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A Case of Regular Controls under Markov Chain Uncertainty

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A Continuous-Time Utility Maximization Problem with Borrowing Constraints in Macroeconomic Heterogeneous Agent Models: A Case of Regular Controls under Markov Chain Uncertainty*

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Abstract

This paper is concerned with the verification of a continuous-time utility maximization problem frequently used in recent macroeconomics. By focusing on Markov chain uncertainty, the problem in this paper can feature many characteristics of a typical consumer's problem in macroeconomics, such as borrowing constraints, endogenous labor supply, unhedgeable labor income, multiple asset choice, stochastic changes in preference, and others. I show that the value function of the problem is actually a constrained viscosity solution to the associated Hamilton–Jacobi–Bellman equation. Furthermore, the value function is continuously differentiable in the interior of its domain. Finally, the candidate optimal control is admissible, unique, and actually optimal.

Key words: Continuous-Time Utility Maximization, Borrowing Constraints, Hamilton–Jacobi–Bellman Equation, Viscosity Solution

JEL Classification: C61, E21, G11

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1 Introduction

A frontier in recent macroeconomic studies adopt the mean-field-game (MFG) approach to describe macroeconomy with ex post heterogeneous agents (e.g., Achdou et al. (2022), Rocheteau et al. (2018), Kaplan et al. (2018), Ahn et al. (2018), Nuño and Moll (2018), Djeutem and Xu (2019), Bornstein (2020), Guerrieri et al. (2020), Kaplan et al. (2020), Alves et al. (2020), McKay and Wieland (2021), Bilal et al. (2021), Nirei and Scheinkman (2021), Laibson et al. (2021), Bilal et al. (2022), and Alvarez and Lippi (2022)). The MFG approach in macroeconomics is a straightforward continuous-time extension of the Bewley–Huggett–Aiyagari models (Bewley (1986), Huggett (1993), and Aiyagari (1994)). However, as discussed in Achdou et al. (2022), continuous-time settings enable more efficient numerical computation compared to discrete-time settings. Hence, we can easily conduct rich economic analyses, including welfare analysis, policy evaluation, and counterfactual simulations, as reported in the aforementioned studies.

In parallel with the above rich macroeconomic applications of the MFG, many macroeconomists are also interested in the theoretical justification for the use of the Hamilton–Jacobi–Bellman (HJB) equation to solve a consumer's utility maximization problem. The HJB equation is a partial differential equation characterizing the value function of the problem, and in the standard of the optimal control literature, we can derive the optimal policies by solving the HJB equation. There is a vast literature in mathematical finance on how to properly apply the HJB equation to economic problems under uncertainty since the pioneering work of Merton (1969). However, comparing consumer problems in the macroeconomic MFG with those in mathematical finance, some are similar but the others are different. Indeed, the theoretical validity of the HJB equation approach in the macroeconomic MFG is less examined, except for some studies such as Rocheteau et al. (2018), Nirei and Scheinkman (2021), and Alvarez and Lippi (2022). In fact, the following two issues are not trivial in general: whether the value function of a consumer's utility maximization problem actually solves the HJB equation and whether a candidate optimal policy derived by the HJB equation is actually optimal.

In this paper, I present the verification result of a continuous-time utility maximization problem in macroeconomic MFGs to close the aforementioned theoretical gaps related to the HJB equation. The macroeconomic utility maximization problem has two typical features: borrowing constraints and unhedgeable uncertainty. A consumer cannot borrow money in excess of an exogenous borrowing constraint. Furthermore, a consumer typically receives unhedgeable labor income; hence, he or she must solve an incomplete market problem. In this paper, I consider a consumer utility maximization problem that not only includes these two main features but also the following typical characteristics of macroeconomic MFGs: a difference between the borrowing rate and lending rate, endogenous labor supply, consumption tax/subsidy, portfolio choice between a liquid asset and an illiquid asset, accumulation of human capital, preference for asset holdings like the money-in-the-utility model or durable goods model, and stochastic changes in preferences. To introduce these rich macroeconomic features, I focus on the uncertainty represented by a Markov chain. However, the Markov chain can describe much idiosyncratic uncertainty in macroeconomics. Hence, this is not restrictive from the perspective of economic theory.

I first show that the value function of the utility maximization problem in the aforementioned setting is actually a constrained viscosity solution to an associated HJB equation (Proposition 7). A viscosity solution is a solution concept of the differential equation;[1] it is broader than the classical solution and does not guarantee its differentiability in the entire domain. Many macroeconomic studies assume that their value functions are a viscosity solution to an associated HJB equation, but in general, sufficient conditions to satisfy this assumption are not clear. In fact, particularly in multiple asset cases, we need to slightly modify the HJB equation from the usual ones for the viscosity solution property, as in (3.3). This modification does not matter in numerical computation because the original HJB equation and the modified one are identical under the assumption that the value function is strictly increasing, concave, and continuously differentiable. However, first of all, we need to show the viscosity solution property for the modified HJB equation.

I next show that the value function is continuously differentiable in the interior of its domain (Proposition 10). Thus, the value function solves the HJB equation in the classical sense in the interior of its domain. Differentiability is crucial in utility maximization problems, as a candidate of optimal consumption is usually identified by the first-order derivative of the value function with respect to a wealth process or a liquid asset process through the first-order

[1] Achdou et al. (2022) provide an economist-friendly introduction of the viscosity solution.

condition. However, a viscosity solution does not guarantee its differentiability everywhere. Thus, in a macroeconomic utility maximization problem, although we first assume or show that the value function is a viscosity solution to the HJB equation, we eventually require the classical solution property of the value function in order to unambiguously identify a candidate of optimal consumption everywhere. Therefore, the result of this study contributes to the justification of the first-order condition to identify optimal consumption in continuous-time macroeconomic MFGs.

Third, I show that a candidate of optimal policies derived by the HJB equation is admissible, actually optimal, and uniquely identified. When we can identify a candidate of optimal policies using the HJB equation, if such a candidate is not admissible in the sense that a consumer cannot follow it physically, a consumer cannot always solve his or her utility maximization problem. For example, if a consumer's wealth process controlled by some policy will blow up in finite time, he or she cannot follow this policy. Therefore, the result of this study contributes to the literature by identifying sufficient conditions for admissibility. Furthermore, the admissibility conditions in this study imply that standard macroeconomic consumers' problems with continuous controls under Markov chain uncertainty are well-posed under mild conditions.

The mathematical analysis in this paper relies on the existing mathematical finance literature. A large body of literature in mathematical finance has solved various continuous-time stochastic optimal control problems. Many studies such as Zariphopoulou (1992), Pham and Tankov (2008), Di Giacinto et al. (2011), and Gassiat et al. (2014) consider constant relative risk aversion (CRRA) utility maximization problems with exogenous borrowing constraints. Since CRRA utility is also a standard specification in macroeconomics, these works can help us understand utility maximization problems in macroeconomics. Indeed, many proofs in this paper are applications in the above studies.

In this paper, I suppose two main assumptions which are somewhat restrictive. The first one is that the instantaneous utility function is bounded above. By this assumption, this paper's result cannot be applied to a CRRA utility whose coefficient of relative risk aversion (RRA) is not larger than one. However, unlike Rocheteau et al. (2018) who suppose that the utility is bounded above and below, I do not restrict the lower boundary of the utility, and hence we can analyze, for example, a CRRA utility whose RRA is larger than one. The next one is that a consumer can always receive a strictly positive and state-dependent income transfer. Therefore, the model in this paper cannot analyze an unemployment status without income support. By the second assumption, a consumer can always choose an identical control (e.g., a constant consumption), and thus it is in his or her admissible set in any situation. This property and the upper boundary of the instantaneous utility by the first assumption imply that the value function is bounded above and below. By the boundedness of the value function, we can easily show many favorable properties, such as the optimality of candidate optimal controls.

This paper's three main takeaways from the viewpoint of mathematical analysis are as follows: (1) the value function satisfies the dynamic programming principle (DPP, i.e., the Bellman equation) if the value function is continuous in the state variables (Theorem 4), (2) the value function is a constrained viscosity solution to the HJB equation if an associated Hamiltonian defined in Section 3 and the value function are continuous on an appropriate state space (Proposition 7), and (3) the value function is continuously differentiable everywhere if it is concave and the Hamiltonian is strictly convex in some variables (Proposition 10). The variables in (3) represent the first-order derivatives of the value function if they exist. Indeed, these conditions and results are quite natural from the viewpoint of optimal stochastic control and consistent with famous textbooks such as Øksendal and Sulem (2007) and Pham (2009). I show that the model in this paper satisfies all the conditions in (1), (2), and (3).

Unlike in the deterministic case, it is not clear whether the value function satisfies the DPP in the stochastic case. Although there exists a version of the DPP without assuming the continuity of the value function (e.g., Bouchard and Touzi (2011)), this differs from our familiar Bellman equation. On the other hand, the continuity of the value function makes the proof of the usual DPP easy. Hence, to introduce the standard DPP, it is necessary to show or assume the continuity of the value function without the DPP or the HJB equation. Furthermore, to show the properties of the value function we usually expect, we need to check the conditions of the value function and the Hamiltonian in (2) and (3). Although it is expected that the standard model satisfies the above conditions, it may be in fact difficult to check the conditions in an interesting and complicated macroeconomic model, if not impossible. In such a situation, we should at least postulate these conditions.[2]

[2]If we can show that there exists a smooth and "nice" solution to the HJB equation, the above

The remainder of this paper is organized as follows. Section 2 formulates a consumer's utility maximization problem, examines the fundamental properties of the value function of the problem, and introduces the DPP. Section 3 characterizes the value function as a constrained viscosity solution to the associated HJB equation and derives the differentiability of the value function everywhere in the interior of its domain. Section 4 proposes a state-constraint boundary condition and verifies the admissibility of candidate optimal policies. Section 5 concludes the paper and discuss future extensions. Appendix A provides the proofs of propositions and lemmas, and Appendix B presents some mathematical results related to asset processes in this paper.

2 Problem Formulation and Value Function

I initially formulate a continuous-time utility maximization problem with borrowing constraints and unhedgeable uncertainty. In the problem, a consumer plans his or her consumption stream, labor supply, deposit to or withdrawal from an illiquid asset, and investment in human capital in an infinite horizon. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,\infty)}$ satisfying the usual conditions. Let $Y := (Y_t)_{t\in[0,\infty)}$ be a K-state, right-continuous, and time-homogeneous Markov chain taking values in a discrete state space $\mathcal{Y} := \{1, 2, \cdots, K\}$, where K is a finite natural number with $K > 1$. I suppose that Y is $\mathbb{F}$-adapted, so that it is also $\mathbb{F}$-progressively measurable. I denote $\lambda_{i,j} \geq 0$ for $i \neq j$ as a constant intensity when Y changes from state i to state j and $\lambda_{i,i} := -\sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}$. The process Y expresses the time-varying and stochastic idiosyncratic characteristics of the consumer, such as labor productivity, labor market employment status, and state of preference.

A consumer can invest his or her money in a liquid asset, an illiquid asset, and human capital. $B := (B_t)_{t\in[0,\infty)}$, $A := (A_t)_{t\in[0,\infty)}$, and $H := (H_t)_{t\in[0,\infty)}$ are a stream of consumer's holding amounts of the liquid asset, illiquid asset, and human capital, respectively. Meanwhile, B, A, and H are governed by the following system of stochastic ordinary differential equations

consideration is not needed. The nice solution means, for example, a closed-form and/or bounded solution. In this case, we can apply the standard verification procedure as in Section 4. For example, a continuous-time version of the so-called Calvo-plus model often has a nice solution such as Nirei and Scheinkman (2021) and Alvarez and Lippi (2022), though it is a firm's problem.

(ODEs):

$$\begin{cases} \mathrm{d}B_t &= \Big(r(B_t, Y_t) + f(L_t, H_t, Y_t) + g(Y_t) - \tau_c(Y_t)C_t - D_tA_t - \chi_A(D_t, Y_t)A_t - \beta_H(Y_t)S_tH_t\Big)\mathrm{d}t, \\ \mathrm{d}A_t &= \Big((r_A(Y_t) + D_t)A_t + g_A(Y_t) - \pi_A(A_t, Y_t)\Big)\mathrm{d}t, \\ \mathrm{d}H_t &= \Big(\alpha_H(S_tH_t, Y_t) - \delta_H(Y_t)H_t\Big)\mathrm{d}t. \end{cases} \tag{2.1}$$

The definitions and interpretations of functions and processes in (2.1) are as follows: $r : \mathbb{R}\times\mathcal{Y} \to \mathbb{R}$ is a measurable function which represents increases/decreases of the liquid asset due to the interest; $L := (L_t)_{t\in[0,\infty)}$ is a stream of the consumer's labor supply that takes values in $[0, \overline{L}]$ for a constant $\overline{L} > 0$; $f : [0, \overline{L}] \times (0, \infty) \times \mathcal{Y} \to [0, \infty)$ is a measurable function which expresses as the labor earnings; $g : \mathcal{Y} \to (0, \infty)$ is a measurable function of income transfers; $\tau_c : \mathcal{Y} \to (0, \infty)$ is a consumption tax/subsidy rate; $C := (C_t)_{t\in[0,\infty)}$ is a stream of the consumer's consumption that takes values in $\mathcal{C} = (0, \infty)$; $D := (D_t)_{t\in[0,\infty)}$ is a stream of the consumer's deposit or withdrawal rate of the illiquid asset taking values in $\mathbb{R}$; $\chi_A : \mathbb{R} \times \mathcal{Y} \to [0, \infty)$ is a measurable transaction cost function to invest in the illiquid asset; $\beta_H : \mathcal{Y} \to (0, \infty)$ is a state-dependent proportional cost to invest in human capital; $S := (S_t)_{t\in[0,\infty)}$ is a stream of investment rates in human capital taking values in $[0, \infty)$; $r_A : \mathcal{Y} \to \mathbb{R}$ is an interest rate on the illiquid asset; $g_A : \mathcal{Y} \to [0, \infty)$ is a measurable function that represents automatic payroll deduction to the illiquid asset; $\pi_A : \mathbb{R} \times \mathcal{Y} \to [0, \infty)$ is a measurable function that represents asset taxation; $\alpha_H : [0, \infty) \times \mathcal{Y} \to [0, \infty)$ is a production technology function of the human capital; and $\delta_H : \mathcal{Y} \to (0, \infty)$ is a state-dependent depreciation rate of human capital. Here, I suppose a borrowing constraint with respect to B. Assume a constant $\underline{B} \geq 0$ such that $B_t \geq -\underline{B}$ for any $t \in [0, \infty)$. In other words, a consumer chooses (C, L, D, S) as $B_t \geq -\underline{B}$, $A_t > 0$, and $H_t > 0$ are satisfied.

Let us consider a consumer's preference, which is expressed as the following time-additive discounted expected utility:

$$\mathrm{E}\left[\int_0^\infty e^{-\rho t}\Big(u(C_t, L_t, A_t, B_t, H_t, Y_t)\Big)\mathrm{d}t\right], \tag{2.2}$$

where E is the expectation operator on $(\Omega, \mathcal{F}, \mathbb{P})$, $\rho \in (0, \infty)$ is a subjective discount rate, $u : \mathcal{C} \times [0, \overline{L}] \times (0, \infty) \times [-\underline{B}, \infty) \times (0, \infty) \times \mathcal{Y} \to \mathbb{R}$ is a instantaneous utility/disutility function of consumption, labor, and the amounts of the assets. Thus, the instantaneous utility u includes a money utility function of B and a utility function of the amount of the durable goods.[3]

Hereafter, for notational simplicity, I write $\overline{\mathcal{X}} = (0,\infty) \times [-\underline{B},\infty) \times (0,\infty)$, $\mathcal{X} = (0,\infty) \times (-\underline{B},\infty) \times (0,\infty)$, and $\partial\mathcal{X} = (0,\infty) \times \{-\underline{B}\} \times (0,\infty)$. Let us introduce the following baseline assumption in this paper:

Assumption 1 *Suppose the following with respect to $r, f, g, \tau_c, \chi_A, r_A, \pi_A, \alpha_H, \beta_H, \delta_H, u$, and $\underline{B}$.*

1. *For any $i \in \mathcal{Y}$, $b \to r(b,i)$ is concave*[4] *and globally Lipschitz continuous on $\mathbb{R}$. Furthermore, there exists $\underline{i} \in \mathcal{Y}$ such that $r(b,\underline{i}) \le r(b,i)$ for any $(b,i) \in \mathbb{R} \times \mathcal{Y}$.*
2. *For any $i \in \mathcal{Y}$, $(l,h) \to f(l,h,i)$ is non-negative, non-decreasing in each argument, continuous, and jointly concave on $[0,\overline{L}] \times (0,\infty)$.*
3. *g and τ_c are bounded above and away from zero. Let $\underline{g} := \min_{i\in\mathcal{Y}} g(i) > 0$, $\overline{g} := \max_{i\in\mathcal{Y}} g(i)$, $\underline{\tau}_c := \min_{i\in\mathcal{Y}} \tau_c(i) > 0$, and $\overline{\tau}_c := \max_{i\in\mathcal{Y}} \tau_c(i)$.*
4. *For any $i \in \mathcal{Y}$, $d \to \chi_A(d,i)$ is non-negative, continuous, and convex on $\mathbb{R}$ with $\chi_A(0,i) = 0$, and a minimizer of $d \to d + \chi_A(d,i)$ exists on $\mathbb{R}$, denoted by $\underline{d}(i) \in (-\infty,0)$, such that $\underline{d}(i) + \chi_A(\underline{d}(i),i) < 0$.*
5. *r_A is bounded. Let $\overline{r}_A := \max_{i\in\mathcal{Y}} r_A(i)$ and $\underline{r}_A := \min_{i\in\mathcal{Y}} r_A(i)$.*
6. *For any $i \in \mathcal{Y}$, $a \to \pi_A(a,i)$ is non-negative, non-decreasing, globally Lipschitz continuous, and convex on $\mathbb{R}$. Additionally, π_A satisfies either (the illiquid asset case) $r_A(i)a - \pi_A(a,i) \ge 0$ for any $(a,i) \in (0,\infty) \times \mathcal{Y}$, or (the durable goods case) $\pi_A(a,i) = 0$ for any $(a,i) \in (0,\infty) \times \mathcal{Y}$.*
7. *For any $i \in \mathcal{Y}$, $x \to \alpha_H(x,i)$ is surjective, strictly increasing, continuous, and concave on $[0,\infty)$ with $\alpha_H(0,i) = 0$. Furthermore, for any fixed $(h,i,s) \in (0,\infty) \times \mathcal{Y} \times [0,\infty)$, the deterministic ODE $\mathrm{d}\widehat{H}_t = (\alpha_H(s\widehat{H}_t,i) - \delta_H(i)\widehat{H}_t)\mathrm{d}t$ with $\widehat{H}_0 = h$ has a unique strictly positive solution.*
8. *β_H and δ_H are bounded above and away from zero. Let $\underline{\delta}_H := \min_{i\in\mathcal{Y}} \delta_H(i) > 0$ and $\overline{\delta}_H(i) := \max_{i\in\mathcal{Y}} \delta_H(i)$.*

[3] We can see the illiquid asset A as the amount of durable goods.

[4] In this paper, I say "concave" or "convex" to refer to weak concavity or weak convexity. If I would refer to strict concavity or convexity, I say "strictly concave" or "strictly convex."

9. *For any $i \in \mathcal{Y}$, $(c, \widetilde{l}, a, b, h) \to u(c, \overline{L} - \widetilde{l}, a, b, h, i)$ is finite, bounded above, non-decreasing in each argument, continuous, and concave on $\mathcal{C} \times [0, \overline{L}] \times \overline{\mathcal{X}}$. Furthermore, there exists a finite measurable function $(c, l, i) \to \underline{u}(c, l, i)$ on $\mathcal{C} \times [0, \overline{L}] \times \mathcal{Y}$ such that $\inf_{(a,b,h) \in \overline{\mathcal{X}}} u(c, l, a, b, h, i) \geq \underline{u}(c, l, i) > -\infty$.*

10. *There exists a constant $\underline{y} \in (0, \underline{g})$ such that $\underline{y} + r(-\underline{B}, i) > 0$ for any $i \in \mathcal{Y}$.*

Throughout this paper, I suppose Assumption 1. I will introduce additional assumptions as needed (Assumptions 8 and 11). Almost all conditions in Assumption 1 are standard in the literature. However, the boundedness of u is restrictive, but frequently used specifications in macroeconomics satisfies this boundedness condition. For example, a CRRA utility with its coefficient of RRA being strictly larger than one or a constant absolute risk aversion utility is bounded above. Hereafter, I suppose that $\sup_{(c,l,a,b,h) \in \mathcal{C} \times [0, \overline{L}] \times \overline{\mathcal{X}}} \{u(c, l, a, b, h, i)\} = 0$ for any $i \in \mathcal{Y}$ without loss of generality.

One important assumption is $\underline{g} = \min_{i \in \mathcal{Y}} g(i) > 0$. This implies that a consumer can obtain money even if he or she does not work. Hence, g represents a type of a income transfer scheme. The reason why $\underline{g} > 0$ is that it allows a consumer to always consume. Thus, we can easily show that the value function has a lower boundary.

The existence of a constant $\underline{y} \in (0, \underline{g})$ in the tenth condition of Assumption 1 implies that the exogenous borrowing constraint $\underline{B}$ is restricted. In particular, the strict inequality $\underline{y} + r(-\underline{B}, i) > 0$ expresses a situation in which the exogenous borrowing constraint does not exceed the natural debt limit when a consumer does not work. The constant $\underline{y}$ yields a nonempty admissible set of controls as well as the condition of $\underline{g} > 0$.

The convexity of the transaction cost and $\chi_A(0, i) = 0$ imply the scaling condition: $\chi_A(cd, i) \leq c\chi_A(d, i)$ for any $(c, d) \in [0, 1] \times \mathbb{R}$ and $i \in \mathcal{Y}$. The scaling condition is needed to show the concavity of the value function with respect to a in the presence of the illiquid asset taxation π_A. Similarly, I suppose that α_H is surjective, strictly increasing, and concave, to show that the value function is concave in h. Under this assumption, there exists the functional inverse of $x \to \alpha_H(x, i)$ for any $i \in \mathcal{Y}$, denoted by $\alpha_H^{-1}(x; i)$. We can easily see that $x \to \alpha_H^{-1}(x; i)$ is convex on $[0, \infty)$.

In the problem formulation of this paper, I assume that the illiquid and human capital asset processes never hit zero. Therefore, the consumer will not strictly live hand-to-mouth. This is because the DPP in three-closed-boundary problems is not trivial. Hence, I avoid technicalities related to three-closed-boundary problems. However, many specifications in the existing literature often virtually imply one-closed-boundary problems. Furthermore, to avoid technicalities regarding the boundary solution in optimization, I suppose that the human capital investment S is not upper bounded. However, I emphasize that we typically numerically compute human capital investment by assuming that its boundaries do not exist. In numerical computation by finite difference schemes, we need to restrict a computational range of (A, B, H). Therefore, optimal human capital investment is eventually bounded in this computational range.

Here, let us consider examples of specification of the functions satisfying Assumption 1. These specifications also satisfy the additional conditions in Assumptions 8 and 11. Indeed, the problem formulation in this paper covers almost all examples of continuous-control problems under the standard utility with continuous controls presented on Benjamin Moll's website.[5] First, let us consider an example of $r(b, i)$, which is a function such as

$$r(b,i) := \begin{cases} r_i^+ b, & \text{if } b \geq 0, \\ r_i^- b, & \text{if } b < 0, \end{cases}$$

where $0 < r_i^+ \leq r_i^- < \infty$ for any $i \in \mathcal{Y}$. Furthermore, suppose that $\underline{i} \in \mathcal{Y}$ exists such that $r_{\underline{i}}^- = \max_{i \in \mathcal{Y}} r_i^-$ and $r_{\underline{i}}^+ = \min_{i \in \mathcal{Y}} r_i^+$. One important generalization is that r allows to not be differentiable everywhere, so that the model can express the difference between the borrowing rate and lending rate as above. Furthermore, the model can express Bardóczy (2017)'s labor-market matching model inspired by Krusell et al. (2010), because, in this case, labor earnings are a concave function of the liquid asset, like as r.

Next, let us consider an example of the labor earning $f(l, h, i)$:

$$f(l,h,i) := f_i \sqrt{hl}, \tag{2.3}$$

where $f_i \geq 0$ for any $i \in \mathcal{Y}$. (2.3) is different from the standard specification $f(l, h, i) = f_i hl$. As will be seen in Proposition 3, the value function is concave if f is jointly concave with respect to

[5] https://benjaminmoll.com/codes/

(l, h). Therefore, I provide the specification of jointly concave f as in (2.3). Note that one can set $f(l, i) = f_i l$ in the absence of the human capital h and also set $f(h, i) = f_i h$ in the absence of the endogenous labor supply l.

