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TRACT HOUSING, THE CORE, AND PENDULUM AUCTIONS

Andrew Mackenzie

Yu Zhou

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Graduate School of Economics
Kyoto University
Yoshida-Hommachi, Sakyo-ku
Kyoto City, 606-8501, Japan

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Tract housing, the core, and pendulum auctions

Andrew Mackenzie* and Yu Zhou†‡

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Abstract

We consider a model of tract housing where buyers and sellers have (i) wealth constraints, and (ii) unit demand over identical indivisible objects represented by a valuation. First, we characterize the strong core. Second, we characterize the bilateral weak core, or the weak core allocations with no side-payments. Finally, when buyer wealth constraints and valuations are private information and when transfers are discrete, we introduce two families of *pendulum auctions*, both of which consist of obviously strategy-proof selections of the bilateral weak core. The *buyer-optimal pendulum auctions* are preferred by the buyers but are inefficient when side-payments are possible, while the *efficient pendulum auctions* are efficient.

Keywords: tract housing, core, pendulum auction, almost-synchronized equilibrium, private wealth constraints, efficiency, obvious strategy-proofness

JEL Codes: C72 (Noncooperative Games), D41 (Perfect Competition), D47 (Market Design), D82 (Asymmetric and Private Information • Mechanism Design)

*Department of Microeconomics and Public Economics, Maastricht University, Maastricht, the Netherlands. Email: a.mackenzie@maastrichtuniversity.nl

†Graduate School of Economics, Kyoto University, Kyoto, Japan. Email: zhouyu_0105@hotmail.com

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1 Introduction

1.1 Overview

During the second half of the twentieth century, the proportion of the United States population living in the suburbs increased from less than one quarter to 50% (U.S. Census Bureau, 2002). As argued by Lane (2015), this phenomenon was driven by entrepreneurial developers building large numbers of *tract houses* during the postwar years in response to high demand, which in turn was driven by the increasing birthrate, the new highway system, low-interest government loans, and persistent prosperity for the working class after World War II. In this article, we analyze the modern day reallocation of these tract houses decades after they were built by modifying the classic housing model of Shapley and Shubik (1972) to accommodate the following two features: (i) the houses are identical, and (ii) a given buyer may be unable to afford a given price.

The model with only the first modification was originally considered by Böhm-Bawerk (1888) in the context of a horse market, and in our context this modification is of course a simplifying assumption: even at the time of construction, houses within a given neighborhood varied by distance from the street corner, indoor appliances, and color, and decades later they further vary based on level of maintenance and later additions to the original constructions. That said, the development of a given tract typically began with the builders constructing a handful of models from which the buyers would choose, then replicating these models in large numbers using techniques that facilitated economies of scale (Lane, 2015), and indeed it is common today for these extremely similar houses to be referred to as *cookie cutter houses*. For these reasons, we argue that this simplifying assumption is a first-order approximation that, particularly for the purposes of our mechanism design analysis, brings our model closer to reality.

The second modification complicates our analysis, and much of our technical contribution consists of addressing these complications. We argue that this modification is particularly important in the context of today's tract houses, as it allows us to distinguish ability to pay from willingness to pay. Indeed, though these houses were quite affordable at the time of their construction, the median inflation-adjusted house price nearly quadrupled between 1940 and 2000 (U.S. Census Bureau, 2000), and more recently the proportion of families in the United States that can afford a modestly priced home—either by paying with cash or by qualifying for a 30-year conventional mortgage with a five percent down payment—has been in decline (U.S. Census Bureau, 2013).

At a high level, we investigate the core when there is complete information and the implementation of the core when there is private information, for both (i) the *continuous model*, where money is infinitely divisible, and (ii) the *discrete model*, where money consists of identical indivisible coins. The latter model is particularly useful for mechanism design, as we design auction-like mechanisms for which the value of a coin is precisely one bid increment. In this case, a common technical assumption in the literature is that each agent's valuation is a multiple of the bid increment, and one of our significant technical contributions is to proceed without this assumption, allowing us to reasonably model auction-like mechanisms where the bid increment is sizable.

First, we show that for both the continuous model and the discrete model, (i) an allocation is in the strong core if and only if it is either a Walrasian equilibrium or efficient no-trade (Theorem 1), and (ii) the associated set of supporting prices forms an interval (Theorem 2). That said, for both models, the strong core may be empty.

Second, we introduce *almost-synchronized equilibria*, where (i) the agents face a common budget set whose price adjusts at a constant rate, and (ii) the agents make their selections within one second of each other. We show that for both the continuous model and the discrete model, (i) an allocation is in the bilateral weak core—that is, the set of weak core allocations with no side-payments—if and only if it is an almost-synchronized equilibrium (Theorem 3), and (ii) the associated set of supporting prices forms an interval (Theorem 4). It follows from our description of this interval that for both models, the bilateral weak core is always nonempty.

Finally, we investigate the implementation of the bilateral weak core in dominant strategies when the wealth constraints and preferences of the buyers are private information while the wealth constraints and preferences of the sellers are common knowledge. Since there is no strategy-proof selection of the weak core in the continuous model (Batziou, Bichler, and Fichtl, 2022; see also Example 7), we focus on the discrete model.

In order to implement the bilateral weak core, we design two families of mechanisms that we call *pendulum auctions*. For each auction in either family, the sellers are arranged from left to right, the buyers iteratively observe the posted prices of the sellers and decide whether to bid or exit, and unmatched sellers become matched from left to right while matched sellers become unmatched (while possibly increasing their posted prices) from right to left; this latter dynamic is what we mean to suggest with the term *pendulum*. The *buyer-optimal pendulum auctions* are preferred by the buyers but are inefficient when side-payments are possible, while the *efficient pendulum auctions* are efficient. Altogether, we find that

- each pendulum auction (i) is obviously strategy-proof (Theorem 5), (ii) selects *cutoff equilibria*, or bilateral weak core allocations that moreover respect the buyer priorities that are used to define the auction (Theorem 6), and (iii) selects allocations that are not strictly dominated for the buyers by any bilateral and individually rational allocation (Theorem 7);
- each buyer-optimal pendulum auction (i) selects cutoff equilibria that dominate all other cutoff equilibria for the buyers (Theorem 8), and (ii) selects allocations that satisfy *constrained efficiency*, or efficiency under the assumption that side-payments are not feasible (Theorem 9); and
- each efficient pendulum auction selects efficient allocations (Theorem 10).

It follows that our pendulum auctions provide two novel families of rules that generalize the minimum Walrasian price rules (Demange, 1982; Leonard, 1983) for economies with wealth constraints; thus in contrast to much of the literature on obviously strategy-proof implementation, we provide implementations of new rules instead of existing ones. Moreover, these new rules are in fact distinct from the minimum Walrasian price rules even when there are not wealth constraints (Example 10).

1.2 Literature

We consider a general equilibrium model where (i) there are identical indivisible objects and money, (ii) each agent owns at most one object, and (iii) each agent has quasi-linear preferences and unit demand. This is a simple model, and as such there are many related matching models and auction models in the literature that generalize along different dimensions—for example, the objects need not be identical, preferences need not be quasi-linear, agents may have multi-unit demand, and matching may be many-to-one or many-to-many—in various combinations. To remain focused, we discuss results from the literature without itemizing the combinations of these dimensions for which they generalize.

To organize our discussion, we consider the relationship of our paper to the literature on the following topics in sequence: (i) the core with wealth constraints, (ii) strategy-proof core selections without wealth constraints, and (iii) auctions with wealth constraints. For clarity, by wealth constraints we mean hard constraints representing the maximum that an agent can feasibly pay; these have also been referred to in the literature as financial constraints, budget constraints, and liquidity constraints.

The core with wealth constraints. For continuous transfers with wealth constraints when side-payments are feasible, under the assumption that no buyer would prefer to purchase an object at the maximum price that he can afford, we have that the weak core is nonempty, and moreover that it coincides with both the set of Walrasian equilibria and the strong core (Quinzii, 1984; Corollary 1). Without this assumption, we find that the strong core coincides with the set of Walrasian equilibria (Corollary 1) but may be empty, while the weak core is always nonempty and thus larger.

For continuous transfers with wealth constraints when side-payments are *not* feasible, three recent contributions investigate stable outcomes: (i) for one-to-one matching, Herings and Zhou (2022) introduce *quantity-constrained competitive equilibria* and show that they coincide with the stable outcomes; (ii) for many-to-one matching, Herings (2020) introduces *expectational equilibria* and shows that they coincide with the stable outcomes;[1] and (iii) for many-to-many matching with wealth constraints, Jagadeesan and Teytelboym (2022) provide conditions under which the *quasi-equilibria* of Debreu (1962) coincide with the stable outcomes. These three contributions include one-to-one matching as a special case for which the set of stable outcomes coincides with the bilateral weak core, and in this case the three solution concepts coincide with our *almost-synchronized equilibria*. The novelty of our contribution is that (i) our characterization allows for side-payments, (ii) our description involves considerably more detail involving supply and demand due to the additional structure of our model, and (iii) this more detailed description applies moreover to the discrete model.

Strategy-proof core selections without wealth constraints. For one-to-one matching without wealth constraints, there are striking similarities between the findings for assignment markets with continuous transfers and for marriage markets with no transfers. First, there are no strategy-proof core selections when there is private information on both sides of the market, both with continuous transfers (Myerson and Satterthwaite, 1983; Matsuo, 1989) and with no transfers (Roth, 1982). That said, there are strategy-proof core selections when there is only private information on one side of the market, both with continuous transfers (Demange, 1982; Leonard, 1983; Demange and Gale, 1985) and with no transfers (Dubins and Freedman, 1981; Roth, 1982), and indeed this is the approach we take in this paper.

Interestingly, the core has a lattice structure such that for each side there is a dominant core allocation, both with continuous transfers (Shapley and Shubik, 1972) and with no transfers (Knuth, 1976, attributed to John Conway), and moreover, each strategy-proof core selection always selects a dominant core allocation for the side with private informa-

[1]The result of Herings (2020) also applies when there are discrete transfers.

tion, both with continuous transfers (Holmström, 1979; Morimoto and Serizawa, 2015) and with no transfers (Alcalde and Barberà, 1994); these are the minimum Walrasian price rules (Demange, 1982; Leonard, 1983) and the deferred acceptance rule (Gale and Shapley, 1962), respectively. That said, in our model the bilateral weak core does not have have a dominant allocation for the buyers, and thus our two families of strategy-proof core selections provide two distinct methods for generalizing the idea of dominant core selection.

Auctions with wealth constraints. To begin, we remark that a large literature investigates auctions that maximize expected revenue as in Myerson (1981) when there are wealth constraints; for example, revenue equivalence does not hold with wealth constraints (Che and Gale, 1998), the order in which objects are sold matters with wealth constraints (Benoît and Krishna, 2001), and a modified all-pay auction can be optimal with wealth constraints (Pai and Vohra, 2014). That said, these contributions are not closely related to our paper because we do not investigate expected revenue maximization, and indeed we restrict the rest of this discussion to contributions about core selection, efficiency, and strategy-proofness.

For the continuous model, it was recently shown that there is no strategy-proof bilateral weak core selection (Batziou, Bichler, and Fichtl, 2022, see also Example 7), reinforcing an earlier impossibility result for the strategy-proof implementation of basic objectives when buyers have multi-unit demand (Dobzinski, Lavi, and Nisan, 2012).[2] As a result, though several auctions for models with continuous transfers and wealth constraints have been proposed (Maskin, 2000; Aggarwal, Muthukrishnan, Pál, and Pál, 2009; Ashlagi, Braverman, Hassidim, Lavi, and Tennenholtz, 2010; Le, 2018), none of these auctions is a strategy-proof core selection.

Though many of the proposed auctions for models with discrete transfers and wealth constraints violate either core selection or strategy-proofness (Ausubel and Milgrom, 2002; Talman and Yang, 2015; van der Laan and Yang, 2016; Zhou, 2017; van der Laan, Talman, and Yang, 2018), three proposals are in fact strategy-proof core selections. First, for matching with contracts, Hatfield and Milgrom (2005) introduce the *cumulative offer process*, which is a strategy-proof selection of stable outcomes and thus in the special case of one-to-one matching is a strategy-proof bilateral weak core selection; since their approach requires each agent to have strict preferences over all sets of contracts he might sign, it is not compatible with our model of multiple identical objects. Second, Milgrom and Segal (2020) introduce a large class of *deferred acceptance clock auctions*, which are obviously strategy-proof implementations that can be extended to accommodate a budget constraint of the auctioneer; by contrast, our pendulum auctions are obviously strategy-proof implementations that accommodate private wealth constraints of the buyers. Finally, allowing for heterogeneous objects but assuming that each valuation is a multiple of the bid increment and that all objects are owned by one seller, Yang and Yu (2022) introduce a strategy-proof and efficient weak core selection together with an associated ex-post Nash implementation through an extensive game form; our paper complements theirs by showing that if we assume all objects are identical, then we can (i) drop their first assumption, (ii) replace their second assumption with the assumption that there are multiple sellers who each own one object, and (iii) design obviously strategy-proof

[2]In response to the impossibility result of Dobzinski, Lavi, and Nisan (2012), several contributions have investigated domain restrictions under which the impossibility result does or does not persist (Ashlagi, Braverman, and Hassidim, 2009; Fiat, Leonardi, Saia, and Sankowski, 2011; Lavi and May, 2012).

implementations of efficient weak core selections.[3]

2 Complete information and the core

2.1 Economies and cooperative axioms

We begin by analyzing the case of complete information. In particular, we consider economies where there are two goods: (i) identical indivisible *objects*, of which each agent has unit demand; and (ii) *money*. We are interested in both *continuous* economies, where the set of possible monetary transfers $\mathbb{T}$ is $\mathbb{R}$, and *discrete* economies, where $\mathbb{T}$ is $\mathbb{Z}$; these are the economies where money is infinitely divisible and the economies where money consists of identical indivisible coins, respectively. In both cases, each agent is either a *buyer* endowed with no object or a *seller* endowed with one object. Moreover, each agent i has a (possibly infinite) *wealth constraint* w_i, as well as a standard quasi-linear preference relation represented by a *valuation* v_i.

Definition: For each set of possible monetary transfers $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, a $\mathbb{T}$-*economy* is a tuple $(N_0, N_1, (\theta_i)_{i \in N_0 \cup N_1})$, where

- N_0 is the nonempty and finite set of *buyers* and N_1 is the nonempty and finite set of *sellers*. Each $i \in N_0$ has an endowment of no object and a zero transfer of money, written $e_i = (0, 0)$, and each $i \in N_1$ has an endowment of one object and a zero transfer of money, written $e_i = (1, 0)$. We let $N \equiv N_0 \cup N_1$ denote the set of *agents*.

- For each $i \in N$, $W_i \equiv \{\infty\} \cup \{w_i \in \mathbb{T} | w_i \geq 0\}$ is the set of admissible *wealth constraints*, and

$$V_i = \begin{cases} \mathbb{R}, & \mathbb{T} = \mathbb{R}, \\ \frac{1}{2}\mathbb{Z} = \{..., -0.5, 0, 0.5, ...\}, & \mathbb{T} = \mathbb{Z} \end{cases}$$

  is the set of admissible *valuations*. It follows from the next item in this definition that (i) the assumption that wealth is non-negative is equivalent to the assumption that it is feasible to consume one's own endowment, and (ii) each valuation represents a preference relation. We use $\frac{1}{2}\mathbb{Z}$ instead of $\mathbb{R}$ for valuations in discrete economies simply to ensure that each preference relation is associated with a unique valuation.[4]

- For each $i \in N$, the *type* $\theta_i = (w_i, v_i)$ specifies (i) the *wealth constraint* $w_i \in W_i$, and (ii) the *valuation* $v_i \in V_i$. The wealth constraint w_i determines the set of possible *(monetary) transfers* $T_i \equiv \{t_i \in \mathbb{T} | t_i \geq -w_i\}$. Moreover, the set T_i determines the set of possible *bundles* $X_i \equiv \{0, 1\} \times T_i$, where each bundle $x_i = (a_i, t_i) \in X_i$ specifies *(object) assignment* a_i and transfer t_i. Finally, the valuation v_i represents

[3] If we impose that (i) the valuation is a multiple of the bid increment, (ii) there is one seller with one object, and (iii) there are no wealth constraints, then the auction of Yang and Yu (2022) coincides with both our efficient pendulum auctions and the minimum Walrasian price rules, but not with our buyer-optimal pendulum auctions. If we relax the first assumption, then (i) the auction of Yang and Yu (2022) is not well-defined, and (ii) the minimum Walrasian price rules, the buyer-optimal pendulum auctions, and the efficient pendulum auctions are all distinct (Example 10).

[4] In a discrete economy, for each $v_i \in \mathbb{Z}$ and each pair $\varepsilon, \varepsilon'$ between 0 and 1, $v_i + \varepsilon$ and $v_i + \varepsilon'$ represent the same preference relation; we simply take $v_i + 0.5$ as the canonical representation.

the associated *standard quasi-linear preference relation* $\succsim_i$ over X_i: for each pair $(a_i, t_i), (a'_i, t'_i) \in X_i$,

$$(a_i, t_i) \succsim_i (a'_i, t'_i) \text{ if and only if } v_i \cdot a_i + t_i \geq v_i \cdot a'_i + t'_i.$$

Informally, w_i is the ability to pay and v_i is the willingness to pay.

- An *allocation* is a list of bundles $x \in \times_{i \in N} X_i$ such that $\sum_{i \in N} x_i = \sum_{i \in N} e_i$. We sometimes write an allocation as $(a, t) \in \{0,1\}^N \times \mathbb{T}^N$, where a is the *(object) assignment profile* and t is the *(monetary) transfer profile*. Given an allocation $x = (a, t)$ and a price $p \in \mathbb{T}$, we define

$$\begin{aligned} N(x) &\equiv \{i \in N | a_i = 1\}, \\ N_0(p|x) &\equiv \{i \in N_0 | x_i = (1, -p)\}, \\ N_0(e|x) &\equiv \{i \in N_0 | x_i = e_i\}, \\ N_1(p|x) &\equiv \{i \in N_1 | x_i = (0, p)\}, \text{ and} \\ N_1(e|x) &\equiv \{i \in N_1 | x_i = e_i\}. \end{aligned}$$

  Moreover, for each $p \in \mathbb{T}$ and each $i \in N_0(p|x) \cup N_1(p|x)$, we say that *$i$ trades in x at price p*. We emphasize that we allow side-payments, and thus it is feasible for an agent who does not trade to consume a non-zero transfer, but we do not introduce notation for this possibility that is analogous to the notation above. Finally, we let $Z \subseteq \times_{i \in N} X_i$ denote the set of allocations.

If $\mathbb{T} = \mathbb{R}$, then we have a *continuous economy*, while if $\mathbb{T} = \mathbb{Z}$, then we have a *discrete economy*; in both cases we have an *economy*. Whenever we refer to an arbitrary economy, we implicitly assume all of this notation.

In our leading application, the objects are the tract houses in a given neighborhood, the sellers are the homeowners who are interested in moving out during some given period, and the buyers are the prospective homeowners who are interested in moving in during that same period. In this case, it is natural to assume that all valuations are non-negative, to therefore (correctly) hypothesize that each seller's wealth will be irrelevant for our solution concepts, and to interpret each buyer's wealth as including not only his current assets but moreover the maximum mortgage loan he is able to secure.

We are interested in describing allocations that would not be blocked by certain groups of agents. In particular, we consider the following standard cooperative axioms.

Definition: Fix an economy and an allocation x. For each nonempty coalition $N' \subseteq N$,

- the set of *internal allocations for N'* is $Z_{N'} \equiv \{x' \in \times_{i \in N'} X_i | \sum_{i \in N'} x'_i = \sum_{i \in N'} e_i\}$,
- *N' strongly blocks x* if and only if there is $x' \in Z_{N'}$ such that for each $i \in N'$, $x'_i \succ_i x_i$, and
- *N' weakly blocks x* if and only if there is $x' \in Z_{N'}$ such that (i) for each $i \in N'$, $x'_i \succsim_i x_i$, and (ii) for some $i \in N'$, $x'_i \succ_i x_i$.

The empty coalition never strongly blocks or weakly blocks. An allocation x satisfies

- *individual rationality* if and only if it is not blocked (strongly, or equivalently, weakly) by any coalition $N' \subseteq N$ such that $|N'| = 1$.

- *weak pairwise stability* if and only if it is not strongly blocked by any coalition $N' \subseteq N$ such that $|N'| \in \{1,2\}$.
- *strong pairwise stability* if and only if it is not weakly blocked by any coalition $N' \subseteq N$ such that $|N'| \in \{1,2\}$.
- *weak core* if and only if it is not strongly blocked by any coalition.
- *strong core* if and only if it is not weakly blocked by any coalition.
- *efficiency* if and only if it is not weakly blocked by N.

We emphasize that pairs are particular important for cooperative analysis in our model. First, the core is equivalent to pairwise stability, both for the weak case and for the strong case.

PROPOSITION 1: Fix an economy. An allocation is (i) *weakly pairwise stable* if and only if it is *weak core*, and (ii) *strongly pairwise stable* if and only if it is *strong core.*

The proof is in Appendix A. Moreover, in order to verify whether an allocation is efficient, it suffices to verify that there are no pairwise improvements.

PROPOSITION 2: Fix an economy. An allocation x violates *efficiency* if and only if there are $i \in N \backslash N(x)$, $j \in N(x)$, and $p \in \mathbb{T}$ such that for $x_i^* \equiv (1, t_i - p)$ and $x_j^* \equiv (0, t_j + p)$, we have (i) $x_i^* \in X_i$ and $x_i^* \succsim_i x_i$, (ii) $x_j^* \in X_j$ and $x_j^* \succsim_j x_j$, and (iii) there is $k \in \{i, j\}$ such that $x_k^* \succ_k x_k$.

The proof is in Appendix A. We conclude this section by observing that while *strong core* trivially implies *efficiency*, there is no logical relationship between *weak core* and *efficiency.*

EXAMPLE 1: *Efficiency does not imply weak core.* The economy may be either continuous or discrete. There are buyers i and j such that $(w_i, v_i) = (\infty, 3)$ and $(w_j, v_j) = (\infty, 2)$, and there is seller k such that $(w_k, v_k) = (\infty, 0)$. The allocation where i trades with k at price 0 is *efficient*, but is not *weak core* as it is strongly blocked by $\{j, k\}$ by having j trade with k at price 1.

EXAMPLE 2: *Weak core does not imply efficiency.* The economy may be either continuous or discrete. There are buyers i and j such that $(w_i, v_i) = (\infty, 1)$ and $(w_j, v_j) = (1, 2)$, and there is seller k such that $(w_k, v_k) = (\infty, 0)$. The allocation where i trades with k at price 1 is *weak core*, but is not *efficient* as it is weakly blocked by N using the allocation where j trades with k at price 1.

2.2 The strong core

We begin by characterizing the strong core. As in other models, our characterization involves the classic notion of Walrasian equilibrium.

Definition: Fix an economy. For each $i \in N$ and each price $p \in \mathbb{T}$, define (i) the *(boundary of the) budget set for i at p,*[5] $B_i(p)$, and (iii) the *demand set for i at p*, $B_i^\delta(p) \subseteq B_i(p)$, by

$$B_i(p) \equiv \begin{cases} X_i \cap \{e_i, (1, -p)\}, & i \in N_0, \\ X_i \cap \{e_i, (0, p)\}, & i \in N_1, \end{cases}$$
$$B_i^\delta(p) \equiv \{x_i \in B_i(p) | \text{ for each } x'_i \in B_i(p), x_i \succsim_i x'_i\}.$$

An allocation x is a *Walrasian equilibrium* if and only if there is $p \in \mathbb{T}$ such that for each $i \in N$, $x_i \in B_i^\delta(p)$. In this case, we say that x is *supported by* p.

That said, our characterization involves some further nuance as there are strong core allocations that are not Walrasian equilibria.

Example 3: *Strong core allocation that is not a Walrasian equilibrium.* The economy is discrete. There is buyer i such that $(w_i, v_i) = (\infty, 0.5)$, and there is seller j such that $(w_j, v_j) = (\infty, 0.5)$. The allocation where both agents consume their endowments is *strong core*, but is not a Walrasian equilibrium.[6]

The allocation in the above example is the no-trade allocation and moreover is *efficient*, and our characterization of the strong core involves this allocation.

Definition: Fix an economy. An allocation x is *efficient no-trade* if and only if (i) x is *efficient*, and (ii) $x = e$.

For continuous economies, the set of Walrasian equilibria includes the no-trade allocation whenever it is efficient, and thus we need not distinguish between the two notions.

Proposition 3: Fix a continuous economy. If an allocation is *efficient no-trade*, then it is a Walrasian equilibrium.

The proof is in Appendix B. Our first theorem states that for both continuous and discrete economies, the strong core consists of (i) the Walrasian equilibria, and (ii) the no-trade allocation provided it is efficient:

Theorem 1: Fix an economy. An allocation satisfies *strong core* if and only if it is a Walrasian equilibrium or *efficient no-trade* (or both).

The proof is in Appendix B. Note that by Proposition 3, the strong core coincides

[5] Our analysis is consistent with standard general equilibrium theory: the full budget set for i at p is $\{(0, t_i) \in X_i | t_i \leq 0\} \cup \{(1, t_i) \in X_i | t_i \leq -p\}$ if $i \in N_0$ and $\{(0, t_i) \in X_i | t_i \leq p\} \cup \{(1, t_i) \in X_i | t_i \leq 0\}$ if $i \in N_1$. For our purposes, we may safely restrict attention to the boundary, which corresponds to the budget line in the textbook model with two infinitely divisible goods.

