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A Factor Pricing Model under Ambiguity: A Multi-Period Framework*

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Abstract

This paper is a multi-period extension of the factor pricing model under ambiguity as developed by Wakai (2018).

Keywords: Ambiguity aversion, asset pricing, factor pricing

JEL Classification Numbers: D81, G11, G12

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1 Introduction

This paper extends the factor pricing model under ambiguity as developed by Wakai (2018) to a multi-period setting. We derive the model by approximating the pricing kernel of a smooth model of decision making under ambiguity introduced by the Klibanoff, Marinacci, and Mukerji (2005, 2009). For the structure of ambiguity, we follow Ju and Miao (2012), where the economy evolves based on a Markov chain but the representative agent has ambiguous beliefs about the hidden states.

Wakai (2018) derives the factor pricing model under ambiguity based on a *rational belief*, a version of rational expectation hypothesis adopted to the smooth model of decision making under ambiguity, which is similar to a notion introduced in Maccheroni, Marinacci, and Ruffino (2013). In this paper, we do not assume a rational belief and allow the representative agent to have biased and ambiguous beliefs throughout the history of state evolution. This generates the difference in the interpretation of an ambiguity premium. In Wakai (2018), the ambiguity premium is generated by ambiguity aversion, but because of the rational belief, it is unclear why the representative agent behaves as if he has an ambiguous belief about fictitious regimes instead of utilizing the objective probability itself. On the other hand, the ambiguity premium of this paper is generated by the presence of ambiguity as well as the aversion to the ambiguity. Because ambiguity itself is *effectively present*, the ambiguity premium of this paper does not separate the premium due to the ambiguity aversion from the premium due to the ambiguity itself. However, the results derived in this paper are more suitable for an empirical estimation of the ambiguity premium based on a regime switching framework.

2 Setting

We consider a multi-period portfolio choice problem with time t varying over $\{0, 1, ..., T\}$, where $1 < T < \infty$. At each t, one of the states s is realized from a finite set S with more than three elements. Let $s^t = (s_0, s_1, ..., s_T)$ be a sequence of the realizations with the fixed initial state s_0.[1] We denote by $\sigma(S^t)$ a power set of S^t, which is embedded in to S^T in the usual fashion. The filtration $\{\mathcal{F}_t\}$ is given by $\mathcal{F}_t \equiv \sigma(S^t)$, where $\sigma(S^t)$ is the power set based on s^t, which is identified with the collection of s^T defined by

$$s^t \equiv \left\{ s^T = \left(s'^t, s_{t+1}..., s_T\right) \middle| s'^t = s^t \right\}.$$

Furthermore, $\mathcal{F}_{t+1}(s^t)$ is the event in $\mathcal{F}_{t+1}$ such that

$$\mathcal{F}_{t+1}(s^t) \equiv \left\{ s^{t+1} = \left(s'^t, s_{t+1}\right) \in S^{t+1} \middle| s'^t = s^t \right\}.$$

Let $(S^T, \sigma(S^T), P)$ be a probability space with nonnul states, and let P_t be the $\sigma(S^t)$-conditional of P with $P_0 = P$. This generates the one-step-ahead probability P_t^{+1}, which is the restriction of P_t to $\sigma(S^{t+1})$. The probability P is regarded as the objective probability.

There is the single representative agent in this economy, who is endowed with the positive and bounded consumption process $\{e_t\}$ adapted to the filtration $\{\mathcal{F}_t\}$, where $e_t(s^t) \in \mathbb{R}_{++}$. There are a finite number $K+1$ of assets that pay a positive amount of consumption good as a dividend, where $1 < K < |S|$. The first K assets are long-lived assets, whose payoffs are not deterministic, while the $(K+1)$th asset is the short-lived risk-free asset that pays one unit of consumption good and matures one-period ahead. Thus, at each time t, $(K+1)$th asset denotes a different short-lived risk free asset. All of assets have net zero supply at all times and states.