Let us consider an example of the transaction cost function to invest in the illiquid asset χ_A and capital taxation function π_A such as

$$\chi_A(d, i) := \frac{\xi_i}{2} d^2, \quad \text{and} \quad \pi_A(a, i) = \begin{cases} 0 & \text{if } a < 0, \\ \frac{\pi_i^{\max}}{2\bar{a}_i} a^2 & \text{if } 0 \leq a \leq \bar{a}_i, \\ \pi_i^{\max} \left(a - \frac{\bar{a}_i}{2} \right) & \text{otherwise,} \end{cases}$$

where $\xi_i > 0$, $\pi_i^{\max} \geq 0$, and $\bar{a}_i > 0$ are constants for all $i \in \mathcal{Y}$. The transaction cost to invest in the illiquid asset is quadratic, which is a frequently used specification. In this paper, I will exclude the kinked cost function like Kaplan et al. (2018), to derive differentiability of the value function. However, as will discuss in the last paragraph in Section 3, the assumption of smooth χ is not too restrictive. Further suppose that $r_A(i) \geq \pi_i^{\max}$ for any $i \in \mathcal{Y}$. Then, $r_A(i)a - \pi_A(a, i) \geq 0$ for any $(a, i) \in (0, \infty) \times \mathcal{Y}$, which is a natural requirement.

Additionally, if $r_A(i) < 0$ and $\pi_A(a, i) = 0$ for any $(a, i) \in (0, \infty) \times \mathcal{Y}$, the illiquid asset can be seen as the amount of durable goods, as in McKay and Wieland (2021), though they suppose that a consumer controls the amount of durable goods by impulse adjustments.

Let us consider an example of the production technology of the human capital α_H such as

$$\alpha_H(x, i) = \theta_i x^{\alpha_i},$$

where $\theta_i > 0$, and $\alpha_i \in (0, 1)$ are constants. The production technology function of the human capital has decreasing returns to scale, which is a standard specification as in Couturier et al. (2020).

Finally, let us consider a utility function u. One standard specification is a linear utility such as

$$u(c, l, a, b, h, i) = \frac{c^{1-\gamma_i}}{1-\gamma_i} - \phi_i \frac{l^{1+1/\nu_i}}{1+1/\nu_i} + \psi_i^a \frac{(a + \zeta_i^a)^{1-\eta_i}}{1-\eta_i} + \psi_i^b \frac{(b + \underline{B} + \zeta_i^b)^{1-\eta_i}}{1-\eta_i} + \psi_i^h \frac{(h + \zeta_i^h)^{1-\eta_i}}{1-\eta_i},$$

where $\gamma_i > 1$, $\nu_i > 0$, $\eta_i > 1$, $\phi_i \geq 0$, $\psi_i^x \geq 0$, and $\zeta_i^x > 0$ are constants for any $i \in \mathcal{Y}$ and $x = a, b$, and h. γ_i expresses a coefficient of relative risk aversion, and ν_i expresses the Frisch elasticity of labor supply. One restriction is $\gamma_i > 1$ and $\eta_i > 1$. Thus, I only consider the case in which a consumer has RRA larger than one. Although RRA is larger than one in the standard specification in macroeconomics, such restriction is necessary to guarantee the upper boundedness of u. Furthermore, it is well known that the utility maximization problem tends to be stable when RRA is larger than one. In addition, the introduction of the utility from b allows us to express the money-in-the-utility model. It can be easily seen that $\inf_{(a,b,h)\in\overline{\mathcal{X}}} u(c,l,a,b,h,i) = \frac{c^{1-\gamma_i}}{1-\gamma_i} - \phi_i \frac{l^{1+1/\nu_i}}{1+1/\nu_i} + \text{constant} > -\infty$

Another specification is a non-linear utility by the CES aggregator such as

$$u(c,l,a,b,h,i) = \frac{1}{1-\gamma_i}\left(\psi_i^c c^{1-\frac{1}{\eta_i}} + \psi_i^l(\overline{L}-l)^{1-\frac{1}{\eta_i}} + \psi_i^a a^{1-\frac{1}{\eta_i}} + \psi_i^b(b+\underline{B})^{1-\frac{1}{\eta_i}} + \psi_i^h h^{1-\frac{1}{\eta_i}}\right)^{\frac{1-\gamma_i}{1-1/\eta_i}},$$

where $\gamma_i > 1$, $\eta_i > 1$, and $\psi_i^x \geq 0$, $x = c, l, a, b, h$ are constants for any $i \in \mathcal{Y}$. The above representation includes frequently used specifications in the endogenous labor supply model and the durable goods model. One restriction is $\eta_i > 1$. Indeed, $\eta_i > 1$ yields $\inf_{(a,b,h)\in\overline{\mathcal{X}}} u(c,l,a,b,h,i) > -\infty$ for any $(c,l) \in \mathcal{C} \times [0,\overline{L}]$. To include the case of $\eta_i \in (0,1)$, for example, we need to modify the term of labor supply to $\psi_i^l(\overline{L}-l+\zeta)^{1-1/\eta_i}$ and the terms of the assets to $\psi_i^x(x+\zeta)^{1-1/\eta_i}$, where $\zeta > 0$ is a sufficiently small constant.[6] The above specifications are standard and frequently used in macroeconomic studies. The results in this paper are true in all the reduced versions: for example, removing the endogenous labor supply, the illiquid asset, or the human capital. In such a reduced model, we need at least the assumptions of u, r, g, τ_c, $\underline{B}$, and the parameters in including features.

To formulate a utility maximization problem in this paper, I initially introduce an admissible set of quadruplets of control processes (C, L, D, S).

Definition 2 (Admissible set with borrowing constraints) *For any $(a,b,h,i) \in \overline{\mathcal{X}} \times \mathcal{Y}$, a quadruplet of stochastic processes taking values in $\mathcal{C} \times [0,\overline{L}] \times \mathbb{R} \times [0,\infty)$, denoted by $(C,L,D,S) = (C_t, L_t, D_t, S_t)_{t\in[0,\infty)}$, is* admissible *under an initial condition (a,b,h,i) if it satisfies the following: (1) It is right-continuous and $\mathbb{F}$-progressively measurable; (2) The system of the stochastic*

[6] In the presence of the endogenous labor supply, we also need $\eta_i \geq 1/\gamma_i$ to satisfy Assumption 11.

ODEs (2.1) *with initial conditions* $B_0 = b, A_0 = a, H_0 = h$, *and* $Y_0 = i$ *has an* $\mathbb{F}$*-adapted solution*[7] *controlled by* (C, L, D, S)*; and (3)* $A_t > 0$, $H_t > 0$ *and* $B_t \geq -\underline{B}$ *hold,* $\mathbb{P}$ *almost surely (*$\mathbb{P}$*-a.s.) for any* $t \in [0, \infty)$*. I denote the set of all admissible processes under* (a, b, h, i) *by* $\mathcal{A}(a, b, h, i)$*. For any* $(a, b, h, i) \in \overline{\mathcal{X}} \times \mathcal{Y}$ *and* $(C, L, D, S) \in \mathcal{A}(a, b, h, i)$*, I denote a liquid asset process, an illiquid asset process, a human capital process, and a Markov chain, starting at* $B_0 = b, A_0 = a, H_0 = h$*, and* $Y_0 = i$ *and controlled by* (C, L, D, S)*, by* $B^{a,b,h,i;C,L,D,S}$, $A^{a,i;D}$, $H^{h,i;S}$*, and* Y^i*, respectively.*

By Definition 2, we can define a value function as follows:

$$V_i(a, b, h) := \sup_{(C,L,D,S)\in\mathcal{A}(a,b,h,i)} \mathrm{E}\left[\int_0^\infty e^{-\rho t}\Big(u(C_t, L_t, A_t^{a,i;D}, B_t^{a,b,h,i;C,L,D,S}, H_t^{h,i;S}, Y_t^i)\Big)\mathrm{d}t\right], \tag{2.4}$$

for all $(a, b, h, i) \in \overline{\mathcal{X}} \times \mathcal{Y}$. By Assumption 1, u is bounded above so that the value function is bounded above as well. As aforementioned, I suppose that u is non-positive without loss of generality. Thus, V is also non-positive. Using standard arguments, several favorable properties of the value function V can be shown only by its definition.

Proposition 3 *$\mathcal{A}(a, b, h, i)$ is not empty for all $(a, b, h, i) \in \overline{\mathcal{X}} \times \mathcal{Y}$. For any $i \in \mathcal{Y}$, V_i is non-decreasing in each argument, jointly concave on $\overline{\mathcal{X}}$, continuous on $\mathcal{X}$, continuous on $\partial\mathcal{X}$, and bounded below* $\min_{j\in\mathcal{Y}}\{\underline{u}((\underline{g} - \underline{y})/\overline{\tau}_c, 0, j)\}/\rho > -\infty$*.*

Proof of Proposition 3. See Appendix A.1. □

Proposition 3 can be shown only by the definition of the value function. I do not use the DPP or the HJB equation. Therefore, these properties are fundamental. Note that, if a maximizer exists in $\mathcal{A}(a, b, h, i)$ for any $(a, b, h, i) \in \overline{\mathcal{X}} \times \mathcal{Y}$ and if $u(c, l, a, b, h, i)$ is strictly concave in $(c, l) \in \mathcal{C} \times [0, \overline{L}]$, the concavity of V_i can be replaced with strict concavity.

The concavity yields two favorable features of the value function. The first is the continuity of the value function in the interior of its domain, which is the most important property in Proposition 3. The continuity guarantees the measurability of the value function and yields the

[7] Based on the concept of a strong solution to stochastic differential equations, I define an $\mathbb{F}$-adapted solution to the stochastic ODE such that it is continuous without Y, $\mathbb{F}$-adapted and satisfies the integral form of (2.1) $\mathbb{P}$-a.s. Definition 19 in Appendix B provides the precise definition of the adapted solution.

DPP. Without assuming the continuity in the interior, we often need to show the "weak" DPP in the sense of Bouchard and Touzi (2011) and consider the "discontinuous" viscosity solution, though we can often show that the discontinuous viscosity solution is in fact continuous by the comparison theorem. The second one is that the value function has concave kinks at most. Under fairly standard conditions, the viscosity solution to the HJB equation only allows convex kinks. Therefore, it can be expected that the viscosity solution is smooth in the interior.

Here, I introduce the DPP to characterize the value function as a solution to the HJB equation.

Theorem 4 (Dynamic programming principle (DPP)) *For any $(a,b,h,i) \in \overline{\mathcal{X}} \times \mathcal{Y}$, $V_i(a,b,h)$ satisfies the following:*

1. *For any $(C,L,D,S) \in \mathcal{A}(a,b,h,i)$ and bounded $\mathbb{F}$-stopping time θ, the following inequality holds:*

$$V_i(a,b,h) \geq \mathrm{E}\left[\int_0^\theta e^{-\rho t} u(C_t, L_t, A_t^{a,i;D}, B_t^{a,b,h,i;C,L,D,S}, H_t^{h,i;S}, Y_t^i)\mathrm{d}t \right.$$
$$\left. + e^{-\rho\theta} V_{Y_\theta^i}(A_\theta^{a,i;D}, B_\theta^{a,b,h,i;C,L,D,S}, H_\theta^{h,i;S})\right]. \quad (2.5)$$

2. *For any $\epsilon > 0$, there exists $(C,L,D,S) \in \mathcal{A}(a,b,h,i)$ such that, for any bounded $\mathbb{F}$-stopping time θ, the following inequality holds:*

$$V_i(a,b,h) - \epsilon \leq \mathrm{E}\left[\int_0^\theta e^{-\rho t} u(C_t, L_t, A_t^{a,i;D}, B_t^{a,b,h,i;C,L,D,S}, H_t^{h,i;S}, Y_t^i)\mathrm{d}t \right.$$
$$\left. + e^{-\rho\theta} V_{Y_\theta^i}(A_\theta^{a,i;D}, B_\theta^{a,b,h,i;C,L,D,S}, H_\theta^{h,i;S})\right]. \quad (2.6)$$

The two DPPs are equivalent to the well-known Bellman equation as follows:

$$V_i(a,b,h) = \sup_{(C,L,D,S) \in \mathcal{A}(a,b,h,i)} \mathrm{E}\left[\int_0^\theta e^{-\rho t} u(C_t, L_t, A_t^{a,i;D}, B_t^{a,b,h,i;C,L,D,S}, H_t^{h,i;S}, Y_t^i)\mathrm{d}t \right.$$
$$\left. + e^{-\rho\theta} V_{Y_\theta^i}(A_\theta^{a,i;D}, B_\theta^{a,b,h,i;C,L,D,S}, H_\theta^{h,i;S})\right], \quad (2.7)$$

for any $(a,b,h,i) \in \overline{\mathcal{X}} \times \mathcal{Y}$ and bounded $\mathbb{F}$-stopping time θ.

On the control problem with closed boundaries, Theorem 4 is not trivial, since we do not know a priori that the value function is continuous up to the boundaries, as discussed in Gassiat et al. (2014). The second DPP (2.6) can be shown easily without the continuity of the value function. In contrast, the continuity on the entire domain makes proving the first DPP (2.6) easy, as in Theorem IV.7.1 of Fleming and Soner (2006). If not, we should use a somewhat technical argument to show the DPP. To apply this, we need the continuity of the value function on $\mathcal{X}$ and on $\partial\mathcal{X}$. These two continuities have been shown in Proposition 3, so that we can show the DPP. Thus, in closed boundary problems not limited to this paper's setting, these two continuities of the value function need to be shown only by its definition if we want to use the DPP shown in the standard manner. The DPP in Theorem 4 can be shown by tracing the proof in Gassiat et al. (2014), so I omit it.[8]

Even though we can show the first DPP (2.6) without the continuity of the value function up to the boundaries, the first DPP (2.6) yields this continuity as follows:

Proposition 5 *For any $i \in \mathcal{Y}$, $V_i(a, b, h)$ is continuous on $\overline{\mathcal{X}}$.*

Proof of Proposition 5. See Appendix A.2. □

By Propositions 3 and 5 and Theorem 4, I characterize the fundamental properties of the value function on the entire domain. The subsequent section shows the viscosity solution property of the value function by these fundamental properties.

[8]The detailed proof of the DPP in closed boundary cases can be found in their working paper version Gassiat et al. (2011).

3 Viscosity Solution Property and Differentiability of the Value Function

The HJB equation in the utility maximization problem (2.4) under the asset dynamics (2.1) is given by

$$\begin{aligned}
\rho v_i(a,b,h) - \sup_{(c,l,d,s)\in\mathcal{C}\times[0,\overline{L}]\times\mathbb{R}\times[0,\infty)} \Big\{ u(c,l,a,b,h,i) \\
+ \partial_b v_i(a,b,h)\Big(r(b,i)+f(l,h,i)+g(i)-\tau_c(i)c-(d+\chi_A(d,i))a-\beta_H(i)sh\Big) \\
+ \partial_a v_i(a,b,h)\Big((r_A(i)+d)a+g_A(i)-\pi_A(a,i)\Big)+\partial_h v_i(a,b,h)\Big(\alpha_H(sh,i)-\delta_H(i)h\Big)\Big\} \\
- \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\Big(v_j(a,b,h)-v_i(a,b,h)\Big) = 0, \quad (3.1)
\end{aligned}$$

for all $(a,b,h,i)\in\overline{\mathcal{X}}\times\mathcal{Y}$, where

$$\partial_b v_i(a,b,h)=\frac{\partial v_i(a,b,h)}{\partial b}, \quad \partial_a v_i(a,b,h)=\frac{\partial v_i(a,b,h)}{\partial a}, \quad \text{and} \quad \partial_h v_i(a,b,h)=\frac{\partial v_i(a,b,h)}{\partial h}.$$

Hereafter, I denote the partial derivative of a function φ with respect to a variable x by $\partial_x\varphi$.

Here, we need to show that the value function actually solves the HJB equation (3.1) in the viscosity sense. Following convention, I use a representation of (3.1) by Hamiltonian:

$$\begin{aligned}
\rho v_i(a,b,h) - \mathcal{H}_i(a,b,h,\partial_a v_i(a,b,h),\partial_b v_i(a,b,h),\partial_h v_i(a,b,h)) \\
- \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\Big(v_j(a,b,h)-v_i(a,b,h)\Big) = 0,
\end{aligned}$$

where $(a,b,h,i,p_a,p_b,p_h)\to\mathcal{H}_i(a,b,h,p_a,p_b,p_h)$ is a Hamiltonian such that

$$\begin{aligned}
\mathcal{H}_i(a,b,h,p_a,p_b,p_h) := \sup_{(c,l,d,s)\in\mathcal{C}\times[0,\overline{L}]\times\mathbb{R}\times[0,\infty)} \Big\{ u(c,l,a,b,h,i) \\
+ p_b\Big(r(b,i)+f(l,h,i)+g(i)-\tau_c(i)c-(d+\chi_A(d,i))a-\beta_H(i)sh\Big) \\
+ p_a\Big((r_A(i)+d)a+g_A(i)-\pi_A(a,i)\Big)+p_h\Big(\alpha_H(sh,i)-\delta_H(i)h\Big)\Big\}. \quad (3.2)
\end{aligned}$$

One important issue of the Hamiltonian (3.2) is that $\mathcal{H}$ is not continuous at $p_b = 0$ in general. To show the viscosity subsolution property of the value function in the standard manner, the continuity of the Hamiltonian on the considered domain is required. Under the frequently used setting in macroeconomics, we can show that $(a, b, h, p_a, p_b, p_h) \to \mathcal{H}_i(a, b, h, p_a, p_b, p_h)$ is jointly continuous on $\overline{\mathcal{X}} \times \mathbb{R} \times (0, \infty)^2$, for any fixed $i \in \mathcal{Y}$. However, it can be easily observed that $\mathcal{H}_i(a, b, h, p_a, 0, p_h) = \infty$ if $p_a \neq 0$ or $p_h > 0$, but $\mathcal{H}_i(a, b, h, 0, 0, 0) = \sup_{c \in \mathcal{C}} u(c, 0, a, b, h, i) < \infty$. We read p_b as the partial derivative of V_i with respect to b if it exists, but Proposition 3 only states that V_i is non-decreasing in b. Therefore, we cannot exclude the possibility of $\partial_b V_i(a, b, h) = 0$ a priori. To address this issue on the boundary, I define the constrained viscosity solution slightly differently than the literature on continuous-time macroeconomics.[9]

Definition 6 (Constrained viscosity solution) *Let us denote by $C^1_K(\mathcal{Z})$ a set of pairs of K functions, which are continuously differentiable on $\mathcal{Z} \subseteq \mathbb{R}^3$. Let $v = (v_i)_{i \in \mathcal{Y}}$ be a pair of locally bounded and continuous functions on a subset $\mathcal{Z}$ of $\overline{\mathcal{X}}$ for all $i \in \mathcal{Y}$.*

1. *v is a* viscosity supersolution *to the HJB equation (3.1) on $\mathcal{Z}$ if, for any $(a, b, h) \in \mathcal{Z}$ and $\varphi \in C^1_K(\mathcal{Z})$, where $v_i - \varphi_i$ attains a local minimum at (a, b, h) for all $i \in \mathcal{Y}$, v and φ satisfy the following inequalities:*

$$\rho v_i(a, b, h) - \mathcal{H}_i(a, b, h, \partial_a \varphi_i(a, b, h), \partial_b \varphi_i(a, b, h), \partial_h \varphi_i(a, b, h)) - \sum_{j \in \mathcal{Y} \setminus \{i\}} \lambda_{i,j} \Big(v_j(a, b, h) - v_i(a, b, h) \Big) \geq 0,$$

   $\partial_b \varphi_i(a, b, h) \geq 0$, and $\partial_h \varphi_i(a, b, h) \geq 0$, for all $i \in \mathcal{Y}$.

2. *v is a* viscosity subsolution *to the HJB equation (3.1) on $\mathcal{Z}$ if, for any $(a, b, h) \in \mathcal{Z}$ and $\varphi \in C^1_K(\mathcal{Z})$, where $v_i - \varphi_i$ attains a local maximum at (a, b, h) for all $i \in \mathcal{Y}$, v and φ satisfy the following inequalities:*

$$\rho v_i(a, b, h) - \mathcal{H}_i(a, b, h, \partial_a \varphi_i(a, b, h), \partial_b \varphi_i(a, b, h), \partial_h \varphi_i(a, b, h)) - \sum_{j \in \mathcal{Y} \setminus \{i\}} \lambda_{i,j} \Big(v_j(a, b, h) - v_i(a, b, h) \Big) \leq 0,$$

[9] Note that the following refinement is not required when the admissible set is bounded.

$\partial_b \varphi_i(a,b,h) \leq 0$, *or* $\partial_h \varphi_i(a,b,h) \leq 0$, *for all* $i \in \mathcal{Y}$.

3. v *is a* constrained viscosity solution *to the HJB equation* (3.1) *on* $\overline{\mathcal{X}}$ *if it is a viscosity supersolution on* $\mathcal{X}$ *and a viscosity subsolution on* $\overline{\mathcal{X}}$.[10]

Thus, Definition 6 considers the following modification of the HJB equation:

$$\min \Big\{ \rho v_i(a,b,h) - \mathcal{H}_i(a,b,h,\partial_a v_i(a,b,h), \partial_b v_i(a,b,h), \partial_h v_i(a,b,h)) \\ - \sum_{j \in \mathcal{Y} \setminus \{i\}} \lambda_{i,j} \Big(v_j(a,b,h) - v_i(a,b,h) \Big),\ \partial_b v_i(a,b,h),\ \partial_h v_i(a,b,h) \Big\} = 0. \quad (3.3)$$

As discussed previously, minimization matters only when $\partial_b v_i(a,b,h) = 0$ or $\partial_h v_i(a,b,h) = 0$. Furthermore, it is important that under the standard assumptions, we can directly solve (3.1) in numerical computation to obtain the value function. Therefore, we do not suffer from minimization in reality. In this sense, minimization is only required for mathematical completion.

Here, we can show the viscosity solution property of the value function in the standard manner through the following proposition:

Proposition 7 (Constrained viscosity solution property) *The value function* $V = (V_i)_{i \in \mathcal{Y}}$ *is a viscosity supersolution to the HJB equation* (3.3) *on* $\mathcal{X}$. *Furthermore, suppose that* $(a,b,h,p_a,p_b,p_h) \to \mathcal{H}_i(a,b,h,p_a,p_b,p_h)$ *is jointly continuous on* $\overline{\mathcal{X}} \times \mathbb{R} \times (0,\infty)^2$ *for all* $i \in \mathcal{Y}$. *Then,* V *is also a viscosity subsolution to* (3.3) *on* $\overline{\mathcal{X}}$. *Therefore,* V *is a constrained viscosity solution to* (3.3) *on* $\overline{\mathcal{X}}$.

Proof of Proposition 7. See Appendix A.3. □

Here, to remove the minimization in (3.3), I introduce the following additional assumption yielding a strict monotonicity of the value function.

Assumption 8 (Limit behavior) *Suppose the followings with respect to* r, f, χ_A, *and* u.

1. $r(0,i) \geq 0$ *for any* $i \in \mathcal{Y}$.
2. $f(l_\epsilon, h, i) \to \infty$ *as* $h \to \infty$ *for any* $l_\epsilon \in (0, \overline{L}]$ *and* $i \in \mathcal{Y}$.
3. $d + \chi_A(d,i) \to \infty$ *as* $d \to \pm\infty$ *for any* $i \in \mathcal{Y}$.

[10] I suppose that the smooth test function can be defined in an open subset $\mathcal{Z} \subseteq \mathbb{R}^3$ such that $\overline{\mathcal{X}} \subseteq \mathcal{Z}$.

4. *$u(c, \overline{L} - \tilde{l}, a, b, h, i)$ is strictly increasing in $(c, \tilde{l}, b, h) \in \mathcal{C} \times [0, \overline{L}] \times [-\underline{B}, \infty) \times (0, \infty)$ for any $(a, i) \in (0, \infty) \times \mathcal{Y}$; $u(c, 0, a, b, h, i) \to 0$ as $(c, b) \to \infty$ for any $(a, h, i) \in (0, \infty)^2 \times \mathcal{Y}$; and there exists a function $l \to \overline{u}^\infty(l)$ on $(0, \overline{L}]$ such that $\lim_{(c,b,h)\to\infty} u(c, l, a, b, h, i) \geq \overline{u}^\infty(l)$ for any $(l, a, i) \in (0, \overline{L}] \times (0, \infty) \times \mathcal{Y}$. Further suppose that there exists a constant $l^* \in [0, \overline{L}]$ such that $\overline{u}^\infty(l) \to 0$ as $l \to l^*$.*

Assumption 8 is standard, and then I have the following lemma.

Lemma 9 *Suppose Assumption 8. Then, $V_i(a, b, h)$ is strictly increasing in b and h on $\overline{\mathcal{X}}$ for any $i \in \mathcal{Y}$. Furthermore, $V_i(a, b, h) \to 0$ as $b \to \infty$ or $h \to \infty$ for any $(a, i) \in (0, \infty) \times \mathcal{Y}$.*

Proof of Lemma 9. See Appendix A.4. □

Lemma 9 assures the strict positivity of the partial derivatives of V if they exist, so that we then can remove the minimization in (3.3).

Here, let us show the classical solution property of the value function in the interior of its domain. I have first considered a viscosity solution in Proposition 7. However, the value function is in fact continuously differentiable everywhere in the interior of its domain. This is because the viscosity solution of a maximization problem "does not admit concave kinks." Achdou et al. (2022) discuss in their Appendix D that the viscosity solution of a maximization problem v only admits the undifferentiability such that $\partial_x^+ v(x) > \partial_x^- v(x)$, because the Hamiltonian of a maximization problem is strictly convex. However, a concave function only admits the undifferentiability such that $\partial_x^+ v(x) < \partial_x^- v(x)$. Therefore, a concave viscosity solution of a maximization problem is continuously differentiable everywhere in the interior of its domain. Rocheteau et al. (2018) use the same idea in the one-asset case. In this study, I generalize their result to a multiple-asset case as the following proposition.

