasi-linear.[20] Thus, by Theorem 3,

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[20] For example, consider a preference relation $R_i \in \mathcal{R}^{Q,B}$ with a valuation function v_i and an income level w_i such that for some $x \in X$, $w_i < v_i(x)$. Such a preference relation is not essentially

we obtain the following result.

Corollary 3. *Let $\mathcal{R} = \mathcal{R}^{Q,B}$. No mechanism on $\mathcal{R}^n$ satisfies efficiency, strategy-proofness, individual rationality, and no subsidy for losers.*

6.2 Implications for the public goods model with transfers

Our results have implications for the public goods model with transfers studied by Ma et al. (2018). The formal model is introduced in Appendix F. Ma et al. (2018) introduce the notion of essential quasi-linearity in the public goods model with transfers.[21] Their main result implies that, in the public goods model with transfers, if the domain is larger than the essentially quasi-linear domain, no mechanism satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy*. As a corollary of Theorem 2, we obtain a stronger impossibility result under additional assumptions on the number of agents and alternatives.

Corollary 4. *In the public goods model with transfers, if there are at least three agents and at least six alternatives, no mechanism satisfies efficiency and strategy-proofness on the essentially quasi-linear domain.*

A formal statement of this corollary, its proof, and a comparison with the results of Ma et al. (2018) are provided in Appendix F.

The key idea underling this result is that the package assignment model can be embedded into the public goods model with transfers. Consider the package assignment model with three agents and two identical objects. The set of package allocations (in which all objects are allocated) is given by

$$(0,1,1), (0,2,0), (0,0,2), (1,1,0), (1,0,1), \text{ and } (2,0,0).$$

If there are six alternatives in the public goods model, we can associate each alternative with one of these package allocations. Then, for each preference relation in the package assignment model, we can construct a corresponding preference relation in the public goods model. Theorem 2 then implies that no mechanism satisfies *efficiency* and *strategy-proofness* on the set of preferences in the public goods model that

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quasi-linear.

[21] The essentially quasi-linear domain is called the *parallel domain* in Ma et al. (2018).

corresponds to the essentially quasi-linear domain in the package assignment model. Since this set of preferences is contained in the essentially quasi-linear domain of the public goods model, the impossibility result follows.

Appendix

A Preliminaries

This section has two parts. First, we define several classes of preferences and state lemmas that guarantee the existence of some preferences that we pick in the proofs. Second, we provide lemmas used in the proofs. We relegate the proofs of all the lemmas to Appendix E.

We introduce some notations we use in the proofs. For each $x \in X$, let $m(x) \equiv \sum_{a \in M} x_a$. That is, $m(x)$ is the number of objects in the package x. Let $\mathcal{X} \equiv \{(x, y) \in X \times X : x > y\}$. Let $\mathbf{t} \equiv (t_x)_{x \in X}$ be our generic notation for a vector in $\mathbb{R}^{|X|}$.

A.1 Preferences

We introduce three types of preferences.

Definition 8. A preference relation R_i is **bounded** if there is a pair $\overline{s}, \underline{s} \in \mathbb{R}_{++}$ such that for each $(x, y) \in \mathcal{X}$ and each $t \in \mathbb{R}$, $\underline{s} < V^{R_i}(x, (y, t)) - t < \overline{s}$.

Note that quasi-linear preferences are bounded. Next, we define a quasi-linear preference relation whose valuation function is "negligibly" small compared with that of another preference relation.

Definition 9. Given a preference relation R_i, $R_j \in \mathcal{R}^Q$ is **negligible with respect to** R_i if for each $x \in X \setminus \{\bar{x}\}$, $v_j(x) < \inf_{(y,y') \in \mathcal{X},\ t \in \mathbb{R}} V^{R_i}(y, (y', t)) - t$, and $v_j(\bar{x}) < \inf_{t \in \mathbb{R}} V^{R_i}(\bar{x}, (0, t)) - t$.

Remark 6. In Definition 9, if R_i is quasi-linear, the inequalities are simplified as follows: for each $x \in X \setminus \{\bar{x}\}$, $v_j(x) < \min_{(y,y') \in \mathcal{X}} v_i(y) - v_i(y')$ and $v_j(\bar{x}) < v_i(\bar{x})$.

Given a preference relation R_i, let $\mathcal{R}^Q(R_i)$ be the class of preferences that are negligible with respect to R_i.[22]

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[22] The right-hand side of the two inequalities in Definition 9 might be zero for some preference relation. For such a preference relation R_i, $\mathcal{R}^Q(R_i) = \emptyset$.

Lemma 1. *For each* $R_i \in \mathcal{R}$ *and each* $R'_i \in \mathcal{R}^Q(R_i)$, $\mathcal{R}^Q(R'_i) \subseteq \mathcal{R}^Q(R_i)$.

Lemma 2. *Let* $i \in N$ *and* $j \in N \setminus \{i\}$. *Let* $R \in \mathcal{R}^n$ *be such that* $R_j \in \mathcal{R}^Q$ *and* $R_{-i,j} \in (\mathcal{R}^Q(R_j))^{n-2}$. *For each* $x \in X$, $\sigma_i(R_{-i}; x) = v_j(\bar{x} - x)$.

Definition 10. Given a vector $\mathbf{t} \in \mathbb{R}^{|X|}$ and $x \in X$, a preference relation R_i is a **monotonic transformation of t at** x if for each $y \in X$ with $y \neq x$, $V^{R_i}(y, (x, t_x)) < t_y$.

In this figure, the vector $\mathbf{t}$ consists of the points on the dotted kinked line. Definition 10 requires that the bundles that are indifferent to (x, t_x) should be to the left of the dotted kinked line. Given a vector $\mathbf{t} \in \mathbb{R}^{|X|}$ and $x \in X$, let $\mathcal{R}^{MT}_{\mathbf{t},x}$ be the set of preferences that are monotonic transformations of $\mathbf{t}$ at x.

A vector $\mathbf{t} \in \mathbb{R}^{|X|}$ is **object monotonic** if for each $(x, y) \in \mathcal{X}$, $t_x > t_y$. In some proofs, we pick an essentially quasi-linear preference relation that is a monotonic transformation of two object monotonic vectors. The following three lemmas give sufficient conditions for the existence of such preferences.

Lemma 3. *Let* $x \in X$. *Let* $\mathbf{t}, \mathbf{s} \in \mathbb{R}^{|X|}$ *be object monotonic vectors such that* $0 \leq t_x < s_x$ *and* $s_{\mathbf{0}} > 0$. *Then, for each* $y \in X \setminus \{x\}$, *there is a bounded preference relation* $R_i \in \mathcal{R}^E$ *such that* $R_i \in \mathcal{R}^{MT}_{\mathbf{t},x} \cap \mathcal{R}^{MT}_{\mathbf{s},y}$.

Lemma 4. *Let* $x \in X$. *Let* $\mathbf{t}, \mathbf{s} \in \mathbb{R}^{|X|}$ *be object monotonic vectors such that* $t_x < s_x$ *and* $t_x < 0$. *Then, for each* $y \in X \setminus \{x\}$, *there is a bounded preference relation* $R_i \in \mathcal{R}^E$ *such that* $R_i \in \mathcal{R}^{MT}_{\mathbf{t},x} \cap \mathcal{R}^{MT}_{\mathbf{s},y}$.

Lemma 5. *Let* $\mathbf{t}, \mathbf{s} \in \mathbb{R}^{|X|}$ *be object monotonic vectors such that* $t_{\bar{x}} < s_{\bar{x}}$ *and* $s_{\mathbf{0}} < 0$. *Then, there is* $R_i \in \mathcal{R}^E$ *such that* $R_i \in \mathcal{R}^{MT}_{\mathbf{t},\bar{x}} \cap \mathcal{R}^{MT}_{\mathbf{s},\mathbf{0}}$.

A.2 Implications of properties of mechanisms

First we state three lemmas related to *efficiency*.

Lemma 6. *Let* f *be an efficient mechanism on* $\mathcal{R}^n$. *Let* $R \in \mathcal{R}^n$ *and* $N' \subseteq N$. *Let* $(x_i)_{i \in N'} \in X^{|N'|}$ *be such that* $\sum_{i \in N'} x_i \leq \sum_{i \in N'} x^f_i(R)$. *Then,* $\sum_{i \in N'} V^{R_i}(x_i, f_i(R)) \leq \sum_{i \in N'} t^f_i(R)$.

The next lemma states that assigning an object to a negligible agent is inefficient.

Lemma 7. *Let* f *be an efficient mechanism on* $\mathcal{R}^n$. *Let* $R \in \mathcal{R}^n$ *and* $i \in N$. *If*

$R_i \in \mathcal{R}^Q(R_j)$ for some $j \in N \setminus \{i\}$, then $x_i^f(R) = \mathbf{0}$.

We introduce the notion of option set. Given a mechanism f, $i \in N$ and $R_{-i} \in \mathcal{R}^{n-1}$, the **option set of agent i under f for R_{-i}** is defined as

$$o_i^f(R_{-i}) \equiv \{z \in X \times \mathbb{R} : \exists R_i \in \mathcal{R} \text{ s.t. } f_i(R_i, R_{-i}) = z\},$$

and let $X_i^f(R_{-i}) \equiv \{x \in X : \exists R_i \in \mathcal{R} \text{ s.t. } x_i^f(R_i, R_{-i}) = x\}$. That is, $o_i^f(R_{-i})$ is the set of bundles available to agent i under f when the other agents report R_{-i}. Similarly, $X_i^f(R_{-i})$ is the set of packages available to agent i under f when the other agents report R_{-i}.

The following lemma states that any package is available to an agent under an *efficient* mechanism if the domain contains the quasi-linear domain and the preferences of the other agents are bounded.

Lemma 8. *Let $\mathcal{R}$ be such that $\mathcal{R} \supseteq \mathcal{R}^Q$ and f be an efficient mechanism on $\mathcal{R}^n$. Let $i \in N$ and $R_{-i} \in \mathcal{R}^{n-1}$ be such that for each $j \in N \setminus \{i\}$, R_j is bounded. Then, $X_i^f(R_{-i}) = X$.*

Next, we state a fact and lemmas related to *strategy-proofness*. We begin with the notion of monotonicity.

Definition 11. A mechanism f on $\mathcal{R}^n$ is **monotonic** if for each $i \in N$, each pair $R_i, R_i' \in \mathcal{R}$, and each $R_{-i} \in \mathcal{R}^{n-1}$, $V^{R_i}(x_i^f(R_i', R_{-i}), f_i(R_i, R_{-i})) \leq V^{R_i'}(x_i^f(R_i', R_{-i}), f_i(R_i, R_{-i}))$.

Remark 7. If R_i and R_i' are quasi-linear, the inequality in Definition 11 is equivalent to $v_i(x_i^f(R_i', R_{-i})) - v_i(x_i^f(R_i, R_{-i})) \leq v_i'(x_i^f(R_i', R_{-i})) - v_i'(x_i^f(R_i, R_{-i}))$.

Monotonicity is known to be a necessary condition for *strategy-proofness*.

Fact 2. *(Kazumura et al., 2020) Each strategy-proof mechanism is monotonic.*

Given a *strategy-proof* mechanism f, $i \in N$, and $R_{-i} \in \mathcal{R}^{n-1}$, for each pair $(x, t), (y, s) \in o_i^f(R_{-i})$, $x = y$ implies $t = s$. Thus, for a *strategy-proof* mechanism f, we can define a mapping $t_i^f(R_{-i}, \cdot) : X_i^f(R_{-i}) \to \mathbb{R}$ such that for each $x \in X_i^f(R_{-i})$, $(x, t_i^f(R_{-i}, x)) \in o_i^f(R_{-i})$. Given a *strategy-proof* mechanism f and $x \in X_i^f(R_{-i})$, let $z_i^f(R_{-i}, x) \equiv (x, t_i^f(R_{-i}, x))$.

Lemma 9. *Let f be a strategy-proof mechanism on $\mathcal{R}^n$. Let $i \in N$ and $R_{-i} \in \mathcal{R}^{n-1}$*

be such that $X_i^f(R_{-i}) = X$*. Then, the vector* $(t_i^f(R_{-i};x))_{x\in X}$ *is object monotonic.*

Lemma 10. *Let* f *be a strategy-proof mechanism on* $\mathcal{R}^n$*. Let* $i \in N$ *and* $R_{-i} \in \mathcal{R}^{n-1}$ *be such that* $X_i^f(R_{-i}) = X$*. Denote* $\mathbf{t} = (t_i^f(R_{-i}, x))_{x\in X}$*. Let* $x \in X$ *and* $R_i \in \mathcal{R}^{MT}_{\mathbf{t},x}$*. Then,* $x_i^f(R_i, R_{-i}) = x$*.*

Finally we provide a fact and a lemma derived from *efficiency* and *strategy-proofness*. If $\mathcal{R} \supseteq \mathcal{R}^Q$, then by Fact 1, an *efficient* and *strategy-proof* mechanism on $\mathcal{R}^n$ coincides with a Groves mechanism on the quasi-linear domain.

Fact 3. *Let* $\mathcal{R}$ *be such that* $\mathcal{R} \supseteq \mathcal{R}^Q$ *and* f *be an efficient and strategy-proof mechanism on* $\mathcal{R}^n$*. For each* $i \in N$*, there is* $h_i : (\mathcal{R}^Q)^{n-1} \to \mathbb{R}$ *such that for each* $R \in \mathcal{R}^n$ *with* $R_{-i} \in (\mathcal{R}^Q)^{n-1}$*,* $t_i^f(R) = h_i(R_{-i}) - \sigma_i(R_{-i}; x_i^f(R))$*.*

The following lemma states that in some specific situations, *efficiency* and *strategy-proofness* give a range of the payment of an agent who receives no object.

Lemma 11. *Let* $\mathcal{R} = \mathcal{R}^E$ *and* f *be an efficient and strategy-proof mechanism on* $\mathcal{R}^n$*.*
(i) Assume $n = 2$*. Let* $i \in N$*,* $j \in N \setminus \{i\}$*, and* $R_j \in \mathcal{R}^Q$*. Then,* $t_i^f(R_j; \mathbf{0}) = 0$*.*
(ii) Assume $n \geq 3$*. Let* $i \in N$*,* $j \in N \setminus \{i\}$*, and* $R_{-i} \in \mathcal{R}^{n-1}$ *be such that* $R_j \in \mathcal{R}^Q$ *and* $R_{-i,j} \in (\mathcal{R}^Q(R_j))^{n-2}$*. Denote*

$$s^* \equiv \max\{\sum_{k\in N\setminus\{i,j\}} v_k(x_k) : (x_k)_{k\in N} \in (X \setminus \{\bar{x}\})^n,\ \sum_{k\in N} x_k \leq \bar{x}\}.$$

Then, $-s^* \leq t_i^f(R_{-i}; \mathbf{0}) \leq 0$*.*

Lemma 11 (i) states that when $n = 2$ and the domain is the essentially quasi-linear domain, the payment of a loser is zero as long as the other agent's preference relation is quasi-linear. The s^* in Lemma 11 (ii) is the maximum of the sum of valuations that the agents other than the agents i and j can achieve under the assumption that no agent receives all the objects. Lemma 11 (ii) states that when $n \geq 3$ and the domain is the essentially quasi-linear domain, the payment of a loser is at least $-s^*$ and at most zero at preference profiles that satisfy the conditions specified in Lemma 11 (ii).

B Proof of Theorem 1

We prove only Theorem 1 (i) because Theorem 1 (ii) is immediate from Theorem 1 (i) and Proposition 1. In the proof of Theorem 1 (i), whenever we take an agent i, the other agent is denoted by j.

Proof of Theorem 1 (i). The proof consists of two parts.

If part. Let $\mathcal{R} \equiv \mathcal{R}^E$ and f be a generalized Vickrey mechanism on $\mathcal{R}^2$. First, we show the following claim.

Claim 1. *Let $R \in \mathcal{R}^n$, $i \in N$, and $x \in X$. If $V^{R_i}(x, f_i(R)) \geq 0$, then $V^{R_i}(x, f_i(R)) - t_i^f(R) = V^{R_i}(x, (\mathbf{0}, 0)) - V^{R_i}(x_i^f(R), (\mathbf{0}, 0))$.*

Proof. By the definition of f, $t_i^f(R) \geq 0$. We also have

$$\begin{aligned} V^{R_i}(x_i^f(R), (\mathbf{0}, 0)) - t_i^f(R) &= V^{R_i}(x_i^f(R), (\mathbf{0}, 0)) - (\sigma_i(R_{-i}; \mathbf{0}) - \sigma_i(R_{-i}; x_i^f(R))) \\ &= \sum_{j \in N} V^{R_j}(x_j^f(R), (\mathbf{0}, 0)) - \sigma_i(R_{-i}; \mathbf{0}). \end{aligned}$$

Thus, by $(x_j^f(R))_{j \in N} \in \text{argmax}_{(x_j)_{j \in N} \in A} V^{R_j}(x_j, (\mathbf{0}, 0))$, $V^{R_i}(x_i^f(R), (\mathbf{0}, 0)) - t_i^f(R) \geq 0$, implying $f_i(R) \; R_i \; (\mathbf{0}, 0)$. Thus, $f_i(R) \in X(R_i)$. Hence, by $R_i \in \mathcal{R}^E$ and $V^{R_i}(x, f_i(R)) \geq 0$, $V^{R_i}(x, f_i(R)) - t_i^f(R) = V^{R_i}(x, (\mathbf{0}, 0)) - V^{R_i}(x_i^f(R), (\mathbf{0}, 0))$. ☐

Strategy-proofness. Without loss of generality, we focus only on agent 1. Let $R \in \mathcal{R}^n$ and $R_1' \in \mathcal{R}$, and denote $R' \equiv (R_1', R_{-1})$. We show $f_1(R) \; R_1 \; f_1(R')$. Since $t_1^f(R') \geq 0$ by the definition of f, if $V^{R_1}(x_1^f(R'), f_1(R)) < 0$, then clearly $f_1(R) \; R_1 \; f_1(R')$. Thus, suppose $V^{R_1}(x_1^f(R'), f_1(R)) \geq 0$. By $(x_i^f(R))_{i \in N} \in \text{argmax}_{(x_i)_{i \in N} \in A} V^{R_i}(x_i, (\mathbf{0}, 0))$,

$$V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) + \sigma_1(R_{-1}; x_1^f(R)) \geq V^{R_1}(x_1^f(R'), (\mathbf{0}, 0)) + \sigma_1(R_{-1}; x_1^f(R')) \quad (1)$$

By this inequality and Claim 1,

$$\begin{aligned}
&V^{R_1}(x_1^f(R'), f_1(R)) \\
&\quad = t_1^f(R) + V^{R_1}(x_1^f(R'), (\mathbf{0}, 0)) - V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) && \text{(by Claim 1)} \\
&\quad \leq t_1^f(R) + \sigma_1(R_{-1}; x_1^f(R)) - \sigma_1(R_{-1}; x_1^f(R')) && \text{(by (1))} \\
&\quad = \sigma_1(R_{-1}; \mathbf{0}) - \sigma_1(R_{-1}; x_1^f(R)) + \sigma_1(R_{-1}; x_1^f(R)) - \sigma_1(R_{-1}; x_1^f(R')) \\
&\quad = \sigma_1(R_{-1}; \mathbf{0}) - \sigma_1(R_{-1}; x_1^f(R')) \\
&\quad = t_1^f(R'). && \text{(by the definition of } f)
\end{aligned}$$

This implies $f_1(R)\ R_1\ f_1(R')$.

Efficiency. Assume for contradiction that $f(R)$ is not *efficient* for some $R \in \mathcal{R}^2$. By Remark 3, there is $((y_1, s_1), (y_2, s_2)) \in Z$ such that

$$(y_i, s_i)\ I_i\ f_i(R) \text{ for each } i \in \{1, 2\} \text{ and } s_1 + s_2 > t_1^f(R) + t_2^f(R). \tag{2}$$

Note that for each $i \in \{1, 2\}$, $s_i = V^{R_i}(y_i, f_i(R))$. Since f is a generalized Vickrey mechanism, for each $i \in \{1, 2\}$, $t_i^f(R) \geq 0$. Thus, by (2), either $s_1 > 0$ or $s_2 > 0$. Without loss of generality, assume $s_1 > 0$. There are two cases.

Case 1. $s_2 > 0$. By $(x_i^f(R))_{i \in N} \in \text{argmax}_{(x_i)_{i \in N} \in A} V^{R_i}(x_i, (\mathbf{0}, 0))$ and Claim 1,

$$\begin{aligned}
&t_1(R) + t_2(R) \\
&\quad = s_1 + V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) - V^{R_1}(y_1, (\mathbf{0}, 0)) + s_2 + V^{R_2}(x_2^f(R), (\mathbf{0}, 0)) - V^{R_2}(y_2, (\mathbf{0}, 0)) \\
&\quad \geq s_1 + s_2,
\end{aligned}$$

contradicting (2).

Case 2. $s_2 \leq 0$. By the definition of f, $t_2^f(R) = V^{R_1}(\bar{x}, (\mathbf{0}, 0)) - V^{R_1}(x_1^f(R), (\mathbf{0}, 0))$. Thus, by $s_1 > 0$ and Claim 1,

$$\begin{aligned}
&t_1^f(R) + t_2^f(R) \\
&\quad = s_1 + V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) - V^{R_1}(y_1, (\mathbf{0}, 0)) + V^{R_1}(\bar{x}, (\mathbf{0}, 0)) - V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) \\
&\quad = s_1 + V^{R_1}(\bar{x}, (\mathbf{0}, 0)) - V^{R_1}(y_1, (\mathbf{0}, 0)) \\
&\quad \geq s_1 + s_2,
\end{aligned}$$

where the last inequality follows from $s_2 \leq 0$ and $V^{R_1}(\bar{x}, (\mathbf{0}, 0)) \geq V^{R_1}(y_1, (\mathbf{0}, 0))$.

This inequality contradicts (2).

Only if part. Let $\mathcal{R} = \mathcal{R}^E$ and let f be a mechanism on $\mathcal{R}^2$ that satisfies *efficiency* and *strategy-proofness*. We do the proof in five steps.

Step 1. *Let $i \in N$ and $R_j \in \mathcal{R}^Q$. Then, for each $x \in X$, $t_i^f(R_j; x) = v_j(\bar{x}) - v_j(\bar{x} - x)$.*[23]

Proof. Without loss of generality, assume $i = 1$. By Fact 3 and $R_2 \in \mathcal{R}^Q$, there exists $h_1 : \mathcal{R}^Q \to \mathbb{R}$ such that for each $x \in X$, $t_1^f(R_2; x) = h_1(R_2) - \sigma_1(R_2; x)$. Note that for each $x \in X$, $\sigma_1(R_2; x) = v_2(\bar{x} - x)$. Further, since R_2 is bounded, Lemma 8 implies $\mathbf{0} \in X_1^f(R_2)$. Thus, by Lemma 11 (i), $h_1(R_2) - v_2(\bar{x}) = t_1^f(R_2; \mathbf{0}) = 0$, which implies $h_1(R_2) = v_2(\bar{x})$. Hence, we obtain the desired result. ■

Step 2. *Let $i \in \{1, 2\}$ and $R \in \mathcal{R}^2$ be such that $R_j \in \mathcal{R}^Q$. Then, $(x_i^f(R), x_j^f(R)) \in$ $\mathrm{argmax}_{(x_1,x_2) \in A} V^{R_i}(x_i, (\mathbf{0}, 0)) + v_j(x_j)$.*

Proof. Without loss of generality, assume $i = 1$. By Step 1 and $R_2 \in \mathcal{R}^Q$, $(\mathbf{0}, 0) \in o_1^f(R_2)$. Thus, by the *strategy-proofness* of f, $f_1(R)\ R_i\ (\mathbf{0}, 0)$. Further, by Step 1 and $R_2 \in \mathcal{R}^Q$, $t_1^f(R) = v_2(\bar{x}) - v_2(x_2^f(R)) \geq 0$. Thus, $f_1(R) \in X(R_1)$.

