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When can we design efficient and strategy-proof rules in package assignment problems?*

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Abstract

We examine the compatibility of efficiency and strategy-proofness in a package assignment model where preferences may not be quasi-linear. Given $r \in \mathbb{R}$, a preference relation is *r-partially quasi-linear* if it is quasi-linear over the set of (consumption) bundles where each bundle is at least as desirable as receiving no object and paying r, and the payment at each bundle is at least r. We show that if a domain includes *r-partially quasi-linear domain*, then no rule is efficient and strategy-proof. We also show that if there is a rule that satisfies efficiency, strategy-proofness, individual rationality, and no subsidy for losers on a domain, the domain must be a subset of the (0-)partially quasi-linear domain. Our results demonstrate that the quasi-linearity of preferences plays an important role to design an efficient and strategy-proof rule.

Keywords. Strategy-proofness, efficiency, non-quasi-linear preferences, partially quasi-linear preferences, generalized Vickrey rule, maximal domain
JEL Classification. D44, D71, D61, D82.

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1 Introduction

Package auctions are widely conducted in many countries to allocate scares resources such as spectrum licenses.[1] The primary goal in most of those auctions is to achieve efficiency. For example, the Federal Communication Commission (FCC), which conducted package auctions intensively in the past 25 years, states that "[t]he auction approach is intended to award the licenses to those who will use them most effectively."[2] In many package auctions in practice, agents (bidders) win packages at very high prices. As we discuss in detail in Section 1.1, an agent's valuation for a package may depend on the payment level in such a large scale auction, i.e., preferences may not be quasi-linear. Though there is a substantial literature on object assignments, most of those papers assume the quasi-linearity of preferences. Without the quasi-linearity assumption, when can we achieve an efficient allocation? Our answer is that the quasi-linearity of preferences is crucial to achieve an efficient allocation.

Formally, we consider a package assignment model with transfers. There are several objects and each agent obtains a package of objects. Our model covers the case where objects are identical, the case where objects are all distinct, and the cases where there are several object types and each object has several copies.[3] A (consumption) bundle consists of a package of objects and a payment. We assume that preferences are *object monotonic*, i.e., at each payment level, more objects are preferred to less. Preferences are not necessarily quasi-linear. A set of preferences is called a *domain*. In particular, the set of object monotonic preferences is called the *object monotonic domain*.

An *(allocation) rule* is a mapping from the set of preference profiles to the set of allocations. We investigate domains on which there is an *efficient* and *strategy-proof* rule. The efficiency of an allocation in this paper means that no other allocation makes

[1] Bikhchandani and Mamer (1997) and Ausubel (2006) discuss other examples of package auctions.

[2] See `https://www.fcc.gov/auctions/about-auctions`.

[3] For example, in the spectrum auction in the UK in 2018, licenses in the 2.3 and 3.4 GHz spectrum bands were allocated. Each spectrum band was divided into small blocks and a license corresponds to a block in one of the spectrum bands. Blocks in the same spectrum band are considered to be identical. Thus, in this auction, there are two types of objects, each of which has several copies. For more details, see `https://www.ofcom.org.uk/consultations-and-statements/category-1/2.3-3.4-ghz-auction-design`.

an agent better off without making any agent worse off or reducing the revenue of the planner. It is well-known that there is an *efficient* and *strategy-proof* rule on the quasi-linear domain (the set of quasi-linear preferences) (Holmström, 1979). Hence, we examine whether an *efficient* and *strategy-proof* rule exists on domains larger than the quasi-linear domain.

Our results indicate that *efficiency* and *strategy-proofness* are incompatible unless preferences are "alomst" quasi-linear. To explain our results, we explain the notion of partially quasi-linear preferences. Take any $r \in \mathbb{R}$. For each preference relation, we can define the set of bundles where each bundle is at least as desirable as no object with payment r and the payment at each bundle is at least r. We call this set of bundles the *r-relevant consumption set*. A preference relation is *r-partially quasi-linear* if it is quasi-linear on the r-relevant consumption set. In particular, if $r = 0$, then we call a 0-partially quasi-linear preference relation a *partially quasi-linear* preference relation.

Our main results depend on the number of agents. Take any $r \in \mathbb{R}$. For two-agent case, we show that the r-partially quasi-linear domain (the set of r-partially quasi-linear preferences) is a *maximal domain* for *efficiency* and *strategy-proofness* (Theorem 1 (ii)). Thus, there is an *efficient* and *strategy-proof* rule on the r-partially quasi-linear domain, and no rule satisfies the two properties on any larger domain. Further, we show that on the r-partially quasi-linear domain, *r-generalized Vickrey rules* (a generalization of the Vickrey rule to non-quasi-linear domains) are the only *efficient* and *strategy-proof* rules (Theorem 1 (i)). On the other hand, for more than two agents, we show that there is no *efficient* and *strategy-proof* rule on any domain including the r-partially quasi-linear domain (Theorem 2).

The above results are strengthened by imposing additional properties. We consider three additional properties. *Individual rationality* is a participation constraint which requires that each agent should receive a bundle which is at least as desirable as she would be if she had received no object and paid nothing. *No subsidy for losers* requires that the payment of losers (agents who receive no object) should be nonnegative. *Common payment for losers* requires that the payment of losers is always the same. *Individual rationality* and *no subsidy for losers* imply *common payment for losers*.

We show that if there is a rule on a domain that satisfies *efficiency*, *strategy-*

proofness, *individual rationality*, and *no subsidy for losers*, then the domain must be a subset of the partially quasi-linear domain (Theorem 3). This result implies that for two agents, the partially quasi-linear domain is the unique maximal domain for the four properties. Further, we show that if there is a rule on a domain that satisfies *efficiency*, *strategy-proofness*, and *common payment for losers*, then there is $r \in \mathbb{R}$ such that the domain is a subset of the r-partially quasi-linear domain (Theorem 4). This result implies that for two agents, the family of r-partially quasi-linear domains coincides with the family of maximal domains for the three properties.

Our results provide a useful tool for verifying the existence of an *efficient* and *strategy-proof* rule on various domains of interest. For example, the object monotonic domain includes the r-partially quasi-linear domain for each $r \in \mathbb{R}$. Thus, our results imply that there is no *efficient* and *strategy-proof* rule on the object monotonic domain (Corollary 1). We also consider other examples of domains studied in the literature on mechanism design without quasi-linearity. The first one is domains of preferences with income effects. The *nonnegative income effect* and *nonpositive income effect domains* include the quasi-linear domain, and further, contain a preference relation that is not r-partially quasi-linear for each $r \in \mathbb{R}$. Thus, our results imply that on those domains no rule satisfies *efficiency*, *strategy-proofness*, and *common payment for losers* (and hence no rule on those domains satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers*) (Corollary 2). Another example is the domain of *quasi-linear preferences with borrowing cost*. This domain includes the quasi-linear domain and contains a preference relation that is not partially quasi-linear. Thus our results imply that there is no rule on this domain that satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* (Corollary 3).

Our results also have some implication to the public goods model with transfers. Indeed, our results imply that in the public goods model, if there are at least three agents and six alternatives, there is no *efficient* and *strategy-proof* rule on the partially quasi-linear domain (Corollary 4).

1.1 Related literature

Most research in mechanism design and auction theory assumes preferences to be quasi-linear. The quasi-linearity of preferences requires that the valuation for a package is independent of the payment, i.e., there is no income effect. As Marshall (1920) argues, this assumption is plausible when the payment is sufficiently small.[4] However, in many practical applications, such as spectrum auctions, agents' payments are typically very high.[5] When payments are high, "[e]xcessive payments for the auctioned objects may damage bidders' budgets to purchase complements for effective uses of the objects and thus, may influence the benefits from the objects" (Morimoto and Serizawa, 2015, p.447). Thus, the quasi-linearity of preferences is not an appropriate assumption in many practical applications. Another source that makes preferences non-quasi-linear is the existence of distortionary frictions (Saitoh and Serizawa, 2008; Fleiner et al., 2019). For example, a bidder may have to borrow money at some interest rate when the payment exceeds her income. The existence of such a borrowing cost makes preferences non-quasi-linear even if the original preferences are quasi-linear.[6]

When preferences are quasi-linear, it is well-known that only Groves rules satisfy *efficiency* and *strategy-proofness* (Holmström, 1979), and further, Vickrey rules are the only rules that satisfy *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* (Holmström, 1979; Chew and Serizawa, 2007). In contrast to these results, our results indicate that when the domain contains enough variety of non-quasi-linear preferences, no rule satisfies those properties.

Similar impossibility results are obtained by Kazumura and Serizawa (2016), Baisa (2020), and Malik and Mishra (2021). However, there are mainly two differences between our paper and those papers. First, those three papers impose *individual rationality* and a no subsidy condition in addition to *efficiency* and *strategy-proofness*. Since we impose only *efficiency* and *strategy-proofness* in our main results (Theorems 1 and 2), their results do not imply our main results. Further, *individual rationality* and *no subsidy for losers* are not plausible in some auctions in practice.

[4] Vives (1987) and Hayashi (2008) give a mathematical foundation for this argument.

[5] For example, most firms that won licenses in the spectrum auction in 2021 in the UK pay around £300 million. For more details, see `https://www.ofcom.org.uk/spectrum/spectrum-management/spectrum-awards/awards-archive/2-3-and-3-4-ghz-auction`

[6] We discuss it formally in Section 6.

For example, entry fees exist in many auctions in practice.[7] Also, entry subsidy is a natural tool for increasing the number of participants.[8] The above three papers do not cover such situations. On the other hand, our paper shows that even when entry fees or subsidies are allowed, it is not possible to design an *efficiency* and *strategy-proof* rule if preferences are not quasi-linear.

Second, those three papers focus on various domains which have no inclusion relation to the domains that we consider. Kazumura and Serizawa (2016) consider domains containing all possible unit-demand preferences. Malik and Mishra (2021) consider the situation where each agent partitions the set of packages into acceptable packages and unacceptable packages, and has preferences such that acceptable packages are all indifferent but an acceptable package is better than any unacceptable package. Preferences in Kazumura and Serizawa (2016) and Malik and Mishra (2021) violate object monotonicity while we focus only on object monotonic preferences. Baisa (2020) considers object monotonic preferences but focus only on preferences with positive income effects of which incremental valuation for an additional object is always non-increasing. On the other hand, we cover domains that do not contain such preferences. For example, we establish impossibility results on the domains of preferences having nonpositive income effects and quasi-linear preferences with borrowing cost. Those domains are not covered by Baisa (2020).

In some situations such as procurement auctions, it is sometimes reasonable to assume that the incremental valuation for an additional object is non-decreasing (Baranov et al., 2017). In this case, generalize Vickrey rules are the only rules that satisfy *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* even when preferences can be non-quasi-linear (Shinozaki et al., 2020).[9]

[7]Chen and Kominers (2021) discuss a lot of practical examples of auctions with entry fees. Further, they show that the planner can sometimes generate more revenue by setting entry fees.

[8]The number of participants is known to be crucial for generating high revenue. Indeed, for the single object case, the revenue maximizing auction generates lower expected revenue than the second price auction with an additional agent (Bulow and Klemperer, 1996). Further, Lu (2009) shows that when entry is costly and endogenous, the revenue maximizing auction involves entry subsidies.

[9]Similarly to Baisa (2020), Shinozaki et al. (2020) also consider the case where incremental valuations of preferences for an additional object is always non-increasing. The family of domains that they consider contains various domains that are not covered by Baisa (2020), and they show that no rule on those domains satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no*

When agents have unit-demand and non-quasi-linear preferences, the *minimum price Walrasian* (MPW) rule satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* (Demange and Gale, 1985), and it is the only rule that satisfies these properties on the *classical domain*—the set of all quasi-linear and non-quasi-linear preferences (Morimoto and Serizawa, 2015).[10] Further, the MPW rule is the ex-post revenue maximizing rule among rules that satisfy *strategy-proofness*, *individual rationality* and a weak fairness condition called *equal treatment of equals*, a weak efficiency condition called *no wastage*, and a no subsidy condition (Kazumura et al., 2020b).[11]

Some papers focus on the existence of a Walrasian equilibrium in package assignment models. It has been known that a Walrasian equilibrium exists when preferences are quasi-linear and satisfy the gross substitutes condition (Kelso and Crawford, 1982). This result is extended to the case where preferences are quasi-linear and satisfy the gross substitutes and complements condition (Sun and Yang, 2006). When preferences are non-quasi-linear, a Walrasian equilibrium exists at each endowment allocation if and only if a Walrasian equilibrium exists in the corresponding quasi-linear economies (Baldwin et al., 2020).

There are papers that study the case where agents have quasi-linear preferences but face hard budget (Che and Gale, 1998, 2000; Pai and Vohra, 2014). Due to the hard budget, the induced preferences are non-quasi-linear. In this setting, there is no rule that satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and a no subsidy condition (Dobzinski et al., 2012; Lavi and May, 2012). Preferences in this model are close to quasi-linear preferences with borrowing cost in our model. The major difference between preferences in this model and quasi-linear preferences with borrowing cost is that in this model, the budget is hard, i.e., payments cannot exceed the budget. On the other hand, since we allow agents to borrow money, payments can exceed the budget. To be more precise, preferences in this model violate continuity whereas we focus only on continuous preferences. Thus, there is no logical relation

subsidy for losers when there are odd number of objects.

[10]This result is extended to the case where there is a common ranking over objects, and agents prefer a higher ranked object than a lower ranked object (Zhou and Serizawa, 2018).

[11]In Kazumura et al. (2020b), it is assumed that the number of agents is larger than that of objects. Sakai and Serizawa (2021) show that the same result holds without this assumption.

between the impossibility results by Dobzinski et al. (2012) and Lavi and May (2012) and our results. Further, we impose only *efficiency* and *strategy-proofness* in our main results, while those papers impose not only these properties but also *individual rationality* and a no subsidy condition.

The partially quasi-linearity is first introduced in the public goods model (Ma et al., 2018). They show that if the domain is larger than the partially quasi-linear domain, *fixed price dictatorships* are the only rules that satisfy *strategy-proofness*, *ontoness*, *individual rationality*, and *no subsidy*, where *ontoness* requires that each alternative should be selected at some preference profile and *no subsidy* requires that the payment of each agent is nonnegative. Since fixed price dictatorships violate *efficiency*, their result implies that if the domain is larger than the partially quasi-linear domain, no rule satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy*. Compared with this result, our result (Corollary 4) shows a stronger impossibility result in a restricted environment. Indeed, we impose only *efficiency* and *strategy-proofness*, whereas Ma et al. (2018) impose *individual rationality* and *no subsidy* as well. On the other hand, we assume that there are at least three agents and six alternatives, while Ma et al. (2018) do not make such an assumption.

Finally, the non-quasi-linearity of preferences is recently introduced in various models. Examples include house allocation model with transfers (Andersson and Svensson, 2014; Andersson et al., 2016; Andersson and Svensson, 2016), matching model with transfers (Morimoto, 2014; Garratt and Pycia, 2020), school choice model (Phan et al., 2021), and trading network model (Fleiner et al., 2019; Schlegel, 2021). A general mechanism design model is studied by Kazumura et al. (2020a), and they give a necessary and sufficient condition for *strategy-proofness* when preferences can be non-quasi-linear.

1.2 Organizations

The rest of this article is organized as follows. In Section 2, we introduce the model and definitions. In Section 3, we define Groves and Vickrey rules and extend Vickrey rules to non-quasi-linear domains. In Section 4, we define partially quasi-linear preferences. We state our main results in Section 5. In Section 6, we state implications of our results. In Section 7, we conclude. All the proofs appear in Appendix.

2 The model and definitions

There are $n \geq 2$ agents and $m \geq 1$ types of objects. We denote the set of agents by $N \equiv \{1, \ldots, n\}$ and the set of object types by $M \equiv \{1, \ldots, m\}$. Each object $a \in M$ has $\bar{x}_a \in \mathbb{N}$ copies. Thus, $\bar{x} \equiv (\bar{x}_a)_{a \in M}$ is the **social endowment**. If $m = 1$, there are only identical objects. If $\bar{x}_a = 1$ for each $a \in M$, then there are only distinct objects. We assume $\sum_{a \in M} \bar{x}_a > 1$.[12] Denote $\mathbf{0} \equiv (0, 0, \ldots, 0) \in \mathbb{R}^m$. A **package** is a vector $x \equiv (x_a)_{a \in M} \in \mathbb{Z}^m$ such that $\mathbf{0} \leq x \leq \bar{x}$.[13] Let X be the set of packages. Each agent receives a package and pays some amount of money. Thus, the **consumption set** is $X \times \mathbb{R}$, and a typical **(consumption) bundle** for an agent is a pair $z \equiv (x, t) \in X \times \mathbb{R}$, where t is interpreted as the amount paid by the agent.

2.1 Preferences and valuations

Each agent i has a complete and transitive preference relation R_i over $X \times \mathbb{R}$. Let P_i and I_i be the strict and indifference relations associated with R_i, respectively. The generic notation for a class of admissible preferences is denoted by $\mathcal{R}$ and we call it a **domain**.[14] The following are standard conditions for a preference relation R_i.

- **Money monotonicity:** For each $x \in X$ and each pair $t, s \in \mathbb{R}$ with $t < s$, $(x, t) \; P_i \; (x, s)$.

- **Possibility of compensation:** For each $(x, t) \in X \times \mathbb{R}$ and each $y \in X$, there are $s, s' \in \mathbb{R}$ such that $(x, t) \; R_i \; (y, s)$ and $(y, s') \; R_i \; (x, t)$.

- **Continuity:** For each $z \in X \times \mathbb{R}$, the **upper contour set** at z, $UC_i(z) \equiv \{z' \in X \times \mathbb{R} : z' \; R_i \; z\}$, and the **lower contour set** at z, $LC_i(z) \equiv \{z' \in X \times \mathbb{R} : z \; R_i \; z'\}$, are both closed.

[12] See Saitoh and Serizawa (2008) and Sakai (2008) for results in the single object case.

[13] Given a pair of vectors $x, y \in \mathbb{Z}^m$, we write $x \geq y$ to mean $x_a \geq y_a$ for each $a \in M$. Similarly, we write $x > y$ to mean $x_a \geq y_a$ for each $a \in M$ and $x_b > y_b$ for some $b \in M$.

[14] As we define in Section 2.2, the domain of an (allocation) rule is a set of preference profiles. In this sense, it is more precise to call a set of preference profiles a domain. However, in this paper, we assume that a rule is defined on a Cartesian product of a set of preferences. Thus, we simply call a set of preferences a domain.

- **Object monotonicity:** For each $(x, t) \in X \times \mathbb{R}$ and each $y \in X$ with $y > x$, $(y, t)\ P_i\ (x, t)$.

We denote the set of preferences that satisfy the above four conditions by $\mathcal{R}^O$, and call it the **object monotonic domain**. Throughout the paper, we assume that preferences satisfy the above four conditions. Thus, whenever we take a domain $\mathcal{R}$, it satisfies $\mathcal{R} \subseteq \mathcal{R}^O$.

A standard class of preferences studied in the literature is the class of quasi-linear preferences.

Definition 1. A preference relation R_i is **quasi-linear** if for each pair $(x, t), (y, s) \in X \times \mathbb{R}$ and each $\delta \in \mathbb{R}$, $(x, t)\ I_i\ (y, s)$ implies $(x, t + \delta)\ I_i\ (y, s + \delta)$.

Let $\mathcal{R}^Q$ be the class of quasi-linear preferences and we call it the **quasi-linear domain**. For each $R_i \in \mathcal{R}^Q$, there is a **valuation function** $v_i : X \to \mathbb{R}_+$ such that $v_i(\mathbf{0}) = 0$, and for each pair $(x, t), (y, s) \in X \times \mathbb{R}$, $(x, t)\ R_i\ (y, s)$ if and only if $v_i(x) - t \geq v_i(y) - s$.

We now extend the notion of valuation to non-quasi-linear preferences. Given a preference relation R_i, $z \in X \times \mathbb{R}$, and $y \in X$, there is a payment $s \in \mathbb{R}$ such that $z\ I_i\ (y, s)$.[15] We call this payment level the **(compensated) valuation of** y **at** z **for** R_i, and denote it by $V^{R_i}(y, z)$. There are two remarks on the notion of valuation that we often use in the rest of the paper.

Remark 1. Given a preference relation R_i and a pair $(x, t), (y, s) \in X \times \mathbb{R}$, we have $(x, t)\ R_i\ (y, s)$ if and only if $V^{R_i}(y, (x, t)) \leq s$.

Remark 2. For each $R_i \in \mathcal{R}^Q$, each $(x, t) \in X \times \mathbb{R}$, and each $y \in X$, $V^{R_i}(y, (x, t)) - t = v_i(y) - v_i(x)$.

Figure 1 is an illustration of the consumption set for $M = \{1, 2\}$ and $\bar{x} = (1, 1)$. Throughout the paper, we use such diagrams to explain and illustrate some definitions and proofs. In this diagram, each of the four horizontal lines represents the set of real numbers, and each point on the lines represents a payment for the package specified on the left side of the line. The vertical dotted line in this diagram connects the points

[15] The existence of such a payment is guaranteed by money monotonicity, possibility of compensation and continuity. For the formal proof of the existence, see Kazumura and Serizawa (2016).

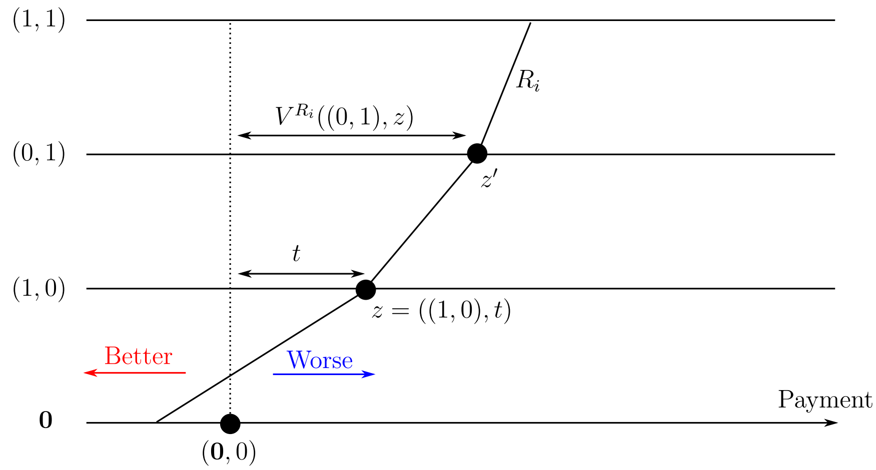

Figure 1: An illustration of the consumption set for $M = \{1,2\}$ and $\bar{x} = (1,1)$.

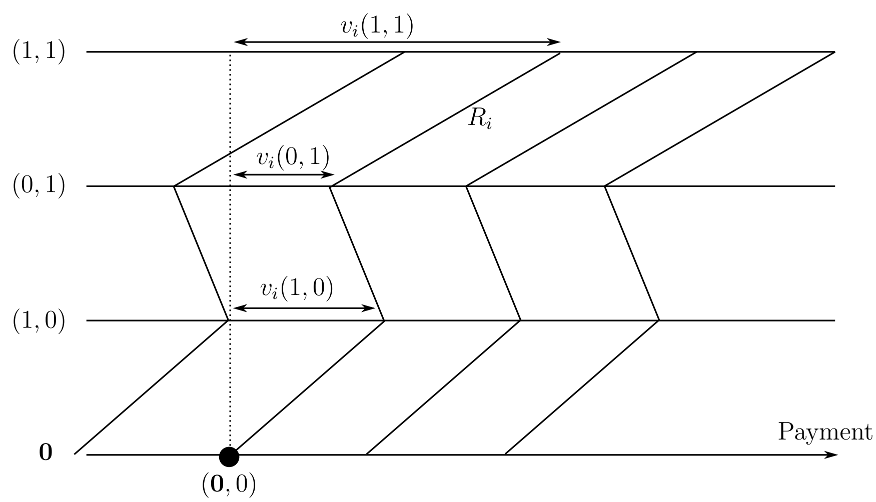

Figure 2: Indifference curves of a quasi-linear preference relation.

where the payment is zero. Then, the consumption set in this example consists of these four horizontal lines. For example, the point z corresponds to the consumption bundle $((1,0),t)$.

One way to describe a preference relation in the diagram is to draw "indifference curves." A typical indifference curve is illustrated in Figure 1. The indifference curve passes through z and z'. This means that z and z' are indifferent for a preference relation R_i. To specify which preference relation an indifference curve belongs to, we sometimes write the notation for the preference relation next to the indifference curve as in Figure 1. By money monotonicity, bundles to the left (resp. right) of an indifference curve are better (resp. worse) than the bundles on the indifference curve. The valuation of a package at a bundle corresponds to the payment at the point that is indifferent to the bundle. Thus, for example, the valuation of $(0,1)$ at z is equal to the payment at z'.

Indifference curves of a quasi-linear preference relation is shown in Figure 2. By Remark 2, indifference curves of a quasi-linear preference relation are parallel to each other. The valuation function of a quasi-linear preference relation corresponds to the the payment levels at the bundles on the indifference curve passing through $(\mathbf{0},0)$ as shown in Figure2.

2.2 Rules and their properties

A **package allocation** is an n-tuple $(x_i)_{i\in N} \in X^n$ such that $\sum_{i\in N} x_i \leq \bar{x}$. We denote the set of package allocations by A. A **(feasible) allocation** is an n-tuple $((x_i,t_i))_{i\in N} \in (X \times \mathbb{R})^n$ such that $(x_i)_{i\in N} \in A$. We denote the set of allocations by Z. A **preference profile** is an n-tuple $R \equiv (R_1,\dots R_n) \in \mathcal{R}^n$. Given $R \in \mathcal{R}^n$ and $i,j \in N$, let $R_{-i} \equiv (R_k)_{k\in N\setminus\{i\}}$ and $R_{-i,j} \equiv (R_k)_{k\in N\setminus\{i,j\}}$.

An **(allocation) rule** on $\mathcal{R}^n$ is a function $f : \mathcal{R}^n \to Z$. Given a rule f and $R \in \mathcal{R}^n$, we denote the bundle assigned to agent i by $f_i(R)$ and we write $f_i(R) \equiv (x_i^f(R), t_i^f(R))$, where $x_i^f(R)$ is the package that agent i receives and $t_i^f(R)$ is her payment.

We mainly focus on rules that satisfy two properties, efficiency and strategy-proofness. The efficiency notion here takes the planner's preferences into account, assuming that she is only interested in her revenue. Formally, an allocation $((x_i,t_i))_{i\in N} \in Z$ is **(Pareto-)efficient** for $R \in \mathcal{R}^n$ if there is no allocation $((y_i,s_i))_{i\in N} \in Z$ such that (i) for each $i \in N$, $(y_i,s_i)\ R_i\ (x_i,t_i)$, (ii) for some $j \in N$, $(y_j,s_j)\ P_i\ (x_j,t_j)$, and (iii) $\sum_{i\in N} s_i \geq \sum_{i\in N} t_i$. Thus, if an allocation is efficient, it is impossible to make an agent better off without harming other agents or reducing the revenue of the planner. By object monotonicity, if an allocation $((x_i,t_i))_{i\in N} \in Z$ is efficient for $R \in \mathcal{R}^n$, then no object remains unassigned, i.e., $\sum_{i\in N} x_i = \bar{x}$.

Remark 3. An allocation $((x_i, t_i))_{i\in N} \in Z$ is efficient for $R \in \mathcal{R}^n$ if and only if there is no allocation $((y_i, s_i))_{i\in N} \in Z$ such that (i) for each $i \in N$, $(y_i, s_i)\ I_i\ (x_i, t_i)$, and (ii) $\sum_{i\in N} s_i > \sum_{i\in N} t_i$.[16]

The efficiency of a rule requires that for each preference profile, an efficient allocation should be selected.

Efficiency: For each $R \in \mathcal{R}^n$, $f(R)$ is efficient for R.

Strategy-proofness means that no agent benefits from misrepresenting her preferences.

Strategy-proofness: For each $R \in \mathcal{R}^n$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $f_i(R)\ R_i\ f_i(R'_i, R_{-i})$.

3 Generalized Vickrey rules

Groves and Vickrey rules are defined on the quasi-linear domain, because they are defined by means of valuation functions. In this section we define those rules and extend Vickrey rules to larger domains.

We first introduce some notations. Given $r \in \mathbb{R}$, $i \in N$, $R_{-i} \in \mathcal{R}^{n-1}$, and $x \in X$, let

$$\sigma_i^r(R_{-i}; x) \equiv \max\left\{\sum_{j\in N\setminus\{i\}} V^{R_j}(x_j, (\mathbf{0}, r)) : (x_j)_{j\in N} \in A, x_i = x\right\}.$$

Thus, $\sigma_i^r(R_{-i}; x)$ is the maximum of the sum of valuations at $(\mathbf{0}, r)$ that the agents other than agent i can achieve when agent i obtains x. When $r = 0$, we sometimes write $\sigma_i(R_{-i}; x)$ instead of $\sigma_i^0(R_{-i}; x)$. Now we define Groves and Vickrey rules on the quasi-linear domain.

Definition 2. A rule f on $(\mathcal{R}^Q)^n$ is a **Groves rule** if for each $R \in (\mathcal{R}^Q)^n$,

$$(x_i^f(R))_{i\in N} \in \text{argmax}_{(x_i)_{i\in N}\in A} \sum_{i\in N} v_i(x_i),$$

[16]This result follows from money monotonicity.

and for each $i \in N$, there is $h_i : (\mathcal{R}^Q)^{n-1} \to \mathbb{R}$ such that

$$t_i^f(R) = h_i(R_{-i}) - \sigma_i(R_{-i}; x_i^f(R)).$$

A rule f on $(\mathcal{R}^Q)^n$ is a **Vickrey rule** if it is a Groves rule and for each $i \in N$, $h_i(\cdot) = \sigma_i(\cdot; \mathbf{0})$.

We now generalize Vickrey rules to larger domains. We do it by means of compensated valuations.

Definition 3. Let $\mathcal{R}$ be an arbitrary domain. Let $r \in \mathbb{R}$. A rule f on $\mathcal{R}^n$ is an r-**generalized Vickrey rule** if for each $R \in \mathcal{R}^n$,

$$(x_i^f(R))_{i \in N} \in \text{argmax}_{(x_i)_{i \in N} \in A} \sum_{i \in N} V^{R_i}(x_i, (\mathbf{0}, r)),$$

and for each $i \in N$,

$$t_i^f(R) = \sigma_i^r(R_{-i}; \mathbf{0}) - \sigma_i^r(R_{-i}; x_i^f(R)) + r.$$

For each $r \in \mathbb{R}$, an r-generalized Vickrey rule is defined in the same manner as Vickrey rules except that compensated valuations at $(\mathbf{0}, r)$ are used instead of valuation functions and a fixed cost r is added to the payment formula. If $r > 0$, we can interpret it as an entry fee. On the other hand, if $r < 0$, then we can interpret it as an entry subsidy. Note that a 0-generalized Vickrey rule coincides with a Vickrey rule on the quasi-linear domain. For simplicity, we sometimes call a 0-generalized Vickrey rule a **generalized Vickrey rule**.