[6] It is tempting to represent the no-trade allocation in this example as a Walrasian equilibrium supported by the non-integer price 0.5, which (i) asks i to pay at least 0.5 and therefore effectively asks him to pay 1, and (ii) asks j to accept at most 0.5 and therefore effectively asks him to accept 0. That said, non-integer prices also support allocations that are not even *efficient*, let alone *strong core*. Indeed, consider buyers i and j such that $(w_i, v_i) = (w_j, v_j) = (\infty, 0.5)$ and seller k such that $(w_k, v_k) = (\infty, 0)$. The no-trade allocation is supported by the price 0.5, but is not *efficient*.

with the set of Walrasian equilibria for continuous economies.

Corollary 1: Fix a continuous economy. An allocation satisfies *strong core* if and only if it is a Walrasian equilibrium.

By definition, each strong core allocation is *efficient*. Moreover, it is easy to see from Theorem 1 that all strong core allocations are *bilateral* in the sense that there are no side-payments: each strong core allocation can be represented by selecting disjoint pairs of buyers and sellers and assigning each pair a price, such that each pair trades at its price and each leftover agent consumes his endowment. This property, which we formalize below, will be focal to our analysis of the weak core in the next section. To avoid confusion, we use *pairwise* for concepts involving a single pair and *bilateral* for concepts involving potentially multiple disjoint pairs throughout this paper.

Definition: Fix an economy. An allocation x is *bilateral* if and only if

- for each $i \in N_0 \backslash N(x)$, we have $t_i = 0$,
- for each $i \in N_1 \cap N(x)$, we have $t_i = 0$, and
- for each $p \in \mathbb{T}$, $|N_0(p|x)| = |N_1(p|x)|$.

Before proceeding, we formalize this important implication of Theorem 1.

Corollary 2: Fix an economy. Each *strong core* allocation is *efficient* and *bilateral*.

Our second theorem describes the structure of the set of Walrasian equilibria, which involves the notions of (i) weak and strict demand, and (ii) weak and strict supply. We remark that our result is analogous to a theorem of Mishra and Talman (2010); the novelty is that we allow for wealth constraints.

Definition: Fix an economy. For each $p \in \mathbb{T}$, we define the

- *weak demanders at* p, $\mathsf{D}(p) \equiv \{i \in N_0 | w_i \geq p \text{ and } v_i \geq p\}$;
- *strict demanders at* p, $\mathsf{D}^!(p) \equiv \{i \in N_0 | w_i \geq p \text{ and } v_i > p\}$;
- *weak suppliers at* p, $\mathsf{S}(p) \equiv \{i \in N_1 | p \geq -w_i \text{ and } p \geq v_i\}$; and
- *strict suppliers at* p, $\mathsf{S}^!(p) \equiv \{i \in N_1 | p \geq -w_i \text{ and } p > v_i\}$.

Theorem 2: Fix an economy, and define $\underline{p}^\star \equiv \inf\{p \in \mathbb{T} || \mathsf{S}(p)| \geq |\mathsf{D}^!(p)|\}$ and $\overline{p}^\star \equiv \sup\{p \in \mathbb{T} || \mathsf{D}(p)| \geq |\mathsf{S}^!(p)|\}$. The Walrasian equilibria are in mutual correspondence with the prices in $[\underline{p}^\star, \overline{p}^\star] \cap \mathbb{T}$ in the following sense:

1. Both $\underline{p}^\star$ and $\overline{p}^\star$ are well-defined. Moreover, if $\mathbb{T} = \mathbb{R}$, then $\underline{p}^\star \leq \overline{p}^\star$.
2. For each Walrasian equilibrium x, there is $p \in [\underline{p}^\star, \overline{p}^\star] \cap \mathbb{T}$ such that x is supported by p.

3. For each $p \in (\underline{p}^\star, \overline{p}^\star) \cap \mathbb{T}$, there is a Walrasian equilibrium that is supported by p. Moreover,
   - if $\min\{p \in \mathbb{T} || \mathsf{S}(p)| \geq |\mathsf{D}^!(p)|\} = \underline{p}^\star < \overline{p}^\star$, then there is a Walrasian equilibrium that is supported by $\underline{p}^\star$,
   - if $\underline{p}^\star < \overline{p}^\star = \max\{p \in \mathbb{T} || \mathsf{D}(p)| \geq |\mathsf{S}^!(p)|\}$, then there is a Walrasian equilibrium that is supported by $\overline{p}^\star$,
   - if $\min\{p \in \mathbb{T} || \mathsf{S}(p)| \geq |\mathsf{D}^!(p)|\} = \underline{p}^\star = \overline{p}^\star = \max\{p \in \mathbb{T} || \mathsf{D}(p)| \geq |\mathsf{S}^!(p)|\}$, then there is a Walrasian equilibrium that is supported by $\underline{p}^\star = \overline{p}^\star$, and
   - if none of the above, then there is no Walrasian equilibrium that is supported by either $\underline{p}^\star$ or $\overline{p}^\star$.

The proof is in Appendix B. It follows from Theorem 2 that the set of Walrasian equilibria may be empty in both the continuous model and the discrete model, but for different reasons. Indeed, in the continuous model, necessarily $\underline{p}^\star \leq \overline{p}^\star$, but a set of reals need not include its infimum (when well-defined) or supremum (when well-defined); see Example 4. By contrast, in the discrete model, necessarily a set of integers includes its infimum (when well-defined) and supremeum (when well-defined), but we need not have $\underline{p}^\star \leq \overline{p}^\star$; see Example 5. For both models, this can occur while the no-trade allocation is inefficient, and thus by Theorem 1 in both models the strong core may be empty.

Example 4: *The strong core is empty with continuous transfers.* The economy is continuous. There are buyers i and j such that $(w_i, v_i) = (w_j, v_j) = (0, 100)$, and there is seller k such that $(w_k, v_k) = (0, 0)$. We have $\underline{p}^\star = \overline{p}^\star = 0$ and $|\mathsf{D}^!(0)| = 2 > 1 = |\mathsf{S}(0)|$, so by Theorem 2 there is no Walrasian equilibrium, so by Corollary 1 there is no *strong core* allocation.

Example 5: *The strong core is empty with discrete transfers.* The economy is discrete. There are buyers i and j such that $(w_i, v_i) = (w_j, v_j) = (\infty, 0.5)$, and there is seller k such that $(w_k, v_k) = (\infty, 0)$. We have $\underline{p}^\star = 1$ and $\overline{p}^\star = 0$, so $\underline{p}^\star = 1 > 0 = \overline{p}^\star$, so by Theorem 2 there is no Walrasian equilibrium. It is easy to verify that no-trade violates *efficiency*, so by Theorem 1 there is no *strong core* allocation.

To conclude this section, we remark that for continuous economies, the strong core is nonempty whenever there are no wealth constraints (Koopmans and Beckmann, 1957), and indeed Example 4 features wealth constraints. By contrast, Example 5 illustrates that for discrete economies, the strong core may be empty even when there are no wealth constraints.

2.3 The bilateral weak core

Because the strong core may be empty, we turn our attention to the weak core. We begin by investigating the relationship between the weak core and bilaterality.

Recall that for both continuous and discrete economies, each strong core allocation is bilateral. Similarly, for continuous economies, each weak core allocation is bilateral.

Proposition 4: Fix a continuous economy. Each *weak core* allocation is *bilateral.*

The proof is in Appendix C. Interestingly, though all weak core allocations for continuous economies are bilateral, nevertheless some weak core allocations for discrete economies involve side-payments.

EXAMPLE 6: *Weak core allocation that is not bilateral.* The economy is discrete. There are buyers i and j such that $(w_i, v_i) = (w_j, v_j) = (\infty, 2)$, and there is seller k such that $(w_k, v_k) = (\infty, 0)$. The allocation where (i) i makes a side-payment of 1 to j in exchange for nothing, and (ii) k gives the object to i in exchange for nothing, is *weak core* but not *bilateral.*

In the above example, the allocation would not occur if i could block by simply refusing to transfer his side-payment to j, but this is not captured by the *weak core* axiom. To disallow such allocations, we simply refine weak core by imposing *bilateral* and *weak core* together.

DEFINITION: Fix an economy. An allocation is *bilateral weak core* if and only if it is *bilateral* and *weak core.*

We characterize the bilateral weak core using a new notion that we call *almost-synchronized equilibrium.* Intuitively, this notion involves a dynamic process with a common budget set whose price adjusts at a constant rate in one of three cases: the price is increasing throughout the process, it is constant throughout, or it is deceasing throughout. Moreover, if the price is not constant, then either (i) transfers are continuous and the price adjusts continuously at a rate of one unit of money per second, or (ii) transfers are discrete and the price is an integer that adjusts discretely by one unit of money once per second. Finally, at each moment, each agent may either select to trade at the current price or consume his endowment.

In an almost-synchronized equilibrium, we loosely require that (i) the agents make these selections within one second of each other, (ii) if an agent selects earliest, then he consumes his most-preferred bundle from the earliest budget set, and (iii) if an agent selects late, then while he may not consume his most-preferred bundle from the earliest budget set, he consumes his most-preferred bundle from any other budget set that is at most one second after the earliest one. This description is loose only because formally, we allow for the possibility that all agents are late.

DEFINITION: Fix an economy. For each $p \in \mathbb{T}$, define $P_{\searrow}(p), P_{\rightarrow}(p), P_{\nearrow}(p) \subseteq \mathbb{T}$ by

$$P_{\searrow}(p) \equiv \begin{cases} [p-1, p) = \{p' \in \mathbb{T} | p > p' \geq p-1\}, & \mathbb{T} = \mathbb{R}, \\ \{p-1\}, & \mathbb{T} = \mathbb{Z}, \end{cases}$$

$$P_{\rightarrow}(p) \equiv \{p\}, \text{ and}$$

$$P_{\nearrow}(p) \equiv \begin{cases} (p, p+1] = \{p' \in \mathbb{T} | p < p' \leq p+1\}, & \mathbb{T} = \mathbb{R}, \\ \{p+1\}, & \mathbb{T} = \mathbb{Z}. \end{cases}$$

An allocation x is an *almost-synchronized equilibrium* if and only if there are $p \in \mathbb{T}$ and $\dashrightarrow \in \{\searrow, \rightarrow, \nearrow\}$ such that for each $i \in N$, either

- $x_i \in B_i^\delta(p)$, or
- for each $p' \in P_{\dashrightarrow}(p)$, $x_i \in B_i^\delta(p')$.

In this case, we say that x is *supported by* $(p, \dashrightarrow)$. Notice that if x is supported by $(p, \rightarrow)$, then x is a Walrasian equilibrium supported by p.

Our next theorem states that the bilateral weak core coincides with the set of almost-synchronized equilibria.

Theorem 3: Fix an economy. An allocation satisfies *bilateral weak core* if and only if it is an almost-synchronized equilibrium.

The proof is in Appendix C. Notice that by Proposition 4, we have characterized the entire weak core for continuous economies.

Corollary 3: Fix a continuous economy. An allocation satisfies *weak core* if and only if it is an almost-synchronized equilibrium.

We conclude this section with a theorem that describes the structure of the bilateral weak core, which involves a novel notion we refer to as *forceful* demand and supply.

Definition: Fix an economy. For each $p \in \mathbb{T}$, we define the

- *forceful demanders at* p, $\mathsf{D}^{!!}(p) \equiv \{i \in N_0 |$ there is $p' \in P_{\nearrow}(p)$ such that $i \in \mathsf{D}^{!}(p')\}$; and
- *forceful suppliers at* p, $\mathsf{S}^{!!}(p) \equiv \{i \in N_1 |$ there is $p' \in P_{\searrow}(p)$ such that $i \in \mathsf{S}^{!}(p')\}$.

Theorem 4: Fix an economy, and define $\underline{p} \equiv \min\{p \in \mathbb{T} | |\mathsf{S}(p)| \geq |\mathsf{D}^{!!}(p)|\}$ and $\overline{p} \equiv \max\{p \in \mathbb{T} | |\mathsf{D}(p)| \geq |\mathsf{S}^{!!}(p)|\}$. The almost-synchronized equilibria are in mutual correspondence with the prices in $[\underline{p}, \overline{p}] \cap \mathbb{T}$ in the following sense:

1. Both $\underline{p}$ and $\overline{p}$ are well-defined with $\underline{p} \leq \overline{p}$.
2. For each almost-synchronized equilibrium x, there is $p \in [\underline{p}, \overline{p}] \cap \mathbb{T}$ such that for some $\dashrightarrow \in \{\searrow, \rightarrow, \nearrow\}$, x is supported by $(p, \dashrightarrow)$.
3. For each $p \in [\underline{p}, \overline{p}] \cap \mathbb{T}$, there is an almost-synchronized equilibrium x such that for some $\dashrightarrow \in \{\searrow, \rightarrow, \nearrow\}$, x is supported by $(p, \dashrightarrow)$.

The proof is in Appendix C. Importantly, it follows from Theorem 4 that the bilateral weak core is always nonempty and thus might be implemented.

Example 4 (Revisited): *The strong core is empty but there is an almost-synchronized equilibrium with continuous transfers.* The economy is continuous. There are buyers i and j such that $(w_i, v_i) = (w_j, v_j) = (0, 100)$, and there is seller k such that $(w_k, v_k) = (0, 0)$. Recall that there is no strong core allocation. That said, there is an almost-synchronized equilibrium where i and k trade at price 0 while j selects late during the ascending process and selects his endowment.

Example 5 (Revisited): *The strong core is empty but there is an almost-synchronized equilibrium with discrete transfers.* The economy is discrete. There are buyers i and j

such that $(w_i, v_i) = (w_j, v_j) = (\infty, 0.5)$, and there is seller k such that $(w_k, v_k) = (\infty, 0)$. Recall that there is no strong core allocation. That said, there is an almost-synchronized equilibrium where i and k trade at price 0 while j selects late during the ascending process and selects his endowment.

3 Incomplete information and pendulum auctions

3.1 Environments and strategy-proof rules

We now analyze the case of incomplete information: we know from Section 2 that the bilateral weak core is always nonempty, and we now investigate whether it can be implemented when the type profile $(\theta_i)_{i\in N}$ is not common knowledge. In particular, we begin by investigating when it is possible to design (i) a *mechanism*, and (ii) an associated *convention* that recommends a strategy to each agent given his type, such that regardless of the true type profile, the strategy profile suggested by the convention is a dominant strategy equilibrium whose outcome is a bilateral weak core allocation. When this is possible, the implemented *rule*—that is, the associated mapping from type profiles to allocations—is said to be *strategy-proof*.

It is well-known that across a variety of one-to-one matching models, there are no *strategy-proof* rules that select core allocations when there is private information on both sides of the market (Roth, 1982; Myerson and Satterthwaite, 1983; Matsuo, 1989). Indeed, it follows from Matsuo (1989) that in our model, even when we have one buyer and one seller with common knowledge that they both have infinite wealth, there is no *strategy-proof* selection from the weak core whether transfers are continuous or discrete. That said, there are often *strategy-proof* rules that select core allocations when there is only private information on one side of the market (Dubins and Freedman, 1981; Roth, 1982; Demange, 1982; Leonard, 1983), and indeed this is the approach we take: we assume that for each buyer i, (w_i, v_i) is private information, while for each seller i, (w_i, v_i) is common knowledge.[7]

Definition: For each set of possible monetary transfers $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, a $\mathbb{T}$-*environment* is a tuple $(N_0, N_1, (\theta_i)_{i\in N_1})$, where

- The following are as in the definition of $\mathbb{T}$-economy: the set of *buyers* N_0; the set of *sellers* N_1; the set of *agents* N; and for each $i \in N$, the *endowment* e_i, the set of admissible *wealth constraints* W_i, and the set of admissible *valuations* V_i. Moreover, for each $i \in N$, a type $\theta_i = (w_i, v_i)$ has the same interpretation as in the definition of $\mathbb{T}$-economy.

- For each $i \in N$, Θ_i is the set of *possible types*. For each buyer $i \in N_0$, the wealth constraint and valuation are both private information, so $\Theta_i \equiv W_i \times V_i$. For each seller $i \in N_1$, the wealth constraint and valuation are both common knowledge, so $\Theta_i = \{\theta_i\} = \{(w_i, v_i)\}$. We define $\Theta \equiv \times_{i\in N}\Theta_i$. When $(\theta_i)_{i\in N_1}$ is clear from context, we sometimes abuse language by referring to $(\theta_i)_{i\in N_0}$ as the type profile in Θ.

[7]Due to the model's symmetry, analogous versions of our results hold if instead each buyer's type is common knowledge and each seller's type is private information, loosely because we can interpret receiving a good as losing a bad, losing a good as receiving a bad, receiving money as losing negative money, and losing money as receiving negative money.

- The following are in the definition of $\mathbb{T}$-economy, but now vary with θ: the set of *allocations* $Z(\theta)$; and for each $i \in N$, the set of possible *transfers* $T_i(\theta_i)$, the set of possible *bundles* $X_i(\theta_i)$, and the *preference relation* $\succsim_i(\theta_i)$.

If $\mathbb{T} = \mathbb{R}$, then we have a *continuous environment*, while if $\mathbb{T} = \mathbb{Z}$, then we have a *discrete environment*; in both cases we have an *environment*. Whenever we refer to an arbitrary environment, we implicitly assume all of this notation.

In our leading application, it is natural to suppose that each seller has a non-negative valuation for his house and thus that his wealth is irrelevant. In this case, we assume that each seller has already listed his house at an asking price equal to his true valuation during an unmodeled *listing stage*,[8] and with this information we consider designing a mechanism for a *bidding stage* where sellers are non-strategic and where each buyer's wealth and valuation are private information.

By the revelation principle, in order to implement the bilateral weak core in dominant strategies, it is necessary for the associated rule to be *strategy-proof* (Gibbard, 1973; Myerson, 1981). This is equivalent to the requirement that in the *direct mechanism* where the buyers simultaneously report their types, it is a dominant strategy for each buyer to report honestly regardless of his type. In our model, the formal statement has some nuance because whether or not a given allocation is feasible generally depends on the private wealth constraints of the buyers.

Definition: Fix an environment. A *rule* is a mapping $\varphi : \Theta \to \cup_{\theta \in \Theta} Z(\theta)$ such that for each $\theta \in \Theta$, $\varphi(\theta) \in Z(\theta)$. We emphasize that a rule always selects an allocation that is feasible according to the reports, and therefore never asks a buyer to pay more than his reported wealth. If a buyer is asked to pay more than his true wealth because he has misreported, then we assume that his transaction simply fails and he instead consumes his endowment, and we represent the associated incentives as follows: for each $i \in N_0$ and each $\theta_i \in \Theta_i$, we define the utility function $u_{i|\theta_i} : \cup_{\theta \in \Theta} Z(\theta) \to \mathbb{R}$ by

$$u_{i|\theta_i}(x) \equiv \begin{cases} v_i \cdot a_i + t_i, & t_i \geq -w_i, \\ 0, & \text{else.} \end{cases}$$

We say that φ is *strategy-proof* if and only if for each $i \in N_0$, each pair $\theta_i, \theta'_i \in \Theta_i$, and each $\theta_{-i} \in \Theta_{-i}$, we have $u_{i|\theta_i}(\varphi(\theta_i, \theta_{-i})) \geq u_{i|\theta_i}(\varphi(\theta'_i, \theta_{-i}))$.

Independently of this paper, it was recently established that with continuous transfers, there are no *strategy-proof* rules that select weak core allocations (Batziou, Bichler, and Fichtl, 2022). For completeness, we illustrate this result with the following example.

Example 7: *Continuous environment; no strategy-proof rule selects from the weak core.* There are buyers i and j and there is seller k such that $(w_k, v_k) = (0, 0)$. Assume, by way of contradiction, that (i) φ is a *strategy-proof* rule, and (ii) for each $\theta \in \Theta$, $\varphi(\theta)$ is a *weak core* allocation.

Consider the type profile (θ_i, θ_j) given by $(w_i, v_i) = (w_j, v_j) = (0, 100)$. At this profile, there are three *weak core* allocations: (i) i trades with k at price 0, (ii) j trades with k at price 0, and (iii) there is no trade and no transfer of money.

[8] As in Demange (1982), it is possible to justify the assumption that the sellers report their true valuations in the listing stage for the rules we will consider on the basis of *maximin strategies*; we omit the straightforward argument for brevity.

Let i^* be a buyer who does not trade in $\varphi(\theta_i, \theta_j)$. If i^* changes his report to $(1, 100)$, then in any *weak core* allocation, i^* trades with k and pays some amount $\varepsilon \in [0, 1]$. We cannot have $\varepsilon > 0$, else if i^* changes his report to $(\frac{\varepsilon}{2}, 100)$, then in any *weak core* allocation, i^* trades with k and pays some amount in $[0, \frac{\varepsilon}{2}]$, so i^* benefits from misreporting $(\frac{\varepsilon}{2}, 100)$ when his true type is $(1, 100)$, contradicting that φ is *strategy-proof*. But then $\varepsilon = 0$, so i^* benefits from misreporting $(1, 100)$ when his true type is $(0, 100)$, contradicting that φ is *strategy-proof*.

We therefore restrict attention to discrete environments for the rest of our analysis. As we will see, in this case it is possible to implement the bilateral weak core not only in dominant strategies, but moreover in *obviously* dominant strategies (Li, 2017). In order to do so, however, we must move beyond direct mechanisms.

3.2 Mechanisms, conventions, and obvious strategy-proofness

At a high level, a strategy is *obviously dominant* if and only if an agent can identify the strategy as dominant without using contingent reasoning. While the classic revelation principle applies for implementation in dominant strategies, it does not apply for implementation in obviously dominant strategies, and thus when investigating such an implementation we cannot safely restrict attention to the direct mechanism (Li, 2017). That said, we can safely restrict attention to extensive game forms with perfect information (Ashlagi and Gonczarowski, 2018; Pycia and Troyan, 2022; Bade and Gonczarowski, 2017; Mackenzie, 2020), which for brevity we refer to simply as *mechanisms*.

Definition: Fix a discrete environment. A *(perfect information) mechanism* is an extensive game form—that is, the result of taking an extensive form game and then deleting the preference profile—with (i) players in N_0, (ii) outcomes in $\cup_{\theta \in \Theta} Z(\theta)$, and (iii) perfect information. We omit the familiar details of the full formal definition as we do not require them; for these omitted details see Mackenzie and Zhou (2022). We do, however, require the following notation:

- H denotes the set of histories, each non-terminal history has a player in N_0 and a nonempty set of available actions $\mathcal{A}(h)$, and for each $i \in N_0$ we let H_i denote the set of histories where i is the player.

- For each $i \in N_0$, a *(pure) strategy for* i is a mapping s_i that associates each $h \in H_i$ with an action $s_i(h) \in \mathcal{A}(h)$. We let S_i denote the set of strategies for i, and define $S \equiv \times_{i \in N_0} S_i$ and $S_{-i} \equiv \times_{j \in N_0 \setminus \{i\}} S_j$.

- $\mathcal{X}$ denotes the mapping that associates each $s \in S$ with the outcome $\mathcal{X}(s)$ that occurs when play proceeds from the initial history according to s, and for each $h \in H$ we let $\mathcal{X}^h$ denote the mapping that associates each $s \in S$ with the outcome $\mathcal{X}^h(s)$ that occurs when play first proceeds from the initial history to h and then proceeds according to s.

Whenever we refer to an arbitrary mechanism, we implicitly assume all of this notation.

In order to describe implementation in a mechanism with incomplete information, we first articulate incentives in a mechanism together with a type profile—that is, in a game with complete information.

Definition: Fix a discrete environment, a mechanism, and a type profile θ. Then

- For each $i \in N_0$ and each $s_i \in S_i$, we say that s_i is *obviously dominant* if and only if for each $h \in H_i$ that can be reached when i plays s_i, each $s'_i \in S_i$ such that $s_i(h) \neq s'_i(h)$, and each pair $s_{-i}, s'_{-i} \in S_{-i}$, we have $u_{i|\theta_i}(\mathcal{X}^h(s_i, s_{-i})) \geq u_{i|\theta_i}(\mathcal{X}^h(s'_i, s'_{-i}))$. In other words, from any history $h \in H_i$ that s_i can reach, the worst-case $u_{i|\theta_i}$ from adhering to s_i must be at least as high as the best-case $u_{i|\theta_i}$ from deviating.

- We say that $s \in S$ is an *obviously dominant strategy equilibrium* if and only if for each $i \in N_0$, s_i is obviously dominant.

We let $\mathbf{OSP}(\theta) \subseteq S$ denote the set of obviously dominant strategy equilibria when the type profile is θ.

Our notion of implementation involves a *convention* that recommends a strategy to each buyer given his type. For example, honesty in a direct mechanism is a convention. The notion we consider requires the strategy profile recommended by the convention to be an obviously dominant strategy equilibrium whose outcome is the one specified by the rule, regardless of the type profile.

Definition: Fix a discrete environment, a rule φ, and a mechanism G. Then

- For each $i \in N_0$, a *type-strategy for i* is a mapping $\mathbb{S}_i : \Theta_i \to S_i$.

- A *convention* is a type-strategy profile $(\mathbb{S}_i)_{i \in N_0}$.

Given a convention $\mathbb{S}$, we say that $(G, \mathbb{S})$ $\mathbf{OSP}$*-implements* φ if and only if for each $\theta \in \Theta$,

- $(\mathbb{S}_i(\theta_i))_{i \in N_0} \in \mathbf{OSP}(\theta)$, and

- $\mathcal{X}((\mathbb{S}_i(\theta_i))_{i \in N_0}) = \varphi(\theta)$.

In this case, we say that G is an *obviously strategy-proof implementation of φ (through $\mathbb{S}$)*.

Altogether, our objective is to design an obviously strategy-proof implementation of a rule that always selects from the bilateral weak core.