Let $(y_1, y_2) \in A$. We show $V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) + v_2(x_2^f(R)) \geq V^{R_1}(y_1, (\mathbf{0}, 0)) - v_2(y_2)$.

Case 1. $V^{R_1}(y_1, f_1(R)) \geq 0$. By Step 1 and the *strategy-proofness* of f, $f_1(R)\ R_i$ $(y_1, v_2(\bar{x}) - v_2(y_2))$, which implies $V^{R_1}(y_1, f_1(R)) \leq v_2(\bar{x}) - v_2(y_2)$. By this inequality and $t_1^f(R) = v_2(\bar{x}) - v_2(x_2^f(R))$, $V^{R_1}(y_1, f_1(R)) - t_1^f(R) \leq v_2(x_2^f(R)) - v_2(y_2)$.

By $f_1(R) \in X(R_1)$, $V^{R_1}(y_1, f_1(R)) \geq 0$, and $R_1 \in \mathcal{R}^E$, $V^{R_1}(y_1, (\mathbf{0}, 0)) - V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) = V^{R_1}(y_1, f_1(R)) - t_1^f(R)$.

Therefore, $V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) + v_2(x_2^f(R)) \geq V^{R_1}(y_1, (\mathbf{0}, 0)) - v_2(y_2)$.

Case 2. $V^{R_1}(y_1, f_1(R)) < 0$. By the object monotonicity of R_1, $(y_1, 0) \in X(R_1)$. By $V^{R_1}(y_1, f_1(R)) < 0$, $f_1(R)\ P_1\ (y_1, 0)$. This implies $V^{R_1}(x_1^f(R), (y_1, 0)) > t_1^f(R) \geq 0$. Thus, by $R_1 \in \mathcal{R}^E$, $V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) - V^{R_1}(y_1, (\mathbf{0}, 0)) = V^{R_1}(x_1^f(R), (y_1, 0)) > t_1^f(R)$.

By $t_1^f(R) = v_2(\bar{x}) - v_2(x_2^f(R))$ and $v_2(\bar{x}) - v_2(y_2) \geq 0$, $v_2(x_2^f(R)) - v_2(y_2) = -(v_2(\bar{x}) - v_2(x_2^f(R))) + v_2(\bar{x}) - v_2(y_2) \leq t_1^f(R)$.

Therefore, $V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) + v_2(x_2^f(R)) > V^{R_1}(y_1, (\mathbf{0}, 0)) + v_2(y_2)$. ■

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[23] It is guaranteed by Lemma 8 that $X_i^f(R_j) = X$.

Remark 8. By Step 2, for each $i \in N$, each $R_j \in \mathcal{R}$, and each $x \in X$, there is $R_i \in \mathcal{R}^Q$ such that $x_i^f(R_i, R_j) = x$. Hence, for each $i \in N$ and each $R_j \in \mathcal{R}$, $X_i^f(R_j) = X$.

Step 3. *Let $i \in \{1, 2\}$ and $R \in \mathcal{R}^2$. Then, $t_i^f(R) = V^{R_j}(\bar{x}, (\mathbf{0}, 0)) - V^{R_j}(x_j^f(R), (\mathbf{0}, 0)) + t_i^f(R_j; \mathbf{0})$.*

Proof. Without loss of generality, assume $i = 1$ and $x_1^f(R) \neq \mathbf{0}$. Further, by Remark 8, we can assume $R_1 \in \mathcal{R}^Q$ without loss of generality. Assume for contradiction that $t_1^f(R) \neq V^{R_2}(\bar{x}, (\mathbf{0}, 0)) - V^{R_2}(x_2^f(R), (\mathbf{0}, 0)) + t_1^f(R_2; \mathbf{0})$.

Let

$$\delta_{\min} \equiv \min\{t_1^f(R) - t_1^f(R_2; \mathbf{0}), V^{R_2}(\bar{x}, (\mathbf{0}, 0)) - V^{R_2}(x_2^f(R), (\mathbf{0}, 0))\}, \text{and}$$

$$\delta_{\max} \equiv \max\{t_1^f(R) - t_1^f(R_2; \mathbf{0}), V^{R_2}(\bar{x}, (\mathbf{0}, 0)) - V^{R_2}(x_2^f(R), (\mathbf{0}, 0))\}.$$

Clearly, $\delta_{\min} < \delta_{\max}$. By $x_1^f(R) \neq \mathbf{0}$ and $X_1^f(R_2) = X$, $t_1^f(R) > t_1^f(R_2; \mathbf{0})$.[24] By object monotonicity, $V^{R_2}(\bar{x}, (\mathbf{0}, 0)) - V^{R_2}(x_2^f(R), (\mathbf{0}, 0)) \geq 0$. Thus, $\delta_{\min} \geq 0$. Let $d \in \mathbb{R}_{++}$ be such that $\delta_{\min} < d < \delta_{\max}$.

Let $(\epsilon_x)_{x \in X} \in \mathbb{R}_+^{|X|}$ be an object monotonic vector that is sufficiently close to 0.[25] Note that since $(\epsilon_x)_{x \in X}$ is object monotonic, for each $x \in X \setminus \{\mathbf{0}\}$, $\epsilon_x > 0$. Let $R_1' \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$v_1'(x) = \begin{cases} d + \epsilon_x & \text{if } x \geq x_1^f(R), \\ \epsilon_x & \text{otherwise.} \end{cases}$$

Since $(\epsilon_x)_{x \in X}$ is object monotonic, R_1' is object monotonic. Denote $R' \equiv (R_1', R_2)$.

Claim 1. $x_1^f(R') = \mathbf{0}$.

Proof. Assume for contradiction that $x_1^f(R') \neq \mathbf{0}$. We have three cases.

Case 1. $x_1^f(R') = x_1^f(R)$. By *strategy-proofness*, $f_1(R') = f_1(R)$. Since $d < \delta_{\max}$ and $\epsilon_{x_1^f(R)}$ is sufficiently close to 0, $v_1'(x_1^f(R)) = d + \epsilon_{x_1^f(R)} < \delta_{\max}$. If $\delta_{\max} = t_1^f(R) - t_1^f(R_2; \mathbf{0})$, then $v_1'(x_1^f(R)) - t_1^f(R) < -t_1^f(R_2; \mathbf{0}))$. This implies $z_1^f(R_2; \mathbf{0})\ P_1'\ f_1(R) =$

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[24] By $X_1^f(R_2) = X$, there is $R_1' \in \mathcal{R}$ such that $x_1^f(R_1', R_2) = \mathbf{0}$. If $t_1^f(R) \leq t_1^f(R_2; \mathbf{0})$, then by the object monotonicity of R_1', $f_1(R)\ P_1'\ (\mathbf{0}, t_1^f(R_2; \mathbf{0})) = f_1(R_1', R_2)$, contradicting *strategy-proofness*. Thus, $t_1^f(R) > t_1^f(R_2; \mathbf{0})$.

[25] For example, take $(\epsilon_x)_{x \in X}$ that satisfies $\epsilon_{\mathbf{0}} = 0$ and $\epsilon_{\bar{x}} < \min\{\delta_{\max} - d, \min_{(x,y) \in \mathcal{X}} v_1(x) - v_1(y), \min_{(x,y) \in \mathcal{X}} V^{R_2}(x, (\mathbf{0}, 0)) - V^{R_2}(y, (\mathbf{0}, 0))\}$. Then the proof of Step 3 works with this $(\epsilon_x)_{x \in X}$.

$f_1(R')$, contradicting *strategy-proofness*. If $\delta_{\max} = V^{R_2}(\bar{x}, (\mathbf{0}, 0)) - V^{R_2}(x_2^f(R), (\mathbf{0}, 0))$, then $v_1'(\mathbf{0}) + V^{R_2}(\bar{x}, (\mathbf{0}, 0)) > v_1'(x_1^f(R)) + V^{R_2}(x_2^f(R), (\mathbf{0}, 0)) = v_1'(x_1^f(R')) + V^{R_2}(x_2^f(R'), (\mathbf{0}, 0))$, contradicting Step 2.

Case 2. $x_1^f(R') > x_1^f(R)$. Since $\epsilon_{x_1^f(R')}$ and $\epsilon_{x_1^f(R)}$ are sufficiently close to 0, $v_1'(x_1^f(R')) - v_1'(x_1^f(R)) = \epsilon_{x_1^f(R')} - \epsilon_{x_1^f(R)} < v_1(x_1^f(R')) - v_1(x_1^f(R))$. This contradicts Fact 2.

Case 3. $x_1^f(R) \not\leq x_1^f(R')$. By $x_1^f(R') \neq \mathbf{0}$, $x_2^f(R') \neq \bar{x}$. Thus, since $\epsilon_{x_1^f(R')}$ is sufficiently small, $v_1'(x_1^f(R')) = \epsilon_{x_1^f(R')} < V^{R_2}(\bar{x}, (\mathbf{0}, 0)) - V^{R_2}(x_2^f(R'), (\mathbf{0}, 0))$. This implies $v_1'(\mathbf{0}) + V^{R_2}(\bar{x}, (\mathbf{0}, 0)) > v_1'(x_1^f(R')) + V^{R_2}(x_2^f(R'), (\mathbf{0}, 0))$, contradicting Step 2. □

We derive a contradiction for each of the following two cases.

Case 1. $\delta_{\min} = t_1^f(R) - t_1^f(R_2; \mathbf{0})$. By $d > \delta_{\min}$, $\epsilon_{x_1^f(R)} > 0$, and Claim 1, $v_1'(x_1^f(R)) - t_1^f(R) = d + \epsilon_{x_1^f(R)} - t_1^f(R) > \delta_{\min} - t_1^f(R) = -t_1^f(R_2; \mathbf{0}) = v_1'(x_1^f(R')) - t_1^f(R')$. This inequality implies $f_1(R)\ P_1'\ f_1(R')$, which contradicts *strategy-proofness*.

Case 2. $\delta_{\min} = V^{R_2}(\bar{x}, (\mathbf{0}, 0)) - V^{R_2}(x_2^f(R), (\mathbf{0}, 0))$. By Claim 1, $x_2^f(R') = \bar{x}$. Since $v_1'(x_1^f(R)) > d > \delta_{\min} = V^{R_2}(\bar{x}, (\mathbf{0}, 0)) - V^{R_2}(x_2^f(R), (\mathbf{0}, 0))$, $v_1'(x_1^f(R)) + V^{R_2}(x_2^f(R), (\mathbf{0}, 0)) > v_1'(x_1^f(R')) + V^{R_2}(x_2^f(R'), (\mathbf{0}, 0))$. This contradicts Step 2. ■

Step 4. *Let* $i \in \{1, 2\}$ *and* $R \in \mathcal{R}^2$. *Then,* $t_i^f(R) = V^{R_j}(\bar{x}, (\mathbf{0}, 0)) - V^{R_j}(x_j^f(R), (\mathbf{0}, 0))$.

Proof. Without loss of generality, assume $i = 1$. By Step 3, it suffices to show $t_1^f(R_2; \mathbf{0}) = 0$. Assume for contradiction that $t_1^f(R_2; \mathbf{0}) \neq 0$.

Claim 1. *There is a bounded* $R_1' \in \mathcal{R}$ *such that* $t_2^f(R_1'; \mathbf{0}) \neq 0$.

Proof. Let $\epsilon \in \mathbb{R}_{++}$ be such that $\epsilon < |t_1^f(R_2; \mathbf{0})|$. Let $R_2' \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$, $v_2'(x) = V^{R_2}(x, (\mathbf{0}, 0)) + \epsilon$.

Let $\mathbf{t}, \mathbf{s} \in \mathbb{R}^{|X|}$ be such that for each $x \in X$, $t_x = t_1^f(R_2; x)$ and $s_x = t_1^f(R_2'; x)$. By Remark 8, they are well-defined, and by Lemma 9, are object monotonic.

By Step 1 and $R_2' \in \mathcal{R}^Q$, $s_{\mathbf{0}} = 0$ and $s_{\bar{x}} = v_2'(\bar{x})$. If $t_{\mathbf{0}} < 0$, then $t_{\mathbf{0}} < s_{\mathbf{0}}$, and thus, $\mathbf{t}$ and $\mathbf{s}$ satisfy the condition of Lemma 4 for $\mathbf{0}$. On the other hand, if $t_{\mathbf{0}} > 0$, then by Step 3, $\epsilon < t_{\mathbf{0}}$, and $s_{\bar{x}} = v_2'(\bar{x})$, $t_{\bar{x}} = V^{R_2}(\bar{x}, (\mathbf{0}, 0)) + t_{\mathbf{0}} > V^{R_2}(\bar{x}, (\mathbf{0}, 0)) + \epsilon = v_2'(\bar{x}) = s_{\bar{x}}$. Hence, in this case, $\mathbf{t}$ and $\mathbf{s}$ satisfy the condition of Lemma 3 for $\bar{x}$.

Therefore, by Lemmas 3 and 4, there is a bounded $R_1' \in \mathcal{R}^E$ such that $R_1' \in \mathcal{R}^{MT}_{\mathbf{t},\mathbf{0}} \cap \mathcal{R}^{MT}_{\mathbf{s},\bar{x}}$. By Lemma 10, $x_1^f(R_1', R_2) = \mathbf{0}$ and $x_1^f(R_1', R_2') = \bar{x}$. Therefore, $x_2^f(R_1', R_2) =$

$\bar{x}$ and $x_2^f(R_1', R_2') = \mathbf{0}$.

Now, we show $t_2^f(R_1';\mathbf{0}) \neq 0$. Assume for contradiction that $t_2^f(R_1';\mathbf{0}) = 0$. By $x_2^f(R_1', R_2') = \mathbf{0}$, $f_2(R_1', R_2') = (\mathbf{0}, 0)$. However, by the definition of R_2', $V^{R_2}(\bar{x}, (\mathbf{0}, 0)) < v_2'(\bar{x})$. By $f_2(R_1', R_2') = (\mathbf{0}, 0)$ and $x_2^f(R_1', R_2) = \bar{x}$, this inequality contradicts Fact 2. □

Since R_1' is bounded by Claim 1, there are $\overline{s}, \underline{s} \in \mathbb{R}_{++}$ such that for each $(x, y) \in \mathcal{X}$ and each $t \in \mathbb{R}$, $\underline{s} < V^{R_1'}(x, (y, t)) - t < \overline{s}$. Let $\mathbf{t} \in \mathbb{R}^{|X|}$ be such that for each $x \in X$, $t_x = t_2^f(R_1', x)$. By Remark 8, $\mathbf{t}$ is well-defined, and by Lemma 9, is object monotonic. There are two cases.

Case 1. $t_{\mathbf{0}} < 0$. Let $R_1'' \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$, $v_1''(x) > \overline{s}$. Let $\mathbf{s} \in \mathbb{R}^{|X|}$ be such that for each $x \in X$, $s_x = t_2^f(R_1''; x)$. By Remark 8, $\mathbf{s}$ is well-defined, and by Lemma 9, is object monotonic.

By Step 1 and $R_1'' \in \mathcal{R}^Q$, $s_{\mathbf{0}} = 0 > t_{\mathbf{0}}$. Thus, $\mathbf{t}$ and $\mathbf{s}$ satisfy the condition of Lemma 4 for $\mathbf{0}$. Therefore, by Lemma 4, there is $R_2'' \in \mathcal{R}^E$ such that $R_2'' \in \mathcal{R}^{MT}_{\mathbf{t},\mathbf{0}} \cap \mathcal{R}^{MT}_{\mathbf{s},\bar{x}}$. By Lemma 10, $x_2^f(R_1', R_2'') = \mathbf{0}$ and $x_2^f(R_1'', R_2'') = \bar{x}$. Therefore, $x_1^f(R_1', R_2'') = \bar{x}$ and $x_1^f(R_1'', R_2'') = \mathbf{0}$. However, $V^{R_1'}(\bar{x}, f_1(R_1'', R_2'')) - t_1^f(R_1'', R_2'') < \overline{s} < v_1''(\bar{x})$. This contradicts Fact 2.

Case 2. $t_{\mathbf{0}} > 0$. Let $R_1'' \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$, $v_1''(x) < \underline{s}$. Let $\mathbf{s} \in \mathbb{R}^{|X|}$ be such that for each $x \in X$, $s_x = t_2^f(R_1''; x)$. By Remark 8, $\mathbf{s}$ is well-defined, and by Lemma 9, is object monotonic.

By Step 1 and $R_1'' \in \mathcal{R}^Q$, $s_{\mathbf{0}} = 0 < t_{\mathbf{0}}$. Thus, the pair t, s satisfy the condition of Lemma 3 for $\mathbf{0}$. Therefore, by Lemma 3, there is $R_2'' \in \mathcal{R}^E$ such that $R_2'' \in \mathcal{R}^{MT}_{\mathbf{t},\bar{x}} \cap \mathcal{R}^{MT}_{\mathbf{s},\mathbf{0}}$. By Lemma 10, $x_2^f(R_1', R_2'') = \bar{x}$ and $x_2^f(R_1'', R_2'') = \mathbf{0}$. Therefore, $x_1^f(R_1', R_2'') = \mathbf{0}$ and $x_1^f(R_1'', R_2'') = \bar{x}$. However, $V^{R_1'}(\bar{x}, f_1(R_1', R_2'')) - t_1^f(R_1', R_2'') > \underline{s} > v_1''(\bar{x})$. This contradicts Fact 2. ■

Step 5. *Completing the proof.*

Using Step 4 and following the proof of Step 2, we can show that for each $R \in \mathcal{R}^2$, $V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) + V^{R_2}(x_2^f(R), (\mathbf{0}, 0)) = \max_{(x_1, x_2) \in A} V^{R_1}(x_1, (\mathbf{0}, 0)) + V^{R_2}(x_2, (\mathbf{0}, 0))$. Hence, by Step 4, we obtain the desired result. ■

C Proof of Theorem 2

Let $\mathcal{R} \equiv \mathcal{R}^E$. Assume for contradiction that there is a mechanism f on $\mathcal{R}^n$ that satisfies *efficiency* and *strategy-proofness.*

We first define three vectors that will be used in the proof. Let $(\epsilon_x)_{x \in X}, (\epsilon'_x)_{x \in X}, (\epsilon''_x)_{x \in X} \in \mathbb{R}_+^{|X|}$ be object monotonic vectors that are sufficiently close to $\mathbf{0}$ and satisfy the following: For each $x \in X$ and each $(y, y') \in \mathcal{X}$,[26]

$$\epsilon'_x < \epsilon_y - \epsilon_{y'} \text{ and } \epsilon''_x < \epsilon'_y - \epsilon'_{y'}. \tag{3}$$

We do the proof in seven steps.

Step 1. *Construction of a preference profile..*

Let $x_1^* \in X \setminus \{\mathbf{0}, \bar{x}\}$ and $x_2^* \equiv \bar{x} - x_1^*$. For each $i \in \{1, 2\}$, we define a preference relation R_i as follows. For each $t \in \mathbb{R}$ and each $x \in X \setminus \{\mathbf{0}\}$, let

$$V^{R_i}(x, (\mathbf{0}, t)) = \begin{cases} 58\alpha + 42 + t & \text{if } x = \bar{x}, \\ 58\alpha + 22 + \epsilon_x + t & \text{if } x_i^* < x \neq \bar{x}, \\ 58\alpha + 22 + t & \text{if } x = x_i^*, \\ \epsilon_x + t & \text{otherwise,} \end{cases}$$

where α is defined by $\alpha \equiv \text{med}\{0, t+2, 1\}$.[27] Note that $\alpha \in [0, 1]$.[28]

Figure 3 illustrates R_i for $i \in \{1, 2\}$. Note that for each $t \in \mathbb{R}$ with $-2 < t < -1$, $V^{R_i}(\cdot, (\mathbf{0}, t))$ is a convex combination of $V^{R_i}(\cdot, (\mathbf{0}, -2))$ and $V^{R_i}(\cdot, (\mathbf{0}, -1))$. By the constructions of R_1 and R_2, R_1 and R_2 are bounded and object monotonic. Moreover, as shown in the following claim, R_1 and R_2 are essentially quasi-linear.

Claim 1. $R_1, R_2 \in \mathcal{R}^E$.

Proof. We prove only that $R_1 \in \mathcal{R}^E$, since $R_2 \in \mathcal{R}^E$ can be shown in the same manner. Let $(x, t) \in X(R_1)$. By Remark 4, it suffices to show that $V^{R_1}(\bar{x}, (x, t)) - t = V^{R_1}(\bar{x}, (\mathbf{0}, 0)) - V^{R_1}(x, (\bar{x}, 0))$.

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[26] For example, let $\epsilon_{\mathbf{0}} = \epsilon'_{\mathbf{0}} = \epsilon''_{\mathbf{0}} = 0$, and for each $x \in X \setminus \{\mathbf{0}\}$, let $\epsilon_x = \frac{m(x)}{m(\bar{x})}$, $\epsilon'_x = \frac{m(x)}{(m(\bar{x}))^2}$, and $\epsilon''_x = \frac{m(x)}{n(m(\bar{x}))^3}$. The proof works with these ϵ, ϵ', and ϵ''.

[27] We denote by $\text{med}\{0, t+2, 1\}$ the median of three numbers, 0, $t+2$, and 1.

[28] If $t \leq -2$, then $\alpha = 0$, if $t \geq -1$, then $\alpha = 1$, and if $-2 < t < -1$, then $\alpha = t + 2 \in [0, 1]$.

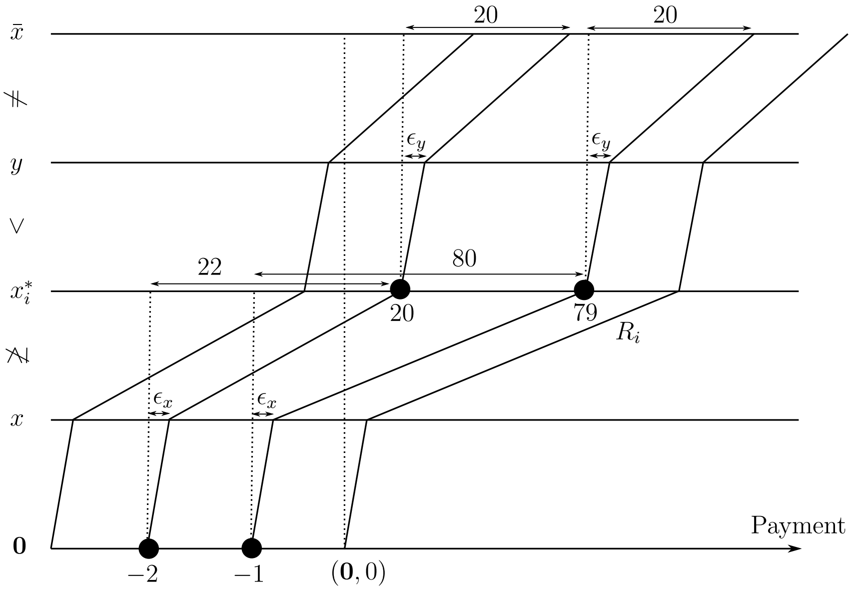

Figure 3: An illustration of R_i for $i \in \{1, 2\}$.