Proposition 10 *Suppose Assumption 8. For any $i \in \mathcal{Y}$, if $(p_a, p_b, p_h) \to H_i(a, b, h, p_a, p_b, p_h)$ is strictly convex on $\mathbb{R} \times (0, \infty)^2$ for any fixed $(a, b, h) \in \mathcal{X}$, then $V_i(a, b, h)$ is continuously differentiable everywhere on $\mathcal{X}$. Therefore, $\partial_b V_i(a, b, h) > 0$ and $\partial_h V_i(a, b, h) > 0$ on $\mathcal{X}$.*

Proof of Proposition 10. See Appendix A.5. □

By Proposition 10, the value function is a classical solution to the HJB equation (3.1) in the interior of its domain. By the differentiability, we can always identify a functional form of candidate optimal policy (C, L, D, S) on $\mathcal{X} \times \mathcal{Y}$. In Propositions 7 and 10, I assume that the Hamiltonian is continuous and strictly convex. Here, I provide a sufficient condition for the continuity and strict convexity of the Hamiltonian.

Assumption 11 (Smoothness) *Suppose the following with respect to f, χ_A, α_H, and u.*

1. *For any $(h, i) \in (0, \infty) \times \mathcal{Y}$, $l \to f(l, h, i)$ is twice continuously differentiable on $(0, \overline{L})$.*
2. *For any $i \in \mathcal{Y}$, $d \to \chi_A(d, i)$ is twice continuously differentiable on $\mathbb{R}$ and $\partial_{dd}\chi_A(d, i) > 0$ for any $d \in \mathbb{R}$. Furthermore, for any $i \in \mathcal{Y}$, $\inf_{d\in\mathbb{R}} \partial_d\chi_A(d, i) < -1$ and $\partial_d\chi_A(d, i) \to \infty$ as $d \to \infty$.*
3. *For any $i \in \mathcal{Y}$, $x \to \alpha_H(x, i)$ is twice continuously differentiable on $(0, \infty)$, and $\partial_{xx}\alpha_H(x, i) < 0$ for any $x \in (0, \infty)$. Furthermore, for any $i \in \mathcal{Y}$, $\partial_x\alpha_H(x, i) \to \infty$ as $x \to 0$ and $\partial_x\alpha_H(x, i) \to 0$ as $x \to \infty$.*
4. *For any $(a, b, h, i) \in \overline{\mathcal{X}} \times \mathcal{Y}$, $(c, l) \to u(c, l, a, b, h, i)$ is twice continuously differentiable on $\mathcal{C} \times (0, \overline{L})$ and strictly concave on $\mathcal{C} \times [0, \overline{L}]$. Furthermore, for any $(c, l, a, b, h, i) \in \mathcal{C} \times (0, \overline{L}) \times \overline{\mathcal{X}} \times \mathcal{Y}$, the Hessian matrix of u with respect to (c, l) is negative definite in the strict sense. In addition, u is twice continuously differentiable with respect to c even if $l = 0$ or $\overline{L}$, and then $\partial_{cc}u(c, l, a, b, h, i) < 0$ for any $(a, b, h, i) \in \overline{\mathcal{X}} \times \mathcal{Y}$. Moreover, for any compact cube subset $\mathcal{K} = [\underline{a}, \overline{a}] \times [\underline{b}, \overline{b}] \times [\underline{h}, \overline{h}] \subset \overline{\mathcal{X}}$, let $\overline{\partial_c u_{\mathcal{K}}}(c, i) := \sup_{(l,a,b,h)\in[0,\overline{L}]\times\mathcal{K}} \partial_c u(c, l, a, b, h, i)$ and $\underline{\partial_c u_{\mathcal{K}}}(c, i) := \inf_{(l,a,b,h)\in[0,\overline{L}]\times\mathcal{K}} \partial_c u(c, l, a, b, h, i)$. Then, $\overline{\partial_c u_{\mathcal{K}}}(c, i) \to 0$ as $c \to \infty$ and $\underline{\partial_c u_{\mathcal{K}}}(c, i) \to \infty$ as $c \to 0$ for any $\mathcal{K}$ and $i \in \mathcal{Y}$. Finally, in the presence of the endogenous labor supply, $(c, l) \to -\partial_l u(c, l, a, b, h, i)/\partial_c u(c, l, a, b, h, i)$ is strictly increasing on $\mathcal{C} \times (0, \overline{L})$ and $\partial_{cl}u(c, l, a, b, h, i) \geq 0$ for any $(c, l, a, b, h, i) \in \mathcal{C} \times (0, \overline{L}) \times \overline{\mathcal{X}} \times \mathcal{Y}$.*

Under Assumption 11, a unique maximizer of the following maximization problem exists for any $(p_b, a, b, h, i) \in (0, \infty) \times \overline{\mathcal{X}} \times \mathcal{Y}$:

$$\mathcal{M}(p_b, a, b, h, i) := \max_{(c,l)\in\mathcal{C}\times[0,\overline{L}]} \{u(c, l, a, b, h, i) + p_b(f(i, h, l) - \tau_c(i)c)\}. \tag{3.4}$$

It can be seen that (3.4) is an optimization problem with respect to (c, l) in the Hamiltonian

$\mathcal{H}$. Hereafter, I denote the unique maximizer of (3.4) by $(c^*(p_b,a,b,h,i), l^*(p_b,a,b,h,i))$. Then, it can be easily seen that $(p_b,a,b,h) \to (c^*(p_b,a,b,h,i), l^*(p_b,a,b,h,i), \mathcal{M}(p_b,a,b,h,i))$ is continuous on $(0,\infty)\times\mathcal{X}$ for any $i\in\mathcal{Y}$. By Assumptions 8 and 11, we can offer the favorable properties of the Hamiltonian, i.e., the continuity and strict convexity.

Lemma 12 *Suppose Assumptions 8 and 11. Then, the Hamiltonian $\mathcal{H}_i(a,b,h,p_a,p_b,p_h)$ is continuous with respect to (a,b,h,p_a,p_b,p_h) on $\overline{\mathcal{X}}\times\mathbb{R}\times(0,\infty)^2$ for any $i\in\mathcal{Y}$, and it is continuously differentiable and strictly convex with respect to (p_a,p_b,p_h) on $\mathbb{R}\times(0,\infty)^2$ for any $(a,b,h,i)\in\overline{\mathcal{X}}\times\mathcal{Y}$. Furthermore, there exists a unique continuous maximizer of the maximization problem in the Hamiltonian.*

Proof of Lemma 12. See Appendix A.6. □

By the envelope theorem, the partial derivatives of the Hamiltonian are

$$\begin{aligned}
\partial_{p_b}\mathcal{H}_i(a,b,h,p_a,p_b,p_h) &= r(b,i) + f(l^*(p_b,a,b,h,i),h,i) + g(i) - \tau_c(i)c^*(p_b,a,b,h,i) \\
&\quad - \left((\partial_d\chi_A)^{-1}\left(\frac{p_a}{p_b}-1;i\right) + \chi_A\left((\partial_d\chi_A)^{-1}\left(\frac{p_a}{p_b}-1;i\right),i\right)\right)a \qquad (3.5)\\
&\quad - \beta_H(i)(\partial_x\alpha_H)^{-1}\left(\frac{\beta_H(i)p_b}{p_h};i\right),\\
\partial_{p_a}\mathcal{H}_i(a,b,h,p_a,p_b,p_h) &= \left(r_A(i) + (\partial_d\chi_A)^{-1}\left(\frac{p_a}{p_b}-1;i\right)\right)a + g_A(i) - \pi_A(a,i), \qquad (3.6)\\
\partial_{p_h}\mathcal{H}_i(a,b,h,p_a,p_b,p_h) &= \alpha_H\left((\partial_x\alpha_H)^{-1}\left(\frac{\beta_H(i)p_b}{p_h};i\right),i\right) - \delta_H(i)h, \qquad (3.7)
\end{aligned}$$

where $(d,i)\to(\partial_d\chi)^{-1}(d;i)$ and $(x,i)\to(\partial_x\alpha_H)^{-1}(x;i)$ are the functional inverse of $d\to\partial_d\chi_A(d,i)$ and $x\to\partial_x\alpha_H(x,i)$, respectively. They are the (candidate) optimal saving/investment rates of B, A, and H, respectively. Note that $\partial_{p_b}\mathcal{H}_i$, $\partial_{p_a}\mathcal{H}_i$, and $\partial_{p_h}\mathcal{H}_i$ are continuous with respect to (a,b,h,p_a,p_b,p_h) on $\overline{\mathcal{X}}\times\mathbb{R}\times(0,\infty)^2$ for any $i\in\mathcal{Y}$. In Assumption 11, I suppose the smooth cost function χ, whereas Kaplan et al. (2018) suppose the kinked cost function at 0. However, Kaplan et al. (2018)'s setting is admissible because their kinked cost function derives a unique maximizer in the Hamiltonian. The Hamiltonian is continuous even in the kinked case, so that the viscosity solution property holds. The partial differentiability of V with respect to a is not clear in the regions where the cost is kinked, but differentiability is required to determine the optimal policy unambiguously, as discussed in the introduction. In this sense, Kaplan et al. (2018)'s model can determine a unique candidate of the optimal deposit and/or withdrawal rate. Furthermore, one can show that the partial derivative of V_i with respect to b is uniquely identified everywhere in the interior domain by the strict convexity of the Hamiltonian with respect to p_b. Therefore, candidates of optimal consumption and labor supply are also uniquely identified.

4 Admissibility and Uniqueness of the Optimal Control

In this section, I discuss the property of a candidate of optimal policies on the boundaries and its admissibility. Herein, I suppose Assumptions 8 and 11. By the concavity of the value function, the right derivative of the value function on the boundaries can be defined, including infinity:

$$\partial_b^+ V_i(a, -\underline{B}, h) := \lim_{b \downarrow -\underline{B}} \frac{V_i(a, b, h) - V_i(a, -\underline{B}, h)}{b + \underline{B}}.$$

However, $\partial_b^+ V_i(a, -\underline{B}, h)$ is indeed locally bounded everywhere.

Lemma 13 *For any $(a, h, i) \in (0, \infty)^2 \times \mathcal{Y}$, $\partial_b^+ V_i(a, -\underline{B}, h)$ is locally bounded. Furthermore, for any $i \in \mathcal{Y}$, $\partial_a V_i(a, -\underline{B}, h)$ and $\partial_h V_i(a, -\underline{B}, h)$ exist for all $(a, h) \in (0, \infty)^2$, and $(a, b, h) \to (\partial_a V_i(a, b, h), \partial_b^+ V_i(a, b, h), \partial_h V_i(a, b, h))$ is continuous on $\overline{\mathcal{X}}$.*

Proof of Lemma 13. See Appendix A.7. □

By Lemma 13, $(\partial_a V_i(a, b, h), \partial_b^+ V_i(a, b, h), \partial_h V_i(a, b, h))$ is continuous on $\overline{\mathcal{X}}$ for any $i \in \mathcal{Y}$. Therefore, we can consider a continuously differentiable extension of V on an extended domain. Let $\mathcal{W} := (0, \infty) \times (-\infty, \infty) \times (0, \infty)$. Let V^* be a function on $\mathcal{W} \times \mathcal{Y}$ such that $V^* \in C_K^1(\mathcal{W})$ and $V_i^*(a, b, h) = V_i(a, b, h)$ on $\overline{\mathcal{X}} \times \mathcal{Y}$. The continuously differentiable extension V^* yields an explicit representation of the state-constraint boundary condition of this problem. As discussed in Soner (1986), the viscosity subsolution property on the boundaries, smoothness of the value function, and continuity of the Hamiltonian imply that

$$
\begin{aligned}
&-\mathcal{H}_i(a,-\underline{B},h,\partial_a\varphi_i(a,-\underline{B},h),\partial_b\varphi_i(a,-\underline{B},h),\partial_h\varphi_i(a,-\underline{B},h)) \\
&\qquad\qquad \le -\mathcal{H}_i(a,-\underline{B},h,\partial_a V_i^*(a,-\underline{B},h),\partial_b V_i^*(a,-\underline{B},h),\partial_h V_i^*(a,-\underline{B},h)),
\end{aligned}
$$

for any $(a,h,i)\in(0,\infty)^2\times\mathcal{Y}$ and smooth function $\varphi=(\varphi_j)_{j\in\mathcal{Y}}$ with $0=V_j^*(a,-\underline{B},h)-\varphi_j(a,-\underline{B},h)=\max_{(a',b',h')\in\overline{\mathcal{X}}}\{V_j^*(a',b',h')-\varphi_j(a',b',h')\}$ for all $j\in\mathcal{Y}$. This inequality yields the following condition:

$$
-\mathbf{n}(a,-\underline{B},h)\cdot\nabla_p\mathcal{H}_i(a,-\underline{B},h,\partial_a V_i^*(a,-\underline{B},h),\partial_b V_i^*(a,-\underline{B},h),\partial_h V_i^*(a,-\underline{B},h))\ge 0, \quad (4.1)
$$

where $\mathbf{n}(a,-\underline{B},h)$ is the exterior normal vector of the region $\overline{\mathcal{X}}$ at $(a,-\underline{B},h)$ and ∇_p is the gradient operator with respect to (p_a,p_b,p_h).[11] The exterior normal vector at $(a,-\underline{B},h)$ is $\mathbf{n}(a,-\underline{B},h)=(0,-1,0)^\top$ for all $(a,h)\in(0,\infty)^2$. Accordingly, the *state-constraint boundary condition* can be expressed as

$$
\begin{aligned}
&\partial_{p_b}\mathcal{H}_i(a,-\underline{B},h,\partial_a V_i^*(a,-\underline{B},h),\partial_b V_i^*(a,-\underline{B},h),\partial_h V_i^*(a,-\underline{B},h)) \\
&\qquad = \partial_{p_b}\mathcal{H}_i(a,-\underline{B},h,\partial_a V_i(a,-\underline{B},h),\partial_b^+ V_i(a,-\underline{B},h),\partial_h V_i(a,-\underline{B},h))\ge 0. \quad (4.2)
\end{aligned}
$$

Since $\partial_{p_b}\mathcal{H}_i$ is the drift of the (candidate) optimally controlled liquid asset process in state i, the inequality (4.2) implies that it must be non-negative on the boundary. Accordingly, we can see that (4.1) is *the* state-constraint boundary condition, which is a necessary condition for optimality, and (4.2) is a reduced form of (4.1).

Let us discuss the admissibility of candidate optimal controls derived by the HJB equation. As is standard, I first define admissible feedback controls.

Definition 14 (Admissible feedback controls) *A quadruplet of measurable functions* $(C_i(a,b,h),L_i(a,b,h),D_i(a,b,h),S_i(a,b,h))$ *on* $\overline{\mathcal{X}}\times\mathcal{Y}$ *is an* admissible feedback control*, if for*

[11] I have taken the directional derivative of the Hamiltonian with respect to (p_a,p_b,p_h) at $(\partial_a V_i^*(a,-\underline{B},h),\partial_b V_i^*(a,-\underline{B},h),\partial_h V_i^*(a,-\underline{B},h))$ in the direction $-\mathbf{n}(a,-\underline{B},h)$. This operation is admissible since $\mathcal{H}_i$ is continuously differentiable and convex with respect to (p_a,p_b,p_h), V_i^* is continuously differentiable at $(a,-\underline{B},h)$, and we can set $\nabla_{(a,b,h)}\varphi_i(a,-\underline{B},h)=\nabla_{(a,b,h)}V_i^*(a,-\underline{B},h)-t\mathbf{n}(a,-\underline{B},h)$, where t is a positive constant.

any $(a,b,h,i) \in \overline{\mathcal{X}} \times \mathcal{Y}$*, the following system of stochastic ODEs with respect to* (A,B,H)*:*

$$\begin{cases} \mathrm{d}B_t = \Big(r(B_t, Y_t^i) + f(L_{Y_t^i}(A_t, B_t, H_t), H_t, Y_t^i) + g(Y_t^i) - \tau_c(Y_t^i) C_{Y_t^i}(A_t, B_t, H_t) \\ \qquad\qquad - \Big(D_{Y_t^i}(A_t, B_t, H_t) + \chi_A(D_{Y_t^i}(A_t, B_t, H_t), Y_t^i) \Big) A_t - \beta_H(Y_t^i) S_{Y_t^i}(A_t, B_t, H_t) H_t \Big)\, \mathrm{d}t, \\ \mathrm{d}A_t = \Big(\Big(r_A(Y_t^i) + D_{Y_t^i}(A_t, B_t, H_t) \Big) A_t + g_A(Y_t^i) - \pi_A(A_t, Y_t^i) \Big) \mathrm{d}t, \\ \mathrm{d}H_t = \Big(\alpha_H(S_{Y_t^i}(A_t, B_t, H_t) H_t, Y_t^i) - \delta_H(Y_t^i) H_t \Big)\, \mathrm{d}t \end{cases}$$

with $(A_0, B_0, H_0) = (a,b,h)$*, has an* $\mathbb{F}$*-adapted solution*
and $(C_{Y_t^i}(A_t,B_t,H_t), L_{Y_t^i}(A_t,B_t,H_t), D_{Y_t^i}(A_t,B_t,H_t), S_{Y_t^i}(A_t,B_t,H_t))_{t\in[0,\infty)}$ *is admissible under* (a,b,h,i)*. I also denote an illiquid asset process, a liquid asset process, and a human capital process starting at* (a,b,h,i) *and controlled by an admissible feedback control* (C,L,D,S) *by* $(A^{a,i;D}, B^{a,b,h,i;C,L,D,S}, H^{h,i;S})$*.*

Under the admissible feedback controls in Definition 14, I have the following result on the optimality conditions of the candidate optimal controls derived by the HJB equation (3.1). In the literature, the result of Proposition 15 is often called a verification theorem. The proof when assuming smoothness of a solution to the HJB equation (3.1) is standard, so I omit it.

Proposition 15 (Verification theorem) *Let* v *be a function in* $C^1_K(\mathcal{W})$ *satisfying the following:*

1. v *is a constrained viscosity solution to the HJB equation* (3.1) *on* $\overline{\mathcal{X}}$*.*
2. *A quadruplet of continuous functions* (C^v, L^v, D^v, S^v)*,*

   $$\begin{aligned} &C_i^v(a,b,h) := c^*(\partial_b v_i(a,b,h), a, b, h, i), \quad L_i^v(a,b,h) := l^*(\partial_b v_i(a,b,h), a, b, h, i), \\ &D_i^v(a,b,h) := (\partial_d \chi_A)^{-1}\left(\frac{\partial_a v_i(a,b,h)}{\partial_b v_i(a,b,h)} - 1; i \right), \quad S_i^v(a,b,h) := \frac{1}{h} (\partial_x \alpha_H)^{-1} \left(\frac{\beta_H(i) \partial_b v_i(a,b,h)}{\partial_h v_i(a,b,h)}; i \right), \end{aligned}$$

   for $(a,b,h,i) \in \overline{\mathcal{X}} \times \mathcal{Y}$*, can be defined, and it is an admissible feedback control.*
3. *For any* $(a,b,h,i) \in \overline{\mathcal{X}} \times \mathcal{Y}$ *and* $(C,L,D,S) \in \mathcal{A}(a,b,h,i)$*, consider the following* $\mathbb{F}$*-local*

martingales:

$$\int_0^t \mathbf{1}\{Y^i_{s-} = j\} e^{-\rho s}\Big(v_k(A^{a,i;D}_s, B^{a,b,h,i;C,L,D,S}_s, H^{h,i;S}_s) \\ - v_j(A^{a,i;D}_s, B^{a,b,h,i;C,L,D,S}_s, H^{h,i;S}_s)\Big)\mathrm{d}\widetilde{N}^{j,k}_s,$$

for $t \in [0,\infty)$ *and* $j,k \in \mathcal{Y}$ *with* $j \neq k$*, where* $(\widetilde{N}^{j,k})_{j\neq k}$ *is a compensated* $K(K-1)$*-dimensional* $\mathbb{F}$*-Poisson process with constant intensities* $(\lambda_{j,k})_{j\neq k}$*, which satisfies* $\{Y_{t-} = j, Y_t = k\} = \{\widetilde{N}^{j,k}_t - \widetilde{N}^{j,k}_{t-} > 0\}$ *for all* $t \in [0,\infty)$ *and* $j,k \in \mathcal{Y}$ *with* $j \neq k$*. Then, all of them are uniformly integrable* $\mathbb{F}$*-martingales.*

4. *The following condition*

$$\lim_{t\to\infty} \mathrm{E}\left[e^{-\rho t} v_{Y^i_t}(A^{a;D}_t, B^{a,b,h,i;C,L,D,S}_t, H^{h;S}_t)\right] = 0$$

is satisfied for any $(a,b,h,i) \in \overline{\mathcal{X}} \times \mathcal{Y}$ *and* $(C,L,D,S) \in \mathcal{A}(a,b,h,i)$*.*

Then, $v = V$ *holds on* $\overline{\mathcal{X}} \times \mathcal{Y}$*, and* $(C^v_{Y^i_t}(A^{a,i;D^v}_t, B^{a,b,h,i;C^v,L^v,D^v,S^v}_t, H^{h,i;S^v}_t),$ $L^v_{Y^i_t}(A^{a,i;D^v}_t, B^{a,b,h,i;C^v,L^v,D^v,S^v}_t, H^{h,i;S^v}_t), D^v_{Y^i_t}(A^{a,i;D^v}_t, B^{a,b,h,i;C^v,L^v,D^v,S^v}_t, H^{h,i;S^v}_t),$ $S^v_{Y^i_t}(A^{a,i;D^v}_t, B^{a,b,h,i;C^v,L^v,D^v,S^v}_t, H^{h,i;S^v}_t))_{t\in[0,\infty)}$ *is an optimal control for any* $(a,b,h,i) \in \overline{\mathcal{X}} \times \mathcal{Y}$*.*

It can be easily seen that Proposition 15 implies the uniqueness of the solution to the HJB equation (3.1). Unfortunately, we have the uniqueness only on a set of functions satisfying the four conditions in Proposition 15. By the comparison theorem, the uniqueness of the viscosity solution is usually established on a larger set of functions. The three assets setting makes it difficult to obtain a general result of the uniqueness. However, a viscosity solution in the one-asset model with linearly separable money utility is unique in a class of continuous functions which vanish at infinity. This can be shown by the standard comparison argument of the viscosity solution to the first order PDE via the doubling-variables method.

Here, let $C^*_i(a,b,h)$, $L^*_i(a,b,h)$, $D^*_i(a,b,h)$, and $S^*_i(a,b,h)$ be functions on $\overline{\mathcal{X}} \times \mathcal{Y}$ such that

$$\begin{aligned} &C^*_i(a,b,h) := c^*(\partial_b V^*_i(a,b,h), a, b, h, i), \quad L^*_i(a,b,h) := l^*(\partial_b V^*_i(a,b,h), a, b, h, i), \\ &D^*_i(a,b,h) := (\partial_d \chi_A)^{-1}\left(\frac{\partial_a V^*_i(a,b,h)}{\partial_b V^*_i(a,b,h)} - 1; i\right), \quad S^*_i(a,b,h) := \frac{1}{h}(\partial_x \alpha_H)^{-1}\left(\frac{\beta_H(i)\partial_b V^*_i(a,b,h)}{\partial_h V^*_i(a,b,h)}; i\right). \end{aligned} \tag{4.3}$$

We already know that the extended value function $V^* \in C^1_K(\mathcal{W})$ solves the HJB equation (3.1) on $\overline{\mathcal{X}}$ in the constrained viscosity sense, and that it is bounded on $\overline{\mathcal{X}}$. Hence, we can easily observe that the value function satisfies conditions 1, 3, and 4 in Proposition 15. Therefore, one can confirm that the verification is completed and (C^*, L^*, D^*, S^*) is an optimal control if one shows the admissibility of (C^*, L^*, D^*, S^*).

To verify whether (C^*, L^*, D^*, S^*) is admissible, the existence of the controlled asset processes is crucially important. In fact, the right continuity and measurability are trivial if the asset processes controlled by (C^*, L^*, D^*, S^*) exist. To demonstrate the existence of the asset processes, it suffices to show that the following system of deterministic ODEs for $(a^i_t, b^i_t, h^i_t)_{t\geq 0}$ has a solution for any $(a, b, h, i) \in \overline{\mathcal{X}} \times \mathcal{Y}$.

$$\mathrm{d}a^i_t = s^a_i(a^i_t, b^i_t, h^i_t)\mathrm{d}t, \quad \mathrm{d}b^i_t = s^b_i(a^i_t, b^i_t, h^i_t)\mathrm{d}t, \quad \mathrm{d}h^i_t = s^h_i(a^i_t, b^i_t, h^i_t)\mathrm{d}t, \tag{4.4}$$

with $(a^i_0, b^i_0, h^i_0) = (a, b, h)$, where $s^b_i(a, b, h) := \partial_{p_b}\mathcal{H}_i(a, b, h, \partial_a V^*_i(a, b, h), \partial_b V^*_i(a, b, h), \partial_h V^*_i(a, b, h))$, $s^a_i(a, b, h) := \partial_{p_a}\mathcal{H}_i(a, b, h, \partial_a V^*_i(a, b, h), \partial_b V^*_i(a, b, h), \partial_h V^*_i(a, b, h))$, and $s^h_i(a, b, h) := \partial_{p_h}\mathcal{H}_i(a, b, h, \partial_a V^*_i(a, b, h), \partial_b V^*_i(a, b, h), \partial_h V^*_i(a, b, h))$. When s^a_i, s^b_i, and s^h_i are Lipschitz on $\overline{\mathcal{X}}$, I can easily prove the existence, following the standard methods. However, s^b_i may not be Lipschitz. To clarify this, I consider the one-asset case presented by Achdou et al. (2022). By Proposition 1 in Achdou et al. (2022), if the subjective discount rate is strictly larger than the interest rate, the saving rate in the lower-income case s^{low}_b exhibits the following asymptotic behavior close to the borrowing constraint:

$$s^{low}_b(b) \sim -k_1\sqrt{b + \underline{B}},$$

where k_1 is a positive constant. Therefore, the saving rate in the lower-income case is not Lipschitz in any neighborhood of the borrowing constraint. The above asymptotic behavior generates a high marginal propensity of consumption among liquid hand-to-mouth consumers, which is often observed in consumption data. Accordingly, this behavior is favorable from the macroeconomic theory viewpoint. Meanwhile, the global existence and uniqueness results of ODEs rely on the local Lipschitz property of their drivers. Hence, this behavior is not favorable mathematically.[12]

Here, I discuss how we should overcome the non-Lipschitz property. I emphasize that a local solution to the deterministic ODEs (4.4) exists when it started everywhere in $\overline{\mathcal{X}}$, because the saving rate functions s_a^i, s_b^i, and s_h^i are continuous on $\overline{\mathcal{X}}$, so that Peano's existence theorem can be applied. Therefore, my verification strategy proceeds as follows: (1) extending a local solution to infinite horizon, (2) showing the optimality of all the solutions, and (3) proving pathwise uniqueness of the solution. First, I consider the extensibility of a local solution. In other words, I shall show that a local solution does not blow up in finite time.

