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Existence of Invariant Measure and Stationary Equilibrium in a Continuous-Time One-Asset Aiyagari Model:

A Case of Regular Controls under Markov Chain Uncertainty

Yuki Shigeta

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Graduate School of Economics
Kyoto University
Yoshida-Hommachi, Sakyo-ku
Kyoto City, 606-8501, Japan

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Existence of Invariant Measure and Stationary Equilibrium in a Continuous-Time One-Asset Aiyagari Model: A Case of Regular Controls under Markov Chain Uncertainty*

Yuki Shigeta†

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Abstract

This paper is concerned with the existence of the invariant measure and stationary equilibrium in a continuous-time Aiyagari model with an endogenous labor supply. First, I demonstrate that the value function, optimal consumption, optimal labor supply, optimal saving rate, and optimally controlled liquid asset process are jointly continuous in parameters such as the interest rate and wage. Second, I show the existence of the ergodic invariant measure of the optimally controlled liquid asset process. Finally, I demonstrate the existence of the stationary equilibrium in a continuous-time one-asset Aiyagari model.

Keywords: Macroeconomic Mean Field Game, Borrowing Constraint, Hamilton–Jacobi–Bellman Equation, Viscosity Solution, Continuity in Parameters, Existence of Invariant Measure

JEL Classification: C62, E21, G11

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†Faculty of Economics, Tokyo Keizai University, 1-7-34 Minami-cho, Kokubunji-shi, Tokyo, Japan. Telephone: +81 42 328 7892, E-mail address: sy46744@gmail.com

1 Introduction

Recent studies on macroeconomic heterogeneous-agent models have employed continuous-time frameworks (e.g., Achdou et al. (2022), Kaplan et al. (2018), Guerrieri et al. (2020), Bornstein (2020), and McKay and Wieland (2021)). This continuous-time approach has been referred to as the mean field game (MFG) (Lasry and Lions (2007)). As discussed in Achdou et al. (2022), the continuous-time approach provides faster and more efficient computation than traditional discrete-time heterogeneous-agent models, which are also known as Bewely–Hugget–Aiyagari models, such as those in Bewley (1986), Huggett (1993), and Aiyagari (1994). Therefore, many macroeconomists have attempted to apply the MFG approach to their models to describe various macroeconomic phenomena.

The existence of the equilibrium of macroeconomic MFGs is an important question in the context of macroeconomic theory. The macroeconomic MFG consists of two core differential equations: the Hamilton–Jacobi–Bellman (HJB) and Kolmogorov forward (KF) equations. First, individual agents in the economy solve their utility maximization problems, the value functions of which are characterized as a solution to the HJB equation. Subsequently, a solution to the KF equation, which expresses the ergodic distribution of the state variables of individuals, is characterized as a cross-sectional distribution of the state variables in the economy. Finally, the equilibrium interest rate, wage, price, and other factors are determined by the cross-sectional distribution of the state variables. The use of the HJB equation to solve various consumer problems can already be mathematically validated owing to the rich literature on optimal controls and mathematical finance. However, less research has examined the cross-sectional distribution of the state variables in continuous-time models and the existence of the equilibrium in the economy. Several issues exist that standard models may naturally satisfy but have not yet been confirmed to demonstrate the existence of the equilibrium, such as the continuity of the asset process in the interest rate and wage. Indeed, no complete proof of these issues is available in the macroeconomic MFG literature.

In this paper, I prove the existence of the equilibrium in a standard one-asset continuous-time Aiyagari model. In my model, there exists a continuum of consumers in the economy who select their consumption and labor supply to maximize their time-additive discounted expected utilities under budget constraints and exogenous borrowing constraints. Further, the consumers are exposed to ex-post heterogeneous idiosyncratic uncertainty, which is represented by a continuous-time Markov chain. Subsequently, the invariant measure of the state variables of the consumers is characterized as the cross-sectional distribution in the economy, and the equilibrium interest rate and wage are determined to satisfy the market clearing condition. I first show the parametric continuities of the value function, optimal controls, and optimally controlled liquid asset process; that is, their continuities in the interest rate, wage, and other parameters. Thereafter, I prove the existence of the unique invariant measure of the state variables. The first step implies that this invariant measure is continuous in the parameters. Hence, I finally demonstrate the existence of the equilibrium in the economy.

In this paper, I first show that the optimal control functions, saving rate, and optimally controlled liquid asset process are continuous in the interest rate, wage, income transfers, consumption tax/subsidy rate, and subjective discount rate (Proposition 4). The continuity of the optimal control functions in these parameters is crucial because the equilibrium parameters in macroeconomic MFGs are determined by the cross-sectional distribution of the optimal policies and assets. Hence, it is necessary to change the parameters to satisfy the market clearing condition, but this operation may fail without continuity in the parameters. To demonstrate the continuity, I apply the so-called doubling-variables method to differential equations. Parametric continuity has seldom been examined, even in mathematical finance studies, as previous works have mainly focused on a partial equilibrium model. Thus, this result contributes to the macroeconomic MFG literature as well as the broader domain.

Second, I show the existence of the unique invariant measure of the liquid asset amount (Proposition 6). This invariant measure is usually numerically computed by the KF equation, but the existence of the equilibrium in macroeconomic MFGs has not been investigated extensively in the literature. Furthermore, based on the result of the parametric continuity, the invariant measure is also continuous in the parameters. Hence, the parameters can be changed gradually for the aggregate demand or supply to satisfy the market clearing condition. As a corollary, I outline the existence of the equilibrium in the one-asset continuous-time Aiyagari model with an endogenous labor supply under several standard assumptions. These two existence results rely on the results of Açıkgöz (2018), in which the author demonstrated the existence of the invariant measure and equilibrium in a discrete-time model. Indeed, I postulate the same assumptions (Assumption 5) as those in Açıkgöz (2018) and many parts of my proof of the two existences can be regarded as a continuous-time version of the proof of Açıkgöz (2018).

One limitation of my existence result is that the utility is restricted to be bounded above. Therefore, my model excludes the log utility or constant relative risk aversion (CRRA) utility, for which the coefficient of relative risk aversion (RRA) is smaller than one. Achdou et al. (2022) showed the uniqueness of the equilibrium in the standard Hugget model in which RRA was smaller than one under the assumption of smoothness of the optimal consumption in the interest rate. Furthermore, a low RRA implies that the substitution effect dominates the income effect. Therefore, these excluded cases are important from the macroeconomic perspective. However, it is well known that in these cases, the value function may blow up if the subjective discount rate is small relative to the interest rate, so that a different treatment to that of the upper bounded utility case is required. Hence, I focus on the upper bounded utility case as the first step.

Some readers may consider that the existence of the equilibrium is trivial because it can be numerically confirmed. However, the continuous-time Aiyagari model presented in this paper is very standard and somewhat pedagogical, and the routine for solving general macroeconomic MFGs is based on the result thereof. In a macroeconomically interesting but complex model, researchers attempt to specify and solve the HJB and KF equations because the standard Aiyagari model derives a plausible equilibrium using this procedure. Indeed, it may be quite difficult, if not impossible, to validate the above routine in the complex model completely. However, the validity of the routine is assured to a certain extent by the result of the standard model. Hence, complete characterizations of the standard model are required as a minimum.

1.1 Literature Review

In the following, I present a brief literature review. In the seminal work by Achdou et al. (2022), a framework of continuous-time macroeconomic MFGs was developed to analyze the income and wealth inequalities more tractably through the heterogeneity of consumers in the economy.

As mentioned previously, many studies, such as Kaplan et al. (2018), Guerrieri et al. (2020), Bornstein (2020), and McKay and Wieland (2021), adopted macroeconomic MFGs and provided rich explanations of various phenomena in a macroeconomy. Achdou et al. (2022) demonstrated the existence and uniqueness of the equilibria in the plain-vanilla Bewley–Aiyagari model under several implicit assumptions. The continuity of the optimally controlled liquid asset process with respect to the interest rate is a very important assumption. By construction, it is difficult to view the process as discontinuous in the interest rate, but a proof of the continuity is not trivial in continuous-time models and no complete proof was provided in Achdou et al. (2022).

Several proofs of the existence of the equilibria have been provided in studies on discrete-time heterogeneous-agent models, such as Kuhn (2013), Acemoglu and Jensen (2015), Açıkgöz (2018), Hu and Shmaya (2019), and Zhu (2020). In particular, Açıkgöz (2018) completely proved the existence of the equilibrium. The author provided the continuity of the optimally controlled asset process by invoking Tychonoff's theorem and Berge's maximum theorem: The author constructed a compact subset of the paths of the asset process in the infinite-dimensional state space using Tychonoff's theorem, and demonstrated that the path of the optimally controlled asset process is unique and in this compact subset. Accordingly, Berge's maximum theorem implies the continuity of the optimally controlled asset process with respect to the interest rate and wage. Although this proof using infinite-dimensional analysis is elegant, the Bellman equation plays no role in the proof.[1] The utility maximization problem as a path choice problem can be converted into a problem to identify the optimal control at each time as a snapshot using the Bellman equation. Hence, it is expected that we can avoid the technical argument of the infinite-dimensional analysis by using the Bellman equation. The HJB equation is a continuous-time counterpart of the Bellman equation. Accordingly, I employ the fact that the value function is a (constrained viscosity) solution to the HJB equation and demonstrate the continuity using the doubling-variables method. Indeed, this proof offers an advantage in simplicity because it only requires standard finite-dimensional calculus.

Regarding continuous-time models, Rocheteau et al. (2018) established the existence of the equilibrium in the monetary equilibrium model with exogenous and heterogeneous timings

[1] As shown by Dutta et al. (1994), the parametric continuity of the value function can be obtained by the Banach fixed point theorem if an admissible set of controls is compact, unlike the feasible set of consumption in usual utility maximization problems.

of lumpy consumption. The model of Rocheteau et al. (2018) is tractable and provides rich economic implications, but the authors assumed that the preference of the consumer is bounded above and below or linear. Furthermore, the authors assumed that the monetary growth rate is a given constant, and hence, they only considered the market clearing in the lump-sum transfers of the aggregate real balance. The continuity of the optimal control in lump-sum transfers is easy to determine because the value function is usually concave therein, whereas the value function may not be concave in the interest rate (i.e., the monetary growth rate in their model). The fact that the value function is usually concave in the transfers is a very useful tool to avoid the continuity issue in the monetary equilibrium model. Indeed, the value function in the discrete-time model in Hu and Shmaya (2019) may not be concave in the price level, but the authors rewrote their model in real terms and discussed the existence and uniqueness of the equilibrium in government transfers.

Bayer et al. (2019) established the existence of the invariant measure in a continuous-time model by demonstrating that the state process is Feller and ergodic. The authors employed the standard theory of stochastic stability of continuous-time stochastic processes by Meyn and Tweedie (1993), with which the discussions in this paper regarding ergodicity overlap. However, Bayer et al. (2019) did not prove the parametric continuity, and hence, they considered a partial equilibrium model.

The stationary equilibrium in this study is an MFG in the economy, in which a continuum of agents lives in an infinite horizon. Meanwhile, the traditional mathematical theory of the MFG has mainly dealt with an equilibrium in a model in which finite agents live in a finite horizon; for example, Lasry and Lions (2007). As discussed in detail in Carmona and Delarue (2018), the forward-backward stochastic differential equation (FBSDE) is a powerful tool for solving the latter MFGs. Indeed, using the FBSDE approach, Carmona and Delarue (2018) showed the existence of the equilibrium in a finite-horizon Aiyagari model with CRRA utility and mean-reverting labor productivities. The FBSDE approach can be regarded as a probabilistic method because the two core equations are stochastic differential equations, whereas the HJB-KF approach can be regarded as analytical because both the HJB and KF equations are deterministic differential equations. The FBSDE approach offers the advantage of easily incorporating common noise, but this is usually a solution method for a finite-horizon model, as mentioned previously. In the study of macroeconomic MFGs, many researchers have adopted the HJB–KF approach under the assumption that the agents in the economy live in an infinite horizon, and the HJB–KF approach is also employed in the current study.

The utility maximization problem in this study is a standard regular control problem, which is a simple extension of the pedagogical model by Achdou et al. (2022) with an endogenous labor supply. This is similar to a traditional consumption-saving problem such as that in Merton (1969), but the problem includes the exogenous borrowing limit and this feature characterizes the problem of the macroeconomic consumer, as emphasized in Achdou et al. (2022). Therefore, as opposed to a problem with a closed-form solution, certain special treatments are required to deal with the problem although it is standard. Shigeta (2022) showed the fundamental properties of the value function, such as the concavity, continuity, and differentiability, and these properties play important roles in the proofs of the propositions in this paper.

The remainder of this paper is organized as follows: Section 2 formulates the consumer utility maximization problem and introduces several basic results as obtained by Shigeta (2022). Section 3 presents a proof of the continuity of the value function, optimal control functions, saving rate function, and optimally controlled liquid asset process in the parameters, such as the interest rate and wage. The existence of the invariant measure is derived in Section 4. As a natural corollary, the existence of the equilibrium of the one-asset Aiyagari model is also derived. Section 5 presents conclusions and a discussion of future extensions. The proofs of the propositions and lemmas in the main text are provided in Appendix A.

2 Preliminaries

Let us first consider a utility maximization problem of a consumer. A consumer plans a consumption plan and labor supply plan in continuous time in an infinite horizon. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space that is endowed with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$. $\mathbb{F}$ satisfies the usual conditions. Furthermore, let $Y := (Y_t)_{t\in[0,\infty)}$ be a continuous-time, finite-state, right-continuous, and $\mathbb{F}$-adapted Markov chain. The finite-state space of Y is denoted by $\mathcal{Y} := \{1, \cdots, K\}$, where $K > 1$ is a finite natural number. Y represents the idiosyncratic uncertainty of the consumers, such as the labor productivity, labor market status, and state of preference. Let $\lambda_{i,j} \geq 0$ be a constant intensity parameter of Y when it changes from state i to state j. Moreover, let $\lambda_{i,i} := -\sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}$.

A consumer has a liquid asset, which is denoted by $B := (B_t)_{t\in[0,\infty)}$. B satisfies the following stochastic ordinary differential equation (ODE):

$$\mathrm{d}B_t = \Big(rB_t + wf(Y_t)L_t + g(Y_t) - \tau_c(Y_t)C_t\Big)\mathrm{d}t, \tag{2.1}$$

where the definitions and interpretations of the functions, parameters, and processes in (2.1) are as follows: $r \in \mathbb{R}$ is a constant interest rate; $w \in [0,\infty)$ is a constant wage; $f : \mathcal{Y} \to [0,\infty)$ is a measurable function that represents the labor productivity of the consumer (for example, a consumer at state $i \in \mathcal{Y}$ does not have any motivation to work if $f(i) = 0$); $L := (L_t)_{t\in[0,\infty)}$ is a labor supply process that takes values in $[0, \overline{L}]$, where $\overline{L} > 0$ is a constant; $g : \mathcal{Y} \to (0,\infty)$ is an income transfer; $\tau_c : \mathcal{Y} \to (0,\infty)$ is a consumption tax/subsidy rate; and $C := (C_t)_{t\in[0,\infty)}$ is a consumption process that takes values in $\mathcal{C} := (0,\infty)$. The consumer selects his or her consumption C and labor supply L to satisfy the following admissibility condition.

Definition 1 (Admissible set with borrowing constraints) *For any $(b,i) \in [0,\infty)\times\mathcal{Y}$, a pair of a consumption process $C = (C_t)_{t\in[0,\infty)}$ and labor supply process $L = (L_t)_{t\in[0,\infty)}$ is* admissible *under an initial condition (b,i) if it satisfies the following: (1) (C,L) is a right-continuous and $\mathbb{F}$-adapted process that takes values in $\mathcal{C}\times[0,\overline{L}]$; (2) for given (C,L), the stochastic ODE (2.1) when starting at $B_0 = b$ and $Y_0 = i$ has an $\mathbb{F}$-adopted solution[2], which is denoted by $B^{b,i;C,L} = (B_t^{b,i;C,L})_{t\in[0,\infty)}$ and $Y^i = (Y_t^i)_{t\in[0,\infty)}$; and (3) $B_t^{b,i;C,L} \geq 0$ for any $t \geq 0$. Let $\mathcal{A}(b,i)$ be a set of all admissible processes under an initial condition $(b,i) \in [0,\infty)\times\mathcal{Y}$.*

According to condition (3) in Definition 1, the exogenous borrowing constraint of the consumer in this study is zero.

The consumer preference can be expressed as a time-separable, discounted, and expected

[2]For an $\mathbb{F}$-progressively measurable function $F : [0,\infty)\times\Omega\times\mathbb{R}\to\mathbb{R}$ and an $\mathcal{F}_0$-measurable real-valued random variable X_0, a stochastic process X is an $\mathbb{F}$-adapted solution to a stochastic ODE $\mathrm{d}X = F(t,\omega,X)\mathrm{d}t$ if it satisfies the following: (1) X is continuous and $\mathbb{F}$-adapted, (2) for an almost sure $\omega\in\Omega$, it satisfies $X_t(\omega) = X_0(\omega) + \int_0^t F(s,\omega,X_s(\omega))\mathrm{d}s$, and (3) $\int_0^t |F(s,\cdot,X_s)|\mathrm{d}s < \infty$, $\mathbb{P}$-a.s., for any $t\in[0,\infty)$.

utility, as follows:

$$U(C,L) := \mathrm{E}\left[\int_0^\infty e^{-\rho t}\Big(u(C_t,L_t,Y_t)+\nu(B_t,Y_t)\Big)\mathrm{d}t\right], \tag{2.2}$$

where $u : \mathcal{C}\times[0,\overline{L}]\times\mathcal{Y}\to\mathbb{R}$ and $\nu : [0,\infty)\times\mathcal{Y}\to\mathbb{R}$ are temporal utility functions, and $\rho\in(0,\infty)$ is a constant subjective discount rate. Subsequently, the utility maximization problem of the consumer is formulated as follows:

$$V_i(b) := \sup_{(C,L)\in\mathcal{A}(b,i)} U(C,L) = \sup_{(C,L)\in\mathcal{A}(b,i)} \mathrm{E}\left[\int_0^\infty e^{-\rho t}\Big(u(C_t,L_t,Y_t^i)+\nu(B_t^{b,i;C,L},Y_t^i)\Big)\mathrm{d}t\right]. \tag{2.3}$$

Thus, V is the value function.

Hereafter, the derivative of a function f with a variable x is written as $\partial_x f$. I now introduce the baseline assumptions in this paper.

Assumption 2

1. *$r\in\mathbb{R}$ and $w\in[0,\infty)$.*
2. *$f:\mathcal{Y}\to[0,\infty)$ is bounded above and below.*
3. *$g:\mathcal{Y}\to(0,\infty)$ is bounded above and away from zero. Let $\underline{g}=\min_{i\in\mathcal{Y}} g(i)>0$.*
4. *$\tau_c:\mathcal{Y}\to(0,\infty)$ is bounded above and away from zero. Let $\overline{\tau}_c=\max_{i\in\mathcal{Y}}\tau_c(i)$.*
5. *For any $i\in\mathcal{Y}$, $(c,l)\to u(c,l,i)$ is finite, bounded above, continuous, strictly increasing in c, non-increasing in l, and strictly concave on $\mathcal{C}\times[0,\overline{L}]$, as well as twice continuously differentiable on $\mathcal{C}\times(0,\overline{L})$. Without loss of generality, I suppose that $\sup_{(c,l,i)\in\mathcal{C}\times[0,\overline{L}]\times\mathcal{Y}} u(c,l,i)=0$. Further, $c\to u(c,l,i)$ is twice continuously differentiable even if $l=\overline{L}$ or 0. For any $(l,i)\in[0,\overline{L}]\times\mathcal{Y}$, $\partial_c u(c,l,i)\to\infty$ as $c\to 0$, and $u(c,0,i)\to 0$ and $\partial_c u(c,l,i)\to 0$ as $c\to\infty$. Moreover, for any $i\in\mathcal{Y}$, the Hessian matrix of u with respect to (c,l) on $\mathcal{C}\times(0,\overline{L})$ is negative definite in the strict sense, and $\partial_{cc}u(c,l,i)<0$ for any $l\in[0,\overline{L}]$. Finally, in the presence of an endogenous labor supply, $(c,l)\to-\partial_l u(c,l,i)/\partial_c u(c,l,i)$ is strictly increasing in each argument for any $i\in\mathcal{Y}$, and $\partial_{cl}u\geq 0$.*
6. *For any $i\in\mathcal{Y}$, $b\to\nu(b,i)$ is bounded above and below, continuous, strictly increasing, and strictly concave on $[0,\infty)$. Without loss of generality, I suppose that $\nu(b,i)\to 0$ as $b\to\infty$ for any $i\in\mathcal{Y}$.*

The upper boundedness of u is a somewhat restrictive condition in Assumption 2. For example, suppose that u is an additive separable iso-elastic utility: $u(c,l,i) = \frac{c^{1-\gamma}}{1-\gamma} - \frac{l^{1+1/\psi}}{1+1/\psi}$, where γ and ψ are positive constants. Thus, according to the fifth condition in Assumption 2, we require $\gamma > 1$. The other conditions with respect to u in Assumption 2 are indeed standard: strict concavity and monotonicity, smoothness, the Inada condition, and the gross substitution between consumption and labor.

