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Ramsey-Cass-Koopmans Model with Declining Population*

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Abstract

This paper investigates how population decline may affect the optimal path in two types of Ramsey-Cass-Koopmans (*RCK*) model with child rearing costs. An optimal path exists in both models under economically plausible conditions and a new type of optimal path exists in the model where the discount rate is only the time preference rate. Under population decline, the existence and properties of an optimal path depend on the range of the rates of population change, regardless of the child rearing costs. First, when population decline is mild, the optimal path is a saddle-point path converging to a finite steady state, as in the standard *RCK* model with increasing population. Second, when population decline is faster, the optimal path is a saddle-point path converging, by *reversible investment*, to a finite steady state (i.e., a balanced growth path (*BGP*)), at which per capita consumption is *larger* than per capita income. Third, when population decline is even faster, the optimal path can be an *asymptotically BGP*, along which both per capita consumption and income keep increasing permanently. We show empirical relevance of these optimal paths by Japanese data and *World Population Prospects 2019*.

Keywords: population decline; dynamic optimization; long-run per capita growth; child rearing cost

JEL Classification: E21; J11; O11; O41

1. Introduction

In recent years population decline has been observed in many developed countries. In the coming 50 years

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even developing countries will experience population decline due to a decline in the fertility rate. Global population today, as Jones (2022) indicates, has a distinct possibility of declining rather than stabilizing in the long run. The world economy could be regarded as entering into the era of population decline.[1]

[Tables 1 and 2 around here]

It is widely believed in real societies with declining population that population decline has negative real effects on economic growth, including negative impacts on national income and thus the pension system and the tax system. However, even if population decline decreases the aggregate real income, *per capita* real income (and consumption), which is decisive for economic welfare, will increase if the increasing effect on per capita capital (or income) due to population decline dominates the decreasing effect on national income. Otherwise, it may decrease per capita income and thus deteriorate economic welfare.

In economics literature, Ritschl (1985) shows in the neoclassical growth model a la Solow (1956) that under a negative rate of population change, a steady state with a positive value of per capita capital stock exists if the exogenous saving rate is negative. Then, the steady state is *unstable*, and therefore, if the initial level of capital lies below its steady-state level, per capita capital and income decrease over time, converging to zero. This may provide a theoretical foundation for the concern that per capita income will decrease due to population decline in the growth process.[2] More recently, Jones (2022) has shown a

[1] The UN *World Population Prospects 2019* provides future predictions of the rates of population change in the main regions of the world, where the middle variants of fertility rate (in Tables 1) and the low variants (in Table 2) are shown. Even in the middle variant scenario, many regions of the world are expected to experience negative population growth. In particular, in 2070, almost all regions and countries presented in the table will experience negative population growth.
[2] Endogenous growth models with scale effect (i.e., the long-run growth rate of per capita output depends positively on the size of population), such as Romer (1990), may lead to a reduction in the growth rate of income when population decreases. In semi-endogenous growth models without scale effect (i.e., the long-run growth rate of per capita output depends positively on the population growth rate, and not on the size of population), such as Jones (1995), the growth

theoretical possibility that living standards stagnate rather than continued exponential growth.[3]

In the recent trend of growth analysis under population decline, it has been intensively studied whether there is an equilibrium growth path along which per capita income and consumption increase. In a simple semi-endogenous growth model with an exogenous saving rate and an aggregate capital-input externality, Christiaans (2011) shows that the long-run growth rate of per capita output exhibits non-monotonous dependency on negative population growth rate: when the negative rates of population change is close to (*resp.* far from) zero, the long-run growth rate of per capita output depends positively (*resp.* negatively) on the rate of population change. In particular, the long-run growth rate of per capita output becomes negative when the population growth rate gets into the negative range. In the semi-endogenous research and development (R&D) growth model by Jones (1995) with an exogenous saving rate, Sasaki and Hoshida (2017) show that when the population growth rate is negative, the rate of technological progress converges to zero and the growth rate of per capita output will be positive. This study uses the Cobb-Douglas (CD) production function, whose elasticity of substitution between labor and capital is unity.[4] Proceeding to a Solow growth model with the CES production function, Sasaki (2019) shows that under population decline, the long-run growth rate of per capita output is equal to the exogenously given rate of technological progress if the elasticity of substitution is less than unity, which is empirically relevant.[5] Then, as long as the rate of technological progress is zero, the growth rate of per

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rate of income tends to be lower when faster population decline occurs. It is not clear, however, whether the growth rate of income can be negative in these models.

[3] Jones (2020, 2022) investigates long-run consequences of population decline in an endogenous growth model whose engine of growth is knowledge production by R&D. When the fertility rate is negative, two steady states exist. One is a steady state in which population, knowledge, and standard of living continues to increase exponentially. The other is a steady state in which population continues to decrease, and knowledge production and standard of living are stagnant. He emphasizes technological progress driven by knowledge production and abstracts capital accumulation.

[4] Sasaki (2015a) builds a small open economy growth model and investigates the relationship between trade patterns and the per capita output growth rate. In this model, he shows that per capita output continues to increase even if the rate of population growth is negative.

[5] For estimations of the elasticity of substitution between labor and capital, see Rowthorn (1999), Antras (2004) Klump, McAdam, and Willman (2007), Chirinko (2008), and Chirinko and Mallick (2017). These studies show that the elasticity of substitution is less than unity.

capita output is zero. By contrast, if the elasticity of substitution is unity (i.e., CD production function), the long-run growth rate of per capita output can be positive even without technological progress. Furthermore, Sasaki (2023) has analyzed effects of population decline in a growth model with automation capital in final goods production, which is based on Prettner (2019). He finds that when the absolute value of negative population growth rate is small, the long-run growth rate of per capita output is positive if the saving rate is high whereas it is zero if the saving rate is lower. It is also found that when the associated absolute value is large, the long-run growth rate of per capita output is positive irrespective of the saving rate.[6]

These recent studies are based on a dynamic macroeconomic model which involves either perfect competition or market failures such as externalities and imperfect competition (or, increasing return to scale). Taking into account the possibility that some marker failures may lead to a reduction in per capita income on an equilibrium growth path, it should be of fundamental importance to ask whether the market economy system, by itself, has an intrinsic mechanism that can generate a reduction in per capita income and consumption in the growth process. For this purpose, we will investigate the existence and properties of an equilibrium path in a perfectly competitive market economy. While Ritschl (1985) analyzed a perfectly competitive market economy using the Solow growth model with an exogenous saving rate and without utility maximization, we will proceed to the Ramsey-Cass-Koopmans optimal growth (*RCK*) model, which endogenizes the saving rate by utility maximization and is thus regarded as the most general framework for the analysis of a competitive market economy system.

This representative dynamic macroeconomic model has never been analyzed under population

[6] Another strand of research involves the production function with non-renewable resources: Sasaki (2021), Sasaki and Mino (2021), Mino and Sasaki (2021, 2022). For example, Sasaki (2021) introduces negative population growth in the semi-endogenous growth model of Groth and Schou (2002) that considers non-renewable resources in final goods production. He shows that even if there are two negative constraints on economic growth (population decline and exhaust of resources), the long-run growth rate of per capita output can be positive.

decline. The reason must be two folds. First, from an empirical view point, the condition for emerging an equilibrium/optimal growth path (we simply call it an "optimal path") which is peculiar to population decline in the *RCK* model does not seem to be satisfied. The associated condition is that the effective depreciation rate of capital is negative, $n+\delta<0$, where n is the rate of population change and δ the depreciation rate of capital. According to Jones (2022), empirically, the (actual and estimated) rates of population decline (the absolute value of $n<0$) are 1% or smaller whereas the depreciation rate (δ) of capital is 3% or 5 % or more. Thus, in most cases, we have $n+\delta>0$ in a real world. Second, from a theoretical viewpoint, it has been believed that properties of the optimal path must be the same or similar even if we should derive it under $n+\delta<0$. The reason will be that even under $n+\delta<0$ a steady state exists and the law of motion is basically the same as in the *RCK* model with increasing population. If so, an explicit analysis of an optimal path under population decline would not provide new economic findings.

In this paper, we introduce child rearing costs into the *RCK* model to address the empirical concern above. Using two types of *RCK* models in which the discount rate in the intertemporal utility is $\rho-n>0$ or $\rho>0$ (time preference rate), we investigate the existence and properties of an optimal path under population decline and find a new type of optimal path in the latter model. We can confirm that an optimal path exists under economically plausible conditions in both models. This means that even under population decline, the *RCK* model remains useful as the theoretical foundation for identifying the socially optimal path in a perfectly competitive market economy.

Under population decline, the existence and properties of an optimal path depends on the range of the rates of population change, independently of the child rearing costs. First, when population decline is mild, the optimal path is a saddle-point path converging to a finite steady state, at which per capita consumption is equal to or smaller than per capita income, as in the standard *RCK* model with increasing population. Second, when population decline is faster (so that the effective depreciation rate of capital is negative), the optimal path is a saddle-point path converging to a finite steady state, i.e., a (degenerate) *balanced growth path (BGP)*, at which per capita consumption is *larger* than per capita income. Then, an economy needs to make *reversible investment* to reach the steady state. Third, when population decline is even faster, the optimal path can be an (degenerate) *asymptotically BGP*, along which both per capita consumption and income keep increasing permanently. We show empirically that these optimal paths specific to population decline can emerge in realistically relevant ranges of the rates of population decline (due to child rearing costs), based on Japanese data and the "low variant" estimation in *World Population Prospects 2019*.