Lemma 16 *For any $(a, b, h, i) \in \overline{\mathcal{X}} \times \mathcal{Y}$, all local solutions to the system of the ODEs (4.4) in state i starting at (a, b, h) do not blow up in finite time. Furthermore, these solutions satisfy $a_t^i \geq ae^{\underline{m}t} > 0, b_t^i \geq -\underline{B}$, and $h_t^i \geq he^{-\overline{\delta}_H t} > 0$ for any $t \in [0, \infty)$, where $\underline{m}$ is a non-positive constant not depending on a, b, h, i, and t.*

Proof of Lemma 16. See Appendix A.8. □

By Lemma 16, we can extend a local solution of candidate optimally controlled asset processes in an infinite horizon. Then, we can show that the candidate optimal control is actually admissible and optimal by the verification theorem.

Proposition 17 *The extended value function V^* satisfies the four conditions in Proposition 15. Therefore, (C^*, L^*, D^*, S^*) is an admissible feedback control, and it is also an optimal control of the utility maximization problem.*

Proof of Proposition 17. See Appendix A.9. □

Let us discuss the uniqueness of optimally controlled asset processes. The uniqueness of the optimally controlled asset processes is a crucial property to evaluate the model. However owing to the possibility of the non-Lipschitzness of saving rates, we cannot apply a standard scheme. Instead, the existence of the optimal control and the strict concavity of u yield the uniqueness of the optimal control. Therefore, I can complete the verification of the utility maximization problem in this paper.

[12] The saving rate is one-sided Lipschitz in a one-asset case with linearly separable asset preferences. In this case, we can show the uniqueness of the solution more easily than in this paper.

Proposition 18 *For any $i \in \mathcal{Y}$, V_i is jointly strictly concave on $\overline{\mathcal{X}}$. Furthermore, the optimal control (C^*, L^*, D^*, S^*) and the asset processes controlled by (C^*, L^*, D^*, S^*) can be uniquely identified up to indistinguishability even when the processes start everywhere in $\overline{\mathcal{X}} \times \mathcal{Y}$.*

Proof of Proposition 18. See Appendix A.10. $\square$

5 Concluding Remarks

In this paper, I presented the verification result of a continuous-time utility maximization problem frequently used in macroeconomics. I demonstrated that the value function actually solves the associated HJB equation in the constrained viscosity sense. Furthermore, the value function is continuously differentiable in the interior of its domain, so that it can solve the HJB equation in the classical sense in the interior. Finally, a candidate optimal control is admissible, optimal, and uniquely identified. Therefore, a standard consumer's problem in heterogeneous-agent macroeconomic models is well-posed, and the HJB equation approach is valid. Subsequently, Shigeta (2022) demonstrates the existence of invariant measure and stationary equilibrium in the one-asset Aiyagari model.

One remaining issue is the richness of models, for example, introducing the models to other types of uncertainty (e.g., diffusion and/or jump-diffusion) and/or discontinuous decision making (e.g., optimal stopping and/or impulse controls). In this paper, I focused on a continuous-control problem with Markov chain uncertainty, but the above extensions are useful for analyzing rich macroeconomic models. However, I expect that this paper's result can be easily extended to a continuous-control model with impulse adjustments at exogenous Poisson timings. In particular, this type of problem has been investigated in Gassiat et al. (2014) and Rocheteau et al. (2018). The concavity of the value function in this problem can be shown easily only by its definition. Therefore, the DPP can be also shown, and hence the viscosity solution property of the value function and the admissibility of candidate optimal controls can be shown under mild assumptions.

The other potential issue is numerical computation. The model in this paper considers four state variables: A, B, H, and Y, which are too many to apply the standard implicit method with fine grids. Accordingly, we need to reduce the state variables or apply another computation method like the machine learning to solve the model numerically (e.g., Fernández-Villaverde et al. (2019)). A more fundamental problem is the decision making about investing to A and H. By continuously investing to A and H, the HJB equation has the terms whose denominators involve $\partial_b V_i$ and $\partial_h V_i$. By the Inada condition of u and α_H, these partial derivatives tend to go to zero when b and h become large. As a result, the numerical computation will be unstable because we need to divide some terms by a very small positive value. Therefore, we may need to change the variables such as $\widetilde{b} = \log(b + \underline{B} + \epsilon)$, where $\epsilon > 0$ is a small constant. The extensions of the models to resolve the above issues are important from the practical viewpoints, but I leave it for future research.

A Proofs

A.1 Proof of Proposition 3

Proof of Proposition 3.

The non-empty admissible set. Fix an arbitrary $i \in \mathcal{Y}$. I will first show the non-empty property of the admissible set. For any $(a, b, h) \in \overline{\mathcal{X}}$, let $(C, L, D, S) = ((g(Y^i) - \underline{y})/\tau_c(Y^i), 0, 0, 0)$. Then, $H^{h,i;0}$ has a strictly positive and unique solution: $H_t^{h,i;0} = he^{-\int_0^t \delta_H(Y_s^i)\mathrm{d}s}$ for $t \in [0, \infty)$. Meanwhile, by the global Lipschitz property of $a \to r_A(i)a - \pi_A(a, i)$, $A^{a,i;0}$ also has a unique solution. Let us check $A^{a,i;0} > 0$. First, let us consider the illiquid asset case. Let $\epsilon \in (0, a)$ be an arbitrary constant, and let $\theta^\epsilon = \inf\{t \in [0, \infty) | A_t^{a,i;0} = \epsilon\}$ be a stopping time. Following convention, set $\inf \emptyset = \infty$. For any fixed constant $T \in (0, \infty)$, I have $A_{T\wedge\theta^\epsilon}^{a,i;0} = a + \int_0^{T\wedge\theta^\epsilon} (r_A(Y_s^i)A_s^{a,i;0} + g_A(Y_s^i) - \pi_A(A_s^{a,i;0}, Y_s^i))\mathrm{d}s \geq a$ since $r_A(i)a - \pi_A(a, i) \geq 0$ and $g_A(i) \geq 0$ for any $(a, i) \in (0, \infty) \times \mathcal{Y}$. Therefore, $\theta^\epsilon = \infty$ and $A_t^{a,i;0} \geq a > 0$ $\mathbb{P}$-a.s. for any $t \in [0, \infty)$. Second, let us consider the durable goods case. Then, I have $A_t^{a,i;0} = ae^{\int_0^t r_A(Y_s^i)\mathrm{d}s} + \int_0^t e^{\int_s^t r_A(Y_r^i)\mathrm{d}r} g_A(Y_s^i)\mathrm{d}s \geq ae^{\underline{r}_A t} > 0$ $\mathbb{P}$-a.s. for any $t \in [0, \infty)$. Thus, in both of the cases, $A_t^{a,i;0}$ is strictly positive. Further, the right-continuity and measurability of (C, L, D, S) are obvious.

Here, consider B's ODE under (C,L,D,S):

$$\mathrm{d}B_t^{a,b,h,i;C,0,0,0} = \Big(r(B_t^{a,b,h,i;C,0,0,0}, Y_t^i) + f(0, H_t^{h,i;0}, Y_t^i) + \underline{y}\Big)\mathrm{d}t,$$

with $B_0^{a,b,h,i;C,0,0,0} = b$. Then, $(B_t^{a,b,h,i;C,0,0,0})_{t\in[0,\infty)}$ uniquely and globally exists since r is globally Lipschitz. Here, suppose that there exists a time $t_1 \in (0,\infty)$ such that $B_{t_1}^{a,b,h,i;C,0,0,0} + \underline{B} < 0$, and let us lead to a contradiction. Then, by the continuity of $B_t^{a,b,h,i;C,0,0,0}$ and $B_0^{a,b,h,i;C,0,0,0} = b \geq -\underline{B}$, there exists a time $t_0 \in [0,t_1)$ such that $B_{t_0}^{a,b,h,i;C,0,0,0} + \underline{B} = 0$ and $B_s^{a,b,h,i;C,0,0,0} + \underline{B} < 0$ for any $s \in (t_0,t_1]$. Here, for any $t_2 \in (t_0,t_1]$, the Lipschitz property of r implies

$$\begin{aligned} B_{t_2}^{a,b,h,i;C,0,0,0} + \underline{B} &= \int_{t_0}^{t_2} \Big(r\left(B_s^{a,b,h,i;C,0,0,0}, Y_s^i\right) + f(0, H_t^{h,i;0}, Y_t^i) + \underline{y}\Big)\,\mathrm{d}s \\ &\geq \int_{t_0}^{t_2} \Big(r\left(B_s^{a,b,h,i;C,0,0,0}, Y_s^i\right) - r(-\underline{B}, Y_s^i) + r(-\underline{B}, Y_s^i) + \underline{y}\Big)\,\mathrm{d}s \\ &\geq \int_{t_0}^{t_2} \left(-L_r\left|B_s^{a,b,h,i;C,0,0,0} + \underline{B}\right|\right)\mathrm{d}s = L_r \int_{t_0}^{t_2} \left(B_s^{a,b,h,i;C,0,0,0} + \underline{B}\right)\mathrm{d}s, \end{aligned}$$

where $L_r > 0$ is a Lipschitz constant of r. I have used $\underline{y} + r(-\underline{B}, j) > 0$ for any $j \in \mathcal{Y}$. Therefore, the Gronwall inequality implies $B_{t_1}^{a,b,h,i;C,0,0,0} + \underline{B} \geq 0$, but this is a contradiction. Hence, $B_t^{a,b,h,i;C,0,0,0} + \underline{B} \geq 0$ for any $t \in [0,\infty)$, and this implies $(C,L,D,S) = ((g(Y^i) - \underline{y})/\tau_c(Y^i), 0, 0, 0) \in \mathcal{A}(a,b,h,i)$.

The non-decreasing property of the value function. Second, I will show the non-decreasing property: fix arbitrary (a,b,h) and (a',b',h') in $\overline{\mathcal{X}}$ with $a \leq a'$, $b \leq b'$, and $h \leq h'$. I choose $(C,L,D,S) \in \mathcal{A}(a,b,h,i)$ arbitrarily. I initially show $\mathcal{A}(a,b,h,i) \subseteq \mathcal{A}(a,b',h,i)$. Let $B := B^{a,b,h,i;C,L,D,S}$ and $B' := B^{a,b',h,i;C,L,D,S}$. Then, the Lipschitz property of r implies the unique existence of B'. Suppose that there exists a time $t_1 \in (0,\infty)$ such that $B'_{t_1} < B_{t_1}$, and let us lead to a contradiction. Then, by the continuity and $b' \geq b$, there exists a time $t_0 \in [0,t_1)$ such that $B'_{t_0} = B_{t_0}$ and $B'_s < B_s$ for any $s \in (t_0,t_1]$. By the liquid asset's ODE, for any $t_2 \in (t_0,t_1]$, I have

$$B'_{t_2} - B_{t_2} = \int_{t_0}^{t_2} \Big(r\Big(B'_s, Y_s^i\Big) - r\Big(B_s, Y_s^i\Big)\Big)\,\mathrm{d}s \geq L_r \int_{t_0}^{t_2} \left(B'_s - B_s\right)\mathrm{d}s.$$

Therefore, the Gronwall inequality yields $B'_{t_1} \geq B_{t_1}$, but this is a contradiction. Hence, $B'_t \geq B_t$ for all $t \in [0,\infty)$, and this implies $\mathcal{A}(a,b,h,i) \subseteq \mathcal{A}(a,b',h,i)$. Thus, $V_i(a,b,h) \leq V_i(a,b',h)$.

I will next show $V_i(a,b,h) \leq V_i(a',b,h)$. By Assumption 1, for any $j \in \mathcal{Y}$, there exists a unique minimizer of $d + \chi_A(d,j)$, denoted by $\underline{d}(j) < 0$, such that $\underline{d}(j) + \chi_A(\underline{d}(j),j) < 0$. Let us consider the ODE: $\mathrm{d}A'_t = ((r_A(Y^i_s) + \underline{d}(Y^i_t))A'_t + g_A(Y^i_t) - \pi_A(A'_t, Y^i_t))\mathrm{d}t$ with $A'_0 = a'$, which has a unique $\mathbb{F}$-adapted solution. Additionally, let θ_A be an $\mathbb{F}$-stopping time such that $\theta_A = \inf\{t \in [0,\infty) | A'_t = A^{a,i;D}_t\}$. Furthermore, let $D' = (D'_t)_{t\in[0,\infty)}$ be a deposit/withdrawal rate process such that

$$D'_t = \begin{cases} \underline{d}(Y^i_t) & \text{if } t \in [0,\theta_A), \\ D_t & \text{if } t \in [\theta_A, \infty). \end{cases}$$

By definition, D' is right-continuous and $\mathbb{F}$-progressively measurable. Then, we have $A^{a',i;D'}_t \geq A^{a,i;D}_t > 0$ $\mathbb{P}$-a.s. for all $t \in [0,\infty)$. This implies that

$$\begin{aligned} D_t A^{a,i;D}_t + \chi_A(D_t, Y^i_t) A^{a,i;D}_t &\geq \underline{d}(Y^i_t) A^{a,i;D}_t + \chi_A(\underline{d}(Y^i_t), Y^i_t) A^{a,i;D}_t \\ &\geq \underline{d}(Y^i_t) A^{a',i;D'}_t + \chi_A(\underline{d}(Y^i_t), Y^i_t) A^{a',i;D'}_t = D'_t A^{a',i;D'}_t + \chi_A(D'_t, Y^i_t) A^{a',i;D'}_t, \quad \text{if } t < \theta_A, \end{aligned}$$

and $D_t A^{a,i;D}_t + \chi_A(D_t, Y^i_t) A^{a,i;D}_t = D'_t A^{a',i;D'}_t + \chi_A(D'_t, Y^i_t) A^{a',i;D'}_t$ if $t \geq \theta_A$. Therefore, $D_t A^{a,i;D}_t + \chi_A(D_t, Y^i_t) A^{a,i;D}_t \geq D'_t A^{a',i;D'}_t + \chi_A(D'_t) A^{a',i;D'}_t$ for all $t \in [0,\infty)$. Similar to the case of the non-decreasing property of V_i with respect to b, we can show $B^{a',b,h,i;C,L,D',S}_t \geq B^{a,b,h,i;C,L,D,S}_t \geq -\underline{B}$. Therefore, all admissible (C,L) under (a,b,h,i) can be financed by (D',S), and (C,L,D',S) is admissible under (a',b,h,i). Furthermore, we can also show $u(C_t, L_t, A^{a',i;D'}_t, B^{a',b,h,i;C,L,D',S}_t, H^{h,i;S}_t, Y^i_t) \geq u(C_t, L_t, A^{a,i;D}_t, B^{a,b,h,i;C,L,D,S}_t, H^{h,i;S}_t, Y^i_t)$. Hence, $V_i(a,b,h) \leq V_i(a',b,h)$.

I will then show $V_i(a,b,h) \leq V_i(a,b,h')$. Let us consider the stochastic ODE: $\mathrm{d}H'_t = -\delta_H(Y_t) H'_t \mathrm{d}t$ with $H'_0 = h'$, which has a unique solution: $H'_t = h' e^{-\int_0^t \delta_H(Y^i_s)\mathrm{d}s}$. Additionally, let θ_H be an $\mathbb{F}$-stopping time such that $\theta_H = \inf\{t \in [0,\infty) | H'_t = H^{h,i;S}_t\}$. Furthermore, let $S' = (S'_t)_{t\in[0,\infty)}$ be an investment rate process such that

$$S'_t = \begin{cases} 0 & \text{if } t \in [0,\theta_H), \\ S_t & \text{if } t \in [\theta_H, \infty). \end{cases}$$

By definition, S' is right-continuous and $\mathbb{F}$-progressively measurable. Then, we have $S'_t H^{h',i;S'}_t \leq S_t H^{h,i;S}_t$ and $H^{h',i;S'}_t \geq H^{h,i;S}_t$ $\mathbb{P}$-a.s. for all $t \in [0,\infty)$. Hence, by the same argument in the case of A, I have $V_i(a,b,h) \le V_i(a,b,h')$.

The concavity of the value function. Third, I will show the concavity. Fix arbitrary $(\widehat{a},\widehat{b},\widehat{h})$ and $(\widetilde{a},\widetilde{b},\widetilde{h})$ in $\overline{\mathcal{X}}$, and take an arbitrary $k \in [0,1]$. I choose arbitrary $(\widehat{C},\widehat{L},\widehat{D},\widehat{S}) \in \mathcal{A}(\widehat{a},\widehat{b},\widehat{h},i)$ and $(\widetilde{C},\widetilde{L},\widetilde{D},\widetilde{S}) \in \mathcal{A}(\widetilde{a},\widetilde{b},\widetilde{h},i)$. Let us consider the following stochastic ODE with respect to A^k:

$$\mathrm{d}A_t^k = \Big(r_A(Y_t^i)A_t^k + k\widehat{D}_t A_t^{\widehat{a},i;\widehat{D}} + (1-k)\widetilde{D}_t A_t^{\widetilde{a},i;\widetilde{D}} + g_A(Y_t^i) - \pi_A(A_t^k, Y_t^i))\Big)\mathrm{d}t,$$

with $A_0^k = a^k := k\widehat{a} + (1-k)\widetilde{a}$. Then, by the global Lipschitz property of $a \to r_A(i)a - \pi_A(a,i)$, there exists a unique solution. Furthermore, since $a \to \pi(a,i)$ is convex, I have

$$\begin{aligned}
A_t^k - \Big(kA_t^{\widehat{a},i;\widehat{D}} + (1-k)A_t^{\widetilde{a},i;\widetilde{D}}\Big) &= \int_0^t \Big(r_A(Y_s^i)\Big(A_s^k - \Big(kA_s^{\widehat{a},i;\widehat{D}} + (1-k)A_s^{\widetilde{a},i;\widetilde{D}}\Big)\Big) \\
&\quad + k\pi_A(A_s^{\widehat{a},i;\widehat{D}}, Y_s^i) + (1-k)\pi_A(A_s^{\widetilde{a},i;\widetilde{D}}, Y_s^i) - \pi_A(A_s^k, Y_s^i)\Big)\mathrm{d}s \\
&\ge \int_0^t \Big(r_A(Y_s^i)\Big(A_s^k - \Big(kA_s^{\widehat{a},i;\widehat{D}} + (1-k)A_s^{\widetilde{a},i;\widetilde{D}}\Big)\Big) \\
&\qquad + \pi_A(kA_s^{\widehat{a},i;\widehat{D}} + (1-k)A_s^{\widetilde{a},i;\widetilde{D}}, Y_s^i) - \pi_A(A_s^k, Y_s^i)\Big)\mathrm{d}s,
\end{aligned}$$

for any $t \in [0,\infty)$. Thus, the non-decreasing property of π_A and the Gronwall argument assuming a contrary imply $A_t^k \ge kA_t^{\widehat{a},i;\widehat{D}} + (1-k)A_t^{\widetilde{a},i;\widetilde{D}} > 0$ for any $t \in [0,\infty)$. Let us define a process $D^k := (D_t^k)_{t\in[0,\infty)}$ and $S^k := (S_t^k)_{t\in[0,\infty)}$ such that

$$\begin{aligned}
D_t^k &:= \frac{kA_t^{\widehat{a},i;\widehat{D}}}{A_t^k}\widehat{D}_t + \frac{(1-k)A_t^{\widetilde{a},i;\widetilde{D}}}{A_t^k}\widetilde{D}_t, \\
S_t^k &:= \frac{\alpha_H^{-1}\Big(k\alpha_H(\widehat{S}_t H_t^{\widehat{h},i;\widehat{S}}, Y_t^i) + (1-k)\alpha_H(\widetilde{S}_t H_t^{\widetilde{h},i;\widetilde{S}}, Y_t^i); Y_t^i\Big)}{kH_t^{\widehat{h},i;\widehat{S}} + (1-k)H_t^{\widetilde{h},i;\widetilde{S}}},
\end{aligned}$$

for all $t \in [0,\infty)$. By the admissibility of $(\widehat{C},\widehat{L},\widehat{D},\widehat{S})$ and $(\widetilde{C},\widetilde{L},\widetilde{D},\widetilde{S})$, D^k and S^k are right-continuous and $\mathbb{F}$-progressively measurable. Let $H^k := kH^{\widehat{h},i;\widehat{S}} + (1-k)H^{\widetilde{h},i;\widetilde{S}} > 0$ and $h^k := k\widehat{h} + (1-k)\widetilde{h}$. Then, I have

$$\begin{aligned}
A_t^k &= a^k + \int_0^t \Big(\Big(r_A(Y_s^i) + D_s^k\Big)A_s^k + g_A(Y_s^i) - \pi_A(A_s^k, Y_s^i)\Big)\mathrm{d}s, \\
H_t^k &= h^k + \int_0^t \Big(\alpha_H\Big(S_s^k H_s^k, Y_s^i\Big) - \delta_H(Y_s^i)H_s^k\Big)\,\mathrm{d}s,
\end{aligned}$$

$\mathbb{P}$-a.s. for all $t \in [0, \infty)$. Hence, A^k and H^k are the illiquid asset process and the human capital process starting at $(A_0^k, H_0^k) = (a^k, h^k)$ and controlled by (D^k, S^k), respectively.

Let $x_t := (kA_t^{\widehat{a},i;\widehat{D}} + (1-k)A_t^{\widetilde{a},i;\widetilde{D}})/A_t^k \in (0,1]$. Then, by the convexity of χ_A and α_H^{-1}, and by the scaling property $\chi(cd, j) \leq c\chi(d, j)$ for any $(c, d, j) \in [0,1] \times \mathbb{R} \times \mathcal{Y}$, I have

$$
\begin{aligned}
(D_t^k + \chi_A(D_t^k, Y_t^i))A_t^k &\leq k\widehat{D}_t A_t^{\widehat{a},i;\widehat{D}} + (1-k)\widetilde{D}_t A_t^{\widetilde{a},i;\widetilde{D}} \\
&\quad + \Big(kA_t^{\widehat{a},i;\widehat{D}}\chi_A(x_t\widehat{D}_t, Y_t^i) + (1-k)A_t^{\widetilde{a},i;\widetilde{D}}\chi_A(x_t\widetilde{D}_t, Y_t^i)\Big)\frac{1}{x_t} \\
&\leq k(\widehat{D}_t + \chi_A(\widehat{D}_t, Y_t^i))A_t^{\widehat{a},i;\widehat{D}} + (1-k)(\widetilde{D}_t + \chi_A(\widetilde{D}_t, Y_t^i))A_t^{\widetilde{a},i;\widetilde{D}}, \quad \text{(A.1)} \\
S_t^k H_t^k &\leq k\widehat{S}_t H_t^{\widehat{h},i;\widehat{S}} + (1-k)\widetilde{S}_t H_t^{\widetilde{h},i;\widetilde{S}}, \quad \text{(A.2)}
\end{aligned}
$$

for any $t \in [0, \infty)$.