Without loss of generality, assume $x \neq \mathbf{0}$. If $x \geq x_1^*$, then it follows directly from the definition of R_1 that $V^{R_1}(\bar{x}, (x, t)) - t = V^{R_1}(\bar{x}, (\mathbf{0}, 0)) - V^{R_1}(x, (\bar{x}, 0))$.

Suppose $x \not\geq x_1^*$. Let $s \equiv V^{R_1}(\mathbf{0}, (x, t))$. If $s < -1$, then since ϵ_x is sufficiently close to 0, $t = V^{R_1}(x, (\mathbf{0}, s)) < V^{R_1}(x, (\mathbf{0}, -1)) = \epsilon_x - 1 < 0$. Thus, by $(x, t) \in X(R_1)$, $s \geq -1$. This implies $V^{R_1}(\bar{x}, (\mathbf{0}, s)) = 100 + s$. Therefore, $V^{R_1}(\bar{x}, (x, t)) - t = V^{R_1}(\bar{x}, (\mathbf{0}, s)) - V^{R_1}(x, (\mathbf{0}, s)) = 100 - \epsilon_x = V^{R_1}(\bar{x}, (\mathbf{0}, 0)) - V^{R_1}(x, (\bar{x}, 0))$. Hence, $R_1 \in \mathcal{R}^E$. $\square$

Let $R_3 \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$v_3(x) = \begin{cases} \epsilon'_x & \text{if } x \neq \bar{x}, \\ 60 & \text{if } x = \bar{x}. \end{cases}$$

For each $i \in N \setminus \{1, 2, 3\}$, let $R_i \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$, $v_i(x) = \epsilon''_x$. Denote $R \equiv (R_1, \ldots, R_n)$.

We conclude this step by stating several properties of R. Some of these properties are immediate and therefore omitted. The first property gives an upper bound for the sum of the valuations that the agents other than agents 1 and 2 can achieve under the assumption that no agent receives $\bar{x}$. This fact will be used in later steps to decide the payments of agents 1 and 2 at some preference profiles.

Property 1. *Let* $(x_i)_{i\in N} \in (X\backslash\{\bar{x}\})^n$ *be such that* $\sum_{i\in N} x_i \leq \bar{x}$. *Then,* $\sum_{i\in N\backslash\{1,2\}} v_i(x_i) < 1$.

Proof. By $x_3 \neq \bar{x}$, $v_3(x_3) = \epsilon'_{x_3}$. For each $i \in N \setminus \{1,2,3\}$, $v_i(x_i) = \epsilon''_{x_i}$. Since $(\epsilon'_x)_{x\in X}$ and $(\epsilon''_x)_{x\in X}$ are sufficiently close to $\mathbf{0}$, $\sum_{i\in N\backslash\{1,2\}} v_i(x_i) < 1$. □

By (3), for each $i \in N \setminus \{1,2,3\}$, R_i is negligible with respect to R_3.

Property 2. *For each* $i \in N \setminus \{1,2,3\}$, $R_i \in \mathcal{R}^Q(R_3)$.

By Property 2 and Lemma 7, we obtain the following.

Property 3. *For each* $(R'_1, R'_2) \in \mathcal{R}^2$ *and each* $i \in N\setminus\{1,2,3\}$, $x^f_i(R'_1, R'_2, R_{-1,2}) = \mathbf{0}$.

The following property is immediate from the definitions of $(\epsilon'_x)_{x\in X}$ and R_3.

Property 4. *Let* $v \in [0, 50]$ *and* $R'_i \in \mathcal{R}^Q$ *be such that for each* $x \in X \setminus \{0, \bar{x}\}$, $v'_i(x) = v + \epsilon_x$, *and* $v'_i(\bar{x}) > 60$. *Then* $R_3 \in \mathcal{R}^Q(R'_i)$.

The final property characterizes the packages that agent 3 can obtain.

Property 5. *Let* $i \in \{1,2\}$ *and* $R'_i \in \mathcal{R}$. *Then,* $x^f_3(R'_i, R_{-i}) = \mathbf{0}$ *or* $\bar{x}$.

Proof. Without loss of generality, assume $i = 1$ and denote $R' \equiv (R'_1, R_{-1})$. Assume for contradiction that $x^f_3(R') \in X \setminus \{\mathbf{0}, \bar{x}\}$. Let $x \equiv x^f_2(R') + x^f_3(R')$. By $x^f_3(R') \neq \mathbf{0}$, $x > x^f_2(R')$. Thus, $V^{R_2}(x, f_2(R')) - t^f_2(R') \geq \epsilon_x - \epsilon_{x^f_2(R')}$.

Therefore,

$$\begin{aligned} V^{R_2}(x, f_2(R')) + V^{R_3}(\mathbf{0}, f_3(R')) &\geq \epsilon_x - \epsilon_{x^f_2(R')} + t^f_2(R') + t^f_3(R') - v_3(x^f_3(R')) \\ &= \epsilon_x - \epsilon_{x^f_2(R')} + t^f_2(R') + t^f_3(R') - \epsilon'_{x^f_3(R')} \\ &> t^f_2(R') + t^f_3(R'), \end{aligned}$$

where the equality follows from the definition of R_3 and the last inequality follows from (3). This inequality contradicts Lemma 6. □

Step 2. *Either* $x^f_3(R) = \bar{x}$, *or* $x^f_1(R) = x^*_1$ *and* $x^f_2(R) = x^*_2$.

Proof. Assume for contradiction that $x_3^f(R) \neq \bar{x}$, and that either $x_1^f(R) \neq x_1^*$ or $x_2^f(R) \neq x_2^*$. By Property 5, $x_3^f(R) = \mathbf{0}$. By Property 3, for each $i \in N \setminus \{1,2,3\}$, $x_i^f(R) = 0$. Thus, $x_1^f(R) = \bar{x} - x_2^f(R)$. This implies $x_1^f(R) \neq x_1^*$ and $x_2^f(R) \neq x_2^*$.

Denote $s_1 \equiv V^{R_1}(\mathbf{0}, f_1(R))$ and $s_2 \equiv V^{R_2}(\mathbf{0}, f_2(R))$. There are three cases.

Case 1. $x_1^f(R) = \bar{x}$ *or* $x_2^f(R) = \bar{x}$. Without loss of generality, we assume $x_1^f(R) = \bar{x}$. By the definition of R_1, $V^{R_1}(x_1^*, f_1(R)) = t_1^f(R) - 20$. By $x_2^f(R) = \mathbf{0}$ and the definition of R_2, $V^{R_2}(x_2^*, f_2(R)) \geq t_2^f(R) + 22$. Thus, $V^{R_1}(x_1^*, f_1(R)) + V^{R_2}(x_2^*, f_2(R)) > t_1^f(R) + t_2^f(R)$, contradicting Lemma 6.

Case 2. $x_1^f(R) \not> x_1^*$ *and* $x_2^f(R) \not> x_2^*$. For each $i \in \{1,2\}$,

$$V^{R_i}(x_i^*, f_i(R)) - t_i^f(R) = V^{R_i}(x_i^*, (\mathbf{0}, s_i)) - V^{R_i}(x_i^f(R), (\mathbf{0}, s_i)) \geq 22 + s_i - (\epsilon_{x_i^f(R)} + s_i) > 0,$$

where the last inequality follows since $\epsilon_{x_i^f(R)}$ is sufficiently close to 0. Thus, $V^{R_1}(x_1^*, f_1(R)) + V^{R_2}(x_2^*, f_2(R)) > t_1^f(R) + t_2^f(R)$, which contradicts Lemma 6.

Case 3. $x_1^f(R) > x_1^*$ *or* $x_2^f(R) > x_2^*$. Without loss of generality, assume $x_1^f(R) > x_1^*$. By Case 1, we can also assume $x_1^f(R) \neq \bar{x}$ without loss of generality. By the definition of R_1, $V^{R_1}(x_1^*, f_1(R)) - t_1^f(R) = V^{R_1}(x_1^*, (\mathbf{0}, s_1)) - V^{R_1}(x_1^f(R), (\mathbf{0}, s_1)) = -\epsilon_{x_1^f(R)}$.

By $x_1^f(R) > x_1^*$, $x_2^f(R) \not> x_2^*$. Thus, as we have shown in Case 2, $V^{R_2}(x_2^*, f_2(R)) - t_2^f(R) \geq 22 - \epsilon_{x_2^f(R)}$. Since $\epsilon_{x_1^f(R)}$ and $\epsilon_{x_2^f(R)}$ are sufficiently close to 0, $V^{R_1}(x_1^*, f_1(R)) + V^{R_2}(x_2^*, f_2(R)) > t_1^f(R) + t_2^f(R)$, which contradicts Lemma 6. ■

Step 3. $x_1^f(R) = x_1^*$ *and* $x_2^f(R) = x_2^*$.

Proof. Assume for contradiction that $x_1^f(R) \neq x_1^*$ or $x_2^f(R) \neq x_2^*$. By Step 2, $x_3^f(R) = \bar{x}$, and thus, $x_1^f(R) = x_2^f(R) = \mathbf{0}$. First, we show the following claim.

Claim 1. $t_i^f(R) < -1$ *for some* $i \in \{1,2\}$.

Proof. Assume for contradiction that $t_1^f(R) \geq -1$ and $t_2^f(R) \geq -1$. By the definitions of R_1 and R_2, $V^{R_1}(x_1^*, f_1(R)) - t_1^f(R) = V^{R_2}(x_2^*, f_2(R)) - t_2^f(R) = 80$. By the definition of R_3, $V^{R_3}(\mathbf{0}, f_3(R)) = t_3^f(R) - v_3(\bar{x}) = t_3^f(R) - 60$. Thus, $V^{R_1}(x_1^*, f_1(R)) + V^{R_2}(x_2^*, f_2(R)) + V^{R_3}(\mathbf{0}, f_3(R)) > t_1^f(R) + t_2^f(R) + t_3^f(R)$, contradicting Lemma 6. □

Without loss of generality, we assume $t_1^f(R) < -1$. Let $R_2' \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$v_2'(x) = \begin{cases} \epsilon_x & \text{if } x \neq \bar{x}, \\ 100 & \text{if } x = \bar{x}. \end{cases}$$

Let $\mathbf{t}, \mathbf{s} \in \mathbb{R}^{|X|}$ be such that for each $x \in X$, $t_x = t_1^f(R_{-1}; x)$ and $s_x = t_1^f(R_2', R_{-1,2}; x)$. By the boundedness of $R_2', R_2, R_3, \ldots, R_n$ and Lemma 8, $\mathbf{t}$ and $\mathbf{s}$ are well-defined and by Lemma 9, are object monotonic.

By Property 4, $R_3 \in \mathcal{R}^Q(R_2')$. Further, by Property 2 and Lemma 1, for each $i \in N \setminus \{1, 2, 3\}$, $R_i \in \mathcal{R}^Q(R_3) \subseteq \mathcal{R}^Q(R_2')$. Thus, by Lemma 11 and Property 1, $-1 < s_{\mathbf{0}} \leq 0$. By $t_{\mathbf{0}} = t_1^f(R)$ and Claim 1, $t_{\mathbf{0}} < s_{\mathbf{0}} \leq 0$.

Thus, $\mathbf{t}$ and $\mathbf{s}$ satisfy the condition of Lemma 4 for $\mathbf{0}$. Therefore, by Lemma 4, there is $R_1' \in \mathcal{R}^E$ such that $R_1' \in \mathcal{R}^{TM}_{\mathbf{t},\mathbf{0}} \cap \mathcal{R}^{TM}_{\mathbf{s},x_1^*}$. By Lemma 10, $x_1^f(R_1', R_{-1}) = \mathbf{0}$ and $x_1^f(R_1', R_2', R_{-1,2}) = x_1^*$. For each $i \in N \setminus \{1, 2\}$, since $R_i \in \mathcal{R}^Q(R_2')$, Lemma 7 implies $x_i^f(R_1', R_2', R_{-1,2}) = \mathbf{0}$. Thus, $x_2^f(R_1', R_2', R_{-1,2}) = x_2^*$.

By Property 3 and $x_1^f(R_1', R_{-1}) = \mathbf{0}$, $x_2^f(R_1', R_{-1}) = \bar{x} - x_3^f(R_1', R_{-1})$. Note that by Property 5 $x_3^f(R_1', R_{-1}) = \mathbf{0}$ or $\bar{x}$. Therefore, $x_2^f(R_1', R_{-1}) = \mathbf{0}$ or $\bar{x}$.

Case 1. $x_2^f(R_1', R_{-1}) = \mathbf{0}$. Since $\epsilon_{x_2^*}$ is sufficiently close to 0, $V^{R_2}(x_2^*, f_2(R_1', R_{-1})) > t_2^f(R_1', R_{-1}) + \epsilon_{x_2^*} = t_2^f(R_1', R_{-1}) + v_2'(x_2^*) = V^{R_2'}(x_2^*, f_2(R_1', R_{-1}))$, which contradicts Fact 2.

Case 2. $x_2^f(R_1', R_{-1}) = \bar{x}$. By $x_2^f(R_1', R_2', R_{-1,2}) = x_2^*$ and the definitions of R_2 and R_2', $V^{R_2}(\bar{x}, f_2(R_1', R_2', R_{-1,2})) = t_2^f(R_1', R_2', R_{-1,2}) + 20 < t_2^f(R_1', R_2', R_{-1,2}) + v_2'(\bar{x}) - v_2'(x_2^*) = V^{R_2'}(\bar{x}, f_2(R_1', R_2', R_{-1,2}))$, which contradicts Fact 2. ■

Step 4. $t_i^f(R) > 20$ *for some* $i \in \{1, 2\}$.

Proof. Assume for contradiction that $t_1^f(R) \leq 20$ and $t_2^f(R) \leq 20$. By Step 3, $x_1^f(R) = x_1^*$ and $x_2^f(R) = x_2^*$. Note that for each $i \in \{1, 2\}$ and each $t \in \mathbb{R}$ with $t > -2$, $V^{R_i}(x_i^*, (\mathbf{0}, t)) > 20$. This implies that $V^{R_1}(\mathbf{0}, f_1(R)) \leq -2$ and $V^{R_2}(\mathbf{0}, f_2(R)) \leq -2$. Thus, by the definitions of R_1 and R_2, $t_1^f(R) - V^{R_1}(\mathbf{0}, f_1(R)) = t_2^f(R) - V^{R_2}(\mathbf{0}, f_2(R)) = 22$.

Therefore, $V^{R_1}(\mathbf{0}, f_1(R)) + V^{R_2}(\mathbf{0}, f_2(R)) + V^{R_3}(\bar{x}, f_3(R)) = t_1^f(R) - 22 + t_2^f(R) - 22 + t_3^f(R) + 60 > t_1^f(R) + t_2^f(R) + t_3^f(R)$, which contradicts Lemma 6. ■

Without loss of generality, we assume $t_1^f(R) > 20$. Let $(\delta_x)_{x\in X} \in \mathbb{R}_+^{|X|}$ be an object monotonic and additive vector that is sufficiently close to $\mathbf{0}$ and satisfies the following: For each $x \in X \setminus \{\mathbf{0}\}$, $\delta_x < \epsilon'_x$.[29,30] Let $R'_1 \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$v'_1(x) = \begin{cases} 20 + \delta_x & \text{if } x_1^* \le x \not\le \bar{x}, \\ 40 + \delta_{x_1^*} & \text{if } x = \bar{x}, \\ \delta_x & \text{otherwise.} \end{cases}$$

Since $\delta_{x_1^*}$ is sufficiently close to 0 and $t_1^f(R) > 20$, we have $v'_1(x_1^*) < t_1^f(R)$.

Step 5. *Either $x_3^f(R'_1, R_{-1}) = \bar{x}$, or $x_1^f(R'_1, R_{-1}) = x_1^*$ and $x_2^f(R'_1, R_{-1}) = x_2^*$.*

Proof. Assume for contradiction that $x_3^f(R'_1, R_{-1}) \neq \bar{x}$, and $x_1^f(R'_1, R_{-1}) \neq x_1^*$ or $x_2^f(R'_1, R_{-1}) \neq x_2^*$. By Property 5, $x_3^f(R'_1, R_{-1}) = \mathbf{0}$. Thus, by Property 3, $x_1^f(R'_1, R_{-1}) = \bar{x} - x_2^f(R'_1, R_{-1})$. This implies $x_1^f(R'_1, R_{-1}) \neq x_1^*$ and $x_2^f(R'_1, R_{-1}) \neq x_2^*$. For simplicity, denote $R' \equiv (R'_1, R_{-1})$ in this step. There are three cases.

Case 1. $x_1^f(R') = \mathbf{0}$. By $x_1^f(R') = \mathbf{0}$ and the definition of R'_1, $V^{R'_1}(x_1^*, f_1(R')) = t_1^f(R') + v'_1(x_1^*) > t_1^f(R') + 20$. By $x_1^f(R') = \bar{x} - x_2^f(R')$, $x_2^f(R') = \bar{x}$. Thus, by the definition of R_2, $V^{R_2}(x_2^*, f_2(R')) = t_2^f(R') - 20$. Thus, $V^{R'_1}(x_1^*, f_1(R')) + V^{R_2}(x_2^*, f_2(R')) > t_1^f(R') + t_2^f(R')$. This contradicts Lemma 6.

Case 2. $x_1^f(R') = \bar{x}$. By $x_3^f(R') = \mathbf{0}$ and the definition of $\delta_{\bar{x}}$, we have $V^{R'_1}(\mathbf{0}, f_1(R')) + V^{R_3}(\bar{x}, f_3(R')) = t_1^f(R') - v'_1(\bar{x}) + t_3^f(R') + v_3(\bar{x}) = t_1^f(R') - 40 - \delta_{\bar{x}} + t_3^f(R') + 60 > t_1^f(R') + t_3^f(R')$. This contradicts Lemma 6.

Case 3. $x_1^* < x_1^f(R') \neq \bar{x}$. By the definition of R'_1, $V^{R'_1}(x_1^*, f_1(R')) = t_1^f(R') + v'_1(x_1^*) - v'_1(x_1^f(R')) = t_1^f(R') + 20 + \delta_{x_1^*} - 20 - \delta_{x_1^f(R')} = t_1^f(R') - \delta_{x_1^f(R') - x_1^*}$, where the last equality follows since $(\delta_x)_{x\in X}$ is additive.

By $x_3^f(R'_1, R_{-1}) = \mathbf{0}$, $x_1^f(R') - x_1^* > \mathbf{0}$, and the definition of R_3, $V^{R_3}(x_1^f(R') - x_1^*, f_3(R')) = t_3^f(R') + v_3(x_1^f(R') - x_1^*) = t_3^f(R') + \epsilon'_{x_1^f(R') - x_1^*}$.

Therefore, by $\delta_{x_1^f(R') - x_1^*} < \epsilon'_{x_1^f(R') - x_1^*}$, $V^{R'_1}(x_1^*, f_1(R')) + V^{R_3}(x_1^f(R') - x_1^*, f_3(R')) >$

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[29] A vector $\mathbf{t} \in \mathbb{R}^{|X|}$ is *additive* if for each pair $x, y \in X$, $t_{x+y} = t_x + t_y$.

[30] For example, take any $\delta' \in \mathbb{R}_{++}$ such that $\delta' < \min\{1, t_1^f(R) - 20\}$, and let $(\delta_x)_{x\in X} \in \mathbb{R}_{++}^{|X|}$ be such that $\delta_{\mathbf{0}} = 0$ and for each $x \in X \setminus \{\mathbf{0}\}$, $\delta_x = \frac{\delta' m(x)}{(m(\bar{x}))^3}$. Then our proof works with this $(\delta_x)_{x\in X}$, and $(\epsilon_x)_{x\in X}$, $(\epsilon'_x)_{x\in X}$, and $(\epsilon''_x)_{x\in X}$ defined in footnote 26.

$t_1^f(R') + t_3^f(R')$. This contradicts Lemma 6.

Case 4. $x_1^* \not\leq x_1^f(R') \neq \mathbf{0}$. By the definition of R'_1, $V^{R'_1}(\mathbf{0}, f_1(R')) = t_1^f(R') - v'_1(x_1^f(R')) = t_1^f(R') - \delta_{x_1^f(R')}$. By the definition of R_3, $V^{R_3}(x_1^f(R'), f_3(R')) = t_3^f(R') + v_3(x_1^f(R')) = t_3^f(R') + \epsilon'_{x_1^f(R')}$.

Therefore, by $\delta_{x_1^f(R')} < \epsilon'_{x_1^f(R')}$, $V^{R'_1}(\mathbf{0}, f_1(R')) + V^{R_3}(x_1^f(R'), f_3(R')) > t_1^f(R') + t_3^f(R')$. This contradicts Lemma 6. ■

Step 6. $x_1^f(R'_1, R_{-1}) = x_1^*$ *and* $x_2^f(R'_1, R_{-1}) = x_2^*$.

Proof. Assume for contradiction that $x_1^f(R'_1, R_{-1}) \neq x_1^*$ or $x_2^f(R'_1, R_{-1}) \neq x_2^*$. By Step 5, $x_3^f(R'_1, R_{-1}) = \bar{x}$, and hence, $x_1^f(R'_1, R_{-1}) = x_2^f(R'_1, R_{-1}) = \mathbf{0}$.

Claim 1. $t_2^f(R'_1, R_{-1}) < -1$.

Proof. Assume for contradiction that $t_2^f(R'_1, R_{-1}) \geq -1$. Then, by $x_2^f(R'_1, R_{-1}) = \mathbf{0}$ and the definition of R_2, $V^{R_2}(x_2^*, f_2(R'_1, R_{-1})) - t_2^f(R'_1, R_{-1}) = 80$. By $x_3^f(R'_1, R_{-1}) = \bar{x}$ and the definition of R_3, $V^{R_3}(\mathbf{0}, f_3(R'_1, R_{-1})) = t_3^f(R'_1, R_{-1}) - v_3(\bar{x}) = t_3^f(R'_1, R_{-1}) - 60$. Thus, $V^{R_2}(x_2^*, f_2(R'_1, R_{-1})) + V^{R_3}(\mathbf{0}, f_3(R'_1, R_{-1})) > t_2^f(R'_1, R_{-1}) + t_3^f(R'_1, R_{-1})$. This contradicts Lemma 6. □

Let $R''_1 \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$v''_1(x) = \begin{cases} \epsilon_x & \text{if } x \neq \bar{x}, \\ 100 & \text{if } x = \bar{x}. \end{cases}$$

Let $\mathbf{t}, \mathbf{s} \in \mathbb{R}^{|X|}$ be such that for each $x \in X$, $t_x = t_2^f(R'_1, R_{-1,2}; x)$ and $s_x = t_2^f(R''_1, R_{-1,2}; x)$. By the boundedness of $R'_1, R''_1, R_3, \ldots, R_n$ and Lemma 8, $\mathbf{t}$ and $\mathbf{s}$ are well-defined, and by Lemma 9, are object monotonic.

By Property 4, $R_3 \in \mathcal{R}^Q(R''_1)$. Further, by Property 2 and Lemma 1, for each $i \in N \setminus \{1, 2, 3\}$, $R_i \in \mathcal{R}^Q(R_3) \subseteq \mathcal{R}^Q(R''_1)$. Thus, by Lemma 11 and Property 1, $-1 < s_{\mathbf{0}} \leq 0$. Since $t_{\mathbf{0}} = t_2^f(R'_1, R_{-1}) \leq -1$ by Claim 1, $t_{\mathbf{0}} < s_{\mathbf{0}} \leq 0$.