The important implication from the results above is that under population decline a competitive market economy does *not* have an intrinsic growth mechanism which can *decrease* per capita income and/or consumption utility over time. Rather, a competitive market economy, even under population decline, has the intrinsic mechanisms which induce a growth path along which per capita income and consumption utility is *constant* (for the BGP case) or *increasing* (for the asymptotically BGP case) over time.

This paper is organized as follows. Section 2 presents the *RCK* model with the discount rate $\rho - n$. Section 3 establishes a theorem on the optimal path for this model. Section 4 analyzes the *RCK* model with the discount rate ρ and presents a theorem on the optimal path similar to Section 3. Section 5 presents a theorem on the new type of optimal path and discuss implications for the three theorems. Section 6 explains empirical relevance of our theoretical results, based on *World Population Prospects 2019*. Section 7 concludes the paper.

2. Ramsey-Cass-Koopmans Model with the Discount Rate $\rho - n > 0$

2.1 The Model

Consider the Ramsey-Cass-Koopmans optimal growth (*RCK*) model of a closed and perfectly competitive market economy. One final good, which can be either consumed or invested, is produced by using labor $L(t)$ and capital $K(t)$. The aggregate production function $F(L(t), K(t))$ satisfies the standard neoclassical assumptions, including $F(L(t),0) = F(0,K(t)) = 0$ and the constant return-to-scale property in $L(t)$ and $K(t)$. Per capita production function is defined by $f(k(t)) \equiv F(1, K(t)/L(t))$ with positive and decreasing marginal product of each factor ($f'(k(t)) > 0$ and $f''(k(t)) < 0$) and Inada conditions ($\lim_{k(t)\to 0} f'(k(t)) = \infty$ and $\lim_{k(t)\to\infty} f'(k(t)) = 0$), where $k(t) = K(t)/L(t)$ is per capita capital. Population, which is equal to total labor, is assumed to be changing at an exogenous rate n at any point in time *t*:

$$\frac{\dot{L}(t)}{L(t)} = n \equiv \lambda - d \qquad (1)$$

where a dot represents a time derivative. The rate n of population change equals the difference between the domestic fertility rate $\lambda \geq 0$ and the mortality rate $d \in [0,1]$[7] and may be either positive or negative. In order to show that the rates of population decline below which an optimal path has different properties from those in a population-increasing economy can be realistically relevant, we introduce child rearing costs.[8]

We assume that the birth and rearing of each child costs an amount $\eta > 0$ at any point in time *t*. Following Barro and Sala-i-Martin (2004, p.413), the child rearing cost is assumed to be positively related to per capita capital $k(t)$, i.e., $\eta(k(t))$ with $\eta'(k(t)) > 0$. This is because the cost η tends to rise with the opportunity cost of parental time, i.e., parents' wage rate $w(t) = f(k(t)) - k(t)f'(k(t))$, which is increasing in $k(t)$. The commodity costs of rearing a child may be either increasing or decreasing in $k(t)$,

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[7] We assume away international labor movements from an exogenous rate of population change.

[8] The child rearing cost may decrease the absolute value of the population declining rates (i.e., make those rates closer to zero) below which an optimal path with different properties from those in a population-increasing economy are induced. In section 3 and 5 we will explain that the existence and properties of an optimal path are independent of the child rearing costs.

which could in general induce a complex form (nonlinearity) of $\eta(k(t))$.[9] Because the aggregate cost of child rearing is $\eta\lambda L(t)$, the aggregate capital stock evolves over time according to $\dot{K}(t) = Y(t) - C(t) - \delta K(t) - \eta\lambda L(t)$, where $C(t)$ is the aggregate consumption and $\delta \in (0,1)$ is the depreciation rate of capital. Thus, per capita capital is accumulated according to $\dot{k}(t) = f(k(t)) - c(t) - (n+\delta)k(t) - \lambda\eta(k(t))$, where $c(t) \equiv C(t)/L(t)$ is per capita consumption. To simplify the analysis, we use a linear child rearing cost function $\eta(k(t)) = bk(t)$ with $b > 0$.

The representative household chooses the time path $\{c(t)\}_{t=0}^{\infty}$ of per capita consumption to maximize its intertemporal utility:

$$U = \int_0^{\infty} u(c(t))e^{-(\rho-n)t}dt \tag{2}$$

subject to per capita capital accumulation function:

$$\dot{k}(t) = f(k(t)) - c(t) - \{\delta - d + (1+b)\lambda\}k(t) \tag{3}$$

given the initial state $k(0) = k_0 > 0$, where $\rho > 0$ is the rate of time preference and $u(c(t))$ is the instantaneous utility with $u'(c(t)) > 0$, $u''(c(t)) < 0$ and Inada conditions ($\lim_{c(t)\to 0} u'(c(t)) = \infty$ and $\lim_{c(t)\to\infty} u'(c(t)) = 0$). We use the intertemporal utility function (2) in which the discount rate is $\rho - n > 0$ because it seems the most common formulation in macroeconomics.[10] We call the term $\delta - d + (1+b)\lambda$ in (3) the "effective depreciation rate (of capital)".

Defining the present-value Hamiltonian function by:

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[9] It is shown that a nonlinear child rearing cost function could induce complex dynamics in the Solow growth model in Daitoh (2020).

[10] The formulation (2) corresponds to the instantaneous utility function which includes the rate of population change in the entire economy, i.e., $u(c(t))e^{nt}$. In section 4 and 5 we analyze the less common *RCK* model where the discount rate is $\rho > 0$. It will be shown that there is an optimal path with theoretically different properties from those in the *RCK* model with the discount rate $\rho - n > 0$.

$$H\left(k(t),c(t),\pi(t),t\right) \equiv u(c(t))e^{-(\rho-n)t} + \pi(t)\left[f\left(k(t)\right) - c(t) - \{\delta - d + (1+b)\lambda\}k(t)\right] \quad (4)$$

the optimal path needs to satisfy the first-order conditions:

$$\frac{\partial H}{\partial c(t)} = u'(c(t))e^{-(\rho-n)t} - \pi(t) = 0 \quad (5)$$

$$\dot{\pi}(t) = -\frac{\partial H}{\partial k(t)} = -\pi(t)\left[f'\left(k(t)\right) - \{\delta - d + (1+b)\lambda\}\right] \quad (6)$$

and the transversality condition ($\lim_{t\to\infty} \pi(t)k(t) = 0$), where $\pi(t)$ is a costate variable. Because the Hamiltonian (4) is concave in $c(t)$ and $k(t)$, these are not only necessary but also sufficient conditions for the optimal path.

The equilibrium dynamics of the model is represented by (3) and the Keynes-Ramsey rule:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon(c(t))}[f'(k(t)) - \{\rho + \delta + b\lambda\}], \quad (7)$$

where $\varepsilon(c(t)) \equiv -c(t)u''(c(t))/u'(c(t))$. A steady state (k^*, c^*) is defined by $\dot{k}(t) = \dot{c}(t) = 0$. In what follows, we will investigate the existence of a steady state and the conditions for it.

We always have $\rho + \delta + b\lambda > 0$ in (7) in the *RCK* model with $\rho - n > 0$. Suppose here that the effective depreciation rate is non-negative, $\delta - d + (1+b)\lambda \geq 0$, in (3). Then, the *kk* curve, which is the combinations of $(k(t), c(t))$ satisfying $\dot{k}(t) = 0$ in (3), can be either an increasing curve $c(t) = f(k(t))$ or an inverted U-shaped curve. The *cc* curve, which is the combinations of $(k(t), c(t))$ satisfying $\dot{c}(t) = 0$ in (7), is a vertical line to the *k* axis in a phase diagram. Therefore, the law of motion indicates that a saddle-point stable steady state uniquely exists given the initial state $k_0 > 0$, as in the standard *RCK* model with increasing population.

2.2 Steady State under Population Decline

Now we investigate a steady state when the effective depreciation rate is negative, $\delta - d + (1+b)\lambda < 0$, in (3). This condition could be represented as the range of fertility or population declining rates:

$$\lambda < -\frac{\delta - d}{1+b} \equiv \lambda_P \quad \text{or} \quad n < n_P \equiv \lambda_P - d = -\frac{\delta + bd}{1+b} \qquad (8).$$

First, let us show the existence of a steady state. On the one hand, the *kk* curve $c(t) = f(k(t)) - \{\delta - d + (1+b)\lambda\}k(t)$ is increasing for all $k(t) \geq 0$.[11] On the other hand, the *cc* curve remains a vertical line at the positive finite value of $k^* > 0$ determined by $f'(k^*) = \rho + \delta + b\lambda$. This is because $\rho + \delta + b\lambda > 0$ always holds in this model. The associated range of fertility or population declining rates is:

$$\lambda > -\frac{\rho + \delta}{b} \equiv \lambda_S \quad \text{or} \quad n > n_S \equiv \lambda_S - d = -\frac{\rho + \delta + bd}{b} \qquad (9).$$

Therefore, as shown in Figure 1, there uniquely exists a saddle-point stable steady state *E* in the positive orthant. The necessary and sufficient conditions for its existence, which is $\delta - d + (1+b)\lambda < 0$ under $\rho + \delta + b\lambda > 0$, could be represented as the range of the population declining rates, $n_S < n < n_P$ (because $n_S < n_P$ holds). In other words, there is no possibility of $n \leq n_S$ in the *RCK* model with the discount rate $\rho - n > 0$. When population declines so rapidly that $n \leq n_S$ holds, this *RCK* model cannot provide any information on the optimal paths of a perfectly competitive market economy.