Let $C^k := k\widehat{C} + (1-k)\widetilde{C}$, $L^k := k\widehat{L} + (1-k)\widetilde{L}$, $b^k := k\widehat{b} + (1-k)\widetilde{b} \geq -\underline{B}$, $B^k := B^{a^k,b^k,h^k,i;C^k,L^k,D^k,S^k}$, $\widehat{B} := B^{\widehat{a},\widehat{b},\widehat{h},i;\widehat{C},\widehat{L},\widehat{D},\widehat{S}}$, and $\widetilde{B} := B^{\widetilde{a},\widetilde{b},\widetilde{h},i;\widetilde{C},\widetilde{L},\widetilde{D},\widetilde{S}}$. Here, suppose that there exists a time $t_1 \in [0, \infty)$ such that $B_{t_1}^k < k\widehat{B}_{t_1} + (1-k)\widetilde{B}_{t_1}$, and let us lead to a contradiction. Then, since Bs are continuous and $b^k = B_0^k = k\widehat{B}_0 + (1-k)\widetilde{B}_0$, there exists a time $t_0 \in [0, t_1)$ such that $B_{t_0}^k = k\widehat{B}_{t_0} + (1-k)\widetilde{B}_{t_0}$ and $B_s^k < k\widehat{B}_s + (1-k)\widetilde{B}_s$ for any $s \in (t_0, t_1]$. Then, since r is concave, f is jointly concave, and the inequalities (A.1) and (A.2) hold, for any $t_2 \in (t_0, t_1]$, I have

$$
\begin{aligned}
B_{t_2}^k - \Big(k\widehat{B}_{t_2} + (1-k)\widetilde{B}_{t_2}\Big) &\geq \int_{t_0}^{t_2} \Big(r\Big(B_s^k, Y_s^i\Big) - r\Big(k\widehat{B}_s + (1-k)\widetilde{B}_s, Y_s^i\Big)\Big)\,\mathrm{d}s \\
&\geq L_r \int_{t_0}^{t_2} \Big(B_s^k - \Big(k\widehat{B}_s + (1-k)\widetilde{B}_s\Big)\Big)\,\mathrm{d}s.
\end{aligned}
$$

Thus, the Gronwall inequality yields $B_{t_1}^k \geq k\widehat{B}_{t_1} + (1-k)\widetilde{B}_{t_1}$, but this is a contradiction. Therefore, $B_t^k \geq k\widehat{B}_t + (1-k)\widetilde{B}_t \geq -\underline{B}$ for any $t \in [0, \infty)$, and I can conclude that $(C^k, L^k, D^k, S^k) \in$

$\mathcal{A}(a^k, b^k, h^k, i)$. By the joint concavity and non-decreasing property of u, I have

$$\begin{aligned}
V_i(a^k, b^k, h^k) &\geq \mathrm{E}\left[\int_0^\infty e^{-\rho t} u(C_t^k, L_t^k, A_t^k, B_t^k, H_t^k, Y_t^i)\mathrm{d}t\right] \\
&\geq k\,\mathrm{E}\left[\int_0^\infty e^{-\rho t} u(\widehat{C}_t, \widehat{L}_t, A_t^{\widehat{a},i;\widehat{D}}, \widehat{B}_t, H_t^{\widehat{h},i;\widehat{S}}, Y_t^i)\mathrm{d}t\right] \\
&\qquad + (1-k)\,\mathrm{E}\left[\int_0^\infty e^{-\rho t} u(\widetilde{C}_t, \widetilde{L}_t, A_t^{\widetilde{a},i;\widetilde{D}}, \widetilde{B}_t, H_t^{\widetilde{h},i;\widetilde{S}}, Y_t^i)\mathrm{d}t\right].
\end{aligned}$$

Since I have chosen $(\widehat{C}, \widehat{L}, \widehat{D}, \widehat{S})$ and $(\widetilde{C}, \widetilde{L}, \widetilde{D}, \widetilde{S})$ arbitrarily, the inequality $V_i(a^k, b^k, h^k) \geq kV_i(\widehat{a}, \widehat{b}, \widehat{h}) + (1-k)V_i(\widetilde{a}, \widetilde{b}, \widetilde{h})$ holds. Thus, V_i is jointly concave on $\overline{\mathcal{X}}$.

<u>*The other claims.*</u> The joint concavity of V_i on $\overline{\mathcal{X}}$ implies the local Lipschitz continuity of V_i on $\mathcal{X}$ (Theorem 10.4 in Rockafellar (1970)). Therefore, V_i is continuous in the interior of its domain, and $(a, h) \to V_i(a, -\underline{B}, h)$ is continuous on $(0, \infty)^2$ since $(a, h) \to V_i(a, -\underline{B}, h)$ is concave on $(0, \infty)^2$. Finally, I check the lower boundary of V_i. For any $(a, b, h) \in \overline{\mathcal{X}}$, $((g(Y^i) - \underline{y})/\tau_c(Y^i), 0, 0, 0)$ is in $\mathcal{A}(a, b, h, i)$. Thus,

$$\begin{aligned}
&V_i(a, b, h) \\
&\geq \mathrm{E}\left[\int_0^\infty e^{-\rho t} u((g(Y_t^i) - \underline{y})/\tau_c(Y_t^i), 0, A_t^{a,i;0}, B_t^{a,b,h,i;(g(Y^i)-\underline{y})/\tau_c(Y^i),0,0,0}, H_t^{h,i;0}, Y_t^i)\mathrm{d}t\right] \\
&\qquad \geq \int_0^\infty e^{-\rho t} \min_{j\in\mathcal{Y}}\{\underline{u}((\underline{g} - \underline{y})/\overline{\tau}_c, 0, j)\}\mathrm{d}t = \frac{\min_{j\in\mathcal{Y}}\{\underline{u}((\underline{g} - \underline{y})/\overline{\tau}_c, 0, j)\}}{\rho}.
\end{aligned}$$

□

A.2 Proof of Proposition 5

Proof of Proposition 5. Fix an arbitrary $i \in \mathcal{Y}$. Let us show that $V_i(a, b, h) \to V_i(a, -\underline{B}, h)$ as $b \downarrow -\underline{B}$ for any fixed $(a, h) \in (0, \infty)^2$. By the non-decreasing property of V_i with respect to b, I have $V_i(a, -\underline{B}, h) \leq \lim_{b\downarrow -\underline{B}} V_i(a, b, h)$ for any fixed $(a, h) \in (0, \infty)^2$. To show the opposite inequality, I use the first DPP (2.5). Fix any $(a, h) \in (0, \infty)^2$ and let $T \in (0, \infty)$ be an arbitrary constant. Then, $((g(Y^i) - \underline{y})/\tau_c(Y^i), 0, 0, 0) \in \mathcal{A}(ae^{-(0\wedge \underline{r}_A)T}, -\underline{B}, he^{\overline{\delta}_H T}, i)$. Furthermore, $A_T^{ae^{-(0\wedge \underline{r}_A)T},i;0} \geq a$ and $H_T^{he^{\overline{\delta}_H T},i;0} \geq h$ hold. For B, I write $\widetilde{B}_t = B_t^{ae^{-(0\wedge \underline{r}_A)T},-\underline{B},he^{\delta_H T},i;(g(Y^i)-\underline{y})/\tau_c(Y^i),0,0,0}$ for notational simplicity. Then, it can be easily seen that $\lim_{T\to 0} \widetilde{B}_T = -\underline{B}$ $\mathbb{P}$-a.s. Let us consider the deterministic ODE $\mathrm{d}\widehat{B}_t = (r(\widehat{B}_t, \underline{i}) + \underline{y})\mathrm{d}t$ with $\widehat{B}_0 = -\underline{B}$. It can be easily seen that $-\underline{B} \le \widehat{B}_t \le \widetilde{B}_t$ for any $t \in [0, \infty)$. Furthermore, I have $\widehat{B}_t > -\underline{B}$ on some interval $(0, t_0]$ by the strict inequality $r(-\underline{B}, \underline{i}) + \underline{y} > 0$ and the continuity of $b \to r(b, \underline{i})$. Therefore, by the first DPP (2.5), I have

$$\begin{aligned} V_i(ae^{-(0\wedge \underline{r}_A)T}, -\underline{B}, he^{\overline{\delta}_H T}) &\ge \mathrm{E}\left[\int_0^T e^{-\rho t}\Big(u(\frac{g(Y_t^i) - \underline{y}}{\tau_c(Y_t^i)}, 0, A_t^{ae^{-(0\wedge \underline{r}_A)T}, i; 0}, \widetilde{B}_t, H_t^{he^{\overline{\delta}_H T}; 0}, Y_t^i)\Big)\mathrm{d}t \right. \\ &\qquad \left. + e^{-\rho T} V_{Y_T^i}(A_T^{ae^{-(0\wedge \underline{r}_A)T}, i; 0}, \widetilde{B}_T, H_T^{he^{\overline{\delta}_H T}, i; 0})\right] \\ &\ge \int_0^T e^{-\rho t} \min_{j\in\mathcal{Y}}\{\underline{u}(\frac{\underline{g} - \underline{y}}{\overline{\tau}_c}, 0, j)\}\mathrm{d}t + \mathrm{E}\left[e^{-\rho T} V_{Y_T^i}(a, \widetilde{B}_T, h)\right] \\ &\ge \frac{1 - e^{-\rho T}}{\rho} \min_{j\in\mathcal{Y}}\{\underline{u}(\frac{\underline{g} - \underline{y}}{\overline{\tau}_c}, 0, j)\} + \mathrm{E}\left[V_{Y_T^i}(a, \widehat{B}_T, h)\right], \end{aligned}$$

where I have used the non-decreasing property of V with respect to a, b, and h. By the definition of the Markov chain, I have

$$\begin{aligned} V_i(ae^{-(0\wedge \underline{r}_D)T}, -\underline{B}, he^{\overline{\delta}_H T}) &\ge \frac{1 - e^{-\rho T}}{\rho} \min_{j\in\mathcal{Y}}\{\underline{u}(\frac{\underline{g} - \underline{y}}{\overline{\tau}_c}, 0, j)\}\} \\ &\quad + \sum_{j\in\mathcal{Y}\setminus\{i\}} \mathbb{P}(Y_T^i = j)\Big(V_j(a, \widehat{B}_T, h) - V_i(a, \widehat{B}_T, h)\Big) + V_i(a, \widehat{B}_T, h). \end{aligned} \tag{A.3}$$

Since V is bounded on $\overline{\mathcal{X}} \times \mathcal{Y}$, I have $\lim_{T\to 0} \sum_{j\in\mathcal{Y}\setminus\{i\}} \mathbb{P}(Y_T^i = j)\Big(V_j(a, \widehat{B}_T, h) - V_i(a, \widehat{B}_T, h)\Big) = 0$. Therefore, I finally obtain

$$V_i(a, -\underline{B}, h) = \lim_{T\to 0} V_i(ae^{-(0\wedge \underline{r}_A)T}, -\underline{B}, he^{\overline{\delta}_H T}) \ge \lim_{T\to 0} V_i(a, \widehat{B}_T, h) = \lim_{b\downarrow -\underline{B}} V_i(a, b, h),$$

where I have used the continuity of $(a, h) \to V_i(a, -\underline{B}, h)$ in the left equality. Hence, $V_i(a, b, h) \to V_i(a, -\underline{B}, h)$ as $b \downarrow -\underline{B}$ holds for any fixed $(a, h) \in (0, \infty)^2$.

Since $V_i(a, b, h)$ is continuous in each argument on $\overline{\mathcal{X}}$ when the other arguments are fixed, and since it is non-decreasing in each argument, we can conclude that $V_i(a, b, h)$ is jointly continuous on $\overline{\mathcal{X}}$ (see Kruse and Deely (1969)). $\square$

A.3 Proof of Proposition 7

Proof of Proposition 7. For simplicity of notations, I write a triplet of the three asset processes as $X = (A, B, H)$ and introduce the infinitesimal generator $\mathcal{L}_X$ at $x = (a, b, h) \in \overline{\mathcal{X}}$ such that

$$\mathcal{L}_X^{c,l,d,s}\varphi_i(x) = \partial_b\varphi_i(x)\Big(r(b,i) + f(l,h,i) + g(i) - \tau_c(i)c - (d + \chi_A(d,i))a - \beta_H(i)sh\Big) \\ + \partial_a\varphi_i(x)\Big((r_A(i) + d)a + g_A(i) - \pi_A(a,i)\Big) + \partial_h\varphi_i(x)\Big(\alpha_H(sh,i) - \delta_H(i)h\Big),$$

for any smooth function φ on $\overline{\mathcal{X}} \times \mathcal{Y}$. Furthermore, let ∇_x be a gradient operator with respect to x such that $\nabla_x\varphi_i(x) = (\partial_a\varphi_i(x), \partial_b\varphi_i(x), \partial_h\varphi_i(x))$. Throughout this proof, for any constant $x \in \overline{\mathcal{X}}$ and $\eta > 0$, let $\mathcal{B}(x,\eta) := \{x' \in \overline{\mathcal{X}} | \|x' - x\| < \eta\}$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^3$. The following proof is standard for the viscosity solution property of the value function in a stochastic environment, as in Gassiat et al. (2014).

<u>*The viscosity supersolution property of the value function.*</u> Fix an arbitrary $x = (a, b, h) \in \mathcal{X}$. Let $\varphi = (\varphi_i)_{i\in\mathcal{Y}} \in C^1_K(\mathcal{X})$ be a smooth test function such that

$$0 = V_i(x) - \varphi_i(x) = \min_{x' \in \mathcal{O}}\{V_i(x') - \varphi_i(x')\},$$

for all $i \in \mathcal{Y}$, where $\mathcal{O}$ is an open subset of $\mathcal{X}$ with $x \in \mathcal{O}$. Note that the non-decreasing property of V_i with respect to b and h implies $\partial_b\varphi_i(x) \geq 0$ and $\partial_h\varphi_i(x) \geq 0$.

Fix an arbitrary $i \in \mathcal{Y}$. I also choose a quadruplet of constants $(c, l, d, s) \in \mathcal{C} \times [0, \overline{L}] \times \mathbb{R} \times [0, \infty)$ arbitrarily. Consider the system of stochastic ODEs as follows:

$$\begin{cases} \mathrm{d}\widehat{B}_t = \Big(r(\widehat{B}_t, Y^i_t) + f(l, \widehat{H}_t, Y^i_t) + g(Y^i_t) - \tau_c(Y^i_t)c - \Big(d + \chi_A(d, Y^i_t)\Big)\widehat{A}_t - \beta_H(Y^i_t)s\widehat{H}_t\Big)\mathrm{d}t, \\ \mathrm{d}\widehat{A}_t = \Big((r_A(Y^i_t) + d)\widehat{A}_t + g_A(Y^i_t) - \pi_A(\widehat{A}_t, Y^i_t)\Big)\mathrm{d}t, \\ \mathrm{d}\widehat{H}_t = \Big(\alpha_H(s\widehat{H}_t, Y^i_t) - \delta_H(Y^i_t)\widehat{H}_t\Big)\mathrm{d}t, \quad \widehat{A}_0 = a, \quad \widehat{B}_0 = b, \quad \widehat{H}_0 = h. \end{cases}$$

It is obvious that the above system has a unique $\mathbb{F}$-adapted solution, and $\widehat{B}_t > -\underline{B}$ holds on some short interval $[0, t']$ by $b > -\underline{B}$. Additionally, let θ be an $\mathbb{F}$-stopping time such that

$\theta := \inf\{t \in [0,\infty) | \widehat{B}_t = -\underline{B} \text{ or } \widehat{A}_t = a/2\}$. Here, I can construct a new control as follows:

$$(C_t, L_t, D_t, S_t) = \begin{cases} (c, l, d, s) & \text{if } t \in [0, \theta), \\ ((\underline{g} - \underline{y})/\overline{\tau}_c, 0, 0, 0) & \text{if } t \in [\theta, \infty). \end{cases}$$

Then, $B_t^{a,b,h,i;C,L,D,S} \geq -\underline{B}$ for all $t \in [0,\infty)$ by the Gronwall argument. Furthermore, the right-continuity and measurability of (C, L, D, S) are immediate, so that it is admissible under (a, b, h, i). Note that $\theta > 0$ $\mathbb{P}$-a.s. by $b > -\underline{B}$ and $a > a/2$.

Fix a sufficiently small $\eta > 0$ such that $b - \eta > -\underline{B}$, $a - \eta > a/2$, and $\mathcal{B}(x, \eta) \subseteq \mathcal{O}$. Let τ be an $\mathbb{F}$-stopping time such that $\tau := \inf\{t \in [0,\infty) | \widetilde{X}_t \notin \mathcal{B}(x, \eta)\}$, where $\widetilde{X} = (A^{a,i;D}, B^{a,b,h,i;C,L,D,S}, H^{h,i;S})$. Additionally, let $(t_n)_{n\geq 1}$ be a sequence of strictly decreasing and positive real numbers such that $t_n \downarrow 0$ as $n \to \infty$. Finally, let $(\xi_n)_{n\geq 1}$ be a sequence of $\mathbb{F}$-stopping times such that $\xi_n = \tau \wedge t_n$ for all $n \geq 1$.

By the first DPP (2.5), I have

$$V_i(x) \geq \mathrm{E}\left[\int_0^{\xi_n} e^{-\rho t} u(C_t, L_t, \widetilde{X}_t, Y_t^i) \mathrm{d}t + e^{-\rho \xi_n} V_{Y^i_{\xi_n}}(\widetilde{X}_{\xi_n})\right].$$

Since $\varphi_j \leq V_j$ for all $j \in \mathcal{Y}$ and since $\varphi_i(x) = V_i(x)$, I obtain

$$\varphi_i(x) \geq \mathrm{E}\left[\int_0^{\xi_n} e^{-\rho t} u(C_t, L_t, \widetilde{X}_t, Y_t^i) \mathrm{d}t + e^{-\rho \xi_n} \varphi_{Y^i_{\xi_n}}(\widetilde{X}_{\xi_n})\right].$$

Applying the generalized Ito formula to $e^{-\rho \cdot} \varphi_{Y^i_\cdot}(\widetilde{X}_\cdot)$ from 0 to ξ_n, I have

$$0 \geq \mathrm{E}\Big[\int_0^{\xi_n} e^{-\rho t}\Big\{u(C_t, L_t, \widetilde{X}_t, Y_t^i) - \rho \varphi_{Y_t^i}(\widetilde{X}_t) + \mathcal{L}_X^{C_t, L_t, D_t, S_t} \varphi_{Y_t^i}(\widetilde{X}_t) + \sum_{j \in \mathcal{Y} \setminus \{Y^i_{t-}\}} \lambda_{Y^i_{t-}, j}\Big(\varphi_j(\widetilde{X}_t) - \varphi_{Y^i_{t-}}(\widetilde{X}_t)\Big)\Big\}\mathrm{d}t\Big],$$

where I have used the fact that φ is bounded on $\mathcal{B}(x, \eta)$ by its continuity. Hence, the stochastic integral minus its compensator in the Ito formula is a true martingale, so that I can apply the optional stopping theorem. Furthermore, the integrand in the above expectation is bounded on $[0, \xi_n)$. Recall that (Y, C, L, D, S) is right-continuous. Thus, on a short interval depending on $\omega \in \Omega$, the integrand is right-continuous $\mathbb{P}$ almost surely. Furthermore, the continuity of

$\widetilde{X}$ implies that, for almost sure $\omega \in \Omega$ and sufficiently large $m \geq N(\omega)$, $\xi_m(\omega) = t_m$ holds. Therefore, by dividing the above expectation by t_n and letting n go to infinity, the bounded convergence theorem and the mean value theorem yield

$$0 \geq u(c,l,x,i) - \rho V_i(x) + \mathcal{L}_X^{c,l,d,s}\varphi_i(x) + \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\Big(V_j(x) - V_i(x)\Big),$$

Since $(x,i) = (a,b,h,i)$ and (c,l,d,s) are chosen arbitrarily, the above inequality implies the viscosity supersolution property of the value function on $\mathcal{X}$.

The viscosity subsolution property of the value function. Fix an arbitrary $x = (a,b,h) \in \overline{\mathcal{X}}$. Let $\overline{\mathcal{O}}$ be a subset of $\overline{\mathcal{X}}$ with $x \in \overline{\mathcal{O}}$. Here, suppose that the closed boundary of $\overline{\mathcal{O}}$, denoted by $\partial\overline{\mathcal{O}}(\subset \overline{\mathcal{O}})$, can only include the lower boundary of b: $\partial\overline{\mathcal{O}} \subset \partial\mathcal{X}$. Let $\varphi = (\varphi_i)_{i\in\mathcal{Y}} \in C_K^1(\overline{\mathcal{X}})$ be a smooth test function such that

$$0 = V_i(x) - \varphi_i(x) = \max_{x'\in\overline{\mathcal{O}}}\{V_i(x') - \varphi_i(x')\},$$

for all $i \in \mathcal{Y}$. Fix an arbitrary $i \in \mathcal{Y}$. Here, I hypothesize that

$$\rho\varphi_i(x) - \mathcal{H}_i(x, \nabla_x\varphi_i(x)) - \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\Big(\varphi_j(x) - \varphi_i(x)\Big) > 0,$$

and $\partial_b\varphi_i(x) > 0$ and $\partial_b\varphi_i(x) > 0$. If the hypothesis is false, then this directly implies the viscosity subsolution property, so that I derive a contradiction to the hypothesis. By the continuity of the Hamiltonian $\mathcal{H}_i$ on $\overline{\mathcal{X}} \times \mathbb{R} \times (0,\infty)^2$, for any small $\epsilon > 0$, there exists a constant $\eta > 0$ such that

$$\rho\varphi_k(x') - \mathcal{H}_k(x', \nabla_x\varphi_k(x')) - \sum_{j\in\mathcal{Y}\setminus\{k\}} \lambda_{k,j}\Big(\varphi_j(x') - \varphi_k(x')\Big) \geq \epsilon,$$

and $\partial_b\varphi_k(x') \geq \epsilon$ and $\partial_h\varphi_k(x') \geq \epsilon$ for all $x' \in \mathcal{B}(x,\eta) \subseteq \overline{\mathcal{O}}$ and $k \in \mathcal{Y}^N$, where $\mathcal{Y}^N$ is a subset of $\mathcal{Y}$ with $i \in \mathcal{Y}^N$. As in Gassiat et al. (2014), I suppose, without loss of generality, that there exists a constant $\delta \geq \epsilon/\rho$ such that $V_k(x') - \varphi_k(x') < -\delta$ for any $x' \notin \mathcal{B}(x,\eta)$ and $k \in \mathcal{Y}$.

By the second DPP (2.6), there exists $(C,L,D,S) \in \mathcal{A}(a,b,h,i)$ such that

$$V_i(x) - \epsilon\frac{1-e^{-\rho}}{2\rho} \leq \mathrm{E}\left[\int_0^{\theta\wedge 1} e^{-\rho t}u(C_t, L_t, \widehat{X}_t, Y_t^i)\mathrm{d}t + e^{-\rho(\theta\wedge 1)}V_{Y^i_{\theta\wedge 1}}(\widehat{X}_{\theta\wedge 1})\right],$$

where $\widehat{X} = (A^{a,i;D}, B^{a,b,h,i;C,L,D,S}, H^{h,i;S})$, and θ is an $\mathbb{F}$-stopping time such that $\theta := \inf\{t \in [0,\infty) | \widehat{X}_t \notin \mathcal{B}(x,\eta) \text{ or } Y^i_t \notin \mathcal{Y}^N\}$. Since $\varphi_k(x') - \delta > V_k(x')$ for $x' \notin \mathcal{B}(x,\eta)$ and $k \in \mathcal{Y}$ and since $\varphi_i(x) = V_i(x)$, I have

$$\varphi_i(x) - \epsilon \frac{1-e^{-\rho}}{2\rho} \leq \mathrm{E}\left[\int_0^{\theta\wedge 1} e^{-\rho t} u(C_t, L_t, \widehat{X}_t, Y^i_t)\mathrm{d}t + e^{-\rho(\theta\wedge 1)}\varphi_{Y^i_{\theta\wedge 1}}(\widehat{X}_{\theta\wedge 1})\right] - \delta\,\mathrm{E}\left[e^{-\rho(\theta\wedge 1)}\mathbb{1}\{\theta \leq 1\}\right].$$

Applying the generalized Ito formula to $e^{-\rho\cdot}\varphi_{Y^i_\cdot}(\widehat{X}_\cdot)$ from 0 to $\theta\wedge 1$, I have

$$\begin{aligned} -\epsilon\frac{1-e^{-\rho}}{2\rho} \leq \mathrm{E}\Big[\int_0^{\theta\wedge 1} & e^{-\rho t}\Big\{u(C_t, L_t, \widehat{X}_t, Y^i_t) - \rho\varphi_{Y^i_t}(\widehat{X}_t) \\ & + \mathcal{L}_X^{C_t,L_t,D_t,S_t}\varphi_{Y^i_t}(\widehat{X}_t) + \sum_{j\in\mathcal{Y}\backslash\{Y^i_{t-}\}} \lambda_{Y^i_{t-},j}\Big(\varphi_j(\widehat{X}_t) - \varphi_{Y^i_{t-}}(\widehat{X}_t)\Big)\Big\}\mathrm{d}t\Big] \\ & - \delta\,\mathrm{E}\left[e^{-\rho(\theta\wedge 1)}\mathbb{1}\{\theta \leq 1\}\right], \end{aligned}$$

where I have used the fact that φ is bounded on $\mathcal{B}(x,\eta)$, and then the stochastic integral minus its compensator in the Ito formula is a true martingale, so that I can apply the optional stopping theorem. By the definition of the Hamiltonian, I have

$$\begin{aligned} 0 \geq -\epsilon\frac{1-e^{-\rho}}{2\rho} + \mathrm{E}\Big[\int_0^{\theta\wedge 1} & e^{-\rho t}\Big\{\rho\varphi_{Y^i_t}(\widehat{X}_t) - \mathcal{H}_{Y^i_t}(\widehat{X}_t, \nabla_x\varphi_{Y^i_t}(\widehat{X}_t)) \\ & - \sum_{j\in\mathcal{Y}\backslash\{Y^i_{t-}\}} \lambda_{Y^i_{t-},j}\Big(\varphi_j(\widehat{X}_t) - \varphi_{Y^i_{t-}}(\widehat{X}_t)\Big)\Big\}\mathrm{d}t\Big] + \delta\,\mathrm{E}\left[e^{-\rho(\theta\wedge 1)}\mathbb{1}\{\theta \leq 1\}\right]. \end{aligned} \tag{A.4}$$

Since $\widehat{X} \in \mathcal{B}(x,\eta)$ and $Y^i_t \in \mathcal{Y}^N$ on $[0, \theta\wedge 1)$, and since $\{Y^i_t \neq Y^i_{t-}\}$ has Lebesgue measure zero on $[0,\infty)$ $\mathbb{P}$-a.s., I have

$$\rho\varphi_{Y^i_t}(\widehat{X}_t) - \mathcal{H}_{Y^i_t}(\widehat{X}_t, \nabla_x\varphi_{Y^i_t}(\widehat{X}_t)) - \sum_{j\in\mathcal{Y}\backslash\{Y^i_{t-}\}} \lambda_{Y^i_{t-},j}\Big(\varphi_j(\widehat{X}_t) - \varphi_{Y^i_{t-}}(\widehat{X}_t)\Big) \geq \epsilon,$$

on $[0,\theta\wedge 1)$ $\mathrm{d}\mathbb{P}\times \mathrm{d}t$-a.e. Thus, (A.4) implies that

$$
\begin{aligned}
0 &\geq -\epsilon\frac{1-e^{-\rho}}{2\rho} + \mathrm{E}\left[\epsilon\frac{1-e^{-\rho(\theta\wedge 1)}}{\rho} + \delta e^{-\rho(\theta\wedge 1)}\mathbb{1}\{\theta\leq 1\}\right]\\
&\geq \frac{\epsilon}{2\rho}(1+e^{-\rho}) - \frac{\epsilon}{\rho}\mathrm{E}\left[\mathbb{1}\{\theta>1\}e^{-\rho(\theta\wedge 1)}\right] = \frac{\epsilon}{2\rho}(1+e^{-\rho}) - \frac{\epsilon}{\rho}\mathbb{P}(\theta>1)e^{-\rho}\\
&\qquad\qquad\qquad\qquad \geq \frac{\epsilon}{2\rho}(1+e^{-\rho}) - \frac{\epsilon}{\rho}e^{-\rho} = \frac{\epsilon}{2\rho}(1-e^{-\rho}) > 0,
\end{aligned}
$$

where I have used the inequality $\epsilon\leq\delta\rho$. However, this is obviously a contradiction. Therefore, I obtain the viscosity subsolution property of the value function on $\overline{\mathcal{X}}$. □

A.4 Proof of Lemma 9

Proof of Lemma 9.