4 Partially quasi-linear preferences

In this section, we define some classes of non-quasi-linear preferences. First, given a preference relation R_i and $r \in \mathbb{R}$, we define the following set:

$$X^r(R_i) \equiv \{(x, t) \in X \times \mathbb{R} : (x, t) \; R_i \; (\mathbf{0}, r) \text{ and } t \geq r\}.$$

This is the set consisting of bundles such that the associated payments are no less than r and they are at least as desirable as $(\mathbf{0}, r)$. We call it the r-**relevant**

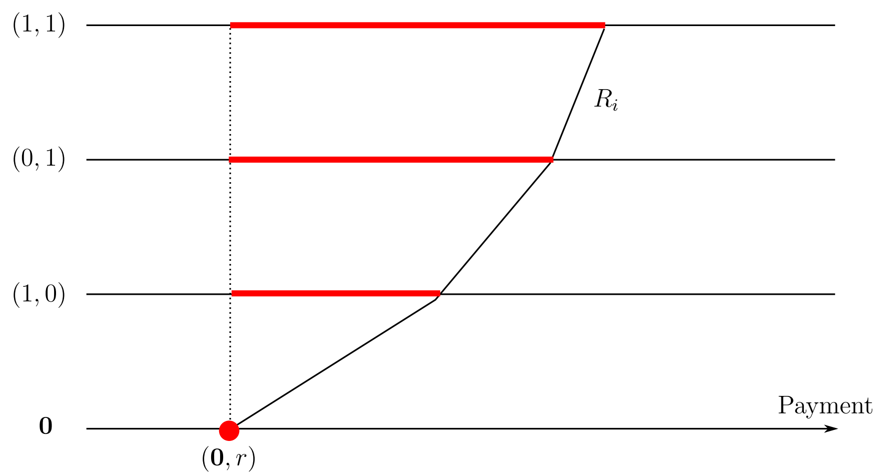

Figure 3: An illustration of the r-relevant consumption set for R_i for $\bar{x} = (1, 1)$.

consumption set for R_i. And in particular when $r = 0$, we sometimes call it the **relevant consumption set for** R_i. Further, we sometimes denote $X(R_i) \equiv X^0(R_i)$. Figure 3 is an illustration of the r-relevant consumption set for a preference relation R_i for $\bar{x} = (1, 1)$. The relevant consumption set for R_i consists of the bundle $(\mathbf{0}, 0)$ and the bundles on the three bold lines.

The preferences defined below are non-quasi-linear preferences that preserve the quasi-linearity on the r-relevant consumption set.

Definition 4. Given $r \in \mathbb{R}$, a preference relation R_i is r**-partially quasi-linear** if for each $(x, t) \in X^r(R_i)$ and each $y \in X$ with $V^{R_i}(y, (x, t)) \geq r$,

$$V^{R_i}(y, (x, t)) - t = V^{R_i}(y, (\mathbf{0}, r)) - V^{R_i}(x, (\mathbf{0}, r)).$$

Remark 4. Given $r \in \mathbb{R}$, a preference relation R_i is r-partially quasi-linear if and only if for each $(x, t) \in X^r(R_i)$, $V^{R_i}(\bar{x}, (x, t)) - t = V^{R_i}(\bar{x}, (\mathbf{0}, r)) - V^{R_i}(x, (\mathbf{0}, r))$.

For each $r \in \mathbb{R}$, let $\mathcal{R}^P(r)$ be the class of r-partially quasi-linear preferences, and call it the r**-partially quasi-linear domain**. For simplicity, we sometimes call a 0-partially quasi-linear preference relation and the 0-partially quasi-linear domain a **partially quasi-linear preference relation** and the **partially quasi-linear domain**, respectively. Further, for simplicity, we sometimes denote $\mathcal{R}^P \equiv \mathcal{R}^P(0)$.

The r-partially quasi-linearity requires a preference relation to be quasi-linear when the consumption set is restricted to the r-relevant consumption set. Thus, for

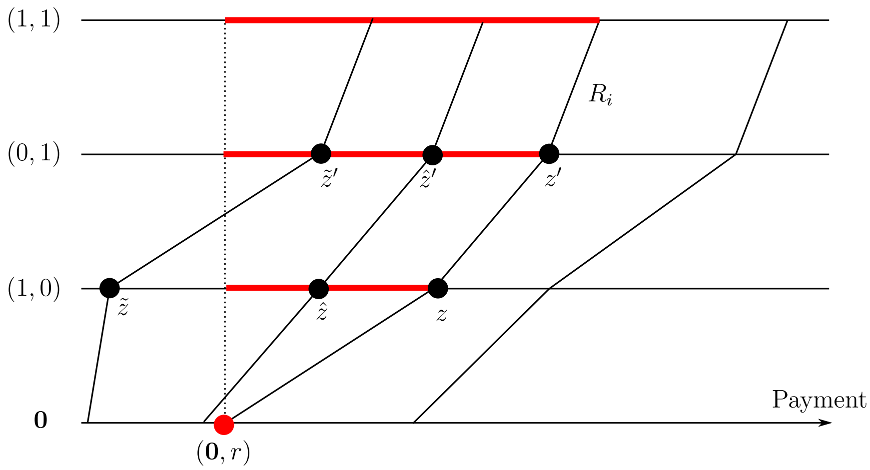

Figure 4: An illustration of an r-partially quasi-linear preference relation.

each $r \in \mathbb{R}$, quasi-linear preferences are r-partially quasi-linear, and hence, $\mathcal{R}^Q \subseteq \mathcal{R}^P(r)$. An r-partially quasi-linear preference relation is shown in Figure 4. It has parallel indifference curves *in* the r-relevant consumption set as shown in the figure. However, the r-partially quasi-linearity does not require that indifference curves that pass through a bundle outside of the r-relevant consumption set should be parallel. Thus, for instance, the indifference curve between z and z' has to be parallel to the one between $\hat{z}$ and $\hat{z}'$. However, it does not have to be parallel to the indifference curve between $\tilde{z}$ and $\tilde{z}'$ because $\tilde{z}$ is not in the r-relevant consumption set.

5 Main results

It is known that on the quasi-linear domain, Groves rules are the only *efficient* and *strategy-proof* rules.

Fact 1 (Holmström (1979)). *Groves rules are the only rules that satisfy efficiency and strategy-proofness on* $(\mathcal{R}^Q)^n$.

As we discussed in Introduction, it is likely that agents may have non-quasi-linear preferences in practice. Thus, we investigate the possibility of designing an *efficient* and *strategy-proof* rule when agents may have non-quasi-linear preferences. More specifically, we investigate how much we can expand the domain from the quasi-linear domain while guaranteeing the existence of an *efficient* and *strategy-proof* rule. Our results depend on the number of agents.

5.1 Two-agent case

First, we define the notion of maximal domain.

Definition 5. A domain $\mathcal{R}$ is a **maximal domain** for a list of properties if
(i) there is a rule on $\mathcal{R}^n$ that satisfies the properties, and
(ii) for each $\mathcal{R}' \supsetneq \mathcal{R}$, no rule on $(\mathcal{R}')^n$ satisfies the properties.

Remark 5. A maximal domain for a list of properties may not be unique. Indeed, the theorem below shows the existence of multiple maximal domains for *efficiency* and *strategy-proofness*.

The following theorem states that for two-agent case, for each $r \in \mathbb{R}$, only the r-generalized Vickrey rules satisfy *efficiency* and *strategy-proofness* on the r-partially quasi-linear domain, and further, the domain is a maximal domain for *efficiency* and *strategy-proofness*.

Theorem 1. *Let $n = 2$ and $r \in \mathbb{R}$.*
(i) On $(\mathcal{R}^P(r))^2$, a rule satisfies efficiency and strategy-proofness if and only if it is an r-generalized Vickrey rule.
(ii) $\mathcal{R}^P(r)$ is a maximal domain for efficiency and strategy-proofness.

Notice that Theorem 1 (i) is not a straightforward extension of Fact 1. First, Theorem 1 (i) holds only for two-agent case, whereas Fact 1 holds for each $n \geq 2$. Indeed, the proof of Theorem 1 (i) depends on the two-agent assumption. Second, for each $r \in \mathbb{R}$, r-generalized Vickrey rules are characterized using only *efficiency* and *strategy-proofness*. On the other hand, on the quasi-linear domain, the class of *efficient* and *strategy-proof* rules is the class of Groves rules which is much larger than the class of r-generalized Vickrey rules.

Theorem 1 (ii) implies that for each $r \in \mathbb{R}$, there is no rule that satisfies *efficiency* and *strategy-proofness* if the domain is larger than the r-partially quasi-linear domain. Theorem 1 (ii) does not exclude the possibility that there is a maximal domain for *efficiency* and *strategy-proofness* such that for each $r \in \mathbb{R}$, it is not the r-partially

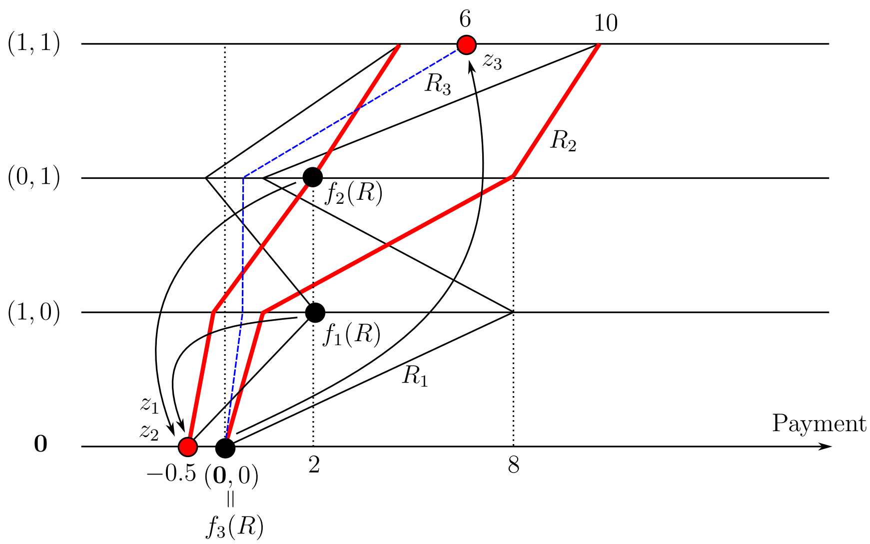

Figure 5: The inefficiency of a generalized Vickrey rule on $(\mathcal{R}^P)^n$ when $n = 3$.

quasi-linear domain. However, as we see in Section 5.3, we can pin down the class of maximal domains for *efficiency*, *strategy-proofness*, and some additional properties.

5.2 More than two-agent case

Even when $n \geq 3$, for each $r \in \mathbb{R}$, r-generalize Vickrey rules are *strategy-proof* on the r-partially quasi-linear domain.

Proposition 1. *For each $r \in \mathbb{R}$, r-generalized Vickrey rules are strategy-proof on $(\mathcal{R}^P(r))^n$.*

However, when $n \geq 3$, for each $r \in \mathbb{R}$, r-generalize Vickrey rules are not *efficient* on the r-partially quasi-linear domain. We give an example for $n = 3$, $\bar{x} = (1, 1)$, and $r = 0$.[17]

Example: (Figure 5.) *When $n = 3$ and $\bar{x} = (1, 1)$, generalized Vickrey rules are not efficient on $(\mathcal{R}^P)^3$.* Let f be a generalized Vickrey rule on $(\mathcal{R}^P)^3$. Let $R_1 \in \mathcal{R}^P$

[17] By modifying this example slightly, we can show that for each $r \in \mathbb{R}$ and each $n \geq 3$, r-generalized Vickrey rules violate *efficiency* on $(\mathcal{R}^P(r))^n$.

be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$V^{R_1}(x, (\mathbf{0}, 0)) = \begin{cases} 8 & \text{if } x = (1,0), \\ 1 & \text{if } x = (0,1), \\ 10 & \text{if } x = (1,1), \end{cases}$$

and $V^{R_1}(\mathbf{0}, ((1,0), 2)) = -0.5$. Let $R_2 \in \mathcal{R}^P$ be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$V^{R_2}(x, (\mathbf{0}, 0)) = \begin{cases} 1 & \text{if } x = (1,0), \\ 8 & \text{if } x = (0,1), \\ 10 & \text{if } x = (1,1), \end{cases}$$

and $V^{R_2}(\mathbf{0}, ((0,1), 2)) = -0.5$.

Note that we can take R_1 and R_2 so that they are partially quasi-linear. Let $R_3 \in \mathcal{R}^Q$ be such that $v_3((1,0)) = v_3((0,1)) = 0.5$ and $v_3((1,1)) = 6$. Denote $R \equiv (R_1, R_2, R_3)$. These preferences are illustrated in Figure 5. The black indifference curves are those of R_1, the bold indifference curves are those of R_2, and the dashed indifference curve is that of R_3. By the definition of the generalized Vickrey rule,

$$f_1(R) = ((1,0), 2), \; f_2(R) = ((0,1), 2), \text{ and } f_3(R) = (\mathbf{0}, 0).$$

Let $((x_i, t_i))_{i \in N} \in Z$ be such that

$$(x_1, t_1) = (x_2, t_2) = (0, -0.5) \text{ and } z_3 = ((1,1), 6).$$

It is easy to show that for each $i \in N$, $(x_i, t_i) \; I_i \; f_i(R)$, and $\sum_{i \in N} t_i = 5 > 4 = \sum_{i \in N} t_i^f(R)$. Hence, by Remark 3, $f(R)$ is not *efficient* for R. □

In contrast to the case of $n = 2$, the following result shows that when $n \geq 3$, *efficiency* and *strategy-proofness* are incompatible even on the r-partially quasi-linear domain for each $r \in \mathbb{R}$.

Theorem 2. *Let $n \geq 3$ and $r \in \mathbb{R}$. No rule on $(\mathcal{R}^P(r))^n$ satisfies efficiency and strategy-proofness.*

Theorem 2 implies that for each $r \in \mathbb{R}$, a maximal domain for *efficiency* and *strategy-proofness* lies between the quasi-linear domain and the r-partially quasi-linear domain. We conclude this subsection by showing that for each $r \in \mathbb{R}$, the

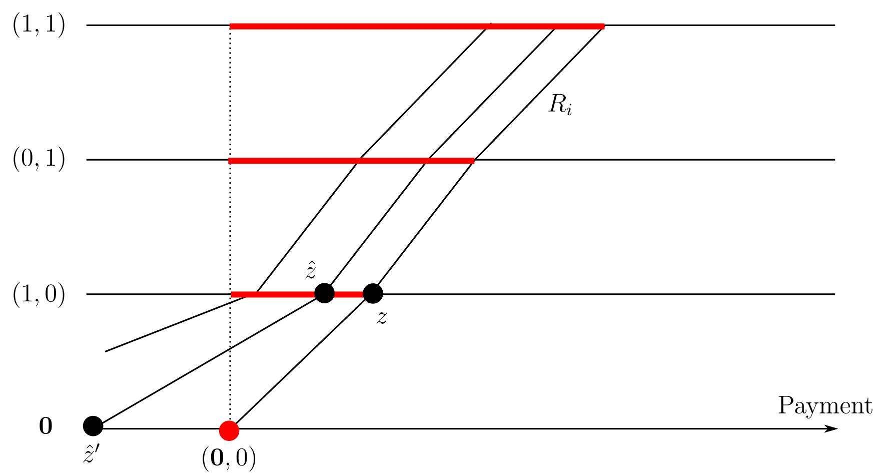

Figure 6: Indifference curves of $R_i \in \hat{\mathcal{R}}^P$.

r-partially quasi-linear domain is "close" to a maximal domain for *efficiency* and *strategy-proofness*. We demonstrate it by showing that for each $r \in \mathbb{R}$, there is a domain that is "close" to the r-partially quasi-linear domain and there is an *efficient* and *strategy-proof* rule on the domain.

For simplicity, we focus only on the (0-)partially quasi-linear domain. The same argument follows for the r-partially quasi-linear domain for each $r \in \mathbb{R}$. Consider a preference relation R_i that satisfies the following conditions: For each $(x,t) \in X(R_i)$ and each $y \in X$, if $V^{R_i}(y,(x,t)) \geq 0$,

$$V^{R_i}(y,(x,t)) - t = V^{R_i}(y,(\mathbf{0},0)) - V^{R_i}(x,(\mathbf{0},0)),$$

and if $V^{R_i}(y,(x,t)) < 0$,

$$t - V^{R_i}(y,(x,t)) \geq V^{R_i}(x,(\mathbf{0},0)) - V^{R_i}(y,(\mathbf{0},0)).$$

Let $\hat{\mathcal{R}}^P$ be the class of preferences that satisfies the above conditions.

Figure 6 illustrates indifference curves of a preference relation $R_i \in \hat{\mathcal{R}}^P$. The first condition of $\hat{\mathcal{R}}^P$ is the same as the requirement of the partially quasi-linearity. Thus, as in Figure 6, indifference curves in the relevant consumption set are parallel. The second condition requires that to change from a bundle $(x,t) \in X(R_i)$ to another package y without making the agent worse off, it is necessary to compensate her at least $V^{R_i}(x,(\mathbf{0},0)) - V^{R_i}(y,(\mathbf{0},0))$. Thus, in Figure 6, the indifference curve between

z and $(\mathbf{0}, 0)$ should be at least as steep as the one between $\hat{z}$ and $\hat{z}'$. It is clear that $\hat{\mathcal{R}}^P \subsetneq \mathcal{R}^P$.

By Proposition 1, generalized Vickrey rules are *strategy-proof* on $(\hat{\mathcal{R}}^P)^n$. Further, it is easy to show that generalized Vickrey rules are *efficient* on $(\hat{\mathcal{R}}^P)^n$. Thus, a maximal domain for *efficiency* and *strategy-proofness* lies between $\mathcal{R}^P$ and $\hat{\mathcal{R}}^P$.

5.3 Further results with additional properties

We have shown that the partially quasi-linearity is the key for the existence of an *efficient* and *strategy-proof* rule. However, there are domains such that for each $r \in \mathbb{R}$, they are not subsets or supersets of the r-partially quasi-linear domain, and there may exist a rule that satisfies *efficiency* and *strategy-proofness* on such a domain. In this section, we impose some additional properties which are reasonable in many settings, and investigate on what domains there is a rule that satisfies *efficiency*, *strategy-proofness*, and the additional properties.

We consider three properties. The first property is a participation constraint. It states that an agent is never assigned a bundle that makes her worse off than she would be if she had received no object and paid nothing.

Individual rationality: For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R)\ R_i\ (\mathbf{0}, 0)$.

The second property requires that the payment of losers (the agents who receive no object) should be nonnegative.

No subsidy for losers: For each $R \in \mathcal{R}^n$ and each $i \in N$, if $x_i^f(R) = \mathbf{0}$, $t_i^f(R) \geq 0$.

No subsidy for losers is a natural requirement satisfied in many allocation problems such as auctions in practice.

On the quasi-linear domain, the class of rules that satisfy *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* is identified.

Fact 2. *(Holmström, 1979; Chew and Serizawa, 2007) Vickrey rules are the only rules that satisfy efficiency, strategy-proofness, individual rationality, and no subsidy for losers on $(\mathcal{R}^Q)^n$.*

Individual rationality implies that the payment of losers is not positive. Thus, *individual rationality* and *no subsidy for losers* imply that the payment of losers is always zero. However, it is sometimes the case that agents pay a fixed amount of money as an entry fee, or receive a fixed amount of money as an entry subsidy—the planner may do so to increase the number of participants. The following property captures these situations.

Common payment for losers: There is $t^* \in \mathbb{R}$ such that for each $R \in \mathcal{R}^n$ and each $i \in N$, if $x_i^f(R) = \mathbf{0}$, then $t_i^f(R) = t^*$.

Of course, *individual rationality* and *no subsidy for losers* imply *common payment for losers*. Before stating the main results in this section, we give a technical result that we use to derive them.

Proposition 2. *Let $r \in \mathbb{R}$. Let $\mathcal{R}$ be such that $\mathcal{R}^Q \subseteq \mathcal{R} \not\subseteq \mathcal{R}^P(r)$. Then, there is no efficient and strategy-proof rule on $\mathcal{R}^n$ that coincides with an r-generalized Vickrey rule on $(\mathcal{R}^Q)^n$.*

By Fact 2, if a rule on $\mathcal{R}^n$ with $\mathcal{R} \supseteq \mathcal{R}^Q$ satisfies *efficiency*, *strategy-proofness*, *individual rationality* and *no subsidy for losers*, then it coincides with a Vickrey rule on $(\mathcal{R}^Q)^n$. Thus, by Proposition 2 and Theorems 1 and 2, we obtain the following result.

Theorem 3. *Let $\mathcal{R}$ be such that $\mathcal{R} \supseteq \mathcal{R}^Q$.*
(i) Let $n = 2$. $\mathcal{R}$ is a maximal domain for efficiency, strategy-proofness, individual rationality and no subsidy for losers if and only if $\mathcal{R} = \mathcal{R}^P$.
(ii) Let $n \geq 3$. If there is a rule on $\mathcal{R}^n$ that satisfies efficiency, strategy-proofness, individual rationality and no subsidy for losers, then $\mathcal{R} \subsetneq \mathcal{R}^P$.

We do not prove Theorem 3 since it is immediate from Proposition 2 and Theorems 1 and 2.

Theorem 3 states that the domain must be a subset of the partially quasi-linear domain for the existence of a rule that satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers*. Further, Theorem 3 (i) states that for two agents, the partially quasi-linear domain is the *unique* maximal domain for the four properties among the domains that include the quasi-linear domain.

By Fact 1, it is easy to see that if a rule on $\mathcal{R}^n$ with $\mathcal{R} \supseteq \mathcal{R}^Q$ satisfies *efficiency*, *strategy-proofness*, and *common payment for losers*, it coincides with an r-generalized Vickrey rule for some $r \in \mathbb{R}$ on $(\mathcal{R}^Q)^n$. Thus, by Proposition 2 and Theorems 1 and 2, we obtain the following result.

Theorem 4. *Let $\mathcal{R}$ be such that $\mathcal{R} \supseteq \mathcal{R}^Q$.*
(i) Let $n = 2$. $\mathcal{R}$ is a maximal domain for efficiency, strategy-proofness, and common payment for losers if and only if $\mathcal{R} = \mathcal{R}^P(r)$ for some $r \in \mathbb{R}$.
(ii) Let $n \geq 3$. If there is a rule on $\mathcal{R}^n$ that satisfies efficiency, strategy-proofness, and common payment for losers, then $\mathcal{R} \subsetneq \mathcal{R}^P(r)$ for some $r \in \mathbb{R}$.

We do not prove Theorem 4 since it is immediate from Proposition 2 and Theorems 1 and 2.

Theorem 4 states that the domain must be a subset of some r-partially quasi-linear domain for the existence of a rule that satisfies *efficiency*, *strategy-proofness*, and *common payment for losers*. Theorem 4 also implies the following: Consider a domain such that for each $r \in \mathbb{R}$, it is not a subset or superset of the r-partially quasi-linear domain. If there is an *efficient* and *strategy-proof* rule on the domain, then either (i) losers pay different amount of money at some preference profile, or (ii) the payment of a loser depends on the other agents' preferences.

6 Implications of our results

6.1 Impossibility results on various domains

Our results are useful to verify whether there is a rule that satisfies *efficiency* and *strategy-proofness* (and *individual rationality* and *no subsidy for losers*, or *common payment for losers*) on various domains of interest. In this section, we consider several reasonable domains and show what out results imply on those domains.

The object monotonic domain includes the partially quasi-linear domain and some non-partially quasi-linear preferences. Thus, there is no *efficient* and *strategy-proof* rule on the object monotonic domain.

Corollary 1. *No rule on* $(\mathcal{R}^O)^n$ *satisfies efficiency and strategy-proofness.*

An important class of preferences studied in the literature is the class of preferences having income effects (Kaneko, 1983; Saitoh and Serizawa, 2008; Baisa, 2020).

Definition 6. A preference relation R_i has **nonnegative income effects** (resp. **nonpositive income effects**) if for each pair $(x,t),(y,s) \in X \times \mathbb{R}$ with $t > s$, $(x,t)\ I_i\ (y,s)$ implies that for each $\delta \in \mathbb{R}_{++}$, $(x,t-\delta)\ R_i\ (y,s-\delta)$ (resp. $(y,s-\delta)\ R_i\ (x,t-\delta)$).

Though income is not modeled explicitly, the zero payment corresponds to the endowed income. Thus, a decrease in payment by $\delta > 0$ can be interpreted as an increase in income by δ. In Definition 6, by $t > s$ and $(x,t)\ I_i\ (y,s)$, the agent prefers x to y at the income level corresponding to the payment s—her willingness to pay for switching from y to x is $t - s > 0$. Then, nonnegative income effects (resp. nonpositive income effects) imply that at the income level corresponding to $s - \delta$, her willingness to pay for switching from y to x is at least (resp. at most) $t - s$. Thus, nonnegative income effects (resp. nonpositive income effects) mean that the willingness to pay for switching from a package to a preferred package is non-decreasing (resp. non-increasing) in income level.[18]

Let $\mathcal{R}^{NNI}$ and $\mathcal{R}^{NPI}$ be the sets of preferences having nonnegative income effects and nonpositive income effects, respectively. A preference relation has both nonnegative and nonpositive income effects if and only if it is quasi-linear. Thus, $\mathcal{R}^Q \subseteq \mathcal{R}^{NNI}$ and $\mathcal{R}^Q \subseteq \mathcal{R}^{NPI}$. It is also clear that $\mathcal{R}^{NNI}$ and $\mathcal{R}^{NPI}$ contain a preference relation such that for each $r \in \mathbb{R}$, it is not r-partially quasi-linear. Thus, by Theorem 4, we obtain the following result.

Corollary 2. *Let* $\mathcal{R} \in \{\mathcal{R}^{NNI}, \mathcal{R}^{NPI}\}$. *No rule on* $\mathcal{R}^n$ *satisfies efficiency, strategy-proofness, and common payment for losers.*

Since *individual rationality* and *no subsidy for losers* imply *common payment for losers*, Corollary 2 implies that no rule satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* on $(\mathcal{R}^{NNI})^n$ or $(\mathcal{R}^{NPI})^n$.

Another class of preferences studied in the literature is the class of quasi-linear preferences with borrowing cost (Saitoh and Serizawa, 2008). Here we model income

[18]For a detailed and graphical explanation of preferences having income effects, see Kazumura et al. (2020a)

explicitly. Suppose that each agent has to borrow money at some interest rate when the payment for a package exceeds her income. For each $i \in N$, let $w_i \in \mathbb{R}_+$ and $r_i \in \mathbb{R}_+$ be agent i's income level and the interest rate that agent i is facing, respectively. Then, when the payment for a package is $t \in \mathbb{R}$, agent i's actual cost is given by a function $c(\cdot; w_i, r_i) : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$c(t; w_i, r_i) = \begin{cases} t & \text{if } t \leq w_i, \\ w_i + (1 + r_i)(t - w_i) & \text{if } t > w_i. \end{cases}$$

That is, if the payment is no more than her income, she just pays that amount. If the payment is higher than her income, she has to borrow the difference between the payment and her income at the interest rate r. We call this function a **borrowing cost function**.

Definition 7. A preference relation R_i is **quasi-linear with borrowing cost** if there are a valuation function $v_i : X \to \mathbb{R}_+$ with $v_i(\mathbf{0}) = 0$, an income level $w_i \in \mathbb{R}_+ \cup \{+\infty\}$, and an interest rate $r_i \in \mathbb{R}_+$ such that for each pair $(x, t), (y, s) \in X \times \mathbb{R}$, $(x, t)\ R_i\ (y, s)$ if and only if $v_i(x) - c(t; w_i, r_i) \geq v_i(y) - c(s; w_i, r_i)$.

Let $\mathcal{R}^{Q,B}$ be the class of quasi-linear preferences with borrowing cost. Here we assume that the income level of an agent and the interest rate that the agent is facing is her private information. An extreme case is that the income is infinity—in this case the agent does not have to borrow money and hence, her preference relation is quasi-linear. Thus, $\mathcal{R}^{Q} \subseteq \mathcal{R}^{Q,B}$. Further, there is also a preference relation in $\mathcal{R}^{QB}$ that is not partially quasi-linear. For example, consider a preference relation $R_i \in \mathcal{R}^{Q,B}$ with a valuation function v_i and an income level w_i such that for some $x \in X$, $w_i < v_i(x)$. Such a preference relation is not partially quasi-linear. Thus, by Theorem 3, we obtain the following result.

Corollary 3. *Let $\mathcal{R} = \mathcal{R}^{Q,B}$. No rule on $\mathcal{R}^n$ satisfies efficiency, strategy-proofness, individual rationality, and no subsidy for losers.*

6.2 Public goods model

Our results also have implications to the public goods model. In this section we consider the public goods model studied by Ma et al. (2018). We do not introduce the public goods model formally. The formal model can be found in the supplementary material (Kazumura, 2022).

Ma et al. (2018) introduce the notion of partially quasi-linearity in the public goods model.[19] Their main result implies that in the public goods model, if the domain is larger than the partially quasi-linear domain, no rule satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy*, where *no subsidy* requires that the payment of each agent is nonnegative. On the other hand, Theorem 2 implies the following.

Corollary 4. *In the public goods model, if there are at least three agents and six alternatives, no rule satisfies efficiency and strategy-proofness on the partially quasi-linear domain.*

A formal statement of this corollary, its proof, and a comparison between this corollary and results by Ma et al. (2018) appear in the supplementary material (Kazumura, 2022).

The key idea for this result is that we can embed the package assignment model to the public goods model. Consider the package assignment model with three agents and two identical objects. The set of package allocation (where all the objects are allocated) consists of the following.

$$(0,1,1),(0,2,0),(0,0,2),(1,1,0),(1,0,1), \text{ and } (2,0,0).$$

If there are six alternatives in the public goods model, we can associate each alternative with one of these package allocations. Then, for each preference relation in the package assignment model, we can find a corresponding preference relation in the public goods model. Theorem 2 then implies that no rule satisfies *efficiency* and *strategy-proofness* on the set of preferences in the public goods model that corresponds to the partially quasi-linear domain in the package assignment model. Since this set of preferences is contained in the partially quasi-linear domain in the public goods model, we obtain the impossibility result.

[19] The partially quasi-linear domain is called the *parallel domain* in Ma et al. (2018).