Thus, $\mathbf{t}$ and $\mathbf{s}$ satisfy the condition of Lemma 4 for $\mathbf{0}$. Therefore, by Lemma 4, there is $R'_2 \in \mathcal{R}^E$ such that $R'_2 \in \mathcal{R}^{MT}_{\mathbf{t},\mathbf{0}} \cap \mathcal{R}^{MT}_{\mathbf{s},x_2^*}$. For simplicity, denote $R' \equiv (R'_1, R'_2, R_{-1,2})$ and $R'' = (R''_1, R'_2, R_{-1,2})$ in this step. By Lemma 10, $x_2^f(R') = \mathbf{0}$ and $x_2^f(R'') = x_2^*$.

For each $i \in N \setminus \{1, 2\}$, since $R_i \in \mathcal{R}^Q(R''_1)$, Lemma 7 implies $x_i^f(R'') = \mathbf{0}$. Thus,

$x_1^f(R'') = x_1^*$. There are three cases.

Case 1. $x_1^f(R') \in X \setminus \{\mathbf{0}, \bar{x}\}$. By $x_1^f(R') \notin \{\mathbf{0}, \bar{x}\}$ and the definitions of R_1', $V^{R_1'}(\bar{x}, f_1(R')) = t_1^f(R') + v_1'(\bar{x}) - v_1'(x_1^f(R')) \geq t_1^f(R') + 20 + \delta_{x_1^*} - \delta_{x_1^f(R')}$.

By Property 3 and $x_2^f(R') = \mathbf{0}$, $x_1^f(R') = \bar{x} - x_3^f(R')$. By $x_1^f(R') \notin \{\mathbf{0}, \bar{x}\}$, $x_3^f(R') \notin \{\mathbf{0}, \bar{x}\}$. Thus by the definition of R_3, $V^{R_3}(\mathbf{0}, f_3(R')) = t_3^f(R') - v_3(x_3^f(R')) = t_3^f(R') - \epsilon'_{x_3}$.

Since $(\delta_x)_{x \in X}$ and $(\epsilon'_x)_{x \in X}$ are sufficiently close to $\mathbf{0}$, $V^{R_1'}(\bar{x}, f_1(R')) + V^{R_3}(\mathbf{0}, f_3(R')) > t_1^f(R') + t_3^f(R')$. This contradicts Lemma 6.

Case 2. $x_1^f(R') = \mathbf{0}$. By the definition of R_1', $V^{R_1'}(x_1^*, f_1(R')) = t_1^f(R') + v_1'(x_1^*) = t_1^f(R') + 20 + \delta_{x_1^*}$ By the definition of R_1'', $V^{R_1''}(x_1^*, f_1(R')) = t_1^f(R') + v_1''(x_1^*) = t_1^f(R') + \epsilon_{x_1^*}$.

Since $\epsilon_{x_1^*}$ is sufficiently close to 0, $V^{R_1'}(x_1^*, f_1(R')) > V^{R_1''}(x_1^*, f_1(R'))$. By $x_1^f(R'') = x_1^*$, contradicting Fact 2.

Case 3. $x_1^f(R') = \bar{x}$. By $x_1^f(R'') = x_1^*$ and the definition of R_1', $V^{R_1'}(\bar{x}, f_1(R'')) = t_1^f(R'') + v_1'(\bar{x}) - v_1'(x_1^*) = t_1^f(R'') + 20$. By $x_1^f(R'') = x_1^*$ and the definition of R_1'', $V^{R_1''}(\bar{x}, f_1(R'')) = t_1^f(R'') + v_1''(\bar{x}) - v_1''(x_1^*) > t_1^f(R'') + 20$. Thus, $V^{R_1'}(\bar{x}, f_1(R'')) < V^{R_1''}(\bar{x}, f_1(R''))$, contradicting Fact 2. ■

Step 7. *Completing the proof.*

By Step 6, $x_1^f(R_1', R_{-1}) = x_1^*$ and $x_2^f(R_1', R_{-1}) = x_2^*$. By Step 3, $x_1^f(R) = x_1^*$. Thus, by *strategy-proofness*, $f_1(R_1', R_{-1}) = f_1(R)$. Let $R_2' \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$v_2'(x) = \begin{cases} 50 + \epsilon_x & \text{if } x \neq \bar{x}, \\ 65 & \text{if } x = \bar{x}. \end{cases}$$

Let $\mathbf{t}, \mathbf{s} \in \mathbb{R}^{|X|}$ be such that for each $x \in X$, $t_x = t_1^f(R_{-1}; x)$ and $s_x = t_1^f(R_2', R_{-1,2}; x)$. By the boundedness of $R_2', R_2, \ldots, R_n$ and Lemma 8, $\mathbf{t}$ and $\mathbf{s}$ are well-defined, and by Lemma 9, are object monotonic.

By Property 4, $R_3 \in \mathcal{R}^Q(R_2')$. Further, by Property 2 and Lemma 1, for each $i \in N \setminus \{1, 2, 3\}$, $R_i \in \mathcal{R}^Q(R_3) \subseteq \mathcal{R}^Q(R_2')$. Thus, by Lemma 11, $s_{\mathbf{0}} \leq 0$. By *strategy-proofness*, $f_1(R_1', R_{-1}) \; R_1' \; z_1^f(R_{-1}; \mathbf{0}) = (\mathbf{0}, t_{\mathbf{0}})$. Thus, by $f_1(R_1', R_{-1}) = f_1(R) = (x_1^*, t_1^f(R))$, $t_{\mathbf{0}} \geq V^{R_1'}(\mathbf{0}, f_1(R_1', R_{-1})) = t_1^f(R) - v_1'(x_1^*) = t_1^f(R) - (20 + \delta_{x_1^*}) > 0$, where the inequality follows since $t_1^f(R) > 20$ by Step 4 and $\delta_{x_1^*}$ is sufficiently close to 0.

If $s_{\mathbf{0}} < 0$, then $\mathbf{t}$ and $\mathbf{s}$ satisfy the condition of Lemma 4 for $\mathbf{0}$. If $s_{\mathbf{0}} = 0$, then $\mathbf{t}$ and $\mathbf{s}$ satisfy the condition of Lemma 3 for $\mathbf{0}$. Thus, by Lemmas 3 and 4, there is $R_1'' \in \mathcal{R}^E$ such that $R_1'' \in \mathcal{R}^{MT}_{\mathbf{t},x_1^*} \cap \mathcal{R}^{MT}_{\mathbf{s},\mathbf{0}}$. By Lemma 10, $x_1^f(R_1'', R_{-1}) = x_1^*$ and $x_1^f(R_1'', R_2', R_{-1,2}) = \mathbf{0}$.

For each $i \in N \setminus \{1,2\}$, by $R_i \in \mathcal{R}^Q(R_2')$ and Lemma 7, $x_i^f(R_1'', R_2', R_{-1,2}) = \mathbf{0}$. Thus, $x_2^f(R_1'', R_2', R_{-1,2}) = \bar{x}$. Further, by Properties 3 and 5, $x_2^f(R_1'', R_{-1}) = x_2^*$.

By the definition of R_2, $V^{R_2}(\bar{x}, f_2(R_1'', R_{-1})) = t_2^f(R_1'', R_{-1}) + 20$. By the definition of R_2', $V^{R_2'}(\bar{x}, f_2(R_1'', R_{-1})) = t_2^f(R_1'', R_{-1}) + v_2'(\bar{x}) - v_2'(x_2^*) < t_2^f(R_1'', R_{-1}) + 20$. Thus, $V^{R_2}(\bar{x}, f_2(R_1'', R_{-1})) > V^{R_2'}(\bar{x}, f_2(R_1'', R_{-1}))$. This contradicts Fact 2. ■

D Proof of Proposition 1

Assume for contradiction that there is an *efficient* and *strategy-proof* mechanism f on $\mathcal{R}^n$ such that it is a (generalized) Vickrey mechanism on $(\mathcal{R}^Q)^n$. We do the proof in four steps.

Step 1. *Constructing a preference profile.*

By $\mathcal{R} \not\subseteq \mathcal{R}^E$, there is $R_1 \in \mathcal{R}$ such that $R_1 \notin \mathcal{R}^E$. By Remark 4, there is $(x^*, t) \in X(R_1)$ such that $V^{R_1}(\bar{x}, (x^*, t)) - t \neq V^{R_1}(\bar{x}, (\mathbf{0}, 0)) - V^{R_1}(x^*, (\mathbf{0}, 0))$.

By the continuity of R_1, we can assume $0 < t < V^{R_1}(x^*, (\mathbf{0}, 0))$ without loss of generality. Also by the continuity of R_1, there is $s \in \mathbb{R}$ such that $0 < s < V^{R_1}(x^*, (\mathbf{0}, 0))$ and $V^{R_1}(\bar{x}, (x^*, t)) - t \neq V^{R_1}(\bar{x}, (x^*, s)) - s$. Without loss of generality, assume

$$V^{R_1}(\bar{x}, (x^*, t)) - t > V^{R_1}(\bar{x}, (x^*, s)) - s. \tag{4}$$

Let $(\epsilon_x)_{x \in X} \in \mathbb{R}_+^{|X|}$ be an object monotonic vector that is sufficiently close to $\mathbf{0}$ and satisfies $\epsilon_{\mathbf{0}} = 0$.[31]

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[31] Formally, the proof works if we take $(\epsilon_x)_{x \in X} \in \mathbb{R}_+^{|X|}$ that is object monotonic and satisfies the following: $\epsilon_{\mathbf{0}} = 0$, and for each $(x, y) \in \mathcal{X}$, $0 < \epsilon_x + \epsilon_y < \min\{\min_{(x',y') \in \mathcal{X}} V^{R_1}(x', (x^*, t)) - V^{R_1}(y', (x^*, t)), \min_{(x',y') \in \mathcal{X}} V^{R_1}(x', (x^*, s)) - V^{R_1}(y', (x^*, s)), V^{R_1}(\bar{x}, (x^*, t)) - t - (V^{R_1}(\bar{x}, (x^*, s)) - s)\}$.

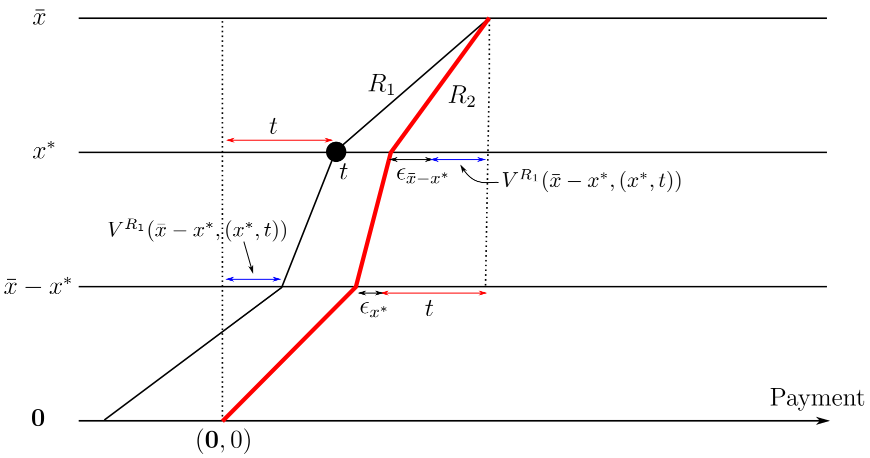

Figure 4: An illustration of R_2.

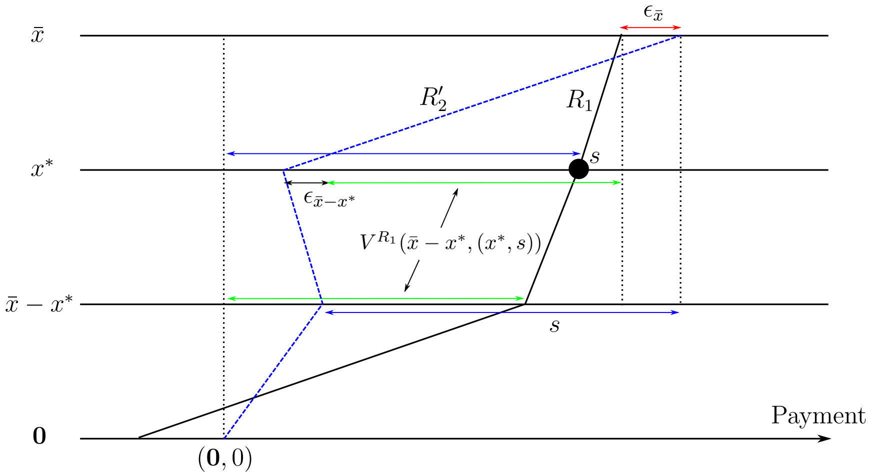

Figure 5: An illustration of R_2'.

Let $R_2 \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$v_2(x) = V^{R_1}(\bar{x},(x^*,t)) - \max\{V^{R_1}(\bar{x}-x,(x^*,t)),0\} - \epsilon_{\bar{x}-x}.$$

Let $R_2' \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$v_2'(x) = \begin{cases} V^{R_1}(\bar{x},(x^*,s)) - \max\{V^{R_1}(\bar{x}-x,(x^*,s)),0\} + \epsilon_{\bar{x}} & \text{if } x \in \{\bar{x}-x^*,\bar{x}\}, \\ V^{R_1}(\bar{x},(x^*,s)) - \max\{V^{R_1}(\bar{x}-x,(x^*,s)),0\} - \epsilon_{\bar{x}-x} & \text{otherwise.} \end{cases}$$

Figures 4 and 5 illustrate R_2 and R'_2, respectively. The following claims show R_2 and R'_2 are object monotonic, and hence $R_2, R'_2 \in \mathcal{R}^Q$.

Claim 1. *R_2 is object monotonic.*

Proof. Let $(x, y) \in \mathcal{X}$. If $y = \mathbf{0}$, then since ϵ_x is sufficiently close to 0 and R_1 is object monotonic, $v_2(x) > 0$.

Suppose $y \neq \mathbf{0}$. By $x > y$, $\bar{x} - x < \bar{x} - y$. Since R_1 is object monotonic, $V^{R_1}(\bar{x} - x, (x^*, t)) < V^{R_1}(\bar{x} - y, (x^*, t))$ and $\epsilon_{\bar{x}-x} < \epsilon_{\bar{x}-y}$. Thus, $v_2(x) = V^{R_1}(\bar{x}, (x^*, t)) - \max\{V^{R_1}(\bar{x} - x, (x^*, t)), 0\} - \epsilon_{\bar{x}-x} > V^{R_1}(\bar{x}, (x^*, t)) - \max\{V^{R_1}(\bar{x} - y, (x^*, t)), 0\} - \epsilon_{\bar{x}-y} = v_2(y)$. □

Claim 2. *R'_2 is object monotonic.*

Proof. Let $(x, y) \in \mathcal{X}$. If $y = \mathbf{0}$, then since ϵ_x is sufficiently close to 0 and R_1 is object monotonic, $v'_2(x) > 0$. Suppose $y \neq \mathbf{0}$. By $x > y$, $\bar{x} - x < \bar{x} - y$. There are two cases.

Case 1. $V^{R_1}(\bar{x} - y, (x^*, s)) > 0$. By $\bar{x} - x < \bar{x} - y$, and $V^{R_1}(\bar{x} - y, (x^*, s)) > 0$, $\max\{V^{R_1}(\bar{x} - x, (x^*, s)), 0\} < V^{R_1}(\bar{x} - y, (x^*, s)) = \max\{V^{R_1}(\bar{x} - y, (x^*, s)), 0\}$. By this inequality and since $(\epsilon_{x'})_{x' \in X}$ is sufficiently close to $\mathbf{0}$, $\max\{V^{R_1}(\bar{x} - x, (x^*, s)), 0\} + \epsilon_{\bar{x}-x} < \max\{V^{R_1}(\bar{x} - y, (x^*, s)), 0\} - \epsilon_{\bar{x}}$.

Therefore, $v'_2(x) \geq V^{R_1}(\bar{x}, (x^*, s)) - \max\{V^{R_1}(\bar{x} - x, (x^*, s)), 0\} - \epsilon_{\bar{x}-x} > V^{R_1}(\bar{x}, (x^*, s)) - \max\{V^{R_1}(\bar{x} - y, (x^*, s)), 0\} + \epsilon_{\bar{x}} \geq v'_2(y)$.

Case 2. $V^{R_1}(\bar{x} - y, (x^*, s)) \leq 0$. By $x > y$, $y \neq \bar{x}$. By $V^{R_1}(x^*, (x^*, s)) = s > 0$, $y \neq \bar{x} - x^*$. Thus, $v'_2(y) = V^{R_1}(\bar{x}, (x^*, s)) - \max\{V^{R_1}(\bar{x} - y, (x^*, s)), 0\} - \epsilon_{\bar{x}-y} = V^{R_1}(\bar{x}, (x^*, s)) - \epsilon_{\bar{x}-y}$, where the last equality follows from $V^{R_1}(\bar{x} - y, (x^*, s)) \leq 0$.

By $\bar{x} - x < \bar{x} - y$, $V^{R_1}(\bar{x} - x, (x^*, s)) < V^{R_1}(\bar{x} - y, (x^*, s)) \leq 0$. Thus, $v'_2(x) \geq V^{R_1}(\bar{x}, (x^*, s)) - \max\{V^{R_1}(\bar{x} - x, (x^*, s)), 0\} - \epsilon_{\bar{x}-x} = V^{R_1}(\bar{x}, (x^*, s)) - \epsilon_{\bar{x}-x}$. Since $(\epsilon_{x'})_{x' \in X}$ is object monotonic, $v'_2(x) > v'_2(y)$. □

Finally we define preferences of the other agents. For each $i \in N \setminus \{1, 2\}$, let $R_i \in \mathcal{R}$ be such that $R_i \in \mathcal{R}^Q(R_2)$ and $R_i \in \mathcal{R}^Q(R'_2)$.[32] Denote $R \equiv (R_1, R_2, \ldots, R_n)$

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[32] We can pick such preferences from $\mathcal{R}$ since $\mathcal{R}^Q \subseteq \mathcal{R}$.

and $R' \equiv (R_1, R'_2, \dots, R_n)$. Since f is *efficient* and $R_2, R'_2, R_3, \dots, R_n \in \mathcal{R}^Q$, Lemma 8 implies $X^f_1(R_{-1}) = X^f_1(R'_{-1}) = X$.

Since f coincides with a Vickrey mechanism on $(\mathcal{R}^Q)^n$ and $R_2, R'_2, R_3, \dots, R_n \in \mathcal{R}^Q$, the option sets of agent 1 under f for R_{-1} and R'_{-1} coincide with the ones under a Vickrey mechanism, respectively. Further, by $R_3, \dots, R_n \in \mathcal{R}^Q(R_2) \cap \mathcal{R}^Q(R'_2)$, Lemma 2 implies that for each $x \in X$, $\sigma_1(R_{-1}; x) = v_2(\bar{x} - x)$ and $\sigma_1(R'_{-1}; x) = v'_2(\bar{x} - x)$. Hence, for each $x \in X$, $t^f_1(R_{-1}; x) = v_2(\bar{x}) - v_2(\bar{x} - x)$ and $t^f_1(R'_{-1}; x) = v'_2(\bar{x}) - v'_2(\bar{x} - x)$.

Step 2. *For each $x \in X \setminus \{\bar{x}\}$, $z^f_1(R_{-1}, \bar{x}) \ P_1 \ z^f_1(R_{-1}, x)$.*

Proof. First note that by $t < V^{R_1}(x^*, (\mathbf{0}, 0))$, $v_2(\bar{x}) = V^{R_1}(\bar{x}, (x^*, t)) < V^{R_1}(\bar{x}, (\mathbf{0}, 0))$. This implies $z^f_1(R_{-1}, \bar{x}) = (\bar{x}, v_2(\bar{x})) \ P_1 \ (\mathbf{0}, 0)$.

Next, let $x \in X \setminus \{\bar{x}, \mathbf{0}\}$. Then, $v_2(\bar{x}) - v_2(\bar{x} - x) = V^{R_1}(\bar{x}, (x^*, t)) - (V^{R_1}(\bar{x}, (x^*, t)) - \max\{V^{R_1}(x, (x^*, t)), 0\} - \epsilon_x) > \max\{V^{R_1}(x, (x^*, t)), 0\} \geq V^{R_1}(x, (x^*, t))$. This implies $(x^*, t) \ P_1 \ (x, v_2(\bar{x}) - v_2(x)) = z^f_1(R_{-1}; x)$. Therefore, $z^f_1(R_{-1}; \bar{x}) = (\bar{x}, v_2(\bar{x})) = (\bar{x}, V^{R_1}(\bar{x}, (x^*, t))) \ I_1 \ (x^*, t) \ P_1 \ z^f_1(R_{-1}; x)$. ∎

Step 3. *For each $x \in X \setminus \{x^*\}$, $z^f_1(R'_{-1}, x^*) \ P_1 \ z^f_1(R'_{-1}, x)$.*

Proof. Note that $t^f_1(R'_{-1}; x^*) = v'_2(\bar{x}) - v'_2(\bar{x} - x^*) = V^{R_1}(\bar{x}, (x^*, s)) + \epsilon_{\bar{x}} - (V^{R_1}(\bar{x}, (x^*, s)) - V^{R_1}(x^*, (x^*, s)) + \epsilon_{\bar{x}}) = s$.

By $s < V^{R_1}(x^*, (\mathbf{0}, 0))$, $z^f_1(R'_{-1}, x^*) = (x^*, s) \ P_1 \ (\mathbf{0}, 0)$. Also, since $v'_2(\bar{x}) > V^{R_1}(\bar{x}, (x^*, s))$, $z^f_1(R'_{-1}, x^*) = (x^*, s) \ P_1 \ (\bar{x}, v'_2(\bar{x})) = z^f_1(R'_{-1}, \bar{x})$.

Now let $x \in X \setminus \{\mathbf{0}, x^*, \bar{x}\}$. Then, $\bar{x} - x \neq \bar{x} - x^*$. Thus,

$$\begin{aligned} v'_2(\bar{x}) - v'_2(\bar{x} - x) &= V^{R_1}(\bar{x}, (x^*, s)) + \epsilon_{\bar{x}} - (V^{R_1}(\bar{x}, (x^*, s)) - \max\{V^{R_1}(x, (x^*, s)), 0\} - \epsilon_x) \\ &= \max\{V^{R_1}(x, (x^*, s)), 0\} + \epsilon_{\bar{x}} + \epsilon_x \\ &> V^{R_1}(x, (x^*, s)). \end{aligned}$$

This implies $z^f_1(R'_{-1}; x^*) = (x^*, s) \ P_1 \ (x, v'_2(\bar{x}) - v'_2(\bar{x} - x)) = z^f_1(R'_{-1}; x)$. ∎

Step 4. *Completing the proof.*

By Steps 2 and 3 and *strategy-proofness*, $x^f_1(R) = \bar{x}$ and $x^f_1(R') = x^*$. For each $i \in N \setminus \{2, 3\}$, since $R_i \in \mathcal{R}^Q(R_2) \cap \mathcal{R}^Q(R'_2)$, Lemma 7 implies $x^f_i(R) = x^f_i(R') = \mathbf{0}$. Therefore, $x^f_2(R) = \mathbf{0}$ and $x^f_2(R') = \bar{x} - x^*$. However, by (4) and since $(\epsilon_x)_{x \in X}$ is sufficiently close to $\mathbf{0}$, $v_2(\bar{x}-x^*) = V^{R_1}(\bar{x},(x^*,t)) - t - \epsilon_{x^*} > V^{R_1}(\bar{x},(x^*,s)) - s + \epsilon_{\bar{x}} = v_2'(\bar{x}-x^*)$. This contradicts Fact 2. $\blacksquare$

E Proofs of Lemmas

We give proofs of Lemmas. Since Lemmas 1 and 10 are straightforward, their proofs are omitted.