For an expositional purpose, we follow the notations and the structure of ambiguity as studied in Ju and Miao (2012). The representative agent believes that at each time t and

[1] We assume the finite state space for an expositional purpose. We can easily extend the finite state S to a general state space S.

state s^t, there are two regimes in this economy and that he is unsure which regime he faces.[2] Each regime z specifies the probability of one-period-ahead state realization, denoted by $\pi_{z,t}$, which is an absolutely continuous with respect to P_0^{+1} and independent of time t and state s^t. At each s^t, the investor believes that he will be in the first regime in the next period with the probability μ_t, where by convention, we abbreviate s^t out of the notation $\mu_t(s^t)$. The agent's belief μ_t evolves as follows: Let Λ be a time-and-state independent 2×2 transition matrix, and let μ_0 be the initial prior.[3] Given μ_t, the posterior μ_{t+1} is updated by the Bayes' rule defined by

$$\mu_{t+1} \equiv \frac{\lambda_{11}\mu_t\pi_{1,t}(s_{t+1}) + \lambda_{21}(1-\mu_t)\pi_{2,t}(s_{t+1})}{\mu_t\pi_{1,t}(s_{t+1}) + (1-\mu_t)\,\pi_{2,t}(s_{t+1})},$$

where $\lambda_{l'l}$ is the (l', l)th element of the transition matrix Λ. Furthermore, at each s^t, for a random variable x measurable with respect to $\mathcal{F}_{t+1}$, we denote by $E_{\pi_{z,t}}[x]$ is the expectation of x under the one-period-ahead probability measure $\pi_{z,t}$ conditional on $\mathcal{F}_{t+1}(s^t)$. In addition, at s^t, we also denote by $E_{\mu_t}[a]$ the expectation of a random variable $a = (a_1, a_2)$ under the probability $(\mu_t, 1-\mu_t)$, where a_z is a value for the regime z.

We assume that at each time t, the representative agent can trade assets without transaction cost and can short and borrow without restrictions. Let $c = \{c_t\}$, $d = \{d_t\}$, $q = \{q_t\}$, $\theta = \{\theta_t\}$ be a feasible consumption, dividends, asset prices, and asset holdings processes adapted to the filtration $\{\mathcal{F}_t\}$, where $c_t(s^t) > 0$, $d_t(s^t) = (d_t^1(s^t), ..., d_t^{K+1}(s^t))$ with $d_t^k(s^t) > 0$, $q_t(s^t) = (q_t^1(s^t), ..., q_t^{K+1}(s^t))$, $\theta_t(s^t) = (\theta_t^1(s^t), ..., \theta_t^{K+1}(s^t))$ for each s^t. Also, let θ_{-1} and $q_T(s^T)$ be defined as the vector of zeros. The budget constraints is: At time t,

$$c_t + \theta_t \cdot q_t = e_t + \theta_{t-1} \cdot (q_t + d_t). \tag{1}$$

The representative agent's preferences follow the *smooth model of decision making under ambiguity* as introduced by Klibanoff et al. (2005, 2009)

$$V_t(c) = u(c_t) + \beta v_{t+1}^{-1}\left(E_{\mu_t}\left[v_{t+1}\left(E_{\pi_{z,t}}\left[V_{t+1}\left(c\right)\right]\right)\right]\right), \tag{2}$$

[2] Wakai (2018) considers L possible regimes, which can be applied to this setting as well.

[3] We can make Λ dependent of ω^t without changing the result of this paper.

where both v_{t+1} and u are strictly increasing and strictly concave on the respective domain. The representative agent decides his consumption process c and asset holdings process θ so as to maximize the representation (2). Then the usual derivation based on the first order conditions leads to the equilibrium price q_t^k that satisfies

$$q_t^k = E_{\mu_t}\left[\left(v_{t+1}^{-1}\right)' \times v_{t+1}' \times E_{\pi_{z,t}}\left[m_{t+1} \times \left(q_{t+1}^k + d_{t+1}^k\right)\right]\right], \tag{3}$$

where

$$\begin{aligned}
\left(v_{t+1}^{-1}\right)' &\equiv \left(v_{t+1}^{-1}\left(E_{\mu_t}[v_{t+1}\left(E_{\pi_{z,t}}\left[V_{t+1}(c)\right]\right)]\right)\right)', \qquad (4)\\
v_{t+1}' &\equiv v_{t+1}'\left(E_{\pi_{z,t}}\left[V_{t+1}(c)\right]\right), \text{ and}\\
m_{t+1} &\equiv \frac{\beta u'\left(c_{t+1}\right)}{u'(c_t)}.
\end{aligned}$$