7 Conclusion

We have demonstrated that the quasi-linearity of preferences plays an important role for the existence of an *efficient* and *strategy-proof* rule in a package assignment model. Further, our results give a useful tool to verify the existence of an *efficient* and *strategy-proof* rule on various domains.

An underling assumption in this paper is that the domain contains all the quasi-linear preferences. In some situations in practice, however, objects are substitutes or complements, and hence, we cannot justify that agents may have any quasi-linear preference relation in such situations. Whether there is an *efficient* and *strategy-proof* rule in those situations is an open question.

Appendix

We provide the proofs of Theorems 1, and 2 and Propositions 1 and 2. Though the results in those theorems and propositions hold for each r-partially quasi-linear domain, we provide only the proofs for the (0-)partially quasi-linear domain. However, the proofs can be easily modified so that they work for each r-partially quasi-linear domain.

A Preliminaries

This section has two parts. First, we define several classes of preferences and state lemmas that guarantee the existence of some preferences that we pick in the proofs. Second, we provide lemmas used in the proofs. We relegate the proofs of all the lemmas to Appendix E so that readers can refer to the lemmas easily.

We now introduce some notations we use in the rest of the paper. For each $x \in X$, let $m(x) \equiv \sum_{a \in M} x_a$. That is, $m(x)$ is the number of objects in the package x. Let $\mathcal{X} \equiv \{(x, y) \in X \times X : x > y\}$. Let $\mathbf{t} \equiv (t_x)_{x \in X}$ be our generic notation for a vector in $\mathbb{R}^{|X|}$.

A.1 Preferences

We introduce three types of preferences.

Definition 8. A preference relation R_i is **bounded** if there is a pair $\overline{s}, \underline{s} \in \mathbb{R}_{++}$ such that for each $(x, y) \in \mathcal{X}$ and each $t \in \mathbb{R}$,

$$\underline{s} < V^{R_i}(x,(y,t)) - t < \overline{s}.$$

Remark 6. Quasi-linear preferences are bounded.

Next, we define a quasi-linear preference relation of which valuation function is "negligibly" small compared with another preference relation.

Definition 9. Given a preference relation R_i, $R_j \in \mathcal{R}^Q$ is **negligible with respect to** R_i if for each $x \in X \setminus \{\bar{x}\}$,

$$\begin{aligned} v_j(x) &< \inf_{(y,y') \in \mathcal{X},\ t \in \mathbb{R}} V^{R_i}(y,(y',t)) - t, \text{ and} \\ v_j(\bar{x}) &< \inf_{t \in \mathbb{R}} V^{R_i}(\bar{x},(0,t)) - t. \end{aligned}$$

Remark 7. In Definition 9, if R_i is quasi-linear, the inequalities are simplified as follows: for each $x \in X \setminus \{\bar{x}\}$,

$$v_j(x) < \min_{(y,y') \in \mathcal{X}} v_i(y) - v_i(y') \text{ and } v_j(\bar{x}) < v_i(\bar{x}).$$

Given a preference relation R_i, let $\mathcal{R}^Q(R_i)$ be the class of preferences that are negligible with respect to R_i.[20]

Lemma 1. *For each $R_i \in \mathcal{R}$ and each $R_i' \in \mathcal{R}^Q(R_i)$, $\mathcal{R}^Q(R_i') \subseteq \mathcal{R}^Q(R_i)$.*

Lemma 1 is straightforward from Definition 9. Thus, we omit the proof.

Lemma 2. *Let $i \in N$ and $j \in N \setminus \{i\}$. Let $R \in \mathcal{R}^n$ be such that $R_j \in \mathcal{R}^Q$ and $R_{-i,j} \in (\mathcal{R}^Q(R_j))^{n-2}$. For each $x \in X$, $\sigma_i(R_{-i}; x) = v_j(\bar{x} - x)$.*

Definition 10. Given a vector $\mathbf{t} \in \mathbb{R}^{|X|}$ and $x \in X$, a preference relation R_i is a **monotonic transformation of t at** x if for each $y \in X$ with $y \neq x$, $V^{R_i}(y,(x,t_x)) < t_y$.

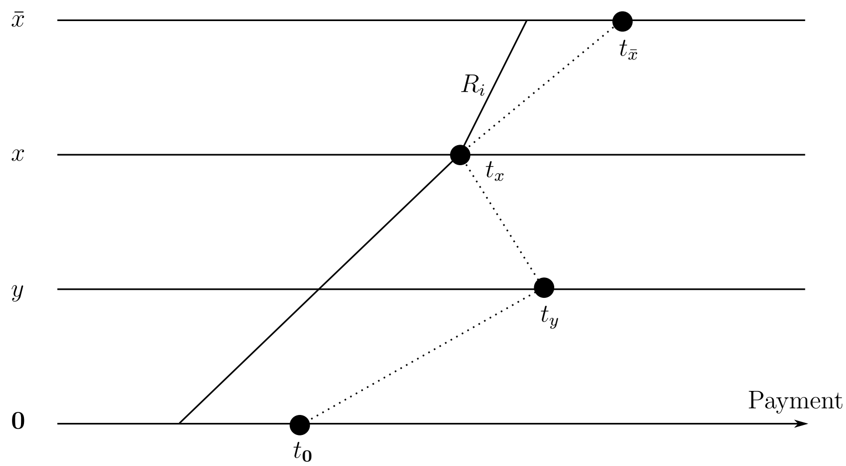

Figure 7: An illustration of a monotonic transformation of $\mathbf{t} \in \mathbb{R}^{|X|}$ at $x \in X$.

Figure 7 is an illustration of a monotonic transformation R_i of $\mathbf{t} \in \mathbb{R}^{|X|}$ at $x \in X$. In this figure, the vector $\mathbf{t}$ consists of the points on the dotted kinked line. Definition 10 requires that the bundles that are indifferent to (x, t_x) should be to the left of the dotted kinked line. Given a vector $\mathbf{t} \in \mathbb{R}^{|X|}$ and $x \in X$, let $\mathcal{R}^{MT}_{\mathbf{t},x}$ be the set of preferences that are monotonic transformations of $\mathbf{t}$ at x.

A vector $\mathbf{t} \in \mathbb{R}^{|X|}$ is **object monotonic** if for each $(x, y) \in \mathcal{X}$, $t_x > t_y$. In some of the proofs, we pick a partially quasi-linear preference relation that is a monotonic transformation of two object monotonic vectors. The following three lemmas give sufficient conditions for the existence of such preferences.

Lemma 3. *Let $x \in X$. Let $\mathbf{t}, \mathbf{s} \in \mathbb{R}^{|X|}$ be object monotonic vectors such that $0 \leq t_x < s_x$ and $s_{\mathbf{0}} > 0$. Then, for each $y \in X \setminus \{x\}$, there is a bounded preference relation $R_i \in \mathcal{R}^P$ such that $R_i \in \mathcal{R}^{MT}_{\mathbf{t},x} \cap \mathcal{R}^{MT}_{\mathbf{s},y}$.*

Lemma 4. *Let $x \in X$. Let $\mathbf{t}, \mathbf{s} \in \mathbb{R}^{|X|}$ be object monotonic vectors such that $t_x < s_x$ and $t_x < 0$. Then, for each $y \in X \setminus \{x\}$, there is a bounded preference relation $R_i \in \mathcal{R}^P$ such that $R_i \in \mathcal{R}^{MT}_{\mathbf{t},x} \cap \mathcal{R}^{MT}_{\mathbf{s},y}$.*

Lemma 5. *Let $\mathbf{t}, \mathbf{s} \in \mathbb{R}^{|X|}$ be object monotonic vectors such that $t_{\bar{x}} < s_{\bar{x}}$ and $s_{\mathbf{0}} < 0$. Then, there is $R_i \in \mathcal{R}^P$ such that $R_i \in \mathcal{R}^{MT}_{\mathbf{t},\bar{x}} \cap \mathcal{R}^{MT}_{\mathbf{s},\mathbf{0}}$.*

[20] The right hand side of the two inequalities in Definition 9 might be zero for some preference relation. For such a preference relation R_i, $\mathcal{R}^Q(R_i) = \emptyset$.

A.2 Implications of properties of rules

First we state three lemmas related to *efficiency*.

Lemma 6. *Let f be an efficient rule on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$ and $N' \subseteq N$. Let $(x_i)_{i \in N'} \in X^{|N'|}$ be such that $\sum_{i \in N'} x_i \leq \sum_{i \in N'} x_i^f(R)$. Then, $\sum_{i \in N'} V^{R_i}(x_i, f_i(R)) \leq \sum_{i \in N'} t_i^f(R)$.*

The following lemma states that it is not *efficient* to assign an object to a negligible agent.

Lemma 7. *Let f be an efficient rule on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$ and $i \in N$. If $R_i \in \mathcal{R}^Q(R_j)$ for some $j \in N \setminus \{i\}$, then $x_i^f(R) = \mathbf{0}$.*

We introduce the notion of option set. Given a rule f, $i \in N$ and $R_{-i} \in \mathcal{R}^{n-1}$, the **option set of agent i under f for R_{-i}** is defined as

$$o_i^f(R_{-i}) \equiv \{z \in X \times \mathbb{R} : \exists R_i \in \mathcal{R} \text{ s.t. } f_i(R_i, R_{-i}) = z\},$$

and let

$$X_i^f(R_{-i}) \equiv \{x \in X : \exists R_i \in \mathcal{R} \text{ s.t. } x_i^f(R_i, R_{-i}) = x\}.$$

That is, $o_i^f(R_{-i})$ is the set of bundles available to agent i under f when the other agents have R_{-i}. Similarly, $X_i^f(R_{-i})$ is the set of packages available to agent i under f when the other agents have R_{-i}.

The following lemma states that any package is available to an agent under an efficient rule if the domain includes the quasi-linear domain and the preferences of the other agents are bounded.

Lemma 8. *Let $\mathcal{R}$ be such that $\mathcal{R} \supseteq \mathcal{R}^Q$ and f be an efficient rule on $\mathcal{R}^n$. Let $i \in N$ and $R_{-i} \in \mathcal{R}^{n-1}$ be such that for each $j \in N \setminus \{i\}$, R_j is bounded. Then, $X_i^f(R_{-i}) = X$.*

Next, we states a fact and lemmas related to *strategy-proofness*. We begin with defining the notion of monotonicity.

Definition 11. A rule f on $\mathcal{R}^n$ is **monotonic** if for each $i \in N$, each pair $R_i, R_i' \in \mathcal{R}$, and each $R_{-i} \in \mathcal{R}^{n-1}$,

$$V^{R_i}(x_i^f(R_i', R_{-i}), f_i(R_i, R_{-i})) \leq V^{R_i'}(x_i^f(R_i', R_{-i}), f_i(R_i, R_{-i})).$$

Remark 8. If R_i and R'_i are quasi-linear, the inequality in Definition 11 is equivalent to the following:

$$v_i(x_i^f(R'_i, R_{-i})) - v_i(x_i^f(R_i, R_{-i})) \leq v'_i(x_i^f(R'_i, R_{-i})) - v'_i(x_i^f(R_i, R_{-i})).$$

The following fact states that monotonicity is a necessary condition for strategy-proofness.

Fact 3. *(Kazumura et al., 2020a) Each strategy-proof rule is monotonic.*

Given a *strategy-proof* rule f, $i \in N$, and $R_{-i} \in \mathcal{R}^{n-1}$, for each pair $(x, t), (y, s) \in o_i^f(R_{-i})$, $x = y$ implies $t = s$. Thus, for a *strategy-proof* rule f, we can define a mapping $t_i^f(R_{-i}, \cdot) : X_i^f(R_{-i}) \to \mathbb{R}$ such that for each $x \in X_i^f(R_{-i})$, $(x, t_i^f(R_{-i}, x)) \in o_i^f(R_{-i})$. Given a *strategy-proof* rule f and $x \in X_i^f(R_{-i})$, let $z_i^f(R_{-i}, x) \equiv (x, t_i^f(R_{-i}, x))$.

Lemma 9. *Let f be a strategy-proof rule on $\mathcal{R}^n$. Let $i \in N$ and $R_{-i} \in \mathcal{R}^{n-1}$ be such that $X_i^f(R_{-i}) = X$. Then, the vector $(t_i^f(R_{-i}; x))_{x \in X}$ is object monotonic.*

Lemma 10. *Let f be a strategy-proof rule on $\mathcal{R}^n$. Let $i \in N$ and $R_{-i} \in \mathcal{R}^{n-1}$ be such that $X_i^f(R_{-i}) = X$. Denote $\mathbf{t} = (t_i^f(R_{-i}, x))_{x \in X}$. Let $x \in X$ and $R_i \in \mathcal{R}_{\mathbf{t},x}^{MT}$. Then, $x_i^f(R_i, R_{-i}) = x$.*

Lemma 10 is straightforward from the definition of *strategy-proofness*. Thus, we omit the proof.

Finally we provide a fact and a lemma derived from *efficiency* and *strategy-proofness*. If $\mathcal{R} \supseteq \mathcal{R}^Q$, then by Fact 1, an *efficient* and *strategy-proof* rule on $\mathcal{R}^n$ coincides with a Groves rule on the quasi-linear domain. This observation is formally documented in the following fact.

Fact 4. *Let $\mathcal{R}$ be such that $\mathcal{R} \supseteq \mathcal{R}^Q$ and f be an efficient and strategy-proof rule on $\mathcal{R}^n$. For each $i \in N$, there is $h_i : (\mathcal{R}^Q)^{n-1} \to \mathbb{R}$ such that for each $R \in \mathcal{R}^n$ with $R_{-i} \in (\mathcal{R}^Q)^{n-1}$, $t_i^f(R) = h_i(R_{-i}) - \sigma_i(R_{-i}; x_i^f(R))$.*

The following lemma states that in some specific situations, *efficiency* and *strategy-proofness* give a range of the payment of an agent who receives no object.

Lemma 11. *Let $\mathcal{R} = \mathcal{R}^P$ and f be an efficient and strategy-proof rule on $\mathcal{R}^n$.*
(i) Assume $n = 2$. Let $i \in N$, $j \in N \setminus \{i\}$, and $R_j \in \mathcal{R}^Q$. Then, $t_i^f(R_j; \mathbf{0}) = 0$.

(ii) Assume $n \geq 3$. Let $i \in N$, $j \in N \setminus \{i\}$, and $R_{-i} \in \mathcal{R}^{n-1}$ be such that $R_j \in \mathcal{R}^Q$ and $R_{-i,j} \in (\mathcal{R}^Q(R_j))^{n-2}$. Denote

$$s^* \equiv \max \left\{ \sum_{k \in N \setminus \{i,j\}} v_k(x_k) : (x_k)_{k \in N} \in (X \setminus \{\bar{x}\})^n,\ \sum_{k \in N} x_k \leq \bar{x} \right\}.$$

Then, $-s^* \leq t_i^f(R_{-i}; \mathbf{0}) \leq 0$.

Lemma 11 (i) states that when $n = 2$ and the domain is the partially quasi-linear domain, the payment of a loser is zero as long as the other agent has a quasi-linear preference relation. The s^* in Lemma 11 (ii) is the maximum of the sum of valuations that the agents other than the agents i and j can achieve under the assumption that no agent receives all the objects. Then Lemma 11 (ii) states that when $n \geq 3$ and the domain is the partially quasi-linear domain, the payment of a loser is at least $-s^*$ and at most zero at preference profiles that satisfy the conditions specified in Lemma 11 (ii).

B Proof of Proposition 1

We prove only that (0-)generalized Vickrey rules are *strategy-proof* on $(\mathcal{R}^P)^n$. However, for each $r \in \mathbb{R}$, we can prove the strategy-proofness of r-generalized Vickrey rules on $(\mathcal{R}^P(r))^n$ in the same manner.

Let $\mathcal{R} = \mathcal{R}^P$ and f be a generalized Vickrey rule on $\mathcal{R}^n$. First we show the following claim.[21]

Claim 1. *Let $R \in \mathcal{R}^n$, $i \in N$, and $x \in X$. If $V^{R_i}(x, f_i(R)) \geq 0$, then*

$$V^{R_i}(x, f_i(R)) - t_i^f(R) = V^{R_i}(x, (\mathbf{0}, 0)) - V^{R_i}(x_i^f(R), (\mathbf{0}, 0)).$$

[21] We use this claim also in the proof of Theorem 1 (i) when we prove the efficiency of r-generalized Vickrey rules.

Proof. By the definition of f, $t_i^f(R) \geq 0$. We also have

$$\begin{aligned} V^{R_i}(x_i^f(R), (\mathbf{0}, 0)) - t_i^f(R) &= V^{R_i}(x_i^f(R), (\mathbf{0}, 0)) - (\sigma_i(R_{-i}; \mathbf{0}) - \sigma_i(R_{-i}; x_i^f(R))) \\ &= \sum_{j \in N} V^{R_j}(x_j^f(R), (\mathbf{0}, 0)) - \sigma_i(R_{-i}; \mathbf{0}) \\ &\geq 0, \end{aligned}$$

where the inequality follows from $(x_j^f(R))_{j \in N} \in \text{argmax}_{(x_j)_{j \in N} \in A} V^{R_j}(x_j, (\mathbf{0}, 0))$. This implies $f_i(R)\ R_i\ (\mathbf{0}, 0)$. Thus, $f_i(R) \in X(R_i)$. Hence, by $R_i \in \mathcal{R}^P$ and $V^{R_i}(x, f_i(R)) \geq 0$, $V^{R_i}(x, f_i(R)) - t_i^f(R) = V^{R_i}(x, (\mathbf{0}, 0)) - V^{R_1}(x_i^f(R), (\mathbf{0}, 0))$. □

Now we show that f is *strategy-proof*. Without loss of generality, we focus only on agent 1. Let $R \in \mathcal{R}^n$ and $R_1' \in \mathcal{R}$, and denote $R' \equiv (R_1', R_{-1})$. We show $f_1(R)\ R_1\ f_1(R')$. Since $t_1^f(R') \geq 0$ by the definition of f, if $V^{R_1}(x_1^f(R'), f_1(R)) < 0$, then clearly $f_1(R)\ R_1\ f_1(R')$. Thus, suppose $V^{R_1}(x_1^f(R'), f_1(R)) \geq 0$.

By $(x_i^f(R))_{i \in N} \in \text{argmax}_{(x_i)_{i \in N} \in A} V^{R_i}(x_i, (\mathbf{0}, 0))$,

$$V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) + \sigma_1(R_{-1}; x_1^f(R)) \geq V^{R_1}(x_1^f(R'), (\mathbf{0}, 0)) + \sigma_1(R_{-1}; x_1^f(R')) \quad (1)$$

By this inequality and Claim 1,

$$\begin{aligned} & V^{R_1}(x_1^f(R'), f_1(R)) \\ &= t_1^f(R) + V^{R_1}(x_1^f(R'), (\mathbf{0}, 0)) - V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) && \text{(by Claim 1)} \\ &\leq t_1^f(R) + \sigma_1(R_{-1}; x_1^f(R)) - \sigma_1(R_{-1}; x_1^f(R')) && \text{(by (1))} \\ &= \sigma_1(R_{-1}; \mathbf{0}) - \sigma_1(R_{-1}; x_1^f(R)) + \sigma_1(R_{-1}; x_1^f(R)) - \sigma_1(R_{-1}; x_1^f(R')) \\ &= \sigma_1(R_{-1}; \mathbf{0}) - \sigma_1(R_{-1}; x_1^f(R')) \\ &= t_1^f(R'). && \text{(by the definition of } f) \end{aligned}$$

This implies $f_1(R)\ R_1\ f_1(R')$. ■

C Proof of Theorem 1

As we mentioned in the beginning of Appendix, we give only the proof for the partially quasi-linear domain. In the proof of Theorem 1, whenever we take an agent i, the other agent is denoted by j.

C.1 Proof of Theorem 1 (i)

If part. Let $\mathcal{R} \equiv \mathcal{R}^P$ and f be a generalized Vickrey rule on $\mathcal{R}^2$. By Proposition 1, f is *strategy-proof*. Thus, we show that f is *efficient*.

Suppose by contradiction that $f(R)$ is not *efficient* for some $R \in \mathcal{R}^2$. By Remark 3, there is $((y_1, s_1), (y_2, s_2)) \in Z$ such that

$$(y_i, s_i) \; I_i \; f_i(R) \text{ for each } i \in \{1, 2\} \text{ and } s_1 + s_2 > t_1^f(R) + t_2^f(R). \tag{2}$$

Note that for each $i \in \{1, 2\}$, $s_i = V^{R_i}(y_i, f_i(R))$. Note also that by Claim 1 of the proof of Proposition 1, for each $\in \{1, 2\}$ with $V^{R_i}(y_i, f_i(R)) \geq 0$,

$$t_i^f(R) = s_i + V^{R_i}(x_i^f(R), (\mathbf{0}, 0)) - V^{R_i}(y_i, (\mathbf{0}, 0)). \tag{3}$$

Since f is a generalized Vickry rule, for each $i \in \{1, 2\}$, $t_i^f(R) \geq 0$. Thus, by (2), either $s_1 > 0$ or $s_2 > 0$. Without loss of generality, assume $s_1 > 0$. There are two cases.

Case 1. $s_2 > 0$. By $(x_i^f(R))_{i \in N} \in \text{argmax}_{(x_i)_{i \in N} \in A} V^{R_i}(x_i, (\mathbf{0}, 0))$ and (3),

$$\begin{aligned}
&t_1(R) + t_2(R) \\
&\quad = s_1 + V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) - V^{R_1}(y_1, (\mathbf{0}, 0)) + s_2 + V^{R_2}(x_2^f(R), (\mathbf{0}, 0)) - V^{R_2}(y_2, (\mathbf{0}, 0)) \\
&\quad \geq s_1 + s_2,
\end{aligned}$$

contradicting (2).

Case 2. $s_2 \leq 0$. By the definition of f, $t_2^f(R) = V^{R_1}(\bar{x}, (\mathbf{0}, 0)) - V^{R_1}(x_1^f(R), (\mathbf{0}, 0))$. Thus, by $s_1 > 0$ and (3),

$$\begin{aligned}
&t_1^f(R) + t_2^f(R) \\
&\quad = s_1 + V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) - V^{R_1}(y_1, (\mathbf{0}, 0)) + V^{R_1}(\bar{x}, (\mathbf{0}, 0)) - V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) \\
&\quad = V^{R_1}(\bar{x}, (\mathbf{0}, 0)) + s_1 - V^{R_1}(y_1, (\mathbf{0}, 0)) \\
&\quad \geq s_1 + s_2,
\end{aligned}$$

where the last inequality follows from $s_2 \leq 0$ and $V^{R_1}(\bar{x}, (\mathbf{0}, 0)) \geq V^{R_1}(y_1, (\mathbf{0}, 0))$. This inequality contradicts (2).

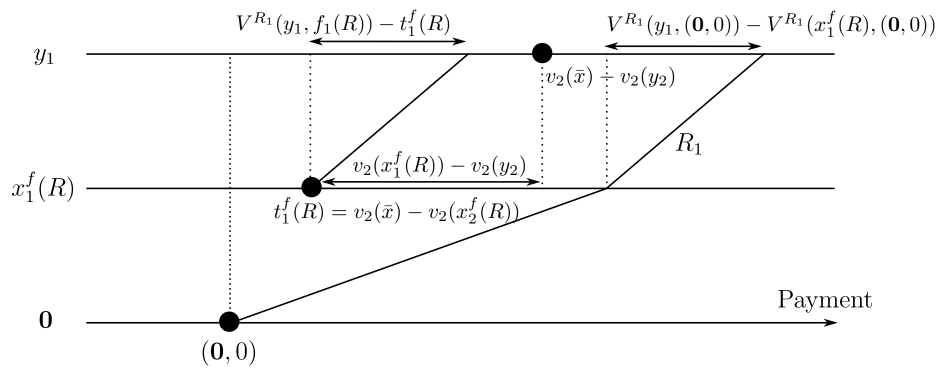

Figure 8: An illustration of Case 1 in the proof of Step 2

Only if part. Let $\mathcal{R} = \mathcal{R}^P$ and let f be a rule on $\mathcal{R}^2$ that satisfies *efficiency* and *strategy-proofness*. We now show that f is a generalized Vickrey rule. We do the proof in five steps.

Step 1. *Let $i \in N$ and $R_j \in \mathcal{R}^Q$. For each $x \in X$, $t_i^f(R_j; x) = v_j(\bar{x}) - v_j(\bar{x} - x)$.*[22]

Proof. Without loss of generality, assume $i = 1$. By Fact 4 and $R_2 \in \mathcal{R}^Q$, there is $h_1 : \mathcal{R}^Q \to \mathbb{R}$ such that for each $x \in X$, $t_1^f(R_2; x) = h_1(R_2) - \sigma_1(R_2; x)$. Note that for each $x \in X$, $\sigma_1(R_2; x) = v_2(\bar{x} - x)$. Further, since R_2 is bounded and by Lemma 8, $\mathbf{0} \in X_1^f(R_2)$. Thus by Lemma 11 (i), $h_1(R_2) - v_2(\bar{x}) = t_1^f(R_2; \mathbf{0}) = 0$, which implies $h_1(R_2) = v_2(\bar{x})$. Hence, we obtain the desired result. ■

Step 2. *Let $i \in \{1, 2\}$ and $R \in \mathcal{R}^2$ be such that $R_j \in \mathcal{R}^Q$. Then, $(x_i^f(R), x_j^f(R)) \in$* argmax$_{(x_1,x_2)\in A} V^{R_i}(x_i, (\mathbf{0}, 0)) + v_j(x_j)$.

Proof. Without loss of generality, assume $i = 1$. By Step 1 and $R_2 \in \mathcal{R}^Q$, $(\mathbf{0}, 0) \in o_1^f(R_2)$. Thus, by the strategy-proofness of f, $f_1(R)\ R_i\ (\mathbf{0}, 0)$. Further, by Step 1 and $R_2 \in \mathcal{R}^Q$, $t_1^f(R) = v_2(\bar{x}) - v_2(x_2^f(R)) \geq 0$. Thus, $f_1(R) \in X(R_1)$.

Let $(y_1, y_2) \in A$. We show $V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) + v_2(x_2^f(R)) \geq V^{R_1}(y_1, (\mathbf{0}, 0)) - v_2(y_2)$. There are two cases

[22] It is guaranteed by Lemma 8 that $X_i^f(R_j) = X$.

Case 1. $V^{R_1}(y_1, f_1(R)) \geq 0$. (Figure 8.) By Step 1 and the strategy-proofness of f, $f_1(R)\ R_i\ (y_1, v_2(\bar{x}) - v_2(y_2))$, which implies $V^{R_1}(y_1, f_1(R)) \leq v_2(\bar{x}) - v_2(y_2)$. By this inequality and $t_1^f(R) = v_2(\bar{x}) - v_2(x_2^f(R))$,

$$V^{R_1}(y_1, f_1(R)) - t_1^f(R) \leq v_2(x_2^f(R)) - v_2(y_2). \tag{4}$$

By $f_1(R) \in X(R_1)$, $V^{R_1}(y_1, f_1(R)) \geq 0$, and $R_1 \in \mathcal{R}^P$,

$$V^{R_1}(y_1, (\mathbf{0}, 0)) - V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) = V^{R_1}(y_1, f_1(R)) - t_1^f(R).$$

Thus,

$$V^{R_1}(y_1, (\mathbf{0}, 0)) - V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) \leq v_2(x_2^f(R)) - v_2(y_2),$$

which implies $V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) + v_2(x_2^f(R)) \geq V^{R_1}(y_1, (\mathbf{0}, 0)) - v_2(y_2)$.

Case 2. $V^{R_1}(y_1, f_1(R)) < 0$. (Figure 9.) By the object monotonicity of R_1, $(y_1, 0) \in X(R_1)$. By $V^{R_1}(y_1, f_1(R)) < 0$, $f_1(R)\ P_1\ (y_1, 0)$. This implies $V^{R_1}(x_1^f(R), (y_1, 0)) > t_1^f(R) \geq 0$. Thus, by $R_1 \in \mathcal{R}^P$,

$$V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) - V^{R_1}(y_1, (\mathbf{0}, 0)) = V^{R_1}(x_1^f(R), (y_1, 0)) > t_1^f(R).$$

By $t_1^f(R) = v_2(\bar{x}) - v_2(x_2^f(R))$ and $v_2(\bar{x}) - v_2(y_2) \geq 0$,

$$v_2(x_2^f(R)) - v_2(y_2) = -(v_2(\bar{x}) - v_2(x_2^f(R))) + v_2(\bar{x}) - v_2(y_2) \leq t_1^f(R)$$

Thus,

$$V^{R_1}(y_1, (\mathbf{0}, 0)) - V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) < v_2(x_2^f(R)) - v_2(y_2),$$

which implies $V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) + v_2(x_2^f(R)) > V^{R_1}(y_1, (\mathbf{0}, 0)) + v_2(y_2)$. ■

Remark 9. By Step 2, for each $i \in N$, each $R_j \in \mathcal{R}$, and each $x \in X$, there is $R_i \in \mathcal{R}^Q$ such that $x_i^f(R_i, R_j) = x$. Hence, for each $i \in N$ and each $R_j \in \mathcal{R}$, $X_i^f(R_j) = X$.

Step 3. *Let $i \in \{1, 2\}$ and $R \in \mathcal{R}^2$. Then,*

$$t_i^f(R) = V^{R_j}(\bar{x}, (\mathbf{0}, 0)) - V^{R_j}(x_j^f(R), (\mathbf{0}, 0)) + t_i^f(R_j; \mathbf{0}).$$

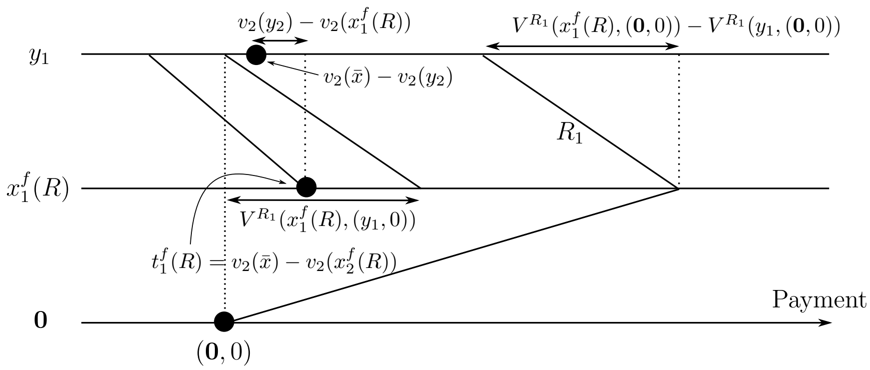

Figure 9: An illustration of Case 2 in the proof of Step 2

Proof. Without loss of generality, assume $i = 1$ and $x_1^f(R) \neq \mathbf{0}$. Further, by Remark 9, we can assume $R_1 \in \mathcal{R}^Q$ without loss of generality. Suppose by contradiction that $t_1^f(R) \neq V^{R_2}(\bar{x}, (\mathbf{0}, 0)) - V^{R_2}(x_2^f(R), (\mathbf{0}, 0)) + t_1^f(R_2; \mathbf{0})$.