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[11] In this case, the marginal product of capital $f'(k(t)) - \{\delta - d + (1+b)\lambda\}$ remains positive even if $k(t)$ goes to infinity. In this sense, the *RCK* model with declining population may have the same characteristic as the Jones-Manuelli (1990,1997) type endogenous growth model.

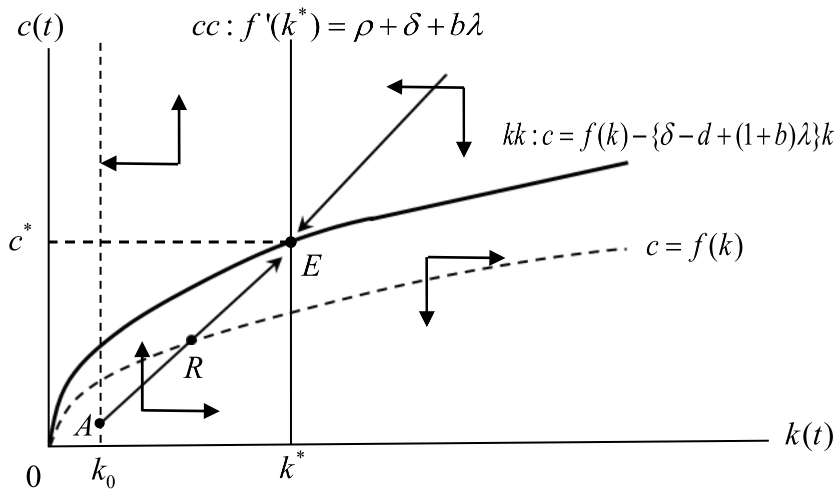

Figure 1. Phase Diagram under Population Decline for $\delta - d + (1+b)\lambda < 0$

Next, we investigate the properties of the steady state under $\delta - d + (1+b)\lambda < 0$ (i.e., $n < n_P$) in comparison to those under $\delta - d + (1+b)\lambda \geq 0$ (i.e., $n_P \leq n$). The most important difference is that the steady-state value of per capita consumption is *larger than* that of per capita income under $\delta - d + (1+b)\lambda < 0$, namely, $c^* = f(k^*) - \{\delta - d + (1+b)\lambda\}k^* > f(k^*)$. This property is shown in Figure 1, where the increasing *kk* curve lies *above* the per capita production function $f(k(t))$. Notice that under $\delta - d + (1+b)\lambda \geq 0$ the steady-state consumption equals or is smaller than the income ($c^* \leq f(k^*)$), because the *kk* curve may be either $c(t) = f(k(t))$ or an inverted U-shaped curve lying below the function $f(k(t))$, which is also true in the standard *RCK* model with increasing population.

The intuitive reason for the higher per capita consumption in the steady state is that under a negative effective depreciation rate ($\delta - d + (1+b)\lambda < 0$) the negative "capital dilution effect" tends to increase per capita capital: a decrease in $L(t)$ raises the value of $k(t) = K(t)/L(t)$ given *K(t)*. Then, in order for per capita capital to be constant over time ($\dot{k}(t) = 0$ in (3)), per capita consumption must be higher so as to cancel out the rise in *k(t)* due to the negative "capital dilution effect".

Given this fact, when the initial level of capital $k_0 > 0$ is lower than k^*, an economy needs to consume more than it produces along the transition path (on *RE* in Figure 1) in order to reach the steady state *E*. Because the final good can be either consumed or invested in the *RCK* model, the economy could consume the existing capital stock by making "reversible investment" (i.e., negative investment/saving) at any point in time along the transition path.

***Proposition 1:* (*Existence and Property of a Steady State under Population Decline in the RCK model with the Discount Rate* $\rho - n > 0$)**

Consider the RCK model with the discount rate $\rho - n > 0$ and *child rearing cost* $b \geq 0$. *Given* $\rho + \delta + b\lambda > 0$ *(i.e.,* $n_S < n$ *), suppose that the effective depreciation rate is negative, or* $\delta - d + (1+b)\lambda < 0$ (*i.e.,* $n < n_P$). *Namely, the population declining rate* $n < 0$ *is assumed to satisfy* $n_S < n < n_P$. *Then,*

(1) *a saddle-point stable steady state* (k^*, c^*) *uniquely exists.*

(2) *Per capita consumption is larger than per capita income in the steady state (*$c^* > f(k^*)$*). Then, if the initial level of capital* $k_0 > 0$ *is lower than the steady-state value* $k^* > 0$*, an economy needs to consume some or all of the existing capital stock by making "reversible investment" before it converges to the steady state along the transition path.*

Notice that this proposition holds in the absence of child rearing cost ($b = 0$). A steady state (k^*, c^*) *always* exists because the positive steady-state value k^* of per capita capital is determined by $f'(k^*) = \rho + \delta > 0$ in (7). Taking $\delta - d + \lambda < 0$ in (3) into account, the steady-state value c^* of per capita consumption exceeds that of per capita income, i.e., $c^* = f(k^*) - \{\delta - d + \lambda\}k^* > f(k^*)$.

2.3 Reversible Investment Constraint

Result (2) in Proposition 1 indicates that we should consider the maximum possible amount of reversible investment at each point in time.[12] The maximum level of per capita consumption an economy can attain is determined by the sum of per capita net income $f(k(t)) - \delta k(t)$ and the existing capital stock $k(t)$, namely, $c(t) \leq f(k(t)) + (1-\delta)k(t)$. We call this inequality a "reversible investment constraint (*RIC*)".[13]

We will investigate whether the *RIC* may be binding or not along the transition path converging to the steady state *E* in Figure 1. The necessary and sufficient condition for the *RIC* to be *non-binding* is that the *RIC* curve $c(t) = f(k(t)) + (1-\delta)k(t)$ (not shown in Figure 1) lies above the *kk* curve $c(t) = f(k(t)) - \{\delta - d + (1+b)\lambda\}k(t)$, which is equivalent to $1-\delta > -\{\delta - d + (1+b)\lambda\}$ or $1 - d + (1+b)\lambda > 0$. The associated range of fertility or population declining rate is:

$$\lambda > -\frac{1-d}{1+b} \equiv \lambda_R \quad \text{or} \quad n > n_R \equiv \lambda_R - d = -\frac{1+bd}{1+b} \qquad (10)$$

Under the assumptions $\lambda \geq 0$ and $d \in [0,1]$, $1 - d + (1+b)\lambda \geq 0$ necessarily holds and thus the *RIC* curve lies above or coincides with the *kk* curve. First, if $1 - d + (1+b)\lambda > 0$ ($n_R < n$) holds and thus the *RIC* curve lies above the *kk* curve, the *RIC* is *non*-binding along the transition path *ARE* starting from $k_0 < k^*$ which converges to the steady state *E* in Figure 1. Second, if $1 - d + (1+b)\lambda = 0$ ($n = n_R$) holds and thus the *RIC* curve coincides with the *kk* curve, the *RIC* is binding at the steady state *E*.[14] This condition holds if and only if both $\lambda = 0$ and $d = 1$ hold.[15]

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[12] Since this is a closed economy, international borrowings are impossible.

[13] The analyses of the *RCK* model with irreversible investment can be found in Arrow and Kurz (1970) and Leonard and Long (1992).

[14] When an economy reaches this steady state, the production ceases because all the aggregate capital is consumed ($K = 0$).

[15] The *RCK* model cannot provide any information on the optimal paths of a perfectly competitive market economy when population declines so rapidly that $n < n_R$ holds. However, this situation does not seem empirically plausible because the absolute value of $n_R < 0$ estimated from the Japanese data in section 6 is unrealistically large.

We summarize the results on the *RIC* in the next proposition.

Proposition 2 **(*Reversible Investment Constraint along a Transition Path toward a Steady State in the RCK model with the Discount Rate*** $\rho - n > 0$**)**

Consider the RCK model with the discount rate $\rho - n > 0$ *and child rearing cost* $b \geq 0$. *Given* $\rho + \delta + b\lambda > 0$ *(i.e.,* $n_S < n$ *), suppose that the effective depreciation rate is negative, or* $\delta - d + (1+b)\lambda < 0$ *(i.e.,* $n < n_P$*), namely,* $n_S < n < n_P$. *Then, a saddle-point stable steady state* (k^*, c^*) *exists.*

(1) The RIC is non-binding along the transition path converging to the steady state which starts from the initial level of capital $k_0 > 0$ *lower than* $k^* > 0$ *if and only if* $1 - d + (1+b)\lambda > 0$ *or* $n_R < n < n_P$ *holds.*

(2) The RIC is binding at the steady state E if and only if $1 - d + (1+b)\lambda = 0$ *or* $n_R = n < n_P$ *holds, which is equivalent to the facts that both* $\lambda = 0$ *and* $d = 1$ *hold.*

3. Optimal Path in the *RCK* Model with Discount Rate $\rho - n > 0$

In this section we derive an optima path, establishing the first theorem. Given that both $n_S < n$ and $n_R \leq n$ necessarily hold in this *RCK* model, we first investigate the existence of an optimal path separately for the case of $n_R < n$ (the *RIC* is *non*-binding) and the case of $n = n_R$ (the *RIC* is binding). Next, we identify the optimal path by separating three cases of $n_R < n_S$, $n_S < n_R$ and $n_R = n_S$. This theorem identifies the optimal path for the *RCK* model with the discount rate $\rho - n > 0$, depending on the range of exogenous rates of population change.