Let R_t^k be the time-t gross return of kth risky asset, and let R_t^f be the time-t gross return of the risk-free asset, that is, $(K+1)$th asset. We denote by $Cov_{\mu_t}[a,b]$ the covariance between two-valued vectors a and b under $(\mu_t, 1-\mu_t)$. A variance of a two-valued vector a under $(\mu_t, 1-\mu_t)$, $Var_{\mu_t}[a]$, is similarly defined. In addition, for random variables x and y measurable with respect to $\mathcal{F}_{t+1}$, let $Cov_{\pi_{z,t}}[x,y]$ be a covariance between x and y under the one-period-ahead probability measure $\pi_{z,t}$ conditional on $\mathcal{F}_{t+1}(s^t)$. We also use $Cov_{P_t}[x,y]$ to denote the covariance between $\mathcal{F}_{t+1}$-measurable x and y under P_t condition on $\mathcal{F}_{t+1}(s^t)$. A variance of $\mathcal{F}_{t+1}$-measurable x under P_t, $Var_{P_t}[x]$, is similarly defined. Note that for $\mathcal{F}_{t+1}$-measurable x and y, $Cov_{P_t^{+1}}[x,y] = Cov_{P_t}[x,y]$ and $Var_{P_t^{+1}}[x] = Var_{P_t}[x]$.

3 Factor Pricing under Ambiguity

To identify the effect of ambiguity, we follow Ju and Miao (2012) and introduce an assumption that links the agent's belief and the objective probability.

Assumption 1: (Regime Switching)

(i) P is generated by a regimen switching model with two regimes, where $\pi_{z,t}$ is the objective probability of state realization under the regime z and ρ_t is the probability of time-$(t+1)$ realization of the first regime.
(ii) φ_t is the Radon-Nykodym derivative where $\mu_t = \varphi_t[1]\rho_t$ and $1-\mu_t = \varphi_t[2]\,(1-\rho_t)$.

Under this assumption, the representative agent is sophisticated enough to know that the economy evolves based on a regime switching model with two regimes. Moreover, the agent knows the one-step-ahead objective probability $\pi_{z,t}$. What is unknown to the agent is the objective probability between these regimes. If each value of φ_t is one, we say that the agent has a *rational belief*.[4]

We first identify the part of returns associated with the variation of m_{t+1} separately from the part of the returns associated with the variation of v'_{t+1}. Let $\widehat{m}_{t+1}$ be the probability-weighted projection of m_{t+1} onto the span of $\left\{R^1_{t+1}, ..., R^K_{t+1}, R^f_{t+1}\right\}$ under P_t. As adopted from Wakai (2018), we then impose the following.

Assumption 2: (Spanning Condition on $\mathbf{m_{t+1}}$)

There exist a set of gross portfolio returns $\left\{R^{RF_1}_{t+1}, ..., R^{RF_I}_{t+1}\right\}$ with $I < K$ such that[5]
(i) $\left\{R^{RF_1}_{t+1}, ..., R^{RF_I}_{t+1}, R^f_{t+1}\right\}$ is linearly independent.
(ii) $\widehat{m}_{t+1} = a^0_t R^f_{t+1} + \sum_{i=1}^{I} a^i_t R^{RF_i}_{t+1}$, where a^i_t are measurable with respect to $\mathcal{F}_t$.

It is well know that Assumption 2 leads to the factor pricing model if the agent is ambiguity neutral.

The following proposition, which is the multi-period extension of Wakai (2018), shows that the regression constant captures a premium related to the presence of ambiguity as well as the aversion to ambiguity (see Appendix A).

[4] In a two-period model of Wakai (2018), we do not assume the existence of the objective regimes.

[5] If the asset market is complete and the number of factors is $|S|-1$, then $\alpha^i_t = 0$ for all k.

Proposition 1:

Suppose that Assumptions 1 and 2 hold. For each k, the gross return of asset k satisfies the factor pricing formula

$$E_{P_t}[R_{t+1}^k - R_{t+1}^f] = \alpha_t^k + \sum_{i=1}^{I} \beta_t^{k,i} E_{P_t}[R_{t+1}^{RF_i} - R_{t+1}^f], \tag{5}$$

or

$$R_{t+1}^k - R_{t+1}^f = \alpha_t^k + \sum_{i=1}^{I} \beta_t^{k,i} \left(R_{t+1}^{RF_i} - R_{t+1}^f \right) + \varepsilon_{t+1}^k, \tag{6}$$

where for each i, $\beta_t^{k,i}$ is a regression coefficient for $R_{t+1}^{RF_i} - R_{t+1}^f$. Furthermore, $E_{P_t}\left[\varepsilon_{t+1}^k\right] = 0$ and α_t^k satisfies

$$\alpha_t^k = -\frac{E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1} \times \varepsilon_{t+1}^k\right]\right]}{E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right]\right]}. \tag{7}$$

Moreover, if the representative agent is ambiguity neutral and has a rational belief, α_t^k is zero.