Shigeta (2022) proved the following properties of the value function and optimal control:

Proposition 3 (Shigeta (2022)) *The value function V satisfies the following:*

1. *For any $i \in \mathcal{Y}$, $b \to V_i(b)$ is bounded above and below, continuous, strictly increasing, and strictly concave on $[0,\infty)$, as well as continuously differentiable on $(0,\infty)$. Furthermore, the upper boundary of V is 0 and the lower boundary is $\frac{1}{\rho}\min_{j\in\mathcal{Y}}\left\{u\Big(\frac{\underline{g}-\underline{y}}{\overline{\tau}_c},0,j\Big)+\nu(0,j)\right\}$, where $\underline{y}$ is an arbitrary constant in $(0,\underline{g})$. Moreover, the derivative of V with respect to b is bounded above, as follows:*

$$-\frac{1}{\underline{y}}\left\{\min_{j\in\mathcal{Y}}\left\{u\Big(\frac{\underline{g}-\underline{y}}{\overline{\tau}_c},0,j\Big)+\nu(0,j)\right\}+(K-1)\overline{\lambda}\min_{j\in\mathcal{Y}}V_j(0)\right\} \geq \partial_b V_i(b), \tag{2.4}$$

   where $\overline{\lambda} = \max_{i,j\in\mathcal{Y}^2}\lambda_{i,j}$. Finally, for any $i \in \mathcal{Y}$, $V_i(b) \to 0$ as $b \to \infty$.

2. *V is a constrained viscosity solution to the following HJB equation with respect to v:*

$$\rho v_i(b) - \mathcal{H}_i(b,\partial_b v_i(b)) - \sum_{j\in\mathcal{Y}\setminus\{i\}}\lambda_{i,j}\Big(v_j(b)-v_i(b)\Big) = 0, \tag{2.5}$$

   where $\mathcal{H} : \mathcal{Y}\times[0,\infty)\times(0,\infty)\to\mathbb{R}$ is the Hamiltonian such that

$$\mathcal{H}_i(b,p) := \sup_{(c,l)\in\mathcal{C}\times[0,\overline{L}]}\left\{u(c,l,i)+\nu(b,i)+p\Big(rb+wf(i)l+g(i)-\tau_c(i)c\Big)\right\}. \tag{2.6}$$

   Further, $(b,p)\to\mathcal{H}_i(b,p)$ is continuous on $[0,\infty)\times(0,\infty)$ for any $i\in\mathcal{Y}$, and $p\to\mathcal{H}_i(b,p)$ is strictly convex and continuously differentiable on $(0,\infty)$ for any $(b,i)\in[0,\infty)\times\mathcal{Y}$. Moreover, for any $i\in\mathcal{Y}$, $(b,p)\to\partial_p\mathcal{H}_i(b,p)$ is continuous on $[0,\infty)\times(0,\infty)$ and $p\to\partial_p\mathcal{H}_i(b,p)$ has at most two non-differentiable points.

3. *For any $(b,i,p)\in[0,\infty)\times\mathcal{Y}\times(0,\infty)$, there exists a unique continuous solution to the*

maximization problem in the Hamiltonian $\mathcal{H}_i(b,p)$, which is denoted by $(c^*(p,i), l^*(p,i))$. Furthermore, for any $i \in \mathcal{Y}$, $p \to c^*(p,i)$ is non-increasing and $p \to l^*(p,i)$ is non-decreasing. $l^*(p,i) = 0$ or $\overline{L}$ if $p \to \partial_p \mathcal{H}_i(b,p)$ is not differentiable at p. Finally, $(c^*(\partial_b V_i(b), i), l^*(\partial_b V_i(b), i))$ is a unique optimal admissible feedback control.[3]

4. *For any $(b,i) \in [0,\infty) \times \mathcal{Y}$, let $B^{*b,i} := (B_t^{*b,i})_{t\in[0,\infty)}$ be an optimally controlled liquid asset process starting at $B_0^{*b,i} = b$ and $Y_0^i = i$. Then, it exists and is unique up to indistinguishability. Furthermore, it satisfies*

$$0 \le B_t^{*b,i} \le \Big(b + \Big(w \max_{j\in\mathcal{Y}} f(j)\overline{L} + \max_{j\in\mathcal{Y}} g(j)\Big)t\Big)e^{|r|t}, \tag{2.7}$$

$\mathbb{P}$-a.s. for any $t \ge 0$.

The constrained viscosity solution in Proposition 3 is a broader concept of the solution to the partial differential equation (PDE). A continuous function $v_i(b)$ is a viscosity supersolution to (2.5) on $(0,\infty)$ (resp. a viscosity subsolution to (2.5) on $[0,\infty)$) if it satisfies the following: for any $b \in (0,\infty)$ (resp. $b \in [0,\infty)$), let $(\widetilde{b}, i) \to \varphi_i(\widetilde{b})$ be a smooth test function on $[0,\infty) \times \mathcal{Y}$ such that for any $i \in \mathcal{Y}$, it satisfies $0 = v_i(b) - \varphi_i(b) = \min_{b'\in\mathcal{B}}\{v_i(b') - \varphi_i(b')\}$ (resp. $0 = v_i(b) - \varphi_i(b) = \max_{b'\in\mathcal{B}}\{v_i(b') - \varphi_i(b')\}$), where $\mathcal{B}$ is a neighborhood of b. Then, v satisfies the following inequality at b for any $i \in \mathcal{Y}$:

$$\rho v_i(b) - \mathcal{H}_i(b, \partial_b \varphi_i(b)) - \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\Big(v_j(b) - v_i(b)\Big) \ge \text{ (resp. } \le) \ 0.$$

A continuous function v is a constrained viscosity solution to (2.5) if it is a viscosity supersolution to (2.5) on $(0,\infty)$ and a viscosity subsolution to (2.5) on $[0,\infty)$.

The constrained viscosity solution property of the value function yields an important necessary condition for optimality; that is, the state-constraint boundary condition. For any

[3] A pair of two measurable functions $(b,i) \to (c^*(b,i), l^*(b,i))$ is an admissible feedback control if, for any $(b,i) \in [0,\infty) \times \mathcal{Y}$, the following stochastic ODE with respect to B,

$$\mathrm{d}B_t = \Big(rB_t + wf(Y_t^i)l^*(B_t, Y_t^i) + g(Y_t^i) - \tau_c(Y_t^i)c^*(B_t, Y_t^i)\Big)\mathrm{d}t,$$

with the initial condition $B_0 = b$, has an $\mathbb{F}$-adapted solution, and $(c^*(B_t, Y_t^i), l^*(B_t, Y_t^i))_{t\in[0,\infty)} \in \mathcal{A}(b,i)$.

$(b,i) \in [0,\infty) \times \mathcal{Y}$, consider the following deterministic ODE:

$$\begin{aligned} \mathrm{d}b_t^i &= s_b^i(b_t^i)\mathrm{d}t \\ &= \partial_p \mathcal{H}_i(b_t^i, \partial_b V_i(b_t^i))\mathrm{d}t \\ &= \Big(rb_t^i + wf(i)l^*(\partial_b V_i(b_t^i), i) + g(i) - \tau_c(i)c^*(\partial_b V_i(b_t^i), i)\Big)\mathrm{d}t, \end{aligned} \tag{2.8}$$

with $b_0^i = b$. For any $(b,i) \in [0,\infty) \times \mathcal{Y}$, the ODE (2.8) has a global solution. The optimally controlled liquid asset process is a combination of the solution to (2.8) in each state $i \in \mathcal{Y}$. The solution to (2.8) is locally bounded. For any $i \in \mathcal{Y}$, the optimal saving rate function $s_b^i(b) = \partial_p \mathcal{H}_i(b, \partial_b V_i(b)) = rb + wf(i)l^*(\partial_b V_i(b), i) + g(i) - \tau_c(i)c^*(\partial_b V_i(b), i)$ satisfies the *state-constraint boundary condition* at $b = 0$ such that

$$s_b^i(0) = wf(i)l^*(\partial_b^+ V_i(0), i) + g(i) - \tau_c(i)c^*(\partial_b^+ V_i(0), i) \geq 0, \tag{2.9}$$

where $\partial_b^+ V_i(0)$ is the right derivative of V_i at 0.

3 Parametric Continuity

Let us discuss the parametric continuity of the optimal controls and assets. The model parameters to be investigated are the interest rate r, wage w, income transfer g, consumption tax/subsidy rate τ_c, and subjective discount rate ρ. I generically denote a quintuplet of parameters (r, w, g, τ_c, ρ) by θ. To emphasize the dependency of the model on the parameters, I express the value function as V_i^θ, the optimal consumption function as c_θ^*, the optimal labor supply as l_θ^*, and the optimal saving rate as $s_b^i(\cdot;\theta)$. Indeed, V^θ, c_θ^*, and l_θ^* are continuous in the income transfer g and subjective discount rate ρ because V^θ is concave in g and ρ, respectively. In contrast, the continuity in the other parameters is not trivial because V^θ may not be concave therein. However, the existence of the equilibrium in the MFG is based on the parametric continuity. For example, the equilibrium interest rate and wage in the Aiyagari model are determined by the cross-sectional distribution of the assets and labor supplies of the individuals. Thus, at a minimum, the parametric continuity in the interest rate and wage should be demonstrated.

As discussed in the first paragraph of Section 1.1, it is difficult to perceive the value function, optimal controls, and optimal saving rate as discontinuous in the parameters, although the proof of the continuity is not trivial. Indeed, the following proposition provides their parametric continuity.

Proposition 4 (Parametric continuity)

1. *The value function V^θ, optimal consumption control function c^*_θ, optimal labor supply control function l^*_θ, and optimal saving rate function s_b are jointly continuous in the amount of the liquid asset b, interest rate r, wage w, income transfer g, consumption tax/subsidy rate τ_c, and subjective discount rate ρ on $[0,\infty) \times \mathbb{R} \times [0,\infty) \times (0,\infty)^K \times (0,\infty)^K \times (0,\infty)$.*
2. *The optimally controlled liquid asset process is continuous in $(b, r, w, g, \tau_c, \rho)$ on $[0,\infty) \times \mathbb{R} \times [0,\infty) \times (0,\infty)^K \times (0,\infty)^K \times (0,\infty)$ in the following sense: let $B^{*b,i;\theta}$ be an optimally controlled liquid asset process starting at $(b,i) \in [0,\infty) \times \mathcal{Y}$ with parameters $\theta = (r, w, g, \tau_c, \rho) \in \mathbb{R} \times [0,\infty) \times (0,\infty)^K \times (0,\infty)^K \times (0,\infty)$. Then, for any fixed $T \in [0,\infty)$ and $(b,\theta) \in [0,\infty) \times \mathbb{R} \times [0,\infty) \times (0,\infty)^K \times (0,\infty)^K \times (0,\infty)$,*

$$\lim_{(b',\theta') \to (b,\theta)} \sup_{t \in [0,T]} \left\| B_t^{*b',i;\theta'} - B_t^{*b,i;\theta} \right\|^2 = 0,$$

   $\mathbb{P}$-a.s.

Proof of Proposition 4. See Appendix A.1. □

I shall explain how I prove Proposition 4. The proof of Proposition 4 is widely applicable. First, the optimal controls and saving rate are a function of the parameters and derivatives of the value function. It can easily be observed that they are continuous in the parameters and in the derivatives of the value function according to Berge's maximum theorem under the Inada condition. Accordingly, it is necessary to show the parametric continuity of the derivatives of the value function. As the value function is concave and continuously differentiable in the state variable, the derivative of the value function is continuous in the parameters if the value function is also continuous in the parameters (see Theorem 25.7 in Rockafellar (1970)). Hence, it suffices to demonstrate the parametric continuity of the value function. To achieve this, I employ the doubling-variables method. The doubling-variables method is a stylized proof technique for the continuity and uniqueness of the viscosity solution, which is also known as the comparison theorem (e.g., Crandall et al. (1987)). It can be applied to demonstrate the comparison theorem if the Hamiltonian is one-sided locally Lipschitz with respect to the state variables. Now, let us consider the Hamiltonian in this paper. It can easily be observed that the Hamiltonian is one-sided locally Lipschitz with respect to the state variable and the parameters of our interest, such as the interest rate, wage, and consumption tax/subsidy rate. Thus, the Hamiltonian with a parameter of interest θ, which is denoted by $\mathcal{H}_i(b, p; \theta)$, locally satisfies

$$\mathcal{H}_i(b', p; \theta') - \mathcal{H}_i(b, p; \theta) \leq p\Big(|r||b' - b| + C_{\mathcal{K}}|\theta' - \theta|\Big) + o(b' - b) + o(\theta' - \theta) \tag{3.1}$$

for any $(b', b, p, \theta', \theta)$ in a compact subset $\mathcal{K}$, where $C_{\mathcal{K}}$ is a non-negative constant that may depend on $\mathcal{K}$. As indicated in the proof, it is sufficient to satisfy (3.1) locally. Therefore, the parametric continuity of the value function can be shown under the following conditions: the value function is concave and continuously differentiable, vanishes at infinity, and is monotone in a parameter of interest. Furthermore, we also need that the derivative of the value function is locally bounded in the state variables and parameter. Indeed, the parametric continuity in the Markov chain case can be demonstrated by the doubling-variables method under (3.1).

4 Existence of Invariant Measure and Stationary Equilibrium in One-Asset Model

Let us discuss the existence of the invariant measure and equilibrium in the one-asset Aiyagari model. To demonstrate the existence of the invariant measure of $(B^{*;\theta}, Y)$, I further suppose the following:

Assumption 5 (For the existence of the invariant probability measure) *Suppose the following with respect to u, r, ρ, f, g, τ_c, and λ.*

1. *There is no money in the utility and u is state independent: $u(c, l, i) = u(c, l)$ and $\nu(b, i) =$*

0. *Further suppose that, for any constant* $\eta > 0, \xi \in \mathbb{R}$, *and* $\zeta \in \mathbb{R}$, $\partial_c u$ *satisfies*

$$\limsup_{x\to\infty} \frac{\partial_c u(\eta x + \xi, \overline{L})}{\partial_c u(\eta x + \zeta, 0)} \leq 1.$$

In the presence of an endogenous labor supply, further suppose the following: u *satisfies either (1)* $\partial_l u(c', l') \to -\infty$ *as* $(c', l') \to (c, \overline{L})$ *for any* $c \in \mathcal{C}$, *or (2) for any* $c \in \mathcal{C}$, $\lim_{(c',l')\to(c,\overline{L})} \partial_{cl} u(c', l')$ *and* $\lim_{(c',l')\to(c,\overline{L})} \partial_{ll} u(c', l')$ *finitely exist, and* $\lim_{(c',l')\to(c,\overline{L})} \left\{\partial_{cc} u(c', l')\partial_{ll} u(c', l') - (\partial_{cl} u(c', l'))^2\right\} \in (0, \infty)$. *Moreover,* u *satisfies either (3)* $\partial_l u(c', l') \to 0$ *as* $(c', l') \to (c, 0)$ *for any* $c \in \mathcal{C}$, *or (4) for any* $c \in \mathcal{C}$, $\lim_{(c',l')\to(c,0)} \partial_{cl} u(c', l')$ *and* $\lim_{(c',l')\to(c,0)} \partial_{ll} u(c', l')$ *finitely exist, and* $\lim_{(c',l')\to(c,0)} \left\{\partial_{cc} u(c', l')\partial_{ll} u(c', l') - (\partial_{cl} u(c', l'))^2\right\} \in (0, \infty)$.

2. $r < \rho$.
3. $f(1) \leq f(i)$ *and* $g(1) < g(i)$ *for any* $i \in \mathcal{Y}$.
4. *There exists a constant* $\tau_c \in (0, \infty)$ *such that* $\tau_c = \tau_c(i)$ *for any* $i \in \mathcal{Y}$.
5. $\lambda_{i,j} > 0$ *for any* $i, j \in \mathcal{Y}$ *with* $i \neq j$.

Let Θ *be a set of parameters* (r, w, g, τ_c, ρ) *in* $\mathbb{R} \times [0, \infty) \times (0, \infty)^K \times (0, \infty) \times (0, \infty)$, *in which all elements satisfy the above conditions.*

The limsup inequality in the first condition is the same as the consequence arising from Assumption 3 in Açıkgöz (2018). Açıkgöz (2018) demonstrated the compactness of the state space in the discrete-time model using the limsup inequality. Indeed, my proof for the compactness is very similar to that in Açıkgöz (2018). Note that iso-elastic utilities usually satisfy the limsup inequality. The third condition implies that all states are not identical. If not, the invariant measure is degenerate. Furthermore, under this assumption, a consumer in the worst state; that is, state 1, always dissaves the liquid asset, even if $r = \rho$.[4] This logic is similar to that of Açıkgöz (2018): the author argues that a consumer in the worst state has an upside potential for future uncertainty, so he or she dissaves and eventually hits at the borrowing constraint. I emphasize that the strict inequality $g(1) < g(i)$ is necessary even if $r < \rho$. Finally, according to the fifth condition, the Markov chain Y is irreducible and aperiodic.

[4] Even if no clear worst state exists, we can confirm that when $r < \rho$, for almost-everywhere $b \geq 0$, there exists at least one state where the saving rate is negative, because of the monotonicity of the resolvent of Y and strict concavity of the value function.

Let us construct a Feller–Dynkin semigroup supporting $(B^{*;\theta}, Y)$ to derive the invariant measure. Let $\mathcal{B}([0,\infty))$ be the Borel sigma-algebra on $[0,\infty)$. Furthermore, let $(\mathcal{S},\mathcal{G}) := ([0,\infty)\times\mathcal{Y}, \mathcal{B}([0,\infty))\otimes 2^{\mathcal{Y}})$ be a measurable space. For any $t \in [0,\infty)$ and $\theta = (r, w, g, \tau_c, \rho) \in \Theta$, let us consider the following functional P_t^θ:

$$P_t^\theta F(b,i) := \mathrm{E}\left[F(B_t^{*b,i;\theta}, Y_t^i)\right], \quad (b,i) \in \mathcal{S}, \quad F \in C_0(\mathcal{S}),$$

where $C_0(\mathcal{S})$ is the set of all continuous functions on $\mathcal{S}$ that vanish at infinity. Thus, P_t^θ is sub-Markov in the sense that $0 \le P_t^\theta F \le 1$ if $0 \le F \le 1$. Hence, there exists a unique sub-Markov kernel, which is also denoted by P_t^θ, such that $P_t^\theta F(b,i) = \int_{\mathcal{S}} F(\widetilde{b},\widetilde{i}) P_t^\theta((b,i),(\mathrm{d}\widetilde{b},\mathrm{d}\widetilde{i}))$. Meanwhile, according to Proposition 4, $(b,i) \to P_t^\theta F(b,i)$ is bounded and continuous for any $F \in C_b(\mathcal{S})$, where $C_b(\mathcal{S})$ is the set of all bounded and continuous functions on $\mathcal{S}$. Moreover, it is easy to observe that $(P_t^\theta)_{t\in[0,\infty)}$ satisfies the semigroup property. Furthermore, the right continuity of $(B^{*;\theta}, Y)$ implies that $(P_t^\theta)_{t\in[0,\infty)}$ satisfies strong continuity. Hence, $(P_t^\theta)_{t\in[0,\infty)}$ is a Feller–Dynkin semigroup and we can consider a canonical Feller process supporting $(P_t^\theta)_{t\in[0,\infty)}$. Furthermore, this canonical Feller process exhibits a strong Markov property. The discussion in this paragraph is standard and I have referenced Section III.6-9 in Rogers and Williams (2000).

Under Assumption 5, the following result regarding the existence of the invariant probability measure of $(P_t^\theta)_{t\in[0,\infty)}$ on $(\mathcal{S},\mathcal{G})$ can be obtained.

Proposition 6 (Existence of invariant probability measure) *Suppose that Assumption 5. For any $\theta \in \Theta$, there exists a probability measure μ^θ on $(\mathcal{S},\mathcal{G})$ such that*

1. *μ^θ is a unique invariant probability measure of $(P_t^\theta)_{t\in[0,\infty)}$: $\mu^\theta = \mu^\theta P_t^\theta$ for any $t \in [0,\infty)$. Furthermore, $(B^{*;\theta}, Y)$ is exponentially ergodic.*
2. *μ^θ has compact support: there exists a constant $\overline{b}^\theta \in [0,\infty)$ such that $\mu^\theta([0,\overline{b}^\theta]\times\mathcal{Y}) = 1$. Furthermore, the smallest $\overline{b}^\theta$ is locally bounded with respect to θ.*
3. *$\theta \to \mu^\theta$ is continuous on Θ in the weak sense: for any $F \in C_b(\mathcal{S}\times\Theta)$, $\int_{\mathcal{S}} F(b,i,\theta')\mathrm{d}\mu^{\theta'}(b,i) \to \int_{\mathcal{S}} F(b,i,\theta)\mathrm{d}\mu^\theta(b,i)$ as $\theta' \to \theta \in \Theta$.*

Proof of Proposition 6. See Appendix A.2. □

Let us discuss the relationship between μ^θ and an associated Kolmogorov forward equation. According to Lebesgue's decomposition theorem, μ^θ can be decomposed into two parts: an absolutely continuous part with respect to the Lebesgue measure $\widehat{\mu}^\theta$ and a purely atomic part $\widetilde{\mu}^\theta$. As discussed in Achdou et al. (2022), the absolutely continuous part $\widehat{\mu}^\theta$ can be computed numerically using the Kolmogorov forward equation:

$$0 = -\frac{\partial}{\partial b}\Big(s_b^i(b;\theta)g(b,i;\theta)\Big) + \sum_{j\in\mathcal{Y}\setminus\{i\}}\Big(\lambda_{j,i}g(b,j;\theta) - \lambda_{i,j}g(b,i;\theta)\Big), \tag{4.1}$$

where $\sum_{i\in\mathcal{Y}} s_b^i(b;\theta)g(b,i;\theta)$ is a constant. The solution to (4.1) is the density function of the absolutely continuous part of μ^θ: $\widehat{\mu}^\theta(\mathrm{d}b,i) = g(b,i;\theta)\mathrm{d}b$.