3.1 Theorem on the Optimal Path

We investigate what is the optimal path for the case of $n_R < n$ with the *RIC non*-binding, using Figure 2. It will be shown that the existence of optimal path depends on the initial level of capital $k_0 > 0$. To explain it precisely, we define the level $\bar{k}$ of per capita capital corresponding to point *B* at which the saddle-point path intersects the *RIC* curve.

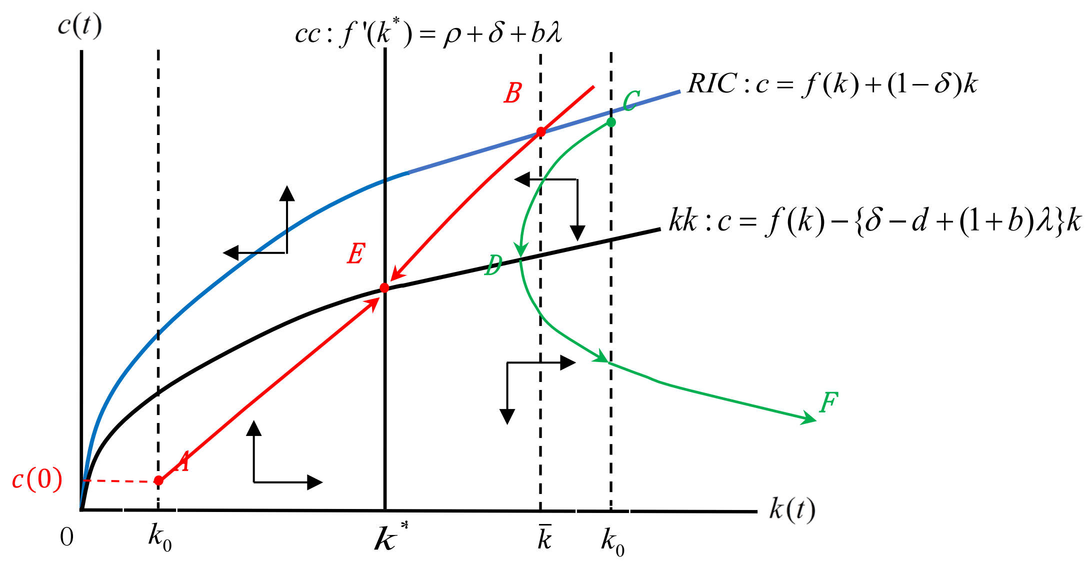

Figure 2. The Optimal Path in the *RCK* Model with the Discount Rate $\rho - n > 0$

First, when the initial condition is $k_0 \leq \bar{k}$, the optimal path is the saddle-point path converging to the steady state *E* (e.g., *AE* or *BE*) because the transversality condition is satisfied along this path. To see this, substituting $f'(k(t)) = \rho + \delta + b\lambda$ derived from (7) into (6) yields:

$$\frac{\dot{\pi}(t)}{\pi(t)} = -\left[f'(k(t)) - \{\delta - d + (1+b)\lambda\} \right] = -(\rho - n) < 0 \qquad (11)$$

Therefore, $\lim_{t\to\infty} \pi(t)k(t) = \lim_{t\to\infty} \pi(t)k^* = 0$ holds.

The other paths are not optimal. On the one hand, any dynamic paths starting from an initial consumption $c(0)$ higher than point *A* (not shown in Figure 2) necessarily reach the *RIC* curve and thus, an economy jumps to the origin (both production and consumption are zero because all the capital is consumed). Because the Keynes-Ramsey rule is not satisfied afterwards, this path is not optimal. On the other hand, any dynamic paths starting from an initial consumption $c(0)$ lower than point *A* (not shown in Figure 2) asymptotically approach the horizontal axis. The transversality condition is not satisfied along this path. To see this, using $\frac{\dot{k}(t)}{k(t)} = \frac{f(k(t))}{k(t)} - \frac{c(t)}{k(t)} - \{\delta - d + (1+b)\lambda\}$ from (3), we obtain:

$$\lim_{t\to\infty}\left[\frac{\dot{\pi}(t)}{\pi(t)} + \frac{\dot{k}(t)}{k(t)}\right] = \lim_{t\to\infty}\left[-f'(k(t)) + \frac{f(k(t))}{k(t)} - \frac{c(t)}{k(t)}\right] = 0 \qquad (12)$$

then, $\pi(t)k(t)$ generally converges to a constant non-zero value, violating the transversality condition. Therefore, the optimal path for the initial condition $k_0 \le \bar{k}$ is only the saddle-point paths converging to the steady state *E*.

Second, when the initial condition is $\bar{k} < k_0$, there does not exist an optimal path. To see this, we should notice that a dynamic path starting from any initial level $c(0)$ of consumption below the *RIC* curve*,* for example, *CDF* in Figure 2, asymptotically approaches the horizontal axis as time goes to infinity. Thus, the transversality condition is not satisfied for the same reason above.

It is found in the case of $n_R < n$ that there exists an optimal path, which is a saddle-point path converging to the steady state, for sufficiently low initial levels of capital $k_0 > 0$ satisfying $0 < k_0 \le \bar{k}$.

Let us turn to the case of $n = n_R$ with the *RIC* binding and investigate an optimal path. The only difference in this case from the case of $n_R < n$ is that the *RIC* curve coincides with the *kk* curve. Thus, when an economy moves along the transition path (e.g., *AE* in Figure 2) and reaches the steady state *E* which lies on the *RIC* curve, all the capital is consumed and the production ceases at *E*. Therefore, the Keynes-Ramsey rule is violated, and thus an optimal path does not exist.

Now we derive the optimal path by separating the three cases of $n_R < n_S$, $n_S < n_R$ and $n_R = n_S$.

First, when $n_R < n_S$ holds, the range of population declining rates ($n < n_P$) where both $n_S < n$ and $n_R \leq n$ hold is $n_S < n$ (then $n_R < n$ also holds). Thus, the optimal path is the saddle-point paths converging to the steady state *E*, which are represented as *AE* and *BE* in Figure 2. Second, when $n_S < n_R$ holds, the range of population declining rates ($n < n_P$) where both $n_S < n$ and $n_R \leq n$ hold is $n_R \leq n$ (then $n_S < n$ also holds). Then, we should separate two subcases: (i) if $n_R < n$ holds, the optimal path is the saddle-point paths converging to *E*, which are represented as *AE* and *BE* in Figure 2, and (ii) If $n_R = n$ holds, the *RIC* curve coincides with the *kk* curve and thus the optimal path does not exist. Third, when $n_R = n_S$ holds, the range of population declining rates ($n < n_P$) where both $n_S < n$ and $n_R \leq n$ hold is $n_S = n_R < n$ only. Thus, the optimal path is the saddle-point paths converging to *E*, which are represented as *AE* and *BE* in Figure 2. To sum up, the optimal path is the saddle-point paths converging to *E* for $\max\{n_S, n_R\} < n < n_P$, while it does not exist for $n_S < n_R = n$.

Therefore, we obtain the first theorem for negative rates of population change.

***Theorem 1:* (*The Optimal Paths in the RCK Model with the Discount Rate* $\rho - n > 0$)**

Consider the Ramsey-Cass-Koopmans model with the discount rate $\rho - n > 0$ *and child rearing cost* $b \geq 0$*, in which* $n_P \equiv -\dfrac{\delta + bd}{1+b}$, $n_S \equiv -\dfrac{\rho + \delta + bd}{b}$ *and* $n_R \equiv -\dfrac{1+bd}{1+b}$.

(1) *When the rate of population change is positive (*$n > 0$*) or negative with* $n_P \leq n \leq 0$*, the optimal path is a saddle-point path converging to the steady state* (k^*, c^*) *with* $c^* \leq f(k^*)$ *for any initial state* $k_0 > 0$.

(2) *When the rate of population decline (the absolute value of* $n < 0$*) is so large that* $\max\{n_S, n_R\} < n < n_P$ *holds, the optimal path is a saddle-point path converging to the steady state* (k^*, c^*) *with* $c^* > f(k^*)$ *if the initial state* k_0 *satisfies* $0 < k_0 \leq \bar{k}$*. If the initial state* k_0 *satisfies*

$\overline{k} < k_0$*, the optimal path does not exist.*

(3) *When the rate of population decline (the absolute value of* $n < 0$*) satisfies* $n_S < n_R = n < n_P$*, the optimal path does not exist.*

Two remarks should be made for Theorem 1. First, the non-existence results of an optimal path in (2) and (3) may not always be a serious problem. On the one hand, the initial condition $\overline{k} < k_0$ in result (2) does not seem realistically relevant. The steady-state value k^* under population decline with $n < n_P$ equals or larger than the steady-state value under population increase.[16] Suppose the exogenous value of n continuously decreases from the positive to the negative range when an economy is moving along the transition path toward a steady state. The initial state k_0 corresponding to the time when the value of n passes $n_P < 0$ into the range of $\max\{n_S, n_R\} < n < n_P$ should be smaller than the steady-state value k^* under population decline with $n < n_P$. Then, $\overline{k} < k_0$ will not occur for $\max\{n_S, n_R\} < n < n_P$. On the other hand, result (3) occurs only in the extreme case of $n = n_R$, or equivalently, when both $\lambda = 0$ and $d = 1$ hold. Thus, we can conclude that result (3) does not occur generically.