Let

$$\delta_{\min} \equiv \min\{t_1^f(R) - t_1^f(R_2; \mathbf{0}), V^{R_2}(\bar{x}, (\mathbf{0}, 0)) - V^{R_2}(x_2^f(R), (\mathbf{0}, 0))\}, \text{and}$$
$$\delta_{\max} \equiv \max\{t_1^f(R) - t_1^f(R_2; \mathbf{0}), V^{R_2}(\bar{x}, (\mathbf{0}, 0)) - V^{R_2}(x_2^f(R), (\mathbf{0}, 0))\}.$$

Clearly, $\delta_{\min} < \delta_{\max}$. By $x_1^f(R) \neq \mathbf{0}$ and $X_1^f(R_2) = X$, $t_1^f(R) > t_1^f(R_2; \mathbf{0})$.[23] By object monotonicity, $V^{R_2}(\bar{x}, (\mathbf{0}, 0)) - V^{R_2}(x_2^f(R), (\mathbf{0}, 0)) \geq 0$. Thus, $\delta_{\min} \geq 0$. Let $d \in \mathbb{R}_{++}$ be such that

$$\delta_{\min} < d < \delta_{\max}.$$

Let $(\epsilon_x)_{x \in X} \in \mathbb{R}_+^{|X|}$ be an object monotonic vector that is sufficiently close to 0.[24] Note that since $(\epsilon_x)_{x \in X}$ is object monotonic, for each $x \in X \setminus \{\mathbf{0}\}$, $\epsilon_x > 0$. Let

[23] By $X_1^f(R_2) = X$, there is $R_1' \in \mathcal{R}$ such that $x_1^f(R_1', R_2) = \mathbf{0}$. If $t_1^f(R) \leq t_1^f(R_2; \mathbf{0})$, then by the object monotonicity of R_1', $f_1(R)\ P_1'\ (\mathbf{0}, t_1^f(R_2; \mathbf{0})) = f_1(R_1', R_2)$, contradicting *strategy-proofness*. Thus, $t_1^f(R) > t_1^f(R_2; \mathbf{0})$.

[24] For example, take $(\epsilon_x)_{x \in X}$ that satisfies $\epsilon_{\mathbf{0}} = 0$ and $\epsilon_{\bar{x}} < \min\{\delta_{\max} - d, \min_{(x,y) \in \mathcal{X}} v_1(x) - v_1(y), \min_{(x,y) \in \mathcal{X}} V^{R_2}(x, (\mathbf{0}, 0)) - V^{R_2}(y, (\mathbf{0}, 0))\}$. Then the proof of Step 3 works with this $(\epsilon_x)_{x \in X}$.

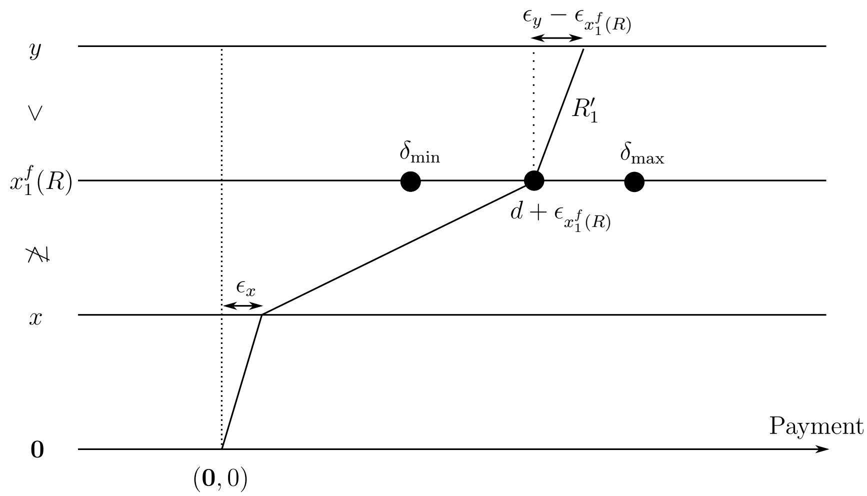

Figure 10: An illustration of R_1'.

$R_1' \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$v_1'(x) = \begin{cases} d + \epsilon_x & \text{if } x \geq x_1^f(R), \\ \epsilon_x & \text{otherwise.} \end{cases}$$

Since $(\epsilon_x)_{x \in X}$ is object monotonic, R_1' is object monotonic. Figures 10 is an illustration of R_1'. Denote $R' \equiv (R_1', R_2)$.

Claim 1. $x_1^f(R') = \mathbf{0}$.

Proof. Suppose by contradiction that $x_1^f(R') \neq \mathbf{0}$. We have three cases.

Case 1. $x_1^f(R') = x_1^f(R)$. By *strategy-proofness*, $f_1(R') = f_1(R)$. Since $d < \delta_{\max}$ and $\epsilon_{x_1^f(R)}$ is sufficiently close to 0, $v_1'(x_1^f(R)) = d + \epsilon_{x_1^f(R)} < \delta_{\max}$. If $\delta_{\max} = t_1^f(R) - t_1^f(R_2; \mathbf{0})$, then $v_1'(x_1^f(R)) - t_1^f(R) < -t_1^f(R_2; \mathbf{0}))$. This implies $z_1^f(R_2; \mathbf{0})\ P_1'\ f_1(R) = f_1(R')$, contradicting *strategy-proofness*. If $\delta_{\max} = V^{R_2}(\bar{x}, (\mathbf{0}, 0)) - V^{R_2}(x_2^f(R), (\mathbf{0}, 0))$, then $v_1'(\mathbf{0}) + V^{R_2}(\bar{x}, (\mathbf{0}, 0)) > v_1'(x_1^f(R)) + V^{R_2}(x_2^f(R), (\mathbf{0}, 0)) = v_1'(x_1^f(R')) + V^{R_2}(x_2^f(R'), (\mathbf{0}, 0))$, contradicting Step 2.

Case 2. $x_1^f(R') > x_1^f(R)$. Since $\epsilon_{x_1^f(R')}$ and $\epsilon_{x_1^f(R)}$ are sufficiently close to 0,

$$v_1'(x_1^f(R')) - v_1'(x_1^f(R)) = \epsilon_{x_1^f(R')} - \epsilon_{x_1^f(R)} < v_1(x_1^f(R')) - v_1(x_1^f(R)).$$

This contradicts Fact 3.

Case 3. $x_1^f(R) \not\leq x_1^f(R')$. By $x_1^f(R') \neq \mathbf{0}$, $x_2^f(R') \neq \bar{x}$. Thus, since $\epsilon_{x_1^f(R')}$ is sufficiently small and $x_1^f(R') \neq \mathbf{0}$,

$$v_1'(x_1^f(R')) = \epsilon_{x_1^f(R')} < V^{R_2}(\bar{x}, (\mathbf{0}, 0)) - V^{R_2}(x_2^f(R'), (\mathbf{0}, 0)).$$

This implies $v_1'(\mathbf{0}) + V^{R_2}(\bar{x}, (\mathbf{0}, 0)) > v_1'(x_1^f(R')) + V^{R_2}(x_2^f(R'), (\mathbf{0}, 0))$, contradicting Step 2. □

We drive a contradiction for each of the following two cases.

Case 1. $\delta_{\min} = t_1^f(R) - t_1^f(R_2; \mathbf{0})$. By $d > \delta_{\min}$, $\epsilon_{x_1^f(R)} > 0$, and Claim 1,

$$v_1'(x_1^f(R)) - t_1^f(R) = d + \epsilon_{x_1^f(R)} - t_1^f(R) > \delta_{\min} - t_1^f(R) = -t_1^f(R_2; \mathbf{0}) = v_1'(x_1^f(R')) - t_1^f(R').$$

This inequality implies $f_1(R)\ P_1'\ f_1(R')$, which contradicts *strategy-proofness*.

Case 2. $\delta_{\min} = V^{R_2}(\bar{x}, (\mathbf{0}, 0)) - V^{R_2}(x_2^f(R), (\mathbf{0}, 0))$. By Claim 1, $x_2^f(R') = \bar{x}$. Since $v_1'(x_1^f(R)) > d > \delta_{\min} = V^{R_2}(\bar{x}, (\mathbf{0}, 0)) - V^{R_2}(x_2^f(R), (\mathbf{0}, 0))$, $v_1'(x_1^f(R)) + V^{R_2}(x_2^f(R), (\mathbf{0}, 0)) > v_1'(x_1^f(R')) + V^{R_2}(x_2^f(R'), (\mathbf{0}, 0))$. This contradicts Step 2. ■

Step 4. *Let $i \in \{1, 2\}$ and $R \in \mathcal{R}^2$. Then,*

$$t_i^f(R) = V^{R_j}(\bar{x}, (\mathbf{0}, 0)) - V^{R_j}(x_j^f(R), (\mathbf{0}, 0)).$$

Proof. Without loss of generality, assume $i = 1$. By Step 3, we only need to show $t_1^f(R_2; \mathbf{0}) = 0$. Suppose by contradiction that $t_1^f(R_2; \mathbf{0}) \neq 0$.

Claim 1. *There is a bounded $R_1' \in \mathcal{R}$ such that $t_2^f(R_1'; \mathbf{0}) \neq 0$.*

Proof. Let $\epsilon \in \mathbb{R}_{++}$ be such that $\epsilon < |t_1^f(R_2; \mathbf{0})|$. Let $R_2' \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$v_2'(x) = V^{R_2}(x, (\mathbf{0}, 0)) + \epsilon.$$

Let $\mathbf{t}, \mathbf{s} \in \mathbb{R}^{|X|}$ be such that for each $x \in X$, $t_x = t_1^f(R_2; x)$ and $s_x = t_1^f(R_2'; x)$. By Remark 9, they are well-defined, and by Lemma 9, are object monotonic.

By Step 1 and $R_2' \in \mathcal{R}^Q$, $s_{\mathbf{0}} = 0$ and $s_{\bar{x}} = v_2'(\bar{x})$. If $t_{\mathbf{0}} < 0$, then $t_{\mathbf{0}} < s_{\mathbf{0}}$, and thus, $\mathbf{t}$ and $\mathbf{s}$ satisfy the condition of Lemma 4 for $\mathbf{0}$. On the other hand, if $t_{\mathbf{0}} > 0$, then by Step 3, $\epsilon < t_{\mathbf{0}}$, and $s_{\bar{x}} = v_2'(\bar{x})$,

$$t_{\bar{x}} = V^{R_2}(\bar{x}, (\mathbf{0}, 0)) + t_{\mathbf{0}} > V^{R_2}(\bar{x}, (\mathbf{0}, 0)) + \epsilon = v_2'(\bar{x}) = s_{\bar{x}}.$$

Hence, in this case, $\mathbf{t}$ and $\mathbf{s}$ satisfy the condition of Lemma 3 for $\bar{x}$.

Therefore, by Lemmas 3 and 4, there is a bounded $R_1' \in \mathcal{R}^P$ such that $R_1' \in \mathcal{R}^{MT}_{\mathbf{t},\mathbf{0}} \cap \mathcal{R}^{MT}_{\mathbf{s},\bar{x}}$. By Lemma 10, $x_1^f(R_1', R_2) = \mathbf{0}$ and $x_1^f(R_1', R_2') = \bar{x}$. Therefore,

$$x_2^f(R_1', R_2) = \bar{x} \text{ and } x_2^f(R_1', R_2') = \mathbf{0}.$$

Now, we show $t_2^f(R_1'; \mathbf{0}) \neq 0$. Suppose by contradiction that $t_2^f(R_1'; \mathbf{0}) = 0$. By $x_2^f(R_1', R_2') = \mathbf{0}$, $f_2(R_1', R_2') = (\mathbf{0}, 0)$. However, by the definition of R_2', $V^{R_2}(\bar{x}, (\mathbf{0}, 0)) < v_2'(\bar{x})$. By $f_2(R_1', R_2') = (\mathbf{0}, 0)$ and $x_2^f(R_1', R_2) = \bar{x}$, this inequality contradicts Fact 3. Hence, $t_2^f(R_1'; \mathbf{0}) \neq 0$. □

Since R_1' is bounded by Claim 1, there are $\overline{s}, \underline{s} \in \mathbb{R}_{++}$ such that for each $(x, y) \in \mathcal{X}$ and each $t \in \mathbb{R}$,

$$\underline{s} < V^{R_1'}(x, (y, t)) - t < \overline{s}.$$

Let $\mathbf{t} \in \mathbb{R}^{|X|}$ be such that for each $x \in X$, $t_x = t_2^f(R_1', x)$. By Remark 9, $\mathbf{t}$ is well-defined, and by Lemma 9, is object monotonic. There are two cases.

Case 1. $t_{\mathbf{0}} < 0$. Let $R_1'' \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$, $v_1''(x) > \overline{s}$. Let $\mathbf{s} \in \mathbb{R}^{|X|}$ be such that for each $x \in X$, $s_x = t_2^f(R_1''; x)$. By Remark 9, $\mathbf{s}$ is well-defined, and by Lemma 9, is object monotonic.

By Step 1 and $R_1'' \in \mathcal{R}^Q$, $s_{\mathbf{0}} = 0 > t_{\mathbf{0}}$. Thus, $\mathbf{t}$ and $\mathbf{s}$ satisfy the condition of Lemma 4 for $\mathbf{0}$. Therefore, by Lemma 4, there is $R_2'' \in \mathcal{R}^P$ such that $R_2'' \in \mathcal{R}^{MT}_{\mathbf{t},\mathbf{0}} \cap \mathcal{R}^{MT}_{\mathbf{s},\bar{x}}$. By Lemma 10, $x_2^f(R_1', R_2'') = \mathbf{0}$ and $x_2^f(R_1'', R_2'') = \bar{x}$. Therefore,

$$x_1^f(R_1', R_2'') = \bar{x} \text{ and } x_1^f(R_1'', R_2'') = \mathbf{0}.$$

However, $V^{R_1'}(\bar{x}, f_1(R_1'', R_2'')) - t_1^f(R_1'', R_2'') < \overline{s} < v_1''(\bar{x})$. This contradicts Fact 3.

Case 2. $t_{\mathbf{0}} > 0$. Let $R_1'' \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$, $v_1''(x) < \underline{s}$.

Let $\mathbf{s} \in \mathbb{R}^{|X|}$ be such that for each $x \in X$, $s_x = t_2^f(R_1''; x)$. By Remark 9, $\mathbf{s}$ is well-defined, and by Lemma 9, is object monotonic.

By Step 1 and $R_1'' \in \mathcal{R}^Q$, $s_{\mathbf{0}} = 0 < t_{\mathbf{0}}$. Thus, the pair t, s satisfy the condition of Lemma 3 for $\mathbf{0}$. Therefore, by Lemma 3, there is $R_2'' \in \mathcal{R}^P$ such that $R_2'' \in \mathcal{R}^{MT}_{\mathbf{t},\bar{x}} \cap \mathcal{R}^{MT}_{\mathbf{s},\mathbf{0}}$. By Lemma 10, $x_2^f(R_1', R_2'') = \bar{x}$ and $x_2^f(R_1'', R_2'') = \mathbf{0}$. Therefore,

$$x_1^f(R_1', R_2'') = \mathbf{0} \text{ and } x_1^f(R_1'', R_2'') = \bar{x}.$$

However, $V^{R_1'}(\bar{x}, f_1(R_1', R_2'')) - t_1^f(R_1', R_2'') > \underline{s} > v_1''(\bar{x})$. This contradicts Fact 3. ■

Step 5. *Completing the proof.*

By using Step 4 and following the proof of Step 2, we can show that for each $R \in \mathcal{R}^2$,

$$V^{R_1}(x_1^f(R), (\mathbf{0}, 0)) + V^{R_2}(x_2^f(R), (\mathbf{0}, 0)) = \max_{(x_1, x_2) \in A} V^{R_1}(x_1, (\mathbf{0}, 0)) + V^{R_2}(x_2, (\mathbf{0}, 0)).$$

Hence, by Step 4, we obtain the desired result. ■

C.2 Proof of Theorem 1 (ii)

This is immediate from Theorem 1 (i) and Proposition 2. ■

D Proof of Theorem 2

As we mentioned in the beginning of Appendix, we give only the proof for the partially quasi-linear domain. Let $\mathcal{R} \equiv \mathcal{R}^P$. Suppose by contradiction that there is a rule f on $\mathcal{R}^n$ that satisfies *efficiency* and *strategy-proofness*.

We first define three vectors we use in the proof. Let $(\epsilon_x)_{x \in X}, (\epsilon_x')_{x \in X}, (\epsilon_x'')_{x \in X} \in \mathbb{R}_+^{|X|}$ be object monotonic vectors that are sufficiently close to $\mathbf{0}$, and satisfy the following: For each $x \in X$ and each $(y, y') \in \mathcal{X}$,[25]

$$\epsilon_x' < \epsilon_y - \epsilon_{y'} \text{ and } \epsilon_x'' < \epsilon_y' - \epsilon_{y'}'. \tag{5}$$

We do the proof in seven steps.

[25]For example, let $\epsilon_{\mathbf{0}} = \epsilon_{\mathbf{0}}' = \epsilon_{\mathbf{0}}'' = 0$, and for each $x \in X \setminus \{\mathbf{0}\}$, let $\epsilon_x = \frac{m(x)}{m(\bar{x})}$, $\epsilon_x' = \frac{m(x)}{(m(\bar{x}))^2}$, and $\epsilon_x'' = \frac{m(x)}{n(m(\bar{x}))^3}$. The proof works with these ϵ, ϵ', and ϵ''.

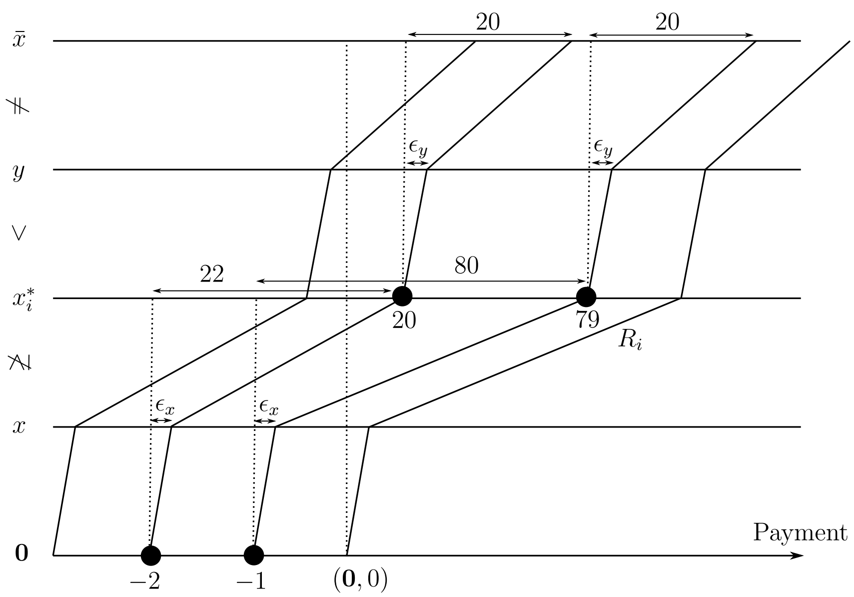

Figure 11: An illustration of R_i for $i \in \{1, 2\}$.

Step 1. *Constructing a preference profile.*

Let $x_1^* \in X \setminus \{\mathbf{0}, \bar{x}\}$ and $x_2^* \equiv \bar{x} - x_1^*$. For each $i \in \{1, 2\}$, we define a preference relation R_i as follows. For each $t \in \mathbb{R}$ and each $x \in X \setminus \{\mathbf{0}\}$, let

$$V^{R_i}(x, (\mathbf{0}, t)) = \begin{cases} 58\alpha + 42 + t & \text{if } x = \bar{x}, \\ 58\alpha + 22 + \epsilon_x + t & \text{if } x_i^* < x \neq \bar{x}, \\ 58\alpha + 22 + t & \text{if } x = x_i^*, \\ \epsilon_x + t & \text{otherwise}, \end{cases}$$

where α is defined as $\alpha \equiv \text{med}\{0, t + 2, 1\}$.[26] Note that $\alpha \in [0, 1]$.[27]

Figure 11 is an illustration of R_i for $i \in \{1, 2\}$. Note that for each $t \in \mathbb{R}$ with $-2 < t < -1$, $V^{R_i}(\cdot, (\mathbf{0}, t))$ is a convex combination of $V^{R_i}(\cdot, (\mathbf{0}, -2))$ and $V^{R_i}(\cdot, (\mathbf{0}, -1))$. By the constructions of R_1 and R_2, R_1 and R_2 are bounded and object monotonic. Further, as we show in the following claim, R_1 and R_2 are partially quasi-linear.

[26] We denote by $\text{med}\{0, t + 2, 1\}$ the median of three numbers, 0, $t + 2$, and 1.

[27] If $t \leq -2$, then $\alpha = 0$, if $t \geq -1$, then $\alpha = 1$, and if $-2 < t < -1$, then $\alpha = t + 2 \in [0, 1]$.

Claim 1. $R_1, R_2 \in \mathcal{R}^P$.

Proof. We prove only that $R_1 \in \mathcal{R}^P$, because we can show $R_2 \in \mathcal{R}^P$ in the same manner. Let $(x,t) \in X(R_1)$. By Remark 4, it is enough to show $V^{R_1}(\bar{x},(x,t)) - t = V^{R_1}(\bar{x},(\mathbf{0},0)) - V^{R_1}(x,(\bar{x},0))$.

Without loss of generality, assume $x \neq \mathbf{0}$. If $x \geq x_1^*$, then it is clear from the definition of R_1 that $V^{R_1}(\bar{x},(x,t)) - t = V^{R_1}(\bar{x},(\mathbf{0},0)) - V^{R_1}(x,(\bar{x},0))$.

Suppose $x \ngeq x_1^*$. Let $s \equiv V^{R_1}(\mathbf{0},(x,t))$. If $s < -1$, then

$$t = V^{R_1}(x,(\mathbf{0},s)) < V^{R_1}(x,(\mathbf{0},-1)) = \epsilon_x - 1 < 0,$$

where the last inequality follows since ϵ_x is sufficiently close to 0. Thus, by $(x,t) \in X(R_1)$, $s \geq -1$. This implies $V^{R_1}(\bar{x},(\mathbf{0},s)) = 100 + s$. Therefore,

$$V^{R_1}(\bar{x},(x,t)) - t = V^{R_1}(\bar{x},(\mathbf{0},s)) - V^{R_1}(x,(\mathbf{0},s)) = 100 - \epsilon_x = V^{R_1}(\bar{x},(\mathbf{0},0)) - V^{R_1}(x,(\bar{x},0)).$$

Hence, $R_1 \in \mathcal{R}^P$. □

Let $R_3 \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$v_3(x) = \begin{cases} \epsilon'_x & \text{if } x \neq \bar{x}, \\ 60 & \text{if } x = \bar{x}. \end{cases}$$

For each $i \in N \setminus \{1,2,3\}$, let $R_i \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$v_i(x) = \epsilon''_x.$$

Denote $R \equiv (R_1, \ldots, R_n)$.

We conclude this step by stating several properties of R. We do not prove some of them because they are trivial. The first property gives an upper bound for the sum of the valuations that the agents other than agents 1 and 2 can achieve under the assumption that no agent receives $\bar{x}$. This fact will be used in later steps to decide the payments of agents 1 and 2 at some preference profiles.

Property 1. *Let $(x_i)_{i\in N} \in (X \setminus \{\bar{x}\})^n$ be such that $\sum_{i\in N} x_i \leq \bar{x}$. Then, $\sum_{i \in N\setminus\{1,2\}} v_i(x_i) < 1$.*

Proof. By $x_3 \neq \bar{x}$, $v_3(x_3) = \epsilon'_{x_3}$. For each $i \in N \setminus \{1,2,3\}$, $v_i(x_i) = \epsilon''_{x_i}$. Since $(\epsilon'_x)_{x \in X}$ and $(\epsilon''_x)_{x \in X}$ are sufficiently close to $\mathbf{0}$, $\sum_{i \in N \setminus \{1,2\}} v_i(x_i) < 1$. □

By (5), for each $i \in N \setminus \{1,2,3\}$, R_i is negligible with respect to R_3.

Property 2. *For each $i \in N \setminus \{1,2,3\}$, $R_i \in \mathcal{R}^Q(R_3)$.*

By Property 2 and Lemma 7, we obtain the following.

Property 3. *For each pair $R'_1, R'_2 \in \mathcal{R}$ and each $i \in N \setminus \{1,2,3\}$, $x_i^f(R'_1, R'_2, R_{-1,2}) = \mathbf{0}$.*

The following property is immediate from the definitions of $(\epsilon'_x)_{x \in X}$ and R_3.

Property 4. *Let $v \in [0, 50]$ and $R'_i \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{0, \bar{x}\}$, $v'_i(x) = v + \epsilon_x$, and $v'_i(\bar{x}) > 60$. Then $R_3 \in \mathcal{R}^Q(R'_i)$.*

The last property states the packages that agent 3 can obtain.

Property 5. *Let $i \in \{1,2\}$ and $R'_i \in \mathcal{R}$. Then, $x_3^f(R'_i, R_{-i}) = \mathbf{0}$ or $\bar{x}$.*

Proof. Without loss of generality, assume $i = 1$ and denote $R' \equiv (R'_1, R_{-1})$. Suppose by contradiction that $x_3^f(R') \in X \setminus \{\mathbf{0}, \bar{x}\}$. Let $x \equiv x_2^f(R') + x_3^f(R')$. By $x_3^f(R') \neq \mathbf{0}$, $x > x_2^f(R')$. Thus, $V^{R_2}(x, f_2(R')) - t_2^f(R') \geq \epsilon_x - \epsilon_{x_2^f(R')}$.

Therefore,

$$\begin{aligned} V^{R_2}(x, f_2(R')) + V^{R_3}(\mathbf{0}, f_3(R')) &\geq \epsilon_x - \epsilon_{x_2^f(R')} + t_2^f(R') + t_3^f(R') - v_3(x_3^f(R')) \\ &= \epsilon_x - \epsilon_{x_2^f(R')} + t_2^f(R') + t_3^f(R') - \epsilon'_{x_3^f(R')} \\ &> t_2^f(R') + t_3^f(R'), \end{aligned}$$

where the equality follows from the definition of R_3 and the last inequality follows from (5). This inequality contradicts Lemma 6. □

Step 2. *Either $x_3^f(R) = \bar{x}$, or $x_1^f(R) = x_1^*$ and $x_2^f(R) = x_2^*$.*

Proof. Suppose by contradiction that $x_3^f(R) \neq \bar{x}$, and $x_1^f(R) \neq x_1^*$ or $x_2^f(R) \neq x_2^*$. By Property 5, $x_3^f(R) = \mathbf{0}$. By Property 3, for each $i \in N \setminus \{1,2,3\}$, $x_i^f(R) = 0$. Thus, $x_1^f(R) = \bar{x} - x_2^f(R)$. This implies $x_1^f(R) \neq x_1^*$ and $x_2^f(R) \neq x_2^*$.

Denote $s_1 \equiv V^{R_1}(\mathbf{0}, f_1(R))$ and $s_2 \equiv V^{R_2}(\mathbf{0}, f_2(R))$. There are three cases.

Case 1. *$x_1^f(R) = \bar{x}$ or $x_2^f(R) = \bar{x}$.* Without loss of generality, we assume $x_1^f(R) = \bar{x}$. By the definition of R_1, $V^{R_1}(x_1^*, f_1(R)) = t_1^f(R) - 20$. By $x_2^f(R) = \mathbf{0}$ and the definition of R_2, $V^{R_2}(x_2^*, f_2(R)) \geq t_2^f(R) + 22$. Thus, $V^{R_1}(x_1^*, f_1(R)) + V^{R_2}(x_2^*, f_2(R)) > t_1^f(R) + t_2^f(R)$, contradicting Lemma 6.

Case 2. *$x_1^f(R) \not> x_1^*$ and $x_2^f(R) \not> x_2^*$.* For each $i \in \{1, 2\}$,

$$V^{R_i}(x_i^*, f_i(R)) - t_i^f(R) = V^{R_i}(x_i^*, (\mathbf{0}, s_i)) - V^{R_i}(x_i^f(R), (\mathbf{0}, s_i)) \geq 22 + s_i - (\epsilon_{x_i^f(R)} + s_i) > 0,$$

where the last inequality follows since $\epsilon_{x_i^f(R)}$ is sufficiently close to 0. Thus, $V^{R_1}(x_1^*, f_1(R)) + V^{R_2}(x_2^*, f_2(R)) > t_1^f(R) + t_2^f(R)$, which contradicts Lemma 6.

Case 3. *$x_1^f(R) > x_1^*$ or $x_2^f(R) > x_2^*$.* Without loss of generality, assume $x_1^f(R) > x_1^*$. By Case 1, We can also assume $x_1^f(R) \neq \bar{x}$ without loss of generality. By the definition of R_1,

$$V^{R_1}(x_1^*, f_1(R)) - t_1^f(R) = V^{R_1}(x_1^*, (\mathbf{0}, s_1)) - V^{R_1}(x_1^f(R), (\mathbf{0}, s_1)) = -\epsilon_{x_1^f(R)}.$$

By $x_1^f(R) > x_1^*$, $x_2^f(R) \not> x_2^*$. Thus, as we have shown in Case 2, $V^{R_2}(x_2^*, f_2(R)) - t_2^f(R) \geq 22 - \epsilon_{x_2^f(R)}$. Since $\epsilon_{x_1^f(R)}$ and $\epsilon_{x_2^f(R)}$ are sufficiently close to 0, $V^{R_1}(x_1^*, f_1(R)) + V^{R_2}(x_2^*, f_2(R)) > t_1^f(R) + t_2^f(R)$, which contradicts Lemma 6. ■

Step 3. *$x_1^f(R) = x_1^*$ and $x_2^f(R) = x_2^*$.*

Proof. Suppose by contradiction that $x_1^f(R) \neq x_1^*$ or $x_2^f(R) \neq x_2^*$. By Step 2, $x_3^f(R') = \bar{x}$, and thus, $x_1^f(R) = x_2^f(R) = \mathbf{0}$. First we show the following claim.