The invariant measure can be regarded as a stationary cross-sectional distribution of the state variable $(B^{*;\theta}, Y)$ in the economy. That is owing to the exact law of large numbers of Sun (2006). Suppose that the consumers in the economy are uniformly populated in a unit interval $[0,1]$ and they are indexed by $n \in [0,1]$. Furthermore, suppose that Y of consumers n and m are mutually independent for all $n, m \in [0,1]$ with $n \neq m$. Thus, $(B^{*;\theta}, Y)$ satisfies the conditions of Sun (2006), and hence, the exact law of large numbers yields

$$\int_0^1 B_t^{*\mu;\theta,n}(\omega)\mathrm{d}n = \mathrm{E}[B_t^{*\mu;\theta,n}],$$

$\mathbb{P}$-a.s. for any $t \in [0,\infty)$, where $B^{*\mu;\theta,n}$ is the optimally controlled liquid asset process of consumer n with an initial distribution μ on $(\mathcal{S},\mathcal{G})$. The left-hand side of the above is a cross-sectional average of the optimally controlled liquid asset processes in the economy at time t. Therefore, Proposition 6 implies

$$\lim_{t\to\infty}\int_0^1 B_t^{*\mu;\theta,n}(\omega)\mathrm{d}n = \lim_{t\to\infty}\mathrm{E}[B_t^{*\mu;\theta,n}] = \int_{\mathcal{S}} b\mathrm{d}\mu^\theta(b,i).$$

Thus, $\int_{\mathcal{S}} b\mathrm{d}\mu^\theta(b,i)$ can be regarded as the stationary asset supply in the economy.

Based on Proposition 6, we can show the existence of the equilibrium in the one-asset Aiyagari model as a natural corollary.

Corollary 7 *Suppose that Assumption 5 holds. Let F be a Cobb—Douglas production function* *such that* $F(k,l) = ak^{\alpha}l^{1-\alpha}$, *where* $\alpha \in (0,1)$ *and* $a \in (0,\infty)$ *are constants. Let* $\delta \in (0,\infty)$ *be a constant depreciation rate of capital. Furthermore, suppose that* $f(i) > 0$ *at least in one state* $i \in \mathcal{Y}$ *and that there exists a negative interest rate* $r \in (-\delta, 0)$ *such that*

$$-\frac{\overline{g}}{r\int_{\mathcal{S}} l^*_{\theta(r)}(\partial_b V_i^{\theta(r)}(b), i)\mathrm{d}\mu^{\theta(r)}(b,i)} - \left(1 + \frac{r+\delta}{r}\frac{1-\alpha}{\alpha}\overline{f}\right)\left(\frac{a\alpha}{r+\delta}\right)^{1/(1-\alpha)} < 0, \tag{4.2}$$

where $\theta(r) := (r, w(r), g, \tau_c, \rho) \in \Theta$ *with* $w(r) := (1-\alpha)a(a\alpha/(r+\delta))^{\alpha/(1-\alpha)}$. *Then, there exists* $(r^*, w^*) \in (-\delta, \rho) \times [0,\infty)$ *such that*

$$\begin{cases} r^* = \partial_k F\Big(\int_{\mathcal{S}} b\mathrm{d}\mu^{\theta^*}(b,i), \int_{\mathcal{S}} l^*_{\theta^*}(\partial_b V_i^{\theta^*}(b), i)\mathrm{d}\mu^{\theta^*}(b,i)\Big) - \delta, \\ w^* = \partial_l F\Big(\int_{\mathcal{S}} b\mathrm{d}\mu^{\theta^*}(b,i), \int_{\mathcal{S}} l^*_{\theta^*}(\partial_b V_i^{\theta^*}(b), i)\mathrm{d}\mu^{\theta^*}(b,i)\Big), \end{cases} \tag{4.3}$$

where $\theta^* = (r^*, w^*, g, \tau_c, \rho) \in \Theta$.

Proof of Corollary 7. See Appendix A.3. □

There are two remaining issues in the existence of the continuous-time one-asset Aiyagari equilibrium. The first is the inequality (4.2). In many cases, it can be expected that (4.2) holds, but it is necessary to assume that this is the case. Ideally, (4.2) should be replaced with a more concrete condition regarding the parameters. However, in the exogenous labor supply case with labor supply $\overline{L} > 0$, the left-hand side of (4.2) can be rewritten as

$$-\frac{\overline{g}}{r\overline{L}} - \left(1 + \frac{r+\delta}{r}\frac{1-\alpha}{\alpha}\overline{f}\right)\left(\frac{a\alpha}{r+\delta}\right)^{1/(1-\alpha)}.$$

This is negative when r is close to $-\delta$, and hence, (4.2) is satisfied. The second issue is the uniqueness of the equilibrium, which I have not derived. As discussed in Açıkgöz (2018), multiplicity of the equilibria may occur in the discrete-time model when the utility is CRRA with an RRA being larger than one, and the continuous-time model in this paper includes such a utility. Hence, the continuous-time model may also have multiple equilibria.

5 Concluding Remarks

This study has demonstrated the existence of the invariant measure and equilibria in a continuous-time, one-asset stationary Aiyagari model. The model presented in this paper is a plain-vanilla version of continuous-time macroeconomic heterogeneous-agent models, but it is also one of the most fundamental models. Several theoretical gaps existed in the literature (e.g., parametric continuity). Although it is expected that standard models can naturally satisfy these, it may be nontrivial to close the gaps mathematically owing to the absence of a closed-form model solution. However, the gaps can be closed by employing the mathematical techniques developed in optimal control literature; for example, the doubling-variables method in Proposition 4. As a result, I obtained the existence of the equilibria, which contributes to the literature on macroeconomic MFGs by providing the theoretical validity of the recently developed approach.

Several issues remain. The first is an extension of the idiosyncratic uncertainty of the consumer. Achdou et al. (2022) found that the cross-sectional distribution of the liquid asset has a Pareto tail if the asset process is governed by a Brownian stochastic differential equation. This Pareto-tailed model is very important in policy discussions within the context of studies in wealth inequality. The HJB equation in the Pareto-tailed model is a second-order PDE, and hence, a different treatment is required from the model under Markov chain uncertainty. Two potential gaps exist in validating the Pareto-tailed model: the parametric continuity and the existence of the stationary distribution. For the parametric continuity, it may be possible to apply the doubling-variables method with the Crandall–Ishii lemma to the second-order HJB equation, although I have not yet confirmed this. For the existence of the stationary distribution, the results from a discrete-time model such as that in Benhabib et al. (2015) will be helpful.

The other issue is the upper boundedness of the instantaneous utility u. The model in this study excludes several important unbounded utilities, such as the CRRA utility whose coefficient of RRA being smaller than one. In the utility maximization problem with such a utility, it is well known that there is no solution to the problem in the range of regular controls if the interest rate is larger than the subjective discount rate. Thus, a different method is required to deal with this case. The above two extensions are important topics for future research.

A Proofs

A.1 Proof of Proposition 4

In this subsection, I consider the parametric continuity of the value function and optimal controls in Proposition 4. The continuity of the optimal controls in the income transfer is easy to determine using the standard concavity argument. However, the joint continuity in the interest rate and wage is not trivial because it is expected that the value function is not concave therein. Indeed, it can easily be observed that the value function of the deterministic consumption-saving problem with the natural debt limit and exogenous labor supply is not concave in the interest rate. Therefore, it is necessary to show the continuity therein without exploiting the concavity.

First, I have obtained the following results:

Proposition 8 *1. The value function is jointly concave in the income transfer g and amount of the liquid asset b on $(0,\infty)^K \times [0,\infty)$. Moreover, the value function is concave in the subjective discount rate ρ on $(0,\infty)$. Furthermore, the value function is non-decreasing in g on $(0,\infty)^K$, in the wage w on $[0,\infty)$, and in ρ on $(0,\infty)$. However, the value function is non-increasing in τ_c on $(0,\infty)^K$.*

2. The value function is non-decreasing in the interest rate r on $\mathbb{R}$. Further suppose that the value function is continuous in r on $\mathbb{R}$, in w on $[0,\infty)$, and in τ_c on $(0,\infty)^K$, while the others are fixed. Then, the value function is jointly continuous in $(b, r, w, g, \tau_c, \rho)$ on $[0,\infty) \times \mathbb{R} \times [0,\infty) \times (0,\infty)^K \times (0,\infty)^K \times (0,\infty)$.

3. Suppose that the value function is jointly continuous in (r, w, g, τ_c, ρ) on $\mathbb{R} \times [0,\infty) \times (0,\infty)^K \times (0,\infty)^K \times (0,\infty)$. Then, the derivative of the value function with respect to b, optimal consumption function, optimal labor supply function, and optimal saving rate are jointly continuous in $(b, r, w, g, \tau_c, \rho)$ on $[0,\infty) \times \mathbb{R} \times [0,\infty) \times (0,\infty)^K \times (0,\infty)^K \times (0,\infty)$.

Proof of Proposition 8. The first claim is obvious when applying the standard argument, as in the proof of Shigeta (2022).

In the second claim, the standard approach is applied as follows: let $\mathcal{A}(b, i; r)$ be an admissible set under the initial condition $(b, i) \in [0,\infty) \times \mathcal{Y}$ and interest rate $r \in \mathbb{R}$. For any fixed $(b, i) \in [0, \infty) \times \mathcal{Y}$, let us select $(C, L) \in \mathcal{A}(b, i; r)$ arbitrarily. Further, let $B^{b,i,r;C,L}$ be an $\mathbb{F}$-adopted solution to the stochastic ODE (2.1) starting at $B_0 = b$ and controlled by (C, L) with interest rate r. Then, there exists $B^{b,i,r;C,L}$ owing to the admissibility of (C, L). Meanwhile, let $r' \in \mathbb{R}$ be a constant with $r' \geq r$. Then, there also exists $B^{b,i,r';C,L}$ according to the Lipschitz property of (2.1). Now, let us hypothesize that a time $t_0 \in (0, \infty)$ exists such that $B^{b,i,r';C,L}_{t_0} < B^{b,i,r;C,L}_{t_0}$ and lead to a contradiction. According to the hypothesis, under the initial condition $B^{b,i,r';C,L}_0 = B^{b,i,r;C,L}_0 = b$ and owing to the continuity of Bs, there also exists a time $t_1 \in (0, t_0)$ such that $B^{b,i,r';C,L}_{t_1} = B^{b,i,r;C,L}_{t_1}$ and $B^{b,i,r';C,L}_s < B^{b,i,r;C,L}_s$ for any $s \in (t_1, t_0]$. Then, I obtain

$$
\begin{aligned}
B^{b,i,r';C,L}_{t_2} - B^{b,i,r;C,L}_{t_2} = \int_{t_1}^{t_2} \left(r' B^{b,i,r';C,L}_t - r B^{b,i,r;C,L}_t \right) \mathrm{d}t \\
\geq |r| \int_{t_1}^{t_2} \left(B^{b,i,r';C,L}_t - B^{b,i,r;C,L}_t \right) \mathrm{d}t
\end{aligned}
$$

for any $t_2 \in [t_1, t_0]$. Hence, the Gronwall inequality implies that $B^{b,i,r';C,L}_{t_0} \geq B^{b,i,r;C,L}_{t_0}$, but this is a contradiction. Therefore, I obtain $B^{b,i,r';C,L}_t \geq B^{b,i,r;C,L}_t \geq 0$ for any $t \in [0, \infty)$, and $(C, L) \in \mathcal{A}(b, i; r')$. This result immediately implies the non-decreasing property of the value function with respect to the interest rate. The joint continuity of the value function can be offered by Kruse and Deely (1969) based on the fact that the value function is monotone and continuous with respect to b, r, w, g, τ_c, and ρ, respectively, when the others are fixed.

In the third claim, using the verification theorem in Shigeta (2022), the optimal consumption function and labor supply function with parameters $\theta = (r, w, g, \tau_c, \rho)$ are defined as

$$
(c^*_\theta(p, i), l^*_\theta(p, i)) \in \arg \max_{(c,l) \in \mathcal{C} \times [0, \overline{L}]} \left\{ u(c, l, i) + p\Big(w f(i) l - \tau_c(i) c\Big) \right\}. \tag{A.1}
$$

Note that $p > 0$. The Inada condition of u with respect to c implies that the feasible set of consumptions in the maximization problem (A.1) (i.e., $\mathcal{C}$) can be restricted to a closed and convex set which is a locally continuous correspondence of θ and p. The strict concavity of u and Berge's maximum theorem (Theorem A.16 and A.17 in Acemoglu (2009)) imply that (c^*_θ, l^*_θ) is unique and continuous in θ and p. Furthermore, as the value function is continuous in parameter θ and concave in the state variable b, Theorem 25.7 in Rockafellar (1970) implies that the derivative of the value function with respect to b is continuous in θ, and this continuity is locally uniform in b: for any closed interval in $[0,\infty)$, which is denoted by $\mathcal{B}$, it holds that $\sup_{b\in\mathcal{B}} |\partial_b V_i^{\theta'}(b) - \partial_b V_i^{\theta}(b)| \to 0$ as $\theta' \to \theta$. Therefore, I conclude that $(b,\theta) \to (c_\theta^*(\partial_b V_i^\theta(b), i), l_\theta^*(\partial_b V_i^\theta(b), i))$ is continuous. Hence, the optimal saving rate is also continuous. $\square$

The first claim in Proposition 8 can be offered in a more general model, such as the case of negative borrowing constraints. However, I restrict the model to demonstrate the second claim. According to Proposition 8, it is sufficient to show that the value function is continuous in the interest rate, wage, and consumption tax/subsidy rate when the others are fixed. For this purpose, a technical discussion via the doubling-variables method is necessary, as follows:

Lemma 9 *The value function is continuous in the interest rate r on $\mathbb{R}$, in the wage w on $[0,\infty)$, and in the consumption tax/subsidy rate τ_c on $(0,\infty)^K$ when the others are fixed.*

Proof of Lemma 9. I sequentially present the continuity of the value function in the interest rate r, wage w, and consumption tax/subsidy rate τ_c. I use V^r to denote the value function with interest rate r.

<u>*Step 1.*</u> Let $\mathcal{H}_i(b,p;r)$ be a Hamiltonian with interest rate r. For any $(r,r',p,b,b',i) \in \mathbb{R}^2 \times (0,\infty) \times [0,\infty)^2 \times \mathcal{Y}$, I obtain

$$\mathcal{H}_i(b',p;r') - \mathcal{H}_i(b,p;r) \le |\nu(b',i) - \nu(b,i)| + p\Big(|r'-r|b + |b'-b||r'|\Big). \tag{A.2}$$

The inequality (A.2) plays an important role in this proof, and I derive another useful inequality as well. Let $\mathcal{R}^2 = \{(r,r') \in [\underline{r},\overline{r}]^2 \mid r' > r\}$ with $-\infty < \underline{r} < \overline{r} < \infty$. Let b, b', and b'' be non-negative constants. If $b' \le b \le b''$, I obtain $V_i^{r'}(b') - V_i^r(b'') \le V_i^{r'}(b) - V_i^r(b)$ for any $(r,r') \in \mathcal{R}^2$ and $i \in \mathcal{Y}$, owing to the monotonicity of $b \to V_i^r(b)$. Meanwhile, suppose that $b'' \le b \le b'$, and let $\overline{v}$ be a uniform upper boundary of $2\partial_b V_i^r(b)$ with respect to $(b,i,r) \in [0,\infty) \times \mathcal{Y} \times [\underline{r},\overline{r}]$. Then, $\overline{v}$ is finite according to inequality (2.4). Furthermore, from the concavity of V, I obtain

$$\begin{aligned} V_i^{r'}(b') - V_i^r(b'') &= V_i^{r'}(b') - V_i^{r'}(b'') + V_i^{r'}(b'') - V_i^r(b') + V_i^r(b') - V_i^r(b'') \\ &\le V_i^{r'}(b'') - V_i^r(b') + \overline{v}(b'-b'') \le V_i^{r'}(b) - V_i^r(b) + \overline{v}(b'-b'') \end{aligned}$$

for any $(r,r') \in \mathcal{R}^2$ and $i \in \mathcal{Y}$. In summary, for any $(b',b'') \in [0,\infty)$, an intermediate value b between b' and b'' satisfies

$$V_i^{r'}(b') - V_i^{r}(b'') \leq V_i^{r'}(b) - V_i^{r}(b) + \overline{v}|b' - b''| \tag{A.3}$$

for any $(r,r') \in \mathcal{R}^2$ and $i \in \mathcal{Y}$. The inequality (A.3) is a workhorse in this proof.

<u>*Step 2.*</u> Fix an arbitrarily small constant $\epsilon > 0$. From the limit behavior of V_i and monotonicity of V_i^r with respect to r,

$$0 \leq \lim_{b\to\infty}\{V_i^{r'}(b) - V_i^{r}(b)\} \leq \lim_{b\to\infty}\{V_i^{\overline{r}}(b) - V_i^{\underline{r}}(b)\} = 0$$

for any $(r,r') \in \mathcal{R}^2$ and $i \in \mathcal{Y}$. Thus, I can choose a constant $\overline{b} \in (0,\infty)$ such that $|V_i^{\overline{r}}(b) - V_i^{\underline{r}}(b)| < \epsilon$ for any $(b,i) \in [\overline{b},\infty) \times \mathcal{Y}$. Meanwhile, let us consider the following maximization problem:

$$\max_{b\in[0,\overline{b}]}\{V_i^{r'}(b) - V_i^{r}(b)\}$$

for any $(r,r') \in \mathcal{R}^2$ and $i \in \mathcal{Y}$. The above problem has a maximizer since $b \to V_i^{r'}(b) - V_i^{r}(b)$ is continuous, which is expressed as $b_{r,r'}^{*i} \in [0,\overline{b}]$.

Let us apply the doubling-variables method as in the proof of the comparison theorem of the viscosity solution. For any $(r,r') \in \mathcal{R}^2$, consider the following functions:

$$\begin{aligned}
\Phi_i(b',b'';r,r') &:= V_i^{r'}(b') - V_i^{r}(b'') - \varphi(b',b'';r,r'),\\
\varphi(b',b'';r,r') &:= \left(\frac{b''-b'}{\sqrt{r'-r}} - \frac{\epsilon}{2}\right)^2 - \frac{\epsilon^2}{4} = \frac{b''-b'}{\sqrt{r'-r}}\left(\frac{b''-b'}{\sqrt{r'-r}} - \epsilon\right).
\end{aligned}$$

It is obvious that φ is continuously differentiable with respect to (b',b''). Let $M_i(r,r') := \max_{(b',b'')\in[0,\overline{b}]^2} \Phi_i(b',b'';r,r')$. From the continuity of $\Phi_i(b',b'';r,r')$, there exists a maximizer of M_i, which is denoted by $(b_{r,r'}^i, \widehat{b}_{r,r'}^i) \in [0,\overline{b}]^2$. Meanwhile, for any $(r,r',i) \in \mathcal{R}^2 \times \mathcal{Y}$ and $b \in [0,\overline{b}]$,

$$\begin{aligned}
0 \leq \Phi_i(b,b;r,r') \leq M_i(r,r') &= V_i^{r'}(b_{r,r'}^i) - V_i^{r}(\widehat{b}_{r,r'}^i) - \varphi(b_{r,r'}^i, \widehat{b}_{r,r'}^i; r,r')\\
&\leq V_i^{r'}(b_{r,r'}^i) - V_i^{r}(\widehat{b}_{r,r'}^i) + \frac{\epsilon^2}{4} \leq V_i^{\overline{r}}(\overline{b}) - V_i^{\underline{r}}(0) + \frac{\epsilon^2}{4} < \infty.
\end{aligned}$$

Thus, $\varphi(b^i_{r,r'}, \widehat{b}^i_{r,r'}; r, r')$ is uniformly bounded with respect to $(r, r') \in \mathcal{R}^2$. This implies that $\varphi(b^i_{r,r'}, \widehat{b}^i_{r,r'}; r, r')$ has a convergent subsequence as $r - r' \to 0$. It can also be observed that $b^i_{r,r'} - \widehat{b}^i_{r,r'} \to 0$ as $r' - r \to 0$ in this convergent subsequence. Hereafter, I study this convergent subsequence.

Suppose that $\varphi(b^i_{r,r'}, \widehat{b}^i_{r,r'}; r, r')$ converges to a strictly positive value $\overline{k} > 0$ as $r' - r \to 0$. Then, $\varphi(b^i_{r,r'}, \widehat{b}^i_{r,r'}; r, r') > \overline{k}/2$ holds when $r' - r$ is sufficiently close to zero. Meanwhile, according to (A.3), there exists $\widetilde{b} \in [0, \overline{b}]$ such that $V^{r'}_i(b^i_{r,r'}) - V^r_i(\widehat{b}^i_{r,r'}) \leq V^{r'}_i(\widetilde{b}) - V^r_i(\widetilde{b}) + \overline{k}/4$ when $r' - r$ is sufficiently close to zero. Thus,

$$M_i(r, r') < V^{r'}_i(b^i_{r,r'}) - V^r_i(\widehat{b}^i_{r,r'}) - \frac{\overline{k}}{2} \leq V^{r'}_i(\widetilde{b}) - V^r_i(\widetilde{b}) - \frac{\overline{k}}{4} < \Phi_i(\widetilde{b}, \widetilde{b}; r, r') \leq M_i(r, r').$$

This is a contradiction. Hence, $\varphi(b^i_{r,r'}, \widehat{b}^i_{r,r'}; r, r')$ converges to a non-positive value. This implies that $(\widehat{b}^i_{r,r'} - b^i_{r,r'})/\sqrt{r' - r}$ converges to a value in $[0, \epsilon]$ up to subsequence if necessary. Therefore, $|\widehat{b}^i_{r,r'} - b^i_{r,r'}|/\sqrt{r' - r} \leq 2\epsilon$ when $r' - r$ is sufficiently close to zero.