3.3 Pendulum auctions

In this section, we introduce two classes of obviously strategy-proof implementations of the bilateral weak core. We refer to all of the mechanisms across the two classes as *pendulum auctions*, we refer to the classes themselves as *versions*, and for each version we distinguish between individual auctions (and their associated implementations) using what we call *auction configurations*. More precisely, for each auction configuration we introduce the *buyer-optimal version* of the pendulum auction and the *efficient version* of the pendulum auction, each of which is an obviously strategy-proof implementation of an associated rule that selects from the bilateral weak core.

We begin by introducing auction configurations. For context, each pendulum auction is such that (i) one buyer plays at a time, and (ii) after bidding, a buyer may be matched to a seller at a price posted by that seller. Throughout the auction, we determine which buyer should play and which seller should receive a match using the auction configuration, which formally consists of a strict ranking of the buyers and a strict ranking of the sellers. To avoid confusion between these rankings, we say that the buyers are ranked on the basis of priority while the sellers are arranged from left to right.

DEFINITION: Fix a discrete environment. An *auction configuration* is a pair $c = (\Pi, \lhd)$, where

- $\Pi : N_0 \to \{1, 2, ..., |N_0|\}$ is a bijection that we call the *priority assignment (for the buyers)*. For each $i \in N_0$, $\Pi(i)$ denotes the *priority of* i, where the first priority 1 is best and the last priority $|N_0|$ is worst. More generally, we say that a priority indexed by an earlier number is *better*.

- For each $i \in N_1$, we define $\underline{p}_i \equiv \max\{-w_i, \lceil v_i \rceil\}$ to be the *minimum* i *is willing and able to accept*.[9]

- $\lhd$ is a strict ranking of the sellers that we call the *seller arrangement*. We require that for each pair $i, j \in N_1$, $i \lhd j$ if and only if (i) $\underline{p}_i < \underline{p}_j$, or (ii) $\underline{p}_i = \underline{p}_j$ and $v_i \leq v_j$; in this case we say that i *is to the left of* j.

We let C denote the set of auction configurations.

In both versions of the pendulum auction for a given auction configuration, unmatched sellers become matched from left to right and matched sellers become unmatched (while sometimes increasing their posted prices) from right to left; this is the dynamic we mean to suggest with the term *pendulum*. The two versions differ in what occurs when (i) all sellers who offer the minimum posted price p are matched to a buyer who has stated that he strictly prefers to trade at this price, (ii) there is an unmatched seller offering $p + 1$ who has not yet been matched, and (iii) a buyer submits a new bid. In particular, the *buyer-optimal version* has the bidder match with an unmatched seller offering $p+1$, while the *efficient version* has one of the sellers who is matched at price p become unmatched while increasing his price to $p + 1$. There are other similarities and differences between the two versions, as we itemize in the following formal definition.

DEFINITION: *Pendulum auctions.* Fix a discrete environment and let $c \in C$. We let $G^{\mathcal{B}|c}$ denote the *buyer-optimal pendulum auction (given c)* and we let $G^{\mathcal{E}|c}$ denote the *efficient pendulum auction (given c)*. Let $G \in \{G^{\mathcal{B}|c}, G^{\mathcal{E}|c}\}$; we define G in both cases as follows.

To begin, H is the set of histories, which we define inductively below. For each $h \in H$, the *state at* h is $(N_Q(h), (N_{M(i)}(h))_{i\in N_1}, N_X(h), (p_i(h))_{i\in N_1}, N_1^{\uparrow}(h))$, where

- $N_Q(h) \subseteq N_0$ is the *queue at* h,

- for each $i \in N_1$, $N_{M(i)}(h) \subseteq N_0$ is the *match of seller* i *at* h,

- $N_X(h) \subseteq N_0$ is the *exited buyers at* h,

- for each $i \in N_1$, $p_i(h)$ is the *price of seller* i *at* h, and

- $N_1^{\uparrow}(h) \subseteq N_1$ is the *rising sellers at* h, giving $N_1 \backslash N_1^{\uparrow}(h)$ as the *resting sellers at* h.

[9]For each $v_i \in \frac{1}{2}\mathbb{Z}$, $\lceil v_i \rceil \equiv \min\{v_i' \in \mathbb{Z} | v_i' \geq v_i\}$ denotes the ceiling function applied to v_i.

The nonempty members of $N_Q(h)$, $\{N_{M(i)}(h)\}_{i\in N_1}$, and $N_X(h)$ form a partition of N_0. Moreover, for each seller $i \in N_1$, the associated match $N_{M(i)}(h)$ is either nobody or one buyer: $|N_{M(i)}(h)| \in \{0,1\}$. We let $p^{\min}(h)$ denote the minimum price at h, $\min_{i\in N_1} p_i(h)$.

There is an initial history $h_\wedge$. At the initial history, (i) all buyers are in the queue; (ii) for each seller i, the initial price is $\underline{p}_i$; and (iii) all sellers are resting.

A history is terminal if and only if the queue is empty. Otherwise, the player is the best-priority buyer in the queue, and the player is asked to select either *bid* or *exit*. The only distinction between the two versions is how the market state updates when the player selects *bid*; we describe the two cases now.

CASE 1: $G = G^{\mathcal{B}|c}$. In this case, if i selects *bid* at history h, then the market state is updated as follows:

- If there is an unmatched seller offering $p^{\min}(h)$, then i matches with the $\lhd$-leftmost such seller and the market state is otherwise unchanged.
- Else if there is a resting matched seller offering $p^{\min}(h)$, then let j denote the $\lhd$-rightmost such seller and let $\mu(j)$ denote the match of j. Then j and $\mu(j)$ become unmatched, j becomes rising without changing his price, and $\mu(j)$ returns to the queue; the market state is otherwise unchanged.
- Else if there is an unmatched seller offering $p^{\min}(h)+1$, then i matches with the $\lhd$-leftmost such seller and the market state is otherwise unchanged.
- Else if there is a resting matched seller offering $p^{\min}(h)+1$, then let j denote the $\lhd$-rightmost such seller and let $\mu(j)$ denote the match of j. Then j and $\mu(j)$ become unmatched, j becomes rising without changing his price, and $\mu(j)$ returns to the queue; the market state is otherwise unchanged.
- Else let j denote the $\lhd$-rightmost seller offering $p^{\min}(h)$ and let $\mu(j)$ denote the match of j. Then j and $\mu(j)$ become unmatched, j remains rising while increasing his price to $p^{\min}(h)+1$, and $\mu(j)$ returns to the queue; the market state is otherwise unchanged.

CASE 2: $G = G^{\mathcal{E}|c}$. In this case, if i selects *bid* at history h, then the market state is updated as follows:

- If there is an unmatched seller offering $p^{\min}(h)$, then i matches with the $\lhd$-leftmost such seller and the market state is otherwise unchanged.
- Else if there is a resting matched seller offering $p^{\min}(h)$, then let j denote the $\lhd$-rightmost such seller and let $\mu(j)$ denote the match of j. Then j and $\mu(j)$ become unmatched, j becomes rising without changing his price, and $\mu(j)$ returns to the queue; the market state is otherwise unchanged.
- Else if there is an unmatched seller offering $p^{\min}(h)+1$ who offered a lower price at $h_\wedge$, then i matches with the $\lhd$-leftmost such seller and the market state is otherwise unchanged.

- Else let j denote the $\lhd$-rightmost seller offering $p^{\min}(h)$ and let $\mu(j)$ denote the match of j. Then j and $\mu(j)$ become unmatched, j becomes resting while increasing his price to $p^{\min}(h)+1$, and $\mu(j)$ returns to the queue; the market state is otherwise unchanged.

For emphasis, unmatched sellers become matched from left to right while matched sellers become unmatched from right to left. From here, the rest of the definition applies to both versions.

Both cases: If i selects *exit*, then (i) i exits, (ii) if i was unmatched from a seller $\mu(i)$ immediately before he exited, then for each buyer j who is matched with a seller to the right of $\mu(i)$, we have that j unmatches from his current seller $\mu(j)$ and then re-matches with the seller immediately to the left of $\mu(j)$, and (iii) the market state is otherwise unchanged.

To complete our description of the mechanism, at each terminal history h, the outcome is the allocation such that (i) for each $i \in N_0$ and each $j \in N_1$ such that $i \in N_{M(j)}(h)$, i transfers $p_j(h)$ to j in exchange for an object, and (ii) everybody else consumes his endowment. Finally, for each infinite play,[10] the outcome is the allocation where each agent consumes his endowment. Whenever we specify $G \in \{G^{\mathcal{B}|c}, G^{\mathcal{E}|c}\}$, we use all of the above notation.

Each pendulum auction has an associated convention that recommends how each buyer should play given his type.

Definition: Fix a discrete environment and let $c \in C$. For $G^{\mathcal{B}|c}$, the *buyer-optimal convention (given c)* is the convention $\mathbb{S}^{\mathcal{B}|c}$ defined as follows. For each $i \in N_0$, each $\theta_i \in \Theta_i$, and each $h \in H_i$,

- if there is a seller offering $p^{\min}(h)$ who is resting unmatched, then bid if and only if $i \in \mathsf{D}(p^{\min}(h))$;
- else if there is a seller offering $p^{\min}(h)$ who is not rising matched, then bid if and only if $i \in \mathsf{D}^!(p^{\min}(h))$;
- else if there is a seller offering $p^{\min}(h)+1$ who is resting unmatched, then bid if and only if $i \in \mathsf{D}(p^{\min}(h)+1)$;
- else bid if and only if $i \in \mathsf{D}^!(p^{\min}(h)+1)$.

For $G^{\mathcal{E}|c}$, the *efficient convention (given c)* is the convention $\mathbb{S}^{\mathcal{E}|c}$ defined as follows. For each $i \in N_0$, each $\theta_i \in \Theta_i$, and each $h \in H_i$,

- if there is a seller offering $p^{\min}(h)$ who is resting unmatched, then bid if and only if $i \in \mathsf{D}(p^{\min}(h))$;
- else if there is a seller offering $p^{\min}(h)$ who is not rising matched, then bid if and only if $i \in \mathsf{D}^!(p^{\min}(h))$;
- else bid if and only if $i \in \mathsf{D}(p^{\min}(h)+1)$.

[10] Recall that a *play* is a completely ordered collection of histories with no such superset.

For each version $\mathcal{V} \in \{\mathcal{B}, \mathcal{E}\}$ and each $c \in C$, when the pendulum auction $G^{\mathcal{V}|c}$ is clear from context, we refer $\mathbb{S}^{\mathcal{V}|c}$ as *the auction's convention*. Moreover, when both the pendulum auction and $\theta \in \Theta$ are clear from context, we let $H_{\mathbb{S}}$ denote the the play that occurs when buyers follow the auction's convention for θ, we let $x_{\mathbb{S}}$ denote the associated outcome, we let $h_{\mathbb{S}}$ denote the terminal history of $H_{\mathbb{S}}$, and we define $p_{\mathbb{S}}^{\min} \equiv p^{\min}(h_{\mathbb{S}})$. By the Pendulum Lemma in Appendix D, both $h_{\mathbb{S}}$ and $p_{\mathbb{S}}^{\min}$ are well-defined.

Finally, each pendulum auction has an associated rule that maps each type profile to the outcome that occurs if all buyers follow its convention.

Definition: Fix a discrete environment and let $c \in C$. Then

- $\varphi^{\mathcal{B}|c}$ is the rule that associates each $\theta \in \Theta$ with the outcome that occurs in $G^{\mathcal{B}|c}$ when all buyers follow the convention $\mathbb{S}^{\mathcal{B}|c}$, and
- $\varphi^{\mathcal{E}|c}$ is the rule that associates each $\theta \in \Theta$ with the outcome that occurs in $G^{\mathcal{E}|c}$ when all buyers follow the convention $\mathbb{S}^{\mathcal{E}|c}$.

It follows from the Pendulum Lemma in Appendix D that for each $\theta \in \Theta$, both $\varphi^{\mathcal{B}|c}(\theta)$ and $\varphi^{\mathcal{E}|c}(\theta)$ belong to $Z(\theta)$, and therefore that both $\varphi^{\mathcal{B}|c}$ and $\varphi^{\mathcal{E}|c}$ are indeed rules. For each version $\mathcal{V} \in \{\mathcal{B}, \mathcal{E}\}$ and each $c \in C$, when the pendulum auction $G^{\mathcal{V}|c}$ is clear from context, we refer $\varphi^{\mathcal{V}|c}$ as *the auction's rule*. Moreover, when both $c \in C$ and $\theta \in \Theta$ are clear from context, we refer to $\varphi^{\mathcal{B}|c}(\theta)$ as *the buyer-optimal pendulum allocation* and refer to $\varphi^{\mathcal{E}|c}(\theta)$ as *the efficient pendulum allocation*.

In the next section, we present our results about the properties of the pendulum auctions. In addition to formalizing our claims about obvious strategy-proof implementation of the bilateral weak core, our results justify our terminology of referring to one version as buyer-optimal and the other as efficient.

3.4 Pendulum auction properties

The results in this section establish properties about the two versions of the pendulum auction across four topics: incentive compatibility, core selection, buyer-optimality, and efficiency. We consider these topics in sequence.

3.4.1 Incentive compatibility

To begin, both versions of the pendulum auction are not only strategy-proof, but moreover obviously strategy-proof.

Theorem 5: Fix a discrete environment. For each auction configuration, each version of the pendulum auction is an obviously strategy-proof implementation of its rule through its convention.

The proof is in Appendix D.

3.4.2 Core selection

As claimed, both versions of the pendulum auction select bilateral weak core allocations. Moreover, both select such allocations that moreover respect the priorities of the buyers specified by the auction configuration, in the sense that a buyer i never envies a worse-priority buyer j who consumes a bundle x_j that i can afford.

Definition: Fix a discrete economy and let $c \in C$. An allocation x satisfies *no justified envy (given c)* if and only if for each pair $i, j \in N_0$ such that (i) $\Pi(i) < \Pi(j)$, and (ii) $x_j \in X_i$, we have $x_i \succsim_i x_j$.

By Theorem 3, each bilateral weak core allocation is an almost-synchronized equilibrium. Such an allocation moreover satisfies *no justified envy* if and only if it is what we call a *cutoff equilibrium*: there are price p and priority cutoff κ such that (i) any buyer with a priority of κ or better is offered price p while the other buyers are offered price $p+1$, and (ii) any seller is offered either p or $p+1$.

Definition: Fix a discrete economy and let $c \in C$. An allocation x is a *cutoff equilibrium (given c)* if and only if there are $p \in \mathbb{Z}$ and $\kappa \in \{0, 1, ..., |N_0|\}$ such that

- for each $i \in N_0$, we have (i) $\Pi(i) \leq \kappa$ implies $x_i \in B_i^\delta(p)$, and (ii) $\Pi(i) > \kappa$ implies $x_i \in B_i^\delta(p+1)$; and
- for each $i \in N_1$, either $x_i \in B_i^\delta(p)$ or $x_i \in B_i^\delta(p+1)$.

In this case, we say that x is *supported by* (p, κ).

Proposition 5: Fix a discrete economy. For each auction configuration, an allocation satisfies *bilateral weak core* and *no justified envy* if and only if it is a cutoff equilibrium.

The proof is in Appendix E. As claimed, both versions of the pendulum auction select allocations that are not only in the bilateral weak core, but that moreover are cutoff equilibria.

Theorem 6: Fix a discrete environment. For each auction configuration and each type profile, both the buyer-optimal pendulum allocation and the efficient pendulum allocation are cutoff equilibria.

The proof is in Appendix E. We remark that while the priorities of the buyers are respected, we cannot analogously say that the seller arrangement is respected: if seller i is to the left of seller j, then it is easy to verify that (i) i might make a sale while j does not, causing j to envy i, and (ii) i might make a sale at price p while j makes a sale at price $p+1$, causing i to envy j.

3.4.3 Buyer-optimality

It is well-known that across a variety of one-to-one matching models, the core has a lattice structure, and moreover for each side there is a dominant core allocation (Shapley and Shubik, 1972; Knuth, 1976, attributed to John Conway). In fact, each *strategy-proof* core selection always selects a dominant core allocation for the side with private information in both assignment markets with continuous transfers (Holmström, 1979; Morimoto and Serizawa, 2015) and marriage markets (Alcalde and Barberà, 1994). That said, in our model with wealth constraints, there need not be a buyer-dominant member of the bilateral weak core.

EXAMPLE 8: *No allocation is buyer-dominant across almost-synchronized equilibria.* The economy may be either continuous or discrete. There are buyers i and j such that $(w_i, v_i) = (w_j, v_j) = (0, 100)$, and there is seller k such that $(w_k, v_k) = (0, 0)$. There are three almost-synchronized equilibria: (i) i trades with k at price 0, (ii) j trades with k with price 0, and (iii) there is no trade. Since i prefers the first to the others while j prefers the second to the others, thus no allocation is buyer-dominant across almost-synchronized equilibria.

We therefore consider two weaker notions of buyer-optimality. First, we require that that the allocation is not dominated for the buyers by any other allocation in a large class that contains the bilateral weak core; we remark that this sort of requirement was first considered in the context of the marriage problem (Roth, 1982). Second, we require that the allocation is dominant for the buyers across all cutoff equilibria.

DEFINITION: Fix a discrete economy and let $c \in C$. An allocation $x \in X$ is

- *not strictly buyer-dominated across bilateral and individually rational allocations* if and only (i) it is *bilateral* and *individually rational*, and (ii) there is no allocation x' that is *bilateral* and *individually rational* such that for each $i \in N_0$, $x'_i \succ_i x_i$; and
- *(weakly) buyer-dominant across cutoff equilibria* if and only if (i) it is a cutoff equilibrium, and (ii) for each cutoff equilibrium x' and each $i \in N_0$, we have $x_i \succsim_i x'_i$.

First, both versions of the pendulum auction select allocations that are not strictly buyer-dominated across bilateral and individual rational allocations.

THEOREM 7: Fix a discrete environment. For each auction configuration and each type profile, neither the buyer-optimal pendulum allocation nor the efficient pendulum allocation is strictly buyer-dominated across bilateral and individually rational allocations.

The proof is in Appendix F. Second, the buyer-optimal pendulum auction always selects an allocation that is buyer-dominant across cutoff equilibria.

THEOREM 8: Fix a discrete environment. For each auction configuration and each type profile, the buyer-optimal pendulum allocation is buyer-dominant across cutoff equilibria.

The proof is in Appendix F. We remark that the efficient pendulum auction does not always select such allocations, as we illustrate after presenting our results about efficiency with Example 9.

3.4.4 Efficiency

Whether or not an allocation is efficient may depend on whether or not side-payments are feasible. To formalize this point, let us say that an allocation is *constrained-efficient* if and only if (i) it is bilateral, and (ii) N does not weakly block it using another bilateral allocation. It is easy to verify that this is equivalent to ordinary efficiency under the additional constraint that side-payments are not feasible, which is a standard assumption in many models (Kelso and Crawford, 1982; Demange and Gale, 1985; Hatfield and Milgrom, 2005).

Definition: Fix a discrete economy. An allocation x satisfies *constrained efficiency* if and only if (i) x is bilateral, and (ii) there is no bilateral $x' \in Z$ such that N weakly blocks x using x'.

An allocation that is buyer-dominant across cutoff equilibria need not satisfy *constrained efficiency*: an indifferent buyer might not match with a seller who strictly prefers to trade, or a buyer might match to a seller who is indifferent when another seller would strictly prefer to trade at the same price. That said, the buyer-optimal pendulum auction always selects an allocation that satisfies *constrained efficiency*.

Theorem 9: Fix a discrete environment. For each auction configuration and each type profile, the buyer-optimal pendulum allocation satisfies *constrained efficiency*.

The proof is in Appendix G. Finally, the efficient pendulum auction always selects an *efficient* allocation.

Theorem 10: Fix a discrete environment. For each auction configuration and each type profile, the efficient pendulum allocation is *efficient*.

The proof is in Appendix G. To conclude this section, we remark that the buyer-optimal pendulum auction does not always select efficient allocations. We illustrate this claim, together with our earlier claim that the efficient pendulum auction does not always select allocations that are buyer-dominant across cutoff equilibria, with the following example.

Example 9: *The buyer-optimal pendulum allocation is not efficient, and the efficient pendulum allocation is not buyer-dominant across cutoff equilibria.* The economy is discrete. There are buyers i and j such that $(w_i, v_i) = (\infty, 4.5)$ and $(w_j, v_j) = (\infty, 5)$, and there is seller k such that $(w_k, v_k) = (\infty, 3)$. The priorities are $\Pi(i) = 1$ and $\Pi(j) = 2$. The buyer-optimal pendulum allocation is such that i trades with k at price 4; this is buyer-dominant across cutoff equilibria, but is not *efficient* because it is weakly blocked by all agents using the allocation where (i) j trades with k at price 4, and (ii) j makes a side-payment of 1 to i. The efficient pendulum allocation is such that j trades with k at price 5; this is *efficient*, but is not buyer-dominant across cutoff equilibria because i prefers the cutoff equilibrium that is the buyer-optimal pendulum allocation.

4 Discussion

We conclude with a discussion of three topics: (i) the relationship of the pendulum auctions to the minimum Walrasian price rules, (ii) the existence of obviously strategy-proof implementations of the bilateral weak core when objects are not identical, and

(iii) additional strategic properties of the pendulum auctions.

First, even when there are no wealth constraints, the buyer-optimal pendulum allocation, the efficient pendulum allocation, and the minimum Walrasian price allocations can all be distinct:

EXAMPLE 10: *Buyers prefer the buyer-optimal pendulum allocation to the efficient pendulum allocation, and prefer this to the unique minimum Walrasian price allocation.* The economy is discrete. There are buyers i, j, and k such that $(w_i, v_i) = (\infty, 2)$, $(w_j, v_j) = (\infty, 1.5)$, and $(w_k, v_k) = (\infty, 2)$, and there are sellers i' and j' such that $(w_{i'}, v_{i'}) = (w_{j'}, v_{j'}) = (\infty, 1)$. The priorities are $\Pi(i) = 1$, $\Pi(j) = 2$, and $\Pi(k) = 3$. In the buyer-optimal pendulum allocation, i and j both trade at price 1; in the efficient pendulum allocation, i trades at price 1 and k trades at price 2; in the unique minimum Walrasian price allocation, i and k both trade at price 2.

Second, for heterogeneous objects with unit demand, our *bilateral* axiom can be suitably generalized to require that there is a matching such that each agent's transfer is the negative of his match's transfer. Unfortunately, in this case there is no obviously strategy-proof implementation of the bilateral weak core even without wealth constraints. Indeed, Theorem 6.1 of Bade and Gonczarowski (2017) states that for heterogeneous objects with *multi-unit demand given by additive preferences* and without wealth constraints, there is no obviously strategy-proof implementation of efficient allocations that assign each loser zero transfer, and their proof involves two objects, two buyers, and a restricted domain with a common set of three possible valuation pairs shared by the bidders. By modifying the argument to involve *three* buyers, by modifying the restricted domain's common set of three possible valuation pairs to be $\{(10, 0), (0, 10), (5, 5)\}$, and by specifying that we have two sellers who value both objects at zero, it is relatively straightforward to adapt their proof to establish our claim. For brevity, we omit the formal details.

Finally, it is not hard to extend the arguments of Theorem 5 to draw a stronger conclusion about the incentives provided by pendulum auctions. In particular, for each type profile, the strategy profile specified by the convention is an *everywhere obviously dominant strategy equilibrium*: for each agent and *each* history where he plays—including those that cannot be reached when he plays according to the convention—the worst-case from adhering is at least as desirable as the best-case from deviating. For brevity, we omit the formal details. As highlighted by Mackenzie and Zhou (2022), any implementation involving such a solution concept preserves its incentives even if agents may make typos.[11]

Appendix

The appendices contain all of our proofs, with each appendix gathering the proofs for a particular topic. The first three appendices cover complete information: Appendix A covers pairwise statements for our cooperative axioms, Appendix B covers the strong core, and Appendix C covers the bilateral weak core. The final four appendices cover incomplete information and pendulum auctions: Appendix D covers obvious strategy-proofness, Appendix E covers cutoff equilibria, Appendix F covers buyer-optimality, and Appendix G covers efficiency. Throughout these appendices, we use i, j, k to index agents,

[11] We remark that another such obviously strategy-proof implementation was recently designed for the division problem with single-peaked preferences (Arribillaga, Massó, and Neme, 2022).

and our convention is to reserve earlier letters for named agents; thus we often let i and j denote specific agents while k denotes an arbitrary agent.

Appendix A: Pairwise results

In this appendix, we prove our results about pairwise statements for our cooperative axioms: Proposition 1 and Proposition 2.

To begin, we first prove the Pair-Blocking Lemma. This lemma states that if N' blocks x using internal allocation x', and if there is a reference internal allocation x^r that all members of N' find at most as desirable as x, then there is a pair of agents in N' who can trade from x^r to make one better off than at x while the other consumes precisely as in x'.

Pair-Blocking Lemma: Fix an economy. Suppose $x \in Z$, $N' \subseteq N$, and $x' \in Z_{N'}$ are such that N' weakly blocks x using x'. If there is $x^r \in Z_{N'}$ such that for each $k \in N'$, $x_k \succsim_k x^r_k$, then there are $i, j \in N'$ and $x^* \in X_i \times X_j$ such that (i) $x^*_i + x^*_j = x^r_i + x^r_j$, (ii) $x^*_i \succ_i x_i$, and (iii) $x^*_j = x'_j$.

Proof: Let x, N', x', and x^r satisfy the hypotheses. Define $N(x') \equiv \{k \in N' | a'_k = 1\}$ and define $N(x^r) \equiv \{k \in N' | a^r_k = 1\}$, extending our notation for allocations to internal allocations. Moreover, define $N_{0\to 1} \equiv N(x') \backslash N(x^r)$, define $N_{1\to 0} \equiv N(x^r) \backslash N(x')$, and define $N_= \equiv N' \backslash (N_{0\to 1} \cup N_{1\to 0})$. Finally, for each $k \in N'$, define $t^{\to}_k \equiv (t'_k - t^r_k)$.