Claim 1. *$t_i^f(R) < -1$ for some $i \in \{1, 2\}$.*

Proof. (Figure 12.) Suppose by contradiction that $t_1^f(R) \geq -1$ and $t_2^f(R) \geq -1$. By the definitions of R_1 and R_2, $V^{R_1}(x_1^*, f_1(R)) - t_1^f(R) = V^{R_2}(x_2^*, f_2(R)) - t_2^f(R) = 80$. By the definition of R_3, $V^{R_3}(\mathbf{0}, f_3(R)) = t_3^f(R) - v_3(\bar{x}) = t_3^f(R) - 60$. Thus,

$$V^{R_1}(x_1^*, f_1(R)) + V^{R_2}(x_2^*, f_2(R)) + V^{R_3}(\mathbf{0}, f_3(R)) > t_1^f(R) + t_2^f(R) + t_3^f(R),$$

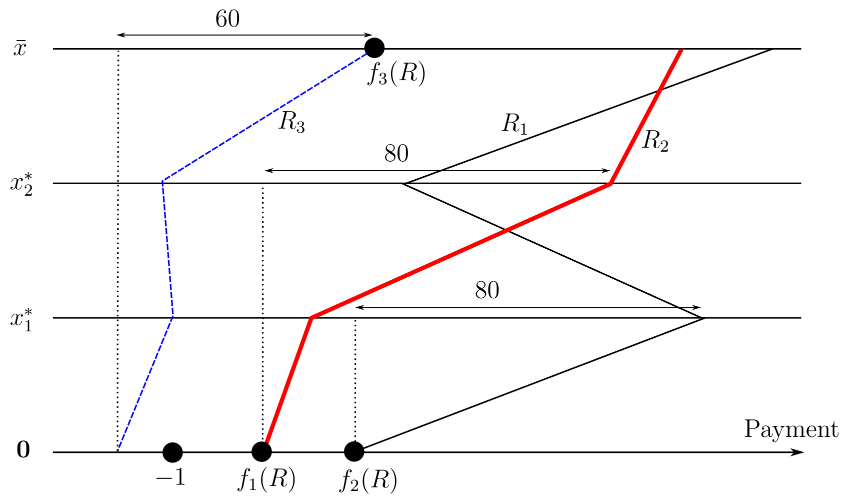

Figure 12: An illustration of R_1, R_2, and R_3.

which contradicts Lemma 6. $\square$

Without loss of generality, we assume $t_1^f(R) < -1$. Let $R_2' \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$v_2'(x) = \begin{cases} \epsilon_x & \text{if } x \neq \bar{x}, \\ 100 & \text{if } x = \bar{x}. \end{cases}$$

Let $\mathbf{t}, \mathbf{s} \in \mathbb{R}^{|X|}$ be such that for each $x \in X$, $t_x = t_1^f(R_{-1}; x)$ and $s_x = t_1^f(R_2', R_{-1,2}; x)$. By the boundedness of $R_2', R_2, R_3, \ldots, R_n$ and Lemma 8, $\mathbf{t}$ and $\mathbf{s}$ are well-defined and by Lemma 9, are object monotonic.

By Property 4, $R_3 \in \mathcal{R}^Q(R_2')$. Further, by Property 2 and Lemma 1, for each $i \in N \setminus \{1,2,3\}$, $R_i \in \mathcal{R}^Q(R_3) \subseteq \mathcal{R}^Q(R_2')$. Thus, by Lemma 11 and Property 1, $-1 < s_{\mathbf{0}} \leq 0$. By $t_{\mathbf{0}} = t_1^f(R)$ and Claim 1,

$$t_{\mathbf{0}} < s_{\mathbf{0}} \leq 0.$$

Thus, $\mathbf{t}$ and $\mathbf{s}$ satisfy the condition of Lemma 4 for $\mathbf{0}$. Therefore, by Lemma 4, there is $R_1' \in \mathcal{R}^P$ such that $R_1' \in \mathcal{R}^{TM}_{\mathbf{t},\mathbf{0}} \cap \mathcal{R}^{TM}_{\mathbf{s},x_1^*}$. By Lemma 10,

$$x_1^f(R_1', R_{-1}) = \mathbf{0} \text{ and } x_1^f(R_1', R_2', R_{-1,2}) = x_1^*.$$

For each $i \in N \setminus \{1,2\}$, since $R_i \in \mathcal{R}^Q(R_2')$, Lemma 7 implies $x_i^f(R_1', R_2', R_{-1,2}) = \mathbf{0}$.

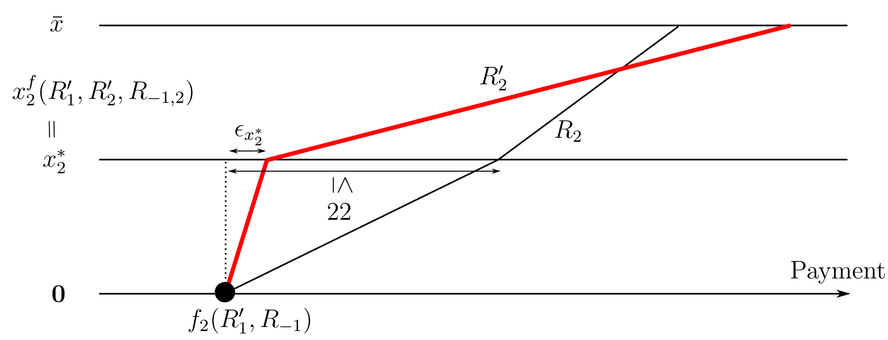

Figure 13: An illustration of R_2 and R_2' in Case 1.

Thus,

$$x_2^f(R_1', R_2', R_{-1,2}) = x_2^*.$$

By Property 3 and $x_1^f(R_1', R_{-1}) = \mathbf{0}$, $x_2^f(R_1', R_{-1}) = \bar{x} - x_3^f(R_1', R_{-1})$. Note that by Property 5 $x_3^f(R_1', R_{-1}) = \mathbf{0}$ or $\bar{x}$. Therefore,

$$x_2^f(R_1', R_{-1}) = \mathbf{0} \text{ or } \bar{x}.$$

Case 1. $x_2^f(R_1', R_{-1}) = \mathbf{0}$. (Figure 13.) Since $\epsilon_{x_2^*}$ is sufficiently close to 0,

$$V^{R_2}(x_2^*, f_2(R_1', R_{-1})) > t_2^f(R_1', R_{-1}) + \epsilon_{x_2^*} = t_2^f(R_1', R_{-1}) + v_2'(x_2^*) = V^{R_2'}(x_2^*, f_2(R_1', R_{-1})),$$

which contradicts Fact 3.

Case 2. $x_2^f(R_1', R_{-1}) = \bar{x}$. (Figure 14.) By $x_2^f(R_1', R_2', R_{-1,2}) = x_2^*$ and the definitions of R_2 and R_2',

$$\begin{aligned} V^{R_2}(\bar{x}, f_2(R_1', R_2', R_{-1,2})) &= t_2^f(R_1', R_2', R_{-1,2}) + 20 \\ &< t_2^f(R_1', R_2', R_{-1,2}) + v_2'(\bar{x}) - v_2'(x_2^*) \\ &= V^{R_2'}(\bar{x}, f_2(R_1', R_2', R_{-1,2})), \end{aligned}$$

which contradicts Fact 3. ■

Step 4. $t_i^f(R) > 20$ *for some* $i \in \{1, 2\}$.

Proof. Suppose by contradiction that $t_1^f(R) \leq 20$ and $t_2^f(R) \leq 20$. By Step 3, $x_1^f(R) = x_1^*$ and $x_2^f(R) = x_2^*$. Note that for each $i \in \{1, 2\}$ and each $t \in \mathbb{R}$ with

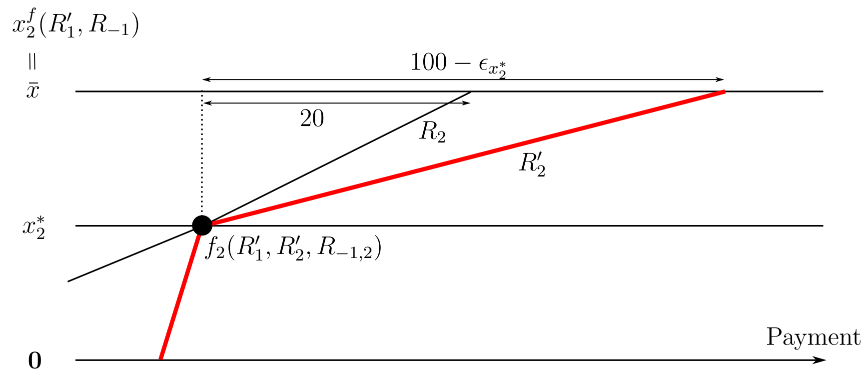

Figure 14: An illustration of R_2 and R_2' in Case 2.

$t > -2$, $V^{R_i}(x_i^*, (\mathbf{0}, t)) > 20$. (See Figure 11.) This implies that $V^{R_1}(\mathbf{0}, f_1(R)) \leq -2$ and $V^{R_2}(\mathbf{0}, f_2(R)) \leq -2$. Thus, by the definitions of R_1 and R_2,

$$t_1^f(R) - V^{R_1}(\mathbf{0}, f_1(R)) = t_2^f(R) - V^{R_2}(\mathbf{0}, f_2(R)) = 22.$$

Therefore,

$$\begin{aligned} V^{R_1}(\mathbf{0}, f_1(R)) + V^{R_2}(\mathbf{0}, f_2(R)) + V^{R_3}(\bar{x}, f_3(R)) &= t_1^f(R) - 22 + t_2^f(R) - 22 + t_3^f(R) + 60 \\ &> t_1^f(R) + t_2^f(R) + t_3^f(R), \end{aligned}$$

which contradicts Lemma 6. ■

Without loss of generality, we assume $t_1^f(R) > 20$. Let $(\delta_x)_{x \in X} \in \mathbb{R}_+^{|X|}$ be an object monotonic and additive vector that is sufficiently close to $\mathbf{0}$ and satisfies the following: For each $x \in X \setminus \{\mathbf{0}\}$, $\delta_x < \epsilon'_x$.[28][29] Let $R_1' \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$v_1'(x) = \begin{cases} 20 + \delta_x & \text{if } x_1^* \leq x \not\leq \bar{x}, \\ 40 + \delta_{x_1^*} & \text{if } x = \bar{x}, \\ \delta_x & \text{otherwise.} \end{cases}$$

Figure 15 is an illustration of R_1'. As is shown in the figure, since $\delta_{x_1^*}$ is sufficiently close to 0 and $t_1^f(R) > 20$, we have $v_1'(x_1^*) < t_1^f(R)$.

[28] A vector $\mathbf{t} \in \mathbb{R}^{|X|}$ is *additive* if for each pair $x, y \in X$, $t_{x+y} = t_x + t_y$.

[29] For example, take any $\delta' \in \mathbb{R}_{++}$ such that $\delta' < \min\{1, t_1^f(R) - 20\}$, and let $(\delta_x)_{x \in X} \in \mathbb{R}_{++}^{|X|}$ be such that $\delta_{\mathbf{0}} = 0$ and for each $x \in X \setminus \{\mathbf{0}\}$, $\delta_x = \frac{\delta' m(x)}{(m(\bar{x}))^3}$. Then our proof works with this $(\delta_x)_{x \in X}$, and $(\epsilon_x)_{x \in X}$, $(\epsilon'_x)_{x \in X}$, and $(\epsilon''_x)_{x \in X}$ defined in footnote 25.

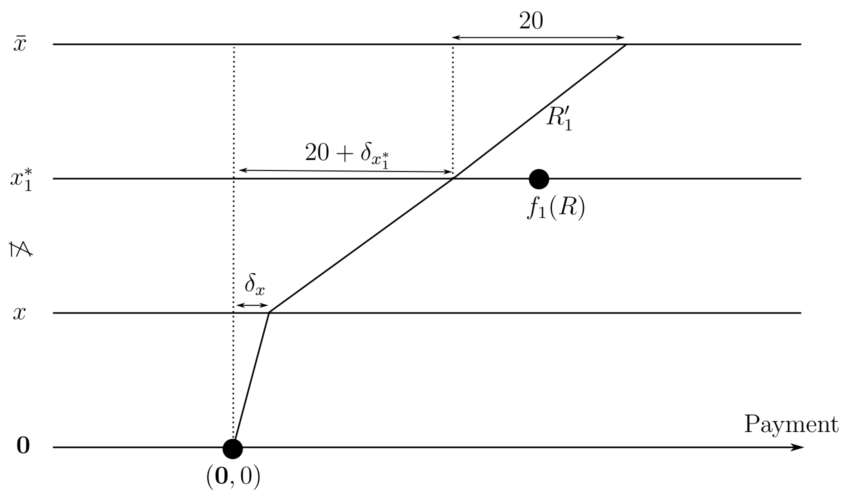

Figure 15: An illustration of R'_1.

Step 5. *Either* $x_3^f(R'_1, R_{-1}) = \bar{x}$, *or* $x_1^f(R'_1, R_{-1}) = x_1^*$ *and* $x_2^f(R'_1, R_{-1}) = x_2^*$.

Proof. Suppose by contradiction that $x_3^f(R'_1, R_{-1}) \neq \bar{x}$, and $x_1^f(R'_1, R_{-1}) \neq x_1^*$ or $x_2^f(R'_1, R_{-1}) \neq x_2^*$. By Property 5, $x_3^f(R'_1, R_{-1}) = \mathbf{0}$. Thus, by Property 3, $x_1^f(R'_1, R_{-1}) = \bar{x} - x_2^f(R'_1, R_{-1})$. This implies $x_1^f(R'_1, R_{-1}) \neq x_1^*$ and $x_2^f(R'_1, R_{-1}) \neq x_2^*$. For simplicity, denote $R' \equiv (R'_1, R_{-1})$. There are three cases.

Case 1. $x_1^f(R') = \mathbf{0}$. (Figure 16.) By $x_1^f(R') = \mathbf{0}$ and the definition of R'_1,

$$V^{R'_1}(x_1^*, f_1(R')) = t_1^f(R') + v'_1(x_1^*) > t_1^f(R') + 20.$$

By $x_1^f(R') = \bar{x} - x_2^f(R')$, $x_2^f(R') = \bar{x}$. Thus, by the definition of R_2, $V^{R_2}(x_2^*, f_2(R')) = t_2^f(R') - 20$. Thus, $V^{R'_1}(x_1^*, f_1(R')) + V^{R_2}(x_2^*, f_2(R')) > t_1^f(R') + t_2^f(R')$. This contradicts Lemma 6.

Case 2. $x_1^* < x_1^f(R')$. (Figure 17.) By the definition of R'_1,

$$\begin{aligned} V^{R'_1}(x_1^*, f_1(R')) &= t_1^f(R') + v'_1(x_1^*) - v'_1(x_1^f(R')) \\ &= t_1^f(R') + 20 + \delta_{x_1^*} - 20 - \delta_{x_1^f(R')} \\ &= t_1^f(R') - \delta_{x_1^f(R') - x_1^*}, \end{aligned}$$

where the last equality follows since $(\delta_x)_{x \in X}$ is additive. By $x_3^f(R'_1, R_{-1}) = \mathbf{0}$, $x_1^f(R')-$

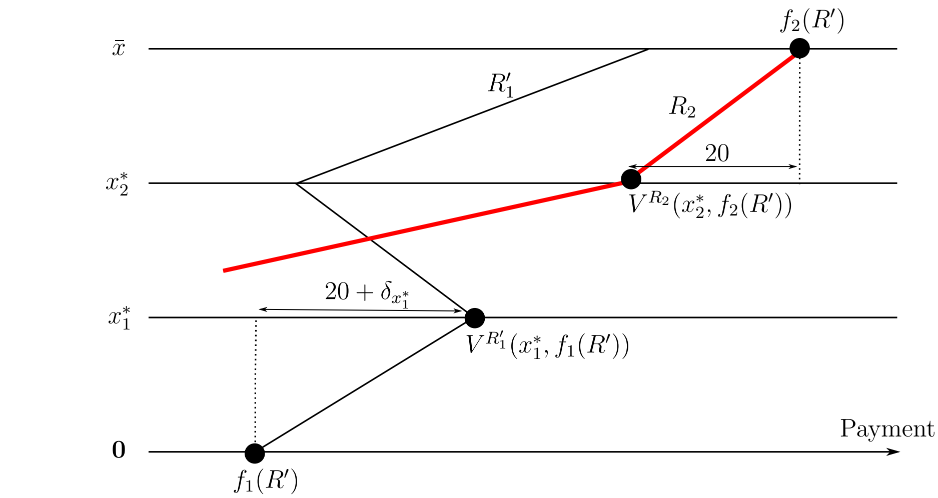

Figure 16: An illustration of R_1' and R_2 in Case 1.

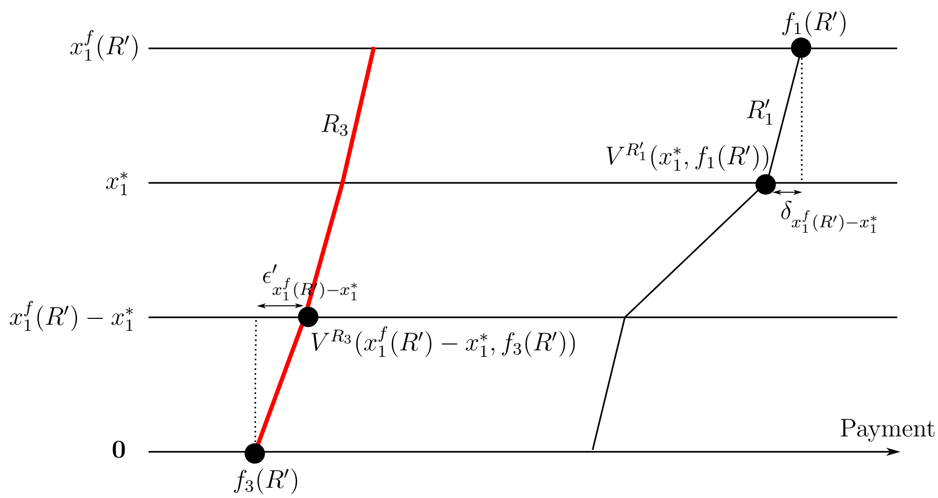

Figure 17: An illustration of R_1' and R_3 in Case 2.

$x_1^* > \mathbf{0}$, and the definition of R_3,

$$V^{R_3}(x_1^f(R') - x_1^*, f_3(R')) = t_3^f(R') + v_3(x_1^f(R') - x_1^*) = t_3^f(R') + \epsilon'_{x_1^f(R')-x_1^*}.$$

By $\delta_{x_1^f(R')-x_1^*} < \epsilon'_{x_1^f(R')-x_1^*}$, $V^{R_1'}(x_1^*, f_1(R')) + V^{R_3}(x_1^f(R') - x_1^*, f_3(R')) > t_1^f(R') + t_3^f(R')$. This contradicts Lemma 6.

Case 3. $x_1^* \not\leq x_1^f(R') \neq \mathbf{0}$. (Figure 18.) By the definition of R_1',

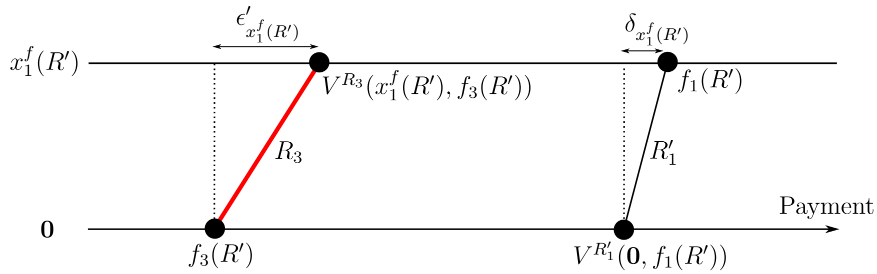

Figure 18: An illustration of R'_1 and R_3 in Case 3.

$$V^{R'_1}(\mathbf{0}, f_1(R')) = t_1^f(R') - v'_1(x_1^f(R')) = t_1^f(R') - \delta_{x_1^f(R')}.$$

By the definition of R_3,

$$V^{R_3}(x_1^f(R'), f_3(R')) = t_3^f(R') + v_3(x_1^f(R')) = t_3^f(R') + \epsilon'_{x_1^f(R')}.$$

By $\delta_{x_1^f(R')} < \epsilon'_{x_1^f(R')}$, $V^{R'_1}(\mathbf{0}, f_1(R')) + V^{R_3}(x_1^f(R'), f_3(R')) > t_1^f(R') + t_3^f(R')$. This contradicts Lemma 6. ■

Step 6. *$x_1^f(R'_1, R_{-1}) = x_1^*$ and $x_2^f(R'_1, R_{-1}) = x_2^*$.*

Proof. Suppose by contradiction that $x_1^f(R'_1, R_{-1}) \neq x_1^*$ or $x_2^f(R'_1, R_{-1}) \neq x_2^*$. By Step 5, $x_3^f(R'_1, R_{-1}) = \bar{x}$, and hence, $x_1^f(R'_1, R_{-1}) = x_2^f(R'_1, R_{-1}) = \mathbf{0}$.

Claim 1. $t_2^f(R'_1, R_{-1}) < -1$.

Proof. Suppose by contradiction that $t_2^f(R'_1, R_{-1}) \geq -1$. Then, by $x_2^f(R'_1, R_{-1}) = \mathbf{0}$ and the definition of R_2, $V^{R_2}(x_2^*, f_2(R'_1, R_{-1})) - t_2^f(R'_1, R_{-1}) = 80$. (See Figure 11.) By $x_3^f(R'_1, R_{-1}) = \bar{x}$ and the definition of R_3, $V^{R_3}(\mathbf{0}, f_3(R'_1, R_{-1})) = t_3^f(R'_1, R_{-1}) - v_3(\bar{x}) = t_3^f(R'_1, R_{-1}) - 60$. Thus, $V^{R_2}(x_2^*, f_2(R'_1, R_{-1})) + V^{R_3}(\mathbf{0}, f_3(R'_1, R_{-1})) > t_2^f(R'_1, R_{-1}) + t_3^f(R'_1, R_{-1})$. This contradicts Lemma 6. □

Let $R''_1 \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$v''_1(x) = \begin{cases} \epsilon_x & \text{if } x \neq \bar{x}, \\ 100 & \text{if } x = \bar{x}. \end{cases}$$

Let $\mathbf{t}, \mathbf{s} \in \mathbb{R}^{|X|}$ be such that for each $x \in X$, $t_x = t_2^f(R_1', R_{-1,2}; x)$ and $s_x = t_2^f(R_1'', R_{-1,2}; x)$. By the boundedness of $R_1', R_1'', R_3, \ldots, R_n$ and Lemma 8, $\mathbf{t}$ and $\mathbf{s}$ are well-defined, and by Lemma 9, are object monotonic.

By Property 4, $R_3 \in \mathcal{R}^Q(R_1'')$. Further, by Property 2 and Lemma 1, for each $i \in N \setminus \{1, 2, 3\}$, $R_i \in \mathcal{R}^Q(R_3) \subseteq \mathcal{R}^Q(R_1'')$. Thus, by Lemma 11 and Property 1, $-1 < s_{\mathbf{0}} \leq 0$. Since $t_{\mathbf{0}} = t_2^f(R_1', R_{-1}) \leq -1$ by Claim 1,

$$t_{\mathbf{0}} < s_{\mathbf{0}} \leq 0.$$

Thus, $\mathbf{t}$ and $\mathbf{s}$ satisfy the condition of Lemma 4 for $\mathbf{0}$. Therefore, by Lemma 4, there is $R_2' \in \mathcal{R}^P$ such that $R_2' \in \mathcal{R}^{MT}_{\mathbf{t},\mathbf{0}} \cap \mathcal{R}^{MT}_{\mathbf{s},x_2^*}$. For simplicity, denote $R' \equiv (R_1', R_2', R_{-1,2})$ and $R'' = (R_1'', R_2', R_{-1,2})$. By Lemma 10,

$$x_2^f(R') = \mathbf{0} \text{ and } x_2^f(R'') = x_2^*.$$

For each $i \in N \setminus \{1, 2\}$, since $R_i \in \mathcal{R}^Q(R_1'')$, Lemma 7 implies $x_i^f(R'') = \mathbf{0}$. Thus,

$$x_1^f(R'') = x_1^*.$$

There are three cases.

Case 1. $x_1^f(R') \in X \setminus \{\mathbf{0}, \bar{x}\}$. By $x_1^f(R') \notin \{\mathbf{0}, \bar{x}\}$ and the definitions of R_1' ,

$$V^{R_1'}(\bar{x}, f_1(R')) = t_1^f(R') + v_1'(\bar{x}) - v_1'(x_1^f(R')) \geq t_1^f(R') + 20 + \delta_{x_1^*} - \delta_{x_1^f(R')}.$$

By Property 3 and $x_2^f(R') = \mathbf{0}$, $x_1^f(R') = \bar{x} - x_3^f(R')$. By $x_1^f(R') \notin \{\mathbf{0}, \bar{x}\}$, $x_3^f(R') \notin \{\mathbf{0}, \bar{x}\}$. Thus by the definition of R_3,

$$V^{R_3}(\mathbf{0}, f_3(R')) = t_3^f(R') - v_3(x_3^f(R')) = t_3^f(R') - \epsilon'_{x_3}.$$

Since $(\delta_x)_{x \in X}$ and $(\epsilon'_x)_{x \in X}$ are sufficiently close to $\mathbf{0}$, $V^{R_1'}(\bar{x}, f_1(R')) + V^{R_3}(\mathbf{0}, f_3(R')) > t_1^f(R') + t_3^f(R')$. This contradicts Lemma 6.

Case 2. $x_1^f(R') = \mathbf{0}$. By the definition of R_1',

$$V^{R_1'}(x_1^*, f_1(R')) = t_1^f(R') + v_1'(x_1^*) = t_1^f(R') + 20 + \delta_{x_1^*}$$

By the definition of R_1'',

$$V^{R_1''}(x_1^*, f_1(R')) = t_1^f(R') + v_1''(x_1^*) = t_1^f(R') + \epsilon_{x_1^*}.$$

Since $\epsilon_{x_1^*}$ is sufficiently close to 0, $V^{R'_1}(x_1^*, f_1(R')) > V^{R''_1}(x_1^*, f_1(R'))$. By $x_1^f(R'') = x_1^*$, this inequality contradicts Fact 3.

Case 3. $x_1^f(R') = \bar{x}$. By $x_1^f(R'') = x_1^*$ and the definition of R'_1,

$$V^{R'_1}(\bar{x}, f_1(R'')) = t_1^f(R'') + v'_1(\bar{x}) - v'_1(x_1^*) = t_1^f(R'') + 20.$$

By $x_1^f(R'') = x_1^*$ and the definition of R''_1,

$$V^{R''_1}(\bar{x}, f_1(R'')) = t_1^f(R'') + v''_1(\bar{x}) - v''_1(x_1^*) > t_1^f(R'') + 20.$$

Thus, $V^{R'_1}(\bar{x}, f_1(R'')) < V^{R''_1}(\bar{x}, f_1(R''))$, contradicting Fact 3. ■

Step 7. *Completing the proof.*

By Step 6, $x_1^f(R'_1, R_{-1}) = x_1^*$ and $x_2^f(R'_1, R_{-1}) = x_2^*$. By Step 3, $x_1^f(R) = x_1^*$. Thus, by *strategy-proofness*, $f_1(R'_1, R_{-1}) = f_1(R)$. Let $R'_2 \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$v'_2(x) = \begin{cases} 50 + \epsilon_x & \text{if } x \neq \bar{x}, \\ 65 & \text{if } x = \bar{x}. \end{cases}$$

Let $\mathbf{t}, \mathbf{s} \in \mathbb{R}^{|X|}$ be such that for each $x \in X$, $t_x = t_1^f(R_{-1}; x)$ and $s_x = t_1^f(R'_2, R_{-1,2}; x)$. By the boundedness of $R'_2, R_2, \ldots, R_n$ and Lemma 8, $\mathbf{t}$ and $\mathbf{s}$ are well-defined, and by Lemma 9, are object monotonic.

By Property 4, $R_3 \in \mathcal{R}^Q(R'_2)$. Further, by Property 2 and Lemma 1, for each $i \in N \setminus \{1, 2, 3\}$, $R_i \in \mathcal{R}^Q(R_3) \subseteq \mathcal{R}^Q(R'_2)$. Thus, by Lemma 11, $s_{\mathbf{0}} \leq 0$. By *strategy-proofness*, $f_1(R'_1, R_{-1})\ R'_1\ z_1^f(R_{-1}; \mathbf{0}) = (\mathbf{0}, t_{\mathbf{0}})$. Thus, by $f_1(R'_1, R_{-1}) = f_1(R) = (x_1^*, t_1^f(R))$,

$$t_{\mathbf{0}} \geq V^{R'_1}(\mathbf{0}, f_1(R'_1, R_{-1})) = t_1^f(R) - v'_1(x_1^*) = t_1^f(R) - (20 + \delta_{x_1^*}) > 0,$$

where the inequality follows since $t_1^f(R) > 20$ by Step 4 and $\delta_{x_1^*}$ is sufficiently close to 0. If $s_{\mathbf{0}} < 0$, then $\mathbf{t}$ and $\mathbf{s}$ satisfy the condition of Lemma 4 for $\mathbf{0}$. If $s_{\mathbf{0}} = 0$, then $\mathbf{t}$ and $\mathbf{s}$ satisfy the condition of Lemma 3 for $\mathbf{0}$. Thus, by Lemmas 3 and 4, there is $R''_1 \in \mathcal{R}^P$ such that $R''_1 \in \mathcal{R}^{MT}_{\mathbf{t}, x_1^*} \cap \mathcal{R}^{MT}_{\mathbf{s}, \mathbf{0}}$. By Lemma 10, $x_1^f(R''_1, R_{-1}) = x_1^*$ and $x_1^f(R''_1, R'_2, R_{-1,2}) = \mathbf{0}$.

For each $i \in N \setminus \{1,2\}$, by $R_i \in \mathcal{R}^Q(R_2')$ and Lemma 7, $x_i^f(R_1'', R_2', R_{-1,2}) = \mathbf{0}$. Thus,

$$x_2^f(R_1'', R_2', R_{-1,2})) = \bar{x}.$$

Further, by Properties 3 and 5,

$$x_2^f(R_1'', R_{-1}) = x_2^*.$$

By the definition of R_2, $V^{R_2}(\bar{x}, f_2(R_1'', R_{-1})) = t_2^f(R_1'', R_{-1}) + 20$. By the definition of R_2',

$$V^{R_2'}(\bar{x}, f_2(R_1'', R_{-1})) = t_2^f(R_1'', R_{-1}) + v_2'(\bar{x}) - v_2'(x_2^*) < t_2^f(R_1'', R_{-1}) + 20.$$

Thus, $V^{R_2}(\bar{x}, f_2(R_1'', R_{-1})) > V^{R_2'}(\bar{x}, f_2(R_1'', R_{-1}))$. This contradicts Fact 3. ■

E Proof of Proposition 2

As we mentioned in the beginning of Appendix, we give only the proof for the partially quasi-linear domain. Thus, $\mathcal{R}$ satisfies $\mathcal{R}^Q \subseteq \mathcal{R} \not\subseteq \mathcal{R}^P$. Suppose by contradiction that there is an *efficient* and *strategy-proof* rule f on $\mathcal{R}^n$ such that it is a (generalized) Vickrey rule on $(\mathcal{R}^Q)^n$. We do the proof in four steps.