Suppose that $\widehat{b}^i_{r,r'} = 0$. Then,

$$-\partial_{b''}\varphi(b^i_{r,r'}, 0; r, r') = \frac{2}{\sqrt{r' - r}} \left(\frac{b^i_{r,r'}}{\sqrt{r' - r}} + \frac{\epsilon}{2} \right) \geq \frac{\epsilon}{\sqrt{r' - r}}.$$

Meanwhile, I obtain $\partial^+_b V^r_i(0) \geq -\partial_{b''}\varphi(b^i_{r,r'}, 0; r, r')$ from the first-order condition. However, this is a contradiction when $r' - r \to 0$ since $\partial^+_b V^r_i(0)$ is bounded uniformly to $r \in [\underline{r}, \overline{r}]$. Thus, we can assume $\widehat{b}^i_{r,r'} > 0$ without loss of generality.

<u>*Step 3.*</u> I first consider the case in which $b^i_{r,r'}$ and $\widehat{b}^i_{r,r'}$ do not converge to $\overline{b}$. Subsequently, we can assume that $b^i_{r,r'} < \overline{b}$ and $\widehat{b}^i_{r,r'} < \overline{b}$ by taking a subsequence if necessary. Furthermore, the first-order condition with respect to b'' is satisfied with equality in this case. Thus, $\partial_{b'}\varphi(b^i_{r,r'}, \widehat{b}^i_{r,r'}; r, r') = -\partial_{b''}\varphi(b^i_{r,r'}, \widehat{b}^i_{r,r'}; r, r') = \partial_b V^r_i(\widehat{b}^i_{r,r'}) > 0$. According to Step 2, $b^i_{r,r'} < \overline{b}$ is a local maximizer of $\widetilde{b} \to V^{r'}_i(\widetilde{b}) - \varphi(\widetilde{b}, \widehat{b}^i_{r,r'}; r, r')$ on $[0, \overline{b}]$. Therefore, using the viscosity subsolution property of V_i, I obtain

$$\rho V^{r'}_i(b^i_{r,r'}) - \sum_{j \in \mathcal{Y} \setminus \{i\}} \lambda_{i,j}(V^{r'}_j(b^i_{r,r'}) - V^{r'}_i(b^i_{r,r'})) \leq \mathcal{H}_i(b^i_{r,r'}, \partial_{b'}\varphi(b^i_{r,r'}, \widehat{b}^i_{r,r'}; r, r'); r').$$

Similarly, since $\widehat{b}^i_{r,r'} \in (0, \overline{b})$ is a local minimizer of $\widetilde{b} \to V_i^r(\widetilde{b}) + \varphi(b^i_{r,r'}, \widetilde{b}; r, r')$ on $[0, \overline{b}]$, the viscosity supersolution property of V_i yields

$$\rho V_i^r(\widehat{b}^i_{r,r'}) - \sum_{j \in \mathcal{Y} \setminus \{i\}} \lambda_{i,j}(V_j^r(\widehat{b}^i_{r,r'}) - V_i^r(\widehat{b}^i_{r,r'})) \geq \mathcal{H}_i(\widehat{b}^i_{r,r'}, -\partial_{b''}\varphi(b^i_{r,r'}, \widehat{b}^i_{r,r'}; r, r'); r).$$

Note that $-\partial_{b''}\varphi(b^i_{r,r'}, \widehat{b}^i_{r,r'}; r, r') = \partial_{b'}\varphi(b^i_{r,r'}, \widehat{b}^i_{r,r'}; r, r')$ by definition. Accordingly, from (A.2),

$$\begin{aligned}
&\left(\rho + \sum_{j \in \mathcal{Y} \setminus \{i\}} \lambda_{i,j}\right)\left(V_i^{r'}(b^i_{r,r'}) - V_i^r(\widehat{b}^i_{r,r'})\right) - \sum_{j \in \mathcal{Y} \setminus \{i\}} \lambda_{i,j}\left(V_j^{r'}(b^i_{r,r'}) - V_j^r(\widehat{b}^i_{r,r'})\right) \\
&\leq \mathcal{H}_i(b^i_{r,r'}, \partial_{b'}\varphi(b^i_{r,r'}, \widehat{b}^i_{r,r'}; r, r'); r') - \mathcal{H}_i(\widehat{b}^i_{r,r'}, -\partial_{b''}\varphi(b^i_{r,r'}, \widehat{b}^i_{r,r'}; r, r'); r) \\
&\leq |\nu(b^i_{r,r'}, i) - \nu(\widehat{b}^i_{r,r'}, i)| + \partial_{b'}\varphi(b^i_{r,r'}, \widehat{b}^i_{r,r'}; r, r')\left(|r' - r|b^i_{r,r'} + |b^i_{r,r'} - \widehat{b}^i_{r,r'}||r'|\right) \\
&\leq \left(2\frac{b^i_{r,r'} - \widehat{b}^i_{r,r'}}{r' - r} + \frac{\epsilon}{\sqrt{r' - r}}\right)\left(|r' - r|\overline{b} + |b^i_{r,r'} - \widehat{b}^i_{r,r'}|(|\overline{r}| \vee |\underline{r}|)\right) + |\nu(b^i_{r,r'}, i) - \nu(\widehat{b}^i_{r,r'}, i)| \\
&= 2\frac{|b^i_{r,r'} - \widehat{b}^i_{r,r'}|^2}{r' - r}(|\overline{r}| \vee |\underline{r}|) + \epsilon\left|\frac{b^i_{r,r'} - \widehat{b}^i_{r,r'}}{\sqrt{r' - r}}\right|(|\overline{r}| \vee |\underline{r}|) + 2|b^i_{r,r'} - \widehat{b}^i_{r,r'}|\overline{b} + \epsilon\overline{b}\sqrt{r' - r} + |\nu(b^i_{r,r'}, i) - \nu(\widehat{b}^i_{r,r'}, i)| \\
&\leq 10(|\overline{r}| \vee |\underline{r}|)\epsilon^2 + \epsilon
\end{aligned}$$

when $r' - r$ is sufficiently close to zero. Moreover, according to (A.3), there exists $\widetilde{b}_j \in [0, \overline{b}]$ such that

$$V_j^{r'}(b^i_{r,r'}) - V_j^r(\widehat{b}^i_{r,r'}) \leq V_j^{r'}(\widetilde{b}_j) - V_j^r(\widetilde{b}_j) + \epsilon \leq \max_{b \in [0, \overline{b}]}\{V_j^{r'}(b) - V_j^r(b)\} + \epsilon = V_j^{r'}(b^{*j}_{r,r'}) - V_j^r(b^{*j}_{r,r'}) + \epsilon$$

for any $j \in \mathcal{Y} \setminus \{i\}$ when $r' - r$ is sufficiently close to zero. Meanwhile, I obtain $V_i^{r'}(b^i_{r,r'}) - V_i^r(\widehat{b}^i_{r,r'}) \geq \Phi_i(b^{*i}_{r,r'}, b^{*i}_{r,r'}; r, r') - \epsilon^2/4 = V_i^{r'}(b^{*i}_{r,r'}) - V_i^r(b^{*i}_{r,r'}) - \epsilon^2/4$. Therefore,

$$\begin{aligned}
\left(\rho + \sum_{j \in \mathcal{Y} \setminus \{i\}} \lambda_{i,j}\right)\left(V_i^{r'}(b^{*i}_{r,r'}) - V_i^r(b^{*i}_{r,r'})\right) - \sum_{j \in \mathcal{Y} \setminus \{i\}} \lambda_{i,j}\left(V_j^{r'}(b^{*j}_{r,r'}) - V_j^r(b^{*j}_{r,r'})\right) \\
\leq (\overline{\lambda} + 1)\epsilon + \left(\frac{\overline{\lambda} + \rho}{4} + 10(|\overline{r}| \vee |\underline{r}|)\right)\epsilon^2
\end{aligned}$$

when $r' - r$ is sufficiently close to zero, where $\overline{\lambda} = \max_{i \in \mathcal{Y}} \sum_{j \in \mathcal{Y} \setminus \{i\}} \lambda_{i,j}$. In the case of $b^i_{r,r'} \to \overline{b}$ or $\widehat{b}^i_{r,r'} \to \overline{b}$, the other also converges to $\overline{b}$. From the inequality (A.3) and $V_i^{\overline{r}}(\overline{b}) - V_i^{\underline{r}}(\overline{b}) < \epsilon$, I obtain

$$
\begin{aligned}
V_i^{r'}(b^{*i}_{r,r'}) - V_i^{r}(b^{*i}_{r,r'}) &\le V_i^{r'}(b^{i}_{r,r'}) - V_i^{r}(\widehat{b}^{i}_{r,r'}) + \frac{\epsilon^2}{4} \\
&\le V_i^{\overline{r}}(b^{i}_{r,r'}) - V_i^{\underline{r}}(\widehat{b}^{i}_{r,r'}) + \frac{\epsilon^2}{4} \le V_i^{\overline{r}}(\widetilde{b}^{i}) - V_i^{\underline{r}}(\widetilde{b}^{i}) + \epsilon + \frac{\epsilon^2}{4} \\
&\le V_i^{\overline{r}}(\overline{b}) - V_i^{\underline{r}}(\overline{b}) + 2\epsilon + \frac{\epsilon^2}{4} \le 3\epsilon + \frac{\epsilon^2}{4}
\end{aligned}
$$

if $r' - r$ is close to zero, where $\widetilde{b}^i$ is an intermediate value between $b^i_{r,r'}$ and $\widehat{b}^i_{r,r'}$. Hence,

$$
\begin{aligned}
&\left(\rho + \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\right)\Big(V_i^{r'}(b^{*i}_{r,r'}) - V_i^{r}(b^{*i}_{r,r'})\Big) - \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\Big(V_j^{r'}(b^{*j}_{r,r'}) - V_j^{r}(b^{*j}_{r,r'})\Big) \\
&\qquad \le \left(\rho + \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\right)\Big(3\epsilon + \frac{\epsilon^2}{4}\Big) \le (\rho + \overline{\lambda})\Big(3\epsilon + \frac{\epsilon^2}{4}\Big)
\end{aligned}
$$

when $r' - r$ is sufficiently close to zero. In summary, for any $i \in \mathcal{Y}$, I obtain

$$
\left(\rho + \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\right)\Big(V_i^{r'}(b^{*i}_{r,r'}) - V_i^{r}(b^{*i}_{r,r'})\Big) - \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\Big(V_j^{r'}(b^{*j}_{r,r'}) - V_j^{r}(b^{*j}_{r,r'})\Big) \le \iota(\epsilon) \tag{A.4}
$$

when $r' - r$ is sufficiently close to zero, where $\epsilon \to \iota(\epsilon)$ is a modulus of continuity: a non-negative, non-decreasing, and continuous function such that $\iota(0) = 0$, and it does not depend on r and r'.

<u>*Step 4.*</u> Consider the stochastic process $Z_t^i := e^{-\rho t}\Big(V^{r'}_{Y_t^i}(b^{*Y_t^i}_{r,r'}) - V^{r}_{Y_t^i}(b^{*Y_t^i}_{r,r'})\Big)$. Then, by the generalized Ito formula and (A.4), I have

$$
\begin{aligned}
V_i^{r'}(b^{*i}_{r,r'}) - V_i^{r}(b^{*i}_{r,r'}) &= Z_0^i \\
&= \mathrm{E}[Z_t^i | Y_0^i = i] + \mathrm{E}\Big[\int_0^t e^{-\rho s}\Big\{\Big(\rho(Y_s^i) + \sum_{j\in\mathcal{Y}\setminus\{Y_s^i\}} \lambda_{Y_s^i,j}\Big)\Big(V^{r'}_{Y_s^i}(b^{*Y_s^i}_{r,r'}) - V^{r}_{Y_s^i}(b^{*Y_s^i}_{r,r'})\Big) \\
&\qquad - \sum_{j\in\mathcal{Y}\setminus\{Y_s^i\}} \lambda_{Y_s^i,j}\Big(V_j^{r'}(b^{*j}_{r,r'}) - V_j^{r}(b^{*j}_{r,r'})\Big)\Big\}\mathrm{d}s\Big| Y_0^i = i\Big] \\
&\qquad\quad \le \mathrm{E}[Z_t^i | Y_0^i = i] + \mathrm{E}\Big[\int_0^t e^{-\rho s}\iota(\epsilon)\mathrm{d}s\Big| Y_0^i = i\Big] \le \mathrm{E}[Z_t^i | Y_0^i = i] + \frac{\iota(\epsilon)}{\rho}.
\end{aligned}
$$

Since $V_i^{r'}(b_{r,r'}^{*i})$ and $V_i^r(b_{r,r'}^{*i})$ are bounded with respect to i, taking the limit as $t \to \infty$ yields

$$0 \le V_i^{r'}(b) - V_i^r(b) \le \max_{\widetilde{b} \in [0,\bar{b}]} \{V_i^{r'}(\widetilde{b}) - V_i^r(\widetilde{b})\} = V_i^{r'}(b_{r,r'}^{*i}) - V_i^r(b_{r,r'}^{*i}) \le \frac{\iota(\epsilon)}{\rho}$$

for any $(b,i) \in [0,\bar{b}] \times \mathcal{Y}$ when $r - r'$ is sufficiently close to zero. Recall that $\bar{b}$ can be considered as an arbitrarily large but finite value. Therefore, I conclude $V_i^{r'}(b) - V_i^r(b) \to 0$ as $r' - r \to 0$ for any $(b,i) \in [0,\infty) \times \mathcal{Y}$ by initially taking an arbitrary subsequence of $\mathcal{R}^2$.

Step 5. Consider the continuity in the wage w. Let $\mathcal{H}_i(b,p;w)$ be a Hamiltonian with wage w. Then,

$$\mathcal{H}_i(b',p;w') - \mathcal{H}_i(b,p;w) \le |\nu(b',i) - \nu(b,i)| + p\Big(|w'-w|f(i)\overline{L} + |b'-b||r|\Big).$$

Thus, steps 1 to 4 can be applied by changing $r' - r$ to $w' - w$ with $w' > w$. Therefore, the value function is continuous in wage w when the others are fixed.

Step 6. Finally, consider the continuity in the consumption tax/subsidy rate τ_c. In this case, I suppose that τ_c differs only in one state $i \in \mathcal{Y}$ and it is identical in the other states. Let $\mathcal{T}_c^2 := \{(\tau_c(i), \tau_c'(i)) \in [\underline{\tau}_c^i, \overline{\tau}_c^i]^2 | \tau_c'(i) > \tau_c(i)\}$ with $0 < \underline{\tau}_c^i < \overline{\tau}_c^i < \infty$. One issue is that the Hamiltonian does not always satisfy the inequality (A.2) for different τ_c' and τ_c. Thus, suppose $p \ge \underline{p}$ for a positive constant $\underline{p}$. On the first-order condition of the Hamiltonian with respect to consumption, I obtain $\partial_c u(c,l,i) - \tau_c p \le \partial_c u(c,\overline{L},i) - \underline{\tau}_c^i \underline{p}$ for any $(c,l) \in \mathcal{C} \times [0,\overline{L}]$. Therefore, the Inada condition with respect to c implies that there exists $\bar{c} > 0$ such that any $c > \bar{c}$ is never optimal in the problem (A.1) for any $p \ge \underline{p}$ and $\tau_c(i) \ge \underline{\tau}_c^i$. Thus, the Hamiltonian can be rewritten such that

$$\mathcal{H}_i(p,b;\tau_c(i)) = \sup_{(c,l) \in (0,\bar{c}] \times [0,\overline{L}]} \Big\{ u(c,l,i) + \nu(b,i) + p\Big(rb + wf(i)l + g(i) - \tau_c(i)c\Big) \Big\}$$

for any $p \ge \underline{p}$ and $\tau_c(i) \in [\underline{\tau}_c^i, \overline{\tau}_c^i]$. Hence, for any $(\tau_c(i), \tau_c'(i), p, b, b') \in \mathcal{T}_c^2 \times [\underline{p},\infty) \times [0,\infty)^2$,

$$\mathcal{H}_i(b',p;\tau_c(i)) - \mathcal{H}_i(b,p;\tau_c'(i)) \le |\nu(b',i) - \nu(b,i)| + p\Big(|\tau_c'(i) - \tau_c(i)|\bar{c} + |b'-b||r|\Big). \tag{A.5}$$

At a state $j \in \mathcal{Y} \setminus \{i\}$, as τ_c is identical at j, I obtain

$$\mathcal{H}_j(b', p; \tau_c(j)) - \mathcal{H}_j(b, p; \tau_c'(j)) \leq |\nu(b', j) - \nu(b, j)| + p|b' - b||r|. \tag{A.6}$$

The other discussions in step 1 remain valid because $\partial_b^+ V_i^{\tau_c}(0)$ is locally bounded with respect to τ_c. I fix $\tau_c(i) \in [\underline{\tau}_c^i, \overline{\tau}_c^i)$ and suppose, without loss of generality, that the constant $\underline{p}$ is not larger than $\partial_b V_i^{\tau_c}(\bar{b})$, where $\bar{b} > 0$ is a finite constant defined in step 2. Thus, $\partial_b V_i^{\tau_c}(b) \geq \underline{p}$ for any $b \in [0, \bar{b}]$. Meanwhile, for any $(b', b'', \tau_c(i), \tau_c'(i)) \in [0, \bar{b}]^2 \times \mathcal{T}_c^2$, define

$$\varphi(b', b''; \tau_c(i), \tau_c'(i)) := \left(\frac{b'' - b'}{\sqrt{\tau_c'(i) - \tau_c(i)}} - \frac{\epsilon}{2} \right)^2 - \frac{\epsilon^2}{4} = \frac{b'' - b'}{\sqrt{\tau_c'(i) - \tau_c(i)}} \left(\frac{b'' - b'}{\sqrt{\tau_c'(i) - \tau_c(i)}} - \epsilon \right).$$

Then, all discussions in step 2 remain valid since they do not depend on the Hamiltonian inequality (A.2).

Let $(b^i_{\tau_c, \tau_c'}, \widehat{b}^i_{\tau_c, \tau_c'})$ be a maximizer of $V_i^{\tau_c}(b') - V_i^{\tau_c'}(b'') - \varphi(b', b''; \tau_c(i), \tau_c'(i))$ on $[0, \bar{b}]^2$. I suppose that $b^i_{\tau_c, \tau_c'}$ and $\widehat{b}^i_{\tau_c, \tau_c'}$ do not converge to $\bar{b}$ as $\tau_c'(i) \downarrow \tau_c(i)$. Then, owing to the first-order condition,

$$\partial_b V_i^{\tau_c'}(\widehat{b}^i_{\tau_c, \tau_c'}) = -\partial_{b''} \varphi(b^i_{\tau_c, \tau_c'}, \widehat{b}^i_{\tau_c, \tau_c'}; \tau_c(i), \tau_c'(i)) = \partial_{b'} \varphi(b^i_{\tau_c, \tau_c'}, \widehat{b}^i_{\tau_c, \tau_c'}; \tau_c(i), \tau_c'(i)) \geq \partial_b V_i^{\tau_c}(b^i_{\tau_c, \tau_c'}) \geq \underline{p}.$$

Thus, the inequality (A.5) can be used, and as a result, I obtain (A.4). In the case of $b^i_{\tau_c, \tau_c'}, \widehat{b}^i_{\tau_c, \tau_c'} \to \bar{b}$, I also obtain (A.4) without using (A.5). At state $j \in \mathcal{Y} \setminus \{i\}$, I also obtain (A.4) using (A.6). Therefore, according to the argument in step 4, $\tau_c(i) \to V_j^{\tau_c}(b)$ is right-continuous for any $(b, j) \in [0, \infty) \times \mathcal{Y}$. Similarly, it can be shown that $\tau_c(i) \to V_j^{\tau_c}(b)$ is left-continuous for any $(b, j) \in [0, \infty) \times \mathcal{Y}$ by setting $\underline{p} \leq \partial_b V_i^{\tau_c'}(\bar{b})$. Hence, $\tau_c(i) \to V_j^{\tau_c}(b)$ is continuous for any $(b, j) \in [0, \infty) \times \mathcal{Y}$. Finally, the result of Kruse and Deely (1969) demonstrates that the value function is continuous in τ_c on $(0, \infty)^K$ when the others are fixed. □

It can easily be observed that the assumption that the exogenous borrowing constraint is zero in Lemma 9 can be removed if the value function is non-decreasing in r. Furthermore, the proof of Lemma 9 can be extended to the continuity in another parameter and the uniqueness of the solution to the HJB equation in the constraint viscosity sense in a class of continuous functions that vanish at infinity. The second claim in Proposition 4 can be proven as follows:

Proof of second claim in Proposition 4. Let $\mathcal{K}$ be a compact subset of $\mathbb{R} \times [0,\infty) \times (0,\infty)^K \times (0,\infty)^K \times (0,\infty)$. Let θ and θ' be parameters in $\mathcal{K}$. Let $b, b' \in [0,\infty)$ be initial values of B^*. Without loss of generality, b and b' are in the closed interval $[0,\bar{b}] \subset [0,\infty)$. For any $t \in [0,T]$ and $i \in \mathcal{Y}$,

$$\|B_t^{*b',i;\theta'} - B_t^{*b,i;\theta}\|^2 = \|b'-b\|^2 + 2\int_0^t \left(B_s^{*b',i;\theta'} - B_s^{*b,i;\theta}\right)\left(s_b^{Y_s^i}(B_s^{*b',i;\theta'};\theta') - s_b^{Y_s^i}(B_s^{*b,i;\theta};\theta)\right)\mathrm{d}s.$$

In this case, I obtain

$$\begin{aligned}
&\left(B_s^{*b',i;\theta'} - B_s^{*b,i;\theta}\right)\left(s_b^{Y_s^i}(B_s^{*b',i;\theta'};\theta') - s_b^{Y_s^i}(B_s^{*b,i;\theta};\theta)\right) \\
&= \left(B_s^{*b',i;\theta'} - B_s^{*b,i;\theta}\right)\Big\{(r'-r)B_s^{*b',i;\theta'} + g'(Y_s^i) - g(Y_s^i) + (w'-w)f(Y_s^i)l_{\theta'}^*(\partial_b V_{Y_s^i}^{\theta'}(B_s^{*b',i;\theta'}), Y_s^i) \\
&\qquad\qquad - (\tau_c'(Y_t^i) - \tau_c(Y_t^i))c_{\theta'}^*(\partial_b V_{Y_s^i}^{\theta'}(B_s^{*b',i;\theta'}), Y_s^i)\Big\} \\
&\quad + wf(Y_s^i)\left(B_s^{*b',i;\theta'} - B_s^{*b,i;\theta}\right)\left\{l_{\theta'}^*(\partial_b V_{Y_s^i}^{\theta'}(B_s^{*b',i;\theta'}), Y_s^i) - l_\theta^*(\partial_b V_{Y_s^i}^{\theta}(B_s^{*b,i;\theta}), Y_s^i)\right\} \\
&\quad - \tau_c(Y_t^i)\left(B_s^{*b',i;\theta'} - B_s^{*b,i;\theta}\right)\left\{c_{\theta'}^*(\partial_b V_{Y_s^i}^{\theta'}(B_s^{*b',i;\theta'}), Y_s^i) - c_\theta^*(\partial_b V_{Y_s^i}^{\theta}(B_s^{*b,i;\theta}), Y_s^i)\right\} \\
&\quad + r\|B_s^{*b',i;\theta'} - B_s^{*b,i;\theta}\|^2
\end{aligned} \tag{A.7}$$

for any $s \in [0,t)$. According to the inequality (2.7), both $B_s^{*b',i;\theta'}$ and $B_s^{*b,i;\theta}$ are bounded on $[0,T]$. Furthermore, the upper boundary is locally bounded in the parameters. Hence, the upper boundary of $B_t^{*b,i;\theta}$ exists for any $(b,t,i,\theta) \in [0,\bar{b}] \times [0,T] \times \mathcal{Y} \times \mathcal{K}$, which is denoted by $\overline{B} < \infty$. Similarly, there exists a finite constant $\overline{C}$ such that $c_\theta^*(\partial_b V_{Y_s^i}^{\theta'}(B_s^{*b',i;\theta'}), Y_s^i) \leq \overline{C}$ for any $(b,t,i,\theta) \in [0,\bar{b}] \times [0,T] \times \mathcal{Y} \times \mathcal{K}$.