Because both the presence of ambiguity and the aversion to ambiguity influence the gross return, we call the term $E_{P_t}[R_{t+1}^{RF_i} - R_{t+1}^f]$ the *gross factor risk premium*. Moreover, if the agent is ambiguity neutral (that is, v_{t+1} is linear) with a rational belief, α_t^k is zero because

$$E_{\mu_t}\left[E_{\pi_{z,t}}\left[m_{t+1} \times \varepsilon_{t+1}^k\right]\right] = E_{\rho_t}\left[\varphi_t \times E_{\pi_{z,t}}\left[m_{t+1} \times \varepsilon_{t+1}^k\right]\right] = E_{P_t}\left[m_{t+1} \times \varepsilon_{t+1}^k\right] = 0.$$

Thus, the regression constant α_t^k captures a premium related to both ambiguity and an attitude toward ambiguity.

Now, we want to identify the part of returns associated with the variation of $\varphi_t \times v'_{t+1}$. First, consider the situation where the agent is risk neutral under a rational belief. Then equation (5) becomes

$$E_{P_t}[R_{t+1}^k - R_{t+1}^f] = \alpha_t^k,$$

where α_t^k satisfies equation (7) and it is zero if the agent is ambiguity neutral. Risk neutrality also implies that

$$\begin{aligned} \alpha_t^k &= -\frac{E_{\rho t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1} \times \varepsilon_{t+1}^k\right]\right]}{E_{\rho t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right]\right]} \\ &= -\frac{E_{\rho t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right] \times E_{\pi_{z,t}}\left[\varepsilon_{t+1}^k\right]\right]}{E_{\rho t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right]\right]} \end{aligned} \tag{8}$$

because m_{t+1} and $E_{\pi_{z,t}}\left[m_{t+1}\right]$ are constant.

We now impose the condition where (8) holds even if the agent is risk averse and does not have a rational belief. This is the key assumption that deviates from the conventional assumption used for the factor pricing model under risk.

Assumption 3:

For each k, $Cov_{\pi_{z,t}}[m_{t+1}, \varepsilon_{t+1}^k] = 0$ for each z.[6]

Assumption 3 is adopted from Wakai (2018), which states that at each regime z, the regression residual does not generate factor risk premium. Thus, one of the channels of the connection between risk and ambiguity is absent, which is sufficient for (8) to hold. However, risk aversion makes $E_{\pi_{z,t}}\left[m_{t+1}\right]$ nonconstant. Thus, unlike the risk neutral case, we cannot isolate the exact effect of ambiguity from the effect of risk aversion.

In (8), the conditional connection between m_{t+1} and ε_{t+1}^k expressed by $E_{\pi_{z,t}}\left[m_{t+1} \times \varepsilon_{t+1}^k\right]$ becomes $E_{\pi_{z,t}}\left[m_{t+1}\right] \times E_{\pi_{z,t}}\left[\varepsilon_{t+1}^k\right]$. Therefore, we can regard $\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right]$ as a *ambiguity pricing kernel* defined on $Z = \{1, 2\}$, which is used to compute the ambiguity premium of random variable $E_{\pi_{z,t}}\left[\varepsilon_{t+1}^k\right]$. We then follow Wakai (2018) and impose the spanning condition, where $R_{t+1}^{RF_1}$ is a gross return of the first risk factor in equation (5).

Assumption 4: (Spanning Condition on $\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right]$)

(i) $\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right] = \widetilde{a}_t^0 E_{\pi_{z,t}}\left[R_{t+1}^f\right] + \widetilde{a}_t^1 E_{\pi_{z,t}}\left[R_{t+1}^{RF_1}\right]$, where $\widetilde{a}_t^i$ are measurable with respect to $\mathcal{F}_t$.

[6] Wakai (2018) investigates a canonical example that guarantees Assumption 3.

Note that for a two-regime model, (i) holds *generically*. Thus, we can safely omit Assumption 4.

In Wakai (2018), because of the rational belief assumption, the spanning condition is defined on $v'_{t+1} \times E_{\pi_{z,t}}[m_{t+1}]$. Here, we no longer assume the rational belief so that the spanning condition is defined on $\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}[m_{t+1}]$, where the effect due to the presence of ambiguity is captured by φ_t.