A second remark is that the existence of an optimal path and its property are independent of the child rearing cost ($b > 0$). This is because result (2) for $\max\{n_S, n_R\} < n < n_P$ holds independently of whether $n_R \le n_S$ or $n_S < n_R$ holds. The necessary and sufficient condition for $n_R \le n_S$ is:

$$\frac{\rho+\delta}{1-d} \le \frac{b}{1+b} \qquad (13).$$

This implies that result (2) holds regardless of the fact that the child rearing cost is large or small relative to the rates of time preference, capital depreciation and mortality.

[16] Recalling $f'(k^*) = \rho + \delta + b\lambda$, this is because the decline in $n \equiv \lambda - d$ may occur due to a decrease in $\lambda > 0$ or an increase in $d \ge 0$.

3.2 Roles of Child Rearing Cost

Let us explain here why we introduce the child rearing cost in the *RCK* model with declining population. If the child rearing cost were absent ($b = 0$), we would have $n_P = -\delta$ and $n_R = -1$, and a steady state would exist because $f'(k^*) = \rho + \delta > 0$ would hold (i.e., $n_S = -\infty$). From an empirical viewpoint, the depreciation rate of capital δ is 3~5% or higher according to Jones (2022). Thus, n_P should be minus 3~5% or lower and n_R minus 100%. However, according to the United Nations' *World Population Prospects (WPP) 2019*, the absolute values of population declining rates which have been observed or estimated in most population-declining countries are 1% or smaller.[17] Given these population declining rates in real societies, we should consider that only the case of $n_P < n < 0$ could actually occur. Then, the optimal path would be qualitatively the same as that in a population-increasing economy.

An introduction of the child rearing cost ($b > 0$) into the model may make the negative values of n_P, n_S and n_R closer to zero.[18] Suppose that the value of $b > 0$ increases. Then, the absolute value of n_S clearly decreases while that of n_R turns out to decrease (except for $d = 1$) because $\partial |n_R| / \partial b = (d-1)/(1+b)^2 \leq 0$ holds. On the other hand, the absolute value of n_P increases if and only

[17] The estimated average annual rates of population change (as medium variants) in Japan, Greece, Italy and Germany for 2020-2025 are -0.40, -0.52, -0.20 and -0.06%, while they are -0.53, -0.47, -0.28 and -0.09% for 2025-2030, respectively.

[18] Daitoh (2020) shows that by introducing child rearing cost depending on the rate of population change ($\eta = bnL$) into the Solow growth model, the rates of population decline below which dynamic paths with different properties from those under population increase are induced could be closer to zero. He also shows numerically by using the semi-endogenous growth model by Christiaans (2011) that such rates of population decline will be sufficiently close to zero when positive externalities from knowledge accumulation are strong enough.

if $\partial|n_P|/\partial b = (d-\delta)/(1+b)^2 > 0$ holds. If $\max\{n_R, n_S\} < n < n_P$, which implies $\delta - d + (1+b)\lambda < 0$, holds, $d - \delta > 0$ must hold and then the value of $n_P < 0$ gets further from zero as $b > 0$ increases. The child rearing cost may thereby make the range of population declining rates closer to zero, in which the optimal path with different properties from those under population increase emerges. Section 6 will show that the optimal paths specific to population decline may emerge in empirically relevant situations in the future covered by *WPP 2019*.

3.3 Implications of Theorem 1

We will now elucidate economic implications of Theorem 1. To begin with, we refer to the conjecture which not a few economists may have in mind that under population decline an equilibrium/optimal path in the standard growth models would probably have the same or similar properties to those in the associated models with increasing population. Even if an equilibrium/optimal path has some different properties, such a path may have been considered to emerge just under realistically irrelevant conditions (e.g., under too large population declining rates, as was illustrated in subsection 3.2 for the *RCK* model).

We find from Theorem 1 that the intuitive conjecture above certainly has a theoretical foundation. Even when the population growth rate $n \equiv \lambda - d$ turns negative, the optimal path under population decline for $n < n_P$ will be a unique saddle-point path converging to the steady state unless both $\lambda = 0$ and $d = 1$ hold. This is the same property as that in the standard *RCK* model with increasing population except that per capita consumption exceeds per capita income in the steady state, namely, $c^* > f(k^*)$ (as you can see in result (1) and (2) in Theorem 1).

A new finding from Theorem 1 is that under population decline the economic mechanism, which is different from the one under population increase, works with reversible investment on the optimal path. When an economy starts from a low initial level of capital $k_0 < k^*$ and moves along a transition path converging to the steady state *E*, it necessarily begins to consume more than it produces (from point *R* in Figure 1) by reversible investment. In this process, per capita capital $k(t) \equiv K(t)/L(t)$ increases because a decrease in $K(t)$ due to the negative saving is dominated by the effect of declining population $L(t)$ on *RE*.[19] This mechanism is in sharp contrast to the one in the standard *RCK* model: per capita capital $k(t) \equiv K(t)/L(t)$ increases because an increase in $K(t)$ by positive saving dominates the decreasing effect of population increase. This new mechanism keeps working permanently when the *RIC* is non-binding in the steady state.

This finding could be regarded as a substantial contribution to the literature on economic growth with declining population in that the saving rate endogenously changes from positive to negative values along the optimal path in the *RCK* model. Existing literature focused only on the Solow-type growth models. The first study that explicitly analyzed the Solow model with declining population ($n < 0$) was Ritschl (1985). Using the Solow model with no depreciation of capital ($\delta = 0$), he found that there does not exist a steady state but per capita capital k increases permanently for $n < 0$ and that if a non-positive saving rate ($s \leq 0$) is assumed a steady state exists but it is unstable.[20] He also showed that a stable steady state exists if a "classical" saving function is introduced instead of the Solow-type saving function. However, the "classical" saving function is not based on solid microeconomic foundation.[21] Introducing simple life-cycle assumptions in a neoclassical framework with no depreciation of capital ($\delta = 0$), Felderer (1988) showed that a steady state exists for any sign of the population growth rate.

In all these studies, the saving rate on an equilibrium path is exogenously given at either positive or negative value. In contrast, our *RCK* model reveals that the saving rate endogenously changes from

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[19] On *AR* in Figure 1, an increase in $K(t)$ by positive saving dominates the increasing effect of declining population $L(t)$.

[20] The capital-labor ratio either converges to zero or grows unboundedly, depending on the initial condition.

[21] In the "classical" saving function, the saving is positive when the return on capital is greater than a certain level, and falls to zero when a minimum rate of return is reached.

positive to negative values along an equilibrium/optimal path.

It is found that explicitly analyzing the effects of population decline on the growth path in the *RCK* model with the discount rate $\rho - n > 0$ not only confirms the intuition but also enables us to understand a new economic mechanism that begins to work under population decline. Thus, it seems of substantial significance to explicitly analyze a growth model with declining population and investigate the existence and properties of an equilibrium/optimal path.

In the next section, we will proceed to explicitly analyze the *RCK* model with the discount rate $\rho > 0$. This model has been used less commonly in macroeconomics but seems worth analyzing under population decline for the reason explained at the beginning of the next section. Section 4 and 5 will provide further new findings, including the non-existence of a steady state and the emergence of an optimal path which is not a saddle-point path converging to a steady state.

4. The Ramsey-Cass-Koopmans Model with the Discount Rate ρ

4.1 The Model

In this section we will analyze the *RCK* model with the intertemporal utility

$$U = \int_0^\infty u(c(t)) e^{-\rho t} dt, \qquad (14)$$

where the discount rate consists only of the time preference rate $\rho > 0$. This less common formulation in macroeconomics has been used in e.g., Blanchard and Fischer (1989) and Sidrauski (1967).

Let us first explain why this model is worth analyzing even though it may appear to ignore population change in the intertemporal utility function. Following Barro and Sala-I-Martin (2004, footnote 4 on pp.87-88), however, we could interpret this intertemporal utility as including population change with the assumption of decreasing marginal utility in the number of children. To be more specific, suppose that the intertemporal utility function is $\int_0^\infty u(c(t)) e^{-\rho^* t} dt$, where $\rho^* > 0$ is the time preference rate at population growth rate $n = 0$. If we add the assumption that the higher the population growth rate n, the higher the discount rate ρ, in a simple form $\rho = \rho^* + n$, the intertemporal utility function is $\int_0^\infty u(c(t))e^{-\rho^* t}dt$ $= \int_0^\infty u(c(t))e^{-(\rho-n)t}dt = \int_0^\infty \{u(c(t))e^{nt}\}e^{-\rho t}dt$. It turns out that this formulation implicitly includes the population change (in $\{u(c(t))e^{nt}\}$) and the stronger decreasing effect on the instantaneous utility of consumption (by $e^{-\rho t}$) due to a higher value of n. Thus, to investigate the effects of population change, this formulation is theoretically no less important than that in the *RCK* model with the discount rate $\rho - n > 0$.

The representative household maximizes the intertemporal utility (14) instead of (2), subject to the same capital accumulation function (3), given the initial state $k(0) = k_0 > 0$. The associated present-value Hamiltonian function is:

$$\tilde{H}\left(k(t), c(t), \pi(t), t\right) \equiv u(c(t))e^{-\rho t} + \pi(t)\left[f\left(k(t)\right) - c(t) - \{\delta - d + (1+b)\lambda\}k(t)\right] \quad (15).$$

Among the first-order conditions for the optimal path, only (5) is replaced with:

$$\frac{\partial \tilde{H}}{\partial c(t)} = u'(c(t))e^{-\rho t} - \pi(t) = 0 \quad (16).$$

The Euler equation is the same as (6). Then, the Keynes-Ramsey rule (7) is modified into:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon(c(t))}[f'(k(t)) - \{\rho + \delta - d + (1+b)\lambda\}] \quad (17).$$

Therefore, the equilibrium dynamics is represented by (3) and (17). What differs in the *RCK* model with the discount rate $\rho > 0$ is that $\rho + \delta - d + (1+b)\lambda$ in (17) is not always positive but can be either positive, zero or negative. We thus need to investigate the existence of a steady state and properties of equilibrium paths by separating these cases.