The second and third lines of (A.7) satisfy

$$\begin{aligned}
&\left(B_s^{*b',i;\theta'} - B_s^{*b,i;\theta}\right)\Big\{(r'-r)B_s^{*b',i;\theta'} + g'(Y_s^i) - g(Y_s^i) + (w'-w)f(Y_s^i)l_{\theta'}^*(\partial_b V_{Y_s^i}^{\theta'}(B_s^{*b',i;\theta'}), Y_s^i) \\
&\qquad\qquad - (\tau_c'(Y_t^i) - \tau_c(Y_t^i))c_{\theta'}^*(\partial_b V_{Y_s^i}^{\theta'}(B_s^{*b',i;\theta'}), Y_s^i)\Big\} \\
&\le |B_s^{*b',i;\theta'} - B_s^{*b,i;\theta}| \left\{|r'-r|\overline{B} + |g'(Y_s^i) - g(Y_s^i)| + |w'-w|\overline{f}\overline{L} + |\tau_c'(Y_t^i) - \tau_c(Y_t^i)|\overline{C}\right\} \\
&\le 2\|B_s^{*b',i;\theta'} - B_s^{*b,i;\theta}\|^2 + \frac{1}{2}\Big\{|r'-r|^2\overline{B}^2 + |g'(Y_s^i) - g(Y_s^i)|^2 + |w'-w|^2\overline{f}^2\overline{L}^2 \\
&\qquad\qquad + |\tau_c'(Y_t^i) - \tau_c(Y_t^i)|^2\overline{C}^2\Big\} \\
&\le 2\|B_s^{*b',i;\theta'} - B_s^{*b,i;\theta}\|^2 + \frac{1}{2}\left\{|r'-r|^2\overline{B}^2 + \|g'-g\|^2 + |w'-w|^2\overline{f}^2\overline{L}^2 + \|\tau_c' - \tau_c\|^2\overline{C}^2\right\}. \qquad \text{(A.8)}
\end{aligned}$$

In the fourth line of (A.7),

$$\begin{aligned}
&wf(Y_s^i)\left(B_s^{*b',i;\theta'} - B_s^{*b,i;\theta}\right)\left\{l_{\theta'}^*(\partial_b V_{Y_s^i}^{\theta'}(B_s^{*b',i;\theta'}), Y_s^i) - l_\theta^*(\partial_b V_{Y_s^i}^{\theta}(B_s^{*b,i;\theta}), Y_s^i)\right\} \\
&= wf(Y_s^i)\left(B_s^{*b',i;\theta'} - B_s^{*b,i;\theta}\right)\left\{l_{\theta'}^*(\partial_b V_{Y_s^i}^{\theta'}(B_s^{*b',i;\theta'}), Y_s^i) - l_{\theta'}^*(\partial_b V_{Y_s^i}^{\theta'}(B_s^{*b,i;\theta}), Y_s^i)\right\} \\
&+ wf(Y_s^i)\left(B_s^{*b',i;\theta'} - B_s^{*b,i;\theta}\right)\left\{l_{\theta'}^*(\partial_b V_{Y_s^i}^{\theta'}(B_s^{*b,i;\theta}), Y_s^i) - l_\theta^*(\partial_b V_{Y_s^i}^{\theta}(B_s^{*b,i;\theta}).Y_s^i)\right\} \qquad \text{(A.9)}
\end{aligned}$$

Since $l_{\theta'}^*(p,i)$ is non-decreasing in p and $\partial_b V_{Y_s^i}^{\theta'}(b)$ is non-increasing in b, the second line in (A.9) is non-positive. In the third line in (A.9), the third claim in Proposition 8 implies that $\theta \to \partial_b V_j^\theta(\widetilde{b})$ is continuous, uniformly to $(\widetilde{b}, j) \in [0, \overline{B}] \times \mathcal{Y}$. Therefore, if θ' is close to θ, for any $j \in \mathcal{Y}$, the distance between $\partial_b V_j^{\theta'}(B_s^{*b,i;\theta})$ and $\partial_b V_j^{\theta}(B_s^{*b,i;\theta})$ is close to zero, uniformly to $s \in [0,T]$. Since $(\theta, p) \to l_\theta^*(p,i)$ is continuous, and hence, is uniformly continuous on a compact set, $l_{\theta'}^*(\partial_b V_{Y_s^i}^{\theta'}(B_s^{*b,i;\theta}), Y_s^i) - l_\theta^*(\partial_b V_{Y_s^i}^{\theta}(B_s^{*b,i;\theta}), Y_s^i)$ becomes close to zero if θ' is sufficiently close to θ. Thus, for any small $\epsilon > 0$, (A.9) satisfies

$$\begin{aligned}
&wf(Y_s^i)\left(B_s^{*b',i;\theta'} - B_s^{*b,i;\theta}\right)\left\{l_{\theta'}^*(\partial_b V_{Y_s^i}^{\theta'}(B_s^{*b',i;\theta'}), Y_s^i) - l_\theta^*(\partial_b V_{Y_s^i}^{\theta}(B_s^{*b,i;\theta}), Y_s^i)\right\} \\
&\le \frac{1}{2}\|B_s^{*b',i;\theta'} - B_s^{*b,i;\theta}\|^2 + \frac{\overline{w}^2\overline{f}^2}{2}\epsilon^2, \qquad \text{(A.10)}
\end{aligned}$$

where $\overline{w} := \max_{\theta \in \mathcal{K}} w < \infty$ if θ' is sufficiently close to θ. Similarly, the fifth line of (A.7) satisfies

$$
\begin{aligned}
&-\tau_c(Y_t^i)\left(B_s^{*b',i;\theta'}-B_s^{*b,i;\theta}\right)\left\{c^*_{\theta'}(\partial_b V^{\theta'}_{Y_s^i}(B_s^{*b',i;\theta'}),Y_s^i)-c^*_{\theta}(\partial_b V^{\theta}_{Y_s^i}(B_s^{*b,i;\theta}),Y_s^i)\right\}\\
&\leq \frac{1}{2}\|B_s^{*b',i;\theta'}-B_s^{*b,i;\theta}\|^2+\frac{\overline{\tau}_c^2}{2}\epsilon^2,
\end{aligned}
\tag{A.11}
$$

where $\overline{\tau}_c=\max_{\theta\in\mathcal{K}}\max_{i\in\mathcal{Y}}\tau_c(i)<\infty$ if θ' is sufficiently close to θ. Let $\mathcal{N}_\epsilon(\theta)$ be a neighborhood of θ on $\mathcal{K}$ in which all $\theta'\in\mathcal{N}_\epsilon(\theta)$ satisfy (A.10) and (A.11). By substituting (A.8), (A.10), and (A.11) into (A.7), I obtain

$$
\begin{aligned}
&\left(B_s^{*b',i;\theta'}-B_s^{*b,i;\theta}\right)\left(s_b^{Y_s^i}(B_s^{*b',i;\theta'};\theta')-s_b^{Y_s^i}(B_s^{*b,i;\theta};\theta)\right)\\
&\leq(3+\overline{r})\|B_s^{*b',i;\theta'}-B_s^{*b,i;\theta}\|^2\\
&\quad+\frac{1}{2}\left\{|r'-r|^2\overline{B}^2+\|g'-g\|^2+|w'-w|^2\overline{f}^2\overline{L}^2+\|\tau_c'-\tau_c\|^2\overline{C}^2+(\overline{\tau}_c^2+\overline{w}^2\overline{f}^2)\epsilon^2\right\},
\end{aligned}
\tag{A.12}
$$

where $\overline{r}:=\max_{\theta\in\mathcal{K}}r\vee 0<\infty$ for any $(s,\theta')\in[0,T]\times\mathcal{N}_\epsilon(\theta)$. Hence, I obtain

$$
\begin{aligned}
&\|B_t^{*b',i;\theta'}-B_t^{*b,i;\theta}\|^2\\
&\leq\|b'-b\|^2+\left(|r'-r|^2\overline{B}^2+\|g'-g\|^2+|w'-w|^2\overline{f}^2\overline{L}^2+\|\tau_c'-\tau_c\|^2\overline{C}^2+(\overline{\tau}_c^2+\overline{w}^2\overline{f}^2)\epsilon^2\right)t\\
&\quad+(6+2\overline{r})\int_0^t\|B_s^{*b',i;\theta'}-B_s^{*b,i;\theta}\|^2\mathrm{d}s
\end{aligned}
$$

for any $(t,\theta')\in[0,T]\times\mathcal{N}_\epsilon(\theta)$. Therefore, the Gronwall inequality yields the following inequality:

$$
\begin{aligned}
\|B_t^{*b',i;\theta'}-B_t^{*b,i;\theta}\|^2&\leq\|b'-b\|^2e^{(6+2\overline{r})t}\\
&\quad+\left(|r'-r|^2\overline{B}^2+\|g'-g\|^2+|w'-w|^2\overline{f}^2\overline{L}^2+\|\tau_c'-\tau_c\|^2\overline{C}^2+(\overline{\tau}_c^2+\overline{w}^2\overline{f}^2)\epsilon^2\right)te^{(6+2\overline{r})t}
\end{aligned}
$$

for any $(t,\theta')\in[0,T]\times\mathcal{N}_\epsilon(\theta)$. Consequently, taking the supremum, I obtain

$$
\begin{aligned}
\sup_{t\in[0,T]}\|B_t^{*b',i;\theta'}-B_t^{*b,i;\theta}\|^2&\leq\|b'-b\|^2e^{(6+2\overline{r})T}\\
&\quad+\left(|r'-r|^2\overline{B}^2+\|g'-g\|^2+|w'-w|^2\overline{f}^2\overline{L}^2+\|\tau_c'-\tau_c\|^2\overline{C}^2+(\overline{\tau}_c^2+\overline{w}^2\overline{f}^2)\epsilon^2\right)Te^{(6+2\overline{r})T},
\end{aligned}
\tag{A.13}
$$

for any $\theta' \in \mathcal{N}_\epsilon(\theta)$. The right-hand side of (A.13) can become arbitrarily small if $(b', \theta') \to (b, \theta)$. Hence, the desired result is obtained. □

A.2 Proof of Proposition 6

This subsection demonstrates Proposition 6. First, the following lemma is required:

Lemma 10 *Suppose Assumption 5. V, s_b, and b^1 satisfy the following for any $\theta \in \Theta$:*

1. *For any $i \in \mathcal{Y}$, $p \to G_i(p;\theta) := wf(i)l^*_\theta(p,i) - \tau_c c^*_\theta(p,i)$ has a strictly positive and left-continuous left derivative on $(0,\infty)$. Furthermore, it has a strictly positive and right-continuous right derivative on $(0,\infty)$.*
2. *For any $i \in \mathcal{Y}$, $b \to V_i^\theta(b)$ is twice continuously differentiable on $(0,\infty)$ if $s_b^i(b;\theta) \neq 0$.*
3. *$s_b^1(b;\theta) < 0$ for any $b \in (0,\infty)$.*
4. *$(b_t^1)_{t\in[0,\infty)}$ reaches zero in finite time whenever it starts everywhere on $[0,\infty)$.*
5. *There exists a sufficiently large but finite $\overline{b} \in (0,\infty)$ such that $s_b^i(b;\theta) < 0$ for any $(b,i) \in (\overline{b},\infty) \times \mathcal{Y}$.*

Furthermore, the first, second, third, and fourth claims hold even if $r = \rho$.

Particularly, the claims 4 and 5 in Lemma 10 are required to show Proposition 6, and the claims 1, 2, and 3 are required to show the claims 4 and 5.

Proof of Lemma 10.

First claim. Here, I show the existence of the strictly positive and left-continuous left derivative of G_i. First, there exist at most two non-differentiable points of G_i.[5] Let $\underline{p}$ and $\overline{p}$ be these non-differentiable points with $0 \leq \underline{p} < \overline{p} \leq \infty$. At the differentiable points, $\partial_p G_i > 0$ according to the strict convexity of the Hamiltonian. Next, I demonstrate that the left derivative of G_i at $\underline{p}$ exists and is left-continuous if $\underline{p} > 0$. Suppose that $p \leq \underline{p}$. Then, $l^*(p,i) = 0$. Therefore,

$$\lim_{p\uparrow\underline{p}} \frac{G_i(p;\theta) - G_i(\underline{p};\theta)}{p - \underline{p}} = -\tau_c \lim_{p\uparrow\underline{p}} \frac{c^*_\theta(p,i) - c^*_\theta(\underline{p},i)}{p - \underline{p}} = -\frac{\tau_c^2}{\partial_{cc}u(c^*_\theta(\underline{p},i),0)} > 0.$$

[5]This is because $G_i(p;\theta) = \partial_p \mathcal{H}_i(b,p;\theta) - rb - g(i)$ and $p \to \partial_p \mathcal{H}_i(b,p;\theta)$ has at most two non-differentiable points (see Proposition 3).

This left derivative is obviously left-continuous. Third, consider the left derivative of G_i at $\overline{p}$. If $\lim_{(c',l')\to(c,\overline{L})} \partial_l u(c',l') = -\infty$, then $\overline{p} = \infty$. In this case, the left derivative of G_i exists and is left-continuous on $(0,\infty)$. Suppose the second case: $\limsup_{(c',l')\to(c,\overline{L})} \partial_l u(c',l') < -\infty$, $\lim_{(c',l')\to(c,\overline{L})} \partial_{cl} u(c',l')$ and $\lim_{(c',l')\to(c,\overline{L})} \partial_{ll} u(c',l')$ finitely exist, and $\lim_{(c',l')\to(c,\overline{L})} (\partial_{cc} u(c',l')\partial_{ll} u(c',l') - (\partial_{cl} u(c',l'))^2) \in (0,\infty)$. According to the first-order condition of the optimality in the Hamiltonian (i.e., $\partial_c u(c^*_\theta, l^*_\theta) - p\tau_c = 0$ and $\partial_l u(c^*_\theta, l^*_\theta) + pwf(i) = 0$) and the implicit function theorem, I obtain

$$\begin{aligned}\partial_p G_i(p;\theta) &= -\begin{pmatrix} \tau_c \\ -wf(i) \end{pmatrix} \cdot \begin{pmatrix} \partial_p c^*_\theta(p,i) \\ \partial_p l^*_\theta(p,i) \end{pmatrix} \\ &= -\begin{pmatrix} \tau_c \\ -wf(i) \end{pmatrix} \cdot \begin{pmatrix} \partial_{cc} u(c^*_\theta, l^*_\theta) & \partial_{cl} u(c^*_\theta, l^*_\theta) \\ \partial_{cl} u(c^*_\theta, l^*_\theta) & \partial_{ll} u(c^*_\theta, l^*_\theta) \end{pmatrix}^{-1} \begin{pmatrix} \tau_c \\ -wf(i) \end{pmatrix} > 0\end{aligned}$$

for any p in a neighborhood of $\overline{p}$ with $p < \overline{p}$. Since $l^*_\theta(p,i) \to \overline{L}$ as $p \to \overline{p}$, the second case assumptions imply that the strictly positive left limit of $\partial_p G_i(p;\theta)$ at $\overline{p}$ finitely exists. Therefore, the left-continuous left derivative of $G_i(p;\theta)$ at $\overline{p}$ exists based on the mean value theorem. The existence of the strictly positive and right-continuous right derivative of G_i can be similarly demonstrated.

<u>*Second claim.*</u> For any $i \in \mathcal{Y}$, $b \to V^\theta_i(b)$ is twice differentiable almost everywhere on $(0,\infty)$ since $b \to V^\theta_i(b)$ is concave and Alexandrov's theorem implies almost-everywhere differentiability. Furthermore, G_i has at most two non-differentiable points and $\partial_b V^\theta_i$ is strictly decreasing. Thus, the HJB equation is differentiable with respect to b almost everywhere. At such a b, by differentiating the HJB equation, I obtain

$$(\rho - r)\partial_b V^\theta_i(b) - s^i_b(b;\theta)\partial_{bb} V^\theta_i(b) - \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\left(\partial_b V^\theta_j(b) - \partial_b V^\theta_i(b)\right) = 0. \tag{A.14}$$

If $s^i_b(b;\theta) \neq 0$, I obtain

$$\partial_{bb} V^\theta_i(b) = \frac{(\rho - r)\partial_b V^\theta_i(b) - \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\left(\partial_b V^\theta_j(b) - \partial_b V^\theta_i(b)\right)}{s^i_b(b;\theta)}.$$

Since the right-hand-side of the above can be defined and is continuous on $(0, \infty)$ if $s_b^i(b;\theta) \neq 0$, $b \to V_i^\theta(b)$ is twice continuously differentiable on $(0, \infty)$ if $s_b^i(b;\theta) \neq 0$.

<u>*Third claim.*</u> Suppose that $s_b^1(b;\theta) > 0$ at $b \in (0, \infty)$ to lead to a contradiction. According to (A.14), $\partial_{bb} V_1^\theta(b) \leq 0$, and $\partial_b V_1^\theta(b) > 0$, I obtain

$$0 \leq \rho - r \leq \sum_{j \in \mathcal{Y} \setminus \{1\}} \lambda_{1,j} \Big(\frac{\partial_b V_j^\theta(b)}{\partial_b V_1^\theta(b)} - 1 \Big).$$

This inequality implies $\partial_b V_j^\theta(b) \geq \partial_b V_1^\theta(b)$ with at least one $j \in \mathcal{Y} \setminus \{1\}$ since $\lambda_{1,j} > 0$ for any $j \in \mathcal{Y} \setminus \{1\}$. Then, $G_j(\partial_b V_j^\theta(b); \theta) \geq G_1(\partial_b V_1^\theta(b); \theta)$ as $p \to G_i(p;\theta)$ is strictly increasing and $G_i(p;\theta) \geq G_1(p;\theta)$ by $f(1) \leq f(i)$. Therefore, I obtain $s_b^j(b;\theta) > s_b^1(b;\theta) > 0$ from $g(j) > g(1)$. For such a j, by applying the same argument, I obtain

$$0 \leq \rho - r \leq \sum_{k \in \mathcal{Y} \setminus \{j\}} \lambda_{j,k} \Big(\frac{\partial_b V_k^\theta(b)}{\partial_b V_j^\theta(b)} - 1 \Big). \tag{A.15}$$

Hence, there exists at least one $k \in \mathcal{Y} \setminus \{1, j\}$ such that $\partial_b V_k^\theta(b) \geq \partial_b V_j^\theta(b) \geq \partial_b V_1^\theta(b)$. Thus, $s_b^k(b;\theta) > s_b^1(b;\theta) > 0$. By applying this argument finitely many times, we can determine a state $k \neq 1$ that satisfies $\partial_b V_k^\theta(b) \geq \partial_b V_j^\theta(b)$ for any $j \in \mathcal{Y}$ and $s_b^k(b;\theta) > 0$. However, such a k does not satisfy (A.15) unless $\partial_b V_i^\theta(b) = \partial_b V_j^\theta(b)$, $\partial_{bb} V_i^\theta(b) = 0$, and $\rho = r$ for any $i, j \in \mathcal{Y}$. In this exceptional case, there exists $\widetilde{b}$ in a neighborhood of b such that $\partial_{bb} V_i^\theta(\widetilde{b}) < 0$ for any $i \in \mathcal{Y}$ and $s_b^1(\widetilde{b};\theta) > 0$, as s_b^1 is continuous and $\partial_b V^\theta$ is strictly decreasing. However, this leads to a contradiction, as follows: At such a $\widetilde{b}$, we can determine a state $k \neq 1$ that satisfies $\partial_b V_k^\theta(\widetilde{b}) \geq \partial_b V_j^\theta(\widetilde{b})$ for any $j \in \mathcal{Y}$ and $s_b^k(\widetilde{b};\theta) > 0$, but $s_b^k(\widetilde{b};\theta) \partial_{bb} V_k^\theta(\widetilde{b}) < 0$. However, in this case, state k does not satisfy (A.15). Thus, the hypothesis is false and $s_b^1(b;\theta) \leq 0$ for any $b \in (0, \infty)$. If $r \leq 0$, $b \to s_b^1(b;\theta)$ is strictly decreasing on $(0, \infty)$. As $s_b^1(0;\theta) = 0$ according to the state-constraint boundary condition, $s_b^1(b;\theta) < 0$ for any $b \in (0, \infty)$. Therefore, hereafter, I suppose that $r > 0$.