Proof of Lemma 2. Let $x \in X$ and $(x_k)_{k\in N} \in A$ be such that $x_i = x$ and $\sigma_i(R_{-i};x) = \sum_{k\in N\setminus\{i\}} v_k(x_k)$. Assume for contradiction that there is $k \in N \setminus \{i,j\}$ such that $x_k > 0$.

Let $(y_\ell)_{\ell\in N} \in A$ be such that $y_i = x_i$, $y_j = x_j + x_k$, $y_k = 0$, and for each $\ell \in N \setminus \{i,j,k\}$, $y_\ell = x_\ell$. By $\sigma_i(R_{-i};x) = \sum_{k\in N\setminus\{i\}} v_k(x_k)$, $\sum_{\ell\in N\setminus\{i\}} v_\ell(y_\ell) \leq \sum_{k\in N\setminus\{i\}} v_k(x_k)$.

However, by $R_k \in \mathcal{R}^Q(R_j)$ and $x_k > 0$, $v_j(x_j) + v_k(x_k) < v_j(x_j + x_k)$. Thus, $\sum_{\ell\in N\setminus\{i\}} v_\ell(y_\ell) = v_j(x_j+x_k) + \sum_{\ell\in N\setminus\{i,j,k\}} v_\ell(x_\ell) > \sum_{k\in N\setminus\{i\}} v_k(x_k)$. This is a contradiction. $\blacksquare$

Proof of Lemma 3. Let $X_+ \equiv \{y \in X : t_y > 0\}$. Let $\mathbf{t}^* \in \mathbb{R}^{|X|}$ be such that for each $y \in X$,

$$t^*_y = \begin{cases} \min\{t_y, s_y\} & \text{if } y \in X_+ \setminus \{\mathbf{0}\}, \\ 0 & \text{otherwise.} \end{cases}$$

Figure 6 illustrates $\mathbf{t}$, $\mathbf{s}$, and $\mathbf{t}^*$. Since $\mathbf{t}$ and $\mathbf{s}$ are object monotonic, for each $(y,y') \in \mathcal{X}$ with $y, y' \in X_+$, $t^*_y > t^*_{y'}$. Further, for each $(y,y') \in \mathcal{X}$, $t^*_y \geq t^*_{y'}$.[33] Let $\bar{\epsilon}, \underline{\epsilon} \in \mathbb{R}_{++}$ be sufficiently close to 0 and satisfy $\bar{\epsilon} > \underline{\epsilon}$.[34]

Fix $y \in X \setminus \{x\}$. We prove that there is $R_i \in \mathcal{R}^E$ such that $R_i \in \mathcal{R}^{MT}_{\mathbf{t},x} \cap \mathcal{R}^{MT}_{\mathbf{s},y}$. We do the proof in three steps.

Step 1. *Construction of a preference relation.*

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[33] Note that by $s_{\mathbf{0}} > 0$ and the object monotonicity of s, for each $y \in X \setminus \{\mathbf{0}\}$, $s_y > 0$.

[34] For example, the proof works if $\bar{\epsilon} > \underline{\epsilon} > 0$ and $2(\bar{\epsilon} + \underline{\epsilon}) < \min\{s_x - t_x, \min_{x'\in X_+} t^*_{x'}, \min_{(x',y')\in\mathcal{X}} s_{x'} - s_{y'}, \min_{(x',y')\in\mathcal{X}, x'\in X_+} t^*_{x'} - t^*_{y'}\}$.

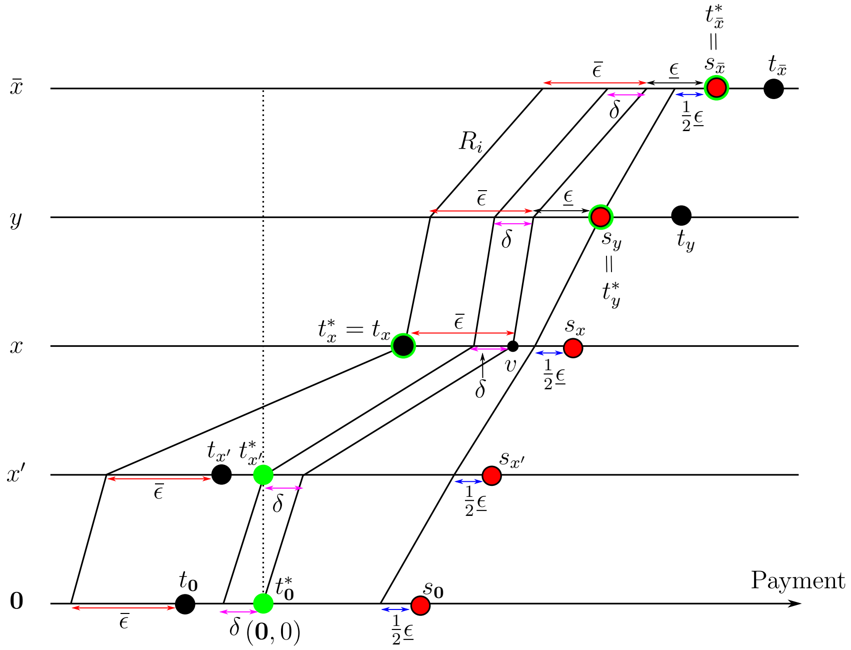

Figure 6: An illustration of $\mathbf{t}$, $\mathbf{s}$, $\mathbf{t}^*$, and R_i.

We define a preference relation R_i as follows: For each $x' \in X \setminus \{\mathbf{0}\}$, let[35]

$$V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) = \begin{cases} t^*_x + \bar{\epsilon} & \text{if } x' = x, \\ t^*_{x'} - \underline{\epsilon} & \text{if } x' \in X_+ \setminus \{x\}, \\ \frac{m(x')}{m(\bar{x})}\underline{\epsilon} & \text{otherwise.} \end{cases}$$

Let

$$\delta \equiv \begin{cases} \max_{x' \in X \setminus X_+} \frac{m(x')}{m(\bar{x})}\underline{\epsilon} & \text{if } X \setminus X_+ \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Note that by $\bar{\epsilon} > \underline{\epsilon}$, $\bar{\epsilon} > \delta$.

For each $t' \in [t^*_{\mathbf{0}} - \delta, t^*_{\mathbf{0}}]$ and each $x' \in X$, let

$$V^{R_i}(x', (\mathbf{0}, t')) = V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) - (t^*_{\mathbf{0}} - t').$$

Let $v \equiv V^{R_i}(x, (\mathbf{0}, t^*_{\mathbf{0}}))$. Note that if $x \neq \mathbf{0}$, then $v = t^*_x + \bar{\epsilon}$, and if $x = \mathbf{0}$, then

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[35] Note that if $x = \mathbf{0}$, the first condition of the definition of $V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}}))$ is redundant since $x' \in X \setminus \{\mathbf{0}\}$.

$v = t^*_{\mathbf{0}}$. Note also that by $\bar{\epsilon} > \delta$, $v - \bar{\epsilon} < V^{R_i}(x, (\mathbf{0}, t^*_{\mathbf{0}} - \delta))$.[36] For each $x' \in X$, let[37]

$$V^{R_i}(x', (x, v - \bar{\epsilon})) = \begin{cases} V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) - \bar{\epsilon} & \text{if } x' \in X_+ \cup \{x\}, \\ \min\{t_{x'}, V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}} - \delta))\} - \bar{\epsilon} & \text{otherwise.} \end{cases}$$

For each $t' \in [V^{R_i}(\mathbf{0}, (x, v - \bar{\epsilon})), t^*_{\mathbf{0}} - \delta]$ and each $x' \in X$, let

$$V^{R_i}(x', (\mathbf{0}, t')) = \alpha \cdot V^{R_i}(x', (x, v - \bar{\epsilon})) + (1 - \alpha) V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}} - \delta)),$$

where $\alpha \in [0, 1]$ is such that $t' = \alpha \cdot V^{R_i}(\mathbf{0}, (x, v - \bar{\epsilon})) - (1 - \alpha)(t^*_{\mathbf{0}} - \delta)$.

For each $x' \in X \setminus \{y\}$, let

$$V^{R_i}(x', (y, s_y)) = s_{x'} - \frac{1}{2}\underline{\epsilon}.$$

Note that since $\bar{\epsilon}$ and $\underline{\epsilon}$ are sufficiently close to 0, for each $x' \in X$, $V^{R_i}(x', (y, s_y)) > V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}}))$.[38]

For each $t \in [t^*_{\mathbf{0}}, V^{R_i}(\mathbf{0}, (y, s_y))]$ and each $x' \in X$, let

$$V^{R_i}(x', (\mathbf{0}, t)) = \alpha \cdot V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) + (1 - \alpha) V^{R_i}(x', (y, s_y)),$$

where $\alpha \in [0, 1]$ is such that $t = \alpha \cdot t^*_{\mathbf{0}} + (1 - \alpha) V^{R_i}(\mathbf{0}, (y, s_y))$.

Finally, for each $t' \in \mathbb{R} \setminus [V^{R_i}(\mathbf{0}, (x, v - \bar{\epsilon})), V^{R_i}(\mathbf{0}, (y, s_y))]$ and each $x' \in X$, let

$$V^{R_i}(x', (\mathbf{0}, t')) = \begin{cases} V^{R_i}(x', (x, v - \bar{\epsilon})) - (V^{R_i}(\mathbf{0}, (x, v - \bar{\epsilon})) - t) & \text{if } t' < V^{R_i}(\mathbf{0}, (x, v - \bar{\epsilon})), \\ V^{R_i}(x', (y, s_y)) + t' - V^{R_i}(\mathbf{0}, (y, s_y)) & \text{if } t' > V^{R_i}(\mathbf{0}, (y, s_y)). \end{cases}$$

Figure 6 illustrates R_i.

Note that by the construction of R_i, R_i is bounded. It is also clear that $R_i \in \mathcal{R}^{MT}_{\mathbf{s}, y}$. Further, we also have $R_i \in \mathcal{R}^{MT}_{\mathbf{t}, x}$. To see this, let $x' \in X \setminus \{x\}$. First, suppose $x = \mathbf{0}$. By $t_{\mathbf{0}} \geq 0$ and the object monotonicity of $\mathbf{t}$, $t_{x'} > 0$ and thus $x' \in X_+ \setminus \{x\}$. Therefore, $V^{R_i}(x', (x, t_x)) = V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}}) = t^*_{x'} - \underline{\epsilon} < t_{x'}$, which

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[36] Formally, $V^{R_i}(x, (\mathbf{0}, t^*_{\mathbf{0}} - \delta)) = V^{R_i}(x, (\mathbf{0}, t^*_{\mathbf{0}})) - \delta = v - \delta > v - \bar{\epsilon}$.

[37] If $x = \mathbf{0}$, then by $t_x \geq 0$ and the object monotonicity of $\mathbf{t}$, $X_+ \supseteq X \setminus \{\mathbf{0}\}$. Thus, in this case, for each $x' \in X \setminus \{\mathbf{0}\}$, $V^{R_i}(x', (x, v - \bar{\epsilon})) = V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) - \bar{\epsilon}$.

[38] To see this, let $x' \in X$. If $x' = x$, then by $t_x < s_x$ and since $\bar{\epsilon}$ and $\underline{\epsilon}$ are sufficiently close to 0, $V^{R_i}(x, (y, s_y)) = s_x - \frac{1}{2}\underline{\epsilon} > t^*_x + \bar{\epsilon} = V^{R_i}(x, (\mathbf{0}, t^*_{\mathbf{0}}))$. Suppose $x' \in X_+ \setminus \{x\}$. Then, by $s_{x'} \geq t^*_{x'}$, $V^{R_i}(x', (y, s_y)) = s_{x'} - \frac{1}{2}\underline{\epsilon} > t^*_{x'} - \underline{\epsilon} = V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}}))$. Suppose $x' \in X \setminus X_+ \cup \{x\}$. By $s_{\mathbf{0}} > 0$ and the object monotonicity of $\mathbf{s}$, $s_{x'} > 0$. Thus, since $\underline{\epsilon}$ is sufficiently close to 0, $V^{R_i}(x', (y, s_y)) \geq s_{x'} - \frac{1}{2}\underline{\epsilon} > \frac{m(x')}{m(\bar{x})} \cdot \underline{\epsilon} = V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}}))$.

implies $(\mathbf{0}, t_{\mathbf{0}})\ P_i\ (x', t_{x'})$. Next, suppose $x \neq \mathbf{0}$. Then, by $t_x < s_x$, $t_x = t^*_x = v - \bar{\epsilon}$. If $x' \in X_+$, then $V^{R_i}(x', (x, t_x)) = V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) - \bar{\epsilon} = t^*_{x'} - \underline{\epsilon} - \bar{\epsilon} < t_{x'}$. If $x' \notin X_+$, then $V^{R_i}(x', (x, t_x)) = \min\{t_{x'}, V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}} - \delta))\} - \bar{\epsilon} < t_{x'}$. Thus, $(\mathbf{0}, t_{\mathbf{0}})\ P_i\ (x', t_{x'})$. Hence, $R_i \in \mathcal{R}^{MT}_{\mathbf{t},x}$.

Step 2. *R_i is object monotonic.*

Proof. To show that R_i is object monotonic, we need to prove that for each $t' \in \mathbb{R}$, the vector $(V^{R_i}(x', (\mathbf{0}, t')))_{x' \in X}$ is object monotonic. By the definition of R_i, for each $t' \in \mathbb{R}$, the vector $(V^{R_i}(x', (\mathbf{0}, t')))_{x' \in X}$ is either (i) obtained by shifting one of the following vectors $(V^{R_i}(x', (x, v - \bar{\epsilon})))_{x' \in X}$, $(V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}} - \delta)))_{x' \in X}$, $(V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})))_{x' \in X}$, and $(V^{R_i}(x', (y, s_y)))_{x' \in X}$, or (ii) a convex combination of two of these four vectors. Thus, we only need to show that these four vectors are object monotonic. Let $(x', y') \in \mathcal{X}$.

We first show that $(V^{R_i}(x'', (\mathbf{0}, t^*_{\mathbf{0}})))_{x'' \in X}$ is object monotonic. Suppose $y' \in X_+ \cup \{x\}$. By $t_x \geq 0$, $x' > y'$, and the object monotonicity of $\mathbf{t}$, we have $x' \in X_+$. Thus, $V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) \geq t^*_{x'} - \underline{\epsilon}$ and $V^{R_i}(y', (\mathbf{0}, t^*_{\mathbf{0}})) \leq t^*_{y'} + \bar{\epsilon}$. By $x' \in X_+$ and $s_x > t_x \geq 0$, $t^*_{x'} = \min\{t_{x'}, s_{x'}\} > \min\{t_{y'}, s_{y'}\} = t^*_{y'}$. Thus, since $\bar{\epsilon}$ and $\underline{\epsilon}$ are sufficiently close to 0, $V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) > V^{R_i}(y', (\mathbf{0}, t^*_{\mathbf{0}}))$. Next, suppose $y' \in X \setminus (X_+ \cup \{x\})$. Then, by $m(x') > m(y')$ and since $\bar{\epsilon}$ is sufficiently close to 0, $V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) \geq \frac{m(x')}{m(\bar{x})}\underline{\epsilon} > \frac{m(y')}{m(\bar{x})}\underline{\epsilon} = V^{R_i}(y', (\mathbf{0}, t^*_{\mathbf{0}}))$. Hence, $(V^{R_i}(x'', (\mathbf{0}, t^*_{\mathbf{0}})))_{x'' \in X}$ is object monotonic.

Since $V^{R_i}(x'', (\mathbf{0}, t^*_{\mathbf{0}} - \delta)) = V^{R_i}(x'', (\mathbf{0}, t^*_{\mathbf{0}})) - \delta$ for each $x'' \in X$ and $(V^{R_i}(x'', (\mathbf{0}, t^*_{\mathbf{0}})))_{x'' \in X}$ is object monotonic, $(V^{R_i}(x'', (\mathbf{0}, t^*_{\mathbf{0}} - \delta)))_{x'' \in X}$ is also object monotonic.

Next, we show that $(V^{R_i}(x'', (x, v - \bar{\epsilon})))_{x'' \in X}$ is object monotonic. Suppose $y' \in X_+ \cup \{x\}$. By $x' > y'$, $x' \in X_+$. Since $(V^{R_i}(x'', (\mathbf{0}, t^*_{\mathbf{0}})))_{x'' \in X}$ is object monotonic as we have shown, $V^{R_i}(x', (x, v - \bar{\epsilon})) = V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}}))) - \bar{\epsilon} > V^{R_i}(y', (\mathbf{0}, t^*_{\mathbf{0}}))) - \bar{\epsilon} = V^{R_i}(y', (x, v - \bar{\epsilon}))$. Next, suppose $y' \notin X_+ \cup \{x\}$. Then, $V^{R_i}(x', (x, v - \bar{\epsilon})) \geq \min\{t_{x'}, V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}} - \delta))\} - \bar{\epsilon}$ and $V^{R_i}(y', (x, v - \bar{\epsilon})) = \min\{t_{x'}, V^{R_i}(y', (\mathbf{0}, t^*_{\mathbf{0}} - \delta))\} - \bar{\epsilon}$. Since $t_{x'} > t_{y'}$ and $(V^{R_i}(x'', (\mathbf{0}, t^*_{\mathbf{0}} - \delta)))_{x'' \in X}$ is object monotonic as we have shown, $V^{R_i}(x', (x, v - \bar{\epsilon})) > V^{R_i}(y', (x, v - \bar{\epsilon}))$. Hence, $(V^{R_i}(x'', (x, v - \bar{\epsilon})))_{x'' \in X}$ is object monotonic.

Finally, since s is object monotonic and $\underline{\epsilon}$ is sufficiently close to 0, it is immediate that $(V^{R_i}(x'', (y, s_y)))_{x'' \in X}$ is object monotonic. Hence, R_i is object monotonic. ■

Step 3. $R_i \in \mathcal{R}^E$.

Proof. Let $(x', t') \in X(R_i)$. By Remark 4, it is enough to show $V^{R_i}(\bar{x}, (x', t')) - t' = V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0))$.

Let $s' \equiv V^{R_i}(\mathbf{0}, (x', t'))$. By $(x', t') \in X(R_i)$ and the definition of t^*, $s' \leq 0 \leq t^*_{\mathbf{0}}$. There are two cases.

Case 1. $X_+ = X$. By the construction of R_i, for each $x'' \in X$ and each $s'' \in \mathbb{R}$ with $s'' \leq t^*_{\mathbf{0}}$,

$$V^{R_i}(x'', (\mathbf{0}, s'')) = V^{R_i}(x'', (\mathbf{0}, t^*_{\mathbf{0}})) - (t^*_{\mathbf{0}} - s'').$$

Thus,

$$\begin{aligned} V^{R_i}(\bar{x}, (x', t')) - t' &= V^{R_i}(\bar{x}, (\mathbf{0}, t^*_{\mathbf{0}})) - (t^*_{\mathbf{0}} - s') - (V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) - (t^*_{\mathbf{0}} - s')) \\ &= V^{R_i}(\bar{x}, (\mathbf{0}, t^*_{\mathbf{0}})) - V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) \\ &= V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0)), \end{aligned}$$

where the last equality follows since $0 < t^*_{\mathbf{0}}$.

Case 2. $X \setminus X_+ \neq \emptyset$. By the object monotonicity of $\mathbf{t}$ and $X \setminus X_+ \neq \emptyset$, we have $\mathbf{0} \notin X_+$. Hence, $t^*_{\mathbf{0}} = 0$.

Suppose $-\delta \leq s' \leq 0$. Then,

$$\begin{aligned} V^{R_i}(\bar{x}, (x', t')) - t' &= V^{R_i}(\bar{x}, (\mathbf{0}, t^*_{\mathbf{0}})) - (t^*_{\mathbf{0}} - s') - (V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) - (t^*_{\mathbf{0}} - s')) \\ &= V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0)). \end{aligned}$$

Suppose $s' \leq V^{R_i}(\mathbf{0}, (x, v - \bar{\epsilon}))$. Note that if $x' \notin X_+ \cup \{x\}$, then $t_{x'} \leq 0$ and thus, $t' = V^{R_i}(x', (\mathbf{0}, s')) \leq V^{R_i}(x', (x, v - \bar{\epsilon})) = \min\{t_{x'}, V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}} - \delta))\} - \bar{\epsilon} < 0$. Thus, by $(x', t') \in X(R_i)$, $x' \in X_+ \cup \{x\}$.

Therefore,

$$\begin{aligned} &V^{R_i}(\bar{x}, (x', t')) - t' \\ &\quad = V^{R_i}(\bar{x}, (x, v - \bar{\epsilon})) - (V^{R_i}(\mathbf{0}, (x, v - \bar{\epsilon})) - s') - (V^{R_i}(x', (x, v - \bar{\epsilon})) - (V^{R_i}(\mathbf{0}, (x, v - \bar{\epsilon})) - s')) \\ &\quad = V^{R_i}(\bar{x}, (x, v - \bar{\epsilon})) - V^{R_i}(x', (x, v - \bar{\epsilon})) \\ &\quad = V^{R_i}(\bar{x}, (\mathbf{0}, t^*_{\mathbf{0}})) - \bar{\epsilon} - (V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) - \bar{\epsilon}) \\ &\quad = V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0)). \end{aligned}$$

Finally suppose $V^{R_i}(\mathbf{0}, (x, v - \bar{\epsilon})) < s' < -\delta$. Let $\alpha \in [0, 1]$ be such that $s' =$

$\alpha \cdot V^{R_i}(\mathbf{0}, (x, v - \bar{\epsilon})) - (1-\alpha)\delta$. Then,

$$
\begin{aligned}
&V^{R_i}(\bar{x}, (x'_i, t')) - t' \\
&= \alpha \cdot V^{R_i}(\bar{x}, (x, v - \bar{\epsilon})) + (1-\alpha)V^{R_i}(\bar{x}, (\mathbf{0}, -\delta)) - (\alpha \cdot V^{R_i}(x', (x, v - \bar{\epsilon})) + (1-\alpha)V^{R_i}(x', (\mathbf{0}, -\delta))) \\
&= V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0)).
\end{aligned}
$$

■

Proof of Lemma 4. Let $\mathbf{t}^* \in \mathbb{R}^{|X|}$ be such that for each $y \in X$,

$$t^*_y = \min\{t_y, s_y, 0\}.$$

Figure 7 illustrates $\mathbf{t}$, $\mathbf{s}$, and $\mathbf{t}^*$. It is easy to see that $\mathbf{t}^*$ is *weakly object monotonic*, i.e., for each $(y, y') \in \mathcal{X}$, $t^*_y \geq t^*_{y'}$. Note also that by $t_x < s_x$ and $t_x < 0$, $t^*_x = t_x$. Let $(\epsilon_y)_{y \in X} \in \mathbb{R}^{|X|}_{++}$ be an object monotonic vector that is sufficiently close to $\mathbf{0}$.[39] Fix $y \in X \setminus \{x\}$. We prove that there is $R_i \in \mathcal{R}^E$ such that $R_i \in \mathcal{R}^{MT}_{\mathbf{t},x} \cap \mathcal{R}^{MT}_{\mathbf{s},y}$. We do the proof in three steps.