Given Assumption 4, the following interpretation is adopted from Wakai (2018): We can use $E_{\pi_{z,t}}\left[R^{RF_1}_{t+1}\right]$ to measure the ambiguity relevant to price. For this part, we introduce an *ambiguity beta*. For each k satisfying $1 \leq k \leq K$,

$$\beta_t^{k,A} \equiv \frac{Cov_{\rho_t}\left[E_{\pi_{z,t}}\left[R^k_{t+1}\right], E_{\pi_{z,t}}\left[R^{RF_1}_{t+1}\right]\right]}{Var_{\rho_t}\left[E_{\pi_{z,t}}\left[R^{RF_1}_{t+1}\right]\right]}, \tag{9}$$

and for each i satisfying $1 \leq i \leq I$,

$$\beta_t^{RF_i,A} \equiv \frac{Cov_{\rho_t}\left[E_{\pi_{z,t}}\left[R^{RF_i}_{t+1}\right], E_{\pi_{z,t}}\left[R^{RF_1}_{t+1}\right]\right]}{Var_{\rho_t}\left[E_{\pi_{z,t}}\left[R^{RF_1}_{t+1}\right]\right]}. \tag{10}$$

As for the interpretation of $\beta_t^{k,A}$, $Var_{\rho_t}\left[E_{\pi_{z,t}}\left[R^{RF_1}_{t+1}\right]\right]$ measures a degree of ambiguity embedded in $E_{\pi_{z,t}}\left[R^{RF_1}_{t+1}\right]$, and $Cov_{P_t^{+1}}\left[E_{\pi_{z,t}}\left[R^k_{t+1}\right], E_{\pi_{z,t}}\left[R^{RF_1}_{t+1}\right]\right]$ measures the contribution of asset k's expected return to this ambiguity. Thus, $\beta_t^{k,A}$ defines the compensation scheme for ambiguity, which is analogous to that for factor risk.

We now define a portfolio that has an exposure only to ambiguity embedded in $E_{\pi_{z,t}}\left[R^{RF_1}_{t+1}\right]$.

Assumption 5: (Ambiguity Factor)

There exists a portfolio AF such that
(i) for each i satisfying $1 \leq i \leq I$, $\beta_t^{AF,i} = 0$,
(ii) $\beta_t^{AF,A} = 1$.

If K is sufficiently large, we can *generically* construct a portfolio AF, which, however, may not be unique.

The following proposition shows that the portfolio AF captures a ambiguity premium, which induces an explicit formula for α_t^k in (7) (see Appendix B).

Proposition 2:

Suppose that Assumptions 1 to 5 hold. Then, for each k, there exists a $\mathcal{F}_t$-measurable process $\left\{\beta_t^{k,AF}\right\}$ such that the gross return of asset k follows the factor pricing formula

$$E_{P_t}[R_{t+1}^k - R_{t+1}^f] = \beta_t^{k,AF} E_{P_t}[R_{t+1}^{AF} - R_{t+1}^f] + \sum_{i=1}^{I} \beta_t^{k,i} E_{P_t}[R_{t+1}^{RF_i} - R_{t+1}^f] \tag{11}$$

or

$$R_{t+1}^k - R_{t+1}^f = \beta_t^{k,AF} E_{P_t}[R_{t+1}^{AF} - R_{t+1}^f] + \sum_{i=1}^{I} \beta_t^{k,i} \left(R_{t+1}^{RF_i} - R_{t+1}^f\right) + \varepsilon_{t+1}^k,$$

where

$$\beta_t^{k,AF} = \left\{\beta_t^{k,A} - \sum_{i=1}^{I} \beta_t^{k,i} \times \beta_t^{RF_i,A}\right\},$$

and $\beta_t^{k,i}$ and ε_{t+1}^k are defined in Proposition 1.

We call R_{t+1}^{AF} and $\beta_t^{k,AF}$ a *ambiguity factor* and *risk-adjusted ambiguity beta*, respectively. Thus, $\beta_t^{k,AF} E_{P_t}[R_{t+1}^{AF} - R_{t+1}^f]$ represents the effect of ambiguity and ambiguity aversion after subtracting the effect of risk aversion, that is, the effect of ambiguity and ambiguity aversion net of risk aversion. Because (11) separates the expected return of ambiguity factor from the expected returns of risk factors, it is the specification most suitable for empirical studies.