Hypothesize that $s_b^1(b;\theta) = 0$ on some interval $(\underline{b}, \overline{b}) \subseteq (0, \infty)$. Then, according to Alexandrov's theorem, I obtain

$$0 \leq \rho - r = \sum_{j \in \mathcal{Y} \setminus \{1\}} \lambda_{1,j} \Big(\frac{\partial_b V_j^\theta(b)}{\partial_b V_1^\theta(b)} - 1 \Big) \tag{A.16}$$

almost everywhere on $(\underline{b}, \bar{b})$. Thus, there exists a state $j \in \mathcal{Y} \setminus \{1\}$ such that $\partial_b V_j^\theta(b) \geq \partial_b V_1^\theta(b)$ and $s_b^j(b;\theta) > s_b^1(b;\theta) = 0$ for some $b \in (\underline{b}, \bar{b})$. The strict inequality in the saving rate is owing to $g(1) < g(j)$. Therefore, based on the same argument as that in the previous paragraph, a contradiction appears, and hence, the hypothesis is false.

Finally, hypothesize that $s_b^1(b;\theta) = 0$ at some $b \in (0,\infty)$ but there exists $\widetilde{b}$ in any neighborhood of b such that $s_b^1(\widetilde{b};\theta) < 0$. Based on the preceding discussion, there exists a point $\widetilde{b} < b$ such that $s_b^1(\widetilde{b};\theta) < 0$ and $\partial_b^- s_b^1(\widetilde{b};\theta) \geq 0$, where ∂^+ and ∂^- are right-derivative and left-derivative operators, respectively. If not, $s_b^1(b;\theta) < 0$, so this is a contradiction. Note that $b \to s_b^1(b;\theta)$ is left-differentiable on $(0,\infty)$ if $s_b^1 \neq 0$, since $\partial_b V_1^\theta$ is strictly decreasing and G_1 has a right-continuous right derivative on $(0,\infty)$ according to the first claim. Hence, $\partial_b^- s_b^1(\widetilde{b};\theta) = r + \partial_p^+ G_1(\partial_b V_1^\theta(\widetilde{b});\theta)\partial_{bb} V_1^\theta(\widetilde{b}) \geq 0$. Furthermore, $\partial_p^+ G_1 > 0$. Thus, $0 \geq \partial_{bb} V_1^\theta(\widetilde{b}) \geq -r/\partial_p^+ G_1(\partial_b V_1^\theta(\widetilde{b});\theta)$. Since $\partial_b V_1^\theta$ is uniformly bounded and bounded away from zero on $[0,b]$, and since $\widetilde{b} \to \partial_p^+ G_1(\partial_b V_1^\theta(\widetilde{b});\theta)$ is left continuous, $\partial_p^+ G_1(\partial_b V_1^\theta(\widetilde{b});\theta)$ is bounded away from zero on a small interval $(\underline{b}, b)$. Therefore, there exists a constant $k_+ > 0$ such that $0 \geq \partial_{bb} V_1^\theta(\widetilde{b}) \geq -rk_+$ on $(\underline{b}, b)$ if $s_b^1(\widetilde{b};\theta) < 0$ and $\partial_b^- s_b^1(\widetilde{b};\theta) \geq 0$. Meanwhile, we can choose a non-decreasing sequence $(b_n)_{n\geq 1}$ on $(\underline{b}, b)$ such that $s_b^1(b_n;\theta) < 0$, $\partial_b^- s_b^1(b_n;\theta) \geq 0$, and $b_n \to b$ as $n \to \infty$. Hence, $0 \geq \partial_{bb} V_1^\theta(b_n) \geq -rk_+$ for any $n \geq 1$. Thus, there exists a subsequence of $(b_n)_{n\geq 1}$, which is also denoted by $(b_n)_{n\geq 1}$, such that $\partial_{bb} V_1^\theta(b_n)$ converges to a finite value. Since (A.14) holds at b_n for any $n \geq 1$, (A.16) is obtained at b by taking the limit as $n \to \infty$. Thus, a contradiction appears, except in the aforementioned case. In the exceptional case, $\rho = r$, $\partial_b V_i^\theta(b) = \partial_b V_j^\theta(b)$ for any $i, j \in \mathcal{Y}$ and $\partial_{bb} V_i^\theta(b) = 0$ for any $i \in \mathcal{Y} \setminus \{1\}$. Furthermore, $s_b^i(b;\theta) > 0$ for any $i \in \mathcal{Y} \setminus \{1\}$. However, this is also a contradiction, as follows: For any $i \in \mathcal{Y} \setminus \{1\}$, I obtain $\sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}(\partial_b V_j^\theta(\widetilde{b}) - \partial_b V_i^\theta(\widetilde{b})) \geq 0$ in a neighborhood of b with $\widetilde{b} \neq b$, since $s_b^i(b;\theta) > 0$ and $\rho = r$. However, I also obtain $\sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}(\partial_b V_j^\theta(b) - \partial_b V_i^\theta(b)) = 0$. Hence,

$$0 \leq \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j} \left(\left(\partial_b V_j^\theta(\widetilde{b}) - \partial_b V_j^\theta(b) \right) - \left(\partial_b V_i^\theta(\widetilde{b}) - \partial_b V_i^\theta(b) \right) \right).$$

Thus,

$$\frac{1}{\lambda_{i,1}} \left\{ \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j} \left(\partial_b V_i^\theta(\widetilde{b}) - \partial_b V_i^\theta(b) \right) - \sum_{j\in\mathcal{Y}\setminus\{1,i\}} \lambda_{i,j} \left(\partial_b V_j^\theta(\widetilde{b}) - \partial_b V_j^\theta(b) \right) \right\} \leq \partial_b V_1^\theta(\widetilde{b}) - \partial_b V_1^\theta(b).$$

If $\widetilde{b} > b$,

$$\frac{1}{\lambda_{i,1}}\left\{\sum_{j\in\mathcal{Y}\setminus\{i\}}\lambda_{i,j}\frac{\partial_b V_i^\theta(\widetilde{b})-\partial_b V_i^\theta(b)}{\widetilde{b}-b}-\sum_{j\in\mathcal{Y}\setminus\{1,i\}}\lambda_{i,j}\frac{\partial_b V_j^\theta(\widetilde{b})-\partial_b V_j^\theta(b))}{\widetilde{b}-b}\right\}\leq\frac{\partial_b V_1^\theta(\widetilde{b})-\partial_b V_1^\theta(b)}{\widetilde{b}-b}<0,$$

where the second inequality is owing to the strictly decreasing property of $\partial_b V_1^\theta$. Hence, the right derivative of $\partial_b V_1^\theta$ at b is zero: $\partial_{bb}^+ V_1^\theta(b) = 0$. This implies that the right derivative of s_b^1 at b is $r > 0$. Hence, $s_b^1(\widetilde{b};\theta)/(\widetilde{b}-b) = (s_b^1(\widetilde{b};\theta) - s_b^1(b;\theta))/(\widetilde{b}-b) > r/2$ if $\widetilde{b}$ is sufficiently close to b but $\widetilde{b} > b$. Thus, I obtain $0 < r(\widetilde{b}-b)/2 < s_b^1(\widetilde{b};\theta)$, but this is a contradiction because $s_b^1 \leq 0$. In summary, $s_b^1(b;\theta) < 0$ for any $b \in (0,\infty)$.

Fourth claim. According to the second and third claims, V_1^θ is twice continuously differentiable everywhere on $(0,\infty)$. Thus,

$$s_b^1(b;\theta)\partial_{bb}V_1^\theta(b) = (\rho-r)\partial_b V_1^\theta(b) - \sum_{j\in\mathcal{Y}\setminus\{1\}}\lambda_{1,j}\left(\partial_b V_j^\theta(b)-\partial_b V_1^\theta(b)\right) \tag{A.17}$$

for any $b \in (0,\infty)$. Meanwhile, I obtain $\partial_b^+ s_b^1(b;\theta) = r+\partial_b^+ G_1(\partial_b V_1^\theta(b);\theta) = r+\partial_p^- G_1(\partial_b V_1^\theta(b);\theta)\partial_{bb}V_1^\theta(b)$. Thus, I obtain $\partial_{bb}V_1^\theta(b) = (\partial_b^+ s_b^1(b;\theta)-r)/\partial_p^- G_1(\partial_b V_1^\theta(b);\theta)$ since $\partial_p^- G_1(\partial_b V_1^\theta(b);\theta) > 0$. By substituting this into (A.17) and rearranging it, I obtain

$$s_b^1(b;\theta)\partial_b^+ s_b^1(b;\theta) = rs_b^1(b;\theta)+\partial_p^- G_1(\partial_b V_1^\theta(b);\theta)\left((\rho-r)\partial_b V_1^\theta(b) - \sum_{j\in\mathcal{Y}\setminus\{1\}}\lambda_{1,j}\left(\partial_b V_j^\theta(b)-\partial_b V_1^\theta(b)\right)\right). \tag{A.18}$$

The following value:

$$(\rho-r)\partial_b^+ V_1^\theta(0) - \sum_{j\in\mathcal{Y}\setminus\{1\}}\lambda_{1,j}\left(\partial_b^+ V_j^\theta(0)-\partial_b^+ V_1^\theta(0)\right)$$

is strictly positive. This can be demonstrated by assuming a contradiction in the same manner as in the third claim. Furthermore, $\partial_p^- G_1$ has a finite and strictly positive left limit at $\partial_b^+ V_1^\theta(0)$ owing to the first claim. Therefore, the right-hand side of (A.18) converges to a finite and strictly positive value as $b \to 0$. Thus, there exists a finite $k^* \in (0,\infty)$ such that

$$\lim_{b\to 0} s_b^1(b;\theta)\partial_b^+ s_b^1(b;\theta) = k^*.$$

As discussed in Proposition 1 in Achdou et al. (2022), this implies that $(s_b^1(b;\theta))^2/b$ is bounded away from zero on an interval $[0, b_1^*]$ with $b_1^* > 0$. Thus, there exists a finite $\overline{k}_1 \in (0, \infty)$ such that $(s_b^1(b;\theta))^2 \geq \overline{k}_1 b$ on the interval $[0, b_1^*]$, and hence, $s_b^1(b;\theta) \leq -\overline{k}_1\sqrt{b}$.

Consider $(b_t^1)_{t\in[0,\infty)}$; i.e., the solution of the ODE (2.8) in state 1. Since $s_b^1(b;\theta) < 0$ for any $b \in (0,\infty)$, $(b_t^1)_{t\in[0,\infty)}$ reaches b_1^* in finite time whenever it starts at $b' \in (b_1^*, \infty)$. Thus, in this case, I consider $(b_t^1)_{t\in[0,\infty)}$ starting at b_1^* without loss of generality. Furthermore, consider an initial value problem of the ODE such that $\mathrm{d}x_t = -\overline{k}_1\sqrt{|x_t|}\mathrm{d}t$ with $x_0 = b_1^* > 0$. A solution to this problem is unique up to the time when x_t reaches zero, and this time is finite. I assume that $x_t = 0$ after x_t reaches zero. Suppose that $x_t < b_t^1$ for some $t \in [0,\infty)$. Since $x_0 = b_0^1 = b_1^*$ and both are continuous, there exists a time $t_1 \in [0, t)$ such that $x_{t_1} = b_{t_1}^1$ and $x_s < b_s^1$ for any $s \in (t_1, t]$. Then, for any $t_2 \in [t_1, t)$, I obtain

$$b_{t_2}^1 - x_{t_2} = \int_{t_1}^{t_2} \Big(s_b^1(b_s^1;\theta) + \overline{k}_1\sqrt{x_s}\Big)\mathrm{d}s \leq \int_{t_1}^{t_2} \Big(-\overline{k}_1\sqrt{b_s^1} + \overline{k}_1\sqrt{x_s}\Big)\mathrm{d}s < 0.$$

However, this is a contradiction, and hence, $x_t \geq b_t^1$. Thus, $(b_t^1)_{t\in[0,\infty)}$ reaches zero in finite time.

Fifth claim. First, consider the case $r \leq 0$. Then, s_b^i is strictly decreasing in b and $s_b^i(b';\theta) - s_b^i(b;\theta) \leq r(b' - b)$ for any $b' \geq b$. Hence, we can obtain the desired result if $r < 0$. In the case of $r = 0$, hypothesize that $\limsup_{b\to\infty} s_b^i(b;\theta) \geq 0$. By the hypothesis, $\limsup_{b\to\infty} G_i(\partial_b V_i^\theta(b);\theta) \geq -g(i) > -\infty$. This implies that there exist two constants $k^* > 0$ and $b^* > 0$ such that $\partial_b V_i^\theta(b) \geq k^*$ for any $b \in [b^*, \infty)$. However, this is a contradiction since V_i^θ is bounded above. Thus, $\limsup_{b\to\infty} s_b^i(b;\theta) < 0$, and hence, we can obtain the desired result.

Next, consider the case $r > 0$. Suppose that there exists a state $i \in \mathcal{Y}$ such that $s_b^i(b;\theta) > 0$ for any $b \in [0,\infty)$. Then, I obtain

$$0 < \rho - r \leq \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\Big(\frac{\partial_b V_j^\theta(b)}{\partial_b V_i^\theta(b)} - 1\Big) \tag{A.19}$$

for any $b \in [0,\infty)$. Now, let $j \in \mathcal{Y}$ be a state where $\partial_b V_j^\theta(b) > \partial_b V_k^\theta(b)$ for any $k \in \mathcal{Y} \setminus \{j\}$. Then, $s_b^j(b;\theta) \leq 0$ because if not, the state j does not satisfy (A.19) and this is contradiction. As $s_b^i(b;\theta) > 0$, I obtain $c_\theta^*(\partial_b V_i^\theta(b), i) < (rb + g(i) + wf(i)l_\theta^*(\partial_b V_i^\theta(b), i))/\tau_c \leq (rb + \overline{g} +$

$w\overline{fL})/\tau_c$. Thus, owing to the first-order condition, $\partial_{cc}u < 0$, and $\partial_{cl}u \geq 0$, I obtain $\partial_b V_i^\theta(b) = \partial_c u(c_\theta^*, l_\theta^*)/\tau_c \geq \partial_c u((rb + \overline{g} + w\overline{fL})/\tau_c, 0)/\tau_c$. Similarly, since $s_b^j(b;\theta) \leq 0$, I obtain $\partial_b V_j^\theta(b) \leq \partial_c u((rb + \underline{g})/\tau_c, \overline{L})/\tau_c$. Thus,

$$0 < \rho - r \leq (K-1)\overline{\lambda} \limsup_{b\to\infty} \Big(\frac{\partial_c u((rb+\underline{g})/\tau_c, \overline{L})}{\partial_c u((rb+\overline{g}+w\overline{fL})/\tau_c, 0)} - 1 \Big) \leq 0,$$

which is a contradiction, and hence, there exists a $\overline{b} \in (0,\infty)$ such that $s_b^i(b;\theta) \leq 0$ for any $(b,i) \in (\overline{b},\infty) \times \mathcal{Y}$. If there exists a $\widetilde{b} \in (\overline{b},\infty)$ such that $s_b^i(b;\theta) = 0$ for any $b \in (\widetilde{b},\infty)$, according to Alexandrov's theorem,

$$0 < \rho - r = \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j} \Big(\frac{\partial_b V_j^\theta(b)}{\partial_b V_i^\theta(b)} - 1 \Big) \tag{A.20}$$

holds almost everywhere on $(\widetilde{b},\infty)$. Thus, a contradiction can also be derived. Consider the case in which there exists an increasing sequence $(b_n)_{n\geq1} \subset (\overline{b},\infty)$ such that $s_b^i(b_n;\theta) = 0$ and $b_n \to \infty$, but for any n, we can choose $\widetilde{b}$ in any neighborhood of b_n such that $s_b^i(\widetilde{b};\theta) < 0$. In this case, by the same argument as in the third claim, it can be observed that (A.20) is satisfied at any b_n, which leads to a contradiction. In summary, there exists a sufficiently large but finite $\overline{b} \in (0,\infty)$ such that $s_b^i(b;\theta) < 0$ for any $(b,i) \in (\overline{b},\infty) \times \mathcal{Y}$. □

Note that in a case of $\rho > r$, there exists at least one state where the saving rate is negative even if no clear worst state exists, i.e., a situation without conditions 3 and 4 in Assumption 5. For any $b \in [0,\infty)$ with $s_b^i(b;\theta) \neq 0$ for any $i \in \mathcal{Y}$, by (A.14) and the generalized Ito formula, I have

$$\partial_b V_i^\theta(b) = \mathrm{E}\left[e^{-(\rho-r)T} \partial_b V_{Y_T^i}^\theta(b) + \int_0^T e^{-(\rho-r)t} s_b^{Y_t^i}(b;\theta) \partial_{bb} V_{Y_t^i}^\theta(b) \mathrm{d}t \right],$$

for any $T \in [0,\infty)$ and $i \in \mathcal{Y}$. Now, hypothesize $s_b^j(b;\theta) > 0$ for any $j \in \mathcal{Y}$. Then, by the bounded convergence theorem and monotone convergence theorem, taking $T \to \infty$ implies

$$\partial_b V_i^\theta(b) = \mathrm{E}\left[\int_0^\infty e^{-(\rho-r)t} s_b^{Y_t^i}(b;\theta) \partial_{bb} V_{Y_t^i}^\theta(b) \mathrm{d}t \right] \leq 0,$$

which is a clear contradiction. Hence, $s_b^j(b;\theta) < 0$ at least one $j \in \mathcal{Y}$. Furthermore, this implies that there exists at least one state where the state constraint boundary condition is binded.

The conditions 3 and 4 in Assumption 5 are required to specify the worst state where the saving rate is always non-positive.

Proposition 6 is demonstrated using Lemma 10.

Proof of Proposition 6.

First and second claims. The proofs of the first and second claims are a continuous-time version of the proof of Proposition 5 in Açıkgöz (2018). Fix an arbitrary $\theta \in \Theta$. I define $\bar{b} := \max_{i \in \mathcal{Y}}\{\sup\{b \in [0,\infty) | s_b^i(b;\theta) = 0\}\}$. According to the fifth claim in Lemma 10, $\bar{b}$ is finite and the measurable space can be restricted such that $(\mathcal{S}_{\bar{b}}, \mathcal{G}_{\bar{b}}) := ([0,\bar{b}] \times \mathcal{Y}, \mathcal{B}([0,\bar{b}]) \otimes 2^{\mathcal{Y}})$. This also implies that there exists a compact set $\mathcal{K}$ on $[0,\bar{b}]$ and time $t^* \in [0,\infty)$ such that $P_t^\theta((b,i), \mathcal{K} \times \mathcal{Y}) = 1$ for any $(b,i,t) \in [0,\bar{b}] \times \mathcal{Y} \times [t^*,\infty)$. Thus, $(B^{*;\theta}, Y)$ is bounded in probability, and hence, is also bounded in probability on average. Furthermore, $(B^{*;\theta}, Y)$ is a Feller process. Therefore, there exists an invariant probability measure of $(P_t^\theta)_{t \in [0,\infty)}$ on $(\mathcal{S}_{\bar{b}}, \mathcal{G}_{\bar{b}})$, which is denoted by μ^θ (Beneš (1968) and Theorem 3.1 in Meyn and Tweedie (1993)).

I demonstrate that μ^θ is unique using the argument of the ergodicity. This can be achieved by showing that there exists an ergodic skeleton chain supporting $(P_t^\theta)_{t \in [0,\infty)}$; that is, a discrete-time ergodic chain supporting $(P_{nT}^\theta)_{n=1}^\infty$ for a fixed time interval T. Hence, the discrete-time argument employed by Açıkgöz (2018) can be applied.