Step 1. *Constructing a preference profile.*

By $\mathcal{R} \not\subseteq \mathcal{R}^P$, there is $R_1 \in \mathcal{R}$ such that $R_1 \notin \mathcal{R}^P$. By Remark 4, there is $(x^*, t) \in X(R_1)$ such that

$$V^{R_1}(\bar{x}, (x^*, t)) - t \neq V^{R_1}(\bar{x}, (\mathbf{0}, 0)) - V^{R_1}(x^*, (\mathbf{0}, 0)).$$

By the continuity of R_1, we can assume $0 < t < V^{R_1}(x^*, (\mathbf{0}, 0))$ without loss of generality. Also by the continuity of R_1, there is $s \in \mathbb{R}$ such that $0 < s < V^{R_1}(x^*, (\mathbf{0}, 0))$ and $V^{R_1}(\bar{x}, (x^*, t)) - t \neq V^{R_1}(\bar{x}, (x^*, s)) - s$. Without loss of generality, assume

$$V^{R_1}(\bar{x}, (x^*, t)) - t > V^{R_1}(\bar{x}, (x^*, s)) - s. \tag{6}$$

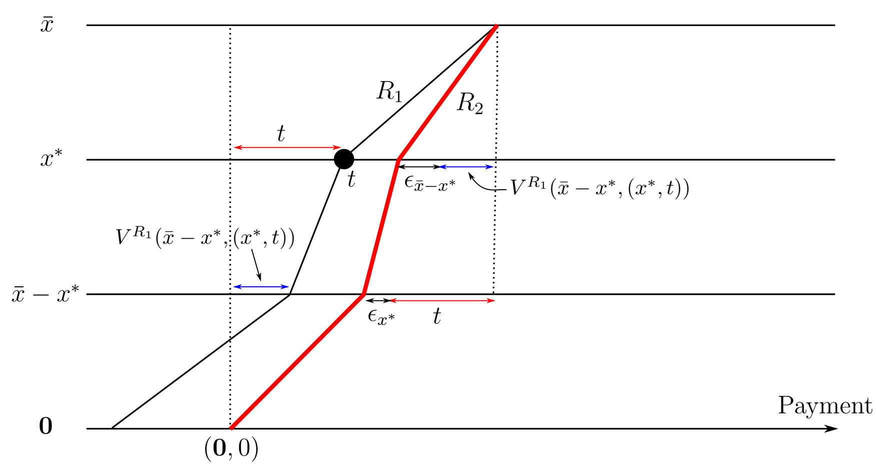

Figure 19: An illustration of R_2.

Let $(\epsilon_x)_{x\in X} \in \mathbb{R}_+^{|X|}$ be an object monotonic vector that is sufficiently close to $\mathbf{0}$ and satisfies $\epsilon_{\mathbf{0}} = 0$.[30]

Let $R_2 \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$v_2(x) = V^{R_1}(\bar{x},(x^*,t)) - \max\{V^{R_1}(\bar{x}-x,(x^*,t)),0\} - \epsilon_{\bar{x}-x}.$$

Let $R_2' \in \mathcal{R}^Q$ be such that for each $x \in X \setminus \{\mathbf{0}\}$,

$$v_2'(x) = \begin{cases} V^{R_1}(\bar{x},(x^*,s)) - \max\{V^{R_1}(\bar{x}-x,(x^*,s)),0\} + \epsilon_{\bar{x}} & \text{if } x \in \{\bar{x}-x^*,\bar{x}\}, \\ V^{R_1}(\bar{x},(x^*,s)) - \max\{V^{R_1}(\bar{x}-x,(x^*,s)),0\} - \epsilon_{\bar{x}-x} & \text{otherwise.} \end{cases}$$

Figures 19 and 20 illustrate R_2 and R_2', respectively. The following two claims show that R_2 and R_2' are object monotonic.

Claim 1. *R_2 is object monotonic.*

Proof. Let $(x,y) \in \mathcal{X}$. If $y = \mathbf{0}$, then since ϵ_x is sufficiently close to 0 and R_1 is object monotonic, $v_2(x) > 0$.

[30] Formally, the proof works if we take $(\epsilon_x)_{x\in X} \in \mathbb{R}_+^{|X|}$ that is object monotonic and satisfies the following: $\epsilon_{\mathbf{0}} = 0$, and for each $(x,y) \in \mathcal{X}$, $0 < \epsilon_x + \epsilon_y < \min\{\min_{(x',y')\in\mathcal{X}} V^{R_1}(x',(x^*,t)) - V^{R_1}(y',(x^*,t)), \min_{(x',y')\in\mathcal{X}} V^{R_1}(x',(x^*,s)) - V^{R_1}(y',(x^*,s)), V^{R_1}(\bar{x},(x^*,t)) - t - (V^{R_1}(\bar{x},(x^*,s)) - s)\}$.

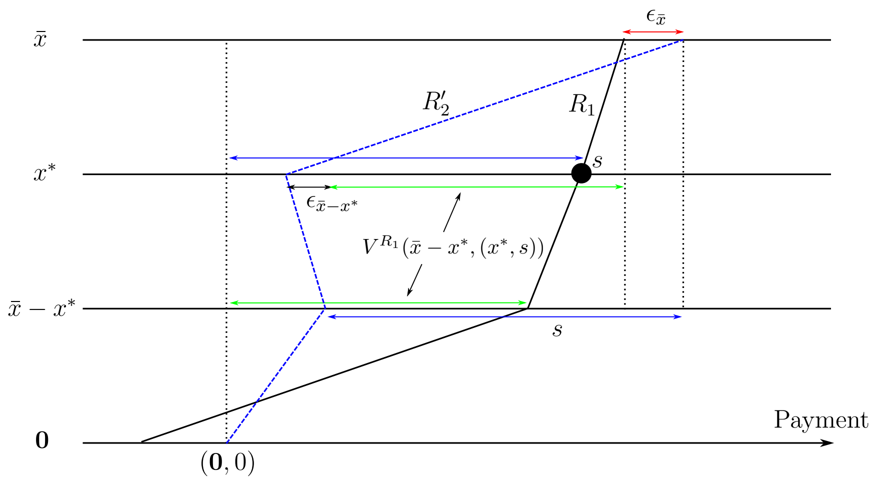

Figure 20: An illustration of R_2'.

Suppose $y \neq \mathbf{0}$. By $x < y$, $\bar{x}-x < \bar{x}-y$. Thus, $V^{R_1}(\bar{x}-x,(x^*,t)) < V^{R_1}(\bar{x}-y,(x^*,t))$ and $\epsilon_{\bar{x}-x} < \epsilon_{\bar{x}-y}$. Therefore,

$$\begin{aligned} v_2(x) &= V^{R_1}(\bar{x},(x^*,t)) - \max\{V^{R_1}(\bar{x}-x,(x^*,t)),0\} - \epsilon_{\bar{x}-x} \\ &> V^{R_1}(\bar{x},(x^*,t)) - \max\{V^{R_1}(\bar{x}-y,(x^*,t)),0\} - \epsilon_{\bar{x}-y} \\ &= v_2(y). \end{aligned}$$

□

Claim 2. *R_2' is object monotonic.*

Proof. Let $(x,y) \in \mathcal{X}$. If $y = \mathbf{0}$, then since ϵ_x is sufficiently close to 0 and R_1 is object monotonic, $v_2'(x) > 0$.

Suppose $y \neq \mathbf{0}$. By $x > y$, $\bar{x}-x < \bar{x}-y$. There are two cases.

Case 1. $V^{R_1}(\bar{x}-y,(x^*,s)) > 0$. By $\bar{x}-x < \bar{x}-y$, and $V^{R_1}(\bar{x}-y,(x^*,s)) > 0$,

$$\max\{V^{R_1}(\bar{x}-x,(x^*,s)),0\} < V^{R_1}(\bar{x}-y,(x^*,s)) = \max\{V^{R_1}(\bar{x}-y,(x^*,s)),0\}.$$

By this inequality and since $(\epsilon_{x'})_{x' \in X}$ is sufficiently close to $\mathbf{0}$, $\max\{V^{R_1}(\bar{x}-x,(x^*,s)),0\} + \epsilon_{\bar{x}-x} < \max\{V^{R_1}(\bar{x}-y,(x^*,s)),0\} - \epsilon_{\bar{x}}$.

Therefore,

$$\begin{aligned} v_2'(x) &\geq V^{R_1}(\bar{x}, (x^*, s)) - \max\{V^{R_1}(\bar{x} - x, (x^*, s)), 0\} - \epsilon_{\bar{x}-x} \\ &> V^{R_1}(\bar{x}, (x^*, s)) - \max\{V^{R_1}(\bar{x} - y, (x^*, s)), 0\} + \epsilon_{\bar{x}} \\ &\geq v_2'(y). \end{aligned}$$

Case 2. $V^{R_1}(\bar{x} - y, (x^*, s)) \leq 0$. By $x > y$, $y \neq \bar{x}$. By $V^{R_1}(x^*, (x^*, s)) = s > 0$, $y \neq \bar{x} - x^*$. Thus,

$$v_2'(y) = V^{R_1}(\bar{x}, (x^*, s)) - \max\{V^{R_1}(\bar{x} - y, (x^*, s)), 0\} - \epsilon_{\bar{x}-y} = V^{R_1}(\bar{x}, (x^*, s)) - \epsilon_{\bar{x}-y},$$

where the last equality follows from $V^{R_1}(\bar{x} - y, (x^*, s)) \leq 0$.

By $\bar{x} - x < \bar{x} - y$, $V^{R_1}(\bar{x} - x, (x^*, s)) < V^{R_1}(\bar{x} - y, (x^*, s)) \leq 0$. Thus,

$$v_2'(x) \geq V^{R_1}(\bar{x}, (x^*, s)) - \max\{V^{R_1}(\bar{x} - x, (x^*, s)), 0\} - \epsilon_{\bar{x}-x} = V^{R_1}(\bar{x}, (x^*, s)) - \epsilon_{\bar{x}-x}.$$

Since $(\epsilon_{x'})_{x' \in X}$ is object monotonic, $v_2'(x) > v_2'(y)$. □

By Claims 1 and 2, $R_2, R_2' \in \mathcal{R}^Q$.

Finally we define preferences of the other agents. For each $i \in N \setminus \{1, 2\}$, let $R_i \in \mathcal{R}$ be such that $R_i \in \mathcal{R}^Q(R_2)$ and $R_i \in \mathcal{R}^Q(R_2')$.[31] Denote $R \equiv (R_1, R_2, \ldots, R_n)$ and $R' \equiv (R_1, R_2', \ldots, R_n)$. Since f is *efficient* and $R_2, R_2', R_3, \ldots, R_n \in \mathcal{R}^Q$, Lemma 8 implies $X_1^f(R_{-1}) = x_1^f(R_{-1}') = X$.

Since f coincides with a Vickrey rule on $(\mathcal{R}^Q)^n$ and $R_2, R_2', R_3, \ldots, R_n \in \mathcal{R}^Q$, the option sets of agent 1 under f for R_{-1} and R_{-1}' coincide with the ones under a Vickrey rule, respectively. Further, by $R_3, \ldots, R_n \in \mathcal{R}^Q(R_2) \cap \mathcal{R}^Q(R_2')$, Lemma 2 implies that for each $x \in X$, $\sigma_1(R_{-1}; x) = v_2(\bar{x} - x)$ and $\sigma_1(R_{-1}'; x) = v_2'(\bar{x} - x)$. Hence, for each $x \in X$,

$$t_1^f(R_{-1}; x) = v_2(\bar{x}) - v_2(\bar{x} - x) \text{ and } t_1^f(R_{-1}'; x) = v_2'(\bar{x}) - v_2'(\bar{x} - x).$$

Step 2. *For each $x \in X \setminus \{\bar{x}\}$, $z_1^f(R_{-1}, \bar{x}) \; P_1 \; z_1^f(R_{-1}, x)$.*

Proof. (Figure 21.) First note that by $t < V^{R_1}(x^*, (\mathbf{0}, 0))$, $v_2(\bar{x}) = V^{R_1}(\bar{x}, (x^*, t)) < V^{R_1}(\bar{x}, (\mathbf{0}, 0))$. This implies $z_1^f(R_{-1}, \bar{x}) = (\bar{x}, v_2(\bar{x})) \; P_1 \; (\mathbf{0}, 0)$.

[31] We can pick such preferences from $\mathcal{R}$ since $\mathcal{R}^Q \subseteq \mathcal{R}$.

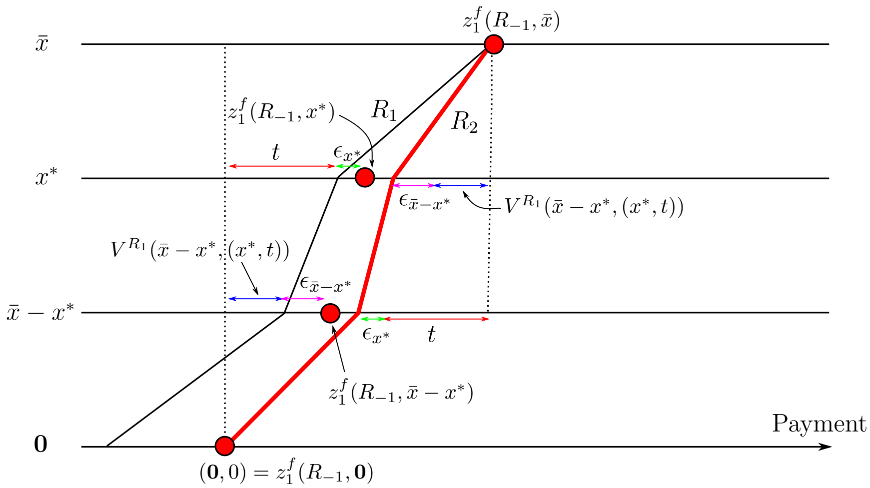

Figure 21: An illustration of the option set of agent 1 for R_{-1}.

Next, let $x \in X \setminus \{\bar{x}, \mathbf{0}\}$. Then,

$$\begin{aligned} &v_2(\bar{x}) - v_2(\bar{x} - x) \\ &\quad = V^{R_1}(\bar{x}, (x^*, t)) - (V^{R_1}(\bar{x}, (x^*, t)) - \max\{V^{R_1}(x, (x^*, t)), 0\} - \epsilon_x) \\ &\quad > \max\{V^{R_1}(x, (x^*, t)), 0\} \\ &\quad \geq V^{R_1}(x, (x^*, t)). \end{aligned}$$

This implies $(x^*, t)\ P_1\ (x, v_2(\bar{x}) - v_2(x)) = z_1^f(R_{-1}; x)$. Therefore, $z_1^f(R_{-1}; \bar{x}) = (\bar{x}, v_2(\bar{x})) = (\bar{x}, V^{R_1}(\bar{x}, (x^*, t)))\ I_1\ (x^*, t)\ P_1\ z_1^f(R_{-1}; x)$. ■

Step 3. *For each $x \in X \setminus \{x^*\}$, $z_1^f(R_{-1}, x^*)\ P_1\ z_1^f(R_{-1}, x)$.*

Proof. (Figure 22.) Note that

$$\begin{aligned} t_1^f(R'_{-1}; x^*) &= v'_2(\bar{x}) - v'_2(\bar{x} - x^*) \\ &= V^{R_1}(\bar{x}, (x^*, s)) + \epsilon_{\bar{x}} - (V^{R_1}(\bar{x}, (x^*, s)) - V^{R_1}(x^*, (x^*, s)) + \epsilon_{\bar{x}}) \\ &= s. \end{aligned}$$

By $s < V^{R_1}(x^*, (\mathbf{0}, 0))$, $z_1^f(R'_{-1}, x^*) = (x^*, s)\ P_1\ (\mathbf{0}, 0)$. Also, since $v'_2(\bar{x}) > V^{R_1}(\bar{x}, (x^*, s))$, $z_1^f(R'_{-1}, x^*) = (x^*, s)\ P_1\ (\bar{x}, v'_2(\bar{x})) = z_1^f(R'_{-1}, \bar{x})$.

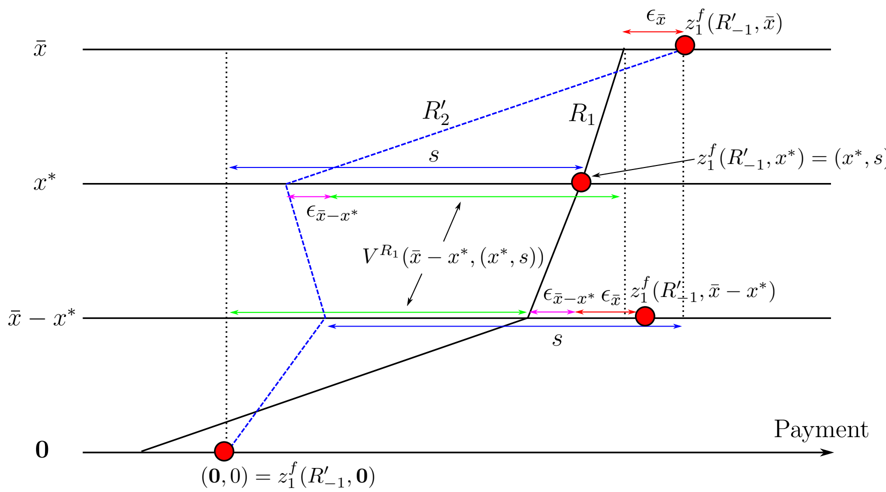

Figure 22: An illustration of the option set of agent 1 for R'_{-1}.

Now let $x \in X \setminus \{\mathbf{0}, x^*, \bar{x}\}$. Then, $\bar{x} - x' \neq \bar{x} - x^*$. Thus,

$$\begin{aligned} v'_2(\bar{x}) - v'_2(\bar{x} - x) &= V^{R_1}(\bar{x}, (x^*, s)) + \epsilon_{\bar{x}} - (V^{R_1}(\bar{x}, (x^*, s)) - \max\{V^{R_1}(x, (x^*, s)), 0\} - \epsilon_x) \\ &= \max\{V^{R_1}(x, (x^*, s)), 0\} + \epsilon_{\bar{x}} + \epsilon_x \\ &> V^{R_1}(x, (x^*, s)). \end{aligned}$$

This implies $z_1^f(R'_{-1}; x^*) = (x^*, s)\ P_1\ (x, v'_2(\bar{x}) - v'_2(\bar{x} - x)) = z_1^f(R'_1; x)$. ■

Step 4. *Completing the proof.*

By Steps 2 and 3 and *strategy-proofness*, $x_1^f(R) = \bar{x}$ and $x_1^f(R') = x^*$. For each $i \in N \setminus \{2, 3\}$, since $R_i \in \mathcal{R}^Q(R_2) \cap \mathcal{R}^Q(R'_2)$, Lemma 7 implies $x_i^f(R) = x_i^f(R') = \mathbf{0}$. Therefore, $x_2^f(R) = \mathbf{0}$ and $x_2^f(R') = \bar{x} - x^*$. However, by (6) and since $(\epsilon_x)_{x \in X}$ is sufficiently close to $\mathbf{0}$,

$$v_2(\bar{x} - x^*) = V^{R_1}(\bar{x}, (x^*, t)) - t - \epsilon_{\bar{x}-x^*} > V^{R_1}(\bar{x}, (x^*, s)) - s + \epsilon_{\bar{x}} = v'_2(\bar{x} - x^*).$$

This contradicts Fact 3. ■

F Proofs of Lemmas

F.1 Proof of Lemma 2

Let $x \in X$ and $(x_k)_{k \in N} \in A$ be such that $x_i = x$ and $\sigma_i(R_{-i}; x) = \sum_{k \in N \setminus \{i\}} v_k(x_k)$. Suppose by contradiction that there is $k \in N \setminus \{i, j\}$ such that $x_k > 0$.

Let $(y_\ell)_{\ell \in N} \in A$ be such that $y_i = x_i$, $y_j = x_j + x_k$, $y_k = 0$, and for each $\ell \in N \setminus \{i, j, k\}$, $y_\ell = x_\ell$. By $\sigma_i(R_{-i}; x) = \sum_{k \in N \setminus \{i\}} v_k(x_k)$, $\sum_{\ell \in N \setminus \{i\}} v_\ell(y_\ell) \leq \sum_{k \in N \setminus \{i\}} v_k(x_k)$.

However, by $R_k \in \mathcal{R}^Q(R_j)$ and $x_k > 0$, $v_j(x_j) + v_k(x_k) < v_j(x_j + x_k)$. Thus,

$$\sum_{\ell \in N \setminus \{i\}} v_\ell(y_\ell) = v_j(x_j + x_k) + \sum_{\ell \in N \setminus \{i,j,k\}} v_\ell(x_\ell) > \sum_{k \in N \setminus \{i\}} v_k(x_k).$$

This is a contradiction. ■

F.2 Proof of Lemma 3

Let

$$X_+ \equiv \{y \in X : t_y > 0\}.$$

Let $\mathbf{t}^* \in \mathbb{R}^{|X|}$ be such that for each $y \in X$,

$$t^*_y = \begin{cases} \min\{t_y, s_y\} & \text{if } y \in X_+, \\ 0 & \text{otherwise.} \end{cases}$$

Figure 23 illustrates $\mathbf{t}$, $\mathbf{s}$, and $\mathbf{t}^*$. Since $\mathbf{t}$ and $\mathbf{s}$ are object monotonic, for each $(y, y') \in \mathcal{X}$ with $y, y' \in X_+$, $t^*_y > t^*_{y'}$. Further, for each $(y, y') \in \mathcal{X}$, $t^*_y \geq t^*_{y'}$.[32] Let $\bar{\epsilon}, \underline{\epsilon} \in \mathbb{R}_{++}$ be sufficiently close to 0 and satisfy $\bar{\epsilon} > \underline{\epsilon}$.[33]

Fix $y \in X \setminus \{x\}$. We prove that there is $R_i \in \mathcal{R}^P$ such that $R_i \in \mathcal{R}^{MT}_{\mathbf{t},x} \cap \mathcal{R}^{MT}_{\mathbf{s},y}$. We do the proof in three steps.

Step 1. *Constructing a preference relation.*

[32] Note that by $s_{\mathbf{0}} > 0$ and the object monotonicity of s, for each $y \in X \setminus \{\mathbf{0}\}$, $s_y > 0$.

[33] For example, the proof works if $\bar{\epsilon} > \underline{\epsilon} > 0$ and $2(\bar{\epsilon} + \underline{\epsilon}) < \min\{s_x - t_x, \min_{x' \in X_+} t^*_{x'}, \min_{(x',y') \in \mathcal{X}} s_{x'} - s_{y'}, \min_{(x',y') \in \mathcal{X}, x' \in X_+} t^*_{x'} - t^*_{y'}\}$.

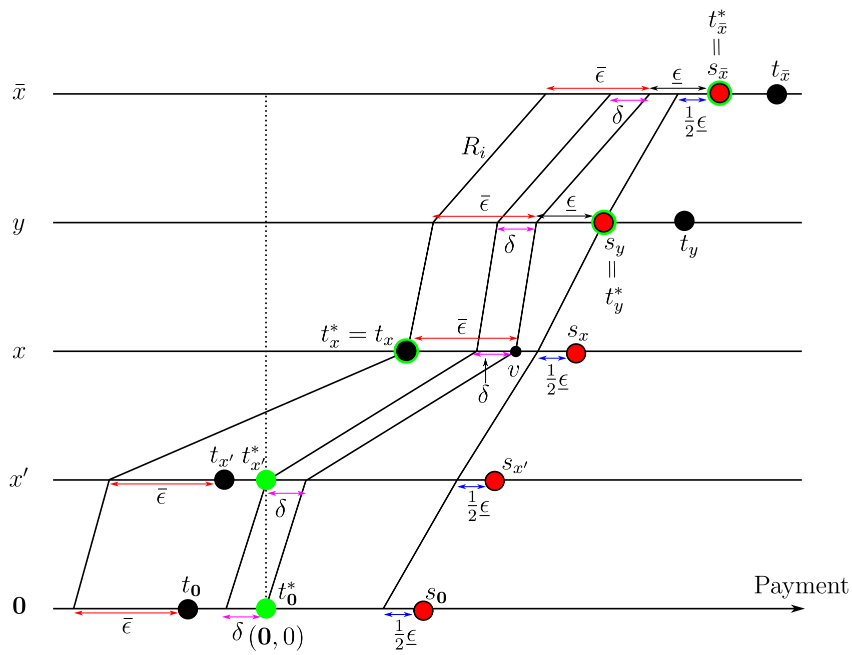

Figure 23: An illustration of $\mathbf{t}$, $\mathbf{s}$, $\mathbf{t}^*$, and R_i.

We define a preference relation R_i as follows: For each $x' \in X \setminus \{\mathbf{0}\}$, let[34]

$$V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) = \begin{cases} t^*_x + \bar{\epsilon} & \text{if } x' = x, \\ t^*_{x'} - \underline{\epsilon} & \text{if } x' \in X_+ \setminus \{x\}, \\ \frac{m(x')}{m(\bar{x})}\underline{\epsilon} & \text{otherwise.} \end{cases}$$

Let

$$\delta \equiv \begin{cases} \max_{x' \in X \setminus X_+} \frac{m(x')}{m(\bar{x})}\underline{\epsilon} & \text{if } X \setminus X_+ \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Note that by $\bar{\epsilon} > \underline{\epsilon}$, $\bar{\epsilon} > \delta$.

For each $t' \in [t^*_{\mathbf{0}} - \delta, t^*_{\mathbf{0}}]$ and each $x' \in X$, let

$$V^{R_i}(x', (\mathbf{0}, t')) = V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) - (t^*_{\mathbf{0}} - t').$$

Let $v \equiv V^{R_i}(x, (\mathbf{0}, t^*_{\mathbf{0}}))$. Note that if $x \neq \mathbf{0}$, then $v = t^*_x + \bar{\epsilon}$, and if $x = \mathbf{0}$, then

[34] Note that if $x = \mathbf{0}$, the first condition of the definition of $V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}}))$ is redundant since $x' \in X \setminus \{\mathbf{0}\}$.

$v = t^*_{\mathbf{0}}$. Note also that by $\bar{\epsilon} > \delta$, $v - \bar{\epsilon} < V^{R_i}(x, (\mathbf{0}, t^*_{\mathbf{0}} - \delta))$.[35] For each $x' \in X$, let[36]

$$V^{R_i}(x', (x, v - \bar{\epsilon})) = \begin{cases} V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) - \bar{\epsilon} & \text{if } x' \in X_+ \cup \{x\}, \\ \min\{t_{x'}, V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}} - \delta))\} - \bar{\epsilon} & \text{otherwise.} \end{cases}$$

For each $t' \in [V^{R_i}(\mathbf{0}, (x, v - \bar{\epsilon})), t^*_{\mathbf{0}} - \delta]$ and each $x' \in X$, let

$$V^{R_i}(x', (\mathbf{0}, t')) = \alpha \cdot V^{R_i}(x', (x, v - \bar{\epsilon})) + (1 - \alpha) V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}} - \delta)),$$

where $\alpha \in [0, 1]$ is such that $t' = \alpha \cdot V^{R_i}(\mathbf{0}, (x, v - \bar{\epsilon})) - (1 - \alpha)(t^*_{\mathbf{0}} - \delta)$.

For each $x' \in X \setminus \{y\}$, let

$$V^{R_i}(x', (y, s_y)) = s_{x'} - \frac{1}{2}\underline{\epsilon}.$$

Note that since $\bar{\epsilon}$ and $\underline{\epsilon}$ are sufficiently close to 0, for each $x' \in X$, $V^{R_i}(x', (y, s_y)) > V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}}))$.[37]

For each $t \in [t^*_{\mathbf{0}}, V^{R_i}(\mathbf{0}, (y, s_y))]$ and each $x' \in X$, let

$$V^{R_i}(x', (\mathbf{0}, t)) = \alpha \cdot V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) + (1 - \alpha) V^{R_i}(x', (y, s_y)),$$

where $\alpha \in [0, 1]$ is such that $t = \alpha \cdot t^*_{\mathbf{0}} + (1 - \alpha) V^{R_i}(\mathbf{0}, (y, s_y))$.

Finally, for each $t' \in \mathbb{R} \setminus [V^{R_i}(\mathbf{0}, (x, v - \bar{\epsilon})), V^{R_i}(\mathbf{0}, (y, s_y))]$ and each $x' \in X$, let

$$V^{R_i}(x', (\mathbf{0}, t')) = \begin{cases} V^{R_i}(x', (x, v - \bar{\epsilon})) - (V^{R_i}(\mathbf{0}, (x, v - \bar{\epsilon})) - t) & \text{if } t' < V^{R_i}(\mathbf{0}, (x, v - \bar{\epsilon})), \\ V^{R_i}(x', (y, s_y)) + t' - V^{R_i}(\mathbf{0}, (y, s_y)) & \text{if } t' > V^{R_i}(\mathbf{0}, (y, s_y)). \end{cases}$$

Figure 23 illustrates R_i.

Note that by the construction of R_i, R_i is bounded. It is also clear that $R_i \in \mathcal{R}^{MT}_{\mathbf{s},y}$. Further, we also have $R_i \in \mathcal{R}^{MT}_{\mathbf{t},x}$. To see this, let $x' \in X \setminus \{x\}$. First, suppose $x = \mathbf{0}$. By $t_{\mathbf{0}} \geq 0$ and the object monotonicity of $\mathbf{t}$, $t_{x'} > 0$ and thus

[35] Formally, $V^{R_i}(x, (\mathbf{0}, t^*_{\mathbf{0}} - \delta)) = V^{R_i}(x, (\mathbf{0}, t^*_{\mathbf{0}})) - \delta = v - \delta > v - \bar{\epsilon}$.

[36] If $x = \mathbf{0}$, then by $t_x \geq 0$ and the object monotonicity of $\mathbf{t}$, $X_+ \supseteq X \setminus \{\mathbf{0}\}$. Thus, in this case, for each $x' \in X \setminus \{\mathbf{0}\}$, $V^{R_i}(x', (x, v - \bar{\epsilon})) = V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) - \bar{\epsilon}$.