Also, we can rewrite (11) as follows

$$\begin{aligned} & E_{P_t}[R_{t+1}^k - R_{t+1}^f] - \beta_t^{k,A} E_{P_t}[R_{t+1}^{AF} - R_{t+1}^f] \\ &= \sum_{i=1}^{I} \beta_t^{k,i} \left\{E_{P_t}[R_{t+1}^{RF_i} - R_{t+1}^f] - \beta_t^{RF_i,A} E_{P_t}[R_{t+1}^{RF_i} - R_{t+1}^f]\right\}. \end{aligned} \tag{12}$$

The left-hand side of (12) is the *ambiguity-adjusted* risk premium or *net* risk premium of the asset k, where $\beta_t^{k,A} E_{P_t}[R_{t+1}^{AF} - R_{t+1}^f]$ represents the gross ambiguity-factor premium associate with R_{t+1}^k. In the right-had side, for each i,

$$E_{P_t}[R_{t+1}^{RF_i} - R_{t+1}^f] - \beta_t^{RF_i,A} E_{P_t}[R_{t+1}^{RF_i} - R_{t+1}^f]$$

is the *ambiguity-adjusted* factor risk premium or *net* factor risk premium, where $\beta_t^{RF_i,A} E_{P_t}[R_{t+1}^{RF_i} - R_{t+1}^f]$ represents the gross ambiguity-factor premium associate with $R_{t+1}^{RF_i}$. Thus, (12) represents the relation between *ambiguity-adjusted* risk premia

Appendix A: The Proof of Proposition 1

By (3),

$$\left(v_{t+1}^{-1}\right)' = \frac{q_t^{K+1}}{E_{\mu_t}\left[v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right]\right]}.$$

Thus,

$$q_t^k = \frac{E_{\mu_t}\left[v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1} \times \left(q_{t+1}^k + d_{t+1}^k\right)\right]\right]}{E_{\mu_t}\left[v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right]\right] \times \dfrac{1}{q_t^{K+1}}}. \tag{13}$$

By using the Radon-Nykodym derivative, (13) is rewritten as

$$q_t^k = \frac{E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1} \times \left(q_{t+1}^k + d_{t+1}^k\right)\right]\right]}{E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right]\right] \times \dfrac{1}{q_t^{K+1}}} \tag{14}$$

By applying the standard statistical relation, (14) leads to

$$1 = \frac{\left\{\begin{array}{c} E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times Cov_{\pi_{z,t}}\left[m_{t+1}, R_{t+1}^k\right]\right] \\ +E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right] E_{\pi_{z,t}}\left[R_{t+1}^k\right]\right] \end{array}\right\}}{E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right]\right] \times R_{t+1}^f}. \tag{15}$$

The second term of the numerator in the right-hand side is rewritten as

$$\begin{aligned} & E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right] E_{\pi_{z,t}}\left[R_{t+1}^k\right]\right] \\ & = Cov_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right], E_{\pi_{z,t}}\left[R_{t+1}^k\right]\right] \\ & + E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right]\right] E_{\rho_t}\left[E_{\pi_{z,t}}\left[R_{t+1}^k\right]\right]. \end{aligned} \tag{16}$$

Because $E_{\rho_t}\left[E_{\pi_{z,t}}\left[R_{t+1}^k\right]\right] = E_{P_t^{+1}}[R_{t+1}^k] = E_{P_t}[R_{t+1}^k]$, (15) and (16) imply

$$\begin{aligned} & E_{P_t}[R_{t+1}^k - R_{t+1}^f] \\ & = -\frac{\left\{\begin{array}{c} E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times Cov_{\pi_{z,t}}\left[m_{t+1}, R_{t+1}^k\right]\right] \\ +Cov_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right], E_{\pi_{z,t}}\left[R_{t+1}^k\right]\right] \end{array}\right\}}{E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right]\right]}. \end{aligned} \tag{17}$$

Given (17), for each portfolio return $R_{t+1}^{RF_i}$

$$\begin{aligned}
&E_{P_t}[R_{t+1}^{RF_i} - R_{t+1}^{f}] \qquad (18)\\
&= -\frac{\left\{\begin{array}{c} E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times Cov_{\pi_{z,t}}\left[m_{t+1}, R_{t+1}^{RF_i}\right]\right] \\ +Cov_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right], E_{\pi_{z,t}}\left[R_{t+1}^{RF_i}\right]\right] \end{array}\right\}}{E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right]\right]}.
\end{aligned}$$