To begin, since $\sum_{k\in N'} x'_k = \sum_{k\in N'} e_k = \sum_{k\in N'} x^r_k$, thus (i) $|N_{0\to 1}| = |N_{1\to 0}|$, and (ii) $\sum_{k\in N'} t^{\to}_k = 0$. Moreover, since for each $k \in N'$ we have $x'_k \succsim_k x_k \succsim_k x^r_k$, thus for each $k \in N_=$ we have $t^{\to}_k \geq 0$. Then we cannot have $N' = N_=$; else each $k \in N'$ has $t^{\to}_k = 0$ and thus $x_k \succsim_k x^r_k = x'_k$, contradicting that N' weakly blocks x using x'. Altogether, then, we have $|N_{0\to 1}| = |N_{1\to 0}| > 0$, so we can define the prices $p_0, p_1 \in \mathbb{T}$ by

$$
\begin{aligned}
p_0 &\equiv \max\{p \in \mathbb{T} | \exists k \in N_{0\to 1} \text{ such that } t^{\to}_k = -p\}, \text{ and} \\
p_1 &\equiv \min\{p \in \mathbb{T} | \exists k \in N_{1\to 0} \text{ such that } t^{\to}_k = p\}.
\end{aligned}
$$

Since

$$
\begin{aligned}
0 &= \sum_{k\in N'} t^{\to}_k \\
&= \sum_{k\in N_{0\to 1}} t^{\to}_k + \sum_{k\in N_{1\to 0}} t^{\to}_k + \sum_{k\in N_=} t^{\to}_k \\
&\geq |N_{0\to 1}| \cdot (-p_0) + |N_{1\to 0}| \cdot p_1 + 0 \\
&= |N_{0\to 1}| \cdot (-p_0) + |N_{0\to 1}| \cdot p_1,
\end{aligned}
$$

thus $p_0 \geq p_1$. We consider two cases.

If $p_0 > p_1$, then let $i \in N_{0\to 1}$ such that $t^{\to}_i = -p_0$, let $j \in N_{1\to 0}$ such that $t^{\to}_j = p_1$, and let $x^* \in X_i \times X_j$ be such that $x^*_i = (1, t^r_i - p_1)$ and $x^*_j = (0, t^r_j + p_1)$. Then $x^*_i \succ_i x'_i \succsim_i x_i$ and $x^*_j = x'_j$, as desired.

If $p_0 = p_1$, then observe that $\sum_{k\in N'} t^{\to}_k = 0$. Since $|N_{0\to 1}| = |N_{1\to 0}|$, this holds if (i) each $k \in N_{0\to 1}$ has $t^{\to}_k = -p_0$, (ii) each $k \in N_{1\to 0}$ has $t^{\to}_k = p_1$, and (iii) each $k \in N_=$ has $t^{\to}_k = 0$. Moreover, (i) no member of $N_{0\to 1}$ has $t^{\to}_k < -p_0$, (ii) no member of $N_{1\to 0}$ has $t^{\to}_k < p_1$, and (iii) no member of $N_=$ has $t^{\to}_k < 0$. Altogether, then, we have (i) each $k \in N_{0\to 1}$ has $t^{\to}_k = -p_0$, (ii) each $k \in N_{1\to 0}$ has $t^{\to}_k = p_1$, and (iii) each $k \in N_=$ has

$t_k^{\rightarrow} = 0$. Since for each $k \in N_=$ we have $x_k \succsim_k x_k^r = x_k'$, thus there is $i \in N_{0\to 1} \cup N_{1\to 0}$ such that $x_i' \succ_i x_i$. To conclude, define $p \equiv p_0 = p_1$. If $i \in N_{0\to 1}$, then let $j \in N_{1\to 0}$ and let $x^* \in X_i \times X_j$ be such that $x_i^* = (1, t_i^r - p)$ and $x_j^* = (0, t_j^r + p)$; if $i \in N_{1\to 0}$, then let $j \in N_{0\to 1}$ and let $x^* \in X_i \times X_j$ be such that $x_i^* = (0, t_i^r + p)$ and $x_j^* = (1, t_j^r - p)$. In both cases, $x_i^* = x_i' \succ_i x_i$ and $x_j^* = x_j'$, as desired. ■

Next, we prove Proposition 1.

PROPOSITION 1: Fix an economy. An allocation is (i) *weakly pairwise stable* if and only if it is *weak core*, and (ii) *strongly pairwise stable* if and only if it is *strong core*.

PROOF: By definition, *weak core* implies *weakly pairwise stability* and *strong core* implies *strong pairwise stability*. For each of the other two implications, we establish the contrapositive.

First, assume that $x \in Z$ violates *weak core*. Then there are $N' \subseteq N$ and $x' \in Z_{N'}$ such that N' strongly (and thus weakly) blocks x using x'. Since x is *individually rational*, thus we can apply the Pair-Blocking Lemma by taking $x^r = (e_k)_{k \in N'}$, so there are $i, j \in N'$ and $x^* \in X_i \times X_j$ such that (i) $x_i^* + x_j^* = e_i + e_j$, (ii) $x_i^* \succ_i x_i$, and (iii) $x_j^* = x_j'$. Moreover, we have $x_j' \succ_j x_j$. Altogether, then, $\{i, j\}$ strongly blocks x using x^*, so x violates *weak pairwise stability*, as desired.

To conclude, assume that $x \in Z$ violates *strong core*. Then there are $N' \subseteq N$ and $x' \in Z_{N'}$ such that N' weakly blocks x using x'. Since x is *individually rational*, thus we can apply the Pair-Blocking Lemma by taking $x^r = (e_k)_{k \in N'}$, so there are $i, j \in N$ and $x^* \in X_i \times X_j$ such that $\{i, j\}$ weakly blocks x using x^*; thus x violates *strong pairwise stability*, as desired. ■

To conclude this appendix, we prove Proposition 2.

PROPOSITION 2: Fix an economy. An allocation x violates *efficiency* if and only if there are $i \in N \backslash N(x)$, $j \in N(x)$, and $p \in \mathbb{T}$ such that for $x_i^* \equiv (1, t_i - p)$ and $x_j^* \equiv (0, t_j + p)$, we have (i) $x_i^* \in X_i$ and $x_i^* \succsim_i x_i$, (ii) $x_j^* \in X_j$ and $x_j^* \succsim_j x_j$, and (iii) there is $k \in \{i, j\}$ such that $x_k^* \succ_k x_k$.

PROOF: We prove both directions in sequence.

[$\Rightarrow$] Suppose $x \in Z$ violates *efficiency*. Then there is $x' \in Z = Z_N$ such that N weakly blocks x using x'. Since the blocking coalition is N, thus we can apply the Pair-Blocking Lemma by taking $x^r = (x_k)_{k \in N} = x$, so there are $i, j \in N$ and $x^* \in X_i \times X_j$ such that (i) $x_i^* + x_j^* = x_i + x_j$, (ii) $x_i^* \succ_i x_i$, and (iii) $x_j^* = x_j' \succsim_j x_j$. We cannot have $a_i^* = a_i$ and $a_j^* = a_j$, else $x_i^* \succ_i x_i$ implies $t_i^* > t_i$ and thus $t_j > t_j^*$ and thus $x_j \succ_j x_j^*$, contradicting $x_j^* \succsim_j x_j$. Then one member of $\{i, j\}$ belongs to $N(x^*) \backslash N(x)$ and the other belongs to $N(x) \backslash N(x^*)$; thus after re-indexing the agents if necessary, the desired conclusion is straightforward.

[$\Leftarrow$] Define $x' \in Z$ by $(x_i^*, x_j^*, x_{-i,j})$. It is trivial to verify that N weakly blocks x using x', so x violates *efficiency*. ■

Appendix B: The strong core

In this appendix, we prove our results about the strong core: Proposition 3, Theorem 1, and Theorem 2.

PROPOSITION 3: Fix a continuous economy. If an allocation is *efficient no-trade*, then it is a Walrasian equilibrium.

PROOF: Let $x \in Z$ be *efficient no-trade*. Define $p_0 \equiv \max_{k \in N_0} \min\{w_k, v_k\}$ and define $p_1 \equiv \min_{k \in N_1} \max\{-w_k, v_k\}$. Observe that the former is the highest price at which some buyer is willing and able to trade, while the latter is the lowest price at which some seller is willing and able to trade.

First, we claim that for each $k \in N_0$, we have $k \notin \mathsf{D}^!(p_1)$. Indeed, if there is $i \in N_0$ such that $i \in \mathsf{D}^!(p_1)$, then by definition there is $j \in N_1$ such that $j \in \mathsf{S}(p_1)$, so for $x' \in Z$ such that i and j trade at price p_1 while all other agents consume their endowments, we have that N weakly blocks x with x', contradicting that x is *efficient*.

Second, we claim that for each $k \in N_1$, we have $k \notin \mathsf{S}^!(p_0)$. Indeed, if there is $i \in N_1$ such that $i \in \mathsf{S}^!(p_0)$, then by definition there is $j \in N_0$ such that $j \in \mathsf{D}(p_0)$, so for $x' \in Z$ such that i and j trade at price p_0 while all other agents consume their endowments, we have that N weakly blocks x with x', contradicting that x is *efficient*.

To conclude, we cannot have $p_0 > p_1$, else there are $i \in N_0$ such that $p_i = p_0$ and $j \in N_1$ such that $p_j = p_1$, so for $x' \in Z$ such that i and j trade at price p_1 while all other agents consume their endowments, we have that N weakly blocks x with x', contradicting that x is *efficient*. If $p_0 = p_1$, then by the first two claims we have that for each $k \in N$, we have $x_k = e_k \in B_k^\delta(p_0) = B_k^\delta(p_1)$, so x is a Walrasian equilibrium supported by $p_0 = p_1$, as desired. Finally, if $p_1 > p_0$, then (i) for each $k \in N_0$, we have $\frac{p_0+p_1}{2} > p_0 \geq \min\{w_k, v_k\}$, so $x_k = e_k \in B_k^\delta(\frac{p_0+p_1}{2})$, and (ii) for each $k \in N_1$, we have $\max\{-w_k, v_k\} \geq p_1 > \frac{p_0+p_1}{2}$, so $x_k = e_k \in B_k^\delta(\frac{p_0+p_1}{2})$; thus x is a Walrasian equilibrium supported by $\frac{p_0+p_1}{2}$, as desired. ■

Next, we prove Theorem 1.

THEOREM 1: Fix an economy. An allocation satisfies *strong core* if and only if it is a Walrasian equilibrium or *efficient no-trade* (or both).

PROOF: We prove both directions in sequence.

[⇒] Let $x \in Z$ satisfy *strong core*. Define $N_{0\to1} \equiv N(x) \backslash N_1$, define $N_{1\to0} \equiv N_1 \backslash N(x)$, and define $N_= \equiv N \backslash (N_{0\to1} \cup N_{1\to0})$. Since $x \in Z$, thus $|N_{0\to1}| = |N_{1\to0}|$.

We first claim that for each $k \in N_=$, we have $t_k = 0$. Indeed, let $i \in N_=$. By *individual rationality*, we have $t_i \geq 0$, and moreover we cannot have $t_i > 0$; else $N\backslash\{i\}$ weakly blocks x with $x' \in \times_{k \in N\backslash\{i\}} X_k$ given by (i) some $j \in N\backslash\{i\}$ receives $(a_j, t_j + t_i)$, and (ii) each $k \in N\backslash\{i, j\}$ receives x_k, contradicting that x is *strong core*. Altogether, then, $t_i = 0$, as desired.

If $|N_{0\to1}| = |N_{1\to0}| = 0$, then by *strong core* and the above claim we have that x is *efficient no-trade*, as desired. Thus let us assume $|N_{0\to1}| = |N_{1\to0}| > 0$.

Define $p \equiv \min\{p' \in \mathbb{T} | \exists k \in N_{1\to0}$ such that $x_k = (0, p')\}$ and assume, by way of contradiction, there is $i \in N_{0\to1}$ such that $-p > t_i$. By definition, there is $j \in N_{1\to0}$ such that $x_j = (0, p)$; let $x' \in X_i \times X_j$ be such that i and j trade at price p. Then $\{i, j\}$ weakly blocks x with x', contradicting that x satisfies *strong core*.

Since $k \in N_=$ implies $t_k = 0$, thus $\sum_{k \in N_{0\to 1} \cup N_{1\to 0}} t_k = 0$. Since $|N_{0\to 1}| = |N_{1\to 0}|$, this holds if (i) each $k \in N_{0\to 1}$ has $t_k = -p$, and (ii) each $k \in N_{1\to 0}$ has $t_k = p$. Moreover, (i) no member of $N_{0\to 1}$ has $t_k < -p$, and (ii) no member of $N_{1\to 0}$ has $t_k < p$. Altogether, then, we have (i) each $k \in N_{0\to 1}$ has $t_k = -p$, and (ii) each $k \in N_{1\to 0}$ has $t_k = p$.

To conclude, assume, by way of contradiction, there is $i \in N$ such that $x_i \notin B_i^\delta(p)$. By *individual rationality*, for each $k \in N_{0\to 1} \cup N_{1\to 0}$ we have $x_k \in B_k^\delta(p)$, so necessarily $i \in N_=$. If $i \in N_0$, then let $j \in N_{1\to 0}$; if $i \in N_1$, then let $j \in N_{0\to 1}$. Finally, let $x' \in X_i \times X_j$ be such that i and j trade at price p. Then $\{i, j\}$ weakly blocks x using x', contradicting that x satisfies *strong core*. Altogether, then, for each $k \in N$ we have $x_k \in B_k^\delta(p)$, so x is a Walrasian equilibrium supported by p, as desired.

[$\Leftarrow$] Let $x \in Z$ be a Walrasian equilibrium or *efficient no-trade* (or both). If x is a Walrasian equilibrium, then each agent consumes a most-preferred bundle in a budget set that includes his endowment, while if x is *efficient no-trade*, then each agent consumes his endowment; thus in both cases x is *individually rational*. Assume, by way of contradiction, that x does not satisfy *strong core*. By Proposition 1, x does not satisfy *strong pairwise stability*.

Since x is *individually rational*, no singleton weakly blocks x, so there are $i, j \in N$ and $x^* \in Z_{\{i,j\}}$ such that $\{i, j\}$ weakly blocks x with x^*. Necessarily i and j trade in x^*; else as x is *individually rational* we have $x_i^* \succsim_i x_i \succsim_i e_i$ and $x_j^* \succsim_j x_j \succsim_j e_j$, so $t_i^* \geq 0$ and $t_j^* \geq 0$, so $t_i^* = 0$ and $t_j^* = 0$ and thus $x_i \succsim_i e_i = x_i^*$ and $x_j \succsim_j e_j = x_j^*$, contradicting that $\{i, j\}$ weakly blocks x using x^*. Thus we can assume, without loss of generality, that $i \in N_0$ and $j \in N_1$. To conclude, we consider two cases (which are not mutually exclusive).

First, suppose x is a Walrasian equilibrium. Then there is $p \in \mathbb{T}$ such that x is supported by p. Let $p^* \in \mathbb{T}$ be the price at which i and j trade in x^*. Since $x_i \in B_i^\delta(p)$, thus $(1, -p^*) = x_i^* \succsim_i x_i \succsim_i (1, -p)$, so $p^* \leq p$. Similarly, since $x_j \in B_j^\delta(p)$, thus $(0, p^*) = x_j^* \succsim_j x_j \succsim_j (0, p)$, so $p^* \geq p$. But then $p^* = p$, so $x_i \succsim_i (1, -p) = x_i^*$ and $x_j \succsim (0, p) = x_j^*$, contradicting that $\{i, j\}$ weakly blocks x using x^*. Altogether, then, x satisfies *strong core*, as desired.

Second, suppose x is *efficient no-trade*. Then for $x' \in Z$ such that i and j consume as in x^* while all other agents consume their endowments, we have that N weakly blocks x with x', contradicting that x is *efficient*. Altogether, then, x satisfies *strong core*, as desired. ■

To conclude this appendix, we prove Theorem 2.

Theorem 2: Fix an economy, and define $\underline{p}^\star \equiv \inf\{p \in \mathbb{T} \,|\, |\mathsf{S}(p)| \geq |\mathsf{D}^!(p)|\}$ and $\overline{p}^\star \equiv \sup\{p \in \mathbb{T} \,|\, |\mathsf{D}(p)| \geq |\mathsf{S}^!(p)|\}$. The Walrasian equilibria are in mutual correspondence with the prices in $[\underline{p}^\star, \overline{p}^\star] \cap \mathbb{T}$ in the following sense:

1. Both $\underline{p}^\star$ and $\overline{p}^\star$ are well-defined. Moreover, if $\mathbb{T} = \mathbb{R}$, then $\underline{p}^\star \leq \overline{p}^\star$.

2. For each Walrasian equilibrium x, there is $p \in [\underline{p}^\star, \overline{p}^\star] \cap \mathbb{T}$ such that x is supported by p.

3. For each $p \in (\underline{p}^\star, \overline{p}^\star) \cap \mathbb{T}$, there is a Walrasian equilibrium that is supported by p. Moreover,

- if $\min\{p \in \mathbb{T} | |\mathsf{S}(p)| \geq |\mathsf{D}^!(p)|\} = \underline{p}^\star < \overline{p}^\star$, then there is a Walrasian equilibrium that is supported by $\underline{p}^\star$,
- if $\underline{p}^\star < \overline{p}^\star = \max\{p \in \mathbb{T} | |\mathsf{D}(p)| \geq |\mathsf{S}^!(p)|\}$, then there is a Walrasian equilibrium that is supported by $\overline{p}^\star$,
- if $\min\{p \in \mathbb{T} | |\mathsf{S}(p)| \geq |\mathsf{D}^!(p)|\} = \underline{p}^\star = \overline{p}^\star = \max\{p \in \mathbb{T} | |\mathsf{D}(p)| \geq |\mathsf{S}^!(p)|\}$, then there is a Walrasian equilibrium that is supported by $\underline{p}^\star = \overline{p}^\star$, and
- if none of the above, then there is no Walrasian equilibrium that is supported by either $\underline{p}^\star$ or $\overline{p}^\star$.

Proof: We prove the three statements in sequence.

Proof of Statement 1: This is a simpler version of the proof of Statement 1 in Theorem 4.

First, we claim that for each $p \in \mathbb{T}$ such that $p < \min_{k \in N}\{w_k, v_k\}$, we have $|\mathsf{D}(p)| = |\mathsf{D}^!(p)| = |N_0|$ and $|\mathsf{S}(p)| = |\mathsf{S}^!(p)| = 0$. Indeed, let $p \in \mathbb{T}$ satisfy the hypothesis. For each $k \in N_0$, we have $w_k > p$ and $v_k > p$, so $k \in \mathsf{D}^!(p) \subseteq \mathsf{D}(p)$. Moreover, for each $k \in N_1$, we have $v_k > p$, so $k \notin \mathsf{S}(p) \supseteq \mathsf{S}^!(p)$. The claim follows immediately.

Second, we claim that for each $p \in \mathbb{T}$ such that $p > \max_{k \in N}\{-w_k, v_k\}$, we have $|\mathsf{D}(p)| = |\mathsf{D}^!(p)| = 0$ and $|\mathsf{S}(p)| = |\mathsf{S}^!(p)| = |N_1|$. Indeed, let $p \in \mathbb{T}$ satisfy the hypothesis. For each $k \in N_0$, we have $p > v_k$, so $k \notin \mathsf{D}(p) \supseteq \mathsf{D}^!(p)$. Moreover, for each $k \in N_1$, we have $p > -w_k$ and $p > v_k$, so $k \in \mathsf{S}^!(p) \subseteq \mathsf{S}(p)$. The claim follows immediately.

It follows directly from the two claims above that both $\underline{p}^\star$ and $\overline{p}^\star$ are well-defined. To conclude, suppose that $\mathbb{T} = \mathbb{R}$. Then for each $p \in \mathbb{R}$ such that $\underline{p}^\star > p$, we have by definition of $\underline{p}^\star$ that $|\mathsf{D}(p)| \geq |\mathsf{D}^!(p)| > |\mathsf{S}(p)| \geq |\mathsf{S}^!(p)|$ and thus by definition of $\overline{p}^\star$ that $\overline{p}^\star \geq p$. Altogether, then, we have $\overline{p}^\star \geq \underline{p}^\star$, as desired.

Proof of Statements 2 and 3: We prove both statements by first establishing a claim analogous to a theorem of Mishra and Talman (2010) and then concluding.

Claim: For each $p \in \mathbb{T}$, there is a Walrasian equilibrium supported by p if and only if both (i) $|\mathsf{S}(p)| \geq |\mathsf{D}^!(p)|$ and (ii) $|\mathsf{D}(p)| \geq |\mathsf{S}^!(p)|$.

We prove both directions of the claim in sequence.

[$\Rightarrow$] Let $p \in \mathbb{T}$ be such that there is a Walrasian equilibrium x supported by p. Then N is partitioned by (the nonempty members of) $N_0(p|x)$, $N_1(p|x)$, $N_0(e|x)$, and $N_1(e|x)$. Since $x \in Z$, thus $|N_0(p|x)| = |N_1(p|x)|$.

First, we claim $|\mathsf{S}(p)| \geq |\mathsf{D}^!(p)|$. Indeed, for each $k \in \mathsf{D}^!(p)$ we have $x_k \in B_k^\delta(p) = \{(1, -p)\}$, so $\mathsf{D}^!(p) \subseteq N_0(p|x)$. Moreover, for each $k \in N_1(p|x)$, we have $(0, p) = x_k \in B_k^\delta(p)$, so $N_1(p|x) \subseteq \mathsf{S}(p)$. Altogether, then, $|\mathsf{D}^!(p)| \leq |N_0(p|x)| = |N_1(p|x)| \leq |\mathsf{S}(p)|$, as desired.

Second, we claim $|\mathsf{D}(p)| \geq |\mathsf{S}^!(p)|$. Indeed, for each $k \in \mathsf{S}^!(p)$ we have $x_k \in B_k^\delta(p) = \{(0, p)\}$, so $\mathsf{S}^!(p) \subseteq N_1(p|x)$. Moreover, for each $k \in N_0(p|x)$, we have $(1, -p) = x_k \in B_k^\delta(p)$, so $N_0(p|x) \subseteq \mathsf{D}(p)$. Altogether, then, $|\mathsf{S}^!(p)| \leq |N_1(p|x)| = |N_0(p|x)| \leq |\mathsf{D}(p)|$, as desired.

[$\Leftarrow$] This is a simpler version of the proof of Statement 3 in Theorem 4. Let $p \in \mathbb{T}$ be such that both (i) $|\mathsf{S}(p)| \geq |\mathsf{D}^!(p)|$ and (ii) $|\mathsf{D}(p)| \geq |\mathsf{S}^!(p)|$. To begin, define $n^* \equiv \max\{|\mathsf{D}^!(p)|, |\mathsf{S}^!(p)|\}$; we will construct a Walrasian equilibrium where there are n^* trades at price p. We consider two cases (which are not mutually exclusive).

First, suppose $n^* = |\mathsf{D}^!(p)| \geq |\mathsf{S}^!(p)|$. Then we have $|\mathsf{S}(p)| \geq |\mathsf{D}^!(p)| \geq |\mathsf{S}^!(p)|$, so we can select $N_1' \subseteq \mathsf{S}(p)$ such that $|N_1'| = n^*$ and $N_1' \supseteq \mathsf{S}^!(p)$. Define $x \in Z$ such that (i) for each $k \in \mathsf{D}^!(p)$, $x_k = (1, -p)$, (ii) for each $k \in N_0 \backslash \mathsf{D}^!(p)$, $x_k = e_k$, (iii) for each $k \in N_1'$, $x_k = (0, p)$, and (iv) for each $k \in N_1 \backslash N_1'$, $x_k = e_k$. It is easy to verify that for each $k \in N$, we have $x_k \in B_k^\delta(p)$; thus x is a Walrasian equilibrium supported by p.

Second, suppose $n^* = |\mathsf{S}^!(p)| \geq |\mathsf{D}^!(p)|$. Then we have $|\mathsf{D}(p)| \geq |\mathsf{S}^!(p)| \geq |\mathsf{D}^!(p)|$, so we can select $N_0' \subseteq \mathsf{D}(p)$ such that $|N_0'| = n^*$ and $N_0' \supseteq \mathsf{D}^!(p)$. Define $x \in Z$ such that (i) for each $k \in N_0'$, $x_k = (1, -p)$, (ii) for each $k \in N_0 \backslash N_0'$, $x_k = e_k$, (iii) for each $k \in \mathsf{S}^!(p)$, $x_k = (0, p)$, and (iv) for each $k \in N_1 \backslash \mathsf{S}^!(p)$, $x_k = e_k$. It is easy to verify that for each $k \in N$, we have $x_k \in B_k^\delta(p)$; thus x is a Walrasian equilibrium supported by p.

Proof from claim: For Statement 2, let x be a Walrasian equilibrium. Then there is $p \in \mathbb{T}$ such that x is supported by p, so by the claim we have both (i) $|\mathsf{S}(p)| \geq |\mathsf{D}^!(p)|$ and (ii) $|\mathsf{D}(p)| \geq |\mathsf{S}^!(p)|$. Since $|\mathsf{S}(p)| \geq |\mathsf{D}^!(p)|$, thus $p \geq \underline{p}^\star$, and since $|\mathsf{D}(p)| \geq |\mathsf{S}^!(p)|$, thus $\overline{p}^\star \geq p$; altogether, then, we have $p \in [\underline{p}^\star, \overline{p}^\star] \cap \mathbb{T}$, as desired.