[37] To see this, let $x' \in X$. If $x' = x$, then by $t_x < s_x$ and since $\bar{\epsilon}$ and $\underline{\epsilon}$ are sufficiently close to 0, $V^{R_i}(x, (y, s_y)) = s_x - \frac{1}{2}\underline{\epsilon} > t^*_x + \bar{\epsilon} = V^{R_i}(x, (\mathbf{0}, t^*_{\mathbf{0}}))$. Suppose $x' \in X_+ \setminus \{x\}$. Then, by $s_{x'} \geq t^*_{x'}$, $V^{R_i}(x', (y, s_y)) = s_{x'} - \frac{1}{2}\underline{\epsilon} > t^*_{x'} - \underline{\epsilon} = V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}}))$. Suppose $x' \in X \setminus X_+ \cup \{x\}$. By $s_{\mathbf{0}} > 0$ and the object monotonicity of $\mathbf{s}$, $s_{x'} > 0$. Thus, since $\underline{\epsilon}$ is sufficiently close to 0, $V^{R_i}(x', (y, s_y)) \geq s_{x'} - \frac{1}{2}\underline{\epsilon} > \frac{m(x')}{m(\bar{x})} \cdot \underline{\epsilon} = V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}}))$.

$x' \in X_+ \setminus \{x\}$. Therefore, $V^{R_i}(x', (x, t_x)) = V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}}) = t^*_{x'} - \underline{\epsilon} < t_{x'}$, which implies $(\mathbf{0}, t_{\mathbf{0}})\ P_i\ (x', t_{x'})$. Next, suppose $x \neq \mathbf{0}$. Then, by $t_x < s_x$, $t_x = t^*_x = v - \bar{\epsilon}$. If $x' \in X_+$, then $V^{R_i}(x', (x, t_x)) = V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) - \bar{\epsilon} = t^*_{x'} - \underline{\epsilon} - \bar{\epsilon} < t_{x'}$. If $x' \notin X_+$, then $V^{R_i}(x', (x, t_x)) = \min\{t_{x'}, V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}} - \delta))\} - \bar{\epsilon} < t_{x'}$. Thus, $(\mathbf{0}, t_{\mathbf{0}})\ P_i\ (x', t_{x'})$. Hence, $R_i \in \mathcal{R}^{MT}_{\mathbf{t},x}$.

Step 2. *R_i is object monotonic.*

Proof. To show that R_i is object monotonic, we need to prove that for each $t' \in \mathbb{R}$, the vector $(V^{R_i}(x', (\mathbf{0}, t')))_{x' \in X}$ is object monotonic. By the definition of R_i, for each $t' \in \mathbb{R}$, the vector $(V^{R_i}(x', (\mathbf{0}, t')))_{x' \in X}$ is either (i) obtained by shifting one of the following vectors $(V^{R_i}(x', (x, v - \bar{\epsilon})))_{x' \in X}$, $(V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}} - \delta)))_{x' \in X}$, $(V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})))_{x' \in X}$, and $(V^{R_i}(x', (y, s_y)))_{x' \in X}$, or (ii) a convex combination of two of these four vectors. Thus, we only need to show that these four vectors are object monotonic. Let $(x', y') \in \mathcal{X}$.

We first show that $(V^{R_i}(x'', (\mathbf{0}, t^*_{\mathbf{0}})))_{x'' \in X}$ is object monotonic. Suppose $y' \in X_+ \cup \{x\}$. By $t_x \geq 0$, $x' > y'$, and the object monotonicity of $\mathbf{t}$, we have $x' \in X_+$. Thus, $V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) \geq t^*_{x'} - \underline{\epsilon}$ and $V^{R_i}(y', (\mathbf{0}, t^*_{\mathbf{0}})) \leq t^*_{y'} + \bar{\epsilon}$. By $x' \in X_+$ and $s_x > t_x \geq 0$, $t^*_{x'} = \min\{t_{x'}, s_{x'}\} > \min\{t_{y'}, s_{y'}\} = t^*_{y'}$. Thus, since $\bar{\epsilon}$ and $\underline{\epsilon}$ are sufficiently close to 0, $V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) > V^{R_i}(y', (\mathbf{0}, t^*_{\mathbf{0}}))$. Next, suppose $y' \in X \setminus (X_+ \cup \{x\})$. Then, by $m(x') > m(y')$ and since $\bar{\epsilon}$ is sufficiently close to 0, $V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) \geq \frac{m(x')}{m(\bar{x})}\underline{\epsilon} > \frac{m(y')}{m(\bar{x})}\underline{\epsilon} = V^{R_i}(y', (\mathbf{0}, t^*_{\mathbf{0}}))$. Hence, $(V^{R_i}(x'', (\mathbf{0}, t^*_{\mathbf{0}})))_{x'' \in X}$ is object monotonic.

Since $V^{R_i}(x'', (\mathbf{0}, t^*_{\mathbf{0}} - \delta)) = V^{R_i}(x'', (\mathbf{0}, t^*_{\mathbf{0}})) - \delta$ for each $x'' \in X$ and $(V^{R_i}(x'', (\mathbf{0}, t^*_{\mathbf{0}})))_{x'' \in X}$ is object monotonic, $(V^{R_i}(x'', (\mathbf{0}, t^*_{\mathbf{0}} - \delta)))_{x'' \in X}$ is also object monotonic.

Next, we show that $(V^{R_i}(x'', (x, v - \bar{\epsilon})))_{x'' \in X}$ is object monotonic. Suppose $y' \in X_+ \cup \{x\}$. By $x' > y'$, $x' \in X_+$. Since $(V^{R_i}(x'', (\mathbf{0}, t^*_{\mathbf{0}})))_{x'' \in X}$ is object monotonic as we have shown, $V^{R_i}(x', (x, v - \bar{\epsilon})) = V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}}))) - \bar{\epsilon} > V^{R_i}(y', (\mathbf{0}, t^*_{\mathbf{0}}))) - \bar{\epsilon} = V^{R_i}(y', (x, v - \bar{\epsilon}))$. Next, suppose $y' \notin X_+ \cup \{x\}$. Then, $V^{R_i}(x', (x, v - \bar{\epsilon})) \geq \min\{t_{x'}, V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}} - \delta))\} - \bar{\epsilon}$ and $V^{R_i}(y', (x, v - \bar{\epsilon})) = \min\{t_{x'}, V^{R_i}(y', (\mathbf{0}, t^*_{\mathbf{0}} - \delta))\} - \epsilon_{\bar{x}}$. Since $t_{x'} > t_{y'}$ and $(V^{R_i}(x'', (\mathbf{0}, t^*_{\mathbf{0}} - \delta)))_{x'' \in X}$ is object monotonic as we have shown, $V^{R_i}(x', (x, v - \bar{\epsilon})) > V^{R_i}(y', (x, v - \bar{\epsilon}))$. Hence, $(V^{R'_i}(x'', (x, v - \bar{\epsilon})))_{x'' \in X}$ is object monotonic.

Finally, since s is object monotonic and $\underline{\epsilon}$ is sufficiently close to 0, it is immediate that $(V^{R_i}(x'', (y, s_y)))_{x'' \in X}$ is object monotonic. Hence, R_i is object monotonic. ■

Step 3. $R_i \in \mathcal{R}^P$.

Proof. Let $(x', t') \in X(R_i)$. By Remark 4, it is enough to show $V^{R_i}(\bar{x}, (x', t')) - t' = V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0))$.

Let $s' \equiv V^{R_i}(\mathbf{0}, (x', t'))$. By $(x', t') \in X(R_i)$ and the definition of t^*, $s' \leq 0 \leq t^*_{\mathbf{0}}$. There are two cases.

Case 1. $X_+ = X$. By the construction of R_i, for each $x'' \in X$ and each $s'' \in \mathbb{R}$ with $s'' \leq t^*_{\mathbf{0}}$,

$$V^{R_i}(x'', (\mathbf{0}, s'')) = V^{R_i}(x'', (\mathbf{0}, t^*_{\mathbf{0}})) - (t^*_{\mathbf{0}} - s'').$$

Thus,

$$\begin{aligned}
V^{R_i}(\bar{x}, (x', t')) - t' &= V^{R_i}(\bar{x}, (\mathbf{0}, t^*_{\mathbf{0}})) - (t^*_{\mathbf{0}} - s') - (V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) - (t^*_{\mathbf{0}} - s')) \\
&= V^{R_i}(\bar{x}, (\mathbf{0}, t^*_{\mathbf{0}})) - V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) \\
&= V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0)),
\end{aligned}$$

where the last equality follows since $0 < t^*_{\mathbf{0}}$.

Case 2. $X \setminus X_+ \neq \emptyset$. By the object monotonicity of $\mathbf{t}$ and $X \setminus X_+ \neq \emptyset$, we have $\mathbf{0} \notin X_+$. Hence, $t^*_{\mathbf{0}} = 0$.

Suppose $-\delta \leq s' \leq 0$. Then,

$$\begin{aligned}
V^{R_i}(\bar{x}, (x', t')) - t' &= V^{R_i}(\bar{x}, (\mathbf{0}, t^*_{\mathbf{0}})) - (t^*_{\mathbf{0}} - s') - (V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) - (t^*_{\mathbf{0}} - s')) \\
&= V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0)).
\end{aligned}$$

Suppose $s' \leq V^{R_i}(\mathbf{0}, (x, v - \bar{\epsilon}))$. Note that if $x' \notin X_+ \cup \{x\}$, then $t_{x'} \leq 0$ and thus, $t' = V^{R_i}(x', (\mathbf{0}, s')) \leq V^{R_i}(x', (x, v - \bar{\epsilon})) = \min\{t_{x'}, V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}} - \delta))\} - \bar{\epsilon} < 0$. Thus, by $(x', t') \in X(R_i)$, $x' \in X_+ \cup \{x\}$.

Therefore,

$$\begin{aligned}
&V^{R_i}(\bar{x}, (x', t')) - t' \\
&\quad = V^{R_i}(\bar{x}, (x, v - \bar{\epsilon})) - (V^{R_i}(\mathbf{0}, (x, v - \bar{\epsilon})) - s') - (V^{R_i}(x', (x, v - \bar{\epsilon})) - (V^{R_i}(\mathbf{0}, (x, v - \bar{\epsilon})) - s')) \\
&\quad = V^{R_i}(\bar{x}, (x, v - \bar{\epsilon})) - V^{R_i}(x', (x, v - \bar{\epsilon})) \\
&\quad = V^{R_i}(\bar{x}, (\mathbf{0}, t^*_{\mathbf{0}})) - \bar{\epsilon} - (V^{R_i}(x', (\mathbf{0}, t^*_{\mathbf{0}})) - \bar{\epsilon}) \\
&\quad = V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0)).
\end{aligned}$$

Finally suppose $V^{R_i}(\mathbf{0},(x,v-\bar{\epsilon})) < s' < -\delta$. Let $\alpha \in [0,1]$ be such that $s' = \alpha \cdot V^{R_i}(\mathbf{0},(x,v-\bar{\epsilon})) - (1-\alpha)\delta$. Then,

$$\begin{aligned}
&V^{R_i}(\bar{x},(x'_i,t')) - t' \\
&=\alpha \cdot V^{R_i}(\bar{x},(x,v-\bar{\epsilon})) + (1-\alpha)V^{R_i}(\bar{x},(\mathbf{0},-\delta)) - (\alpha \cdot V^{R_i}(x',(x,v-\bar{\epsilon})) + (1-\alpha)V^{R_i}(x',(\mathbf{0},-\delta))) \\
&=V^{R_i}(\bar{x},(\mathbf{0},0)) - V^{R_i}(x',(\mathbf{0},0)).
\end{aligned}$$

■

F.3 Proof of Lemma 4

Let $\mathbf{t}^* \in \mathbb{R}^{|X|}$ be such that for each $y \in X$,

$$t^*_y = \min\{t_y, s_y, 0\}.$$

Figure 24 illustrates $\mathbf{t}$, $\mathbf{s}$, and $\mathbf{t}^*$. It is easy to see that $\mathbf{t}^*$ is *weakly object monotonic*, i.e., for each $(y,y') \in \mathcal{X}$, $t^*_y \geq t^*_{y'}$. Note also that by $t_x < s_x$ and $t_x < 0$, $t^*_x = t_x$. Let $(\epsilon_y)_{y \in X} \in \mathbb{R}^{|X|}_{++}$ be an object monotonic vector that is sufficiently close to $\mathbf{0}$.[38] Fix $y \in X \setminus \{x\}$. We prove that there is $R_i \in \mathcal{R}^P$ such that $R_i \in \mathcal{R}^{MT}_{\mathbf{t},x} \cap \mathcal{R}^{MT}_{\mathbf{s},y}$. We do the proof in three steps.

Step 1. *Constructing a preference relation.*

We define a preference relation R_i as follows. For each $x' \in X \setminus \{x\}$, let

$$V^{R_i}(x',(x,t_x)) = t^*_x - \epsilon_{\bar{x}-x'}.$$

Let $\delta \in \mathbb{R}_{++}$ be sufficiently close to 0.[39] For each $x' \in X \setminus \{y\}$, let

$$V^{R_i}(x',(y,s_y)) = s_{x'} - \delta.$$

Note that since δ is sufficiently close to 0, for each $x' \in X$, $V^{R_i}(x',(x,t_x)) < V^{R_i}(x',(y,s_y))$.

---

[38] Formally, the proof works if $(\epsilon_y)_{y \in X}$ is object monotonic and satisfies $0 < \epsilon_{\mathbf{0}}$ and $\epsilon_{\bar{x}} < \min\{-t_x, \min_{(x',y') \in \mathcal{X}} t_{x'} - t_{y'}\}$.

[39] Formally, the proof works if δ satisfies $0 < \delta < \epsilon_{\mathbf{0}}$, $\delta < \min\{s_{x'} - t_{x'} : s_{x'} > t_{x'}, x' \in X\}$, and $\delta < \min_{(x',y') \in \mathcal{X}} s_{x'} - s_{y'}$.

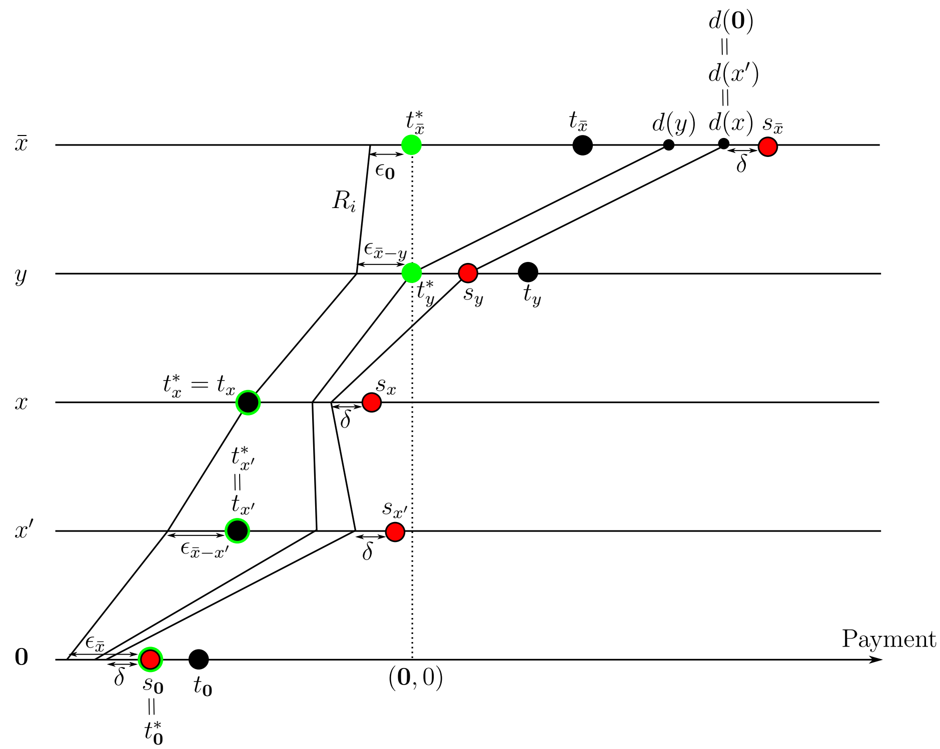

Figure 24: An illustration of $\mathbf{t}$, $\mathbf{s}$, $\mathbf{t}^*$, and R_i.

Let $d : X \to \mathbb{R}$ be such that for each $x' \in X$,

$$d(x') = V^{R_i}(\bar{x}, (y, s_y)) - \max\{V^{R_i}(x', (y, s_y)), 0\}.$$

Note that for each $(x', y') \in \mathcal{X}$, $d(x') \leq d(y')$.

For each $t' \in [V^{R_i}(\bar{x}, (x, t_x)), V^{R_i}(\bar{x}, (y, s_y))]$ and each $x' \in X$, let

$$V^{R_i}(x', (\bar{x}, t')) = \begin{cases} t' - d(x') & \text{if } t' > d(x'), \\ \alpha \cdot \min\{V^{R_i}(x', (y, s_y)), 0\} + (1-\alpha)V^{R_i}(x', (x, t_x)) & \text{if } t' \leq d(x'), \end{cases}$$

where $\alpha \in [0, 1]$ is such that $t' = \alpha \cdot d(x') + (1-\alpha)V^{R_i}(\bar{x}, (x, t_x))$.

Finally, for each $t' \in \mathbb{R} \setminus [V^{R_i}(\mathbf{0}, (x, t_x)), V^{R_i}(\mathbf{0}, (y, s_y))]$ and each $x' \in X$, let

$$V^{R_i}(x', (\mathbf{0}, t')) = \begin{cases} V^{R_i}(x', (x, t_x)) - (V^{R_i}(\mathbf{0}, (x, t_x)) - t') & \text{if } t' < V^{R_i}(\mathbf{0}, (x, t_x)), \\ V^{R_i}(x', (y, s_y)) + (t' - V^{R_i}(\mathbf{0}, (y, s_y))) & \text{if } t' > V^{R_i}(\mathbf{0}, (y, s_y)). \end{cases}$$

Figure 24 illustrates R_i. Note that by the construction of R_i, R_i is bounded. Further, it is clear that $R_i \in \mathcal{R}^{MT}_{\mathbf{t},x} \cap \mathcal{R}^{MT}_{\mathbf{s},y}$.

Step 2. R_i *is object monotonic.*

Proof. To show that R_i is object monotonic, we need to prove the object monotonicity of $(V^{R_i}(x', (x, t_x)))_{x' \in X}$, $(V^{R_i}(x', (y, s_y)))_{x' \in X}$, and $(V^{R_i}(x', (\bar{x}, t')))_{x' \in X}$ for each $t' \in [V^{R_i}(\bar{x}, (x, t_x)), V^{R_i}(\bar{x}, (y, s_y))]$. First observe that since $\mathbf{t}^*$ is weakly object monotonic and $(\epsilon_{x'})_{x' \in X}$ is object monotonic, $(V^{R_i}(x', (x, t_x)))_{x' \in X}$ is object monotonic. Note also that since s is object monotonic and δ is sufficiently close to 0, $(V^{R_i}(x', (y, s_y)))_{x' \in X}$ is object monotonic.

Let $t' \in [V^{R_i}(\bar{x}, (x, t_x)), V^{R_i}(\bar{x}, (y, s_y))]$. Now we show that $(V^{R_i}(x', (\bar{x}, t')))_{x' \in X}$ is object monotonic. Let $(x', y') \in \mathcal{X}$. Observe that if $V^{R_i}(y', (y, s_y)) > 0$, then by the object monotonicity of $(V^{R_i}(x'', (y, s_y)))_{x'' \in X}$, we have $d(x') = V^{R_i}(\bar{x}, (y, s_y)) - V^{R_i}(x', (y, s_y))$ and $d(y') = V^{R_i}(\bar{x}, (y, s_y)) - V^{R_i}(y', (y, s_y))$, and hence, $d(x') < d(y')$. There are three cases.

Case 1. $t' > d(y')$. By $t' \leq V^{R_i}(\bar{x}, (y, s_y))$, $d(y') < V^{R_i}(\bar{x}, (y, s_y))$. This inequality and the definition of d imply $V^{R_i}(y', (y, s_y)) > 0$. Thus, $d(x') < d(y') < t'$. Therefore,

$$V^{R_i}(x', (\bar{x}, t')) = t' - d(x') > t' - d(y') = V^{R_i}(y', (\bar{x}, t')).$$

Case 2. $d(x') < t' \leq d(y')$.[40] By $d(x') < t'$, $V^{R_i}(x', (\bar{x}, t')) = t' - d(x') > 0$. On the other hand, by $t' \leq d(y')$ and $V^{R_i}(y', (x, t_x)) < 0$, $V^{R_i}(y', (\bar{x}, t')) < 0$. Thus, $V^{R_i}(x', (\bar{x}, t')) > V^{R_i}(y', (\bar{x}, t'))$.

Case 3. $t' \leq d(x')$ *and* $t' \leq d(y')$. By the definition of R_i,

$$\begin{aligned} V^{R_i}(x', (\bar{x}, t')) &= \alpha \cdot \min\{V^{R_i}(x', (y, s_y)), 0\} + (1 - \alpha) V^{R_i}(x', (x, t_x)), \text{ and} \\ V^{R_i}(y', (\bar{x}, t')) &= \beta \cdot \min\{V^{R_i}(y', (y, s_y)), 0\} + (1 - \beta) V^{R_i}(y', (x, t_x)), \end{aligned}$$

where $\alpha \in [0, 1]$ and $\beta \in [0, 1]$ are such that $t' = \alpha \cdot d(x') + (1 - \alpha) V^{R_i}(\bar{x}, (x, t_x))$ and $t' = \beta \cdot d(y') + (1 - \beta) V^{R_i}(\bar{x}, (x, t_x))$, respectively. Note that by $d(x') \leq d(y')$, $\alpha \geq \beta$.

Suppose $V^{R_i}(x', (y, s_y)) > 0$. Then, $V^{R_i}(x', (\bar{x}, t')) = (1 - \alpha) V^{R_i}(x', (x, t_x))$. By $x' > y'$ and the definition of $(V^{R_i}(x'', (x, t_x)))_{x'' \in X}$, $0 > V^{R_i}(x', (x, t_x)) > V^{R_i}(y', (x, t_x))$.

[40] By $d(x') \leq d(y')$, it cannot be the case that $d(y') < t' \leq d(x')$.

Thus, by $(1-\alpha) \leq (1-\beta)$,

$$V^{R_i}(x', (\bar{x}, t')) = (1-\alpha)V^{R_i}(x', (x, t_x)) > (1-\beta)V^{R_i}(y', (x, t_x)) \geq V^{R_i}(y', (\bar{x}, t')).$$

Suppose $V^{R_i}(x', (y, s_y)) \leq 0$. By the object monotonicity of $(V^{R_i}(x'', (y, s_y)))_{x'' \in X}$, $V^{R_i}(y', (y, s_y)) < V^{R_i}(x', (y, s_y)) \leq 0$. Also, by the object monotonicity of $(V^{R_i}(x'', (x, t_x)))_{x'' \in X}$, $V^{R_i}(x', (x, t_x)) > V^{R_i}(y', (x, t_x))$. Thus,

$$\begin{aligned} V^{R_i}(x', (\bar{x}, t')) &= \alpha \cdot V^{R_i}(x', (y, s_y)) + (1-\alpha)V^{R_i}(x', (x, t_x)) \\ &> \alpha \cdot V^{R_i}(y', (y, s_y)) + (1-\alpha)V^{R_i}(y', (x, t_x)) \\ &\geq \beta \cdot V^{R_i}(y', (y, s_y)) + (1-\beta)V^{R_i}(y', (x, t_x)) \\ &= V^{R_i}(y', (\bar{x}, t')), \end{aligned}$$

where the second inequality follows from $\alpha \geq \beta$ and $V^{R_i}(y', (y, s_y)) > V^{R_i}(y', (x, t_x))$.
■

Step 3. $R_i \in \mathcal{R}^P$.

Proof. First we show that for each $x' \in X$, $V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0)) = V^{R_i}(\bar{x}, (y, s_y)) - V^{R_i}(x', (y, s_y))$. If $V^{R_i}(\mathbf{0}, (y, s_y)) \leq 0$, it is clear that this equality holds. Suppose $V^{R_i}(\mathbf{0}, (y, s_y)) > 0$. Then, $V^{R_i}(\mathbf{0}, (\bar{x}, d(\mathbf{0}))) = 0$, and this implies $V^{R_i}(\bar{x}, (\mathbf{0}, 0)) = d(\mathbf{0})$. By $V^{R_i}(\mathbf{0}, (y, s_y)) > 0$, for each $x' \in X \setminus \{\mathbf{0}\}$, $V^{R_i}(x', (y, s_y)) > 0$ and thus $d(x') < d(\mathbf{0})$. Thus, for each $x' \in X$,

$$\begin{aligned} V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0)) &= d(\mathbf{0}) - V^{R_i}(x', (\bar{x}, d(\mathbf{0}))) \\ &= d(\mathbf{0}) - (d(x') - d(\mathbf{0})) \\ &= V^{R_i}(\bar{x}, (y, s_y)) - V^{R_i}(x', (y, s_y)). \end{aligned}$$

Let $(x', t') \in X(R_i)$. By Remark 4, it is enough to show $V^{R_i}(\bar{x}, (x', t')) - t' = V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0))$.

If $V^{R_i}(\mathbf{0}, (x', t')) \geq V^{R_i}(\mathbf{0}, (y, s_y))$, then by the definition of R_i, $V^{R_i}(\bar{x}, (x', t')) - t' = V^{R_i}(\bar{x}, (y, s_y)) - V^{R_i}(x', (y, s_y)) = V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0))$.

Suppose $V^{R_i}(\mathbf{0}, (x', t')) < V^{R_i}(\mathbf{0}, (y, s_y))$. Let $s' \equiv V^{R_i}(\bar{x}, (x', t'))$. There are three cases.

Case 1. $s' < d(x')$. By the definition of R_i, $t' = V^{R_i}(x', (\bar{x}, s')) < 0$. This contradicts the fact that $(x', t') \in X(R_i)$.

Case 2. $s' = d(x')$. By the definition of R_i, $t' = V^{R_i}(x', (\bar{x}, s')) = \min\{V^{R_i}(x', (y, s_y)), 0\} \leq 0$. By $t' \geq 0$, $t' = 0$ and $V^{R_i}(x', (y, s_y)) \geq 0$. Thus,

$$V^{R_i}(\bar{x}, (x', t')) - t' = d(x') = V^{R_i}(\bar{x}, (y, s_y)) - V^{R_i}(x', (y, s_y)) = V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0)).$$

Case 3. $s' > d(x')$. By $V^{R_i}(\mathbf{0}, (x', t')) < V^{R_i}(\mathbf{0}, (y, s_y))$, $s' < V^{R_i}(\bar{x}, (y, s_y))$. Thus, by $s' > d(x')$, $d(x') < V^{R_i}(\bar{x}, (y, s_y))$, which implies $V^{R_i}(x', (y, s_y)) > 0$. Therefore,

$$\begin{aligned} V^{R_i}(\bar{x}, (x', t')) - t' &= s' - (s' - d(x')) \\ &= V^{R_i}(\bar{x}, (y, s_y)) - V^{R_i}(x', (y, s_y)) \\ &= V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x', (\mathbf{0}, 0)). \end{aligned}$$

■

F.4 Proof of Lemma 5

We do the proof in three steps.

Step 1. *Constructing a preference relation.*

We define a preference relation R_i as follows. Let $t^* \in \mathbb{R}$ be such that $t^* < \min\{t_{\mathbf{0}}, s_{\mathbf{0}}\}$. Let $(\epsilon_x)_{x \in X} \in \mathbb{R}_+^{|X|}$ be an object monotonic vector such that $\epsilon_{\mathbf{0}} = 0$ and for each $x \in X \setminus \{\mathbf{0}\}$, $\epsilon_x > 0$ but sufficiently close to 0.[41] In particular, we take $(\epsilon_x)_{x \in X}$ so that it satisfies $s_{\mathbf{0}} + \epsilon_{\bar{x}} < 0$. For each $x \in X \setminus \{\bar{x}\}$, let

$$V^{R_i}(x, (\bar{x}, t_{\bar{x}})) = t^* + \epsilon_x.$$

For each $x \in X \setminus \{\mathbf{0}\}$, let

$$V^{R_i}(x, (\mathbf{0}, s_{\mathbf{0}})) = \begin{cases} s_{\mathbf{0}} + \epsilon_x & \text{if } x \neq \bar{x}, \\ \max\{t_{\bar{x}}, s_{\mathbf{0}}\} + \epsilon_{\bar{x}} & \text{if } x = \bar{x}. \end{cases}$$

[41] Formally, the proof works if $(\epsilon_x)_{x \in X} \in \mathbb{R}_+^{|X|}$ is an object monotonic vector that satisfies $\epsilon_{\mathbf{0}} = 0$ and $\epsilon_{\bar{x}} < \min\{t_{\mathbf{0}} - t^*, \min_{x \in X \setminus \{\mathbf{0}\}} s_x - s_{\mathbf{0}}, s_{\bar{x}} - t_{\bar{x}}, -s_{\mathbf{0}}\}$.

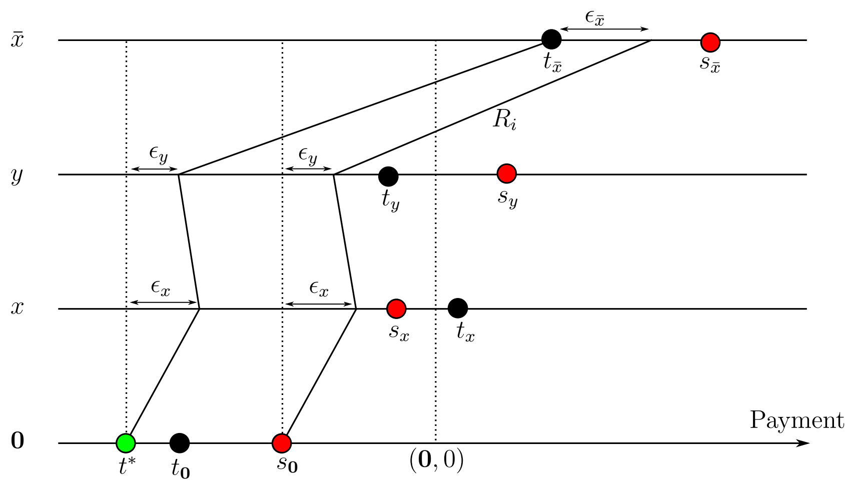

Figure 25: An illustration of R_i.

For each $t' \in \mathbb{R}$ with $t^* < t' < s_{\mathbf{0}}$, and each $x \in X$, let

$$V^{R_i}(x, (\mathbf{0}, t')) = \alpha \cdot V^{R_i}(x, (\mathbf{0}, t^*)) + (1 - \alpha) V^{R_i}(x, (\mathbf{0}, s_{\mathbf{0}})),$$

where $\alpha \in [0, 1]$ is such that $t' = \alpha \cdot t^* + (1 - \alpha) s_{\mathbf{0}}$.