Suppose that the effective depreciation rate is non-negative, $\delta - d + (1+b)\lambda \geq 0$, in (3).[22] If

[22] Under this condition with an equality, $c^* \leq f(k^*)$ holds in the steady state and thus an economy can reach the steady state along a transition path without reversible investment.

$\rho+\delta-d+(1+b)\lambda>0$ holds in (17), a saddle-point stable steady state (k^*,c^*) uniquely exists, as in the *RCK* model with the discount rate $\rho-n>0$. Whenever the former inequality holds the latter necessarily holds, but the opposite is not always true.

4.2 Steady State and Equilibrium Path under Population Decline

In this section we investigate a steady state when the effective depreciation rate is negative, $\delta-d+(1+b)\lambda<0$, in (3), which is equivalent to (8) $n<n_P$. A new possibility in the *RCK* model with the discount rate $\rho>0$ is that not only $\rho+\{\delta-d+(1+b)\lambda\}>0$ but also $\rho+\{\delta-d+(1+b)\lambda\}\le 0$ may hold in (17) even in this case.[23] To precisely examine the existence of a steady state, we should separate the latter case into $\rho+\{\delta-d+(1+b)\lambda\}=0$ and $\rho+\{\delta-d+(1+b)\lambda\}<0$.

First, suppose $\rho+\delta-d+(1+b)\lambda>0$, which is equivalent to

$$\lambda>-\frac{\rho+\delta-d}{1+b}\equiv\tilde{\lambda}_S \quad \text{or} \quad n>\tilde{n}_S\equiv\tilde{\lambda}_S-d=-\frac{\rho+\delta+bd}{1+b}.$$

Then, there is a positive finite value of $k^*>0$ satisfying $\dot{c}(t)=0$ in (17) and thus a finite steady state (k^*,c^*) uniquely exists. The phase diagram for the *RCK* model with the discount rate $\rho>0$ is the same as Figure 1 except that the *cc* curve is represented by $f'(k^*)=\rho+\delta-d+(1+b)\lambda>0$. Thus, the finite steady state is saddle-point stable. Here again, because the *kk* curve is $c=f(k)-\{\delta-d+(1+b)\lambda\}k$, per capita consumption is larger than per capita income in the finite steady state, i.e., $c^*=f(k^*)-\{\delta-d+(1+b)\lambda\}k^*>f(k^*)$.

Second, suppose $\rho+\delta-d+(1+b)\lambda=0$, or equivalently, $\lambda=\tilde{\lambda}_S$ or $n=\tilde{n}_S$. Then, recalling Inada condition $\lim_{t\to\infty}f'(k(t))=0$, $f'(k^*)=0$ for $\dot{c}(t)=0$ in (17) is satisfied at $k^*=+\infty$, implying that the *cc*

[23] We separate these cases depending on whether or not the *cc* curve ($\dot{c}(t)=0$) can be drawn in the positive orthant in a phase diagram.

curve does not appear in the positive orthant as shown in Figure 3. Substituting $\delta - d + (1+b)\lambda = -\rho$ into (3), we have the *kk* curve $c = f(k) + \rho k$. This could be interpreted as saying that a steady state defined by $\dot{c}(t) = \dot{k}(t) = 0$ exists on the *kk* curve at a point with an infinitely long distance from the origin, i.e., $k^* = +\infty$ and $c^* = f(k^*) + \rho k^* = +\infty$. In other words, there exists a steady state (k^*, c^*) in the limit*, or there is an infinite steady state.*[24]

Third, suppose $\rho + \delta - d + (1+b)\lambda < 0$, or equivalently,

$$\lambda < -\frac{\rho + \delta - d}{1+b} \equiv \tilde{\lambda}_S \quad \text{or}^{25} \quad n < \tilde{n}_S \equiv \tilde{\lambda}_S - d = -\frac{\rho + \delta + bd}{1+b} \qquad (18).$$

Then, a steady state (k^*, c^*) does not exist for any $k(t) \geq 0$ because there exist neither a finite nor an infinite value of $k^* > 0$ which satisfies $f'(k^*) = \rho + \delta - d + (1+b)\lambda < 0$. To sum up, given $\tilde{n}_S < n_P$, a finite steady state exists when n satisfies $\tilde{n}_S < n < n_P$ while it does not exist when n satisfies $n \leq \tilde{n}_S$.

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[24] A steady state in the limit should be regarded as plausible because there are an infinite number of combinations $(\rho, \delta, d, b, \lambda)$ of parameter values satisfying $\rho + \delta - d + (1+b)\lambda = 0$.

[25] In this case $\rho + \delta < d$ must hold for $\tilde{\lambda}_S > 0$. Then, an increase in $b > 0$ decreases the fertility rate $\tilde{\lambda}_S$, making the absolute value of $\tilde{n}_S = \tilde{\lambda}_S - d < 0$ larger.

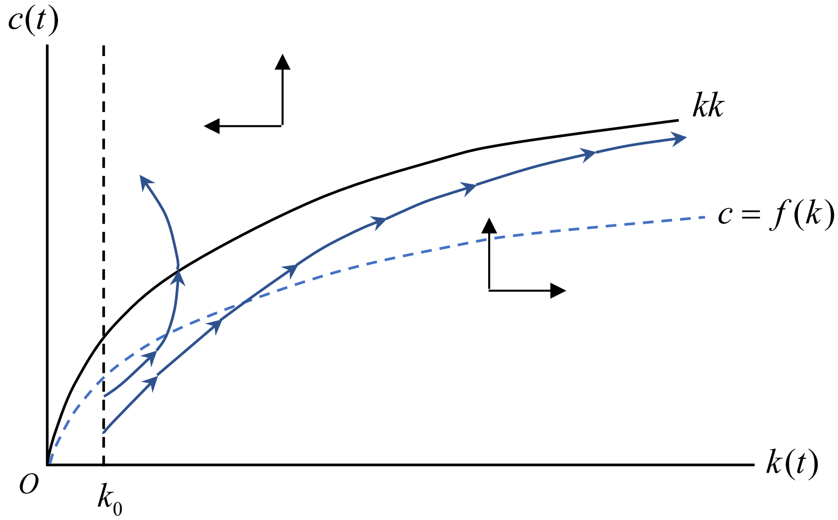

Figure 3. Property of Equilibrium Paths under $\rho + \{\delta - d + (1+b)\lambda\} \leq 0$ (*i.e.*, $n \leq \tilde{n}_S$)

We find an important property of equilibrium paths[26] under $n \leq \tilde{n}_S$ ($\rho + \{\delta - d + (1+b)\lambda\} \leq 0$) by Figure 3. The kk curve is $c = f(k) + \rho k$ for $\rho + \delta - d + (1+b)\lambda = 0$ while it is $c = f(k) - \{\delta - d + (1+b)\lambda\}k$ for $\rho + \delta - d + (1+b)\lambda < 0$. In both cases, the law of motion indicates that per capita consumption comes to be larger than per capita income (e.g., $c = f(k) + \rho k > f(k)$) along any equilibrium path when time goes to infinity.

We thus obtain the next proposition. Result (1) for the existence of a steady state is different from Proposition 1 while result (2) for the property of equilibrium paths remains the same.

Proposition 3: (Existence of a Steady State and the Property of Equilibrium Paths in the RCK model with the Discount Rate $\rho > 0$**)**

Consider the RCK model with the discount rate $\rho > 0$ *and child rearing cost* $b \geq 0$. *Suppose that the*

[26] We call a dynamic path following the law of motion an "equilibrium path". The optimal path will be an equilibrium path along which the intertemporal utility is maximized.

effective depreciation rate is negative, i.e. $\delta - d + (1+b)\lambda < 0$ *(i.e.,* $n < n_P$) holds. Then,

(1) *(i) when* $\rho + \delta - d + (1+b)\lambda > 0$ *(i.e.,* $\tilde{n}_S < n < n_P$*) holds, a saddle-point stable finite steady state* (k^*, c^*) *uniquely exists. (ii) When* $\rho + \delta - d + (1+b)\lambda = 0$ *(i.e.,* $n = \tilde{n}_S$*), a steady state* (k^*, c^*) *exists in the limit, or there is an infinite steady state, i.e.,* $k^* = c^* = +\infty$ *.(iii) When* $\rho + \delta - d + (1+b)\lambda < 0$ *(i.e.,* $n < \tilde{n}_S$*) holds, a steady state does not exist.*

(2) *In all the three cases above, when time goes to infinity, per capita consumption comes to be larger than per capita income along any equilibrium path starting from the initial level of capital* $k_0 > 0$. *Thus, an economy needs to consume some or all of the existing capital stock by making "reversible investment" in the long-run.*

Result (2) indicates that we should consider the *RIC* when identifying the optimal path. As in subsection *2.3*, the *RIC* curve lies above or coincides with the *kk* curve because $1 - d + (1+b)\lambda \geq 0$ always holds (recall that the *RIC* curve lies above the *kk* curve unless both $\lambda = 0$ and $d = 1$ hold).