The strictly increasing property of the value function in b. It is obvious that $V_i(a,b,h)\leq 0$ for any $(a,b,h,i)\in\overline{\mathcal{X}}\times\mathcal{Y}$. Fix $(a,b,h,i)\in\overline{\mathcal{X}}\times\mathcal{Y}$. Here, I hypothesize $V_i(a,b,h)=V_i(a,b',h)$ for some $b'>b$, and let us lead to a contradiction. Then, since V_i is non-decreasing in b, $V_i(a,b,h)=V_i(a,\widetilde{b},h)$ for any $\widetilde{b}\in[b,b']$. This implies that V_i is partially differentiable with respect to b on (b,b'). The partial derivative is $\partial_b V_i(a,\widetilde{b},h)=0$ for any $\widetilde{b}\in(b,b')$. Then, since V_i is concave and non-decreasing, $\partial_b V_i(a,\widetilde{b},h)=0$ implies that $\partial_b^+ V_i(a,\widehat{b},h)=\partial_b^- V_i(a,\widehat{b},h)=0$ for any $\widehat{b}>b$, where $\partial_b^+ V_i$ and $\partial_b^- V_i$ are the right and left partial derivative of V_i with respect to b. Therefore, $V_i(a,b',h)=V_i(a,b,h)$ holds for any $b'\geq b$.

I will show that $V_i(a,b,h)=0$ under the hypothesis. For any $(b',i)\in[0,\infty)\times\mathcal{Y}$, since $r(0,i)\geq 0$ and r is globally Lipschitz, I have $r(b',i)\geq r(b',i)-r(0,i)\geq -L_r b'$, where $L_r>0$ is a Lipschitz constant. Let $m>L_r$ be a finite constant. Then, for any $(t,b',i)\in[0,\infty)^2\times\mathcal{Y}$, I have $r(e^{-mt}b',i)+me^{-mt}b'\geq 0$ and $r(e^{-mt}b',i)+me^{-mt}b'\to\infty$ as $b'\to\infty$. Here, I choose $(C,L,D,S)=((g(Y^i)+r(e^{-mt}b',Y^i)+me^{-mt}b')/\tau_c(Y^i),0,0,0)$ for any $b'\geq b\vee 0$. Then, $C_t=(g(Y_t^i)+r(e^{-mt}b',Y_t^i)+me^{-mt}b')/\tau_c(Y_t^i)\in\mathcal{C}$. Meanwhile, by the Gronwall inequality, we can easily see that $B_t^{a,b',h,i;(g(Y^i)+r(e^{-mt}b',Y^i)+me^{-t}b')/\tau_c(Y^i),0,0,0}-e^{-mt}b'\geq 0$ for any $t\in[0,\infty)$, and this implies $((g(Y^i)+r(e^{-mt}b',Y^i)+me^{-mt}b')/\tau_c(Y^i),0,0,0)\in\mathcal{A}(a,b',h,i)$. Therefore, the definition of the value function yields

$$
\begin{aligned}
V_i(a,b,h) &= V_i(a,b',h) \\
&\geq \mathrm{E}\left[\int_0^\infty e^{-\rho t} u\left(\frac{g(Y_t^i)+r(e^{-mt}b',Y_t^i)+me^{-mt}b'}{\tau_c(Y_t^i)}, 0, A_t^{a,i;0}, e^{-mt}b', H_t^{h,i;0}, Y_t^i\right) \mathrm{d}t\right].
\end{aligned}
$$

Hence, the bounded convergence theorem yields $V_i(a,b,h) = \lim_{b'\to\infty} V_i(a,b',h) \geq 0$. Thus, I have $V_i(a,b,h) = 0$.

Here, I derive a contradiction from $V_i(a,b,h) = 0$. I have

$$
0 = V_i(a,b,h) = \sup_{(C,L,D,S)\in\mathcal{A}(a,b,h,i)} \mathrm{E}\left[\int_0^\infty e^{-\rho t} u(C_t, L_t, A_t^{a,i:D}, B_t^{a,b,h,i;C,L,D,S}, H_t^{h,i;S}, Y_t^i)\mathrm{d}t\right].
$$

Thus, there exists a sequence $(C^n, L^n, D^n, S^n)_{n\geq 1} \subseteq \mathcal{A}(a,b,h,i)$ such that

$$
0 = \limsup_{n\to\infty} \mathrm{E}\left[\int_0^\infty e^{-\rho t}\Big(u(C_t^n, L_t^n, A_t^{a,i;D^n}, B_t^{a,b,h,i;C^n,L^n,D^n,S^n}, H_t^{h,i;S^n}, Y_t^i)\Big)\mathrm{d}t\right].
$$

The above equality implies that $C_t^n \to \infty$ as $n\to\infty$ $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-a.e. However, such plans are not admissible because B_t must become smaller than $-\underline{B}$ $\mathbb{P}$-a.s. since an admissible L_t is bounded and $d+\chi_A(d)\to\infty$ as $d\to-\infty$. In the case that A or H tends to go to infinity $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-a.e., we need that D^n or S^n tends to go to infinity, but it is not admissible as well. Therefore, the hypothesis is false, and $V_i(a,b,h)$ is strictly increasing in b.

The strictly increasing property of the value function in h. Next, I will show that $h\to V_i(a,b,h)$ is strictly increasing on $(0,\infty)$ for any $(a,b,i)\in(0,\infty)\times[-\underline{B},\infty)\times\mathcal{Y}$. I hypothesize $V_i(a,b,h) = V_i(a,b,h')$ for some $(a,b,h)\in\overline{\mathcal{X}}$ and $h'\in(0,\infty)$ with $h'>h$, and lead to a contradiction. By the same argument as in the case of $b\to V_i(a,b,h)$, this hypothesis implies $V_i(a,b,h') = V_i(a,b,h)$ for any $h'\geq h$. Fix an arbitrary constant $l_\epsilon\in(0,\overline{L})$. For any $h'\geq h$, let $(C,L,D,S) = ((g(Y^i)+f(l_\epsilon, H^{h',i;0}, Y^i)/2-\underline{y})/\tau_c(Y^i), l_\epsilon, 0, 0)$. Then, it can be easily seen that $B_t^{a,b,h',i;C,L,D,S}\geq -\underline{B}$ and $B_t^{a,b,h',i;C,L,D,S}\to\infty$ as $h'\to\infty$ for any $t\in[0,\infty)$. Thus, $(C,L,D,S)\in\mathcal{A}(a,b,h',i)$.

Furthermore, $H_t^{h',i;0} \geq h' e^{-\overline{\delta}_H t}$. Therefore, by the definition of the value function, I have

$$
\begin{aligned}
&V_i(a,b,h) = V_i(a,b,h') \\
&\geq \mathrm{E}\left[\int_0^\infty e^{-\rho t} u\left(\frac{g(Y_t^i) + 0.5 f(l_\epsilon, h' e^{-\overline{\delta}_H t}, Y_t^i) - \underline{y}}{\tau_c(Y_t^i)}, l_\epsilon, A_t^{a,i;0}, B_t^{a,b,h',i;C,L,D,S}, h' e^{-\overline{\delta}_H t}, Y_t^i\right) \mathrm{d}t\right],
\end{aligned}
$$

for any $h' \geq h$. Then, the bounded convergence theorem and monotone convergence theorem yield

$$
\begin{aligned}
&V_i(a,b,h) \\
&\geq \lim_{h' \to \infty} \mathrm{E}\left[\int_0^\infty e^{-\rho t} u\left(\frac{g(Y_t^i) + 0.5 f(l_\epsilon, h' e^{-\overline{\delta}_H t}, Y_t^i) - \underline{y}}{\tau_c(Y_t^i)}, l_\epsilon, A_t^{a,i;0}, B_t^{a,b,h',i;C,L,D,S}, h' e^{-\overline{\delta}_H t}, Y_t^i\right) \mathrm{d}t\right] \\
&\qquad \geq \mathrm{E}\left[\int_0^\infty e^{-\rho t} \overline{u}^\infty(l_\epsilon) \mathrm{d}t\right] \geq \frac{\overline{u}^\infty(l_\epsilon)}{\rho},
\end{aligned}
$$

where I have used $f(l_\epsilon, h', j) \to \infty$ as $h' \to \infty$ for any $j \in \mathcal{Y}$ and $l_\epsilon \in (0, \overline{L}]$. Since $\overline{u}^\infty(l_\epsilon) \to 0$ as $l_\epsilon \to l^*$, the above inequality implies $V_i(a,b,h) \geq 0$, so that $V_i(a,b,h) = 0$ holds. However, this is a contradiction as observed in the case of $b \to V_i(a,b,h)$. Thus, $V_i(a,b,h)$ is strictly increasing in h for any $i \in \mathcal{Y}$. □

A.5 Proof of Proposition 10

Proof of Proposition 10. In this proof, I employ the same notations as those in the proof of Proposition 7 for simplicity.

Step 1. Fix $(x,i) = (a,b,h,i) \in \mathcal{X} \times \mathcal{Y}$. Since V_i is jointly concave, it satisfies the following supergradient inequality:

$$V_i(x') - V_i(x) \leq p \cdot (x' - x),$$

for any $x' \in \mathcal{X}$ and $p = (p_a, p_b, p_h) \in \partial V_i(x)$, where $\partial V_i(x)$ is a closed, bounded, and convex subset of $\mathbb{R}^3$ (Theorem 23.4 in Rockafellar (1970), which is a version for a convex function). $\partial V_i(x)$ is typically referred to as the superdifferential of V_i at x.

Here, let us construct the continuous envelope functions of the partial derivatives of V_i. Since the region of undifferentiable points of V_i has Lebesgue measure zero by its concavity (Theorem 25.5 in Rockafellar (1970)), I can arbitrarily choose two sequences $(x_n)_{n\geq 1} = (a_n, b_n, h_n)_{n\geq 1}$ and $(x'_n)_{n\geq 1} = (a'_n, b'_n, h'_n)_{n\geq 1}$ on $\mathcal{X}$ such that they converge to $x \in \mathcal{X}$, and V_i is differentiable at x_n and x'_n for all $n \geq 1$. Furthermore, without loss of generality, I can suppose that there exists $(\underline{a}, \underline{b}, \underline{h}), (\overline{a}, \overline{b}, \overline{h}) \in \mathcal{X}$ with $\underline{a} < \overline{a}$, $\underline{b} < \overline{b}$, and $\underline{h} < \overline{h}$ such that $a_n, a'_n \in (\underline{a}, \overline{a})$, $b_n, b'_n \in (\underline{b}, \overline{b})$, $h_n, h'_n \in (\underline{h}, \overline{h})$ for all $n \geq 1$. Let $\mathcal{O} := (\underline{a}, \overline{a}) \times (\underline{b}, \overline{b}) \times (\underline{h}, \overline{h})$, and suppose $x \in \mathcal{O}$.

By the local Lipschitz property of concave functions, $(\nabla_x V_i(x_n))_{n\geq 1}$ and $(\nabla_x V_i(x'_n))_{n\geq 1}$ are bounded. Their boundedness implies the existence of their convergent subsequences. Hereafter, I write these arbitrary convergent subsequences as $(\nabla_x V_i(\widetilde{x}_n))_{n\geq 1}$ and $(\nabla_x V_i(\widetilde{x}'_n))_{n\geq 1}$. Furthermore, I write their limits as $\nabla_x \widetilde{V}_i = (\partial_a \widetilde{V}_i, \partial_b \widetilde{V}_i, \partial_h \widetilde{V}_i)$ and $\nabla_x \widetilde{V}'_i = (\partial_a \widetilde{V}'_i, \partial_b \widetilde{V}'_i, \partial_h \widetilde{V}'_i)$, respectively.

Suppose that $\partial_b V_i(\widetilde{x}_n) \to 0$ as $n \to 0$. Then, by the supergradient inequality, I have

$$0 = \lim_{n\to\infty} \partial_b V_i(\widetilde{x}_n)(b' - \widetilde{b}_n) \geq \lim_{n\to\infty} \left\{ V_i(\widetilde{a}_n, b', \widetilde{h}_n) - V_i(\widetilde{x}_n) \right\} = V_i(a, b', h) - V_i(a, b, h),$$

for $b' > b$. However, this is a contradiction to the strictly increasing property of V_i with respect to b. Therefore, $\partial_b V_i(\widetilde{x}_n)$ converges to a strictly positive value. Similarly, we can see that $\partial_h V_i(\widetilde{x}_n)$, $\partial_b V_i(\widetilde{x}'_n)$, and $\partial_h V_i(\widetilde{x}'_n)$ also converge to strictly positive values.

Since V_i is differentiable at $\widetilde{x}_n$ for all $n \geq 1$, the supergradient inequality holds: $\nabla_x V_i(\widetilde{x}_n) \cdot (x' - \widetilde{x}_n) \geq V_i(x') - V_i(\widetilde{x}_n)$, for any $x' \in \mathcal{X}$. Taking a limit as $n \to \infty$, I have $\nabla_x \widetilde{V}_i \cdot (x' - x) \geq V_i(x') - V_i(x)$, for any $x' \in \mathcal{X}$. Thus, $\nabla_x \widetilde{V}_i \in \partial V_i(x)$. Similarly, I have $\nabla_x \widetilde{V}'_i \in \partial V_i(x)$.

Step 2. Here, for all $n \geq 1$, let us consider a smooth test function $\varphi^n = (\varphi^n_j)_{j\in\mathcal{Y}}$ such that $0 = \min_{x'\in\mathcal{O}}\{V_j(x') - \varphi^n_j(x')\} = V_j(\widetilde{x}_n) - \varphi^n_j(\widetilde{x}_n)$ for any $j \in \mathcal{Y}$. Note that, by the differentiability of V at $\widetilde{x}_n$ and by the first-order condition, $\nabla_x \varphi^n_j(\widetilde{x}_n)$ is equal to $\nabla_x V_j(\widetilde{x}_n)$ for all $j \in \mathcal{Y}$ and $n \geq 1$. Then, the viscosity supersolution property of V yields $\rho V_i(\widetilde{x}_n) - \mathcal{H}_i(\widetilde{x}_n, \nabla_x V_i(\widetilde{x}_n)) - \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j} \Big(V_j(\widetilde{x}_n) - V_i(\widetilde{x}_n) \Big) \geq 0$. Letting n go to infinity yields

$$\rho V_i(x) - \mathcal{H}_i \left(x, \nabla_x \widetilde{V}_i \right) - \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j} \Big(V_j(x) - V_i(x) \Big) \geq 0, \tag{A.5}$$

where I have used the continuity of the Hamiltonian and value function. Similarly, I have

$$\rho V_i(x) - \mathcal{H}_i\left(x, \nabla_x \widetilde{V}_i'\right) - \sum_{j \in \mathcal{Y}\setminus\{i\}} \lambda_{i,j}\Big(V_j(x) - V_i(x)\Big) \geq 0. \tag{A.6}$$

I randomly choose $p = (p_a, p_b, p_h) \in \partial V_i(x)$. Let $\varphi \in C^1_K(\mathcal{X})$ be a test function such that

$$\varphi_i(x' = (a', b', h')) := V_i(x) + p \cdot (x' - x) + \|x' - x\|^2$$

and φ_j with $j \in \mathcal{Y}\setminus\{i\}$ is a smooth function satisfying $0 = \max_{x' \in \mathcal{O}}\{V_j(x') - \varphi_j(x')\} = V_j(x) - \varphi_j(x)$. Then, by the supergradient inequality, I have

$$V_i(x') - \varphi_i(x') = V_i(x') - V_i(x) - p \cdot (x' - x) - \|x' - x\|^2 \leq 0,$$

and the equality holds only when $x' = x$. Thus, $0 = \max_{x' \in \mathcal{O}}\{V_i(x') - \varphi_i(x')\} = V_i(x) - \varphi_i(x)$. Furthermore, the gradient of φ_i at x is $\nabla_x \varphi_i(x) = p$. Hence, the viscosity subsolution property of V yields

$$\rho V_i(x) - \mathcal{H}_i(x, p) - \sum_{j \in \mathcal{Y}\setminus\{i\}} \lambda_{i,j}\Big(V_j(x) - V_i(x)\Big) \leq 0. \tag{A.7}$$

Here, suppose that $\nabla_x \widetilde{V}_i \neq \nabla_x \widetilde{V}_i'$ and let us lead to a contradiction. Then, for any $k \in (0, 1)$, one can choose $p \in \partial V_i(x)$ such that $p = k\nabla_x \widetilde{V}_i + (1-k)\nabla_x \widetilde{V}_i'$. By the inequalities (A.5) to (A.7) and the strict convexity of $\mathcal{H}$ with respect to p, I have

$$\begin{aligned}
\rho V_i(x) &- \sum_{j \in \mathcal{Y}\setminus\{i\}} \lambda_{i,j}\Big(V_j(x) - V_i(x)\Big) \leq \mathcal{H}_i(x, p) \\
&< k\mathcal{H}_i(x, \nabla_x \widetilde{V}_i) + (1-k)\mathcal{H}_i(x, \nabla_x \widetilde{V}_i') \leq \rho V_i(x) - \sum_{j \in \mathcal{Y}\setminus\{i\}} \lambda_{i,j}\Big(V_j(x) - V_i(x)\Big).
\end{aligned}$$

This is a contradiction, and thus,

$$(\partial_a \widetilde{V}_i, \partial_b \widetilde{V}_i, \partial_h \widetilde{V}_i) = \nabla_x \widetilde{V}_i = \nabla_x \widetilde{V}_i' = (\partial_a \widetilde{V}_i', \partial_b \widetilde{V}_i', \partial_h \widetilde{V}_i'). \tag{A.8}$$

If $(x_n)_{n\geq 1} = (x_n')_{n\geq 1}$, the equality (A.8) implies that every convergent subsequence of $(\nabla_x V_i(x_n))_{n\geq 1}$ has the same limit. Hence, the original sequence converges to this limit. Furthermore, since

$(x_n)_{n\geq 1}$ and $(x'_n)_{n\geq 1}$ are chosen arbitrarily, I can define the following continuous envelope function of $\nabla_x V_i$ unambiguously.

$$\overline{\nabla}_x V_i(x) = (\overline{\partial}_a V_i(x), \overline{\partial}_b V_i(x), \overline{\partial}_h V_i(x)) := \begin{cases} \nabla_x V_i(x), & \text{if } V_i \text{ is differentiable at } x, \\ \lim_{x' \to x} \nabla_x V_i(x'), & \text{otherwise,} \end{cases}$$

for any $x \in \mathcal{X}$. By construction, $\overline{\nabla}_x V_i(x) \in \partial V_i(x)$ for any $x \in \mathcal{X}$, and it is continuous.

Step 3. I shall show that the continuous envelope function is actually the partial derivative of V_i. I denote the directional derivative of a function F at $x = (a, b, h)$ in the direction $d = (q, r, s)$ by $\mathrm{d}^+ F(x; d)$ such that

$$\mathrm{d}^+ F(x; d) := \lim_{t \downarrow 0} \frac{F(x + td) - F(x)}{t}.$$

Fix an arbitrary $x \in \mathcal{X}$. Note that I do not require the differentiability of V_i at x. Let $d = (q, r, s) \in \mathbb{R}^3 \setminus \{0\}$ be an arbitrary direction. Then, since $\overline{\nabla}_x V_i(x) \in \partial V_i(x)$, Theorem 23.2 in Rockafellar (1970) yields

$$\mathrm{d}^+ V_i(x; d) \leq \overline{\nabla}_x V_i(x) \cdot d.$$

Let us show the opposite inequality. Let us define a set such that $L(x; d) := \{x' \in \mathcal{X} | x' = x + kd, \text{ for some } k \in \mathbb{R}\}$. I initially suppose that I can select a sequence $(x_n)_{n\geq 1} = (x + t_n d)_{n\geq 1} \subseteq L(x; d)$ such that $(t_n)_{n\geq 1}$ is a strictly decreasing and positive sequence with $t_n \downarrow 0$ as $n \to \infty$, and V_i is differentiable at x_n for all $n \geq 1$. Here, for any $n \geq 1$ and $t > 0$, let $\lambda_n := t/(t + t_n) \in (0, 1)$ and it satisfies $x + td = (1 - \lambda_n)x + \lambda_n(x_n + td)$ and $x_n = \lambda_n x + (1 - \lambda_n)(x_n + td)$. Since V_i is concave, I have

$$V_i(x + td) \geq (1 - \lambda_n)V_i(x) + \lambda_n V_i(x_n + td) \Leftrightarrow \frac{V_i(x + td) - V_i(x)}{t} \geq \frac{\lambda_n}{t}\Big(V_i(x_n + td) - V_i(x)\Big),$$
$$V_i(x_n) \geq \lambda_n V_i(x) + (1 - \lambda_n)V_i(x_n + td) \Leftrightarrow \frac{\lambda_n}{t}\Big(V_i(x_n + td) - V_i(x)\Big) \geq \frac{V_i(x_n + td) - V_i(x_n)}{t}.$$

Thus, for all $n \geq 1$, I have

$$\mathrm{d}^+ V_i(x; d) = \lim_{t \downarrow 0} \frac{V_i(x + td) - V_i(x)}{t} \geq \lim_{t \downarrow 0} \frac{V_i(x_n + td) - V_i(x_n)}{t} = \mathrm{d}^+ V_i(x_n; d) = \nabla_x V_i(x_n) \cdot d.$$

Hence, taking the limit as $n \to \infty$, I obtain $\mathrm{d}^+ V_i(x; d) \geq \overline{\nabla}_x V_i(x) \cdot d$. Therefore, I have

$d^+V_i(x;d) = \overline{\nabla}_x V_i(x) \cdot d$.

If we cannot choose a differentiable sequence $(x_n)_{n\geq 1} \subseteq L(x;d)$, I can choose, instead, a sequence of directions $(d_m)_{m\geq 1} \subseteq \mathbb{R}^3 \setminus \{0\}$ such that, for all $m \geq 1$, there exists a sequence $(x_n^m)_{n\geq 1} = (x + t_n^m d_m)_{n\geq 1} \subseteq L(x;d_m)$, where $t_n^m \downarrow 0$ as $n \to \infty$, and V_i is differentiable at x_n^m for all $n \geq 1$. Furthermore, we can suppose that $d_m \to d$. If not, V_i is not differentiable on some measurable subset of $\mathcal{X}$ having positive mass with respect to the Lebesgue measure, so it is a contradiction. Subsequently, I have $d^+V_i(x;d_m) = \overline{\nabla}_x V_i(x) \cdot d_m$ for all $m \geq 1$. Since the directional derivative of a concave function is continuous in direction,[13] I have $d^+V_i(x;d) = \lim_{m\to\infty} d^+V_i(x;d_m) = \overline{\nabla}_x V_i(x) \cdot d$.

In summary, for any $x = (a,b,h) \in \mathcal{X}$ and any direction $d = (q,r,s) \in \mathbb{R}^3 \setminus \{0\}$, I have

$$d^+V_i(x;d) = \overline{\nabla}_x V_i(x) \cdot d.$$

Therefore, V_i is linearly differentiable everywhere on $\mathcal{X}$ in the Gâteaux sense. Since $x \to \overline{\nabla}_x V_i(x)$ is continuous, we can conclude that V_i is continuously differentiable everywhere on $\mathcal{X}$. □

A.6 Proof of Lemma 12

Proof of Lemma 12.