Açıkgöz (2018) demonstrated the existence of the uniformly ergodic invariant measure by employing Theorem 16.0.2 in Meyn et al. (2009). To apply this, it is necessary to choose a nontrivial measure v on $(\mathcal{S}_{\bar{b}}, \mathcal{G}_{\bar{b}})$ and time $T^* \in [0,\infty)$ such that $P_{T^*}^\theta((b,i), \mathcal{S}_{\bar{b}}) \geq v(\mathcal{S}_{\bar{b}})$ for any $(b,i) \in \mathcal{S}_{\bar{b}}$. Let T_1 be a time when $(b_t^1)_{t \in [0,\infty)}$ starting at $\bar{b}$ reaches zero. According to the fourth claim in Lemma 10, T_1 is finite. Furthermore, $(b_t^1)_{t \in [0,\infty)}$ starting at $\bar{b}$ is always not smaller than $(b_t^1)_{t \in [0,\infty)}$ starting at $\widetilde{b} \in [0,\bar{b})$. Therefore, by the time T_1, $(b_t^1)_{t \in [0,\infty)}$ reaches zero even if it starts everywhere on $[0,\bar{b}]$. Moreover, let $\widehat{q} := \min_{i \in \mathcal{Y} \setminus \{1\}} \mathbb{P}(Y_1^i = 1) > 0$ and $\widetilde{q} := \mathbb{P}(Y_s^1 = 1, \text{ for any } s \in [0,T_1]) > 0$. Then, by the Markov property of $(B^{*;\theta}, Y)$, I obtain $P_{T_1+1}^\theta((b,i), \{(0,1)\}) \geq \widehat{q}\widetilde{q} =: q > 0$ for any $(b,i) \in \mathcal{S}_{\bar{b}}$. Let us define the following measure on $(\mathcal{S}_{\bar{b}}, \mathcal{G}_{\bar{b}})$:

$$v(A) := \begin{cases} q, & \text{if } (0,1) \in A, \\ 0, & \text{if } (0,1) \notin A, \end{cases} \qquad A \in \mathcal{G}_{\bar{b}}.$$

Subsequently, let $T^* := T_1 + 1$, and I obtain $P^\theta_{T^*}((b,i),\mathcal{S}_{\bar{b}}) \geq v(\mathcal{S}_{\bar{b}})$ for any $(b,i) \in \mathcal{S}_{\bar{b}}$. Since v is not trivial, there exists a uniformly ergodic invariant probability measure of $(P^\theta_{nT^*})_{n=1}^\infty$ on $(\mathcal{S}_{\bar{b}}, \mathcal{G}_{\bar{b}})$. Furthermore, the ergodicity implies that such an invariant probability measure is unique. Since the skeleton chain supporting $(P^\theta_{nT^*})_{n=1}^\infty$ is uniformly ergodic, the original $(B^{*;\theta}, Y)$ supporting $(P^\theta_t)_{t\in[0,\infty)}$ is exponentially ergodic, which is a simple application of Theorem 5.3 in Down et al. (1995).

<u>*Third claim.*</u> I demonstrate that the smallest upper boundary of the support of μ^θ is locally bounded with respect to θ. I denote this upper boundary with θ as $\bar{b}^\theta$. Let $\mathcal{K}$ be a compact subset of Θ. Suppose that there exists a sequence $(\theta_n)_{n=1}^\infty \subseteq \mathcal{K}$ such that $\bar{b}^{\theta_n} \to \infty$ as $n \to \infty$. Since $(\theta_n)_{n=1}^\infty$ is bounded, there exists a convergent subsequence, which is also denoted by $(\theta_n)_{n=1}^\infty$, and let $\theta^* \in \mathcal{K}$ be its limit. According to the fifth claim in Lemma 10, $\bar{b}^{\theta^*}$ is finite. The hypothesis implies that for some $b \in (\bar{b}^{\theta^*}, \infty)$, there exists a sufficiently large natural number n^* and state $i \in \mathcal{Y}$ such that $s^i_b(b;\theta_n) \geq 0$ for any $n \geq n^*$. Hence, owing to the parametric continuity of s^i_b, I obtain $s^i_b(b;\theta^*) \geq 0$, but this is a contradiction since $b > \bar{b}^{\theta^*}$. Therefore, $\bar{b}^\theta$ is locally bounded with respect to θ.

Fix an arbitrary compact subset $\mathcal{K}$ of Θ. Furthermore, let $\bar{b}^{\mathcal{K}} := \sup_{\theta\in\mathcal{K}} \bar{b}^\theta < \infty$. In the following discussion, I work on a measurable space $(\mathcal{S}_{\bar{b}^{\mathcal{K}}}, \mathcal{G}_{\bar{b}^{\mathcal{K}}})$. Proposition 4 implies that for any $t \in [0,\infty)$, $(b,i,\theta) \to P^\theta_t((b,i),\cdot)$ is weak continuous on $[0,\bar{b}^{\mathcal{K}}] \times \mathcal{Y} \times \mathcal{K}$. Therefore, Theorem 12.13 in Stokey et al. (1989) implies that μ^θ on $(\mathcal{S}_{\bar{b}^{\mathcal{K}}}, \mathcal{G}_{\bar{b}^{\mathcal{K}}})$ is also weak continuous in θ on $\mathcal{K}$. This can be achieved through a skeleton chain, as defined in the proof of the first and second claims. Thus, for any $F \in C_b(\mathcal{S}\times\Theta)$ and $\theta, \theta' \in \mathcal{K}$, I obtain

$$
\begin{aligned}
&\Big| \int_{\mathcal{S}_{\bar{b}^{\mathcal{K}}}} F(b,i,\theta')\mathrm{d}\mu^{\theta'}(b,i) - \int_{\mathcal{S}_{\bar{b}^{\mathcal{K}}}} F(b,i,\theta)\mathrm{d}\mu^\theta(b,i) \Big| \\
&\quad \leq \sup_{(b,i)\in\mathcal{S}_{\bar{b}^{\mathcal{K}}}} |F(b,i,\theta') - F(b,i,\theta)| + \Big| \int_{\mathcal{S}_{\bar{b}^{\mathcal{K}}}} F(b,i,\theta)\mathrm{d}\mu^{\theta'}(b,i) - \int_{\mathcal{S}_{\bar{b}^{\mathcal{K}}}} F(b,i,\theta)\mathrm{d}\mu^\theta(b,i) \Big| \to 0
\end{aligned}
$$

as $\theta' \to \theta$. The convergence $\sup_{(b,i)\in\mathcal{S}_{\bar{b}^{\mathcal{K}}}} |F(b,i,\theta') - F(b,i,\theta)| \to 0$ is owing to the uniform continuity of F on the compact set $\mathcal{S}_{\bar{b}^{\mathcal{K}}} \times \mathcal{K}$. Since $\mathcal{K}$ is selected arbitrarily, I can conclude that μ^θ is weak continuous in θ on Θ. □

A.3 Proof of Corollary 7

It is necessary to check the limit behavior of the aggregate liquid asset to demonstrate Corollary 7, as follows:

Lemma 11 *1. For any $i \in \mathcal{Y}$, $\mathcal{V}^{i,\theta} := (\mathcal{V}_t^{i,\theta})_{t\in[0,\infty)} = (e^{-(\rho-r)t}\partial_b V^\theta_{Y_t^i}(B_t^{*0,i;\theta}))_{t\in[0,\infty)}$ is a bounded and non-negative $\mathbb{F}$-supermartingale if $r \in (0,\rho]$.*

2. For any $i \in \mathcal{Y}$, $\mathcal{V}_t^{i,\theta} \to 0$ as $t \to \infty$ $\mathbb{P}$-a.s. if $r \in (0,\rho]$.

3. $\liminf_{r\uparrow\rho} \int_{\mathcal{S}} b \mathrm{d}\mu^\theta(b,i) = \infty$.

4. $\int_{\mathcal{S}} l^*_\theta(\partial_b V_i^\theta(b), i)\mathrm{d}\mu^\theta(b,i) > 0$ if $wf(i) > -\tau_c\partial_l u(g(i)/\tau_c, 0)/\partial_c u(g(i)/\tau_c, 0)$ for some $i \in \mathcal{Y}$.

Proof of Lemma 11.

<u>*First claim.*</u> I refer to a stationary point as a point $b \in [0,\infty)$ with $s^i_b(b;\theta) = 0$. I first demonstrate that, for any interior stationary point $\bar{b}_i \in (0,\infty)$ with $s^b_i(\bar{b}_i;\theta) = 0$, the following equality is satisfied:

$$(\rho - r)\partial_b V_i^\theta(\bar{b}_i) - \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\left(\partial_b V_j^\theta(\bar{b}_i) - \partial_b V_i^\theta(\bar{b}_i)\right) = 0. \tag{A.21}$$

At a stationary point $\bar{b}_i$, the saving rate s^i_b satisfies one of the following four cases: (1) there exists an interval $[b_-, \bar{b}_i]$ or $[\bar{b}_i, b^+]$ such that $s^i_b(b;\theta) = 0$ for any $b \in [b_-, \bar{b}_i] \cup [\bar{b}_i, b^+]$, (2) there exists a point b in any neighborhood of $\bar{b}_i$ such that $s^i_b(b;\theta) < 0$ with $b > \bar{b}_i$, (3) there exists a point b in any neighborhood of $\bar{b}_i$ such that $s^i_b(b;\theta) > 0$ with $b < \bar{b}_i$, or (4) there exist b^+ and b^- in any neighborhood of $\bar{b}_i$ in which $s^i_b(b^+;\theta) > 0$ with $b^+ > \bar{b}_i$ and $s^i_b(b^-;\theta) < 0$ with $b^- < \bar{b}_i$. In case (1), using Alexandrov's theorem, I obtain

$$(\rho - r)\partial_b V_i^\theta(b) - \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\left(\partial_b V_j^\theta(b) - \partial_b V_i^\theta(b)\right) = 0$$

almost everywhere on $[b_-, \bar{b}_i]$ or $[\bar{b}_i, b^+]$. Thus, (A.21) holds from the continuity of $\partial_b V^\theta$. In case (2), there are three possible cases: (2-1) there exists an interval $[b_-, \bar{b}_i]$ such that $s^i_b(b;\theta) = 0$ for any $b \in [b_-, \bar{b}_i]$, (2-2) there exists a point b in any neighborhood of $\bar{b}_i$ such that $s^i_b(b;\theta) > 0$ with $b < \bar{b}_i$, or (2-3) there exists a point b in any neighborhood of $\bar{b}_i$ such that $s^i_b(b;\theta) < 0$ for any $b < \bar{b}_i$. It can be observed that case (2-1) also holds (A.21) according to the same argument as in case (1). In case (2-2), there exist two sequences

$(b_n^+)_{n\geq 1}$ and $(b_n^-)_{n\geq 1}$ such that $(b_n^+)_{n\geq 1}$ is non-increasing (resp. $(b_n^-)_{n\geq 1}$ is non-decreasing) with $b_n^+ \downarrow \bar{b}_i$ (resp. $b_n^- \uparrow \bar{b}_i$) as $n \to \infty$ and $s_b^i(b_n^+;\theta) < 0$ (resp. $s_b^i(b_n^-;\theta) > 0$) for any $n \geq 1$. Then, for any $n \geq 1$, $(\rho - r)\partial_b V_i^\theta(b_n^-) - \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\Big(\partial_b V_j^\theta(b_n^-) - \partial_b V_i^\theta(b_n^-)\Big) \leq 0$ and $(\rho - r)\partial_b V_i^\theta(b_n^+) - \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\Big(\partial_b V_j^\theta(b_n^+) - \partial_b V_i^\theta(b_n^+)\Big) \geq 0$. Thus, taking the limit yields (A.21). In case (2-3), in the same manner as in the proof of the third claim in Lemma 10, a non-decreasing sequence $(b_n)_{n\geq 1}$ can be determined such that $b_n \uparrow \bar{b}_i$ as $n \to \infty$, $\partial_b V_i$ is differentiable at b_n for any $n \geq 1$, and $\partial_{bb} V_i^\theta(b_n)$ converges to a finite value as $n \to \infty$. Thus,

$$\begin{aligned}(\rho - r)\partial_b V_i^\theta(\bar{b}_i) &- \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\Big(\partial_b V_j^\theta(\bar{b}_i) - \partial_b V_i^\theta(\bar{b}_i)\Big)\\ &= \lim_{n\to\infty}\left\{(\rho - r)\partial_b V_i^\theta(b_n) - \sum_{j\in\mathcal{Y}\setminus\{i\}} \lambda_{i,j}\Big(\partial_b V_j^\theta(b_n) - \partial_b V_i^\theta(b_n)\Big)\right\}\\ &= \lim_{n\to\infty}\Big\{s_i^b(b_n;\theta)\partial_{bb} V_i^\theta(b_n)\Big\} = 0.\end{aligned}$$

This implies that (A.21). In case (3), there are two possible cases: (3-1) s_b^i satisfies (1) or (2), or (3-2) there exists a point b in any neighborhood of $\bar{b}_i$ such that $s_b^i(b;\theta) > 0$ with $b > \bar{b}_i$. Based on the above discussion, case (3-1) satisfies (A.21); thus, let us consider case (3-2). However, in case (3-2), we can employ the argument in the proof of the third claim in Lemma 10 similarly to case (2-3). Case (4) holds (A.21) as well as case (2-2).

Subsequently, consider the saving rate at a boundary stationary point. Suppose that $s_b^i(0;\theta) = 0$. Consider that there exists a point $b > 0$ in any neighborhood of zero such that $s_b^i(b;\theta) > 0$. However, in this case, the same argument as those in cases (2-3) and (3-2) can be applied in interior stationary points. Thus, I obtain (A.21). In the case where there exists $b > 0$ in any neighborhood of zero such that $s_b^i(b;\theta) \leq 0$, (A.21) holds with the inequality $\geq$.

According to the above discussion, the drift of $\mathcal{V}^{i,\theta}$ satisfies the following: In a region $\{(b,i) \in [0,\infty) \times \mathcal{Y} | s_b^i(b;\theta) \neq 0\}$, the drift of $\mathcal{V}^{i,\theta}$ is as follows:

$$\begin{aligned}\mathcal{L}^{\mathcal{V}^{i,\theta}}\mathcal{V}_t^{i,\theta} = -e^{(\rho-r)t}\Big(&(\rho - r)\partial_b V_{Y_t^i}^\theta(B_t^{*0,i;\theta}) - s_b^{Y_t^i}(B_t^{*0,i;\theta})\partial_{bb} V_{Y_t^i}^\theta(B_t^{*0,i;\theta})\\ &- \sum_{j\in\mathcal{Y}\setminus\{Y_t^i\}} \lambda_{i,j}\Big(\partial_b V_j^\theta(B_t^{*0,i;\theta}) - \partial_b V_{Y_t^i}^\theta(B_t^{*0,i;\theta})\Big)\Big) = 0. \quad \text{(A.22)}\end{aligned}$$

The final equality is owing to (A.14). At a stationary point b, $B^{*0,i;\theta}$ does not move from b unless Y^i changes. Thus,

$$\mathcal{L}^{\mathcal{V}^{i,\theta}}\mathcal{V}_t^{i,\theta} = -e^{(\rho-r)t}\Big((\rho-r)\partial_b V^\theta_{Y_t^i}(B_t^{*0,i;\theta}) \\ - \sum_{j\in\mathcal{Y}\setminus\{Y_t^i\}} \lambda_{i,j}\Big(\partial_b V_j^\theta(B_t^{*0,i;\theta}) - \partial_b V^\theta_{Y_t^i}(B_t^{*0,i;\theta})\Big)\Big) \le 0 \quad \text{(A.23)}$$

and the strict inequality holds if b is zero and if there exists b in any neighborhood of zero such that $s_b^i(b;\theta) < 0$. For example, $(0,1)$ is a boundary stationary point with strict inequality.

Finally, I show the supermartingale property of $\mathcal{V}^{i,\theta}$. Let $t, s \in [0,\infty)$ with $s \le t$. Consider the following sequence of stopping times $(\tau_n)_{n\ge1}$:

$$\begin{aligned}
\tau_1 &= \inf\{t \in [s,\infty) | s_b^{Y_t^i}(B_t^{*0,i;\theta};\theta) = 0\}, \\
\tau_{2n} &= \inf\{t \in [\tau_{2n-1},\infty) | Y_t^i \neq Y_{t-}^i, \quad s_b^{Y_t^i}(B_t^{*0,i;\theta};\theta) \neq 0\}, \\
\tau_{2n+1} &= \inf\{t \in [\tau_{2n},\infty) | s_b^{Y_t^i}(B_t^{*0,i;\theta};\theta) = 0\},
\end{aligned}$$

for any $n = 1, 2, \cdots$. On $[s,\tau_1)$ and $[\tau_{2n},\tau_{2n+1})$ for any $n \ge 1$, the drift of $\mathcal{V}^{i,\theta}$ is zero according to (A.22). Meanwhile, on $[\tau_{2n-1},\tau_{2n})$ for any $n \ge 1$, the drift of $\mathcal{V}^{i,\theta}$ is non-positive according to (A.23). Furthermore, it can easily be observed that the drift of $\mathcal{V}^{i,\theta}$ is right-continuous. On $[s,\tau_1 \wedge t)$, $s_b^{Y_t^i}(B_t^{*0,i;\theta};\theta) \neq 0$ and $\partial_b V^\theta$ is differentiable. Hence, the drift of $\mathcal{V}^{i,\theta}$ is zero. Therefore, applying the generalized Ito formula to $\mathcal{V}^{i,\theta}$ on $[s,\tau_1 \wedge t)$ and taking the expectation, I obtain $\mathrm{E}[\mathcal{V}^{i,\theta}_{\tau_1\wedge t}] = \mathrm{E}[\mathcal{V}^{i,\theta}_s]$.

On event $\{\tau_2 \le t\}$, using (A.23) and the optional stopping theorem, I obtain $\mathrm{E}_{\tau_1\wedge t}[\mathcal{V}^{i,\theta}_{\tau_2\wedge t}] = \mathrm{E}_{\tau_1\wedge t}[\mathcal{V}^{i,\theta}_{\tau_2}] \le \mathcal{V}^{i,\theta}_{\tau_1} = \mathcal{V}^{i,\theta}_{\tau_1\wedge t}$. Similarly, on event $\{\tau_1 \le t < \tau_2\}$, I obtain $\mathrm{E}_{\tau_1\wedge t}[\mathcal{V}^{i,\theta}_{\tau_2\wedge t}] = \mathrm{E}_{\tau_1\wedge t}[\mathcal{V}^{i,\theta}_t] \le \mathcal{V}^{i,\theta}_{\tau_1} = \mathcal{V}^{i,\theta}_{\tau_1\wedge t}$. On event $\{t < \tau_1\}$, I obtain $\mathrm{E}_{\tau_1\wedge t}[\mathcal{V}^{i,\theta}_{\tau_2\wedge t}] = \mathrm{E}_{\tau_1\wedge t}[\mathcal{V}^{i,\theta}_t] = \mathcal{V}^{i,\theta}_{\tau_1\wedge t}$. Therefore,

$$\mathrm{E}[\mathcal{V}^{i,\theta}_{\tau_2\wedge t}] = \mathrm{E}[\mathbb{1}_{\{\tau_1\le t\}}\mathrm{E}_{\tau_1\wedge t}[\mathcal{V}^{i,\theta}_{\tau_2\wedge t}] + \mathbb{1}_{\{\tau_1>t\}}\mathrm{E}_{\tau_1\wedge t}[\mathcal{V}^{i,\theta}_{\tau_2\wedge t}]] \le \mathrm{E}[\mathbb{1}_{\{\tau_1\le t\}}\mathcal{V}^{i,\theta}_{\tau_1\wedge t} + \mathbb{1}_{\{\tau_1>t\}}\mathcal{V}^{i,\theta}_{\tau_1\wedge t}] = \mathrm{E}[\mathcal{V}^{i,\theta}_s].$$

Since $s_b^{Y_t^i}(B_t^{*0,i;\theta};\theta) \neq 0$ and $\partial_b V^\theta$ is differentiable on $[\tau_2,\tau_3)$, according to the strong Markov property of (B^*, Y), I obtain $\mathbb{1}_{\{\tau_2 \le t\}} \mathrm{E}_{\tau_2 \wedge t}[\mathcal{V}^{i,\theta}_{\tau_2 \wedge t}] = \mathbb{1}_{\{\tau_2 \le t\}} \mathrm{E}_{\tau_2 \wedge t}[\mathcal{V}^{i,\theta}_{\tau_3 \wedge t}]$. Hence,

$$\mathrm{E}[\mathcal{V}^{i,\theta}_s] \ge \mathrm{E}[\mathbb{1}_{\{\tau_2 > t\}} \mathcal{V}^{i,\theta}_{\tau_2 \wedge t} + \mathbb{1}_{\{\tau_2 \le t\}} \mathrm{E}_{\tau_2 \wedge t}[\mathcal{V}^{i,\theta}_{\tau_2 \wedge t}]] = \mathrm{E}[\mathbb{1}_{\{\tau_2 > t\}} \mathcal{V}^{i,\theta}_{\tau_3 \wedge t} + \mathbb{1}_{\{\tau_2 \le t\}} \mathrm{E}_{\tau_2 \wedge t}[\mathcal{V}^{i,\theta}_{\tau_3 \wedge t}]] = \mathrm{E}[\mathcal{V}^{i,\theta}_{\tau_3 \wedge t}].$$

By applying this argument finitely many times, I obtain $\mathrm{E}[\mathcal{V}^{i,\theta}_s] \ge \mathrm{E}[\mathcal{V}^{i,\theta}_{\tau_n \wedge t}]$ for any $n \ge 1$. On the other hand, it is clear that $\tau_n \to \infty$ as $n \to \infty$ $\mathbb{P}$-a.s. Hence, for an almost sure $\omega \in \Omega$, there exists a natural number $n^*(\omega)$ such that $\tau_n(\omega) > t$ for any $n \ge n^*(\omega)$. Therefore, $\mathcal{V}^{i,\theta}_{\tau_n \wedge t} \to \mathcal{V}^{i,\theta}_t$ as $n \to \infty$ $\mathbb{P}$-a.s. From the bounded convergence theorem, I obtain $\mathrm{E}[\mathcal{V}^{i,\theta}_s] \ge \mathrm{E}[\mathcal{V}^{i,\theta}_t]$. Thus, $\mathcal{V}^{i,\theta}$ is an $\mathbb{F}$-supermartingale.