4.3 The Optimal Path in the Presence of a Finite Steady State

We will derive the optimal path by separating the following four cases based on the condition for a finite steady state to exist and the condition for the *RIC* curve to lie above the *kk* curve:

Case A-1: A finite steady state ($\tilde{n}_S < n < n_P$) and the *RIC* above the *kk* curve ($n_R < n$)

Case A-2: A finite steady state ($\tilde{n}_S < n < n_P$) and the *RIC* = the *kk* curve ($n = n_R$)

Case B-1: No finite steady state ($n \leq \tilde{n}_S$) and the *RIC* above the *kk* curve ($n_R < n$)

Case B-2: No finite steady state ($n \leq \tilde{n}_S$) and the *RIC* = the *kk* curve ($n = n_R$)

Given $\tilde{n}_S < n_P$ and $n_R < n_P$, we should separate the cases of $\tilde{n}_S < n_R$ and $n_R < \tilde{n}_S$. In section 4 we derive the optimal path and investigate its property in the case of $\tilde{n}_S < n_R$ while we will analyze the case of $n_R < \tilde{n}_S$ in section 5.[27] The necessary and sufficient condition for $\tilde{n}_S < n_R$ is:

$$\rho + \delta > 1 \tag{24}$$

Note that this condition does not depend on the child rearing cost (b).

Looking at Figure 4 for $\tilde{n}_S < n_R$, the effective range for an optimal path, in which the *RIC* curve lies above or coincides with the *kk* curve, is $n_R \le n < n_P$. In this range, a finite steady state always exists (because $\tilde{n}_S < n$), and thus, either $n_R < n$ (Case A-1) or $n = n_R$ (Case A-2) may occur. When n lies in the range $n_R < n < n_P$, Case A-1 occurs. The derivation of an optimal path is based on (3) and (17) and thus the same as that in the *RCK* model with the discount rate $\rho - n > 0$ except that the *cc* curve is replaced with $f'(k^*) = \rho + \delta - d + (1+b)\lambda > 0$. Therefore, the optimal path is a saddle-point path converging to the steady state, which is qualitatively the same as those (*AE* and *BE*) in Figure 2 and Theorem 1.

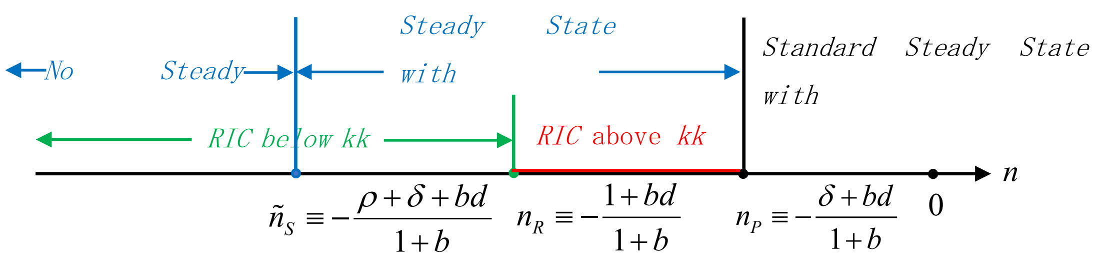

Figure 4. Effective Range for Optimal Path in the Case of $\tilde{n}_S < n_R$ or $\rho + \delta > 1$

When $n = n_R$ holds, Case A-2 occurs. The only difference in Case A-2 from Case A-1 is that the *RIC* curve coincides with the *kk* curve in Figure 2 and thus the steady state *E* lies on the *RIC* curve. Then, all the capital stock is consumed and production ceases at *E*. Because the economy jumps to the origin

[27] We make a comment on the case of $n_R = \tilde{n}_S$ (or, $\rho + \delta = 1$) at the end of subsection 4.2.

and the Keynes-Ramsey rule does not hold afterwards, there exists no optimal path. However, this non-existence result is not a serious problem because it occurs only in the exceptional case where both $\lambda = 0$ and $d = 1$ hold.[28]

This establishes Theorem 2 for $\tilde{n}_S < n_R$ or $\rho + \delta > 1$. Although $\rho + \delta > 1$ is less likely to be satisfied empirically, Theorem 2 is meaningful because it shows that the *RCK* model with the discount rate $\rho > 0$ may induce qualitatively the same results on the optimal paths as Theorem 1 for the *RCK* model with the discount rate $\rho - n > 0$.

Theorem 2 (The Optima Path in the Absence of a Finite Steady State)

Consider the Ramsey-Cass-Koopmans model with the discount rate $\rho > 0$ *and child rearing cost* $b > 0$. *If* $\tilde{n}_S < n_R$ or $\rho + \delta > 1$ *holds, the optimal path changes depending on the range of exogenous rates* n *of population change, where* $n_P \equiv -\dfrac{\delta + bd}{1+b}$, $\tilde{n}_S \equiv -\dfrac{\rho + \delta + bd}{1+b}$ *and* $n_R \equiv -\dfrac{1+bd}{1+b}$.

(1) *When the rate of population change is positive (*$n > 0$*) or negative with* $n_P \le n \le 0$*, the optimal path is a saddle-point path converging to the steady state* (k^*, c^*) *with* $c^* \le f(k^*)$ *for any initial state* $k_0 > 0$.

(2) *When the rate of population decline (the absolute value of* $n < 0$*) is so large that* $n_R < n < n_P$ *holds, the optimal path is a saddle-point path converging to the steady state* (k^*, c^*) *with* $c^* > f(k^*)$ *if the initial state* k_0 *satisfies* $0 < k_0 \le \overline{k}$. *If the initial state* k_0 *satisfies* $\overline{k} < k_0$*, the optimal path does not exist.*

(3) *When the rate of population decline satisfies* $n = n_R$*, the optimal path does not exist.*

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[28] In other words, this non-existence case does not occur generically. Recall the analysis in subsection 2.3.

We should be careful to derive the optimal path when $\tilde{n}_S = n_R$ or $\rho + \delta = 1$ holds.[29] If $\tilde{n}_S = n_R < n \leq n_P$ holds, then Case A-1 occurs and thus we obtain the qualitatively the same result as (2) in Theorem 2. If $\tilde{n}_S = n_R = n$ holds, then Case B-2 occurs. We will explain the optimal path for this case in the next section.

5. New Findings *in the RCK Model with the Discount Rate* $\rho > 0$

In this section we derive the optimal path in the case of $n_R < \tilde{n}_S$ and show new findings in the *RCK* model with the discount rate $\rho > 0$.

5.1 The Optimal Paths in the Absence of a Finite Steady State

Looking at Figure 5 for $n_R < \tilde{n}_S$, the effective range for an optimal path, in which the *RIC* curve lies above or coincides with the *kk* curve, is $n_R \leq n < n_P$. It is found that a finite steady state exists for $\tilde{n}_S < n < n_P$ while no finite steady state exists for $n_R \leq n \leq \tilde{n}_S$. The optimal path in the former case is qualitatively the same as in result (2) of Theorem 2. We should thus explore what will be the optimal path in the cases of $n_R < n \leq \tilde{n}_S$ and $n_R = n \leq \tilde{n}_S$ when no finite steady state exists.

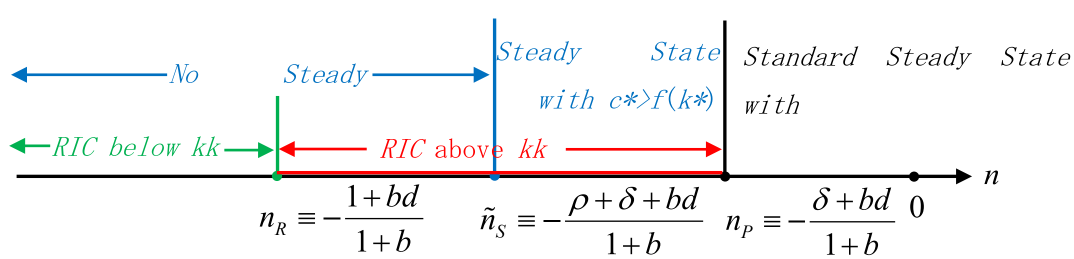

Figure 5. Effective Ranges for Optimal Path in the Case of $n_R < \tilde{n}_S$ or $\rho + \delta < 1$

[29] There are an infinite number of combinations (ρ, δ) of parameters satisfying $\rho + \delta = 1$.

Let us first explore the optimal path for $n_R < n \leq \tilde{n}_S$ corresponding to Case B-1, using Figure 6. Because $n \leq \tilde{n}_S$ or $\rho + \delta - d + (1+b)\lambda \leq 0$ holds, the *cc* curve does not appear in the positive orthant. Instead, by (17), per capita consumption $c(t)$ is increasing for all $k(t) > 0$ over time. Thus, all dynamic paths that intersect the *kk* curve from below, for example, *CDF,* are not optimal because they necessarily reach the *RIC* curve, violating the Keynes-Ramsey rule, when time goes to infinity.