The continuity of the Hamiltonian. Fix an arbitrary $i \in \mathcal{Y}$. First, let us consider the problem (3.4). As in Lemma SI.10 of Rocheteau et al. (2018), we can restrict a feasible set of consumption $\mathcal{C}$ as being a locally compact correspondence with respect to (a,b,h,p_b). Let $\mathcal{K}$ be an arbitrary compact cube subset of $\overline{\mathcal{X}}$. Further, let $\mathcal{P}$ be an arbitrary closed interval on $(0,\infty)$. For any $(c,l,p_b,a,b,h) \in \mathcal{C} \times [0,\overline{L}] \times \mathcal{P} \times \mathcal{K}$, I have

$$\partial_c u(c,l,a,b,h,i) - \tau_c(i)p_b \leq \overline{\partial_c u_{\mathcal{K}}}(c,i) - \underline{\tau}_c \inf_{p_b\in\mathcal{P}} p_b \to -\underline{\tau}_c \inf_{p_b\in\mathcal{P}} p_b < 0 \quad \text{as } c \to \infty.$$

[13]That is a consequence of the fact that the directional derivative of a concave function is concave in direction. Refer to Theorem 23.1 in Rockafellar (1970).

Therefore, there exists $\overline{c}_{\mathcal{K}} \in \mathcal{C}$ such that all $c > \overline{c}_{\mathcal{K}}$ are not optimal. Similarly, I have

$$\partial_c u(c,l,a,b,h,i) - \tau_c(i)p_b \geq \underline{\partial_c u_{\mathcal{K}}}(c,i) - \overline{\tau}_c \sup_{p_b \in \mathcal{P}} p_b \to \infty \quad \text{as } c \to 0.$$

Therefore, there exists $\underline{c}_{\mathcal{K}} \in \mathcal{C}$ such that all $c < \underline{c}_{\mathcal{K}}$ are not optimal. Both of $\underline{c}_{\mathcal{K}}$ and $\overline{c}_{\mathcal{K}}$ only depends on $\mathcal{P}$ and $\mathcal{K}$. Thus,

$$\begin{aligned}\mathcal{M}(p_b,a,b,h,i) &= \sup_{(c,l)\in\mathcal{C}\times[0,\overline{L}]} \{u(c,l,a,b,h,i) + p_b(f(l,h,i) - \tau_c(i)c)\} \\ &= \max_{(c,l)\in[\underline{c}_{\mathcal{K}},\overline{c}_{\mathcal{K}}]\times[0,\overline{L}]} \{u(c,l,a,b,h,i) + p_b(f(l,h,i) - \tau_c(i)c)\},\end{aligned}$$

for any $(p_b,a,b,h) \in \mathcal{P}\times\mathcal{K}$. Hence, Berge's maximum theorem implies that a pair of the maximizer $(c^*(p_b,a,b,h,i), l^*(p_b,a,b,h,i))$ is continuous on $\mathcal{P}\times\mathcal{K}$. We can easily extend the continuity in $\mathcal{P}\times\mathcal{K}$ to that in $(0,\infty)\times\overline{\mathcal{X}}$. Therefore, $(p_b,a,b,h) \to \mathcal{M}(p_b,a,b,h,i)$ is continuous on $(0,\infty)\times\overline{\mathcal{X}}$ for any $i \in \mathcal{Y}$. The above also implies that c^* is always an interior solution.

Here, let us consider the Hamiltonian which can be expressed as follows:

$$\begin{aligned}\mathcal{H}_i(a,b,h,p_a,p_b,p_h) &= \mathcal{M}(p_b,a,b,h,i) \\ &\quad + \Big((p_a - p_b)d^*(p_a,p_b,i) + p_b\chi_A(d^*(p_a,p_b,i),i)\Big)a \\ &\quad + p_h\alpha_H\Big((sh)^*(p_b,p_h,i),i\Big) - p_b\beta_H(i)(sh)^*(p_b,p_h,i) \\ &\quad + p_a(r_A(i)a + g_A(i) - \pi_A(a,i)) + p_b(r(b,i)+g(i)) - p_h\delta_H(i)h, \quad \text{(A.9)}\end{aligned}$$

where $d^*(p_a,p_b,i) = (\partial_d\chi_A)^{-1}(p_a/p_b - 1; i)$, and $(sh)^*(p_b,p_h,i) = (\partial_x\alpha_H)^{-1}(\beta_H(i)p_b/p_h; i)$. The continuity on $\overline{\mathcal{X}}\times\mathbb{R}\times(0,\infty)^2$ is obvious, so I will check the strict convexity. Since it suffices to show the convexity of the Hamiltonian with respect to (p_a,p_b,p_h), I here fix $(a,b,h,i) \in \overline{\mathcal{X}}\times\mathcal{Y}$ arbitrarily.

The convexity of the Hamiltonian. First, let us consider the convexity of $p_b \to \mathcal{M}(p_b,a,b,h,i)$. I initially consider the interior solution case, and next consider the boundary solution case. In the interior solution case, c^* and $l^* \in (0,\overline{L})$ satisfy the first order condition such that

$$\partial_c u(c^*,l^*,a,b,h,i) - \tau_c(i)p_b = 0, \quad \text{and} \quad \partial_l u(c^*,l^*,a,b,h,i) + p_b\partial_l f(l^*,h,i) = 0.$$

Since $\partial_c u$, $\partial_l u$, and $\partial_l f$ are continuous in (c^*, l^*) and since $\partial_{ll} f \leq 0$ and the Hessian matrix of u with respect to (c, l), denoted by $H(u)$, is negative definite in $(c, l) \in \mathcal{C} \times (0, \overline{L})$, the implicit function theorem implies that there exists a neighborhood of p_b, denoted by $\widehat{\mathcal{P}}$, such that $(c^*(p_b, a, b, h, i), l^*(p_b, a, b, h, i)) \in \mathcal{C} \times (0, \overline{L})$ is differentiable with respect to p_b on $\widehat{\mathcal{P}}$, and the derivatives are

$$\begin{pmatrix} \partial_{p_b} c^*(p_b, \cdots) \\ \partial_{p_b} l^*(p_b, \cdots) \end{pmatrix} = \underbrace{\left(H(u)(c^*, l^*, \cdots) + p_b \begin{pmatrix} 0 & 0 \\ 0 & \partial_{ll} f(l^*, \cdots) \end{pmatrix} \right)^{-1}}_{=:A(p_b)} \underbrace{\begin{pmatrix} \tau_c(i) \\ -\partial_l f(l^*, \cdots) \end{pmatrix}}_{=:x(p_b)}.$$

Then, $A(p_b)$ is negative definite. Meanwhile, by the envelope theorem, I have $\partial_{p_b} \mathcal{M}(p_b, a, b, h, i) = f(l^*(p_b, a, b, h, i), h, i) - \tau_c(i) c^*(p_b, a, b, h, i)$. Therefore, I have

$$\partial_{p_b p_b} \mathcal{M}(p_b, a, b, h, i) = - \begin{pmatrix} \tau_c(i) \\ -\partial_l f(l^*, \cdots) \end{pmatrix} \cdot \begin{pmatrix} \partial_{p_b} c^*(p_b, \cdots) \\ \partial_{p_b} l^*(p_b, \cdots) \end{pmatrix} = -x(p_b) \cdot (A(p_b) x(p_b)) > 0.$$

Hence, $\mathcal{M}$ is strictly convex on $\widehat{\mathcal{P}}$.

Here, let us show that $p_b \to l^*(p_b, \cdots)$ is non-decreasing if l^* is an interior solution. By the first order condition, I have

$$\frac{\partial_l f(l^*, h, i)}{\tau_c(i)} = -\frac{\partial_l u(c^*, l^*, a, b, h, i)}{\partial_c u(c^*, l^* a, b, h, i)}.$$

Let us consider the situation that p_b changes p_b' in $\widehat{\mathcal{P}}$. Then, the above equality holds even when p_b'. If $c^*(p_b', \cdots) > c^*(p_b, \cdots)$, then since $(c, l) \to -\partial_l u(c, l, \cdots)/\partial_c u(c, l, \cdots)$ is strictly increasing and $l \to \partial_l f(l, \cdots)$ is non-increasing, I have $l^*(p_b', \cdots) \leq l^*(p_b, \cdots)$. Similarly, if $c^*(p_b', \cdots) < c^*(p_b, \cdots)$, then $l^*(p_b', \cdots) \geq l^*(p_b, \cdots)$. Here, suppose that $p_b' < p_b$. Then, by the first order condition with respect to c, I have $\partial_c u(c^*(p_b, \cdots), l^*(p_b, \cdots), \cdots) > p_b'$. If $c^*(p_b', \cdots) < c^*(p_b, \cdots)$, then since $\partial_{cc} u < 0$, I have $\partial_c u(c^*(p_b', \cdots), l^*(p_b, \cdots), \cdots) > p_b'$. Further, since $l^*(p_b', \cdots) \geq l^*(p_b, \cdots)$ and $\partial_{cl} u \geq 0$, I also have $\partial_c u(c^*(p_b', \cdots), l^*(p_b', \cdots), \cdots) > p_b'$, which is a contradiction. Therefore, $c^*(p_b', \cdots) \geq c^*(p_b, \cdots)$. If $c^*(p_b', \cdots) > c^*(p_b, \cdots)$, then $l^*(p_b', \cdots) \leq l^*(p_b, \cdots)$. If $c^*(p_b', \cdots) = c^*(p_b, \cdots)$, then since $\partial_c u(c^*(p_b', \cdots), l^*(p_b, \cdots), \cdots) > p_b'$ and $\partial_{cl} u \geq 0$, I have $l^*(p_b', \cdots) < l^*(p_b, \cdots)$. Hence, $p_b \to l^*(p_b, \cdots)$ is non-decreasing if l^* is an interior solution. Thus, by the continuity of l^*, there exists an interval $(\underline{p_b}, \overline{p_b}) \subseteq (0, \infty)$ such that $l^* \in (0, \overline{L})$ if $p_b \in (\underline{p_b}, \overline{p_b})$, $l^* = 0$ if $\underline{p_b} > 0$ and $p_b \leq \underline{p_b}$, and $l^* = \overline{L}$ if $\underline{p_b} < \infty$ and $p_b \geq \underline{p_b}$. Therefore, l^* is non-decreasing on $(0, \infty)$, and $\mathcal{M}$ is strictly convex on $(\underline{p_b}, \overline{p_b})$.

I here consider the boundary solution case, i.e., $l^* = 0$ or $\overline{L}$. I suppose $p_b \neq \underline{p_b}$ and $p_b \neq \overline{p_b}$. Since c^* is always an interior solution, it satisfies the first order condition. Therefore, the implicit function theorem yields $\partial_{p_b} c^* = \tau_c(i)/\partial_{cc} u(c^*, l^*, a, b, h, i)$, and hence $\partial_{p_b p_b} \mathcal{M}(p_b, a, b, h, i) = -(\tau_c(i))^2/\partial_{cc} u(c^*, l^*, a, b, h, i) > 0$. Thus, $\mathcal{M}$ is strictly convex on $(0, \underline{p_b})$ and on $(\overline{p_b}, \infty)$. Furthermore, since $\partial_{p_b} \mathcal{M}$ is continuous, $\partial_{p_b} \mathcal{M}$ is strictly increasing on $(0, \infty)$. Therefore, $p_b \to \mathcal{M}(p_b, \cdots)$ is strictly convex on $(0, \infty)$.

Next, I shall verify the convexity of the terms related to p_a (the second line in (A.9)). I write $\mathcal{H}_A(p_a, p_b)$ as these terms. Then, I have

$$\partial_{p_a} \mathcal{H}_A(p_a, p_b) = (r_A(i) + d^*(p_a, p_b, i))\, a, \quad \partial_{p_a p_a} \mathcal{H}_A(p_a, p_b) = \frac{a}{p_b \partial_{dd} \chi_A(d^*(p_a, p_b, i), i)} > 0,$$
$$\partial_{p_b} \mathcal{H}_A(p_a, p_b) = -\left(d^*(p_a, p_b, i) + \chi_A(d^*(p_a, p_b, i), i)\right) a,$$
$$\partial_{p_b p_b} \mathcal{H}_A(p_a, p_b) = \frac{p_a^2 a}{p_b^3 \partial_{dd} \chi_A(d^*(p_a, p_b, i), i)} \geq 0, \quad \partial_{p_a p_b} \mathcal{H}_A(p_a, p_b) = -\frac{p_a a}{p_b^2 \partial_{dd} \chi_A(d^*(p_a, p_b, i), i)}.$$

In addition, $\partial_{p_a p_a} \mathcal{H}_A \partial_{p_b p_b} \mathcal{H}_A - \left(\partial_{p_a p_b} \mathcal{H}_A\right)^2 = 0$. Therefore, $(p_a, p_b) \to \mathcal{H}_A(p_a, p_b)$ is (at least weakly) convex on $\mathbb{R} \times (0, \infty)$, and $p_a \to \mathcal{H}_A(p_a, p_b)$ is strictly convex on $\mathbb{R}$ for any $p_b \in (0, \infty)$.

Finally, I check the convexity of the terms related to p_h (the third line in (A.9)). I write $\mathcal{H}_H(p_b, p_h)$ as these terms. Then, I have

$$\partial_{p_b} \mathcal{H}_H(p_b, p_h) = -\beta_H(i)(sh)^*(p_b, p_h, i), \quad \partial_{p_b p_b} \mathcal{H}_H(p_b, p_h) = -\frac{(\beta_H(i))^2}{p_h \partial_{xx} \alpha_H((sh)^*(p_b, p_h, i), i)} > 0,$$
$$\partial_{p_h} \mathcal{H}_H(p_b, p_h) = \alpha_H((sh)^*(p_b, p_h, i), i),$$
$$\partial_{p_h p_h} \mathcal{H}_H(p_b, p_h) = -\frac{(\beta_H(i))^2 p_b^2}{p_h^3 \partial_{xx} \alpha_H((sh)^*(p_b, p_h, i), i)} > 0, \quad \partial_{p_b p_h} \mathcal{H}_H(p_b, p_h) = \frac{(\beta_H(i))^2 p_b}{p_h^2 \partial_{xx} \alpha_H((sh)^*(p_b, p_h, i), i)}.$$

In addition, $\partial_{p_b p_b} \mathcal{H}_H \partial_{p_h p_h} \mathcal{H}_H - \left(\partial_{p_b p_h} \mathcal{H}_H\right)^2 = 0$. Thus, $(p_b, p_h) \to \mathcal{H}_H(p_b, p_h)$ is (at least weakly) convex on $(0, \infty)^2$, and $p_h \to \mathcal{H}_H(p_b, p_h)$ is strictly convex on $(0, \infty)$ for any $p_b \in (0, \infty)$. In summary, $(p_a, p_b, p_h) \to \mathcal{H}_A(p_a, p_b) + \mathcal{H}_H(p_b, p_h)$ is convex on $\mathbb{R} \times (0, \infty)^2$, and $(p_a, p_h) \to \mathcal{H}_A(p_a, p_b) + \mathcal{H}_H(p_b, p_h)$ is strictly convex on $\mathbb{R} \times (0, \infty)$ for any $p_b \in (0, \infty)$. Accordingly, the

Hamiltonian, $\mathcal{H}_i(a,b,h,p_a,p_b,p_h) = \mathcal{M} + \mathcal{H}_A + \mathcal{H}_H +$ linear terms, is strictly convex with respect to (p_a,p_b,p_h) on $\mathbb{R}\times(0,\infty)^2$, for any fixed $(a,b,h,i)\in\overline{\mathcal{X}}\times\mathcal{Y}$. $\square$

A.7 Proof of Lemma 13

Proof of Lemma 13.

Step 1. Fix an arbitrary $(a,h,i)\in(0,\infty)^2\times\mathcal{Y}$. The inequality (A.3) is

$$\begin{aligned} V_i(ae^{-(0\wedge \underline{r}_A)T}, -\underline{B}, he^{\overline{\delta}_H T}) \geq & \frac{1-e^{-\rho T}}{\rho}\min_{j\in\mathcal{Y}}\{\underline{u}(\frac{g-y}{\overline{\tau}_c},0,j)\} \\ & + \sum_{j\in\mathcal{Y}\setminus\{i\}}\mathbb{P}(Y_T^i=j)\Big(V_j(a,\widehat{B}_T,h)-V_i(a,\widehat{B}_T,h)\Big)+V_i(a,\widehat{B}_T,h), \end{aligned}$$

for any $T>0$, where $\widehat{B}$ is a solution to the deterministic ODE $\mathrm{d}\widehat{B}_t = (r(\widehat{B}_t,\underline{i})+\underline{y})\mathrm{d}t$ with $\widehat{B}_0=-\underline{B}$. Here, by the mean value theorem, I have $\lim_{T\to 0}\frac{\widehat{B}_T+\underline{B}}{T}=\lim_{T\to0}\frac{1}{T}\int_0^T(r(\widehat{B}_t,\underline{i})+\underline{y})\mathrm{d}t = r(-\underline{B},\underline{i})+\underline{y}>0$. Meanwhile, I also have $\lim_{T\to0}\frac{1-e^{-\rho T}}{\rho T}=1$, and $\lim_{T\to0}\frac{\mathbb{P}(Y_T^i=j)}{T}=\lambda_{i,j}$, for any $j\in\mathcal{Y}\setminus\{i\}$. Furthermore, since $(a,h)\to V_i(a,-\underline{B},h)$ is concave and then locally Lipschitz, I have

$$\lim_{T\to0}\frac{V_i(ae^{-(0\wedge\underline{r}_A)T},-\underline{B},he^{\overline{\delta}_H T})-V_i(a,-\underline{B},h)}{T}\leq \overline{\delta}_H h\partial_h^- V_i(a,-\underline{B},h)-(0\wedge\underline{r}_A)a\partial_a^-V_i(a,-\underline{B},h)<\infty.$$

Therefore, dividing (A.3) by $\widehat{B}_T+\underline{B}>0$ and taking the limit as $T\to0$, I obtain

$$\begin{aligned} &\frac{1}{r(-\underline{B},\underline{i})+\underline{y}}\Big(-\min_{j\in\mathcal{Y}}\{\underline{u}(\frac{g-y}{\overline{\tau}_c},0,j)\}-\sum_{j\in\mathcal{Y}\setminus\{i\}}\lambda_{i,j}\Big(V_j(a,-\underline{B},h)-V_i(a,-\underline{B},h)\Big) \\ &\qquad + \overline{\delta}_H h\partial_h^-V_i(a,-\underline{B},h)-(0\wedge\underline{r}_A)a\partial_a^-V_i(a,-\underline{B},h)\Big) \\ &\geq \lim_{T\to0}\frac{V_i(a,\widehat{B}_T,h)-V_i(a,-\underline{B},h)}{\widehat{B}_T+\underline{B}}=\lim_{b\downarrow-\underline{B}}\frac{V_i(a,b,h)-V_i(a,-\underline{B},h)}{b+\underline{B}}=\partial_b^+V_i(a,-\underline{B},h). \end{aligned} \quad \text{(A.10)}$$

Thus, $\partial_b^+V_i(a,-\underline{B},h)$ is finite. Furthermore, since the value function is bounded, the upper boundary of $\partial_b^+V_i(a,-\underline{B},h)$ only depends on the values of $h\partial_h^-V_i(a,-\underline{B},h)$ and $a\partial_a^-V_i(a,-\underline{B},h)$. Meanwhile, $(a,h)\to V_i(a,-\underline{B},h)$ is concave on $(0,\infty)^2$. Thus, the locally Lipschitz property of V_i implies that $\partial_h^-V_i(a,-\underline{B},h)$ and $\partial_a^-V_i(a,-\underline{B},h)$ are locally bounded and $\partial_b^+V_i(a,-\underline{B},h)$ is also locally bounded on $(0,\infty)^2$. The locally bounded property of $\partial_b^+ V_i(a,-\underline{B},h)$ implies that $\partial_b V_i(a,b,h) \uparrow \partial_b^+ V_i(a,-\underline{B},h)$ as $b \downarrow \underline{B}$ since V_i is concave. Hence, the right derivative is right continuous.

<u>*Step 2.*</u> Next, let us consider $\partial_a V_i(a,-\underline{B},h)$ and $\partial_h V_i(a,-\underline{B},h)$. By the concavity of $(a,h) \to V_i(a,-\underline{B},h)$ on $(0,\infty)^2$, $V_i(a,-\underline{B},h)$ is differentiable almost everywhere. Here, fix an arbitrary $(a,h) \in (0,\infty)^2$. Then, the supergradient inequality yields $\partial_a V_i(a,b,h)(a'-a)+\partial_b V_i(a,b,h)(b'-b) \geq V_i(a',b',h) - V_i(a,b,h)$, for all $(a',b') \in (0,\infty) \times (-\underline{B},\infty)$ and $b \in (0,\infty)$. Letting (b,b') go to $-\underline{B}$, by the continuity of V_i, I have

$$\left(\lim_{b\downarrow -\underline{B}} \partial_a V_i(a,b,h)\right)(a'-a) \geq V_i(a',-\underline{B},h) - V_i(a,-\underline{B},h),$$

for any $a' \in (0,\infty)$. Therefore, this implies

$$\begin{aligned}
\lim_{b\downarrow -\underline{B}} \partial_a V_i(a,b,h) &\geq \frac{V_i(a',-\underline{B},h) - V_i(a,-\underline{B},h)}{a'-a}, \quad \text{if } a' > a,\\
\lim_{b\downarrow -\underline{B}} \partial_a V_i(a,b,h) &\leq \frac{V_i(a',-\underline{B},h) - V_i(a,-\underline{B},h)}{a'-a}, \quad \text{if } a' < a
\end{aligned}$$

Thus, I obtain $\partial_a^+ V_i(a,-\underline{B},h) \leq \lim_{b\downarrow -\underline{B}} \partial_a V_i(a,b,h) \leq \partial_a^- V_i(a,-\underline{B},h)$. Therefore, $\partial_a V_i(a,b,h) \to \partial_a V_i(a,-\underline{B},h)$ as $b \downarrow -\underline{B}$ if $\partial_a V_i(a,-\underline{B},h)$ exists. Similarly, $\partial_h V_i(a,b,h) \to \partial_h V_i(a,-\underline{B},h)$ as $b \downarrow -\underline{B}$ if $\partial_h V_i(a,-\underline{B},h)$ exists.

<u>*Step 3.*</u> Finally, let us show that $\partial_a V_i(a,-\underline{B},h)$ and $\partial_h V_i(a,-\underline{B},h)$ exist for all $(a,h) \in (0,\infty)^2$ and that $(a,h) \to (\partial_a V_i(a,-\underline{B},h), \partial_b^+ V_i(a,-\underline{B},h), \partial_h V_i(a,-\underline{B},h))$ is continuous on $(0,\infty)^2$. Fix an arbitrary $(a,h) \in (0,\infty)^2$, and let $(a_n,h_n)_{n\geq 1}$ be an arbitrary sequence such that $a_n \to a$ and $h_n \to h$ as $n \to \infty$, and $\partial_a V_i(a_n,-\underline{B},h_n)$ and $\partial_h V_i(a_n,-\underline{B},h_n)$ exist for all $n \geq 1$. Without loss of generality, there exists $(\underline{a},\overline{a},\underline{h},\overline{h}) \in (0,\infty)^4$ with $\underline{a} < \overline{a}$ and $\underline{h} < \overline{h}$ such that $\underline{a} < a_n < \overline{a}$ and $\underline{h} < h_n < \overline{h}$ for all $n \geq 1$. Then, by the locally bounded property of $\partial_b^+ V_i(a,-\underline{B},h)$, $(\partial_b^+ V_i(a_n,-\underline{B},h_n))_{n\geq 1}$ is bounded. Similarly, by the local Lipschitz property of $(a,h) \to V_i(a,-\underline{B},h)$, $(\partial_a V_i(a_n,-\underline{B},h_n))_{n\geq 1}$ and $(\partial_h V_i(a_n,-\underline{B},h_n))_{n\geq 1}$ are bounded. Therefore, a convergent subsequence of $(\partial_a V_i(a_n,-\underline{B},h_n), \partial_b^+ V_i(a_n,-\underline{B},h_n), \partial_h V_i(a_n,-\underline{B},h_n))_{n\geq 1}$ exists. I also write $(\partial_a V_i(a_n,-\underline{B},h_n), \partial_b^+ V_i(a_n,-\underline{B},h_n), \partial_h V_i(a_n,-\underline{B},h_n))_{n\geq 1}$ as an arbitrary convergent subsequence.

Since the Hamiltonian is continuous when the partial derivatives of V_i with respect to b and h are positive, their positivity needs to be checked. The limit of $(\partial_b^+ V_i(a_n, -\underline{B}, h_n))_{n\geq 1}$ is strictly positive because $\partial_b^+ V_i(a_n, -\underline{B}, h_n) \geq \partial_b V_i(a_n, b, h_n)$ for any $b > -\underline{B}$ and $(a, h) \to \partial_b V_i(a, b, h)$ has a strictly positive minimum on $[\underline{a}, \overline{a}] \times [\underline{h}, \overline{h}]$ by its continuity. Meanwhile, by the strict increasing property of V_i with respect to h, I have $\partial_h V_i(a_n, -\underline{B}, h_n) > 0$ for all $n \geq 1$. Here, suppose $\partial_h V_i(a_n, -\underline{B}, h_n) \to 0$ as $n \to \infty$. Since $(a, h) \to V_i(a, -\underline{B}, h)$ is concave and is differentiable at (a_n, h_n), the supergradient inequality yields

$$0 = \lim_{n\to\infty} \left\{\partial_a V_i(a_n, -\underline{B}, h_n)(a_n - a_n) + \partial_h V_i(a_n, -\underline{B}, h_n)(h' - h_n)\right\}$$
$$\geq \lim_{n\to\infty} \left\{V_i(a_n, -\underline{B}, h') - V_i(a_n, -\underline{B}, h_n)\right\} = V_i(a, -\underline{B}, h') - V_i(a, -\underline{B}, h).$$

If $h' > h$, this is a contradiction, and hence $\partial_h V_i(a_n, -\underline{B}, h_n)$ converges to a strictly positive value.