Now, we consider the following regression

$$E_{P_t}[R_{t+1}^{k} - R_{t+1}^{f}] = \alpha_t^k + \sum_{i=1}^{I} \beta_t^{k,i} E_{P_t}[R_{t+1}^{RF_i} - R_{t+1}^{f}], \qquad (19)$$

or

$$R_{t+1}^{k} - R_{t+1}^{f} = \alpha_t^k + \sum_{i=1}^{I} \beta_t^{k,i} \left(R_{t+1}^{RF_i} - R_{t+1}^{f}\right) + \varepsilon_{t+1}^{k}, \qquad (20)$$

where $E_{P_t}\left[\varepsilon_{t+1}^k\right] = 0$. By applying (20) to (17), Assumptions 1 and 2 and (18) imply that

$$E_{P_t}[R_{t+1}^{k} - R_{t+1}^{f}] = \widehat{\alpha}_t^k + \sum_{i=1}^{I} \beta_t^{k,i} E_{P_t}[R_{t+1}^{RF_i} - R_{t+1}^{f}], \qquad (21)$$

where

$$\widehat{\alpha}_t^k = -\frac{\left\{\begin{array}{c} E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times Cov_{\pi_{z,t}}\left[m_{t+1}, \varepsilon_{t+1}^{k}\right]\right] \\ +Cov_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right], E_{\pi_{z,t}}\left[\varepsilon_{t+1}^{k}\right]\right] \end{array}\right\}}{E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right]\right]}.$$

By comparing (19) and (21),

$$\begin{aligned}
\alpha_t^k &= \widehat{\alpha}_t^k \qquad (22)\\
&= -\frac{\left\{\begin{array}{c} E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times Cov_{\pi_{z,t}}\left[m_{t+1}, \varepsilon_{t+1}^{k}\right]\right] \\ +Cov_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right], E_{\pi_{z,t}}\left[\varepsilon_{t+1}^{k}\right]\right] \end{array}\right\}}{E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right]\right]}.
\end{aligned}$$

The first term of the numerator of the right-hand side becomes

$$\begin{aligned}
&E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times Cov_{\pi_{z,t}}\left[m_{t+1}, \varepsilon_{t+1}^{k}\right]\right] \qquad (23)\\
&= E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1} \times \varepsilon_{t+1}^{k}\right]\right]\\
&- E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right] \times E_{\pi_{z,t}}\left[\varepsilon_{t+1}^{k}\right]\right],
\end{aligned}$$

and the second term of the numerator of the right-hand side becomes

$$\begin{aligned}
&Cov_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right], E_{\pi_{z,t}}\left[\varepsilon^k_{t+1}\right]\right] \qquad (24)\\
&= E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right] \times E_{\pi_{z,t}}\left[\varepsilon^k_{t+1}\right]\right]\\
&- E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right]\right] E_{\rho_t}\left[E_{\pi_{z,t}}\left[\varepsilon^k_{t+1}\right]\right],
\end{aligned}$$

where the last term is zero because $E_{\rho_t}\left[E_{\pi_{z,t}}\left[\varepsilon^k_{t+1}\right]\right] = E_{P_t^{+1}}\left[\varepsilon^k_{t+1}\right] = E_{P_t}\left[\varepsilon^k_{t+1}\right] = 0$. Given (23) and (24), (22) is rewritten as

$$\alpha^k_t = -\frac{E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1} \times \varepsilon^k_{t+1}\right]\right]}{E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right]\right]},$$

which completes the proof. ■

Appendix B: The Proof of Proposition 2

By Assumptions 3 and 4, (7) is rewritten as

$$\alpha^k_t = E_{\rho_t}\left[\left\{\gamma^0_t E_{\pi_{z,t}}\left[R^f_{t+1}\right] + \gamma^1_t E_{\pi_{z,t}}\left[R^{RF_1}_{t+1}\right]\right\} \times E_{\pi_{z,t}}\left[\varepsilon^k_{t+1}\right]\right], \qquad (25)$$

where $\gamma^0_t \equiv -\frac{\widetilde{a}^0_t}{E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right]\right]}$ and $\gamma^1_t \equiv -\frac{\widetilde{a}^1_t}{E_{\rho_t}\left[\varphi_t \times v'_{t+1} \times E_{\pi_{z,t}}\left[m_{t+1}\right]\right]}$.