Second claim. Since $\mathcal{V}^{i,\theta}$ is a bounded $\mathbb{F}$-supermartingale (i.e., a uniformly integrable $\mathbb{F}$-supermartingale), the martingale convergence theorem implies that there exists an integrable random variable $\mathcal{V}^{i,\theta}_\infty$ such that $\mathcal{V}^{i,\theta}_t \to \mathcal{V}^{i,\theta}_\infty$ as $t \to \infty$ $\mathbb{P}$-a.s. It is necessary to verify $\mathcal{V}^{i,\theta}_\infty = 0$, $\mathbb{P}$-a.s to demonstrate the second claim. The case of $\rho > r$ is immediate. Accordingly, I suppose that $\rho = r$.

First, for a given $\epsilon > 0$, I hypothesize that $\mathbb{P}(\mathcal{V}^{i,\theta}_t > \epsilon$ for any $t \ge 0) > 0$, which leads to a contradiction. Let $\tau_\epsilon := \inf\{t \in [0,\infty) | \mathcal{V}^{i,\theta}_t \le \epsilon\}$ be the first hitting time of $\mathcal{V}^{i,\theta}$ at ϵ. Then, the hypothesis can be rewritten as $\mathbb{P}(\tau_\epsilon = \infty) > 0$. Meanwhile, from $\mathcal{V}^{i,\theta} \ge 0$ and $\mathcal{L}^{\mathcal{V}^{i,\theta}} \mathcal{V}^{i,\theta} \le 0$, I obtain

$$0 \le \mathrm{E}[\mathcal{V}^{i,\theta}_{\tau_\epsilon \wedge t}] = \mathcal{V}^{i,\theta}_0 + \mathrm{E}\left[\int_0^{\tau_\epsilon \wedge t} \mathcal{L}^{\mathcal{V}^{i,\theta}} \mathcal{V}^{i,\theta}_s \mathrm{d}s\right] \le \partial_b^+ V_i^\theta(0) - k^+ \mathrm{E}\left[\int_0^{\tau_\epsilon \wedge t} \mathbb{1}_{\{(B^{*0,i;\theta}_s, Y^i_s) = (0,1)\}} \mathrm{d}s\right],$$

where $k^+ := -\sum_{j \in \mathcal{Y} \setminus \{1\}} \lambda_{1,j} \left(\partial_b^+ V_j^\theta(0) - \partial_b^+ V_1^\theta(0)\right) > 0$. The positivity of k^+ has been demonstrated in the proof of the fourth claim in Lemma 10. Accordingly, I obtain

$$\int_0^t \mathbb{P}(\tau_\epsilon = \infty,\ (B^{*0,i;\theta}_s, Y^i_s) = (0,1)) \mathrm{d}s \le \frac{\partial_b^+ V_i^\theta(0)}{k^+} < \infty.$$

Hence,

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{P}(\tau_\epsilon = \infty,\ (B^{*0,i;\theta}_s, Y^i_s) = (0,1)) \mathrm{d}s = 0.$$

This implies that there exists a strictly positive and increasing sequence $(t_n)_{n \ge 1}$ with $t_n \to \infty$

such that

$$\lim_{n\to\infty}\mathbb{P}(\tau_\epsilon=\infty,\ (B^{*0,i;\theta}_{t_n},Y^i_{t_n})=(0,1))=\lim_{n\to\infty}\mathbb{P}((B^{*0,i;\theta}_{t_n},Y^i_{t_n})=(0,1)|\tau_\epsilon=\infty)\mathbb{P}(\tau_\epsilon=\infty)=0,$$

in which I have used the hypothesis $\mathbb{P}(\tau_\epsilon=\infty)>0$. Therefore, for any small $\eta>0$, there exists a time $t^*\in[0,\infty)$ such that $\mathbb{P}((B^{*0,i;\theta}_{t_n},Y^i_{t_n})=(0,1)|\tau_\epsilon=\infty)<\eta$ for any $t_n\geq t^*$. However, s^1_b satisfies the third and fourth claims in Lemma 10 even if $r=\rho$, and $B^{*0,i;\theta}_t$ is bounded on the event $\{\tau_\epsilon=\infty\}$. Hence, by applying the same argument as in the proof of the first and second claims in Proposition 6, there exists a deterministic time interval $T^*>0$ and a probability $q>0$ such that $\mathbb{P}((B^{*0,i;\theta}_{t_n+T^*},Y^i_{t_n+T^*})=(0,1)|\tau_\epsilon=\infty)\geq q$ for any $n\geq 1$. If $\eta<q$, this is a contradiction. Hence, $\mathbb{P}(\tau_\epsilon=\infty)=0$. Thus, $\mathcal{V}^{i,\theta}$ can reach any level $\epsilon>0$ in finite time.

Next, I show the stochastic asymptotic stability of $\mathcal{V}^{i,\theta}$. Fix an arbitrarily small $\epsilon\in(0,1)$ and $\eta>0$. Let $\tau_{\epsilon\eta}$ be the first hitting time of $\mathcal{V}^{i,\theta}$ at $\epsilon\eta$. Based on the preceding discussion, $\mathbb{P}(\tau_{\epsilon\eta}<\infty)=1$. Furthermore, I obtain $\epsilon\eta\geq\mathcal{V}^{i,\theta}_{\tau_{\epsilon\eta}}$ from the right continuity of $\mathcal{V}^{i,\theta}$. Let $\tau_\eta:=\inf\{t\geq\tau_{\epsilon\eta}|\mathcal{V}^{i,\theta}_t\geq\eta\}$. Thus, τ_η is the first hitting time of $\mathcal{V}^{i,\theta}$ at η after $\tau_{\epsilon\eta}$. Since $\mathcal{V}^{i,\theta}$ is a uniformly integrable supermartingale, I obtain $\mathrm{E}_{\tau_{\epsilon\eta}}[\mathcal{V}^{i,\theta}_{\tau_\eta\wedge(t+\tau_{\epsilon\eta})}]\leq\mathcal{V}^{i,\theta}_{\tau_{\epsilon\eta}}\leq\epsilon\eta$ for any $t\geq 0$, $\mathbb{P}$-a.s. from the optional stopping theorem. Moreover, I obtain

$$\epsilon\eta\geq\mathrm{E}_{\tau_{\epsilon\eta}}[\mathcal{V}^{i,\theta}_{\tau_\eta\wedge(t+\tau_{\epsilon\eta})}]\geq\mathrm{E}_{\tau_{\epsilon\eta}}[\mathbb{1}_{\{\tau_\eta\leq t+\tau_{\epsilon\eta}\}}\mathcal{V}^{i,\theta}_{\tau_\eta\wedge(t+\tau_{\epsilon\eta})}]\geq\eta\mathbb{P}(\tau_\eta\leq t+\tau_{\epsilon\eta}|\mathcal{F}_{\tau_{\epsilon\eta}})$$

for any $t\geq 0$, $\mathbb{P}$-a.s. Hence, setting the limit as $t\to\infty$ yields $\mathbb{P}(\tau_\eta<\infty|\mathcal{F}_{\tau_{\epsilon\eta}})\leq\epsilon$, $\mathbb{P}$-a.s. This implies that $\mathbb{P}(\mathcal{V}^{i,\theta}_t\leq\eta\text{ for all }t\geq\tau_{\epsilon\eta}|\mathcal{F}_{\tau_{\epsilon\eta}})=\mathbb{P}(\tau_\eta=\infty|\mathcal{F}_{\tau_{\epsilon\eta}})\geq 1-\epsilon$, $\mathbb{P}$-a.s. Hence, I obtain

$$\begin{aligned}\mathbb{P}\left(\limsup_{t\to\infty}\mathcal{V}^{i,\theta}_t\leq\eta\right)&\geq\mathrm{E}\left[\mathbb{1}\{\tau_{\epsilon\eta}<\infty\}\mathbb{1}\{\mathcal{V}^{i,\theta}_t\leq\eta\text{ for all }t\geq\tau_{\epsilon\eta}\}\right]\\&=\mathrm{E}\left[\mathbb{1}\{\tau_{\epsilon\eta}<\infty\}\,\mathrm{E}_{\tau_{\epsilon\eta}}[\mathbb{1}\{\mathcal{V}^{i,\theta}_t\leq\eta\text{ for all }t\geq\tau_{\epsilon\eta}\}]\right]\\&=\mathrm{E}\left[\mathbb{1}\{\tau_{\epsilon\eta}<\infty\}\mathbb{P}(\mathcal{V}^{i,\theta}_t\leq\eta\text{ for all }t\geq\tau_{\epsilon\eta}|\mathcal{F}_{\tau_{\epsilon\eta}})\right]\geq\mathbb{P}(\tau_{\epsilon\eta}<\infty)(1-\epsilon)=1-\epsilon.\end{aligned}$$

This implies that $\limsup_{t\to\infty}\mathcal{V}^{i,\theta}_t=0$, $\mathbb{P}$-a.s., which is the second claim.

<u>*Third claim.*</u> Let $\theta=(r,w,g,\tau_c,\rho)\in\Theta$ and let $\overline{\theta}=(\rho,w,g,\tau_c,\rho)$. Without loss of generality, I suppose that $r>0$. From the second claim, I obtain $\liminf_{t\to\infty}B^{*0,i;\overline{\theta}}_t=\infty$ $\mathbb{P}$-a.s. Furthermore,

if $0 \leq b \leq b'$ and $r > 0$, $B_t^{*b,i;\theta} \leq B_t^{*b',i;\theta}$. This inequality can be demonstrated by applying the Gronwall inequality to $((B_t^{*b',i;\theta} - B_t^{*b,i;\theta}) \wedge 0)^2$. According to Proposition 6, I obtain

$$\int_{\mathcal{S}} b\mathrm{d}\mu^\theta(b,i) = \int_{\mathcal{S}^2} bP_t^\theta((\widetilde{b},\widetilde{i}),(\mathrm{d}b,\mathrm{d}i))\mu^\theta(\mathrm{d}\widetilde{b},\mathrm{d}\widetilde{i}) = \mathrm{E}[B_t^{*\widetilde{b},\widetilde{i};\theta}]$$

for any $t \geq 0$, where the distribution of $(\widetilde{b},\widetilde{i})$ is μ^θ. Furthermore, I obtain

$$\int_{\mathcal{S}} b\mathrm{d}\mu^\theta(b,i) = \mathrm{E}[B_t^{*\widetilde{b},\widetilde{i};\theta}] \geq \mathrm{E}[B_t^{*0,\widehat{i};\theta}],$$

where the distribution of $\widehat{i}$ is the stationary distribution of Y. Since $B_t^{*;\theta}$ is continuous with respect to θ (note that this continuity is not restricted to the region Θ) and since $B_t^{*;\theta}$ is bounded, the bounded convergence theorem yields

$$\liminf_{r\uparrow\rho} \int_{\mathcal{S}} b\mathrm{d}\mu^\theta(b,i) \geq \liminf_{r\uparrow\rho} \mathrm{E}[B_t^{*0,\widehat{i};\theta}] = \mathrm{E}[B_t^{*0,\widehat{i};\overline{\theta}}].$$

Thus, when taking the limit as $t \to \infty$, the Fatou lemma yields

$$\liminf_{r\uparrow\rho} \int_{\mathcal{S}} b\mathrm{d}\mu^\theta(b,i) \geq \liminf_{t\to\infty} \mathrm{E}[B_t^{*0,\widehat{i};\overline{\theta}}] \geq \mathrm{E}[\liminf_{t\to\infty} B_t^{*0,\widehat{i};\overline{\theta}}] = \infty.$$

This is the desired result of the third claim.

Fourth claim. Suppose that $l_\theta^*(\partial_b^+ V_i^\theta(0), i) = 0$ under the assumption in the fourth claim. Based on the first-order condition, I obtain $\partial_l u(c_\theta^*, 0) + wf(i)\partial_b^+ V_i^\theta(0) \leq 0$. Meanwhile, the optimal consumption is always an interior solution. Thus, the first-order condition with respect to the consumption holds with the equality: $\partial_c u(c_\theta^*, 0) - \tau_c \partial_b^+ V_i^\theta(0) = 0$. Furthermore, from the state-constraint boundary condition, I obtain $g(i)/\tau_c \geq c_\theta^*(\partial_b^+ V_i^\theta(0), i)$. Therefore, I have

$$wf(i) \leq -\tau_c \frac{\partial_l u(c_\theta^*(\partial_b^+ V_i^\theta(0), i), 0)}{\partial_c u(c_\theta^*(\partial_b^+ V_i^\theta(0), i), 0)} \leq -\tau_c \frac{\partial_l u(g(i)/\tau_c, 0)}{\partial_c u(g(i)/\tau_c, 0)},$$

where the second inequality of the above is owing to the monotonicity of $c \to -\partial_l u(c,0)/\partial_c u(c,0)$. However, this is a contradiction, and hence, $l_\theta^*(\partial_b^+ V_i^\theta(0), i) > 0$. Owing to the continuity of l_θ^* and $\partial_b^+ V_i^\theta$, there exist a positive constant $\underline{l}^* \in (0, \overline{L}]$ and time $t^* \in (0, \infty)$ such that

$l_\theta^*(\partial_b^+ V_i^\theta(b_t^{0,i}), i) > \underline{l}^*$ on $[0, t^*]$, where $b_t^{0,i}$ is a solution to the ODE (2.8) in state i starting at 0.

Consider the probability $q > 0$ and time interval $T^* > 0$ defined in the proof of the first and second claims in Proposition 6. Based on the Markov property of $(B^{*;\theta}, Y)$, I obtain

$$\int_{\mathcal{S}} l_\theta^*(\partial_b V_j^\theta(b), j) \mathrm{d}\mu^\theta(b, j) \geq \underline{l}^* \mu^\theta([0, b_{t^*}^{0,i}] \times i) = \underline{l}^* \int_{\mathcal{S} \times ([0, b_{t^*}^{0,i}] \times i)} P_{T^*+t^*}^\theta((\widetilde{b}, \widetilde{i}), (\mathrm{d}b, \mathrm{d}j)) \mu^\theta(\mathrm{d}\widetilde{b}, \mathrm{d}\widetilde{i})$$
$$\geq \underline{l}^* q \mathbb{P}(Y_t^1 \in \{1, i\} \text{ for any } t \in [0, t^*) \text{ and } Y_{t^*}^1 = i) > 0.$$

Thus, the desired result is obtained. □

The first, second, and third claims are inspired by the result of Chamberlain and Wilson (2000) in the discrete-time model. It can easily be observed that the third claim in Lemma 11 holds even if w is simultaneously changed on a compact set when $r \uparrow \rho$. The second claim in Lemma 11 can be interpreted as a sufficient condition of the transversality condition. It can easily be observed that $e^{-rt} B_t^{*0,i;\theta} \leq (\overline{g} + w\overline{fL})(1 - e^{-rt})/r$ when $r > 0$. Hence, $0 \leq \limsup_{t \to \infty} e^{-\rho t} \partial_b V_{Y_t^i}^\theta(B_t^{*0,i;\theta}) B_t^{*0,i;\theta} = \limsup_{t \to \infty} \mathcal{V}_t^{i,\theta} e^{-rt} B_t^{*0,i;\theta} \leq \limsup_{t \to \infty} \mathcal{V}_t^{i,\theta}(\overline{g} + w\overline{fL})(1 - e^{-rt})/r = 0$ if $r > 0$. Meanwhile, a simple calculation yields

$$0 = \lim_{t \to \infty} \mathrm{E}\left[\mathcal{V}_t^{i,\theta} e^{-rt} B_t^{*0,i;\theta}\right] = \mathrm{E}\left[\int_0^\infty e^{-rt} \mathcal{V}_t^{i,\theta}\Big(g(Y_t^i) + G_{Y_t^i}(\partial_b V_{Y_t^i}^\theta(B_t^{*0,i;\theta}))\Big)\mathrm{d}t\right].$$

Thus, $\mathcal{V}_t^{i,\theta}$ can also be interpreted as a Lagrange multiplier (the shadow price of the liquid asset). Note that the fourth claim is only a sufficient condition for a positive aggregate labor supply, so the aggregate labor supply may be positive even if the condition is not satisfied. Let us demonstrate Corollary 7 using Lemma 11.

Proof of Corollary 7. According to the first-order condition of the profit maximization of the representative firm with respect to the capital demand, I obtain

$$r = \partial_K F(K_D, L_D) - \delta = a\alpha \left(\frac{K_D}{L_D}\right)^{\alpha - 1} - \delta = a\alpha k_D^{\alpha - 1} - \delta,$$

where K_D is the amount of capital demand, L_D is the amount of labor demand, and $k_D = K_D/L_D$. Therefore, I obtain $k_D(r) := (a\alpha/(r + \delta))^{1/(1-\alpha)}$. Meanwhile, from the first-order condition with respect to the labor demand, I obtain

$$w(r) := w = \partial_L F(K_D, L_D) = (1-\alpha)a\left(\frac{K_D}{L_D}\right)^{\alpha} = (1-\alpha)a(k_D(r))^{\alpha} = (1-\alpha)a\left(\frac{a\alpha}{r+\delta}\right)^{\alpha/(1-\alpha)}.$$

It is clear that $r \to w(r) \in (0,\infty)$ is strictly decreasing.

For fixed g, τ_c, and ρ, let $\theta(r) := (r, w(r), g, \tau_c, \rho)$. Then, $\theta(r) \in \Theta$ if $r \in (-\delta, \rho)$. Furthermore, consider the following supply function:

$$k_S(r) := \frac{\int_{\mathcal{S}} b\mathrm{d}\mu^{\theta(r)}(b,i)}{\int_{\mathcal{S}} l^*_{\theta(r)}(\partial_b V_i^{\theta(r)}(b), i)\mathrm{d}\mu^{\theta(r)}(b,i)}$$

for any $r \in (-\delta, \rho)$. Since $w(r) \in (0,\infty)$ if $r \in (-\delta, \rho]$, $w(r) \to \infty$ as $r \downarrow -\delta$, and $f(i) > 0$ in some state $i \in \mathcal{Y}$, $\int_{\mathcal{S}} l^*_{\theta(r)}(\partial_b V_i^{\theta(r)}(b), i)\mathrm{d}\mu^{\theta(r)}(b,i)$ is strictly positive when r is sufficiently small based on the fourth claim in Lemma 11, and it is bounded above $\overline{L} > 0$. Hence, k_S can be defined when r is sufficiently small. By the third claim in Lemma 11, I obtain $\liminf_{r\uparrow\rho} k_S(r) = \infty$.

Suppose $r < 0$, and for any $i \in \mathcal{Y}$, let $\overline{b}_i^{\theta(r)}$ be the upper boundary of the support of $\mu^{\theta(r)}(\cdot, i)$. Then, $s_b^i(\overline{b}_i^{\theta(r)}; \theta(r)) = r\overline{b}_i^{\theta(r)} + g(i) + w(r)f(i)l^*_{\theta(r)}(\partial_b V_i^{\theta(r)}(\overline{b}_i^{\theta(r)}), i) - \tau_c(i)c^*_{\theta(r)}(\partial_b V_i^{\theta(r)}(\overline{b}_i^{\theta(r)}), i) = 0$. This implies $\overline{b}_i^{\theta(r)} < -(g(i) + w(r)f(i)l^*_{\theta(r)}(\partial_b V_i^{\theta(r)}(\overline{b}_i^{\theta(r)}), i))/r$. Furthermore, $b < -(\overline{g} + w(r)\overline{f}l^*_{\theta(r)}(\partial_b V_i^{\theta(r)}(b), i))/r$ for any $b \in [0, \overline{b}_i^{\theta(r)}]$ since $b \to l^*_{\theta(r)}(\partial_b V_i^{\theta(r)}(b), i)$ is non-increasing. Thus,

$$\int_{\mathcal{S}} b\mathrm{d}\mu^{\theta(r)}(b,i) \leq -\frac{\overline{g} + w(r)\overline{f}\int_{\mathcal{S}} l^*_{\theta(r)}(\partial_b V_i^{\theta(r)}(b), i)\mathrm{d}\mu^{\theta(r)}(b,i)}{r}.$$

Hence, if $r < 0$,

$$k_S(r) \leq -\frac{\overline{g}}{r\int_{\mathcal{S}} l^*_{\theta(r)}(\partial_b V_i^{\theta(r)}(b), i)\mathrm{d}\mu^{\theta(r)}(b,i)} - \frac{w(r)}{r}\overline{f}.$$

Let us consider the excess supply function $k_X(r) := k_S(r) - k_D(r)$. Then, I obtain $\liminf_{r\uparrow\rho} k_X(r) = \infty$. Meanwhile, according to (4.2), there exists $r \in (-\delta, 0)$ such that

$$k_X(r) \leq -\frac{\overline{g}}{r\int_{\mathcal{S}} l^*_{\theta(r)}(\partial_b V_i^{\theta(r)}(b), i)\mathrm{d}\mu^{\theta(r)}(b,i)} - \left(1 + \frac{r+\delta}{r}\frac{1-\alpha}{\alpha}\overline{f}\right)\left(\frac{a\alpha}{r+\delta}\right)^{1/(1-\alpha)} < 0.$$

Therefore, from the continuity of $r \to k_X(r)$, the intermediate value theorem implies that there exists at least one $r^* \in (-\delta, \rho)$ such that $k_X(r^*) = 0$. Accordingly, $(r^*, w(r^*))$ satisfies (4.3). □

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