For any $b \in (-\underline{B}, \infty)$, by the viscosity supersolution property of V_i on $\mathcal{X}$, I have

$$\rho V_i(a_n, b, h_n) - \mathcal{H}_i(a_n, b, h_n, \partial_a V_i(a_n, b, h_n), \partial_b V_i(a_n, b, h_n), \partial_h V_i(a_n, b, h_n))$$
$$- \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\Big(V_j(a_n, b, h_n) - V_i(a_n, b, h_n)\Big) \geq 0,$$

for all $n \geq 1$. First letting b go to $-\underline{B}$ and next letting n tend to go to infinity, I have

$$\rho V_i(a, -\underline{B}, h) - \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\Big(V_j(a, -\underline{B}, h) - V_i(a, -\underline{B}, h)\Big)$$
$$\geq \mathcal{H}_i\left(a, -\underline{B}, h, \lim_{n\to\infty} \partial_a V_i(a_n, -\underline{B}, h_n), \lim_{n\to\infty} \partial_b^+ V_i(a_n, -\underline{B}, h_n), \lim_{n\to\infty} \partial_h V_i(a_n, -\underline{B}, h_n)\right).$$

Then, as in the proof of Proposition 10, the argument of "does not admit concave kinks" yields the unique continuous envelope function of $(\partial_a V_i(a, -\underline{B}, h), \partial_b^+ V_i(a, -\underline{B}, h), \partial_h V_i(a, -\underline{B}, h))$. Therefore, I can conclude that $(a, h) \to V_i(a, -\underline{B}, h)$ is differentiable everywhere. The continuity of the partial derivatives on $\overline{\mathcal{X}}$ can be shown by the "does not admit concave kinks" again. $\square$

A.8 Proof of Lemma 16

Proof of Lemma 16. I arbitrarily fix $(a, b, h, i) \in \overline{\mathcal{X}} \times \mathcal{Y}$. Let us denote by (a_t^i, b_t^i, h_t^i) a local solution to the system of the ODEs (4.4) in state i starting at (a, b, h). Furthermore, let $T_e > 0$ be a time by when (a_t^i, b_t^i, h_t^i) can be defined. Then, one can easily observe that $h_t^i \geq he^{-\overline{\delta}_H t} > 0$ for all $t \in [0, T_e)$. Additionally, since $\partial_a V^*/\partial_b V^* \geq 0$ and $x \to (\partial_x \chi)^{-1}(x; i)$ is non-decreasing, I have $D^* \geq \underline{(\partial_x \chi)^{-1}}(-1) := \min_{j \in \mathcal{Y}} (\partial_x \chi)^{-1}(-1; j) > -\infty$. Thus, in both of the illiquid asset case and the durable goods case, I have $a_t^i \geq ae^{(\underline{r}_A \wedge 0 + \underline{(\partial_x \chi)^{-1}}(-1))t} > 0$ for all $t \in [0, T_e)$.

Here, let us show $b_t^i \geq -\underline{B}$ for all $t \in [0, T_e)$. I define the saving rate s_b^i out of $\overline{\mathcal{X}}$ such that

$$s_b^i(a', b', h') := s_b^i(a', -\underline{B}, h') + r(b', i) - r(-\underline{B}, i)$$

for any $(a', b', h') \in (0, \infty) \times (-\infty, -\underline{B}) \times (0, \infty)$. Then, by the state-constrained boundary condition (4.2), I have

$$s_b^i(a', b', h') = s_b^i(a', -\underline{B}, h') + r(b', i) - r(-\underline{B}, i) \geq r(b', i) - r(-\underline{B}, i), \tag{A.11}$$

for any $(a', b', h', i) \in (0, \infty) \times (-\infty, -\underline{B}) \times (0, \infty) \times \mathcal{Y}$. Here, suppose that $b_{t_1}^i < -\underline{B}$ for some $t_1 \in [0, T_e)$. Then, there exists a time $t_0 \in [0, t_1)$ such that $b_{t_0}^i = -\underline{B}$ and $b_s^i < -\underline{B}$ for all $s \in (t_0, t_1]$, since b^i is continuous and $b \geq -\underline{B}$. By the inequality (A.11), for any $s \in (t_0, t_1]$, I have

$$\frac{\mathrm{d}\left(b_s^i + \underline{B}\right)}{\mathrm{d}s} = s_b^i(a_s^i, b_s^i, h_s^i) \geq r(b_s^i, i) - r(-\underline{B}, i) \geq L_r\left(b_s^i + \underline{B}\right).$$

Thus, the Gronwall inequality yields $b_{t_1}^i + \underline{B} \geq 0$, but this is a contradiction. Therefore, $b_t^i \geq -\underline{B}$ for all $t \in [0, T_e)$.

Here, I derive the upper boundary of $a^i + b^i + h^i$. Since $f(l, h, i)$ is non-decreasing in l and concave in h, there exist two constants $L_{\overline{f}} > 0$ and $K_{\overline{f}}$ such that $f(l, h, i) \leq f(\overline{L}, h, i) \leq K_{\overline{f}} + L_{\overline{f}} h$ for any $(l, h, i) \in [0, \overline{L}] \times (0, \infty) \times \mathcal{Y}$. Let $\overline{f}(h) := K_{\overline{f}} + L_{\overline{f}} h$. Furthermore, since $\partial_x \alpha_H(x, i) \to 0$ as $x \to \infty$ for any $i \in \mathcal{Y}$, I have $h_M := (\max_{(x,i) \in [0,\infty) \times \mathcal{Y}} \{\alpha_H(x, i) - \beta_H(i)x\}) \vee 0 < \infty$. Then, without loss of generality, I take the Lipschitz constant of r, denoted by $L_r > 0$, being larger than $\overline{r}_A \vee 0$ and $L_{\overline{f}}$. By the system of the ODEs (4.4), I have

$$\begin{aligned}\frac{\mathrm{d}(a_t^i + b_t^i + h_t^i)}{\mathrm{d}t} &\leq r(b_t^i, i) + f(l_t^*, h_t^i, i) + g(i) - \beta_H(i) s_t^* h_t^i + r_A(i) a_t^i + g_A(i) + \alpha_H(s_t^* h_t^i, i) \\ &\leq L_r(a_t^i + b_t^i) + L_r \underline{B} + r(-\underline{B}, i) + \overline{f}(h_t^i) + \overline{g} + \overline{g}_A + h_M \\ &\leq L_r(a_t^i + b_t^i + h_t^i) + L_r \underline{B} + \overline{r}_{-\underline{B}} + K_{\overline{f}} + \overline{g} + \overline{g}_A + h_M,\end{aligned}$$

for any $t \in [0, T_e)$, where $\overline{g}_A := \max_{j \in \mathcal{Y}} g_A(j)$, $\overline{r}_{-\underline{B}} := \max_{j \in \mathcal{Y}} |r(-\underline{B}, j)|$, and $(l_t^*, s_t^*) := (L_i^*(a_t^i, b_t^i, h_t^i), S_i^*(a_t^i, b_t^i, h_t^i))$. Hence, a simple integration exercise yields

$$a_t^i + b_t^i + h_t^i \leq \Big(a + b + h + \Big(L_r \underline{B} + \overline{g} + \overline{g}_A + \overline{r}_{-\underline{B}} + K_{\overline{f}} + h_M\Big) t\Big) e^{L_r t}, \tag{A.12}$$

for any $t \in [0, T_e)$. Thus, $a^i + b^i + h^i$ is dominated from above by a finite process that can be defined on $[0, \infty)$. Since a^i, b^i, and h^i are bounded below, they do not blow up in finite time. $\square$

A.9 Proof of Proposition 17

Proof of Proposition 17. It suffices to show that the extension of the value function V^* satisfies the four conditions in Proposition 15. First, the extension of the value function is in $C_K^1(\mathcal{W})$, and it is a constrained viscosity solution to the HJB equation (3.1) on $\overline{\mathcal{X}}$ by Proposition 7. Second, since the value function is bounded, the martingale condition and the limit growth condition are satisfied. Therefore, I need to show the admissibility of (C^*, L^*, D^*, S^*).

I show the existence of asset processes starting at any $(a, b, h) \in \overline{\mathcal{X}}$ and controlled by the candidate optimal control (C^*, L^*, D^*, S^*). Fix an arbitrary $i \in \mathcal{Y}$. Then, a local solution to the ODEs (4.4) exists by Peano's existence theorem. Let T_e be the largest time by when a local solution can be defined. Here, hypothesize that T_e is finite and let us lead to a contradiction. Then, it can be easily seen that there exists a finite limit of the local solution $(a_{T_e}^i, b_{T_e}^i, h_{T_e}^i) = \lim_{t \to T_e}(a_t^i + b_t^i + h_t^i) \in \overline{\mathcal{X}}$ since the local solution is bounded in finite time by Lemma 16. For this convergence, I refer to Lemma 2.14 in Teschl (2012) as well. Therefore, at the time T_e, we can obtain a new local solution to the ODEs (4.4) starting at $(a_{T_e}^i, b_{T_e}^i, h_{T_e}^i)$. However, this is a contradiction, and hence $T_e = \infty$. This implies that the ODEs (4.4) has a global solution. For the system of stochastic ODEs, we can also identify its solution by extending the solution to (4.4). For example, when $Y_0 = i$, we can obtain the solution to (4.4) starting at (a, b, h) in state i, up to the first changing time of Y. If Y changes from i to j at time τ, then we can obtain the solution to (4.4) starting at $(a^i_\tau, b^i_\tau, h^i_\tau)$ in state j. Repeatedly extending the solution as above, we can obtain a solution to the stochastic ODE. By construction, this extension satisfies $A_t > 0$, $H_t > 0$, and $B_t \geq -\underline{B}$, $\mathbb{P}$-a.s. for any $t \in [0, \infty)$. The continuity of the extension is immediate, but we need some technical arguments to show the adaptedness. For any given $i \in \mathcal{Y}$, let $(\Phi(t, x, i))_{t \in [0,\infty)}$ be a solution to the ODEs (4.4) in state i starting at $x \in \overline{\mathcal{X}}$. Invoking the measurable selection argument as in Zubelevich (2012), we can show that the mapping $(t, x) \to \Phi(t, x, i)$ is Borel-measurable. Then, let τ_1 be the first changing time of Y^i, and let us consider an at-most-one-change solution such that

$$\phi^1(t, x, i) := \mathbb{1}\{\tau_1 > t\}\Phi(t, x, i) + \mathbb{1}\{\tau_1 \leq t\}\Phi(t - \tau_1, \Phi(\tau_1, x, i), Y^i_{\tau_1}),$$

for any $(t, x, i) \in [0, \infty) \times \overline{\mathcal{X}} \times \mathcal{Y}$. $\phi^1(t, x, i)$ is $\mathcal{F}_t$-measurable by the Borel-measurability of Φ. Here, let us define an at-most-n-times-changes solution recursively:

$$\phi^{n+1}(t, x, i) := \mathbb{1}\{\tau_{n+1} > t\}\phi^n(t, x, i) + \mathbb{1}\{\tau_{n+1} \leq t\}\Phi(t - \tau_{n+1}, \phi^n(\tau_{n+1}, x, i), Y^i_{\tau_{n+1}}),$$

for any $(t, x, i) \in [0, \infty) \times \overline{\mathcal{X}} \times \mathcal{Y}$, where τ_n is the n-th changing time of Y^i. By the mathematical induction, we can prove that $\phi^n(t, x, i)$ is $\mathcal{F}_t$-measurable for any $n \geq 1$ and $t \in [0, \infty)$. Since the Markov chain Y almost surely changes at most finitely many times in finite time, we can easily see that $\phi^n(t, x, i)$ point-wisely converges to the aforementioned extended solution as $n \to \infty$, $\mathbb{P}$-a.s. Thus, the value of the extended solution at time t is $\mathcal{F}_t$-measurable for any $t \in [0, \infty)$, and hence the extended solution is $\mathbb{F}$-adapted. The right continuity and measurability of (C^*, L^*, D^*, S^*) are evident. Therefore, (C^*, L^*, D^*, S^*) is an admissible feedback control, and it is an optimal control of the utility maximization problem. $\square$

A.10 Proof of Proposition 18

Proof of Proposition 18. The strict concavity of V_i^* on $\overline{\mathcal{X}}$ is immediate, due to the existence of optimal controls and the strict concavity of u. I demonstrate the uniqueness of the optimally controlled asset processes. Fix an arbitrary $(a,b,h,i) \in \overline{\mathcal{X}} \times \mathcal{Y}$. Let us denote optimally controlled asset processes starting at (a,b,h,i) by $(\widetilde{A}, \widetilde{B}, \widetilde{H})$ and $(\widehat{A}, \widehat{B}, \widehat{H})$, respectively. Let us also denote the optimal controls based on $(\widetilde{A}, \widetilde{B}, \widetilde{H})$ and $(\widehat{A}, \widehat{B}, \widehat{H})$ by $(\widetilde{C}, \widetilde{L}, \widetilde{D}, \widetilde{S})$ and $(\widehat{C}, \widehat{L}, \widehat{D}, \widehat{S})$, respectively. I hypothesize that $(\widetilde{C}, \widetilde{L}, \widetilde{D}, \widetilde{S}) \neq (\widehat{C}, \widehat{L}, \widehat{D}, \widehat{S})$ on some measurable subset of $\Omega \times [0,\infty)$ having positive mass with respect to $\mathrm{d}\mathbb{P} \times \mathrm{d}t$, and lead to a contradiction. Here, as in the proof of Proposition 3, I choose $k \in (0,1)$ arbitrarily and introduce a new control as follows:

$$
\begin{aligned}
&C_t^k := k\widetilde{C}_t + (1-k)\widehat{C}_t, \quad L_t^k := k\widetilde{L}_t + (1-k)\widehat{L}_t, \quad D_t^k := \frac{k\widetilde{A}_t}{A_t^k}\widetilde{D}_t + \frac{(1-k)\widehat{A}_t}{A_t^k}\widehat{D}_t, \\
&S_t^k := \frac{\alpha_H^{-1}\Big(k\alpha_H(\widetilde{S}_t\widetilde{H}_t, Y_t^i) + (1-k)\alpha_H(\widehat{S}_t\widehat{H}_t, Y_t^i); Y_t^i\Big)}{k\widetilde{H}_t + (1-k)\widehat{H}_t},
\end{aligned}
$$

for $t \in [0,\infty)$, where A^k is a solution to the following stochastic ODE:

$$
\mathrm{d}A_t^k = \Big(r_A(Y_t^i)A_t^k + k\widetilde{A}_t\widetilde{D}_t + (1-k)\widehat{A}_t\widehat{D}_t + g_A(Y_t^i) - \pi(A_t^k, Y_t^i)\Big)\,\mathrm{d}t,
$$

with $A_0^k = a$. Note that as in the proof of Proposition 3, we can see $A_t^k \geq k\widetilde{A}_t + (1-k)\widehat{A}_t > 0$ $\mathbb{P}$-a.s. on $[0,\infty)$. Furthermore, let $H^k := k\widetilde{H} + (1-k)\widehat{H}$. Then, A^k and H^k are an illiquid asset process and a human capital process controlled by (D^k, S^k), respectively, and

$$
\begin{aligned}
\Big(D_t^k + \chi_A(D_t^k, Y_t^i)\Big)A_t^k &\leq k\Big(\widetilde{D}_t + \chi_A(\widetilde{D}_t, Y_t^i)\Big)\widetilde{A}_t + (1-k)\Big(\widehat{D}_t + \chi_A(\widehat{D}_t, Y_t^i)\Big)\widehat{A}_t, \\
S_t^k H_t^k &\leq k\widetilde{S}_t\widetilde{H}_t + (1-k)\widehat{S}_t\widehat{H}_t,
\end{aligned}
$$

for all $t \in [0,\infty)$. Here, let

$$
\begin{aligned}
C_t^+ := k\Big(\widetilde{D}_t + \chi_A(\widetilde{D}_t, Y_t^i)\Big)\widetilde{A}_t + (1-k)\Big(\widehat{D}_t + \chi_A(\widehat{D}_t, Y_t^i)\Big)\widehat{A}_t - \Big(D_t^k + \chi_A(D_t^k, Y_t^i)\Big)A_t^k \\
+ \beta_H(Y_t^i)\Big(k\widetilde{S}_t\widetilde{H}_t + (1-k)\widehat{S}_t\widehat{H}_t - S_t^k H_t^k\Big) \geq 0,
\end{aligned}
$$

for all $t \in [0,\infty)$. One can easily observe that $(C^k + C^+/\tau_c(Y^i), L^k, D^k, S^k)$ is admissible under (a,b,h,i). Meanwhile, let $B^k = B^{a,b,h,i;C^k+C^+/\tau_c(Y^i),L^k,D^k,S^k}$ and it can be easily observed that $B_t^k \geq k\widetilde{B}_t + (1-k)\widehat{B}_t$ for any $t \in [0,\infty)$. By the strict concavity and strictly increasing property of u, I have

$$
\begin{aligned}
V_i(a,b,h) &\geq \mathrm{E}\left[\int_0^\infty e^{-\rho t}\Big(u(C_t^k + C_t^+/\tau_c(Y_t^i), L_t^k, A_t^k, B_t^k, H_t^k, Y_t^i)\Big)\mathrm{d}t\right] \\
&\geq \mathrm{E}\left[\int_0^\infty e^{-\rho t} u(C_t^k, L_t^k, k\widetilde{A}_t + (1-k)\widehat{A}_t, k\widetilde{B}_t + (1-k)\widehat{B}_t, H_t^k, Y_t^i)\mathrm{d}t\right] \\
&\geq k\,\mathrm{E}\left[\int_0^\infty e^{-\rho t} u(\widetilde{C}_t, \widetilde{L}_t, \widetilde{A}_t, \widetilde{B}_t, \widetilde{H}_t, Y_t^i)\mathrm{d}t\right] + (1-k)\,\mathrm{E}\left[\int_0^\infty e^{-\rho t} u(\widehat{C}_t, \widehat{L}_t, \widehat{A}_t, \widehat{B}_t, \widehat{H}_t, Y_t^i)\mathrm{d}t\right] \\
&= kV_i(a,b,h) + (1-k)V_i(a,b,h) = V_i(a,b,h).
\end{aligned}
$$

The second or third inequality is a strict inequality by the hypothesis, but this is a contradiction. Therefore, $(\widetilde{C}, \widetilde{L}, \widetilde{D}, \widetilde{S}) = (\widehat{C}, \widehat{L}, \widehat{D}, \widehat{S})$ $\mathrm{d}\mathbb{P} \times \mathrm{d}t$-a.e.

Here, we can suppose that a unique optimal control (C^*, L^*, D^*, S^*) satisfies (4.3) for a triplet of optimally controlled asset processes (A^*, B^*, H^*), and it is locally bounded by definition. Furthermore, $\inf_{t\in[0,T]} S_t^* > 0$ $\mathbb{P}$-a.s. for any finite $T > 0$. Given (C^*, L^*, D^*, S^*), I consider the following system of stochastic ODEs with respect to (A^{**}, B^{**}, H^{**}):

$$
\begin{cases}
\mathrm{d}B_t^{**} = \Big(r(B_t^{**}, Y_t^i) + f(L_t^*, H_t^{**}, Y_t^i) + g(Y_t^i) - \tau_c(Y^i)C_t^* \\
\qquad\qquad -\Big(D_t^* + \chi_A(D_t^*, Y_t^i)\Big)A_t^{**} - \beta_H(Y_t^i)S_t^* H_t^{**}\Big)\mathrm{d}t, \\
\mathrm{d}A_t^{**} = \Big((r_A(Y_t^i) + D_t^*)A_t^{**} + g_A(Y_t^i) - \pi_A(A_t^{**}, Y_t^i)\Big)\mathrm{d}t, \\
\mathrm{d}H_t^{**} = \Big(\alpha_H(S_t^* H_t^{**}, Y_t^i) - \delta_H(Y_t^i)H_t^{**}\Big)\mathrm{d}t,
\end{cases}
\tag{A.13}
$$

with $(A_0^{**}, B_0^{**}, H_0^{**}) = (a,b,h)$. An $\mathbb{F}$-adapted solution to the initial value problem (A.13) is unique since the driver with respect to (A^{**}, B^{**}, H^{**}) is locally Lipschitz.[14] Further, all the optimally controlled asset processes solve (A.13). Therefore, the optimally controlled asset processes are unique up to indistinguishability. □

[14]We can first show the uniqueness of (A^{**}, H^{**}) and then the uniqueness of B^{**}.

B Adapted Solution to Stochastic ODEs

This section introduces the formal definition of an $\mathbb{F}$-adapted solution to stochastic ODEs. This definition follows from a concept of the strong solution to stochastic differential equations (SDEs). Suppose that Z is an $\mathbb{F}$-progressively measurable and right-continuous process taking values in $\mathbb{R}^N$, where N is a finite natural number. Following the main text, let us denote a right-continuous, K-state, and $\mathbb{F}$-adapted Markov chain by Y. As mentioned in the main text, I consider the following type of a stochastic ODE with respect to $X \in \mathbb{R}$ on $[0, T]$.

$$\mathrm{d}X_t = F(t, X_t, Y_t, Z_t)\mathrm{d}t, \tag{B.1}$$

with $X_0 = x_0$, where x_0 is an $\mathcal{F}_0$-measurable and finite random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and F is a measurable function on $[0, \infty) \times \mathbb{R} \times \mathcal{Y} \times \mathbb{R}^N$. Here, I suppose that $(t, x, z) \to F(t, x, j, z)$ is continuous for any fixed $j \in \mathcal{Y}$, and $(F(t, x, Y_t, Z_t))_{t\in[0,\infty)}$ is right-continuous and $\mathbb{F}$-progressively measurable for any fixed $x \in \mathbb{R}$. Then, I can define the $\mathbb{F}$-adapted solution as follows:

Definition 19 ($\mathbb{F}$-adapted solution to stochastic ODEs) *For any finite $T > 0$, a stochastic process $X := (X_t)_{t\in[0,T]}$ is an* $\mathbb{F}$-adapted solution *to the stochastic ODE* (B.1) *on $[0, T]$ starting at x_0 if (a) X is continuous $\mathbb{P}$-a.s. and $\mathbb{F}$-adapted; (b) for any $t \in [0, T]$, the following inequality holds $\mathbb{P}$-a.s.:*

$$\Big| \int_0^t F(s, X_s, Y_s, Z_s)\mathrm{d}s \Big| < \infty,$$

and (c) for any $t \in [0, T]$, X satisfies the following $\mathbb{P}$-a.s.:

$$X_t = x_0 + \int_0^t F(s, X_s, Y_s, Z_s)\mathrm{d}s.$$

Then, I obtain the following standard properties:

Proposition 20 *Fix an arbitrary $T > 0$.*

1. (Existence) *Suppose that there exist a constant $C_F > 0$ and a measurable function ϕ :*

$[0,T]\times\mathbb{R}^N\to[0,\infty)$ *such that*

$$|F(t,x,i,z)-F(t,y,i,z)|\leq C_F|x-y|,$$
$$|F(t,x,Y_t,Z_t)|\leq C_F(1+|x|)+\sup_{s\in[0,T]}\phi(s,Z_s)<\infty,$$

for any $(t,x,y,i,z)\in[0,T]\times\mathbb{R}^2\times\mathcal{Y}\times\mathbb{R}^N$, $\mathbb{P}$*-a.s. Then, there exists a unique* $\mathbb{F}$*-adapted solution to the stochastic ODE* (B.1) *on* $[0,T]$ *starting at* x_0.

2. (Uniqueness) *Fix an arbitrary* $w\in\mathbb{R}$ *and let* $\mathcal{N}_w$ *be an arbitrary neighborhood of* w*. Further, suppose that there exists a finitely positive measurable function* $M_w:[0,T]\times\mathbb{R}^N\to(0,\infty)$*, which may depend on* w *and* $\mathcal{N}_w$ *such that*

$$|F(t,x,i,z)-F(t,y,i,z)|\leq M_w(t,z)|x-y|,\quad\text{and}\quad\sup_{s\in[0,T]}M_w(s,Z_s)<\infty,$$

for any $(t,x,y,i,z)\in[0,T]\times\mathcal{N}_w^2\times\mathcal{Y}\times\mathbb{R}^N$, $\mathbb{P}$*-a.s. Then, an* $\mathbb{F}$*-adapted solution to the stochastic ODE* (B.1) *on* $[0,T]$ *starting at* x_0 *is unique up to indistinguishability if it exists.*

The proof of Proposition 20 is obvious by applying the same argument as the deterministic ODEs, so I omit it. In the main text, I consider the three-dimensional stochastic ODE with respect to (A,B,H). However, given (Y,C,L,D,S), A and H are solvable alone, and B's stochastic ODE is one-dimensional given (A,H,Y,C,L,D,S). Therefore, it suffices to consider the one-dimensional stochastic ODE as in the equation (B.1).

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