We rewrite (25) as

$$\begin{aligned}
\alpha^k_t &= E_{\rho_t}\left[\left\{\gamma^0_t E_{\pi_{z,t}}\left[R^f_{t+1}\right] + \gamma^1_t E_{\pi_{z,t}}\left[R^{RF_1}_{t+1}\right]\right\} \times E_{\pi_{z,t}}\left[\varepsilon^k_{t+1}\right]\right]\\
&= \gamma^1_t E_{\rho_t}\left[E_{\pi_{z,t}}\left[R^{RF_1}_{t+1}\right] \times E_{\pi_{z,t}}\left[\varepsilon^k_{t+1}\right]\right]\\
&= \gamma^1_t \left\{Cov_{\rho_t}\left[E_{\pi_{z,t}}\left[R^{RF_1}_{t+1}\right], E_{\pi_{z,t}}\left[\varepsilon^k_{t+1}\right]\right] + E_{\rho_t}\left[E_{\pi_{z,t}}\left[R^{RF_1}_{t+1}\right]\right] E_{\rho_t}\left[E_{\pi_{z,t}}\left[\varepsilon^k_{t+1}\right]\right]\right\}\\
&= \gamma^1_t Cov_{\rho_t}\left[E_{\pi_{z,t}}\left[R^{RF_1}_{t+1}\right], E_{\pi_{z,t}}\left[\varepsilon^k_{t+1}\right]\right],
\end{aligned}$$

where the last line follows from $E_{\rho_t}\left[E_{\pi_{z,t}}\left[\varepsilon^k_{t+1}\right]\right] = E_{P_t^{+1}}\left[\varepsilon^k_{t+1}\right] = E_{P_t}\left[\varepsilon^k_{t+1}\right] = 0$. By definition of ε_{t+1}^{k},

$$\begin{aligned}
&\gamma_t^1 Cov_{\rho_t}\left[E_{\pi_{z,t}}\left[R_{t+1}^{RF_1}\right], E_{\pi_{z,t}}\left[\varepsilon_{t+1}^{k}\right]\right] \\
=&\gamma_t^1 Cov_{\rho_t}\left[E_{\pi_{z,t}}\left[R_{t+1}^{RF_1}\right], E_{\pi_{z,t}}\left[R_{t+1}^{k}-R_{t+1}^{f}-\alpha_t^k-\sum_{i=1}^{I}\beta_t^{k,i}\left(R_{t+1}^{RF_i}-R_{t+1}^{f}\right)\right]\right] \\
=&\gamma_t^1\left\{Cov_{\rho_t}\left[E_{\pi_{z,t}}\left[R_{t+1}^{RF_1}\right], E_{\pi_{z,t}}\left[R_{t+1}^{k}\right]\right]-\sum_{i=1}^{I}\beta_t^{k,i}Cov_{\rho_t}\left[E_{\pi_{z,t}}\left[R_{t+1}^{RF_1}\right], R_{t+1}^{RF_i}\right]\right\} \\
&=\gamma_t^1 Var_{\rho_t}\left[E_{\pi_{z,t}}\left[R_{t+1}^{RF_1}\right]\right]\left\{\beta_t^{k,A}-\sum_{i=1}^{I}\beta_t^{k,i}\times\beta_t^{RF_i,A}\right\}.
\end{aligned}$$

Therefore,

$$\alpha_t^k=\gamma_t^1 Var_{\rho_t}\left[E_{\pi_{z,t}}\left[R_{t+1}^{RF_1}\right]\right]\left\{\beta_t^{k,A}-\sum_{i=1}^{I}\beta_t^{k,i}\times\beta_t^{RF_i,A}\right\}. \tag{26}$$

Now, for the ambiguity factor, by Assumption 5, (5) and (26) lead to

$$E_{P_t}[R_{t+1}^{AF}-R_{t+1}^{f}]=\alpha_t^{AF}=\gamma_t^1 Var_{\rho_t}\left[E_{\pi_{z,t}}\left[R_{t+1}^{RF_1}\right]\right]. \tag{27}$$

Then, by (27), (26) becomes

$$\alpha_t^k=\beta_t^{k,AF}E_{P_t}[R_{t+1}^{AF}-R_{t+1}^{f}],$$

where

$$\beta_t^{k,AF}=\left\{\beta_t^{k,A}-\sum_{i=1}^{I}\beta_t^{k,i}\times\beta_t^{RF_i,A}\right\},$$

as desired. ■

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