For Statement 3, observe that for each $p \in \mathbb{T}$, we have (i) $p > \underline{p}^\star$ if and only if there is $p' < p$ such that $|\mathsf{S}(p)| \geq |\mathsf{S}(p')| \geq |\mathsf{D}^!(p')| \geq |\mathsf{D}^!(p)|$, and (ii) $p < \overline{p}^\star$ if and only if there is $p' > p$ such that $|\mathsf{D}(p)| \geq |\mathsf{D}(p')| \geq |\mathsf{S}^!(p')| \geq |\mathsf{S}^!(p)|$. The conclusion is straightforward from this observation and the claim. ■

Appendix C: The bilateral weak core

In this appendix, we prove our results about the bilateral weak core: Proposition 4, Theorem 3, and Theorem 4.

Proposition 4: Fix a continuous economy. Each *weak core* allocation is *bilateral.*

Proof: This argument is similar to the proof of the [$\Rightarrow$] part of Theorem 1. To begin, let $x \in Z$ satisfy *weak core*. Define $N_{0\to1} \equiv N(x) \backslash N_1$, define $N_{1\to0} \equiv N_1 \backslash N(x)$, and define $N_= \equiv N \backslash (N_{0\to1} \cup N_{1\to0})$. Since $x \in Z$, thus $|N_{0\to1}| = |N_{1\to0}|$.

We first claim that for each $k \in N_=$, we have $t_k = 0$. Indeed, let $i \in N_=$. By *individual rationality*, we have $t_i \geq 0$, and moreover we cannot have $t_i > 0$; else $N \backslash \{i\}$ strongly blocks x by assigning to each $k \in N \backslash \{i\}$ the bundle $(a_k, t_k + \frac{t_i}{n-1})$, contradicting that x is *weak core*. Altogether, then, $t_i = 0$, as desired.

If $|N_{0\to1}| = |N_{1\to0}| = 0$, then by the above claim we have that x is *bilateral*, as desired. Thus let us assume $|N_{0\to1}| = |N_{1\to0}| > 0$.

Define $p \equiv \min\{p' \in \mathbb{T} | \exists k \in N_{1\to0} \text{ such that } x_k = (0, p')\}$ and assume, by way of contradiction, there is $i \in N_{0\to1}$ such that $-p > t_i$. By definition, there is $j \in N_{1\to0}$ such that $x_j = (0, p)$; let $x' \in X_i \times X_j$ be such that i and j trade at price $\frac{p - t_i}{2}$. Then $\{i, j\}$ strongly blocks x with x', contradicting that x satisfies *weak core*.

Since $k \in N_=$ implies $t_k = 0$, thus $\sum_{k \in N_{0\to1} \cup N_{1\to0}} t_k = 0$. Since $|N_{0\to1}| = |N_{1\to0}|$, this holds if (i) each $k \in N_{0\to1}$ has $t_k = -p$, and (ii) each $k \in N_{1\to0}$ has $t_k = p$. Moreover, (i) no member of $N_{0\to1}$ has $t_k < -p$, and (ii) no member of $N_{1\to0}$ has $t_k < p$. Altogether, then, we have (i) each $k \in N_{0\to 1}$ has $t_k = -p$, and (ii) each $k \in N_{1\to 0}$ has $t_k = p$; thus x is *bilateral*, as desired. ■

Next, we prove Theorem 3.

THEOREM 3: Fix an economy. An allocation satisfies *bilateral weak core* if and only if it is an almost-synchronized equilibrium.

PROOF: We prove both directions in sequence.

[$\Rightarrow$] Let $x \in Z$ satisfy *bilateral weak core*. Define

$$P \equiv \{p' \in \mathbb{T} | \exists k \in N_1 \text{ such that } x_k = (0, p')\}.$$

We consider three cases.

CASE 1: $|P| = 0$. If $\mathbb{T} = \mathbb{R}$, then define $p \equiv \max_{k \in N_0} \min\{w_k, v_k\}$; if $\mathbb{T} = \mathbb{Z}$, then define $p \equiv \max_{k \in N_0} \min\{w_k, \lfloor v_k \rfloor\}$. In both cases, we clearly have $p = \max\{p' \in \mathbb{T} || \mathsf{D}(p')| \geq 1\}$. By *bilaterality*, we have $x = e$.

First, suppose $|\mathsf{D}^!(p)| > 0$. By construction of p, for each $k \in N_0$ and each $p' \in P_{\nearrow}(p)$, $x_k = e_k \in B_k^\delta(p')$. Moreover, we must have $|\mathsf{S}^!(p)| = 0$, as otherwise there would be a buyer and a seller who strongly block x by trading at p, contradicting *weak core*. Then for each $k \in N_1$, $x_k = e_k \in B_k^\delta(p)$. Altogether, then, x is an almost-synchronized equilibrium supported by $(p, \nearrow)$.

Second, suppose $|\mathsf{D}^!(p)| = 0$. Then for each $k \in N_0$, $x_k = e_k \in B_k^\delta(p)$. Moreover, let $p' \in P_{\searrow}(p)$. Since $|\mathsf{D}(p)| > 0$, thus $|\mathsf{D}^!(p')| > 0$, so we must have $|\mathsf{S}^!(p')| = 0$, as otherwise there would be a buyer and a seller who strongly block x by trading at p', contradicting *weak core*. Since $p' \in P_{\searrow}(p)$ was arbitrary, thus for each $k \in N_1$ and each $p' \in P_{\searrow}(p)$, we have $x_k = e_k \in B_k^\delta(p')$. Altogether, then, x is an almost-synchronized equilibrium supported by $(p, \searrow)$.

CASE 2: $|P| = 1$. Let p denote the unique member of P. By *bilaterality*, (i) at least one buyer and one seller trade, (ii) each agent who trades in x does so at price p, and (iii) each agent who does not trade in x receives zero transfer. To begin, observe that either (i) $N_0(e|x) \cap \mathsf{D}^!(p) = \emptyset$, or (ii) $N_1(e|x) \cap \mathsf{S}^!(p) = \emptyset$, as otherwise there would be a buyer and a seller who strongly block x by trading at p, contradicting *weak core*. There are therefore two sub-cases (which are not mutually exclusive).

If $N_0(e|x) \cap \mathsf{D}^!(p) = \emptyset$, then we claim x is an almost-synchronized equilibrium supported by $(p, \searrow)$. Indeed, for each $k \in N_0(e|x)$, we have $x_k \in B_k^\delta(p)$. Moreover, for each $k \in N_1(e|x)$ and each $p' \in P_{\searrow}(p)$, we have $x_k \in B_k^\delta(p')$, as otherwise k and a buyer who trades at p would strongly block x by trading at p', contradicting *weak core*. Finally, each $k \in N$ who trades in x does so at price p and moreover has $x_k \in B_k^\delta(p)$, as otherwise k strongly blocks x by consuming e_k, contradicting *weak core*. This completes the proof of our claim.

If $N_1(e|x) \cap \mathsf{S}^!(p) = \emptyset$, then we claim x is an almost-synchronized equilibrium supported by $(p, \nearrow)$. Indeed, for each $k \in N_1(e|x)$, we have $x_k \in B_k^\delta(p)$. Moreover, for each $k \in N_0(e|x)$ and each $p' \in P_{\nearrow}(p)$, we have $x_k \in B_k^\delta(p')$, as otherwise k and a seller who trades at p would strongly block x by trading at p', contradicting *weak core*. Finally, each $k \in N$ who trades in x does so at price p and moreover has $x_k \in B_k^\delta(p)$, as otherwise k

strongly blocks x by consuming e_k, contradicting *weak core*. This completes the proof of our claim.

CASE 3: $|P| > 1$. In this case, the economy must be discrete. Indeed, assume by way of contradiction the economy is continuous. By *bilaterality*, there are a buyer who pays p_2 and a seller who receives p_1 such that $p_2 > p_1$. But then these agents can strongly block x by trading at $\frac{p_1+p_2}{2}$, contradicting *weak core*.

We claim that there is $p \in \mathbb{T} = \mathbb{Z}$ such that $P = \{p, p+1\}$. Indeed, for each pair $p, p' \in P$ such that $p' > p$, we must have $p' = p+1$; else by *bilaterality*, there are a buyer who pays $p' > p+1$ and a seller who receives p; but then these agents strongly block x by trading at $p+1$, contradicting *weak core*. The claim follows immediately.

To conclude, we claim that x is supported by $(p, \nearrow)$. Indeed, by *bilaterality*, (i) at least one buyer and one seller trade in x at price p, (ii) at least one buyer and one seller trade in x at price $p+1$, (iii) each agent who trades in x does so at either price p or price $p+1$, and (iv) each agent who does not trade in x receives zero transfer. First, each $k \in N$ who trades in x is assigned a member of either $B_k^\delta(p)$ or $B_k^\delta(p+1)$, as otherwise he strongly blocks x by consuming e_k, contradicting *weak core*. Moreover, each buyer k who does not trade in x receives a member of $B_k^\delta(p+1)$, as otherwise k and a seller who trades at p strongly block x by trading at $p+1$, contradicting *weak core*. Finally, each seller k who does not trade in x receives a member of $B_k^\delta(p)$, as otherwise k and a buyer who trades at $p+1$ strongly block x by trading at p, contradicting *weak core*. This completes the proof of our claim.

[$\Leftarrow$] Let $x \in Z$ be an almost-synchronized equilibrium. Then there are $p \in \mathbb{T}$ and $\dashrightarrow \in \{\searrow, \rightarrow, \nearrow\}$ such that x is supported by $(p, \dashrightarrow)$. We consider two cases.

CASE 1: The economy is continuous. In this case, (i) for each $k \in N_0$, $\cap_{p' \in P_{\dashrightarrow}(p)} B_k(p')$ is either $\{e_k\}$ or $\{e_k, (1, -p)\}$ and (ii) for each $k \in N_1$, $\cap_{p' \in P_{\dashrightarrow}(p)} B_k(p')$ is either $\{e_k\}$ or $\{e_k, (0, p)\}$; thus by definition of almost-synchronized equilibrium, we have that (i) each agent who trades in x does so at price p, and (ii) x is *bilateral*. Moreover, since each agent k receives a $\succsim_k$-optimal member of a budget set that includes e_k, thus x is *individually rational*, so no pair strongly blocks x without trading.

To conclude, we consider three cases. First, if $\dashrightarrow = \searrow$, then we have that $\mathsf{D}^!(p) \subseteq N_0(p|x)$ and thus no buyer can strongly block by trading at p or higher, and no seller can strongly block by trading at a price below p. Second, if $\dashrightarrow = \rightarrow$, then no buyer can strongly block by trading at p or higher, and no seller can strongly block by trading at p or lower. Finally, if $\dashrightarrow = \nearrow$, then no buyer can strongly block by trading at a price above p, and we have that $\mathsf{S}^!(p) \subseteq N_1(p|x)$ and thus no seller can strongly block by trading at p or lower. Altogether, then, in all three cases no pair strongly blocks x without trading and no pair strongly blocks x by trading, so x satisfies *weak pairwise stability*; thus by Proposition 1, x satisfies *weak core*, as desired.

CASE 2: The economy is discrete. In this case, it follows from the definition of almost-synchronized equilibrium that we can select $p \in \mathbb{Z}$ such that x is supported by $(p, \nearrow)$.

We first claim that x satisfies *weak core*. Indeed, since each agent k receives a $\succsim_k$-optimal member of a budget set that includes e_k, thus x is *individually rational*, so no pair strongly blocks x without trading. Moreover, each buyer consumes a bundle that he finds at least as desirable as trading at $p+1$, and each seller consumes a bundle that he finds at least as desirable as trading at p; it follows that no buyer and seller strongly block x by trading at some price. Altogether, then, x satisfies *weak pairwise stability*, so by Proposition 1 x satisfies *weak core*, as desired.

To conclude, we claim that x is *bilateral*. Indeed, since x is an almost-synchronized equilibrium, thus each agent who does not trade receives a zero transfer. Moreover, let a denote the number of buyers who trade at p, let b denote the number of buyers who trade at $p+1$, let a' denote the number of sellers who trade at p, and let b' denote the number of sellers who trade at $p+1$. Since $x \in Z$, thus we have (i) $ap+b(p+1) = a'p+b'(p+1)$, and (ii) $a+b = a'+b'$; thus by (i) we have $(a+b)p+b = (a'+b')p+b'$, so by (ii) we have $b = b'$ and thus $a = a'$. This completes the proof of our claim. ■

Before proving Theorem 4, we first prove the Directional Continuity Lemma. This lemma provides the leftward-continuity and rightward-continuity of (i) weak and forceful demand, and (ii) weak and forceful supply, for continuous economies.

Directional Continuity Lemma: For each continuous economy, (i) $|\mathsf{D}(p)|$ and $|\mathsf{S}^{!!}(p)|$ are rightward-continuous in p, and (ii) $|\mathsf{S}(p)|$ and $|\mathsf{D}^{!!}(p)|$ are leftward-continuous in p.[12]

Proof: For each $i \in N_0$, each $j \in N_1$, and each $p \in \mathbb{T} = \mathbb{R}$, define

$$\mathbb{1}_i(p) = \begin{cases} 1, & i \in \mathsf{D}(p), \\ 0, & \text{else}, \end{cases} \qquad \mathbb{1}^{!!}_i(p) = \begin{cases} 1, & i \in \mathsf{D}^{!!}(p), \\ 0, & \text{else}, \end{cases}$$
$$\mathbb{1}_j(p) = \begin{cases} 1, & j \in \mathsf{S}(p), \\ 0, & \text{else, and} \end{cases} \qquad \mathbb{1}^{!!}_j(p) = \begin{cases} 1, & j \in \mathsf{S}^{!!}(p), \\ 0, & \text{else.} \end{cases}$$

It is straightforward to verify that for each $i \in N_0$ and each $j \in N_1$, we have

$$(\mathbb{1}_i)^{-1}(1) = (-\infty, \min\{w_i, v_i\}], \qquad (\mathbb{1}^{!!}_i)^{-1}(1) = (-\infty, \min\{w_i, v_i\}),$$
$$(\mathbb{1}_j)^{-1}(1) = [\max\{-w_j, v_j\}, \infty), \text{ and} \qquad (\mathbb{1}^{!!}_j)^{-1}(1) = (\max\{-w_j, v_j\}, \infty).$$

It follows that for each $i \in N_0$ and each $j \in N_1$, we have that (i) $\mathbb{1}_i(p)$ and $\mathbb{1}^{!!}_j(p)$ are rightward-continuous in p, and (ii) $\mathbb{1}^{!!}_i(p)$ and $\mathbb{1}_j(p)$ are leftward-continuous in p. As rightward-continuity and leftward-continuity are properties of functions that are preserved under summation, it follows that (i) $|\mathsf{D}(p)|$ and $|\mathsf{S}^{!!}(p)|$ are rightward-continuous in p, and (ii) $|\mathsf{S}(p)|$ and $|\mathsf{D}^{!!}(p)|$ are leftward-continuous in p, as desired. ■

To conclude this appendix, we prove Theorem 4.

Theorem 4: Fix an economy, and define $\underline{p} \equiv \min\{p \in \mathbb{T} | |\mathsf{S}(p)| \geq |\mathsf{D}^{!!}(p)|\}$ and $\overline{p} \equiv \max\{p \in \mathbb{T} | |\mathsf{D}(p)| \geq |\mathsf{S}^{!!}(p)|\}$. The almost-synchronized equilibria are in mutual correspondence with the prices in $[\underline{p}, \overline{p}] \cap \mathbb{T}$ in the following sense:

1. Both $\underline{p}$ and $\overline{p}$ are well-defined with $\underline{p} \leq \overline{p}$.

[12] Recall that for each function $f : \mathbb{R} \to \mathbb{R}$ and each $x \in X$, we say that $y \in \mathbb{R}$ is a *rightward limit at* x if and only if for each $\varepsilon > 0$, there is $\delta > 0$ such that for each $x' \in (x-\delta, x]$, $|f(x') - y| < \varepsilon$. If x has a rightward limit, then it is unique and we denote it by $\lim_{x' \twoheadrightarrow x} f(x')$. We say that $f(x)$ is *rightward-continuous* if and only if for each $x \in \mathbb{R}$, (i) $\lim_{x' \twoheadrightarrow x} f(x')$ exists, and moreover (ii) $f(x) = \lim_{x' \twoheadrightarrow x} f(x')$. The analogous statements hold for *leftward limits* and *leftward-continuity*; we denote the leftward limit at x by $\lim_{x \twoheadleftarrow x'} f(x')$.

2. For each almost-synchronized equilibrium x, there is $p \in [\underline{p}, \overline{p}] \cap \mathbb{T}$ such that for some $\dashrightarrow \in \{\searrow, \rightarrow, \nearrow\}$, x is supported by $(p, \dashrightarrow)$.

3. For each $p \in [\underline{p}, \overline{p}] \cap \mathbb{T}$, there is an almost-synchronized equilibrium x such that for some $\dashrightarrow \in \{\searrow, \rightarrow, \nearrow\}$, x is supported by $(p, \dashrightarrow)$.

Proof: We prove the three statements in sequence.

Proof of Statement 1: To begin, define $P_{1,0} \equiv \{p \in \mathbb{T} | |\mathsf{S}(p)| \geq |\mathsf{D}^{!!}(p)|\}$ and $P_{0,1} \equiv \{p \in \mathbb{T} | |\mathsf{D}(p)| \geq |\mathsf{S}^{!!}(p)|\}$.

First, we claim that for each $p \in \mathbb{T}$ such that $p < \min_{k \in N}\{w_k - 1, v_k - 1\}$, we have $|\mathsf{D}(p)| = |\mathsf{D}^{!!}(p)| = |N_0|$ and $|\mathsf{S}(p)| = |\mathsf{S}^{!!}(p)| = 0$. Indeed, let $p \in \mathbb{T}$ satisfy the hypothesis. For each $k \in N_0$, we have (i) $p + 1 \in P_{\nearrow}(p)$; (ii) $w_k - 1 > p$, so $w_k > p + 1$; and (iii) $v_k - 1 > p$, so $v_k > p + 1$; thus $k \in \mathsf{D}^{!}(p + 1) \subseteq \mathsf{D}^{!!}(p) \subseteq \mathsf{D}(p)$. Moreover, for each $k \in N_1$, we have $v_k > v_k - 1 > p$, so $k \notin \mathsf{S}(p) \supseteq \mathsf{S}^{!!}(p)$. The claim follows immediately.

Second, we claim that for each $p \in \mathbb{T}$ such that $p > \max_{k \in N}\{-w_k + 1, v_k + 1\}$, we have $|\mathsf{D}(p)| = |\mathsf{D}^{!!}(p)| = 0$ and $|\mathsf{S}(p)| = |\mathsf{S}^{!!}(p)| = |N_1|$. Indeed, let $p \in \mathbb{T}$ satisfy the hypothesis. For each $k \in N_0$, we have $p > v_k + 1 > v_k$, so $k \notin \mathsf{D}(p) \supseteq \mathsf{D}^{!!}(p)$. Moreover, for each $k \in N_1$, we have (i) $p - 1 \in P_{\searrow}(p)$, (ii) $p > -w_k + 1$, so $p - 1 > -w_k$; and (iii) $p > v_k + 1$, so $p - 1 > v_k$; thus $k \in \mathsf{S}^{!}(p - 1) \subseteq \mathsf{S}^{!!}(p) \subseteq \mathsf{S}(p)$. The claim follows immediately.

It follows directly from the two claims above that both $\inf P_{1,0}$ and $\sup P_{0,1}$ are well-defined. If the economy is continuous, then by the Directional Continuity Lemma, since rightward-continuity and leftward-continuity are properties of functions that are preserved under multiplication by a constant and summation, thus $|\mathsf{D}(p)| - |\mathsf{S}^{!!}(p)|$ is rightward-continuous in p and $|\mathsf{S}(p)| - |\mathsf{D}^{!!}(p)|$ is leftward-continuous in p; it follows that $\sup P_{0,1} = \max P_{0,1}$ and $\inf P_{1,0} = \min P_{1,0}$. If the economy is discrete, then it is immediate that $\sup P_{0,1} = \max P_{0,1}$ and $\inf P_{1,0} = \min P_{1,0}$. Altogether, then, in both cases $\underline{p}$ and $\overline{p}$ are well-defined.

To conclude, we first claim that $\underline{p} \in P_{0,1}$. Indeed, if the economy is continuous, then by definition of $\underline{p}$, we have that for each $p < \underline{p}$, $|\mathsf{D}(p)| \geq |\mathsf{D}^{!!}(p)| > |\mathsf{S}(p)| \geq |\mathsf{S}^{!!}(p)|$, so $|\mathsf{D}(p)| - |\mathsf{S}^{!!}(p)| > 0$. In this case, as argued above, $|\mathsf{D}(p)| - |\mathsf{S}^{!!}(p)|$ is rightward-continuous in p; thus $|\mathsf{D}(\underline{p})| - |\mathsf{S}^{!!}(\underline{p})| \geq 0$, so $\underline{p} \in P_{0,1}$, as desired. If the economy is discrete, then by definition of $\underline{p}$, we have $|\mathsf{D}(\underline{p})| \geq |\mathsf{D}^{!}(\underline{p})| = |\mathsf{D}^{!!}(\underline{p} - 1)| > |\mathsf{S}(\underline{p} - 1)| \geq |\mathsf{S}^{!}(\underline{p} - 1)| = |\mathsf{S}^{!!}(\underline{p})|$, so $\underline{p} \in P_{0,1}$, as desired. Since in both cases we have $\underline{p} \in P_{0,1}$, thus by definition of $\overline{p}$, we have $\underline{p} \leq \overline{p}$, as desired.

Proof of Statement 2: Let $x \in Z$ be an almost-synchronized equilibrium. Then there are $p' \in \mathbb{T}$ and $\dashrightarrow' \in \{\searrow, \rightarrow, \nearrow\}$ such that x is supported by $(p', \dashrightarrow')$. We consider two cases.

Case 1: The economy is continuous. In this case, we have that (i) for each $k \in N_0$, $\cap_{p'' \in P_{\dashrightarrow}(p')} B_k(p'')$ is either $\{e_k\}$ or $\{e_k, (1, -p')\}$ and (ii) for each $k \in N_1$, $\cap_{p'' \in P_{\dashrightarrow}(p')} B_k(p'')$ is either $\{e_k\}$ or $\{e_k, (0, p')\}$; thus by definition of almost-synchronized equilibrium, each agent who trades in x does so at price p'. Then we must have $p' \geq \underline{p}$; else (i) by definition of $\underline{p}$, $|\mathsf{D}^{!!}(p')| > |\mathsf{S}(p')|$, (ii) at least $|\mathsf{D}^{!!}(p')|$ buyers trade in x, and (iii) at most $|\mathsf{S}(p')|$ sellers trade in x; contradicting $x \in Z$. Similarly, we must have $p' \leq \overline{p}$; else (i) by definition of $\overline{p}$, $|\mathsf{S}^{!!}(p')| > |\mathsf{D}(p')|$, (ii) at most $|\mathsf{D}(p')|$ buyers trade in x, and (iii) at least $|\mathsf{S}^{!!}(p')|$ sellers trade in x; contradicting $x \in Z$. Then $p' \in [\underline{p}, \overline{p}]$, as desired.

Case 2: The economy is discrete. In this case, define $p_1 \equiv \min(\{p'\} \cup P_{\dashrightarrow'}(p'))$ and define $p_2 \equiv \max(\{p'\} \cup P_{\dashrightarrow'}(p'))$; since $\mathbb{T} = \mathbb{Z}$ we have $p_2 - p_1 \in \{0, 1\}$. First, we must have $p_2 \geq \underline{p}$; else (i) by definition of $\underline{p}$, $|\mathsf{D}^{!!}(p_2)| > |\mathsf{S}(p_2)|$, (ii) at least $|\mathsf{D}^{!!}(p_2)|$ buyers trade in x, and (iii) at most $|\mathsf{S}(p_2)|$ sellers trade in x; contradicting $x \in Z$. Second, we must have $p_1 \leq \overline{p}$; else (i) by definition of $\overline{p}$, $|\mathsf{S}^{!!}(p_1)| > |\mathsf{D}(p_1)|$, (ii) at most $|\mathsf{D}(p_1)|$ buyers trade in x, and (iii) at least $|\mathsf{S}^{!!}(p_1)|$ sellers trade in x; contradicting $x \in Z$.

To conclude, assume, by way of contradiction, that $\{p_1, p_2\} \cap ([\underline{p}, \overline{p}] \cap \mathbb{T}) = \emptyset$. By the previous paragraph, (i) $p_2 \neq \underline{p}$, so $p_2 > \underline{p}$, and (ii) $p_1 \neq \overline{p}$, so $p_1 < \overline{p}$. Then we cannot have $\overline{p} \geq p_2$, else $p_2 \in [\underline{p}, \overline{p}] \cap \mathbb{T}$, contradicting $\{p_1, p_2\} \cap ([\underline{p}, \overline{p}] \cap \mathbb{T}) = \emptyset$. But then $p_2 > \overline{p} > p_1$ and $p_2 - p_1 \in \{0, 1\}$, contradicting $\overline{p} \in \mathbb{T} = \mathbb{Z}$. It follows directly that we can select $p \in \{p_1, p_2\} \cap ([\underline{p}, \overline{p}] \cap \mathbb{T})$ and $\dashrightarrow \in \{\searrow, \rightarrow, \nearrow\}$ such x is supported by $(p, \dashrightarrow)$, as desired.

Proof of Statement 3: Let $p \in [\underline{p}, \overline{p}] \cap \mathbb{T}$. By definition of $\underline{p}$, we have $|\mathsf{S}(p)| \geq |\mathsf{D}^{!!}(p)|$, and by definition of $\overline{p}$, we have $|\mathsf{D}(p)| \geq |\mathsf{S}^{!!}(p)|$. We consider two cases (which are not mutually exclusive).