Finally, for each $t' \in \mathbb{R} \setminus [t^*, s_{\mathbf{0}}]$ and each $x \in X$, let

$$V^{R_i}(x, (\mathbf{0}, t')) = \begin{cases} V^{R_i}(x, (\mathbf{0}, t^*)) - (t^* - t') & \text{if } t' < t^*, \\ V^{R_i}(x, (\mathbf{0}, s_{\mathbf{0}})) + (t' - s_{\mathbf{0}}) & \text{if } t' > s_{\mathbf{0}}. \end{cases}$$

Figure 25 illustrates R_i. Note that by the construction of R_i, R_i is bounded. For each $x \in X \setminus \{\bar{x}\}$, since $t^* < t_{\mathbf{0}}$ and ϵ_x is sufficiently close to 0, $V^{R_i}(x, (\bar{x}, t_{\bar{x}})) = t^* + \epsilon_x < t_x$. Thus, $R_i \in \mathcal{R}^{MT}_{\mathbf{t}, \bar{x}}$. Since $t_{\bar{x}} < s_{\bar{x}}$ and $\epsilon_{\bar{x}}$ is sufficiently close to 0, $V^{R_i}(\bar{x}, (\mathbf{0}, s_{\mathbf{0}})) = \max\{t_{\bar{x}}, s_{\mathbf{0}}\} + \epsilon_{\bar{x}} < s_{\bar{x}}$. Further, for each $x \in X \setminus \{\mathbf{0}, \bar{x}\}$, since ϵ_x is sufficiently close to 0, $V^{R_i}(x, (\mathbf{0}, s_{\mathbf{0}})) = s_{\mathbf{0}} + \epsilon_x < s_x$. Thus, $R_i \in \mathcal{R}^{MT}_{\mathbf{s}, \mathbf{0}}$.

Step 2. R_i *is object monotonic.*

Proof. To show R_i is object monotonic, we only need to prove the object monotonicity of $(V^{R_i}(x, (\bar{x}, t_x)))_{x \in X}$ and $(V^{R_i}(x, (\mathbf{0}, s_{\mathbf{0}})))_{x \in X}$. Let $(x, y) \in \mathcal{X}$.

First we show that $(V^{R_i}(x', (\bar{x}, t_x)))_{x' \in X}$ is object monotonic. If $x = \bar{x}$, then since $t^* < t_{\bar{x}}$ and ϵ_y is sufficiently close to 0, $V^{R_i}(\bar{x}, (\bar{x}, t_{\bar{x}})) = t_{\bar{x}} > t^* + \epsilon_y = V^{R_i}(y, (\bar{x}, t_{\bar{x}}))$.

Suppose $x \neq \bar{x}$. Then, since $(\epsilon_{x'})_{x' \in X}$ is object monotonic, $V^{R_i}(x, (\bar{x}, t_{\bar{x}})) = t^* + \epsilon_x > t^* + \epsilon_y = V^{R_i}(y, (\bar{x}, t_{\bar{x}}))$. Hence, $(V^{R_i}(x', (\bar{x}, t_x)))_{x' \in X}$ is object monotonic.

Next, we show that $(V^{R_i}(x', (\mathbf{0}, s_{\mathbf{0}})))_{x' \in X}$ is object monotonic. Note that $V^{R_i}(x, (\mathbf{0}, s_{\mathbf{0}})) \geq s_{\mathbf{0}} + \epsilon_x$, and since $y \neq \bar{x}$, $V^{R_i}(y, (\mathbf{0}, s_{\mathbf{0}})) = s_{\mathbf{0}} + \epsilon_y$. Thus, since $(\epsilon_{x'})_{x' \in X}$ is object monotonic, $V^{R_i}(x, (\mathbf{0}, s_{\mathbf{0}})) > V^{R_i}(y, (\mathbf{0}, s_{\mathbf{0}}))$. Therefore, $(V^{R_i}(x', (\mathbf{0}, s_{\mathbf{0}})))_{x' \in X}$ is object monotonic, and hence, R_i is object monotonic. ■

Step 3. $R_i \in \mathcal{R}^P$.

Note that by $s_{\mathbf{0}} < 0$, for each $x \in X$, $V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x, (\mathbf{0}, 0)) = V^{R_i}(\bar{x}, (\mathbf{0}, s_{\mathbf{0}})) - V^{R_i}(x, (\mathbf{0}, s_{\mathbf{0}}))$. Let $(x, t') \in X(R_i)$. By Remark 4, it is enough to show $V^{R_i}(\bar{x}, (x, t')) - t' = V^{R_i}(\bar{x}, (\mathbf{0}, 0)) - V^{R_i}(x, (\mathbf{0}, 0))$. Without loss of generality, assume $x \neq \bar{x}$. Let $s' \equiv V^{R_i}(\mathbf{0}, (x, t'))$.

By $(x, t') \in X(R_i)$, $t' \geq 0$. Since $s_{\mathbf{0}} < 0$ and ϵ_x is sufficiently close to 0, $V^{R_i}(x, (\mathbf{0}, s_{\mathbf{0}})) = s_{\mathbf{0}} + \epsilon_x < 0 \leq t'$. Thus, $s' > s_{\mathbf{0}}$. Therefore, by the definition of R_i,

$$\begin{aligned} V^{R_i}(\bar{x}, (x, t')) - t' &= V^{R_i}(\bar{x}, (\mathbf{0}, s')) - V^{R_i}(x, (\mathbf{0}, s')) \\ &= V^{R_i}(\bar{x}, (\mathbf{0}, s_{\mathbf{0}})) + (s' - s_{\mathbf{0}}) - (V^{R_i}(x, (\mathbf{0}, s_{\mathbf{0}})) + (s' - s_{\mathbf{0}})) \\ &= V^{R_i}(\bar{x}, (\mathbf{0}, s_{\mathbf{0}})) - V^{R_i}(x, (\mathbf{0}, s_{\mathbf{0}})) \\ &= V^{R_i}(\bar{x}, (\mathbf{0}, s_{\mathbf{0}})) - V^{R_i}(x, (\mathbf{0}, s_{\mathbf{0}})). \end{aligned}$$

Hence, $R_i \in \mathcal{R}^P$. ■

F.5 Proof of Lemma 6

Suppose by contradiction that $\sum_{i \in N'} V^{R_i}(x_i, f_i(R)) > \sum_{i \in N'} t_i^f(R)$. Let $((y_i, s_i))_{i \in N} \in Z$ be such that for each $i \in N$

$$(y_i, s_i) = \begin{cases} (x_i, V^{R_i}(x_i, f_i(R))) & \text{if } i \in N', \\ f_i(R) & \text{otherwise.} \end{cases}$$

It is clear that for each $i \in N$, $(y_i, s_i)\ I_i\ f_i(R)$. Moreover,

$$\sum_{i \in N} s_i = \sum_{i \in N'} V^{R_i}(x_i, f_i(R)) + \sum_{i \in N \setminus N'} t_i^f(R) > \sum_{i \in N} t_i^f(R).$$

By Remark 3, this contradicts *efficiency*. ■

F.6 Proof of Lemma 7

Suppose by contradiction that $x_i^f(R) \neq \mathbf{0}$ and there is $j \in N \setminus \{i\}$ such that $R_i \in \mathcal{R}^Q(R_j)$. Denote $x \equiv x_i^f(R) + x_j^f(R)$. By $x_i^f(R) \neq \mathbf{0}$, $x > x_j^f(R)$. By $R_i \in \mathcal{R}^Q(R_j)$,

$$v_i(x_i^f(R)) < V^{R_j}(x, f_j(R)) - t_j^f(R).$$

Since $V^{R_i}(\mathbf{0}, f_i(R)) = t_i^f(R) - v_i(x_i^f(R))$,

$$\begin{aligned} V^{R_i}(\mathbf{0}, f_i(R)) + V^{R_j}(x, f_j(R)) &= t_i^f(R) - v_i(x_i^f(R)) + V^{R_j}(x, f_j(R)) \\ &> t_i^f(R) + t_j^f(R). \end{aligned}$$

This contradicts Lemma 6. ■

F.7 Proof of Lemma 8

Since R_j is bounded for each $j \in N \setminus \{i\}$, there is a pair $\bar{s}, \underline{s} \in \mathbb{R}_{++}$ such that for each $j \in N \setminus \{i\}$, each $(x, y) \in \mathcal{X}$, and each $t \in \mathbb{R}$,

$$\underline{s} < V^{R_j}(x, (y, t)) - t < \bar{s}.$$

Let $(\epsilon_x)_{x \in X} \in \mathbb{R}_+^{|X|}$ be an object monotonic vector such that $\epsilon_{\bar{x}} < \underline{s}$. Let $x \in X$. We show that there is $R_i \in \mathcal{R}^Q$ such that $x_i(R_i, R_{-i}) = x$.

Let $R_i \in \mathcal{R}^Q$ be such that for each $y \in X \setminus \{\mathbf{0}\}$,

$$v_i(y) = \begin{cases} n \cdot \bar{s} + \epsilon_y & \text{if } y \geq x, \\ \epsilon_y & \text{otherwise.} \end{cases}$$

Since $(\epsilon_x)_{x \in X}$ is object monotonic, R_i is object monotonic. For simplicity, denote $R \equiv (R_i, R_{-i})$. Suppose by contradiction that $x_i^f(R) \neq x$. There are two cases.

Case 1. $x_i^f(R) > x$. Take any $j \in N \setminus \{i\}$. Denote $y \equiv x_j^f(R) + (x_i^f(R) - x)$. Then, $y + x = x_i^f(R) + x_j^f(R)$, and by $x_i^f(R) > x$, $y > x_j^f(R)$. By the definition of $(\epsilon_{y'})_{y' \in X}$,

$$\begin{aligned} V^{R_i}(x, f_i(R)) + V^{R_j}(y, f_j(R)) &= t_i^f(R) + v_i(x) - v_i(x_i^f(R)) + V^{R_j}(y, f_j(R)) - t_j^f(R) + t_j^f(R) \\ &> t_i^f(R) + \epsilon_x - \epsilon_{x_i^f(R)} + \underline{s} + t_j^f(R) \\ &> t_i^f(R) + t_j^f(R), \end{aligned}$$

which contradicts Lemma 6.

Case 2. $x_i^f(R) \not\gtrsim x$. By $x_i^f(R) \neq \bar{x}$, $v_i(\bar{x}) - v_i(x_i^f(R)) = n \cdot \bar{s} + \epsilon_{\bar{x}} - \epsilon_{x_i^f(R)} > n \cdot \bar{s}$. Thus,

$$\begin{aligned} V^{R_i}(\bar{x}, f_i(R)) + \sum_{j \in N \setminus \{i\}} V^{R_j}(\mathbf{0}, f_j(R)) \\ &= t_i^f(R) + v_i(\bar{x}) - v_i(x_i^f(R)) + \sum_{j \in N \setminus \{i\}} (V^{R_j}(\mathbf{0}, f_j(R)) - t_j^f(R) + t_j^f(R)) \\ &> n \cdot \bar{s} - (n-1) \cdot \bar{s} + \sum_{j \in N} t_j^f(R) \\ &> \sum_{j \in N} t_j^f(R), \end{aligned}$$

which contradicts Lemma 6. ■

F.8 Proof of Lemma 9

Let $(x, y) \in \mathcal{X}$. By $X_i^f(R_{-i}) = X$, there are $R_i, R_i' \in \mathcal{R}$ such that $x_i^f(R_i, R_{-i}) = x$ and $x_i^f(R_i', R_{-i}) = y$. By *strategy-proofness*, $f_i(R_i', R_{-i})\ R_i'\ f_i(R_i, R_{-i})$. This implies $V^{R_i'}(x, f_i(R_i', R_{-i})) \leq t_i(R_i, R_{-i})$. By this and the object monotonicity of R_i',

$$t_i^f(R_{-i}; y) = t_i^f(R_i', R_{-i}) < V^{R_i'}(x, f_i(R_i', R_{-i})) \leq t_j^f(R_i', R_{-i}) = t_j^f(R_{-i}; x).$$

Thus, $(t_i^f(R_{-i}; x'))_{x' \in X}$ is object monotonic. ■

F.9 Proof of Lemma 11

We prove only (ii), because we can prove (i) by setting $s = 0$ and following the proof of (ii).

Without loss of generality, assume $i = 1$ and $j = 2$. By $R_3, \ldots, R_n \in \mathcal{R}^Q$ and Lemma 8, for each $R_2' \in \mathcal{R}^Q$, $X_1^f(R_2', R_{-1,2}) = X$. By Lemma 2, for each $R_2' \in \mathcal{R}^Q$ with $R_3, \ldots, R_n \in \mathcal{R}^Q(R_2')$ and each $x \in X$, $\sigma_1(R_2', R_{-1,2}; x) = v_2'(\bar{x} - x)$. Thus, by $R_3, \ldots, R_n \in \mathcal{R}^Q$ and Fact 4, there is $h_1 : (\mathcal{R}^Q)^{n-1} \to \mathbb{R}$ such that for each $R_2' \in \mathcal{R}^Q$ with $R_3, \ldots, R_n \in \mathcal{R}^Q(R_2')$, and each $x \in X$, $t_1^f(R_2', R_{-1,2}; x) = h_1(R_2', R_{-1,2}) - v_2'(\bar{x} - x)$. This implies that for each $R_2' \in \mathcal{R}^Q$ with $R_3, \ldots, R_n \in \mathcal{R}^Q(R_2')$, and each pair

$x, y \in X$

$$t_1^f(R_2', R_{-1,2}; x) = h_1(R_2', R_{-1,2}) - v_2'(\bar{x} - x) + v_2'(\bar{x} - y) - v_2'(\bar{x} - y)$$
$$= t_1^f(R_2', R_{-1,2}; y) + v_2'(\bar{x} - y) - v_2'(\bar{x} - x). \tag{7}$$

Take any $a \in M$ and let $e^a = (e_1^a, \ldots, e_m^a) \in X$ be such that for each $\ell \in M$, $e_\ell^a = 1$ if $\ell = a$ and $e_\ell^a = 0$ otherwise. Let $x \equiv \bar{x} - e^a$. Let

$$\mathcal{R}^* = \{R_2' \in \mathcal{R}^Q : \text{for each } (y, y') \in \mathcal{X}, v_2'(y) - v_2'(y') > v_2(y) - v_2(y')\}.$$

Note that since $R_3, \ldots, R_n \in \mathcal{R}^Q(R_2)$, for each $R_2' \in \mathcal{R}^*$, $R_3, \ldots, R_n \in \mathcal{R}^Q(R_2')$.

Let $\mathbf{t} \in \mathbb{R}^{|X|}$ be such that for each $y \in X$, $t_y = t_1^f(R_{-1}; y)$. By $X_1^f(R_{-1}) = X$ and Lemma 9, $\mathbf{t}$ is object monotonic.

Step 1. $t_{\mathbf{0}} \geq -s^*$.

Proof. Suppose by contradiction that $t_{\mathbf{0}} < -s^*$. We first show that we can assume $t_{e^a} < 0$ without loss of generality. To show this, we prove that there is $R_2^* \in \mathcal{R}^*$ such that $t_1^f(R_2^*, R_{-1,2}; \mathbf{0}) < -s^*$ and $t_1^f(R_2^*, R_{-1,2}; e^a) < 0$.

Let $R_2^* \in \mathcal{R}^Q$ be such that $v_2^*(\bar{x}) > v_2(\bar{x})$ and $v_2^*(\bar{x}) - v_2^*(x) < -t_{\mathbf{0}}$. Let $\mathbf{s} \in \mathbb{R}^{|X|}$ be such that for each $y \in X$, $s_y = t_1^f(R_2^*, R_{-1,2}; y)$. We first prove $s_{\mathbf{0}} \leq t_{\mathbf{0}}$.

Suppose by contradiction that $s_{\mathbf{0}} > t_{\mathbf{0}}$. By $t_{\mathbf{0}} < 0$, $\mathbf{t}$ and $\mathbf{s}$ satisfy the condition of Lemma 4 for $\mathbf{0}$. Thus, by Lemma 4, there is $R_1' \in \mathcal{R}^P$ such that $R_1' \in \mathcal{R}^{MT}_{\mathbf{t},\mathbf{0}} \cap \mathcal{R}^{MT}_{\mathbf{s},\bar{x}}$. By Lemma 10, $x_1^f(R_1', R_{-1}) = \mathbf{0}$ and $x_1^f(R_1', R_2^*, R_{-1,2}) = \bar{x}$. Then, $x_2^f(R_1', R_2^*, R_{-1,2}) = \mathbf{0}$. Further, by $R_3, \ldots, R_n \in \mathcal{R}^Q(R_2)$, Lemma 7 implies that for each $j \in N \setminus \{1, 2\}$, $x_j^f(R_1', R_{-1}) = \mathbf{0}$. This implies $x_2^f(R_1', R_{-1}) = \bar{x}$. However, by $v_2^*(\bar{x}) > v_2(\bar{x})$, this contradicts Fact 3. Hence, $s_{\mathbf{0}} \leq t_{\mathbf{0}}$.

By $t_{\mathbf{0}} < -s^*$ and $s_{\mathbf{0}} < -s^*$. Further, by $s_{\mathbf{0}} \leq t_{\mathbf{0}} < 0$, $v_2^*(\bar{x}) - v_2^*(x) < -t_{\mathbf{0}}$, and (7),

$$s_{e^a} = s_{\mathbf{0}} + v_2^*(\bar{x}) - v_2^*(x) < s_{\mathbf{0}} - t_{\mathbf{0}} \leq 0.$$

Hence, we can assume $t_{e^a} < 0$ without loss of generality.

Let $R_2', R_2'' \in \mathcal{R}^*$ be such that

$$v_2''(x) < v_2'(x) \text{ and } v_2'(\bar{x}) < v_2''(\bar{x}).$$

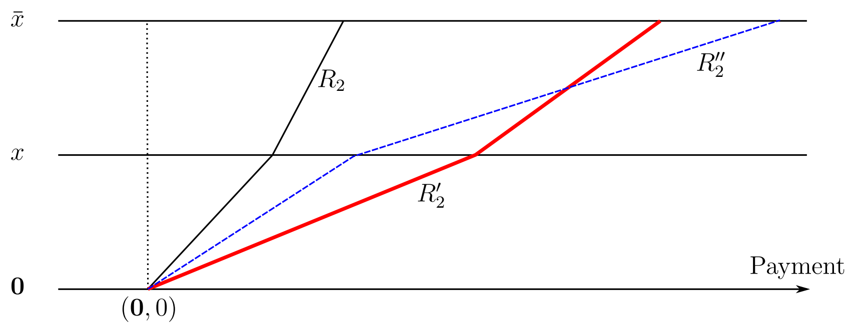

Figure 26: An illustration of R_2, R_2', and R_2''.

Figure 26 illustrates R_2, R_2', and R_2''. Let $\mathbf{s}', \mathbf{s}'' \in \mathbb{R}^{|X|}$ be such that for each $y \in X$, $s_y' = t_1^f(R_2', R_{-1,2}; y)$ and $s_y'' = t_1^f(R_2'', R_{-1,2}; y)$, respectively. By $X_1^f(R_2', R_{-1,2}) = X_1^f(R_2'', R_{-1,2}) = X$ and Lemma 9, $\mathbf{s}'$ and $\mathbf{s}''$ are object monotonic.

Note that by $R_2', R_2'' \in \mathcal{R}^*$, $R_3, \ldots, R_n \in \mathcal{R}^Q(R_2) \cap \mathcal{R}^Q(R_2') \cap \mathcal{R}^Q(R_2'')$. Thus, for each $R_1' \in \mathcal{R}$ and each $i \in N \setminus \{1, 2\}$, $x_i^f(R_1', R_{-1}) = x_i^f(R_1', R_2', R_{-1,2}) = x_i^f(R_1', R_2'', R_{-1,2}) = \mathbf{0}$.

Claim 1. $s_{e^a}' = s_{e^a}''$.

Proof. To complete the proof, it is enough to show $s_{e^a}' = t_{e^a}$ and $s_{e^a}'' = t_{e^a}$. We focus only on the proof of $s_{e^a}' = t_{e^a}$ because the same argument holds for $s_{e^a}'' = t_{e^a}$. Suppose by contradiction that $s_{e^a}' \neq t_{e^a}$. There are two cases.

Case 1. $s_{e^a}' < t_{e^a}$. By $t_{e^a} < 0$, $s_{e^a}' < 0$. Thus, $\mathbf{t}$ and $\mathbf{s}'$ satisfy the condition of Lemma 4 for e^a. Therefore, by Lemma 4, there is $R_1' \in \mathcal{R}^P$ such that $R_1' \in \mathcal{R}_{\mathbf{t},\mathbf{0}}^{MT} \cap \mathcal{R}_{\mathbf{s}',e^a}^{MT}$. By Lemma 10, $x_1^f(R_1', R_{-1}) = \mathbf{0}$ and $x_1^f(R_1', R_2', R_{-1,2}) = e^a$. Thus, $x_2^f(R_1', R_{-1}) = \bar{x}$ and $x_2^f(R_1', R_2', R_{-1,2}) = x$. However, by $R_2' \in \mathcal{R}^*$, $v_2'(\bar{x}) - v_2'(x) > v_2(\bar{x}) - v_2(x)$. This contradicts Fact 3.

Case 2. $s_{e^a}' > t_{e^a}$. By $t_{e^a} < 0$, $\mathbf{t}$ and $\mathbf{s}'$ satisfy the condition of Lemma 4 for e^a. Thus, by Lemma 4, there is $R_1' \in \mathcal{R}^P$ such that $R_1' \in \mathcal{R}_{\mathbf{t},e^a}^{MT} \cap \mathcal{R}_{\mathbf{s}',\bar{x}}^{MT}$. By Lemma 10, $x_1^f(R_1', R_{-1}) = e^a$ and $x_1^f(R_1', R_2', R_{-1,2}) = \bar{x}$. Thus, $x_2^f(R_1', R_{-1}) = x$ and $x_2^f(R_1', R_2', R_{-1,2}) = \mathbf{0}$. However, by $R_2' \in \mathcal{R}^*$, $v_2'(x) > v_2(x)$. This contradicts

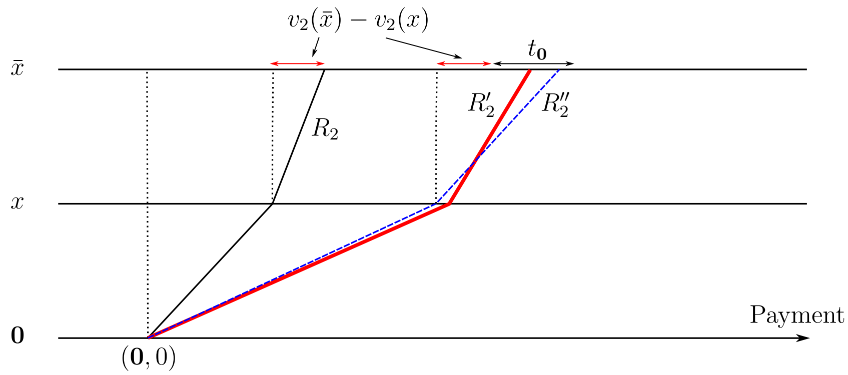

Figure 27: An illustration of R_2, R_2', and R_2''.

Fact 3. □

By Claim 1, (7), and the definitions of R_2' and R_2'',

$$s''_{\bar{x}} = s''_{e^a} + v''_2(x) < s'_{e^a} + v'_2(x) = s'_{\bar{x}}.$$

By $s'_{e^a} = t_{e^a} < 0$ and the object monotonicity of $\mathbf{s}'$, $s'_{\mathbf{0}} < 0$. Thus, $\mathbf{s}'$ and $\mathbf{s}''$ satisfy the condition of Lemma 5. Therefore, by Lemma 5, there is $R_1' \in \mathcal{R}^P$ such that $R_1' \in \mathcal{R}^{MT}_{\mathbf{s}',\mathbf{0}} \cap \mathcal{R}^{MT}_{\mathbf{s}'',\bar{x}}$. By Lemma 10, $x_1^f(R_1', R_2', R_{-1,2}) = \mathbf{0}$ and $x_1^f(R_1', R_2'', R_{-1,2}) = \bar{x}$. Thus, $x_2^f(R_1', R_2', R_{-1,2}) = \bar{x}$ and $x_2^f(R_1'', R_2', R_{-1,2}) = \mathbf{0}$. However, by the definitions of R_2'' and R_2'', $v_2'(\bar{x}) < v_2''(\bar{x})$, which contradicts Fact 3. ■

Step 2. $t_{\mathbf{0}} \leq 0$.

Proof. Suppose by contradiction that $t_{\mathbf{0}} > 0$. Let $R_2', R_2'' \in \mathcal{R}^*$ be such that

$$v''_2(x) < v'_2(x) \text{ and } v'_2(\bar{x}) < v''_2(\bar{x}) < v''_2(x) + (v_2(\bar{x}) - v_2(x) + t_{\mathbf{0}}).$$

Note that we can define such preferences since $t_{\mathbf{0}} > 0$. Note also that $v'_2(\bar{x}) - v'_2(x) < v''_2(\bar{x}) - v''_2(x)$. Figure 27 is an illustration of R_2, R_2', and R_2''. Let $\mathbf{s}', \mathbf{s}'' \in \mathbb{R}^{|X|}$ be such that for each $y \in X$, $s'_y = t_1^f(R_2', R_{-1,2}; y)$ and $s''_y = t_1^f(R_2'', R_{-1,2}; y)$. By $X_1^f(R_2', R_{-1,2}) = X_1^f(R_2'', R_{-1,2}) = X$ and Lemma 9, $\mathbf{s}'$ and $\mathbf{s}''$ are object monotonic.

Notice that by $R_2', R_2'' \in \mathcal{R}^*$, $R_3, \ldots, R_n \in \mathcal{R}^Q(R_2) \cap \mathcal{R}^Q(R_2') \cap \mathcal{R}^Q(R_2'')$. Thus, for each $R_1' \in \mathcal{R}$ and each $i \in N \setminus \{1, 2\}$, $x_i^f(R_1', R_{-1}) = x_i^f(R_1', R_2', R_{-1,2}) = x_i^f(R_1', R_2'', R_{-1,2}) = \mathbf{0}$.

Claim 1. $s'_{e^a} = s''_{e^a} = t_{e^a}$.

Proof. We only prove $s'_{e^a} = t_{e^a}$ because the same argument holds for $s''_{e^a} = t_{e^a}$. Suppose by contradiction that $s'_{e^a} \neq t_{e^a}$. There are two cases.

Case 1. $s'_{e^a} < t_{e^a}$. If $s'_{e^a} \geq 0$, $\mathbf{t}$ and $\mathbf{s}'$ satisfy the condition of Lemma 3 for e^a. Further, if $s'_{e^a} < 0$, then $\mathbf{t}$ and $\mathbf{s}'$ satisfy the condition of Lemma 4 for e^a. Thus, by Lemmas 3 and 4, there is $R'_1 \in \mathcal{R}$ such that $R'_1 \in \mathcal{R}^{MT}_{\mathbf{t},\mathbf{0}} \cap \mathcal{R}^{MT}_{\mathbf{s}',e^a}$. By Lemma 10, $x^f_1(R'_1, R_{-1}) = \mathbf{0}$ and $x^f_1(R'_1, R'_2, R_{-1,2}) = e^a$. Thus, $x^f_2(R'_1, R_{-1}) = \bar{x}$ and $x^f_2(R'_1, R'_2, R_{-1,2}) = x$. However, by $R'_2 \in \mathcal{R}^*$, $v'_2(\bar{x}) - v'_2(x) > v_2(\bar{x}) - v_2(x)$, which contradicts Fact 3.

Case 2. $s'_{e^a} > t_{e^a}$. By the definitions of R'_2 and R''_2, $v'_2(\bar{x}) - v'_2(x) < v''_2(\bar{x}) - v''_2(x) < v_2(\bar{x}) - v_2(x) + t_\mathbf{0}$. Thus, by (7),

$$s'_\mathbf{0} = s'_{e^a} - (v'_2(\bar{x}) - v'_2(x)) > t_{e^a} - (v_2(\bar{x}) - v_2(x) + t_\mathbf{0}) = t_\mathbf{0} - t_\mathbf{0} = 0.$$

By $t_\mathbf{0} > 0$ and the object monotonicity of $\mathbf{t}$, $t_{e^a} > 0$. Thus, $\mathbf{t}$ and $\mathbf{s}'$ satisfy the condition of Lemma 3 for e^a. Therefore, by Lemma 3, there is $R'_1 \in \mathcal{R}^P$ such that $R'_1 \in \mathcal{R}^{MT}_{\mathbf{t},e^a} \cap \mathcal{R}^{MT}_{\mathbf{s}',\bar{x}}$. By Lemma 10, $x^f_1(R'_1, R_{-1}) = e^a$ and $x^f_1(R'_1, R'_2, R_{-1,2}) = \bar{x}$. Thus, $x^f_2(R'_1, R_{-1}) = x$ and $x^f_2(R'_1, R'_2, R_{-1,2}) = \mathbf{0}$. However, by $R'_2 \in \mathcal{R}^*$, $v'_2(x) > v_2(x)$. This contradicts Fact 3. □

By Claim 1, (7), and $v''_2(x) < v'_2(x)$,

$$s''_{\bar{x}} = s''_{e^a} + v''_2(x) < s'_{e^a} + v'_2(x) = s'_{\bar{x}}.$$

By Claim 1, $t_\mathbf{0} > 0$, and the object monotonicity of $\mathbf{t}$ and $\mathbf{s}''$,

$$s''_{\bar{x}} > s''_{e^a} = t_{e^a} > t_\mathbf{0} > 0.$$

Moreover, by Claim 1, (7), and the definition of R'_2,

$$s'_\mathbf{0} = s'_{e^a} - (v'_2(\bar{x}) - v'_2(x)) > t_{e^a} - (v_2(\bar{x}) - v_2(x) - t_\mathbf{0}) = t_\mathbf{0} - t_\mathbf{0} = 0.$$

Thus, $\mathbf{s}'$ and $\mathbf{s}''$ satisfy the condition of Lemma 3 for $\bar{x}$. Therefore, by Lemma 3, there is $R'_1 \in \mathcal{R}^P$ such that $R'_1 \in \mathcal{R}^{MT}_{\mathbf{s}',\mathbf{0}} \cap \mathcal{R}^{MT}_{\mathbf{s}'',\bar{x}}$. By Lemma 10, $x^f_1(R'_1, R'_2, R_{-1,2}) = \mathbf{0}$

and $x_1^f(R_1', R_2'', R_{-1,2}) = \bar{x}$. Thus, $x_2^f(R_1', R_2', R_{-1,2}) = \bar{x}$ and $x_2^f(R_1', R_2'', R_{-1,2}) = \mathbf{0}$. However, by the definitions of R_2' and R_2'', $v_2'(\bar{x}) > v_2''(\bar{x})$. This contradicts Fact 3. ■

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