Step 1. *Construction of a preference relation.*

We define a preference relation R_i as follows. For each $x' \in X \setminus \{x\}$, let

$$V^{R_i}(x', (x, t_x)) = t^*_x - \epsilon_{\bar{x}-x'}.$$

Let $\delta \in \mathbb{R}_{++}$ be sufficiently close to 0.[40] For each $x' \in X \setminus \{y\}$, let

$$V^{R_i}(x', (y, s_y)) = s_{x'} - \delta.$$

Note that since δ is sufficiently close to 0, for each $x' \in X$, $V^{R_i}(x', (x, t_x)) < V^{R_i}(x', (y, s_y))$.

Let $d : X \to \mathbb{R}$ be such that for each $x' \in X$,

$$d(x') = V^{R_i}(\bar{x}, (y, s_y)) - \max\{V^{R_i}(x', (y, s_y)), 0\}.$$

Note that for each $(x', y') \in \mathcal{X}$, $d(x') \leq d(y')$.

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[39] Formally, the proof works if $(\epsilon_y)_{y \in X}$ is object monotonic and satisfies $0 < \epsilon_{\mathbf{0}}$ and $\epsilon_{\bar{x}} < \min\{-t_x, \min_{(x',y') \in \mathcal{X}} t_{x'} - t_{y'}\}$.

[40] Formally, the proof works if δ satisfies $0 < \delta < \epsilon_{\mathbf{0}}$, $\delta < \min\{s_{x'} - t_{x'} : s_{x'} > t_{x'}, x' \in X\}$, and $\delta < \min_{(x',y') \in \mathcal{X}} s_{x'} - s_{y'}$.

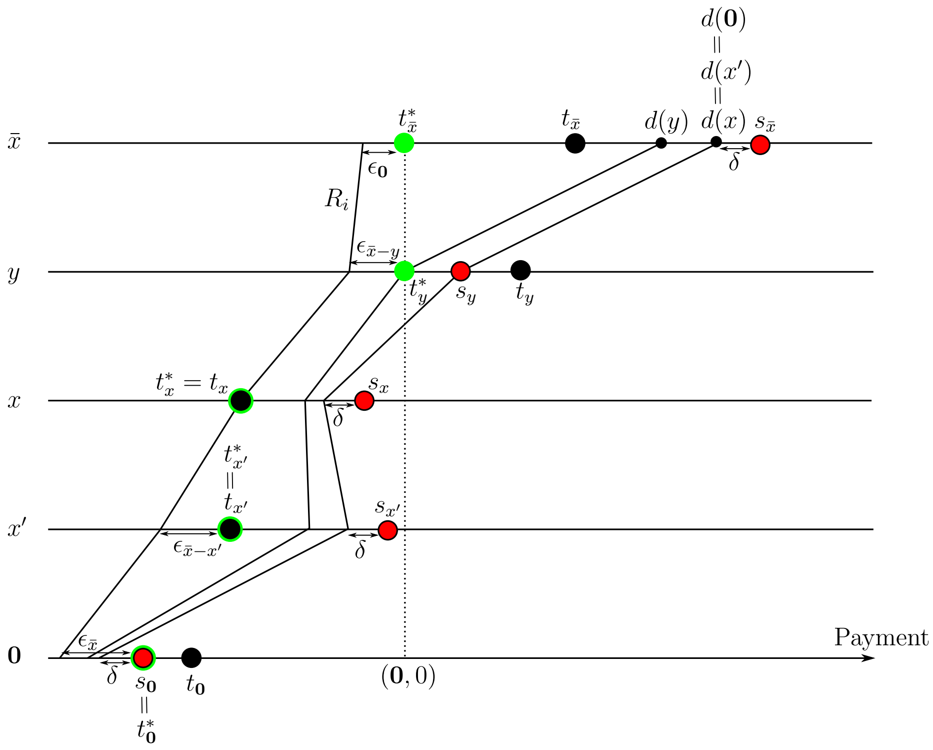

Figure 7: An illustration of $\mathbf{t}$, $\mathbf{s}$, $\mathbf{t}^*$, and R_i.

For each $t' \in [V^{R_i}(\bar{x}, (x, t_x)), V^{R_i}(\bar{x}, (y, s_y))]$ and each $x' \in X$, let

$$V^{R_i}(x', (\bar{x}, t')) = \begin{cases} t' - d(x') & \text{if } t' > d(x'), \\ \alpha \cdot \min\{V^{R_i}(x', (y, s_y)), 0\} + (1-\alpha) V^{R_i}(x', (x, t_x)) & \text{if } t' \leq d(x'), \end{cases}$$

where $\alpha \in [0, 1]$ is such that $t' = \alpha \cdot d(x') + (1-\alpha) V^{R_i}(\bar{x}, (x, t_x))$.

Finally, for each $t' \in \mathbb{R} \setminus [V^{R_i}(\mathbf{0}, (x, t_x)), V^{R_i}(\mathbf{0}, (y, s_y))]$ and each $x' \in X$, let

$$V^{R_i}(x', (\mathbf{0}, t')) = \begin{cases} V^{R_i}(x', (x, t_x)) - (V^{R_i}(\mathbf{0}, (x, t_x)) - t') & \text{if } t' < V^{R_i}(\mathbf{0}, (x, t_x)), \\ V^{R_i}(x', (y, s_y)) + (t' - V^{R_i}(\mathbf{0}, (y, s_y))) & \text{if } t' > V^{R_i}(\mathbf{0}, (y, s_y)). \end{cases}$$

Figure 7 illustrates R_i. Note that by the construction of R_i, R_i is bounded. Further, it is clear that $R_i \in \mathcal{R}^{MT}_{\mathbf{t},x} \cap \mathcal{R}^{MT}_{\mathbf{s},y}$.

Step 2. *R_i is object monotonic.*

Proof. To show that R_i is object monotonic, we need to prove the object monotonicity of $(V^{R_i}(x', (x, t_x)))_{x' \in X}$, $(V^{R_i}(x', (y, s_y)))_{x' \in X}$, and $(V^{R_i}(x', (\bar{x}, t')))_{x' \in X}$ for each $t' \in$

$[V^{R_i}(\bar{x},(x,t_x)), V^{R_i}(\bar{x},(y,s_y))]$. First observe that since $\mathbf{t}^*$ is weakly object monotonic and $(\epsilon_{x'})_{x'\in X}$ is object monotonic, $(V^{R_i}(x',(x,t_x)))_{x'\in X}$ is object monotonic. Note also that since s is object monotonic and δ is sufficiently close to 0, $(V^{R_i}(x',(y,s_y)))_{x'\in X}$ is object monotonic.

Let $t' \in [V^{R_i}(\bar{x},(x,t_x)), V^{R_i}(\bar{x},(y,s_y))]$. Now we show that $(V^{R_i}(x',(\bar{x},t')))_{x'\in X}$ is object monotonic. Let $(x',y') \in \mathcal{X}$. Observe that if $V^{R_i}(y',(y,s_y)) > 0$, then by the object monotonicity of $(V^{R_i}(x'',(y,s_y)))_{x''\in X}$, we have $d(x') = V^{R_i}(\bar{x},(y,s_y)) - V^{R_i}(x',(y,s_y))$ and $d(y') = V^{R_i}(\bar{x},(y,s_y)) - V^{R_i}(y',(y,s_y))$, and hence, $d(x') < d(y')$. There are three cases.

Case 1. *$t' > d(y')$.* By $t' \leq V^{R_i}(\bar{x},(y,s_y))$, $d(y') < V^{R_i}(\bar{x},(y,s_y))$. This inequality and the definition of d imply $V^{R_i}(y',(y,s_y)) > 0$. Thus, $d(x') < d(y') < t'$. Therefore,

$$V^{R_i}(x',(\bar{x},t')) = t' - d(x') > t' - d(y') = V^{R_i}(y',(\bar{x},t')).$$

Case 2. *$d(x') < t' \leq d(y')$.*[41] By $d(x') < t'$, $V^{R_i}(x',(\bar{x},t')) = t' - d(x') > 0$. On the other hand, by $t' \leq d(y')$ and $V^{R_i}(y',(x,t_x)) < 0$, $V^{R_i}(y',(\bar{x},t')) < 0$. Thus, $V^{R_i}(x',(\bar{x},t')) > V^{R_i}(y',(\bar{x},t'))$.

Case 3. *$t' \leq d(x')$ and $t' \leq d(y')$.* By the definition of R_i,

$$V^{R_i}(x',(\bar{x},t')) = \alpha \cdot \min\{V^{R_i}(x',(y,s_y)),0\} + (1-\alpha)V^{R_i}(x',(x,t_x)), \text{ and}$$
$$V^{R_i}(y',(\bar{x},t')) = \beta \cdot \min\{V^{R_i}(y',(y,s_y)),0\} + (1-\beta)V^{R_i}(y',(x,t_x)),$$

where $\alpha \in [0,1]$ and $\beta \in [0,1]$ are such that $t' = \alpha \cdot d(x') + (1-\alpha)V^{R_i}(\bar{x},(x,t_x))$ and $t' = \beta \cdot d(y') + (1-\beta)V^{R_i}(\bar{x},(x,t_x))$, respectively. Note that by $d(x') \leq d(y')$, $\alpha \geq \beta$.

Suppose $V^{R_i}(x',(y,s_y)) > 0$. Then, $V^{R_i}(x',(\bar{x},t')) = (1-\alpha)V^{R_i}(x',(x,t_x))$. By $x' > y'$ and the definition of $(V^{R_i}(x'',(x,t_x)))_{x''\in X}$, $0 > V^{R_i}(x',(x,t_x)) > V^{R_i}(y',(x,t_x))$. Thus, by $(1-\alpha) \leq (1-\beta)$,

$$V^{R_i}(x',(\bar{x},t')) = (1-\alpha)V^{R_i}(x',(x,t_x)) > (1-\beta)V^{R_i}(y',(x,t_x)) \geq V^{R_i}(y',(\bar{x},t')).$$

Suppose $V^{R_i}(x',(y,s_y)) \leq 0$. By the object monotonicity of $(V^{R_i}(x'',(y,s_y)))_{x''\in X}$, $V^{R_i}(y',(y,s_y)) < V^{R_i}(x',(y,s_y)) \leq 0$. Also, by the object monotonicity of $(V^{R_i}(x'',(x,t_x)))_{x''\in X}$,

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[41] By $d(x') \leq d(y')$, it cannot be the case that $d(y') < t' \leq d(x')$.

$V^{R_i}(x', (x, t_x)) > V^{R_i}(y', (x, t_x))$. Thus,

$$\begin{aligned} V^{R_i}(x', (\bar{x}, t')) &= \alpha \cdot V^{R_i}(x', (y, s_y)) + (1-\alpha) V^{R_i}(x', (x, t_x)) \\ &> \alpha \cdot V^{R_i}(y', (y, s_y)) + (1-\alpha) V^{R_i}(y', (x, t_x)) \\ &\geq \beta \cdot V^{R_i}(y', (y, s_y)) + (1-\beta) V^{R_i}(y', (x, t_x)) \\ &= V^{R_i}(y', (\bar{x}, t')), \end{aligned}$$

where the second inequality follows from $\alpha \geq \beta$ and $V^{R_i}(y', (y, s_y)) > V^{R_i}(y', (x, t_x))$.
■

Step 3. $R_i \in \mathcal{R}^E$.

Proof. First we show that for each $x' \in X$, $V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0)) = V^{R_i}(\bar{x}, (y, s_y)) - V^{R_i}(x', (y, s_y))$. If $V^{R_i}(\mathbf{0}, (y, s_y)) \leq 0$, it is clear that this equality holds. Suppose $V^{R_i}(\mathbf{0}, (y, s_y)) > 0$. Then, $V^{R_i}(\mathbf{0}, (\bar{x}, d(\mathbf{0}))) = 0$, and this implies $V^{R_i}(\bar{x}, (\mathbf{0}, 0)) = d(\mathbf{0})$. By $V^{R_i}(\mathbf{0}, (y, s_y)) > 0$, for each $x' \in X \setminus \{\mathbf{0}\}$, $V^{R_i}(x', (y, s_y)) > 0$ and thus $d(x') < d(\mathbf{0})$. Thus, for each $x' \in X$,

$$\begin{aligned} V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0)) &= d(\mathbf{0}) - V^{R_i}(x', (\bar{x}, d(\mathbf{0}))) \\ &= d(\mathbf{0}) - (d(x') - d(\mathbf{0})) \\ &= V^{R_i}(\bar{x}, (y, s_y)) - V^{R_i}(x', (y, s_y)). \end{aligned}$$

Let $(x', t') \in X(R_i)$. By Remark 4, it is enough to show $V^{R_i}(\bar{x}, (x', t')) - t' = V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0))$.

If $V^{R_i}(\mathbf{0}, (x', t')) \geq V^{R_i}(\mathbf{0}, (y, s_y))$, then by the definition of R_i, $V^{R_i}(\bar{x}, (x', t')) - t' = V^{R_i}(\bar{x}, (y, s_y)) - V^{R_i}(x', (y, s_y)) = V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0))$.

Suppose $V^{R_i}(\mathbf{0}, (x', t')) < V^{R_i}(\mathbf{0}, (y, s_y))$. Let $s' \equiv V^{R_i}(\bar{x}, (x', t'))$. There are three cases.

Case 1. $s' < d(x')$. By the definition of R_i, $t' = V^{R_i}(x', (\bar{x}, s')) < 0$. This contradicts the fact that $(x', t') \in X(R_i)$.

Case 2. $s' = d(x')$. By the definition of R_i, $t' = V^{R_i}(x', (\bar{x}, s')) = \min\{V^{R_i}(x', (y, s_y)), 0\} \leq 0$. By $t' \geq 0$, $t' = 0$ and $V^{R_i}(x', (y, s_y)) \geq 0$. Thus,

$$V^{R_i}(\bar{x}, (x', t')) - t' = d(x') = V^{R_i}(\bar{x}, (y, s_y)) - V^{R_i}(x', (y, s_y)) = V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0)).$$

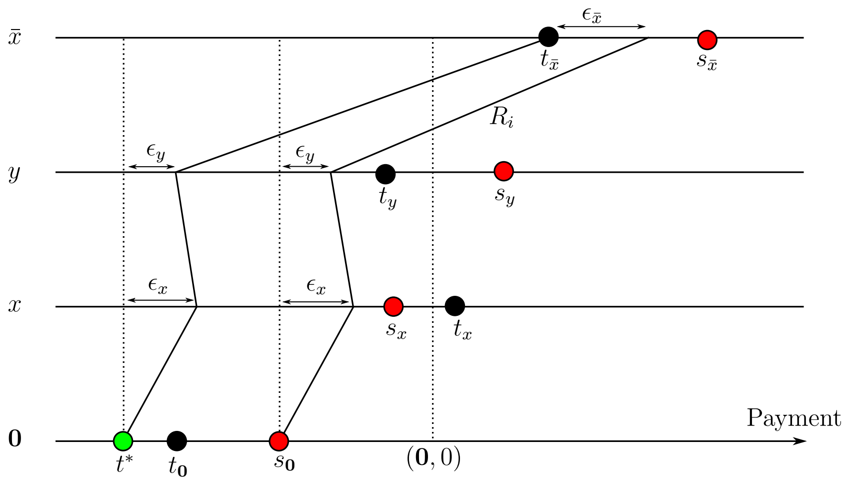

Figure 8: An illustration of R_i.

Case 3. $s' > d(x')$. By $V^{R_i}(\mathbf{0}, (x', t')) < V^{R_i}(\mathbf{0}, (y, s_y))$, $s' < V^{R_i}(\bar{x}, (y, s_y))$. Thus, by $s' > d(x')$, $d(x') < V^{R_i}(\bar{x}, (y, s_y))$, which implies $V^{R_i}(x', (y, s_y)) > 0$. Therefore,

$$\begin{aligned} V^{R_i}(\bar{x}, (x', t')) - t' &= s' - (s' - d(x')) \\ &= V^{R_i}(\bar{x}, (y, s_y)) - V^{R_i}(x', (y, s_y)) \\ &= V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0)). \end{aligned}$$

■

Proof of Lemma 5. We do the proof in three steps.

Step 1. *Construction of a preference relation.*

We define a preference relation R_i as follows. Let $t^* \in \mathbb{R}$ be such that $t^* < \min\{t_{\mathbf{0}}, s_{\mathbf{0}}\}$. Let $(\epsilon_x)_{x \in X} \in \mathbb{R}_+^{|X|}$ be an object monotonic vector such that $\epsilon_{\mathbf{0}} = 0$ and for each $x \in X \setminus \{\mathbf{0}\}$, $\epsilon_x > 0$ but sufficiently close to 0.[42] In particular, we take $(\epsilon_x)_{x \in X}$ so that it satisfies $s_{\mathbf{0}} + \epsilon_{\bar{x}} < 0$. For each $x \in X \setminus \{\bar{x}\}$, let

$$V^{R_i}(x, (\bar{x}, t_{\bar{x}})) = t^* + \epsilon_x.$$

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[42] Formally, the proof works if $(\epsilon_x)_{x \in X} \in \mathbb{R}_+^{|X|}$ is an object monotonic vector that satisfies $\epsilon_{\mathbf{0}} = 0$ and $\epsilon_{\bar{x}} < \min\{t_{\mathbf{0}} - t^*, \min_{x \in X \setminus \{\mathbf{0}\}} s_x - s_{\mathbf{0}}, s_{\bar{x}} - t_{\bar{x}}, -s_{\mathbf{0}}\}$.

For each $x \in X \setminus \{\mathbf{0}\}$, let

$$V^{R_i}(x,(\mathbf{0},s_{\mathbf{0}})) = \begin{cases} s_{\mathbf{0}} + \epsilon_x & \text{if } x \neq \bar{x}, \\ \max\{t_{\bar{x}}, s_{\mathbf{0}}\} + \epsilon_{\bar{x}} & \text{if } x = \bar{x}. \end{cases}$$

For each $t' \in \mathbb{R}$ with $t^* < t' < s_{\mathbf{0}}$, and each $x \in X$, let

$$V^{R_i}(x,(\mathbf{0},t')) = \alpha \cdot V^{R_i}(x,(\mathbf{0},t^*)) + (1-\alpha)V^{R_i}(x,(\mathbf{0},s_{\mathbf{0}})),$$

where $\alpha \in [0,1]$ is such that $t' = \alpha \cdot t^* + (1-\alpha)s_{\mathbf{0}}$.

Finally, for each $t' \in \mathbb{R} \setminus [t^*, s_{\mathbf{0}}]$ and each $x \in X$, let

$$V^{R_i}(x,(\mathbf{0},t')) = \begin{cases} V^{R_i}(x,(\mathbf{0},t^*)) - (t^* - t') & \text{if } t' < t^*, \\ V^{R_i}(x,(\mathbf{0},s_{\mathbf{0}})) + (t' - s_{\mathbf{0}}) & \text{if } t' > s_{\mathbf{0}}. \end{cases}$$

Figure 8 illustrates R_i. Note that by the construction of R_i, R_i is bounded. For each $x \in X \setminus \{\bar{x}\}$, since $t^* < t_{\mathbf{0}}$ and ϵ_x is sufficiently close to 0, $V^{R_i}(x,(\bar{x},t_{\bar{x}})) = t^* + \epsilon_x < t_x$. Thus, $R_i \in \mathcal{R}^{MT}_{\mathbf{t},\bar{x}}$. Since $t_{\bar{x}} < s_{\bar{x}}$ and $\epsilon_{\bar{x}}$ is sufficiently close to 0, $V^{R_i}(\bar{x},(\mathbf{0},s_{\mathbf{0}})) = \max\{t_{\bar{x}}, s_{\mathbf{0}}\} + \epsilon_{\bar{x}} < s_{\bar{x}}$. Further, for each $x \in X \setminus \{\mathbf{0},\bar{x}\}$, since ϵ_x is sufficiently close to 0, $V^{R_i}(x,(\mathbf{0},s_{\mathbf{0}})) = s_{\mathbf{0}} + \epsilon_x < s_x$. Thus, $R_i \in \mathcal{R}^{MT}_{\mathbf{s},\mathbf{0}}$.

Step 2. *R_i is object monotonic.*

Proof. To show R_i is object monotonic, it suffices to show the object monotonicity of $(V^{R_i}(x,(\bar{x},t_x)))_{x \in X}$ and $(V^{R_i}(x,(\mathbf{0},s_{\mathbf{0}})))_{x \in X}$. Let $(x,y) \in \mathcal{X}$.

First we show that $(V^{R_i}(x',(\bar{x},t_x)))_{x' \in X}$ is object monotonic. If $x = \bar{x}$, then since $t^* < t_{\bar{x}}$ and ϵ_y is sufficiently close to 0, $V^{R_i}(\bar{x},(\bar{x},t_{\bar{x}})) = t_{\bar{x}} > t^* + \epsilon_y = V^{R_i}(y,(\bar{x},t_{\bar{x}}))$. Suppose $x \neq \bar{x}$. Then, since $(\epsilon_{x'})_{x' \in X}$ is object monotonic, $V^{R_i}(x,(\bar{x},t_{\bar{x}})) = t^* + \epsilon_x > t^* + \epsilon_y = V^{R_i}(y,(\bar{x},t_{\bar{x}}))$. Hence, $(V^{R_i}(x',(\bar{x},t_x)))_{x' \in X}$ is object monotonic.

Next, we show that $(V^{R_i}(x',(\mathbf{0},s_{\mathbf{0}})))_{x' \in X}$ is object monotonic. Note that $V^{R_i}(x,(\mathbf{0},s_{\mathbf{0}})) \geq s_{\mathbf{0}} + \epsilon_x$, and since $y \neq \bar{x}$, $V^{R_i}(y,(\mathbf{0},s_{\mathbf{0}})) = s_{\mathbf{0}} + \epsilon_y$. Thus, since $(\epsilon_{x'})_{x' \in X}$ is object monotonic, $V^{R_i}(x,(\mathbf{0},s_{\mathbf{0}})) > V^{R_i}(y,(\mathbf{0},s_{\mathbf{0}}))$. Therefore, $(V^{R_i}(x',(\mathbf{0},s_{\mathbf{0}})))_{x' \in X}$ is object monotonic, and hence, R_i is object monotonic. ■

Step 3. $R_i \in \mathcal{R}^E$.

Note that by $s_{\mathbf{0}} < 0$, for each $x \in X$, $V^{R_i}(\bar{x},(\mathbf{0},0)) - V^{R_i}(x,(\mathbf{0},0)) = V^{R_i}(\bar{x},(\mathbf{0},s_{\mathbf{0}})) - V^{R_i}(x,(\mathbf{0},s_{\mathbf{0}}))$. Let $(x,t') \in X(R_i)$. By Remark 4, it is enough to show $V^{R_i}(\bar{x},(x,t')) -$

$t' = V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x, (\mathbf{0}, 0))$. Without loss of generality, assume $x \neq \bar{x}$. Let $s' \equiv V^{R_i}(\mathbf{0}, (x, t'))$.