Case 1: $|\mathsf{D}^{!}(p)| \geq |\mathsf{S}^{!}(p)|$. To begin, define $n^* \equiv \max\{|\mathsf{D}^{!!}(p)|, |\mathsf{S}^{!}(p)|\}$; we will construct an almost-synchronized equilibrium where there are n^* trades at price p.

First, since (i) $\mathsf{D}^{!}(p) \supseteq \mathsf{D}^{!!}(p)$, and (ii) $|\mathsf{D}^{!}(p)| \geq |\mathsf{S}^{!}(p)|$, thus we have $|\mathsf{D}^{!}(p)| \geq n^* \geq |\mathsf{D}^{!!}(p)|$, so we can select $N_0' \subseteq \mathsf{D}^{!}(p)$ such that $|N_0'| = n^*$ and $N_0' \supseteq \mathsf{D}^{!!}(p)$.

Second, since (i) $\mathsf{S}(p) \supseteq \mathsf{S}^{!}(p)$, and (ii) $|\mathsf{S}(p)| \geq |\mathsf{D}^{!!}(p)|$, thus we have $|\mathsf{S}(p)| \geq n^* \geq |\mathsf{S}^{!}(p)|$, so can select $N_1' \subseteq \mathsf{S}(p)$ such that $|N_1'| = n^*$ and $N_1' \supseteq \mathsf{S}^{!}(p)$.

To conclude, define $x \in Z$ such that (i) for each $k \in N_0'$, $x_k = (1, -p)$, (ii) for each $k \in N_0 \backslash N_0'$, $x_k = e_k$, (iii) for each $k \in N_1'$, $x_k = (0, p)$, and (iv) for each $k \in N_1 \backslash N_1'$, $x_k = e_k$. It is easy to verify that (i) for each $k \in N_0' \cup N_1$, we have $x_k \in B_k^{\delta}(p)$, and (ii) for each $k \in N_0 \backslash N_0'$ and each $p' \in P_{\nearrow}(p)$, we have $x_k \in B_k^{\delta}(p')$; thus x is an almost-synchronized equilibrium supported by $(p, \nearrow)$.

Case 2: $|\mathsf{S}^{!}(p)| \geq |\mathsf{D}^{!}(p)|$. To begin, define $n^* \equiv \max\{|\mathsf{D}^{!}(p)|, |\mathsf{S}^{!!}(p)|\}$; we will construct an almost-synchronized equilibrium where there are n^* trades at price p. The argument is symmetric to that in Case 1; we include the details for completeness.

First, since (i) $\mathsf{D}(p) \supseteq \mathsf{D}^{!}(p)$, and (ii) $|\mathsf{D}(p)| \geq |\mathsf{S}^{!!}(p)|$, thus we have $|\mathsf{D}(p)| \geq n^* \geq |\mathsf{D}^{!}(p)|$, so we can select $N_0' \subseteq \mathsf{D}(p)$ such that $|N_0'| = n^*$ and $N_0' \supseteq \mathsf{D}^{!}(p)$.

Second, since (i) $\mathsf{S}^{!}(p) \supseteq \mathsf{S}^{!!}(p)$, and (ii) $|\mathsf{S}^{!}(p)| \geq |\mathsf{D}^{!}(p)|$, thus we have $|\mathsf{S}^{!}(p)| \geq n^* \geq |\mathsf{S}^{!!}(p)|$, so can select $N_1' \subseteq \mathsf{S}^{!}(p)$ such that $|N_1'| = n^*$ and $N_1' \supseteq \mathsf{S}^{!!}(p)$.

To conclude, define $x \in Z$ such that (i) for each $k \in N_0'$, $x_k = (1, -p)$, (ii) for each $k \in N_0 \backslash N_0'$, $x_k = e_k$, (iii) for each $k \in N_1'$, $x_k = (0, p)$, and (iv) for each $k \in N_1 \backslash N_1'$, $x_k = e_k$. It is easy to verify that (i) for each $k \in N_0 \cup N_1'$, we have $x_k \in B_k^{\delta}(p)$, and (ii) for each $k \in N_1 \backslash N_1'$ and each $p' \in P_{\searrow}(p)$, we have $x_k \in B_k^{\delta}(p')$; thus x is an almost-synchronized equilibrium supported by $(p, \searrow)$. ■

Appendix D: Incentive compatibility

In this appendix, we prove our result about the incentive compatibility of the pendulum auctions: Theorem 5.

To begin, we first prove the Pendulum Lemma. This lemma provides some observations about both versions of the pendulum auction that will be useful throughout our remaining appendices. Moreover, this lemma establishes two claims made in the main text: for each pendulum lemma, we have that (i) for each $\theta \in \Theta$, both $h_{\mathbb{S}}$ and $p_{\mathbb{S}}^{\min}$ are well-defined, as claimed in the definition of the auction's convention (see Statement 2 below); and (ii) for each $\theta \in \Theta$, we have $x_{\mathbb{S}} \in Z(\theta)$, so the auction's rule is indeed a rule as claimed in its definition (see Statement 5 and Statement 8 below).

Pendulum Lemma: Fix a discrete environment. Let $\mathcal{V}$ be a version, let c be an auction configuration, let $\theta \in \Theta$, and for brevity define $x \equiv x_{\mathbb{S}}$. Then

1. For each $h \in H$, (i) the priorities of matched buyers get worse from left to right, and (ii) prices are non-decreasing from left to right.

2. $H_{\mathbb{S}}$ has a terminal history, so both $h_{\mathbb{S}}$ and $p_{\mathbb{S}}^{\min}$ are well-defined.

3. For each pair $i, j \in N_0$, each $p_j \in \mathbb{Z}$, and each three $h_i^-, h_i, h_i^+ \in H$ such that (i) at h_i^- we have that i is matched, (ii) h_i is the history immediately after h_i^-, (iii) at h_i we have that i is the player and j is matched at price p_j, and (iv) h_i^+ is the history immediately after i selects *exit* at h_i, we have the following: (i) at h_i^+ we have that j is matched at price p_j, (ii) if at h_i we have that j is matched with a resting seller, then we have this at h_i^+ as well, and (iii) if at h_i we have that j is matched with a rising seller, then we have this at h_i^+ as well.

4. For each $i \in N_0(e|x)$, we have $x_i \in B_i^\delta(p_{\mathbb{S}}^{\min} + 1)$. Moreover, let h_i be the final history in $H_{\mathbb{S}}$ where i plays. Then (i) i selects *exit* at h_i to remain in $H_{\mathbb{S}}$, and (ii) $x_i \in B_i^\delta(p^{\min}(h_i) + 1)$.

5. For each $p \in \mathbb{Z}$ and each $i \in N_0(p|x)$, we have $p \in \{p_{\mathbb{S}}^{\min}, p_{\mathbb{S}}^{\min} + 1\}$ and $x_i \in B_i^\delta(p)$. Moreover, let h_i be the final history in $H_{\mathbb{S}}$ where i plays. Then (i) i selects *bid* at h_i to remain in $H_{\mathbb{S}}$, (ii) $p \in \{p^{\min}(h_i), p^{\min}(h_i) + 1\}$, and (iii) at each history in $H_{\mathbb{S}}$ after h_i, i is matched with a seller at price p.

6. For each $p \in \mathbb{Z}$ and each $i \in N_1$, we have $i \in \mathsf{S}(p)$ if and only if $p \geq p_i(h_\wedge)$.

7. For each $i \in N_1(e|x)$, we have $x_i \in B_i^\delta(p_{\mathbb{S}}^{\min} - 1)$.

8. For each $p \in \mathbb{Z}$ and each $i \in N_1(p|x)$, we have $p \in \{p_{\mathbb{S}}^{\min}, p_{\mathbb{S}}^{\min} + 1\}$ and $x_i \in B_i^\delta(p)$.

9. For each $p \in \mathbb{Z}$ and each pair $i, j \in N_1$ such that i is to the left of j, we have (i) $j \in \mathsf{S}(p)$ implies $i \in \mathsf{S}(p)$, and (ii) $j \in \mathsf{S}^!(p)$ implies $i \in \mathsf{S}^!(p)$.

Proof: Fix a version $\mathcal{V}$ and an auction configuration c, let $\theta \in \Theta$, and for brevity define $x \equiv x_{\mathbb{S}}$. We prove the seven statements in sequence.

Proof of Statement 1: To begin, we claim that at each history, the priorities of the matched buyers get worse from left to right. Indeed, this follows directly from the following observations: (i) buyers are called to play for the first time from the queue in order of worsening priority, (ii) unmatched sellers are matched from left to right, (iii) if a matched seller becomes unmatched and then his previous match bids, then his previous match returns to him, and (iv) if a matched seller i becomes unmatched and then his previous match exits, then any buyer matched with a seller to the right of i moves one seller to the left.

To conclude, we use induction to prove that at each history, prices are non-decreasing (from left to right). For the base step, we have non-decreasing prices at $h_\wedge$. For the inductive step, let $h, h' \in H$ be such that (i) we have non-decreasing prices at h, and (ii) h' is an immediate successor of h. If prices are the same at h and h', then prices are non-decreasing at h' and we may proceed; thus let us assume there is $i \in N_1$ such that $p_i(h') \neq p_i(h)$. Then at h, we have that prices are non-decreasing and moreover that i is the rightmost seller offering $p^{\min}(h)$, so (i) each seller to the left of i offers $p^{\min}(h)$, and (ii) any sellers to the right of i offer non-decreasing prices that are each at least $p^{\min}(h)+1$. It follows that at h', we have that (i) each seller to the left of i offers $p^{\min}(h)$, (ii) i offers $p^{\min}(h)+1$, and (iii) any sellers to the right of i offer non-decreasing prices that are each at least $p^{\min}(h)+1$. Altogether, then, prices are non-decreasing at h'. Since each history can be reached from $h_\wedge$ through a path of immediate successors, thus by induction we are done.

Proof of Statement 2: It follows from the definition of the pendulum auctions that the minimum price offered by sellers always rises after at most $5|N_1|-1$ bids; thus as the convention specifies that each buyer should exit if this price exceeds his valuation, necessarily $H_{\mathbb{S}}$ has a terminal history $h_{\mathbb{S}}$ with minimum price $p^{\min}_{\mathbb{S}} \equiv p^{\min}(h_{\mathbb{S}})$, as desired.

Proof of Statement 3: Let i, j, p_j, h_i^-, h_i, and h_i^+ satisfy the hypotheses. Moreover, let $\mu(i)$ denote the match of i at h_i^-, let p_i denote the price that $\mu(i)$ offers at h_i^-, and let $\mu(j)$ denote the match of j at h_i. If $\mu(j)$ is to the left of $\mu(i)$, then j does not unmatch and re-match immediately after i exits and we are done; thus let us assume that $\mu(j)$ is to the right of $\mu(i)$.

We begin with three observations. First, by Statement 1 we have $p_i \leq p_j$. Second, since $\mu(i)$ becomes unmatched between h_i^- and h_i, thus at h_i^- we have that j is matched with $\mu(j)$ at price p_j. Third, since at h_i^- we have that $\mu(i)$ is matched at price p_i, thus by the first two observations and the definition of the pendulum auctions we have $p_j \in \{p_i, p_i+1\}$. To conclude, we consider three cases.

First, if $p_j = p_i$, then by definition of the pendulum auctions, at h_i^- we have that $\mu(i)$ is the rightmost resting matched seller offering p_i; thus at both h_i and h_i^+ we have that all sellers from $\mu(i)$ to $\mu(j)$ offer $p_j = p_i$ while rising, including both the match of j at h_i and the match of j at h_i^+, as desired.

Second, if $p_j = p_i + 1$ and we have fixed the buyer-optimal pendulum auction, then by definition of this auction, at h_i^- we have that (i) $\mu(i)$ is the rightmost seller offering p_i, (ii) since there is a matched seller offering p_i+1, thus $\mu(i)$ is rising matched, and (iii) each seller offering p_i+1 is rising matched; thus by definition of this auction, at both h_i and h_i^+ we have that all sellers from $\mu(i)$ to $\mu(j)$ offer $p_j = p_i+1$ while rising, including both the match of j at h_i and the match of j at h_i^+, as desired.

Third, if $p_j = p_i + 1$ and we have fixed the efficient pendulum auction, then by definition of this auction, at h_i^- we have that (i) $\mu(i)$ is the rightmost seller offering p_i, (ii) since there is a matched seller offering p_i+1, thus $\mu(i)$ is rising matched, and (iii) each seller offering p_i+1 is resting; thus by definition of this auction, at both h_i and h_i^+ we have that all sellers from $\mu(i)$ to $\mu(j)$ offer $p_j = p_i+1$ while resting, including both the match of j at h_i and the match of j at h_i^+, as desired.

Proof of Statement 4: Let i and h_i satisfy the hypotheses. Since $i \in N_X(h_{\mathbb{S}})$, thus i selects *exit* at h_i to remain in $H_{\mathbb{S}}$. We cannot have $i \in \mathsf{D}^!(p^{\min}(h_i)+1)$, else for both auctions the convention recommends *bid* at h_i, contradicting that i follows the convention during $H_{\mathbb{S}}$. Then $x_i = e_i \in B_i^\delta(p^{\min}(h_i)+1)$. Since prices do not decrease during $H_{\mathbb{S}}$, thus $p_{\mathbb{S}}^{\min} \geq p^{\min}(h_i)$, so $x_i = e_i \in B_i^\delta(p_{\mathbb{S}}^{\min}+1)$, as desired.

Proof of Statement 5: Let p, i, and h_i satisfy the hypotheses. Since i is matched at $h_{\mathbb{S}}$, thus i selects *bid* at h_i to remain in $H_{\mathbb{S}}$. Moreover, since i does not play in $H_{\mathbb{S}}$ after h_i, thus there is $p_i \in \mathbb{Z}$ such that at the history in $H_{\mathbb{S}}$ immediately after h_i, i is matched with a seller at price p_i. By Statement 3, whenever the exit of another buyer causes i to unmatch and re-match, we have that i re-matches at the same price; thus by induction on the histories in $H_{\mathbb{S}}$ after h_i, at $h_{\mathbb{S}}$ we have that i is matched at price p_i, so $p = p_i$. It follows that from the definition of the pendulum auctions and our inductive argument that (i) $p \in \{p_{\mathbb{S}}^{\min}, p_{\mathbb{S}}^{\min}+1\}$, (ii) $p = p_i \in \{p^{\min}(h_i), p^{\min}(h_i)+1\}$, and (iii) at each history in $H_{\mathbb{S}}$ after h_i, i is matched with a seller at price $p = p_i$. Finally, by the convention, we have that $x_i = (1, -p_i) \in B_i^\delta(p_i) = B_i^\delta(p)$, as desired.

Proof of Statement 6: Let $p \in \mathbb{Z}$ and let $i \in N_1$. First, if $i \in \mathsf{S}(p)$, then (i) $p \geq -w_i$, and (ii) $p \geq v_i$ and thus $p \geq \lceil v_i \rceil$; thus $p \geq \max\{-w_i, \lceil v_i \rceil\} = p_i(h_\wedge)$, as desired. Second, if $p \geq p_i(h_\wedge)$, then $p \geq \max\{-w_i, \lceil v_i \rceil\}$, so (i) $p \geq -w_i$, and (ii) $p \geq \lceil v_i \rceil \geq v_i$; thus $i \in \mathsf{S}(p)$, as desired.

Proof of Statement 7: Let $i \in N_1(e|x)$ and assume, by way of contradiction, that $p_{\mathbb{S}}^{\min} > p_i(h_\wedge)$. Then $p_i(h_{\mathbb{S}}) \geq p_{\mathbb{S}}^{\min} > p_i(h_\wedge)$, so the price of i increases during $H_{\mathbb{S}}$, so there is a history in $H_{\mathbb{S}}$ where i has a match; but then by the convention i has a match at $h_{\mathbb{S}}$, contradicting $i \in N_1(e|x)$. Thus $p_i(h_\wedge) \geq p_{\mathbb{S}}^{\min} > p_{\mathbb{S}}^{\min} - 1$, so by Statement 6 we have that $i \notin \mathsf{S}(p_{\mathbb{S}}^{\min}-1)$, so $x_i = e_i \in B_i^\delta(p_{\mathbb{S}}^{\min}-1)$, as desired.

Proof of Statement 8: Let p and i satisfy the hypotheses. By definition of the pendulum auctions, we have $p \in \{p_{\mathbb{S}}^{\min}, p_{\mathbb{S}}^{\min}+1\}$. Moreover, $p = p_i(h_{\mathbb{S}}) \geq p_i(h_\wedge)$, so by Statement 6 we have that $i \in \mathsf{S}(p)$, so $x_i = (0,p) \in B_i^\delta(p)$, as desired.

Proof of Statement 9: Let p, i, and j satisfy the hypotheses. Since i is to the left of j, thus either (i) $p_i(h_\wedge) < p_j(h_\wedge)$, or (ii) $p_i(h_\wedge) = p_j(h_\wedge)$ and $v_i \leq v_j$.

If $j \in \mathsf{S}(p)$, then by Statement 6 we have that $p \geq p_j(h_\wedge) \geq p_i(h_\wedge)$, so by Statement 6 we have that $i \in \mathsf{S}(p)$, as desired.

If moreover $j \in \mathsf{S}^!(p)$, then we consider two cases. In the first case where $\underline{p}_i < \underline{p}_j$, we have $p \geq \underline{p}_j > \underline{p}_i \geq \lceil v_i \rceil \geq v_i$, so $i \in \mathsf{S}^!(p)$, as desired. In the second case where $\underline{p}_i = \underline{p}_j$ and $v_i \leq v_j$, we have $p > v_j \geq v_i$, so $i \in \mathsf{S}^!(p)$, as desired. ■

To conclude this appendix, we prove Theorem 5.

Theorem 5: Fix a discrete environment. For each auction configuration, each version of the pendulum auction is an obviously strategy-proof implementation of its rule through its convention.

Proof: Fix a version $\mathcal{V}$ and an auction configuration c, let $\theta \in \Theta$, and for brevity let φ denote the auction's rule. By definition of φ, we have $\mathcal{X}((\mathbb{S}_i(\theta_i))_{i \in N_0}) = \varphi(\theta)$; thus it remains to show that $(\mathbb{S}_i(\theta_i))_{i \in N_0} \in \mathbf{OSP}(\theta)$. Let $i \in N_0$ and let $h \in H_i$ be a history that can be reached when i plays $\mathbb{S}_i(\theta_i)$.

To begin, for each $h' \in H_i$, define the prices $p^?(h'), p^*(h') \in \mathbb{Z}$ as follows. First, let $p^?(h')$ denote the unique price $p \in \{p^{\min}(h'), p^{\min}(h') + 1\}$ such that either (i) the convention recommends that i bid at h' if and only if $i \in \mathsf{D}(p)$, or (ii) the convention recommends that i bid at h' if and only if $i \in \mathsf{D}^!(p)$; thus $p^?(h')$ is the price that i is asked about at h'. Second, let $p^*(h')$ denote the minimum price at which i can be matched across terminal histories that follow h', which is well-defined because we reach a terminal history where i is matched when from h' onward we have that i always bids while his peers always exit; thus $p^*(h')$ is the best price that i can be assigned at the end of the auction following h'. From here, we prove three claims and then conclude.

Claim 1: For each finite play $H' \subseteq H$, if h_i denotes the last history in H' where i plays and if x_i denotes the bundle that i is assigned at the terminal history of H', then (i) if i selects *exit* at h_i to remain in H', then $x_i = e_i$, and (ii) if i selects *bid* at h_i to remain in H', then $x_i = (1, -p^?(h_i))$. Indeed, the first part of this claim is trivial. For the second part, since (i) by definition of the pendulum auctions and their conventions, at the history in H' immediately after h_i, i is matched with a seller at price $p^?(h_i)$; and (ii) by the Pendulum Lemma, whenever the exit of another buyer causes i to unmatch and re-match, we have that i re-matches at the same price; thus by induction on the histories in H' after h_i, at the terminal history of H' we have that i is matched at price $p^?(h_i)$, so $x_i = (1, -p^?(h_i))$, as desired.

Claim 2: $p^*(h) \geq p^?(h)$. Indeed, it follows from the definition of the pendulum auctions and their conventions that along each finite play, $p^?(h')$ is non-decreasing; thus the desired conclusion follows immediately from Claim 1. (In fact, with some additional effort it can be verified that $p^*(h) = p^?(h)$, but we do not require this observation for our proof.)

Claim 3: At h, the worst-case scenario from adhering to the convention is at least as desirable as the best-case scenario from deviating. We consider two cases.

First, suppose the convention recommends that i exit at h. Then $i \notin \mathsf{D}^!(p^?(h))$, and by Claim 2 we have $p^*(h) \geq p^?(h)$, so altogether $i \notin \mathsf{D}^!(p^*(h))$. Since (i) the worst-case scenario from adhering is e_i, (ii) the best-case scenario from deviating, taken across infinite plays, is e_i, and (iii) the best-case scenario from deviating, taken across finite plays, is either e_i or $(1, -p^*(h))$, we are done.

Second, suppose the convention recommends that i bid at h. We claim that for each finite play where i follows the convention at his last history by bidding, i receives a bundle that is at least as desirable as e_i at the terminal history. Indeed, let $H' \subseteq H$ be such a play and let h_i denote the last history in H' where i plays. Since the convention recommends *bid* at h_i, thus $i \in \mathsf{D}(p^?(h_i))$, and by Claim 1 we have that i receives $(1, -p^?(h_i))$ at the terminal history; thus altogether i receives a bundle that is at least as desirable as e_i at the terminal history, as desired. Since (i) the worst-case scenario from adhering, taken across infinite plays, is e_i, (ii) by our previous claim, the worst-case scenario from adhering, taken across finite plays where i follows the convention at his last history by bidding, is at least as desirable as e_i, (iii) by Claim 1, the worst-case scenario from adhering, taken across finite plays where i follows the convention at his last history by exiting, is e_i, and (iv) the best-case scenario from deviating is e_i, we are done.

To conclude, since $h \in H_i$ was an arbitrary history that can be reached when i plays $\mathbb{S}_i(\theta_i)$, thus $\mathbb{S}_i(\theta_i)$ is obviously dominant. Since $i \in N_0$ was arbitrary, thus $(\mathbb{S}_i(\theta_i))_{i \in N_0} \in$

OSP(θ). Finally, since $\theta \in \Theta$ was arbitrary, we are done. ■

Appendix E: Core selection

In this appendix, we prove our results about the core selection of the pendulum auctions: Proposition 5 and Theorem 6.

PROPOSITION 5: Fix a discrete economy. For each auction configuration, an allocation satisfies *bilateral weak core* and *no justified envy* if and only if it is a cutoff equilibrium.

PROOF: Let $c \in C$. We prove both directions in sequence.

[$\Rightarrow$] Let x satisfy the hypotheses. By Theorem 3, x is an almost-synchronized equilibrium, so there is $p \in \mathbb{Z}$ such that for each $k \in N$, either $x_k \in B^\delta_k(p)$ or $x_k \in B^\delta_k(p+1)$. We consider two cases.

CASE 1: $N_0(p|x) = \emptyset$. In this case, x is clearly a cutoff equilibrium supported by $(p, 0)$.

CASE 2: $N_0(p|x) \neq \emptyset$. In this case, let i denote the worst-priority buyer in $N_0(p|x)$ and define $\kappa \equiv \Pi(i)$. We claim that x is supported by (p, κ).

Indeed, let $k \in N_0$. First, if $\Pi(k) = \kappa$, then $k = i$ and clearly $x_k \in B^\delta_k(p)$. Second, if $\Pi(k) > \kappa$, then either (i) $x_k = (1, -(p+1)) \in B^\delta_k(p+1)$, or (ii) $x_k = e_k \in B^\delta_k(p) \cup B^\delta_k(p+1)$ and thus $x_k = e_k \in B^\delta_k(p+1)$. Finally, if $\Pi(k) < \kappa$, then we cannot have $x_k = (1, -(p+1))$, else we have $\Pi(k) < \Pi(i)$, $x_i \in X_k$, and $x_i \succ_k x_k$, contradicting *no justified envy*. Then $x_k = e_k$ or $x_k = (1, -p)$. If $x_k = e_k$, then since $\Pi(k) < \Pi(i)$ and $x_i = (1, -p)$, thus by *no justified envy* we have $x_k \in B^\delta_k(p)$, while if $x_k = (1, -p)$, then clearly we have $x_k \in B^\delta_k(p)$. Altogether, then, for each $k \in N_0$ we have (i) $\Pi(k) \leq \kappa$ implies $x_k \in B^\delta_k(p)$, and (ii) $\Pi(k) > \kappa$ implies $x_k \in B^\delta_k(p+1)$, so x is a cutoff equilibrium supported by (p, κ), as desired.

[$\Leftarrow$] Let x be a cutoff equilibrium. Then there are $p \in \mathbb{Z}$ and $\kappa \in \{0, 1, ..., |N_0|\}$ such that x is a cutoff equilibrium supported by (p, κ), so x is an almost-synchronized equilibrium supported by $(p, \nearrow)$, so by Theorem 3 we have that x satisfies *bilateral weak core*. Moreover, since a buyer i who is offered price p_i cannot envy a buyer j who is offered price $p_j \geq p_i$ and consumes $x_j \in X_i$, thus it follows directly from the definition of cutoff equilibrium that x satisfies *no justified envy*, as desired. ■

To conclude this appendix, we prove Theorem 6.

THEOREM 6: Fix a discrete environment. For each auction configuration and each type profile, both the buyer-optimal pendulum allocation and the efficient pendulum allocation are cutoff equilibria.

PROOF: Fix a version $\mathcal{V}$ and an auction configuration c, let $\theta \in \Theta$, and for brevity define $x \equiv x_{\mathbb{S}}$. We consider two cases.