By $(x, t') \in X(R_i)$, $t' \geq 0$. Since $s_{\mathbf{0}} < 0$ and ϵ_x is sufficiently close to 0, $V^{R_i}(x, (\mathbf{0}, s_{\mathbf{0}})) = s_{\mathbf{0}} + \epsilon_x < 0 \leq t'$. Thus, $s' > s_{\mathbf{0}}$. Therefore, by the definition of R_i,

$$\begin{aligned}
V^{R_i}(\bar{x}, (x, t')) - t' &= V^{R_i}(\bar{x}, (\mathbf{0}, s')) - V^{R_i}(x, (\mathbf{0}, s')) \\
&= V^{R_i}(\bar{x}, (\mathbf{0}, s_{\mathbf{0}})) + (s' - s_{\mathbf{0}}) - (V^{R_i}(x, (\mathbf{0}, s_{\mathbf{0}})) + (s' - s_{\mathbf{0}})) \\
&= V^{R_i}(\bar{x}, (\mathbf{0}, s_{\mathbf{0}})) - V^{R_i}(x, (\mathbf{0}, s_{\mathbf{0}})) \\
&= V^{R_i}(\bar{x}, (\mathbf{0}, s_{\mathbf{0}})) - V^{R_i}(x, (\mathbf{0}, s_{\mathbf{0}})).
\end{aligned}$$

Hence, $R_i \in \mathcal{R}^E$. ■

Proof of Lemma 6. Assume for contradiction that $\sum_{i \in N'} V^{R_i}(x_i, f_i(R)) > \sum_{i \in N'} t_i^f(R)$. Let $((y_i, s_i))_{i \in N} \in Z$ be such that for each $i \in N$

$$(y_i, s_i) = \begin{cases} (x_i, V^{R_i}(x_i, f_i(R))) & \text{if } i \in N', \\ f_i(R) & \text{otherwise.} \end{cases}$$

It is clear that for each $i \in N$, $(y_i, s_i)\ I_i\ f_i(R)$. Moreover, $\sum_{i \in N} s_i = \sum_{i \in N'} V^{R_i}(x_i, f_i(R)) + \sum_{i \in N \setminus N'} t_i^f(R) > \sum_{i \in N} t_i^f(R)$. By Remark 3, this contradicts *efficiency*. ■

Proof of Lemma 7. Assume for contradiction that $x_i^f(R) \neq \mathbf{0}$ and there is $j \in N \setminus \{i\}$ such that $R_i \in \mathcal{R}^Q(R_j)$. Denote $x \equiv x_i^f(R) + x_j^f(R)$. By $x_i^f(R) \neq \mathbf{0}$, $x > x_j^f(R)$. By $R_i \in \mathcal{R}^Q(R_j)$, $v_i(x_i^f(R)) < V^{R_j}(x, f_j(R)) - t_j^f(R)$.

Since $V^{R_i}(\mathbf{0}, f_i(R)) = t_i^f(R) - v_i(x_i^f(R))$, $V^{R_i}(\mathbf{0}, f_i(R)) + V^{R_j}(x, f_j(R)) = t_i^f(R) - v_i(x_i^f(R)) + V^{R_j}(x, f_j(R)) > t_i^f(R) + t_j^f(R)$. This contradicts Lemma 6. ■

Proof of Lemma 8. Since R_j is bounded for each $j \in N \setminus \{i\}$, there is a pair $\bar{s}, \underline{s} \in \mathbb{R}_{++}$ such that for each $j \in N \setminus \{i\}$, each $(x, y) \in \mathcal{X}$, and each $t \in \mathbb{R}$, $\underline{s} < V^{R_j}(x, (y, t)) - t < \bar{s}$.

Let $(\epsilon_x)_{x \in X} \in \mathbb{R}_+^{|X|}$ be an object monotonic vector such that $\epsilon_{\bar{x}} < \underline{s}$. Let $x \in X$. We show that there is $R_i \in \mathcal{R}^Q$ such that $x_i(R_i, R_{-i}) = x$.

Let $R_i \in \mathcal{R}^Q$ be such that for each $y \in X \setminus \{\mathbf{0}\}$,

$$v_i(y) = \begin{cases} n \cdot \bar{s} + \epsilon_y & \text{if } y \geq x, \\ \epsilon_y & \text{otherwise.} \end{cases}$$

Since $(\epsilon_x)_{x \in X}$ is object monotonic, R_i is object monotonic. For simplicity, denote $R \equiv (R_i, R_{-i})$. Assume for contradiction that $x_i^f(R) \neq x$. There are two cases.

Case 1. $x_i^f(R) > x$. Take any $j \in N \setminus \{i\}$. Denote $y \equiv x_j^f(R) + (x_i^f(R) - x)$. Then, $y + x = x_i^f(R) + x_j^f(R)$, and by $x_i^f(R) > x$, $y > x_j^f(R)$. By the definition of $(\epsilon_{y'})_{y' \in X}$,

$$\begin{aligned} V^{R_i}(x, f_i(R)) + V^{R_j}(y, f_j(R)) &= t_i^f(R) + v_i(x) - v_i(x_i^f(R)) + V^{R_j}(y, f_j(R)) - t_j^f(R) + t_j^f(R) \\ &> t_i^f(R) + \epsilon_x - \epsilon_{x_i^f(R)} + \underline{s} + t_j^f(R) \\ &> t_i^f(R) + t_j^f(R), \end{aligned}$$

which contradicts Lemma 6.

Case 2. $x_i^f(R) \not\geq x$. By $x_i^f(R) \neq x$, $v_i(x) - v_i(x_i^f(R)) = n \cdot \bar{s} + \epsilon_x - \epsilon_{x_i^f(R)} > n \cdot \bar{s}$. Thus,

$$\begin{aligned} & V^{R_i}(x, f_i(R)) + \sum_{j \in N \setminus \{i\}} V^{R_j}(\mathbf{0}, f_j(R)) \\ &\quad = t_i^f(R) + v_i(x) - v_i(x_i^f(R)) + \sum_{j \in N \setminus \{i\}} (V^{R_j}(\mathbf{0}, f_j(R)) - t_j^f(R) + t_j^f(R)) \\ &\quad > n \cdot \bar{s} - (n-1) \cdot \bar{s} + \sum_{j \in N} t_j^f(R) \\ &\quad > \sum_{j \in N} t_j^f(R), \end{aligned}$$

which contradicts Lemma 6. ■

Proof of Lemma 9. Let $(x, y) \in \mathcal{X}$. By $X_i^f(R_{-i}) = X$, there are $R_i, R_i' \in \mathcal{R}$ such that $x_i^f(R_i, R_{-i}) = x$ and $x_i^f(R_i', R_{-i}) = y$. By *strategy-proofness*, $f_i(R_i', R_{-i})$ R_i' $f_i(R_i, R_{-i})$. This implies $V^{R_i'}(x, f_i(R_i', R_{-i})) \leq t_i(R_i, R_{-i})$. By this and the object monotonicity of R_i',

$$t_i^f(R_{-i}; y) = t_i^f(R_i', R_{-i}) < V^{R_i'}(x, f_i(R_i', R_{-i})) \leq t_i^f(R_i, R_{-i}) = t_i^f(R_{-i}; x).$$

Thus, $(t_i^f(R_{-i}; x'))_{x' \in X}$ is object monotonic. ■

Proof of Lemma 11. We prove only (ii), because we can prove (i) by setting $s = 0$ and following the proof of (ii).

Without loss of generality, assume $i = 1$ and $j = 2$. By $R_3, \ldots, R_n \in \mathcal{R}^Q$ and Lemma 8, for each $R'_2 \in \mathcal{R}^Q$, $X_1^f(R'_2, R_{-1,2}) = X$. By Lemma 2, for each $R'_2 \in \mathcal{R}^Q$ with $R_3, \ldots, R_n \in \mathcal{R}^Q(R'_2)$ and each $x \in X$, $\sigma_1(R'_2, R_{-1,2}; x) = v'_2(\bar{x} - x)$. Thus, by $R_3, \ldots, R_n \in \mathcal{R}^Q$ and Fact 4, there is $h_1 : (\mathcal{R}^Q)^{n-1} \to \mathbb{R}$ such that for each $R'_2 \in \mathcal{R}^Q$ with $R_3, \ldots, R_n \in \mathcal{R}^Q(R'_2)$, and each $x \in X$, $t_1^f(R'_2, R_{-1,2}; x) = h_1(R'_2, R_{-1,2}) - v'_2(\bar{x} - x)$. This implies that for each $R'_2 \in \mathcal{R}^Q$ with $R_3, \ldots, R_n \in \mathcal{R}^Q(R'_2)$, and each pair $x, y \in X$

$$\begin{aligned} t_1^f(R'_2, R_{-1,2}; x) &= h_1(R'_2, R_{-1,2}) - v'_2(\bar{x} - x) + v'_2(\bar{x} - y) - v'_2(\bar{x} - y) \\ &= t_1^f(R'_2, R_{-1,2}; y) + v'_2(\bar{x} - y) - v'_2(\bar{x} - x). \end{aligned} \tag{5}$$

Take any $a \in M$ and let $e^a = (e_1^a, \ldots, e_m^a) \in X$ be such that for each $\ell \in M$, $e_\ell^a = 1$ if $\ell = a$ and $e_\ell^a = 0$ otherwise. Let $x \equiv \bar{x} - e^a$. Let

$$\mathcal{R}^* = \{R'_2 \in \mathcal{R}^Q : \text{for each } (y, y') \in \mathcal{X}, v'_2(y) - v'_2(y') > v_2(y) - v_2(y')\}.$$

Note that since $R_3, \ldots, R_n \in \mathcal{R}^Q(R_2)$, for each $R'_2 \in \mathcal{R}^*$, $R_3, \ldots, R_n \in \mathcal{R}^Q(R'_2)$.

Let $\mathbf{t} \in \mathbb{R}^{|X|}$ be such that for each $y \in X$, $t_y = t_1^f(R_{-1}; y)$. By $X_1^f(R_{-1}) = X$ and Lemma 9, $\mathbf{t}$ is object monotonic.

Step 1. $t_{\mathbf{0}} \geq -s^*$.

Proof. Assume for contradiction that $t_{\mathbf{0}} < -s^*$. We first show that we can assume $t_{e^a} < 0$ without loss of generality. To show this, we prove that there is $R_2^* \in \mathcal{R}^*$ such that $t_1^f(R_2^*, R_{-1,2}; \mathbf{0}) < -s^*$ and $t_1^f(R_2^*, R_{-1,2}; e^a) < 0$.

Let $R_2^* \in \mathcal{R}^Q$ be such that $v_2^*(\bar{x}) > v_2(\bar{x})$ and $v_2^*(\bar{x}) - v_2^*(x) < -t_{\mathbf{0}}$. Let $\mathbf{s} \in \mathbb{R}^{|X|}$ be such that for each $y \in X$, $s_y = t_1^f(R_2^*, R_{-1,2}; y)$. We first prove $s_{\mathbf{0}} \leq t_{\mathbf{0}}$.

Assume for contradiction that $s_{\mathbf{0}} > t_{\mathbf{0}}$. By $t_{\mathbf{0}} < 0$, $\mathbf{t}$ and $\mathbf{s}$ satisfy the condition of Lemma 4 for $\mathbf{0}$. Thus, by Lemma 4, there is $R'_1 \in \mathcal{R}^E$ such that $R'_1 \in \mathcal{R}^{MT}_{\mathbf{t},\mathbf{0}} \cap \mathcal{R}^{MT}_{\mathbf{s},\bar{x}}$. By Lemma 10, $x_1^f(R'_1, R_{-1}) = \mathbf{0}$ and $x_1^f(R'_1, R_2^*, R_{-1,2}) = \bar{x}$. Then, $x_2^f(R'_1, R_2^*, R_{-1,2}) = \mathbf{0}$. Further, by $R_3, \ldots, R_n \in \mathcal{R}^Q(R_2)$, Lemma 7 implies that for each $j \in N \setminus \{1, 2\}$, $x_j^f(R'_1, R_{-1}) = \mathbf{0}$. This implies $x_2^f(R'_1, R_{-1}) = \bar{x}$. However, by $v_2^*(\bar{x}) > v_2(\bar{x})$, this contradicts Fact 3. Hence, $s_{\mathbf{0}} \leq t_{\mathbf{0}}$.

By $t_{\mathbf{0}} < -s^*$ and $s_{\mathbf{0}} < -s^*$. Further, by $s_{\mathbf{0}} \leq t_{\mathbf{0}} < 0$, $v_2^*(\bar{x}) - v_2^*(x) < -t_{\mathbf{0}}$, and (5),

$$s_{e^a} = s_{\mathbf{0}} + v_2^*(\bar{x}) - v_2^*(x) < s_{\mathbf{0}} - t_{\mathbf{0}} \leq 0.$$

Hence, we can assume $t_{e^a} < 0$ without loss of generality.

Let $R_2', R_2'' \in \mathcal{R}^*$ be such that

$$v_2''(x) < v_2'(x) \text{ and } v_2'(\bar{x}) < v_2''(\bar{x}).$$

Let $\mathbf{s}', \mathbf{s}'' \in \mathbb{R}^{|X|}$ be such that for each $y \in X$, $s_y' = t_1^f(R_2', R_{-1,2}; y)$ and $s_y'' = t_1^f(R_2'', R_{-1,2}; y)$, respectively. By $X_1^f(R_2', R_{-1,2}) = X_1^f(R_2'', R_{-1,2}) = X$ and Lemma 9, $\mathbf{s}'$ and $\mathbf{s}''$ are object monotonic.

Note that by $R_2', R_2'' \in \mathcal{R}^*$, $R_3, \ldots, R_n \in \mathcal{R}^Q(R_2) \cap \mathcal{R}^Q(R_2') \cap \mathcal{R}^Q(R_2'')$. Thus, for each $R_1' \in \mathcal{R}$ and each $i \in N \setminus \{1, 2\}$, $x_i^f(R_1', R_{-1}) = x_i^f(R_1', R_2', R_{-1,2}) = x_i^f(R_1', R_2'', R_{-1,2}) = \mathbf{0}$.

Claim 1. $s_{e^a}' = s_{e^a}''$.

Proof. To complete the proof, it is enough to show $s_{e^a}' = t_{e^a}$ and $s_{e^a}'' = t_{e^a}$. We focus only on the proof of $s_{e^a}' = t_{e^a}$ because the same argument holds for $s_{e^a}'' = t_{e^a}$. Assume for contradiction that $s_{e^a}' \neq t_{e^a}$. There are two cases.

Case 1. $s_{e^a}' < t_{e^a}$. By $t_{e^a} < 0$, $s_{e^a}' < 0$. Thus, $\mathbf{t}$ and $\mathbf{s}'$ satisfy the condition of Lemma 4 for e^a. Therefore, by Lemma 4, there is $R_1' \in \mathcal{R}^E$ such that $R_1' \in \mathcal{R}_{\mathbf{t},\mathbf{0}}^{MT} \cap \mathcal{R}_{\mathbf{s}',e^a}^{MT}$. By Lemma 10, $x_1^f(R_1', R_{-1}) = \mathbf{0}$ and $x_1^f(R_1', R_2', R_{-1,2}) = e^a$. Thus, $x_2^f(R_1', R_{-1}) = \bar{x}$ and $x_2^f(R_1', R_2', R_{-1,2}) = x$. However, by $R_2' \in \mathcal{R}^*$, $v_2'(\bar{x}) - v_2'(x) > v_2(\bar{x}) - v_2(x)$. This contradicts Fact 3.

Case 2. $s_{e^a}' > t_{e^a}$. By $t_{e^a} < 0$, $\mathbf{t}$ and $\mathbf{s}'$ satisfy the condition of Lemma 4 for e^a. Thus, by Lemma 4, there is $R_1' \in \mathcal{R}^E$ such that $R_1' \in \mathcal{R}_{\mathbf{t},e^a}^{MT} \cap \mathcal{R}_{\mathbf{s}',\bar{x}}^{MT}$. By Lemma 10, $x_1^f(R_1', R_{-1}) = e^a$ and $x_1^f(R_1', R_2', R_{-1,2}) = \bar{x}$. Thus, $x_2^f(R_1', R_{-1}) = x$ and $x_2^f(R_1', R_2', R_{-1,2}) = \mathbf{0}$. However, by $R_2' \in \mathcal{R}^*$, $v_2'(x) > v_2(x)$. This contradicts Fact 3. $\square$

By Claim 1, (5), and the definitions of R_2' and R_2'',

$$s_{\bar{x}}'' = s_{e^a}'' + v_2''(x) < s_{e^a}' + v_2'(x) = s_{\bar{x}}'.$$

By $s'_{e^a} = t_{e^a} < 0$ and the object monotonicity of $\mathbf{s}'$, $s'_{\mathbf{0}} < 0$. Thus, $\mathbf{s}'$ and $\mathbf{s}''$ satisfy the condition of Lemma 5. Therefore, by Lemma 5, there is $R'_1 \in \mathcal{R}^E$ such that $R'_1 \in \mathcal{R}^{MT}_{\mathbf{s}',\mathbf{0}} \cap \mathcal{R}^{MT}_{\mathbf{s}'',\bar{x}}$. By Lemma 10, $x^f_1(R'_1, R'_2, R_{-1,2}) = \mathbf{0}$ and $x^f_1(R'_1, R''_2, R_{-1,2}) = \bar{x}$. Thus, $x^f_2(R'_1, R'_2, R_{-1,2}) = \bar{x}$ and $x^f_2(R'_1, R''_2, R_{-1,2}) = \mathbf{0}$. However, by the definitions of R'_2 and R''_2, $v'_2(\bar{x}) < v''_2(\bar{x})$, which contradicts Fact 3. ■

Step 2. $t_{\mathbf{0}} \leq 0$.

Proof. Assume for contradiction that $t_{\mathbf{0}} > 0$. Let $R'_2, R''_2 \in \mathcal{R}^*$ be such that

$$v''_2(x) < v'_2(x) \text{ and } v'_2(\bar{x}) < v''_2(\bar{x}) < v''_2(x) + (v_2(\bar{x}) - v_2(x) + t_{\mathbf{0}}).$$

Note that we can define such preferences since $t_{\mathbf{0}} > 0$. Note also that $v'_2(\bar{x}) - v'_2(x) < v''_2(\bar{x}) - v''_2(x)$. Let $\mathbf{s}', \mathbf{s}'' \in \mathbb{R}^{|X|}$ be such that for each $y \in X$, $s'_y = t^f_1(R'_2, R_{-1,2}; y)$ and $s''_y = t^f_1(R''_2, R_{-1,2}; y)$. By $X^f_1(R'_2, R_{-1,2}) = X^f_1(R''_2, R_{-1,2}) = X$ and Lemma 9, $\mathbf{s}'$ and $\mathbf{s}''$ are object monotonic.

Notice that by $R'_2, R''_2 \in \mathcal{R}^*$, $R_3, \ldots, R_n \in \mathcal{R}^Q(R_2) \cap \mathcal{R}^Q(R'_2) \cap \mathcal{R}^Q(R''_2)$. Thus, for each $R'_1 \in \mathcal{R}$ and each $i \in N \setminus \{1, 2\}$, $x^f_i(R'_1, R_{-1}) = x^f_i(R'_1, R'_2, R_{-1,2}) = x^f_i(R'_1, R''_2, R_{-1,2}) = \mathbf{0}$.

Claim 1. $s'_{e^a} = s''_{e^a} = t_{e^a}$.

Proof. We only prove $s'_{e^a} = t_{e^a}$ because the same argument holds for $s''_{e^a} = t_{e^a}$. Assume for contradiction that $s'_{e^a} \neq t_{e^a}$. There are two cases.

Case 1. $s'_{e^a} < t_{e^a}$. If $s'_{e^a} \geq 0$, $\mathbf{t}$ and $\mathbf{s}'$ satisfy the condition of Lemma 3 for e^a. Further, if $s'_{e^a} < 0$, then $\mathbf{t}$ and $\mathbf{s}'$ satisfy the condition of Lemma 4 for e^a. Thus, by Lemmas 3 and 4, there is $R'_1 \in \mathcal{R}$ such that $R'_1 \in \mathcal{R}^{MT}_{\mathbf{t},\mathbf{0}} \cap \mathcal{R}^{MT}_{\mathbf{s}',e^a}$. By Lemma 10, $x^f_1(R'_1, R_{-1}) = \mathbf{0}$ and $x^f_1(R'_1, R'_2, R_{-1,2}) = e^a$. Thus, $x^f_2(R'_1, R_{-1}) = \bar{x}$ and $x^f_2(R'_1, R'_2, R_{-1,2}) = x$. However, by $R'_2 \in \mathcal{R}^*$, $v'_2(\bar{x}) - v'_2(x) > v_2(\bar{x}) - v_2(x)$, which contradicts Fact 3.

Case 2. $s'_{e^a} > t_{e^a}$. By the definitions of R'_2 and R''_2, $v'_2(\bar{x}) - v'_2(x) < v''_2(\bar{x}) - v''_2(x) < v_2(\bar{x}) - v_2(x) + t_{\mathbf{0}}$. Thus, by (5),

$$s'_{\mathbf{0}} = s'_{e^a} - (v'_2(\bar{x}) - v'_2(x)) > t_{e^a} - (v_2(\bar{x}) - v_2(x) + t_{\mathbf{0}}) = t_{\mathbf{0}} - t_{\mathbf{0}} = 0.$$

By $t_{\mathbf{0}} > 0$ and the object monotonicity of $\mathbf{t}$, $t_{e^a} > 0$. Thus, $\mathbf{t}$ and $\mathbf{s}'$ satisfy the condition of Lemma 3 for e^a. Therefore, by Lemma 3, there is $R'_1 \in \mathcal{R}^E$ such that $R'_1 \in \mathcal{R}^{MT}_{\mathbf{t},e^a} \cap \mathcal{R}^{MT}_{\mathbf{s}',\bar{x}}$. By Lemma 10, $x^f_1(R'_1, R_{-1}) = e^a$ and $x^f_1(R'_1, R'_2, R_{-1,2}) = \bar{x}$. Thus,

$x_2^f(R_1', R_{-1}) = x$ and $x_2^f(R_1', R_2', R_{-1,2}) = \mathbf{0}$. However, by $R_2' \in \mathcal{R}^*$, $v_2'(x) > v_2(x)$. This contradicts Fact 3. $\square$

By Claim 1, (5), and $v_2''(x) < v_2'(x)$,

$$s''_{\bar{x}} = s''_{e^a} + v_2''(x) < s'_{e^a} + v_2'(x) = s'_{\bar{x}}.$$

By Claim 1, $t_{\mathbf{0}} > 0$, and the object monotonicity of $\mathbf{t}$ and $\mathbf{s}''$,

$$s''_{\bar{x}} > s''_{e^a} = t_{e^a} > t_{\mathbf{0}} > 0.$$

Moreover, by Claim 1, (5), and the definition of R_2',

$$s'_{\mathbf{0}} = s'_{e^a} - (v_2'(\bar{x}) - v_2'(x)) > t_{e^a} - (v_2(\bar{x}) - v_2(x) - t_{\mathbf{0}}) = t_{\mathbf{0}} - t_{\mathbf{0}} = 0.$$

Thus, $\mathbf{s}'$ and $\mathbf{s}''$ satisfy the condition of Lemma 3 for $\bar{x}$. Therefore, by Lemma 3, there is $R_1' \in \mathcal{R}^E$ such that $R_1' \in \mathcal{R}^{MT}_{\mathbf{s}',\mathbf{0}} \cap \mathcal{R}^{MT}_{\mathbf{s}'',\bar{x}}$. By Lemma 10, $x_1^f(R_1', R_2', R_{-1,2}) = \mathbf{0}$ and $x_1^f(R_1', R_2'', R_{-1,2}) = \bar{x}$. Thus, $x_2^f(R_1', R_2', R_{-1,2}) = \bar{x}$ and $x_2^f(R_1', R_2'', R_{-1,2}) = \mathbf{0}$. However, by the definitions of R_2' and R_2'', $v_2'(\bar{x}) < v_2''(\bar{x})$. This contradicts Fact 3. $\blacksquare$

F Public goods model with transfers

We introduce the formal public goods model with transfers studied in Ma et al. (2018), and provide a proof of Corollary 4. We then compare our result with the results in Ma et al. (2018).

There are $\hat{n} \geq 2$ agents and $\hat{m} \geq 1$ alternatives. We denote the set of agents by $\hat{N} \equiv \{1, \ldots, \hat{n}\}$ and the set of alternatives by $\hat{A} \equiv \{1, \ldots, \hat{m}\}$. Each agent $i \in \hat{N}$ has a complete and transitive preference relation $\succeq_i$ over $\hat{A} \times \mathbb{R}$. Let $\succ_i$ and $\sim_i$ be the strict and indifference relations associated with $\succeq_i$, respectively. We assume that preferences satisfy the following three conditions.

- **Money monotonicity:** For each $a \in \hat{A}$ and each pair $t, s \in \mathbb{R}$ with $t < s$, $(a, t) \succ_i (a, s)$.
- **Possibility of compensation:** For each $(a, t) \in \hat{A} \times \mathbb{R}$ and each $b \in \hat{A}$, there are $s, s' \in \mathbb{R}$ such that $(a, t) \succeq_i (b, s)$ and $(b, s') \succeq_i (a, t)$.
- **Continuity:** For each $z \in \hat{A} \times \mathbb{R}$, the **upper contour set** at z, $UC_i(z) \equiv \{z' \in$

$\hat{A} \times \mathbb{R} : z' \succeq_i z\}$, and the **lower contour set** at z, $LC_i(z) \equiv \{z' \in \hat{A} \times \mathbb{R} : z \succeq_i z'\}$, are both closed.

The generic notation for a class of admissible preferences (that satisfy the above conditions) is denoted by $\mathcal{D}$ and we call it a **domain**. Denote by $\bar{\mathcal{D}}$ the class of preferences that satisfy the above three conditions.

We introduce the notion of (compensated) valuation as in the package assignment model. Given a preference relation $\succeq_i$, $z \in \hat{A} \times \mathbb{R}$, and $b \in \hat{A}$, there is a payment $s \in \mathbb{R}$ such that $z \sim_i (b, s)$. We call the payment level the **(compensated) valuation of** b **at** z **for** $\succeq_i$, and denote it by $V^{\succeq_i}(b, z)$.

For each preference relation $\succeq_i$, let $a(\succeq_i) \in \hat{A}$ be such that for each $a \in \hat{A}$, $(a, 0) \succeq_i (a(\succeq_i), 0)$. That is, $a(\succeq_i)$ is the worst alternative at zero payment.

An **allocation** is a pair $(a, (t_1, \ldots, t_{\hat{n}}))$ of an alternative $a \in \hat{A}$ and a **payment vector** $(t_1, \ldots, t_{\hat{n}}) \in \mathbb{R}^{\hat{n}}$. Let $\hat{Z}$ be the set of allocations. A preference profile is an n-tuple $\succeq \equiv (\succeq_1, \ldots, \succeq_{\hat{n}}) \in \mathcal{D}^{\hat{n}}$. Given $\succeq \in \mathcal{D}^{\hat{n}}$, $i \in \hat{N}$, and $\hat{N}' \subseteq \hat{N}$, let $\succeq_{-i} \equiv (\succeq_k)_{k \in \hat{N} \setminus \{i\}}$ and $\succeq_{-\hat{N}'} \equiv (\succeq_k)_{k \in \hat{N} \setminus \hat{N}'}$.

An **mechanism** on $\mathcal{D}^{\hat{n}}$ is a function $\varphi : \mathcal{D}^{\hat{n}} \to \hat{Z}$. For each $\succeq \in \mathcal{D}^{\hat{n}}$, let $\alpha(\succeq)$ be the alternative chosen by φ at $\succeq$, and for each $i \in \hat{N}$, let $\tau_i(\succeq)$ be the payment of agent i at $\varphi(\succeq)$.

An allocation $(a, (t_1, \ldots, t_n)) \in \hat{Z}$ is **(Pareto) efficient** for $\succeq \in \mathcal{D}^{\hat{n}}$ if there is no allocation $(b, (s_1, \ldots, s_n)) \in \hat{Z}$ such that (i) for each $i \in \hat{N}$, $(b, s_i) \succeq_i (a, t_i)$, (ii) for some $j \in \hat{N}$, $(b, s_j) \succ_j (a, t_j)$, and (iii) $\sum_{i \in \hat{N}} s_i \geq \sum_{i \in \hat{N}} t_i$.

We consider mechanisms that satisfy the following two properties.

Efficiency: For each $\succeq \in \mathcal{D}^{\hat{n}}$, $\varphi(\succeq)$ is efficient for $\succeq$.

Strategy-proofness: For each $\succeq \in \mathcal{D}^{\hat{n}}$, each $i \in \hat{N}$, and each $\succeq'_i \in \mathcal{D}$, $\varphi_i(\succeq) \succeq_i \varphi_i(\succeq'_i, \succeq_{-i})$.

We also introduce three additional properties to compare our result with those in Ma et al. (2018).

Ontoness: For each $a \in \hat{A}$, there is $\succeq \in \mathcal{D}^{\hat{n}}$ such that $\alpha(\succeq) = a$.

Individual rationality: For each $\succeq \in \mathcal{D}^{\hat{n}}$ and each $i \in \hat{N}$, $\varphi_i(\succeq) \succeq_i (a(\succeq_i), 0)$.

No subsidy: For each $\succeq \in \mathcal{D}^{\hat{n}}$ and each $i \in \hat{N}$, $\tau_i(\succeq) \geq 0$.

We introduce the notion of essentially quasi-linear preferences in the public goods model. Given $\succeq_i \in \mathcal{D}$, let

$$\hat{A}(\succeq_i) \equiv \{(a,t) \in \hat{A} \times \mathbb{R} : (a,t) \succeq_i (a(\succeq_i), 0) \text{ and } t \geq 0\}.$$

Definition 12. A preference relation $\succeq_i$ is **essentially quasi-linear** if for each $(a,t) \in \hat{A}(\succeq_i)$ and each $b \in \hat{A}$ with $V^{\succeq_i}(b,(a,t)) \geq 0$,

$$V^{\succeq_i}(b,(a,t)) - t = V^{\succeq_i}(b,(a(\succeq_i),0)) - V^{\succeq_i}(a,(a(\succeq_i),0)).$$

Let $\mathcal{D}^E$ be the class of essentially quasi-linear preferences, and call it the **essentially quasi-linear domain**. Note that the essentially quasi-linear domain coincides with the *parallel domain* introduced by Ma et al. (2018).

Formally, Corollary 4 states the following: Assume $\hat{n} \geq 3$ and $|\hat{A}| \geq 6$. Let $\mathcal{D} = \mathcal{D}^E$. Then, no mechanism on $\mathcal{D}^{\hat{n}}$ satisfies *efficiency* and *strategy-proofness*.

F.1 Proof of Corollary 4

Assume for contradiction that there is a mechanism φ on $\mathcal{D}^{\hat{n}}$ that satisfies *efficiency* and *strategy-proofness*. We do the proof in five steps.

Step 1. *Defining classes of preferences.*

Take six distinct alternatives $a_1, a_2, a_3, a_4, a_5, a_6 \in \hat{A}$. Let $\mathcal{D}_1 \subseteq \mathcal{D}^E$ be the class of preferences that satisfy the following properties:

1. For each $t \in \mathbb{R}$, $(a_6,t) \succ_1 (a_5,t) \sim_1 (a_4,t) \succ_1 (a_3,t) \sim_1 (a_2,t) \sim_1 (a_1,t)$.
2. For each $a \in \hat{A} \setminus \{a_1, \ldots, a_6\}$ and each $t \in \mathbb{R}$, $(a,t) \sim_1 (a_1,t)$.

Let $\mathcal{D}_2 \subseteq \mathcal{D}^E$ be the class of preferences that satisfy the following properties:

1. For each $t \in \mathbb{R}$, $(a_2,t) \succ_2 (a_1,t) \sim_2 (a_4,t) \succ_2 (a_3,t) \sim_2 (a_5,t) \sim_2 (a_6,t)$.
2. For each $a \in \hat{A} \setminus \{a_1, \ldots, a_6\}$ and each $t \in \mathbb{R}$, $(a,t) \sim_2 (a_6,t)$.

Let $\mathcal{D}_3 \subseteq \mathcal{D}^E$ be the class of preferences that satisfy the following properties:

1. For each $t \in \mathbb{R}$, $(a_3,t) \succ_3 (a_1,t) \sim_3 (a_5,t) \succ_3 (a_2,t) \sim_3 (a_4,t) \sim_3 (a_6,t)$.

2. For each $a \in \hat{A} \setminus \{a_1, \ldots, a_6\}$ and each $t \in \mathbb{R}$, $(a, t) \sim_3 (a_6, t)$.

Let $\succeq_{-\{1,2,3\}} \in (\mathcal{D}^E)^{\hat{n}-3}$ be such that for each $i \in \hat{N} \setminus \{1, 2, 3\}$, $\succeq_i$ satisfies the following:

1. For each $t \in \mathbb{R}$, $(a_1, t) \sim_i (a_2, t) \sim_i (a_3, t) \sim_i (a_4, t) \sim_i (a_5, t) \sim_i (a_6, t)$.
2. For each $a \in \hat{A} \setminus \{a_1, \ldots, a_6\}$ and each $t \in \mathbb{R}$, $(a_1, t) \succ_i (a, t)$.

Step 2. *For each* $(\succeq_1, \succeq_2, \succeq_3) \in \mathcal{D}_1 \times \mathcal{D}_2 \times \mathcal{D}_3$, $\alpha(\succeq_1, \succeq_2, \succeq_3, \succeq_{-\{1,2,3\}}) \in \{a_1, \ldots, a_6\}$.

Proof. Assume for contradiction that there is $(\succeq_1, \succeq_2, \succeq_3) \in \mathcal{D}_1 \times \mathcal{D}_2 \times \mathcal{D}_3$ such that $\alpha(\succeq_1, \succeq_2, \succeq_3, \succeq_{-\{1,2,3\}}) = a$ for some $a \in \hat{A} \setminus \{a_1, \ldots, a_6\}$. Let $(a_1, (t_i)_{i \in \hat{N}}) \in \hat{Z}$ be such that for each $i \in \hat{N}$, $t_i = \tau_i(\succeq_1, \succeq_2, \succeq_3, \succeq_{-\{1,2,3\}})$.

It is clear that $\sum_{i \in \hat{N}} t_i = \sum_{i \in \hat{N}} \tau_i(\succeq_1, \succeq_2, \succeq_3, \succeq_{-\{1,2,3\}})$. By the definition of $\succeq_1$, $(a_1, t_1) \sim_1 \varphi_1(\succeq_1, \succeq_2, \succeq_3, \succeq_{-\{1,2,3\}})$. For each $i \in \hat{N} \setminus \{1\}$, by the definition of $\succeq_i$, $(a_1, t_i) \succ_i \varphi_i(\succeq_1, \succeq_2, \succeq_3, \succeq_{-\{1,2,3\}})$. Thus, $\varphi_i(\succeq_1, \succeq_2, \succeq_3, \succeq_{-\{1,2,3\}})$ is not *efficient* for $(\succeq_1, \succeq_2, \succeq_3, \succeq_{-\{1,2,3\}})$, a contradiction. ■

Step 3. *Embedding the package assignment model to the public goods model.*

Consider the package assignment model studied in this paper with three agents and two identical objects. That is, $n = 3$, $m = 1$, and $\bar{x} = 2$. Let $\phi : \{a_1, \ldots, a_6\} \to A$ be such that for each $a \in \{a_1, \ldots, a_6\}$,

$$\phi(a) = \begin{cases} (0,1,1) & \text{if } a = a_1, \\ (0,2,0) & \text{if } a = a_2, \\ (0,0,2) & \text{if } a = a_3, \\ (1,1,0) & \text{if } a = a_4, \\ (1,0,1) & \text{if } a = a_5, \\ (2,0,0) & \text{if } a = a_6. \end{cases}$$

For each $\succeq_1 \in \mathcal{D}_1$, let $R_1^{\succeq_1}$ be a preference relation in the package assignment model such that for each pair $(a, t), (a', s) \in \{a_1, \ldots, a_6\} \times \mathbb{R}$, $(a, t) \succeq_1 (a', s)$ if and only if $(\phi_1(a), t)\ R_1^{\succeq_1}\ (\phi_1(a'), s)$. Let

$$\mathcal{R}_1 \equiv \{R_1^{\succeq_1} : \succeq_1 \in \mathcal{D}_1\}.$$

For each $\succeq_2 \in \mathcal{D}_2$, let $R_2^{\succeq_2}$ be a preference relation in the package assignment model such that for each pair $(a,t),(a',s) \in \{a_1,\dots,a_6\} \times \mathbb{R}$, $(a,t) \ \succeq_2 \ (a',s)$ if and only if $(\phi_2(a),t) \ R_2^{\succeq_2} \ (\phi_2(a'),s)$. Let

$$\mathcal{R}_2 \equiv \{R_2^{\succeq_2} : \succeq_2 \in \mathcal{D}_2\}.$$

For each $\succeq_3 \in \mathcal{D}_3$, let $R_3^{\succeq_3}$ be a preference relation in the package assignment model such that for each pair $(a,t),(a',s) \in \{a_1,\dots,a_6\} \times \mathbb{R}$, $(a,t) \ \succeq_3 \ (a',s)$ if and only if $(\phi_3(a),t) \ R_3^{\succeq_3} \ (\phi_3(a'),s)$. Let

$$\mathcal{R}_3 \equiv \{R_3^{\succeq_3} : \succeq_3 \in \mathcal{D}_3\}.$$

Note that for each $i \in \{1,2,3\}$ and each $\succeq_i \in \mathcal{D}_i$, since $\succeq_i \in \mathcal{D}^E$, $R_i^{\succeq_i} \in \mathcal{R}^E$. Further, for each $i \in \{1,2,3\}$ and each $R_i \in \mathcal{R}^E$, there is $\succeq_i \in \mathcal{D}_i$ such that $R_i = R_i^{\succeq_i}$. Therefore, $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}_3 = \mathcal{R}^E$.

Let f be the mechanism in the package assignment model on $(\mathcal{R}^E)^3$ such that for each $R \in (\mathcal{R}^E)^3$,

$$f(R) = ((\phi_i(\alpha(\succeq_1,\succeq_2,\succeq_3,\succeq_{-\{1,2,3\}})), \tau_i(\succeq_1,\succeq_2,\succeq_3,\succeq_{-\{1,2,3\}})))_{i\in\{1,2,3\}},$$

where for each $i \in \{1,2,3\}$, $\succeq_i$ is such that $R_i = R_i^{\succeq_i}$.

Step 4. *f is efficient and strategy-proof in the package assignment model.*

Proof. Efficiency. Assume for contradiction that f is not *efficient*. Then, there are $R \in (\mathcal{R}^E)^3$ and $((x_i,t_i))_{i\in\{1,2,3\}} \in Z$ such that for each $i \in N$, $(x_i,t_i) \ I_i \ f_i(R)$, and $\sum_{i\in\{1,2,3\}} t_i > \sum_{i\in\{1,2,3\}} t_i^f(R)$. For each $i \in \{1,2,3\}$, let $\succeq_i \in \mathcal{D}_i$ be such that $R_i = R_i^{\succeq_i}$. By the definition of f, for each $i \in \{1,2,3\}$, $\varphi_i(\succeq_1,\succeq_2,\succeq_3,\succeq_{-\{1,2,3\}}) = (\phi^{-1}(x^f(R)), t^f(R))$.

Let $(a,(s_i)_{i\in\hat{N}}) \in \hat{Z}$ be such that $a = \phi^{-1}(x_1,x_2,x_3)$, and for each $i \in \hat{N}$,

$$s_i = \begin{cases} t_i & \text{if } i \in \{1,2,3\}, \\ \tau_i(\succeq_1,\succeq_2,\succeq_3,\succeq_{-\{1,2,3\}}) & \text{otherwise.} \end{cases}$$

By $a = \phi^{-1}(x_1,x_2,x_3)$, $a \in \{a_1,\dots,a_6\}$.

Let $i \in \{1,2,3\}$. By $\phi_i(a) = x_i$, $\varphi_i(\succeq_1,\succeq_2,\succeq_3,\succeq_{-\{1,2,3\}}) = \phi^{-1}(x^f(R), t_i^f(R))$, and

$(x_i, t_i)\ I_i\ f_i(R)$,

$$(a, s_i) \sim_i \varphi_i(\succeq_1, \succeq_2, \succeq_3, \succeq_{-\{1,2,3\}}).$$

Let $i \in \hat{N} \setminus \{1, 2, 3\}$. By the definition of $\succeq_i$ and $a, \alpha(\succeq_1, \succeq_2, \succeq_3, \succeq_{-\{1,2,3\}}) \in \{a_1, \ldots, a_6\}$, $(a, s_i) \sim_i \varphi_i(\succeq_1, \succeq_2, \succeq_3, \succeq_{-\{1,2,3\}})$.

By $\sum_{i\in\{1,2,3\}} t_i > \sum_{i\in\{1,2,3\}} t_i^f(R)$, $\sum_{i\in\hat{N}} s_i > \sum_{i\in\hat{N}} \tau_i(\succeq_1, \succeq_2, \succeq_3, \succeq_{-\{1,2,3\}})$. Thus, φ is not *efficient*, a contradiction.

Strategy-proofness. Without loss of generality, we focus on agent 1. Let $R \in (\mathcal{R}^E)^3$ and $R'_1 \in \mathcal{R}^E$. For each $i \in \{1, 2, 3\}$, there is $\succeq_i \in \mathcal{D}_i$ such that $R_i = R_i^{\succeq_i}$. Also, there is $\succeq'_1 \in \mathcal{D}_1$ such that $R'_1 = R_1^{\succeq'_1}$. By the definition of f, $f_1(R) = (\phi_1(\alpha(\succeq_1, \succeq_2, \succeq_3, \succeq_{-\{1,2,3\}})), \tau_1(\succeq_1, \succeq_2, \succeq_3, \succeq_{-\{1,2,3\}}))$ and $f_1(R'_1, R_{-1}) = (\phi_1(\alpha(\succeq'_1, \succeq_2, \succeq_3, \succeq_{-\{1,2,3\}})), \tau_1(\succeq'_1, \succeq_2, \succeq_3, \succeq_{-\{1,2,3\}}))$. Since φ is *strategy-proof*, $\varphi_1(\succeq_1, \succeq_2, \succeq_3, \succeq_{-\{1,2,3\}}) \succeq_1 \varphi_1(\succeq'_1, \succeq_2, \succeq_3, \succeq_{-\{1,2,3\}})$. Thus, by $R_1 = R_1^{\succeq_1}$, $f_1(R)\ R_1\ f_1(R'_1, R_{-1})$. ■

Step 5. *Completing the proof.*

By Theorem 2 there is no *efficient* and *strategy-proof* mechanism on $(\mathcal{R}^E)^3$. This contradicts Step 4. ■

F.2 Comparison to Ma et al. (2018)

In their main result, Ma et al. (2018) show that if the domain is larger than the essentially quasi-linear domain, *fixed price dictatorships* are the only mechanisms that satisfy *strategy-proofness*, *ontoness*, *individual rationality*, and *no subsidy*.

Definition 13. A mechanism φ on $\mathcal{D}^{\hat{n}}$ is a **fixed price dictatorship** if there are $i \in \hat{N}$ and a price vector $(p_a)_{a\in\hat{A}} \in \mathbb{R}^{\hat{m}}$ such that for each $\succeq \in \mathcal{D}^{\hat{n}}$,

1. for each $a \in \hat{A}$, $(\alpha(\succeq), p_{\alpha(\succeq)}) \succeq_i (a, p_a)$,
2. $\tau_i(\succeq) = p_{\alpha(\succeq)}$, and
3. for each $j \in \hat{N} \setminus \{i\}$, $\tau_j(\succeq) = 0$.

Fact 4 (Theorem 3 in Ma et al. (2018))**.** *Assume $\hat{m} \geq 3$. Let $\mathcal{D}$ be such that $\mathcal{D} \supsetneq \mathcal{D}^E$. If a mechanism on $\mathcal{D}^{\hat{n}}$ satisfies strategy-proofness, ontoness, individual rationality,* *and no subsidy, then it is a fixed price dictatorship.*[43]

Note that *efficiency* implies *ontoness*. Further, it is easy to show that fixed price dictatorships are not *efficient*. Thus, as a corollary, the following result is obtained.

Fact 5 (Ma et al. (2018))**.** *Assume $\hat{m} \geq 3$. Let $\mathcal{D}$ be such that $\mathcal{D} \supsetneq \mathcal{D}^E$. Then, no mechanism on $\mathcal{D}^{\hat{n}}$ satisfies efficiency, strategy-proofness, individual rationality, and no subsidy.*

There are mainly three differences between Corollary 4 and Facts 4 and 5. First, we impose only *efficiency* and *strategy-proofness*. Thus, Corollary 4 implies Fact 5 for the case where there are at least three agents and six alternatives. In contrast, since *ontoness* is imposed instead of *efficiency* in Fact 4, Corollary 4 does not imply Fact 4. Second, we obtain an impossibility result on the essentially quasi-linear domain while in Facts 4 and 5, the domain is required to be larger than the essentially quasi-linear domain. Third, in Corollary 4, we assume that there are at least six alternatives. Thus, it is not clear if there is an *efficient* and *strategy-proof* mechanism on the essentially quasi-linear domain when the number of alternatives is less than six.

Ma et al. (2018) also have a result without *individual rationality* and *no subsidy*. To establish a result, they require another domain richness condition.

Definition 14. A preference relation$\succeq_i$ is with **two slopes** if there exists a utility function u_i of $\succeq$ with the following property: there exist $a, b \in \mathbb{R}_{++}$ and a valuation function $v_i : A \to \mathbb{R}$ such that for each $a \in A$, either $u_i(a, t) = v_i(a) - at$ for each $t \in \mathbb{R}$ or $u_i(a, t) = v_i(a) - bt$ for each $t \in \mathbb{R}$.

Let $\mathcal{D}^{TS}$ be the set of preferences with two slopes.

Fact 6 (Theorem 4 in Ma et al. (2018))**.** *Assume $\hat{m} \geq 3$. If a mechanism on $(\mathcal{D}^{TS})^{\hat{n}}$ satisfies strategy-proofness and ontoness, then it is a fixed price dictatorship.*

Fact 7 (Ma et al. (2018))**.** *Assume $\hat{m} \geq 3$. No mechanism on $(\mathcal{D}^{TS})^{\hat{n}}$ satisfies strategy-proofness and ontoness.*

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[43] To be precise, Ma et al. (2018) allows agents to have different domains. They show that if the domain of each agent contains the essentially quasi-linear domain and the domains of at least $\hat{n} - 1$ agents are larger than the essentially quasi-linear domain, then the mechanisms that satisfy the four properties are fixed price serial dictatorships.

Note that if the two parameters a and b for a preference relation in $\mathcal{D}^{TS}$ satisfies $a = b$, then it is quasi-linear. Thus, $\mathcal{D}^{TS}$ contains the quasi-linear domain. If $a \neq b$, the preference relation exhibits income effects. Thus, $\mathcal{D}^{TS}$ must contains a variety of preferences with income effects, which are not contained in the essentially quasi-linear domain.[44] However, the essentially quasi-linear domain contains preferences that do not have two slopes. Thus, the essentially quasi-linear domain in Corollary 4 and the domain of preferences with two slopes have no set inclusion relation.

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[44] Note that preferences with two slopes can be defined in the package assignment model. Suppose that a preference relation R_i in the package assignment model is with two slopes, a and b. Suppose also that the slop for $\mathbf{0}$ is a and that for a package $x \in X \setminus \{\mathbf{0}\}$ is b. If $a > b$, then this preference exhibits positive income effects for x. In contrast, if $a < b$, then this preference exhibits negative income effects.

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