CASE 1: There is $i \in \mathsf{S}^!(p^{\min}_{\mathbb{S}})$ such that $x_i = e_i$. In this case, by the Pendulum Lemma we have $p^{\min}_{\mathbb{S}} \geq p_i(h_\wedge)$. Moreover, since $x_i = e_i$, thus by the convention i never increases his price during $H_{\mathbb{S}}$, so $p_i(h_\wedge) = p_i(h_{\mathbb{S}}) \geq p_{\mathbb{S}}^{\min}$. Altogether, then, $p_i(h_\wedge) = p_i(h_{\mathbb{S}}) = p_{\mathbb{S}}^{\min}$.

In this case, we claim that x is a cutoff equilibrium supported by $(p_{\mathbb{S}}^{\min} - 1, 0)$. Indeed, by definition of the pendulum auctions, any matched sellers at $h_{\mathbb{S}}$ are to the left of i, so by the Pendulum Lemma any matched seller at $h_{\mathbb{S}}$ offers $p_{\mathbb{S}}^{\min}$; thus each agent who trades in x does so at price $p_{\mathbb{S}}^{\min}$. By the Pendulum Lemma, each $k \in N_0(p_{\mathbb{S}}^{\min}|x) \cup N_1(p_{\mathbb{S}}^{\min}|x)$ consumes a member of $B_k^\delta(p_{\mathbb{S}}^{\min})$. Moreover, for each $k \in N_0(e|x) = N_0 \backslash N_0(p_{\mathbb{S}}^{\min}|x)$, there is a history where k selects *exit* while i is unmatched and offering $p_{\mathbb{S}}^{\min}$, so by the convention k consumes a member of $B_k^\delta(p_{\mathbb{S}}^{\min})$. Finally, by the Pendulum Lemma, each seller $k \in N_1(e|x) = N_1 \backslash N_1(p_{\mathbb{S}}^{\min}|x)$ consumes a member of $B_k^\delta(p_{\mathbb{S}}^{\min} - 1)$. Altogether, then, x is a cutoff equilibrium supported by $(p_{\mathbb{S}}^{\min} - 1, 0)$, as desired.

Case 2: There is no $k \in \mathsf{S}^!(p_{\mathbb{S}}^{\min})$ such that $x_k = e_k$. In this case, we first claim that x is an almost-synchronized equilibrium supported by $(p_{\mathbb{S}}^{\min}, \nearrow)$. Indeed, by the Pendulum Lemma, we have that each $k \in N_0$ consumes a member of either $B_k^\delta(p_{\mathbb{S}}^{\min})$ or $B_k^\delta(p_{\mathbb{S}}^{\min} + 1)$. Moreover, by the assumption of this case, we have that each $k \in N_1(e|x)$ consumes a member of $B_k^\delta(p_{\mathbb{S}}^{\min})$; thus by the Pendulum Lemma, each $k \in N_1$ consumes a member of either $B_k^\delta(p_{\mathbb{S}}^{\min})$ or $B_k^\delta(p_{\mathbb{S}}^{\min} + 1)$. Altogether, then, x is an almost-synchronized equilibrium supported by $(p_{\mathbb{S}}^{\min}, \nearrow)$, as desired.

If $N_0(p_{\mathbb{S}}^{\min}|x) = \emptyset$, then clearly x is a cutoff equilibrium supported by $(p_{\mathbb{S}}^{\min}, 0)$ and we are done; thus let us assume $N_0(p_{\mathbb{S}}^{\min}|x) \neq \emptyset$. In this case, let i denote the worst-priority buyer in $N_0(p_{\mathbb{S}}^{\min}|x)$ and define $\kappa \equiv \Pi(i)$. We claim that x is supported by $(p_{\mathbb{S}}^{\min}, \kappa)$. To prove this claim, we first consider the buyers in $N_0(e|x)$ with better priority than i, then consider the other buyers.

To begin, let $k \in N_0(e|x)$ such that $\Pi(k) < \kappa$ and let $h_k \in H_{\mathbb{S}}$ denote the history in $H_{\mathbb{S}}$ where k selects *exit*. Since prices do not decrease during $H_{\mathbb{S}}$, thus $p_{\mathbb{S}}^{\min} \geq p^{\min}(h_k)$. Assume, by way of contradiction, that (i) $p_{\mathbb{S}}^{\min} = p^{\min}(h_k)$, and (ii) at h_k we have that each seller offering $p_{\mathbb{S}}^{\min}$ is rising matched. In this case, if at h_k we have that i is matched to a seller $\mu(i)$, then by the Pendulum Lemma we have that at the previous history k is matched to a seller $\mu(k)$ to the left of $\mu(i)$; it follows that at h_k, we have that $\mu(k)$ is unmatched, so $\mu(k)$ does not offer $p_{\mathbb{S}}^{\min} = p^{\min}(h_k)$, so $\mu(k)$ offers a price higher than $p_{\mathbb{S}}^{\min}$, so by the Pendulum Lemma $\mu(i)$ offers a price higher than $p_{\mathbb{S}}^{\min}$. But then at h_k, we have that (i) each seller offering $p_{\mathbb{S}}^{\min}$ is rising matched, (ii) each seller who does not offer $p_{\mathbb{S}}^{\min}$ is offering a higher price, and (iii) i is not matched at price $p_{\mathbb{S}}^{\min}$; thus by the Pendulum Lemma we have that i cannot become matched after h_k to a seller offering $p_{\mathbb{S}}^{\min}$ by bidding or by unmatching and re-matching when another buyer exits, so $i \notin N_0(p_{\mathbb{S}}^{\min}|x)$, contradicting $i \in N_0(p_{\mathbb{S}}^{\min}|x)$. Altogether, then, either (i) $p^{\min}(h_k) < p_{\mathbb{S}}^{\min}$, or (ii) at h_k there is a seller offering $p_{\mathbb{S}}^{\min}$ who is not rising matched; thus by the convention we have $k \notin \mathsf{D}^!(p)$ and thus $x_k = e_k \in B_k^\delta(p_{\mathbb{S}}^{\min})$, as desired.

To conclude, we consider the other buyers. Indeed, first, by the Pendulum Lemma, we have that $k \in N_0(p_{\mathbb{S}}^{\min}|x)$ implies $\Pi(k) \leq \kappa$ and $x_k \in B_k^\delta(p_{\mathbb{S}}^{\min})$. Second, by the Pendulum Lemma, we have that $k \in N_0(p_{\mathbb{S}}^{\min} + 1|x)$ implies (i) at $h_{\mathbb{S}}$ we have that the match of k is to the right of the match of i , so $\Pi(k) > \Pi(i) = \kappa$; and (ii) $x_k \in B_k^\delta(p_{\mathbb{S}}^{\min} + 1)$. Finally, since x is an almost-synchronized equilibrium supported by $(p_{\mathbb{S}}^{\min}, \nearrow)$, thus for each $k \in N_0(e|x)$ such that $\Pi(k) > \kappa$, we have that $x_k = e_k \in B_k^\delta(p_{\mathbb{S}}^{\min}) \cup B_k^\delta(p_{\mathbb{S}}^{\min} + 1)$ and thus $x_k = e_k \in B_k^\delta(p_{\mathbb{S}}^{\min} + 1)$. Altogether, then, x is a cutoff equilibrium supported by $(p_{\mathbb{S}}^{\min}, \kappa)$, as desired. ■

Appendix F: Buyer-optimality

In this appendix, we prove our results about the buyer-optimality of the pendulum auctions: Theorem 7 and Theorem 8.

THEOREM 7: Fix a discrete environment. For each auction configuration and each type profile, neither the buyer-optimal pendulum allocation nor the efficient pendulum allocation is strictly buyer-dominated across bilateral and individually rational allocations.

PROOF: Fix a version $\mathcal{V}$ and an auction configuration c, let $\theta \in \Theta$, and for brevity define $x \equiv x_{\mathbb{S}}$. By Theorem 6, x is a cutoff equilibrium, so by Proposition 5, x is *bilateral* and *individually rational*.

Assume, by way of contradiction, there is an allocation x' that is *bilateral* and *individually rational* such that for each $k \in N_0$, $x'_k \succ_k x_k$. Since x is *individually rational*, thus for each $k \in N_0$ we have $x'_k \succ_k x_k \succsim_k e_k$. Then $N_0(e|x') = \emptyset$, so as x' is *bilateral* we have $|N_1| \geq |\cup_{p' \in \mathbb{Z}} N_1(p'|x')| = |\cup_{p' \in \mathbb{Z}} N_0(p'|x')| = |N_0|$. To complete the proof, we establish that there is a contradiction in two cases whose arguments share a common structure but differ in the details.

CASE 1: $N_0(e|x) \neq \emptyset$. To begin, in this case we must have $N_1(e|x) \neq \emptyset$, else as x is *bilateral* we have $|N_1| = |\cup_{p' \in \mathbb{Z}} N_1(p'|x)| = |\cup_{p' \in \mathbb{Z}} N_0(p'|x)| < |N_0|$, contradicting $|N_1| \geq |N_0|$. Thus we can define $p \equiv \min_{k \in N_1(e|x)} p_k(h_\wedge)$. Moreover, it follows from the convention that for each $k \in N_1(e|x)$, k is never matched during $H_{\mathbb{S}}$; thus there is $i \in N_1$ who offers p while resting unmatched at each history in $H_{\mathbb{S}}$.

First, we claim $|\mathsf{S}(p-1)| < |N_0|$. Indeed, for each $k \in N_1(e|x)$, we have that $p_k(h_\wedge) \geq p > p-1$ and thus by the Pendulum Lemma that $k \notin \mathsf{S}(p-1)$. Then $\mathsf{S}(p-1) \subseteq N_1 \backslash N_1(e|x)$, so as x is *bilateral* and $N_0(e|x) \neq \emptyset$ we have $|\mathsf{S}(p-1)| \leq |N_1 \backslash N_1(e|x)| = |\cup_{p' \in \mathbb{Z}} N_1(p'|x)| = |\cup_{p' \in \mathbb{Z}} N_0(p'|x)| < |N_0|$, as desired.

Next, we claim $|N_0| = |\cup_{p' \leq p-1} N_0(p'|x')|$. Indeed, since x is *bilateral*, thus each $k \in N_0$ belongs to either $N_0(e|x)$ or $\cup_{p' \in \mathbb{Z}} N_0(p'|x)$. First, for each $k \in N_0(e|x)$, we have that k selects *exit* at a history where i offers p while resting unmatched, so by the convention $k \notin \mathsf{D}(p)$; thus as $x'_k \succ_k e_k$ and x' is *bilateral*, we have that $k \in \cup_{p' \leq p-1} N_0(p'|x')$. Second, for each $k \in \cup_{p' \in \mathbb{Z}} N_0(p'|x)$, by definition of the pendulum auctions we have that the match of k at $h_{\mathbb{S}}$ is to the left of i, so by the Pendulum Lemma there is $p_k \leq p$ such that $x_k = (1, -p_k)$; thus as $x'_k \succ_k x_k \succsim_k e_k$ and x' is *bilateral*, we have that $k \in \cup_{p' \leq p-1} N_0(p'|x')$. Altogether, then, $|N_0| = |\cup_{p' \leq p-1} N_0(p'|x')|$, as desired.

To conclude, by the two claims we have that $|\mathsf{S}(p-1)| < |N_0| = |\cup_{p' \leq p-1} N_0(p'|x')|$, so as x' is *bilateral* there are $p' \leq p-1$ and $j \in N_1 \backslash \mathsf{S}(p-1)$ such that $j \in N_1(p'|x')$. But then $j \notin \mathsf{S}(p')$ and $x'_j = (0, p')$, contradicting that $x' \in X$ is *individually rational*.

CASE 2: $N_0(e|x) = \emptyset$. To begin, since $|N_1| \geq |N_0|$, thus we can let i denote the seller who has $|N_0| - 1$ sellers to his left and define $p \equiv p_i(h_\wedge)$. Moreover, since $N_0(e|x) = \emptyset$, thus by definition of the pendulum auctions, i becomes matched during $H_{\mathbb{S}}$, and moreover this occurs only when all sellers to his left are matched, at which point the queue is empty and the auction ends. Then by the Pendulum Lemma, we have $N_0 \subseteq \cup_{p' \leq p} N_0(p'|x)$.

First, we claim $|\mathsf{S}(p-1)| < |N_0|$. Indeed, by definition of the pendulum auctions, for each $k \in N_1$ who is either i or to the right of i, we have $p_k(h_\wedge) \geq p_i(h_\wedge) = p > p-1$ and thus by the Pendulum Lemma that $k \notin \mathsf{S}(p-1)$. Then each member of $\mathsf{S}(p-1)$ is to the left of i, so $|\mathsf{S}(p-1)| < |N_0|$, as desired.

Next, we claim $|N_0| = |\cup_{p' \leq p-1} N_0(p'|x')|$. Indeed, for each $k \in N_0$, since (i) $k \in \cup_{p' \leq p} N_0(p'|x)$, (ii) $x'_k \succ_k x_k \succsim_k e_k$, and (iii) x' is *bilateral*, thus $k \in \cup_{p' \leq p-1} N_0(p'|x')$. Then $|N_0| = |\cup_{p' \leq p-1} N_0(p'|x')|$, as desired.

To conclude, by the two claims we have that $|\mathsf{S}(p-1)| < |N_0| = |\cup_{p' \leq p-1} N_0(p'|x')|$, so as x' is *bilateral* there are $p' \leq p-1$ and $j \in N_1 \backslash \mathsf{S}(p-1)$ such that $j \in N_1(p'|x')$. But then $j \notin \mathsf{S}(p')$ and $x'_j = (0, p')$, contradicting that $x' \in X$ is *individually rational.* ■

To conclude this appendix, we prove Theorem 8.

THEOREM 8: Fix a discrete environment. For each auction configuration and each type profile, the buyer-optimal pendulum allocation is buyer-dominant across cutoff equilibria.

PROOF: Let $c \in C$ and fix the associated buyer-optimal pendulum auction, let $\theta \in \Theta$, and for brevity define $x \equiv x_{\mathbb{S}}$. By Theorem 6, x is a cutoff equilibrium. Assume, by way of contradiction, there are cutoff equilibrium x' and $i \in N_0$ such that $x'_i \succ_i x_i$. By Proposition 5, both x and x' satisfy *bilateral weak core* and *no justified envy.*

By the Pendulum Lemma, $x'_i \succ_i x_i \succsim_i e_i$ and $x'_i \succ_i x_i \succsim_i (1, -(p^{\min}_{\mathbb{S}} + 1))$; thus as x' is *bilateral*, we have $i \in \cup_{p' \leq p^{\min}_{\mathbb{S}}} N_0(p'|x')$. Then we can define $p \in \mathbb{Z}$ to be the minimum price such that $|N_0(p|x')| \geq 1$; it follows that $p \leq p^{\min}_{\mathbb{S}}$ and $i \in \cup_{p' \in \{p, p+1, \ldots, p^{\min}_{\mathbb{S}}\}} N_0(p'|x')$. To complete the proof, we first establish a claim loosely stating that the final history in $H_{\mathbb{S}}$ where $|N_0(p|x')|$ sellers offer p satisfies certain conditions, then conclude.

CLAIM: There is non-terminal $h' \in H_{\mathbb{S}}$ such that from left to right, (i) the first $|N_0(p|x')|$ sellers offer p while rising matched, (ii) the next $|\mathsf{S}(p+1)| - |N_0(p|x')|$ sellers offer $p+1$ while rising matched, and (iii) the player selects *bid.*[13]

To begin, (i) since x' is *bilateral*, we have $|N_0(p|x')| = |N_1(p|x')|$, (ii) since x' is *individually rational*, we have $N_1(p|x') \subseteq \mathsf{S}(p)$, and (iii) by the Pendulum Lemma, for each $k \in \mathsf{S}(p)$ we have $p \geq p_k(h_\wedge)$; thus there are at least $|N_0(p|x')|$ sellers who offer price p or less at $h_\wedge$. Then by the Pendulum Lemma and the definition of the buyer-optimal pendulum auction, in order to establish the claim it suffices to prove that fewer than $|N_0(p|x')|$ sellers offer price p or less at $h_{\mathbb{S}}$. If $p < p^{\min}_{\mathbb{S}}$, then (i) no seller offers price p or less at $h_{\mathbb{S}}$, and (ii) $|N_0(p|x')| \geq 1$, so we are done; thus let us assume that $p = p^{\min}_{\mathbb{S}}$.

In this case, since $i \in \cup_{p' \in \{p, p+1, \ldots, p^{\min}_{\mathbb{S}}\}} N_0(p'|x')$ and $p = p^{\min}_{\mathbb{S}}$, thus $i \in N_0(p|x')$. Let h_i denote the last history where i plays in $H_{\mathbb{S}}$ and let $N_0(p|h_i)$ denote the set of buyers who are matched at price p at h_i. Since $(1, -p) = x'_i \in X_i$ and $(1, -p) = x'_i \succ_i x_i \succsim_i e_i$, thus $i \in \mathsf{D}^!(p)$. Then $i \in \mathsf{D}^!(p)$ and $(1, -p) \succ_i x_i$, so by the Pendulum Lemma and the definition of the buyer-optimal pendulum auction and its convention, (i) each member of $N_0(p|h_i)$ is matched with a rising matched seller at h_i and has better priority than i, and thus moreover (ii) $N_0(p|x) \subseteq N_0(p|h_i) \subseteq \mathsf{D}^!(p)$, and (iii) there are no unmatched sellers offering p or less at $h_{\mathbb{S}}$. Then since x' satisfies *no justified envy* and *bilaterality*, and since $i \in N_0(p|x')$, thus each member of $N_0(p|x)$ belongs to $N_0(p|x')$. Altogether, then, $|N_0(p|x')| \geq |N_0(p|x) \cup \{i\}| = |N_0(p|x)| + 1 > |N_0(p|x)|$, so as there are no unmatched

[13] We remark that there need not be any such $h' \in H_{\mathbb{S}}$ for the efficient pendulum auction. Indeed, the efficient pendulum auction does not select allocations that are buyer-dominant across cutoff equilibria (Example 9).

sellers offering p or less at $h_{\mathbb{S}}$, we have that there are fewer than $|N_0(p|x')|$ sellers offering price p or less at $h_{\mathbb{S}}$, as desired.

Proof from claim: Let $N_0(p|h')$ denote the set of buyers who are matched at price p at h', let $N_0(p+1|h')$ denote the set of buyers who are matched at price $p+1$ at h', and let i' denote the player at h'. By the claim and the Pendulum Lemma, we have $|N_0(p|h')| = |N_0(p|x')|$ and $|N_0(p+1|h')| = |\mathsf{S}(p+1)| - |N_0(p|x')|$. To complete the proof, we make two observations and then conclude.

First, we observe that each member of $N_0(p+1|h') \cup \{i'\}$ trades in x'. Indeed, by the buyer-optimal convention we have (i) $N_0(p+1|h') \subseteq \mathsf{D}^!(p+1) = \mathsf{D}^{!!}(p)$, and (ii) $i' \in \mathsf{D}^{!!}(p)$. Moreover, since x' is a cutoff equilibrium and $|N_0(p|x')| \geq 1$, thus each member of $\mathsf{D}^{!!}(p)$ trades in x'. Altogether, then, each member of $N_0(p+1|h') \cup \{i'\}$ trades in x', as desired.

Second, we observe that $|N_0(p+1|x')| < |N_0(p+1|h') \cup \{i'\}|$. Indeed, since x' is a cutoff equilibrium and $|N_0(p|x')| \geq 1$, thus each seller who trades in x' belongs to $\mathsf{S}(p+1)$, and thus moreover $|N_0(p|x')| + |N_0(p+1|x')| = |\cup_{p' \in \mathbb{Z}} N_0(p'|x')| = |\cup_{p' \in \mathbb{Z}} N_1(p'|x')| \leq |\mathsf{S}(p+1)|$, so $|N_0(p+1|x')| \leq |\mathsf{S}(p+1)| - |N_0(p|x')| = |N_0(p+1|h')| < |N_0(p+1|h') \cup \{i'\}|$, as desired.

To conclude, by the two observations we have that there is $i^* \in N_0(p+1|h') \cup \{i'\}$ who also belongs to $N_0(p|x')$. By the Pendulum Lemma and by the definition of the buyer-optimal pendulum auction and its convention, (i) each member of $N_0(p|h')$ has better priority than i^*, and (ii) $N_0(p|h') \subseteq \mathsf{D}^!(p)$. Then since x' satisfies *no justified envy* and *bilaterality*, and since $i^* \in N_0(p|x')$, thus each member of $N_0(p|h')$ belongs to $N_0(p|x')$. But then $|N_0(p|x')| \geq |N_0(p|h') \cup \{i^*\}| = |N_0(p|x')| + 1$, contradicting $|N_0(p|x')| = |N_0(p|x')|$. ■

Appendix G: Efficiency

In this appendix, we prove our results about the efficiency of the pendulum auctions: Theorem 9 and Theorem 10.

Theorem 9: Fix a discrete environment. For each auction configuration and each type profile, the buyer-optimal pendulum allocation satisfies *constrained efficiency*.

Proof: Let $c \in C$ and fix the associated buyer-optimal pendulum auction, let $\theta \in \Theta$, and for brevity define $x \equiv x_{\mathbb{S}}$. By Theorem 6, x is a cutoff equilibrium, so by Proposition 5, x is *bilateral* and *individually rational*. Assume, by way of contradiction, there is a *bilateral* allocation x' such that N weakly blocks x using x'. Then (i) since x is *individually rational*, thus for each $k \in N$ we have $x'_k \succsim_k x_k \succsim_k e_k$, so x' is *individually rational*, and (ii) there is $i \in N$ such that $x'_i \succ_i x_i$. We consider two cases.

Case 1: There is $j \in \mathsf{S}^!(p^{\min}_{\mathbb{S}})$ such that $x_j = e_j$. In this case, by the Pendulum Lemma we have $p^{\min}_{\mathbb{S}} \geq p_j(h_\wedge)$. Moreover, since $x_j = e_j$, thus by the convention j never increases his price during $H_{\mathbb{S}}$, so $p_j(h_\wedge) = p_j(h_{\mathbb{S}}) \geq p^{\min}_{\mathbb{S}}$. Altogether, then, $p_j(h_\wedge) = p_j(h_{\mathbb{S}}) = p^{\min}_{\mathbb{S}}$. Moreover, by the convention, at each history in $H_{\mathbb{S}}$ we have that j offers $p^{\min}_{\mathbb{S}}$ while resting unmatched.

By the Pendulum Lemma, we have $N_0 = N_0(e|x) \cup N_0(p^{\min}_{\mathbb{S}}|x) \cup N_0(p^{\min}_{\mathbb{S}} + 1|x)$ and $N_1 = N_1(e|x) \cup N_1(p^{\min}_{\mathbb{S}}|x) \cup N_1(p^{\min}_{\mathbb{S}} + 1|x)$. We make the following *itemized claim* consisting of six statements about what the agents consume in x':

1. No member of $N_0(e|x)$ trades in x' at price $p^{\min}_{\mathbb{S}}$ or higher.

2. No member of $N_0(p^{\min}_{\mathbb{S}}|x)$ trades in x' at a price above $p^{\min}_{\mathbb{S}}$.

3. $N_0(p^{\min}_{\mathbb{S}}+1|x) = \emptyset$.

4. No member of $N_1(e|x)$ trades in x' at a price below $p^{\min}_{\mathbb{S}}$.

5. Each member of $N_1(p^{\min}_{\mathbb{S}}|x)$ trades in x' at price $p^{\min}_{\mathbb{S}}$ or higher.

6. $N_1(p^{\min}_{\mathbb{S}}+1|x) = \emptyset$.

We prove these statements in sequence, then conclude.

Proof of Statement 1: For each $k \in N_0(e|x)$, there is a history where k selects *exit* while j offers $p^{\min}_{\mathbb{S}}$ while resting unmatched, so by the buyer-optimal convention $k \notin \mathsf{D}(p^{\min}_{\mathbb{S}})$, so as x' is *individually rational* we have that k does not trade in x' at price $p^{\min}_{\mathbb{S}}$ or higher, as desired.

Proof of Statement 2: This is a direct consequence of the fact that N weakly blocks x using x'.

Proof of Statement 3: Since at each history in $H_{\mathbb{S}}$ we have that j offers $p^{\min}_{\mathbb{S}}$ while unmatched, thus by definition of the buyer-optimal pendulum auction we have that $N_0(p^{\min}_{\mathbb{S}}+1|x) = \emptyset$.

Proof of Statement 4: For each $k \in N_1(e|x)$, we have by the buyer-optimal convention that k never increases his price during $H_{\mathbb{S}}$, so $p_k(h_\wedge) = p_k(h_{\mathbb{S}}) \geq p^{\min}_{\mathbb{S}} > p^{\min}_{\mathbb{S}}-1$; thus by the Pendulum Lemma we have that $k \notin \mathsf{S}(p^{\min}_{\mathbb{S}}-1)$, so as x' is *individually rational* we have that k does not trade in x' at price $p^{\min}_{\mathbb{S}}-1$ or lower, as desired.

Proof of Statement 5: Let $k \in N_1(p^{\min}_{\mathbb{S}}|x)$. By definition of the buyer-optimal pendulum auction and its convention, we have that k is to the left of j, so since $j \in \mathsf{S}^!(p^{\min}_{\mathbb{S}})$, thus by the Pendulum Lemma we have that $k \in \mathsf{S}^!(p^{\min}_{\mathbb{S}})$. Then since N weakly blocks x using x', thus $x'_k \succsim_k x_k = (0, p^{\min}_{\mathbb{S}}) \succ_k e_k$, so as x' is *bilateral* we have that k trades in x' at price $p^{\min}_{\mathbb{S}}$ or higher, as desired.

Proof of Statement 6: This is a direct consequence of Statement 3 and the fact that x is *bilateral*.

Conclusion: By the itemized claim we have that (i) $N_0(p^{\min}_{\mathbb{S}}|x') \subseteq N_0(p^{\min}_{\mathbb{S}}|x)$; and (ii) no buyer trades in x' at a price above $p^{\min}_{\mathbb{S}}$, and thus as x' is *bilateral* we have $N_1(p^{\min}_{\mathbb{S}}|x) \subseteq N_1(p^{\min}_{\mathbb{S}}|x')$. Moreover, since x and x' are *bilateral*, thus we have that $|N_0(p^{\min}_{\mathbb{S}}|x')| \leq |N_0(p^{\min}_{\mathbb{S}}|x)| = |N_1(p^{\min}_{\mathbb{S}}|x)| \leq |N_1(p^{\min}_{\mathbb{S}}|x')| = |N_0(p^{\min}_{\mathbb{S}}|x')|$. Altogether, then, we have $N_0(p^{\min}_{\mathbb{S}}|x') = N_0(p^{\min}_{\mathbb{S}}|x)$ and $N_1(p^{\min}_{\mathbb{S}}|x') = N_1(p^{\min}_{\mathbb{S}}|x)$. Moreover, by the itemized claim we have that (i) no buyer trades in x' at a price above $p^{\min}_{\mathbb{S}}$, and (ii) no seller trades in x' at a price below $p^{\min}_{\mathbb{S}}$; thus as x' is *bilateral* we have that no agent trades in x' at a price other than $p^{\min}_{\mathbb{S}}$. But then since x' is *bilateral* we have that $x' = x$, contradicting that N weakly blocks x using x'.

Case 2: There is no $k \in \mathsf{S}^!(p^{\min}_{\mathbb{S}})$ such that $x_k = e_k$. In this case, we begin with two *initial claims*, then make an *itemized claim* and conclude.

First, we claim there is $j \in N_0$ such that $x_j \notin B_j^\delta(p_{\mathbb{S}}^{\min})$. Indeed, assume by way of contradiction this is not the case. Then as x is *bilateral* we have $N_0(p_{\mathbb{S}}^{\min}+1|x) = N_1(p_{\mathbb{S}}^{\min}+1|x) = \emptyset$, so for each $k \in N_1$, either (i) $x_k = (0, p_{\mathbb{S}}^{\min})$ and thus as x is *individually rational* we have $x \in B_k^\delta(p_{\mathbb{S}}^{\min})$, or (ii) $x_k = e_k$ and so by the assumption of this case we have $k \notin \mathsf{S}^!(p_{\mathbb{S}}^{\min})$ and thus $x_k \in B_k^\delta(p_{\mathbb{S}}^{\min})$. Altogether, then, for each $k \in N_0 \cup N_1$ we have $x_k \in B_k^\delta(p_{\mathbb{S}}^{\min})$. But then as x' is *bilateral* and $x_i' \succ_i x_i$, we have that (i) if $i \in N_0$, then there is $p' < p_{\mathbb{S}}^{\min}$ such that $(1, -p') = x_i'$, so as x' is *bilateral* there is $k \in N_1$ such that $x_k \succsim_k (0, p_{\mathbb{S}}^{\min}) \succ_k (0, p') = x_k'$, contradicting $x_k' \succsim_k x_k$, and similarly (ii) if $i \in N_1$, then there is $p' > p_{\mathbb{S}}^{\min}$ such that $(1, -p') = x_i'$, so as x' is *bilateral* there is $k \in N_0$ such that $x_k \succsim_k (1, -p_{\mathbb{S}}^{\min}) \succ_k (1, -p') = x_k'$, contradicting $x_k' \succsim_k x_k$.

Second, let h_j denote the last history in $H_{\mathbb{S}}$ where j plays; we claim (i) $p^{\min}(h_j) = p_{\mathbb{S}}^{\min}$ and (ii) at h_j, each seller offering $p_{\mathbb{S}}^{\min}$ is rising matched. To begin, by the Pendulum Lemma we have $x_j \in \{e_j, (1, -p_{\mathbb{S}}^{\min}), (1, -(p_{\mathbb{S}}^{\min}+1))\}$, and moreover (i) if $x_j = e_j$, then since $x_j \notin B_j^\delta(p_{\mathbb{S}}^{\min})$ we have $j \in \mathsf{D}^!(p_{\mathbb{S}}^{\min})$, (ii) since x is *individually rational* and $x_j \notin B_j^\delta(p_{\mathbb{S}}^{\min})$, thus $x_j \neq (1, -p_{\mathbb{S}}^{\min})$, and (iii) if $x_j = (1, -(p_{\mathbb{S}}^{\min}+1))$, then since x is *individually rational* we have $j \in \mathsf{D}(p_{\mathbb{S}}^{\min}+1)$ and thus $j \in \mathsf{D}^!(p_{\mathbb{S}}^{\min})$. Then (i) $j \in \mathsf{D}^!(p_{\mathbb{S}}^{\min})$, and (ii) by the Pendulum Lemma, at h_j we have that j either exits or bids and then is immediately matched at price $p_{\mathbb{S}}^{\min}+1$; thus by the convention we have $p^{\min}(h_j) \geq p_{\mathbb{S}}^{\min}$, so since prices do not decrease during $H_{\mathbb{S}}$ we have $p^{\min}(h_j) = p_{\mathbb{S}}^{\min}$. To conclude, we have (i) $j \in \mathsf{D}^!(p_{\mathbb{S}}^{\min})$, (ii) $p^{\min}(h_j) = p_{\mathbb{S}}^{\min}$, and (iii) at h_j, j either exits or bids and then is immediately matched at price $p_{\mathbb{S}}^{\min}+1$; thus by definition of the buyer-optimal pendulum auction and its convention we have that at h_j, each seller offering $p_{\mathbb{S}}^{\min}$ is rising matched, as desired.

By the Pendulum Lemma, we have $N_0 = N_0(e|x) \cup N_0(p_{\mathbb{S}}^{\min}|x) \cup N_0(p_{\mathbb{S}}^{\min}+1|x)$ and $N_1 = N_1(e|x) \cup N_1(p_{\mathbb{S}}^{\min}|x) \cup N_1(p_{\mathbb{S}}^{\min}+1|x)$. We make the following *itemized claim* consisting of six statements about what the agents consume in x':

1. No member of $N_0(e|x)$ trades in x' at a price above $p_{\mathbb{S}}^{\min}+1$.

2. Each member of $N_0(p_{\mathbb{S}}^{\min}|x)$ trades in x' at price $p_{\mathbb{S}}^{\min}$ or lower.

3. No member of $N_0(p_{\mathbb{S}}^{\min}+1|x)$ trades in x' at a price above $p_{\mathbb{S}}^{\min}+1$.

4. No member of $N_1(e|x)$ trades in x' at price $p_{\mathbb{S}}^{\min}$ or lower.

5. No member of $N_1(p_{\mathbb{S}}^{\min}|x)$ trades in x' at a price below $p_{\mathbb{S}}^{\min}$.

6. No member of $N_1(p_{\mathbb{S}}^{\min}+1|x)$ trades in x' at price $p_{\mathbb{S}}^{\min}$ or lower.

We prove these statements in sequence, then conclude.

Proof of Statement 1: Let $k \in N_0(e|x)$. Then there is a history h_k where k selects *exit* when the minimum price is $p^{\min}(h_k) \leq p_{\mathbb{S}}^{\min}$, so by the buyer-optimal convention $k \notin \mathsf{D}^!(p^{\min}(h_k)+1)$ and thus $k \notin \mathsf{D}^!(p_{\mathbb{S}}^{\min}+1)$, so $k \notin \mathsf{D}(p_{\mathbb{S}}^{\min}+2)$, so as x' is *individually rational* we have that k does not trade in x' at a price above $p_{\mathbb{S}}^{\min}+1$, as desired.

Proof of Statement 2: Let $k \in N_0(p_{\mathbb{S}}^{\min}|x)$, let h_k denote the final history in $H_{\mathbb{S}}$ where k plays, and let h_k' denote the history in $H_{\mathbb{S}}$ immediately after h_k. By the Pendulum Lemma, we have that at each history in $H_{\mathbb{S}}$ after h_k, k is matched with a seller at price $p_{\mathbb{S}}^{\min}$.

First, we claim that (i) h_j is after h_k, and (ii) at h_j we have that k is matched with a rising seller. Indeed, since (i) at h_j we have that each seller offering $p_{\mathbb{S}}^{\min}$ is rising matched, and (ii) at $h_{\mathbb{S}}$ we have that k is matched with a seller at price $p_{\mathbb{S}}^{\min}$, thus by definition of the buyer-optimal pendulum auction we have that h_j is after h_k. Then at h_j we have that k is matched with a seller offering $p_{\mathbb{S}}^{\min}$, so at h_j we have that k is matched with a rising seller, as desired.

Second, we claim that at h'_k we have that k is matched with a rising seller at price $p_{\mathbb{S}}^{\min}$. Since at h'_k we have that k is matched with a seller at price $p_{\mathbb{S}}^{\min}$, thus we need only show that at h'_k this seller is rising. Indeed, assume by way of contradiction that at h'_k this seller is resting. By the Pendulum Lemma, whenever the exit of another buyer causes k to unmatch from a resting seller and re-match, we have that k re-matches with a resting seller. But then by induction on the histories in H' after h_k, at h_j we have that k is matched with a resting seller, contradicting the first claim.

To conclude, since k is matched with a rising seller at price $p_{\mathbb{S}}^{\min}$ immediately after bidding at h_k, thus by the convention we have that $k \in \mathsf{D}^!(p_{\mathbb{S}}^{\min})$, so since N weakly blocks x using x' and since x' is *bilateral* we have that k trades in x' at price p or lower, as desired.

Proof of Statement 3: This is a direct consequence of the fact that N weakly blocks x using x'.

Proof of Statement 4: Let $k \in N_1(e|x)$. By the buyer-optimal convention, we have that k never increases his price or becomes matched during $H_{\mathbb{S}}$. Moreover, at h_j we have that each seller offering $p^{\min}(h_j) = p_{\mathbb{S}}^{\min}$ is matched. Altogether, then, $p_k(h_\wedge) = p_k(h_j) > p^{\min}(h_j) = p_{\mathbb{S}}^{\min}$; thus by the Pendulum Lemma we have that $k \notin \mathsf{S}(p_{\mathbb{S}}^{\min})$, so as x' is *individually rational* we have that k does not trade in x' at price $p_{\mathbb{S}}^{\min}$ or lower, as desired.

Proof of Statement 5: This is a direct consequence of the fact that N weakly blocks x using x'.

Proof of Statement 6: This is a direct consequence of the fact that N weakly blocks x using x'.

Conclusion: To complete the proof, we make two *concluding claims* and then conclude.

First, we claim that $N_0(p_{\mathbb{S}}^{\min}|x') = N_0(p_{\mathbb{S}}^{\min}|x)$ and $N_1(p_{\mathbb{S}}^{\min}|x') = N_1(p_{\mathbb{S}}^{\min}|x)$. Indeed, by the itemized claim we have that (i) no seller trades in x' at a price below $p_{\mathbb{S}}^{\min}$, and thus as x' is *bilateral* we have $N_0(p_{\mathbb{S}}^{\min}|x) \subseteq N_0(p_{\mathbb{S}}^{\min}|x')$; and (ii) $N_1(p_{\mathbb{S}}^{\min}|x') \subseteq N_1(p_{\mathbb{S}}^{\min}|x)$. Moreover, since x and x' are *bilateral*, thus we have that $|N_1(p_{\mathbb{S}}^{\min}|x')| \leq |N_1(p_{\mathbb{S}}^{\min}|x)| = |N_0(p_{\mathbb{S}}^{\min}|x)| \leq |N_0(p_{\mathbb{S}}^{\min}|x')| = |N_1(p_{\mathbb{S}}^{\min}|x')|$. Altogether, then, we have $N_0(p_{\mathbb{S}}^{\min}|x') = N_0(p_{\mathbb{S}}^{\min}|x)$ and $N_1(p_{\mathbb{S}}^{\min}|x') = N_1(p_{\mathbb{S}}^{\min}|x)$, as desired.

Second, we claim that (i) $i \in N_0(e|x) \cup N_1(e|x)$, and (ii) i trades in x' at price $p_{\mathbb{S}}^{\min} + 1$. Indeed, since $x'_i \succ_i x_i$, thus by the first concluding claim we have that $i \notin N_0(p_{\mathbb{S}}^{\min}|x) \cup N_1(p_{\mathbb{S}}^{\min}|x)$. Moreover, by the itemized claim we have that (i) no buyer trades in x' at a price above $p_{\mathbb{S}}^{\min}+1$, and (ii) no seller trades in x' at a price below $p_{\mathbb{S}}^{\min}$; thus as x' is *bilateral* we have that if an agent trades in x' at a price other than $p_{\mathbb{S}}^{\min}$, then this agent trades in x' at price $p_{\mathbb{S}}^{\min} + 1$. Since x' is *bilateral*, thus by the first concluding claim we have that for each $k \in N_0(p_{\mathbb{S}}^{\min} + 1|x) \cup N_1(p_{\mathbb{S}}^{\min} + 1|x)$, either (i) $x'_k = e_k$ and thus as x is *individually rational* we have $x_k \succsim_k x'_k$, or (ii) $x'_k = x_k$; thus $i \notin N_0(p_{\mathbb{S}}^{\min} + 1|x) \cup N_1(p_{\mathbb{S}}^{\min} + 1|x)$.

Altogether, then, we have $i \in N_0(e|x) \cup N_1(e|x)$. Finally, since (i) $x'_i \succ_i x_i$, (ii) x' is *bilateral*, and (iii) by the first concluding claim we have that i does not trade in x' at price $p^{\min}_{\mathbb{S}}$, thus i trades in x' at price $p^{\min}_{\mathbb{S}} + 1$, as desired.

To conclude, we consider two sub-cases. First, if $i \in N_0(e|x)$ and i trades in x' at price $p^{\min}_{\mathbb{S}} + 1$, then since $x'_i \succ_i x_i$ we have $i \in \mathsf{D}^!(p^{\min}_{\mathbb{S}} + 1)$. But then since $x_i = e_i$, there is a history h_i where i selects *exit* when the minimum price is $p^{\min}(h_i) \leq p^{\min}_{\mathbb{S}}$, so by the buyer-optimal convention $i \notin \mathsf{D}^!(p^{\min}(h_i) + 1)$ and thus $i \notin \mathsf{D}^!(p^{\min}_{\mathbb{S}} + 1)$, contradicting $i \in \mathsf{D}^!(p^{\min}_{\mathbb{S}} + 1)$. Second, if $i \in N_1(e|x)$ and i trades in x' at price $p^{\min}_{\mathbb{S}} + 1$, then since $x'_i \succ_i x_i$ we have $i \in \mathsf{S}^!(p^{\min}_{\mathbb{S}} + 1)$. By the Pendulum Lemma, we have that (i) $p^{\min}_{\mathbb{S}} + 1 \geq p_i(h_\wedge)$, and (ii) each seller to the left of i belongs to $\mathsf{S}^!(p^{\min}_{\mathbb{S}} + 1)$. Moreover, since (i) at h_j, we have that each seller offering $p^{\min}(h_j) = p^{\min}_{\mathbb{S}}$ is matched; and (ii) by the buyer-optimal convention, at each history in $H_{\mathbb{S}}$ we have that i offers $p_i(h_\wedge)$ while resting unmatched; thus we have $p_i(h_\wedge) \geq p^{\min}_{\mathbb{S}} + 1$, so altogether $p_i(h_\wedge) = p^{\min}_{\mathbb{S}} + 1$. Then for each $k \in N_0(e|x)$, k selects *exit* at a history where i offers $p^{\min}_{\mathbb{S}} + 1$ while resting unmatched, so by the buyer-optimal convention $k \notin \mathsf{D}(p^{\min}_{\mathbb{S}} + 1)$; thus as x' is *individually rational*, it follows from the first concluding claim that $N_0(p^{\min}_{\mathbb{S}} + 1|x') \subseteq N_0(p^{\min}_{\mathbb{S}} + 1|x)$. Moreover, for each $k \in N_1(p^{\min}_{\mathbb{S}} + 1|x)$, by the buyer-optimal convention we have that k is to the left of i and thus that $k \in \mathsf{S}^!(p^{\min}_{\mathbb{S}} + 1)$, so as $x'_k \succsim_k x_k$ and x' is *bilateral* we have $k \in N_1(p^{\min}_{\mathbb{S}} + 1|x')$; thus $N_1(p^{\min}_{\mathbb{S}} + 1|x) \cup \{i\} \subseteq N_1(p^{\min}_{\mathbb{S}} + 1|x')$. But then as x is *bilateral*, we have $|N_0(p^{\min}_{\mathbb{S}} + 1|x')| \leq |N_0(p^{\min}_{\mathbb{S}} + 1|x)| = |N_1(p^{\min}_{\mathbb{S}} + 1|x)| < |N_1(p^{\min}_{\mathbb{S}} + 1|x) \cup \{i\}| \leq |N_1(p^{\min}_{\mathbb{S}} + 1|x')|$, contradicting $|N_0(p^{\min}_{\mathbb{S}} + 1|x')| = |N_1(p^{\min}_{\mathbb{S}} + 1|x')|$. ■

To conclude this appendix, we prove Theorem 10.

THEOREM 10: Fix a discrete environment. For each auction configuration and each type profile, the efficient pendulum allocation is *efficient*.

PROOF: Let $c \in C$ and fix the associated efficient pendulum auction, let $\theta \in \Theta$, and for brevity define $x \equiv x_{\mathbb{S}}$. Assume, by way of contradiction, that x violates *efficiency*. For each $p' \in \mathbb{Z}$, define

$$
\begin{aligned}
\mathsf{D}(p'|x) &\equiv \{k \in N \backslash N(x) | t_k - p' \geq -w_k \text{ and } v_k \geq p'\}, \\
\mathsf{D}^!(p'|x) &\equiv \{k \in N \backslash N(x) | t_k - p' \geq -w_k \text{ and } v_k > p'\}, \\
\mathsf{S}(p'|x) &\equiv \{k \in N(x) | t_k + p' \geq -w_k \text{ and } p' \geq v_k\}, \text{ and} \\
\mathsf{S}^!(p'|x) &\equiv \{k \in N(x) | t_k + p' \geq -w_k \text{ and } p' > v_k\}.
\end{aligned}
$$

By Proposition 2, there are $i \in N \backslash N(x)$, $j \in N(x)$, and $p \in \mathbb{Z}$ such that (i) $i \in \mathsf{D}(p|x)$, (ii) $j \in \mathsf{S}(p|x)$, and (iii) either $i \in \mathsf{D}^!(p|x)$ or $j \in \mathsf{S}^!(p|x)$.

First, we claim that we have $N \backslash N(x) = N_0(e|x) \cup N_1(p^{\min}_{\mathbb{S}}|x) \cup N_1(p^{\min}_{\mathbb{S}} + 1|x)$ and $N(x) = N_1(e|x) \cup N_0(p^{\min}_{\mathbb{S}}|x) \cup N_0(p^{\min}_{\mathbb{S}} + 1|x)$. Indeed, this follows directly from the Pendulum Lemma.

Second, we claim that $p \leq p^{\min}_{\mathbb{S}}$. Indeed, first, for each $k \in N_0(e|x)$, there is a history h_k where k selects *exit* when the minimum price is $p^{\min}(h_k) \leq p^{\min}_{\mathbb{S}}$, so by the efficient convention we have $k \notin \mathsf{D}(p^{\min}(h_k) + 1)$, so $k \notin \mathsf{D}(p^{\min}_{\mathbb{S}} + 1)$, so as $t_k = 0$ we have $k \notin \mathsf{D}(p^{\min}_{\mathbb{S}} + 1|x)$.[14] Second, for each $k \in N_1(p^{\min}_{\mathbb{S}}|x)$, by the Pendulum Lemma we have $p^{\min}_{\mathbb{S}} + 1 > p^{\min}_{\mathbb{S}} \geq v_k$, so $k \notin \mathsf{D}(p^{\min}_{\mathbb{S}} + 1|x)$. Finally, for each $k \in N_1(p^{\min}_{\mathbb{S}} + 1|x)$,

[14]We remark that there may be $k \in N_0(e|x)$ such that $k \in \mathsf{D}(p^{\min}_{\mathbb{S}} + 1)$ in the buyer-optimal pendulum auction. Indeed, the buyer-optimal pendulum auction does not select efficient allocations (Example 9).

it follows from the definition of the efficient pendulum auction that k increased his price during $H_{\mathbb{S}}$, so $p_{\mathbb{S}}^{\min}+1 = p_k(h_{\mathbb{S}}) > p_k(h_\wedge) \geq v_k$; thus $k \notin \mathsf{D}(p_{\mathbb{S}}^{\min}+1|x)$. Altogether, then, by the first claim we have that for each $k \in N \backslash N(x)$ we have $k \notin \mathsf{D}(p_{\mathbb{S}}^{\min}+1|x)$; thus $i \notin \mathsf{D}(p_{\mathbb{S}}^{\min}+1|x)$, so since $i \in \mathsf{D}(p|x)$ we have $p \leq p_{\mathbb{S}}^{\min}$, as desired.

To complete the proof, we establish a contradiction in three cases that, by the first claim, are collectively exhaustive.

Case 1: $j \in N_1(e|x)$. In this case, (i) since $t_j = 0$ and $j \in \mathsf{S}(p|x)$, thus we have $j \in \mathsf{S}(p)$; and (ii) by the convention, j never increases his price during $H_{\mathbb{S}}$, so $p_j(h_\wedge) = p(h_{\mathbb{S}})$; thus by the Pendulum Lemma we have $p \geq p_j(h_\wedge) = p_j(h_{\mathbb{S}}) \geq p_{\mathbb{S}}^{\min}$, so by the second claim we have $p = p_{\mathbb{S}}^{\min} = p_j(h_\wedge)$. Then by the efficient convention, at each history in $H_{\mathbb{S}}$ we have that j offers p while resting unmatched.

To conclude, we use the first claim to argue that $i \notin N \backslash N(x)$, contradicting that $i \in N \backslash N(x)$. Indeed, first, for each $k \in N_0(e|x)$, there is a history h_k where k selects *exit* and j offers p while resting unmatched, so by the efficient convention we have $k \notin \mathsf{D}(p)$, so as $t_k = 0$ we have $k \notin \mathsf{D}(p|x)$; thus $i \notin N_0(e|x)$. Second, for each $k \in N_1 \backslash N_1(e|x)$, (i) as argued above we have $j \in \mathsf{S}(p)$; and (ii) by the efficient convention we have that k is to the left of j; thus by the Pendulum Lemma we have $k \in \mathsf{S}(p)$, so $p \geq v_k$. Then we cannot have $i \in N_1 \backslash N_1(e|x)$; else $i \notin \mathsf{D}^!(p|x)$, so $j \in \mathsf{S}^!(p|x)$, so since $t_j = 0$ we have $j \in \mathsf{S}^!(p)$, so since i is to the left of j thus by the Pendulum Lemma we have $i \in \mathsf{S}^!(p)$, so $p > v_i$ and thus $i \notin \mathsf{D}(p|x)$, contradicting $i \in \mathsf{D}(p|x)$. Thus as claimed we have $i \notin N \backslash N(x)$, contradicting that $i \in N \backslash N(x)$.

Case 2: $j \in N_0(p_{\mathbb{S}}^{\min}|x)$. In this case, (i) by the Pendulum Lemma we have $v_j \geq p_{\mathbb{S}}^{\min}$, and (ii) by the second claim and by $j \in \mathsf{S}(p|x)$ we have $p_{\mathbb{S}}^{\min} \geq p \geq v_j$; thus $v_j = p_{\mathbb{S}}^{\min} = p$. Then by the efficient convention, at $h_{\mathbb{S}}$ we have that the match of j is resting, so at each history in $H_{\mathbb{S}}$ this seller either (i) offers a price less than p, or (ii) offers p while resting. Moreover, we have that $j \notin \mathsf{S}^!(p|x)$, and thus that $i \in \mathsf{D}^!(p|x)$.

To conclude, we use the first claim to argue that $i \notin N \backslash N(x)$, contradicting that $i \in N \backslash N(x)$. Indeed, first, for each $k \in N_0(e|x)$, there is a history h_k where k selects *exit* and either (i) a seller offers a price less than p, or (ii) a seller offers p while resting, so by the efficient convention we have $k \notin \mathsf{D}^!(p)$, so as $t_k = 0$ we have $k \notin \mathsf{D}^!(p|x)$; thus $i \notin N_0(e|x)$. Second, for each $k \in N_1(p_{\mathbb{S}}^{\min}|x)$, by the Pendulum Lemma we have $p = p_{\mathbb{S}}^{\min} \geq v_k$, so $k \notin \mathsf{D}^!(p|x)$; thus $i \notin N_1(p|x)$. Finally, we must have $N_1(p_{\mathbb{S}}^{\min}+1|x) \neq \emptyset$; else by definition of the efficient pendulum auction, at $h_{\mathbb{S}}$ we have that j is matched at price $p_{\mathbb{S}}^{\min}$ while another buyer is matched at price $p_{\mathbb{S}}^{\min}+1$, so at $h_{\mathbb{S}}$ we have that the match of j is rising, contradicting that at $h_{\mathbb{S}}$ the match of j is resting. Thus as claimed we have $i \notin N \backslash N(x)$, contradicting that $i \in N \backslash N(x)$.

Case 3: $j \in N_0(p_{\mathbb{S}}^{\min}+1|x)$. In this case, since (i) by the Pendulum Lemma, we have $v_j \geq p_{\mathbb{S}}^{\min}+1 > p_{\mathbb{S}}^{\min}$, and (ii) by the second claim we have $p_{\mathbb{S}}^{\min} \geq p$; thus we have $v_j > p$, so $j \notin \mathsf{S}(p|x)$, contradicting $j \in \mathsf{S}(p|x)$. ■

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