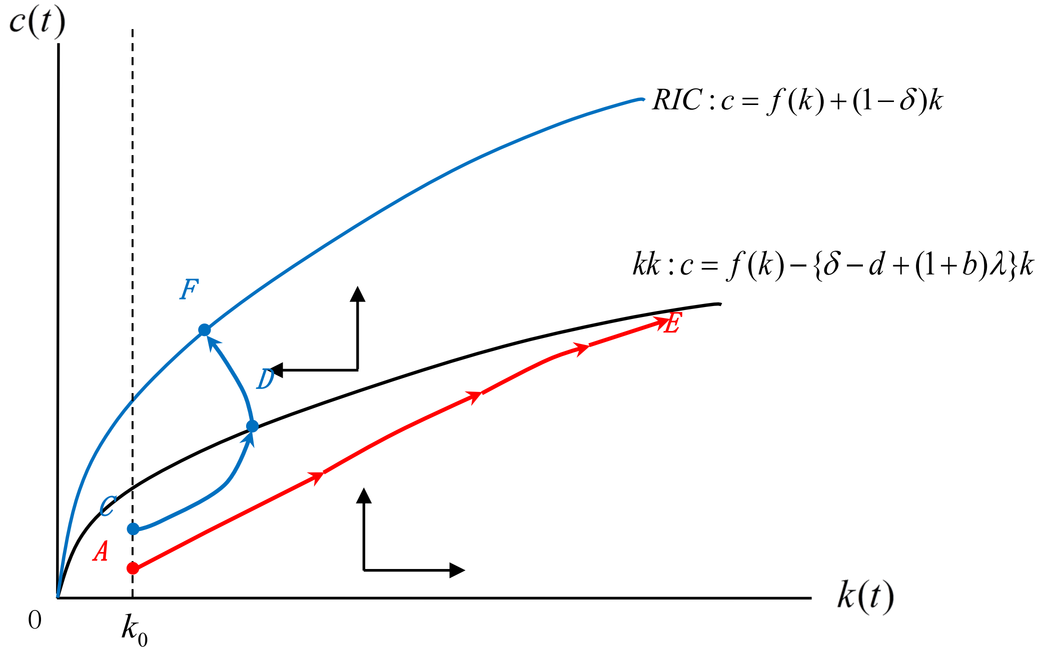

Figure 6. Phase Diagram in the Absence of a Finite Steady State（Case B-1）

In contrast, we can generally consider equilibrium paths which always lie below the *kk* curve and asymptotically approach the *kk* curve as time goes to infinity. Among them, the path *AE* with the highest consumption $c(t)$ for the same level of capital $k(t)$ is optimal because the instantaneous utility $u(c(t))$ is maximized at any point in time from $t = 0$ to $t = \infty$. Along this path, the transversality condition is satisfied. To see this, because the slope of the *kk* curve asymptotically approaches

$-\{\delta - d + (1+b)\lambda\} > 0$,[30] we obtain from (3) and (6):

$$\lim_{t\to\infty}\left[\frac{\dot{\pi}(t)}{\pi(t)}+\frac{\dot{k}(t)}{k(t)}\right]=\lim_{t\to\infty}\left[-f'(k(t))+\frac{f(k(t))}{k(t)}-\frac{c(t)}{k(t)}\right]$$

$$=\lim_{t\to\infty}\left[-\frac{c(t)}{k(t)}\right]=\delta-d+(1+b)\lambda<0 \qquad (19)$$

Therefore, $\lim_{t\to\infty}\pi(t)k(t)=0$ holds.

The optimal path for $n_R = n \le \tilde{n}_S$ corresponding to Case B-2 is derived in the same way except that the *RIC* curve coincides with the *kk* curve. Thus, the optimal path is the equilibrium path (like *AE* in Figure 6) with the highest consumption $c(t)$ for the same level of capital $k(t)$, and asymptotically approaches the *kk* curve from below.

Proposition 4 （***Optimal Path in the Absence of a Finite Steady State***）

Consider the RCK model with the discount rate $\rho > 0$ and child rearing cost $b \ge 0$. Suppose that the effective depreciation rate is negative, i.e., $\delta - d + (1+b)\lambda < 0$ $(n < n_P)$ holds and *a finite steady state does not exist ($\rho + \delta - d + (1+b)\lambda \le 0$ or $n \le \tilde{n}_S$). Then, whether the RIC curve lies above or coincides with the kk curve ($n_R \le n$), the optimal path is the equilibrium path which always lies below the kk curve with the highest consumption for the same level of capital, and asymptotically approaches the kk curve from below as time goes to infinity.*

5.2 Properties of an Asymptotically Balanced Growth Path

Next, we investigate the properties of these optimal paths, charactering them in three respects. In what follows, we will show that the optimal path turns out to have common properties for

[30] Under $\rho + \delta - d + (1+b)\lambda = 0$, $\{\delta - d + (1+b)\lambda\} = -\rho < 0$ holds. The analysis below remains unchanged.

$\rho + \delta - d + (1+b)\lambda = 0$ ($n = \tilde{n}_S$) and $\rho + \delta - d + (1+b)\lambda < 0$ ($n < \tilde{n}_S$) while the state to which an economy converges is different between these cases.

First, the consumption-capital ratio $c(t)/k(t)$ asymptotically approaches a positive constant $-\{\delta - d + (1+b)\lambda\} > 0$ along this path. Second, the asymptotic growth rates of per capita consumption and capital along the optimal path turn out to be zero. To see this, we have from (3):

$$\frac{\dot{k}(t)}{k(t)} = \frac{f(k(t))}{k(t)} - \frac{c(t)}{k(t)} - \{\delta - d + (1+b)\lambda\} \tag{20}$$

By (19) and (20), we obtain along the asymptotic growth path:

$$\lim_{t\to\infty}\left[\frac{\dot{c}(t)}{c(t)}\right] = \lim_{t\to\infty}\left[\frac{\dot{k}(t)}{k(t)}\right] = 0 \tag{21}$$

The asymptotic growth rate of per capita income $y(t)$ is also zero because $\dot{y}(t)/y(t) = [k(t)f'(k(t))/f(k(t))](\dot{k}(t)/k(t)) \to 0$ as $t \to \infty$.

Third, the state to which the optimal path asymptotically approaches differs, depending on whether either $\rho + \delta - d + (1+b)\lambda = 0$ or $\rho + \delta - d + (1+b)\lambda < 0$ holds. When $\rho + \delta - d + (1+b)\lambda = 0$ holds, a steady state (k^*, c^*) with $\dot{c}(t) = \dot{k}(t) = 0$ exists in the limit (recall $k^* = c^* = +\infty$). Using $y(t) = f(k(t))$, $\dot{y}(t) = 0$ also holds. Then, the optimal path asymptotically approaches the steady state with $\dot{c}(t) = \dot{k}(t) = 0$, that is, it is a (*degenerate*) *asymptotically balanced* growth path (BGP). This is a theoretically different result from that in the presence of a finite steady state: the optimal path converges to the steady state with $\dot{c}(t) = \dot{k}(t) = 0$, that is, it is a (*degenerate*) *balanced* growth path (BGP).[31] Despite

[31] Palivos *et al.* (1997) define a degenerate and nondegenerate BGP and asymptotically BGP. A dynamic path is said to be a BGP if $\eta_c = (\dot{c}/c)$ and $\eta_k = (\dot{k}/k)$ are constant over time, and "nondegenerate" if $\eta_c > 0$ and $\eta_k > 0$ hold. A degenerate BGP corresponds to a stationary state with $\eta_c = \eta_k = 0$. A dynamic path is said to be an asymptotically BGP if

that, we could regard the property of the optimal path as similar between these cases in the sense that an economy converges to the steady state where $c(t)$ and $k(t)$ are constant over time as time goes to infinity.[32] Note that the limiting values of (c/k) are different between the case of a BGP ((c/k)= the slope of the line connecting the origin and the intersection of the *cc* and the *kk* curves) and the case of an asymptotically BGP （(c/k)=the limit of the slope of the *kk* curve $= \rho > 0$）.

In contrast, when $\rho + \delta - d + (1+b)\lambda < 0$ holds, the optimal path does *not* approach the steady state with $\dot{c}(t) = \dot{k}(t) = 0$ because there exist neither a finite nor an infinite value of k^* which satisfy $f'(k^*) = \rho + \delta - d + (1+b)\lambda < 0$. Instead, an economy asymptotically approaches a state satisfying (21), where the *growth rates* of per capita consumption and capital both approach zero when time goes to infinity. Then, we should further clarify such economic mechanism behind this property that the growth rate of $k(t)$ is higher than the growth rate of the change in $k(t)$ along the transition path. To see this, on the one hand, differentiating (3) with respect to time and dividing both sides of the resulting equation by $\dot{k}(t)$, we have:

$$\frac{d\dot{k}(t)/dt}{\dot{k}(t)} = = [f'(k(t)) - \{\delta - d + (1+b)\lambda\}] - \frac{\dot{c}(t)}{\dot{k}(t)}$$

It has been found in (21) that as $t \to \infty$, $\dot{c}(t)/c(t) = \dot{k}(t)/k(t) \to 0$. Thus, $\dot{c}(t)/\dot{k}(t) = c(t)/k(t)$ holds in the limit. Therefore, as $t \to \infty$, we have:

$$\frac{d\{\dot{k}(t)\}/dt}{\dot{k}(t)} = = f'(k(t)) - \{\delta - d + (1+b)\lambda\} - \frac{c(t)}{k(t)} \tag{22}$$

On the other hand, we obtain from (3):

$$\frac{\dot{k}(t)}{k(t)} = \frac{f(k(t))}{k(t)} - \{\delta - d + (1+b)\lambda\} - \frac{c(t)}{k(t)} \tag{23}$$

$\overline{\eta}_c = \lim_{t\to\infty}(\dot{c}/c)$ and $\overline{\eta}_k = \lim_{t\to\infty}(\dot{k}/k)$ exist and are finite, and "nondegenerate" if $\overline{\eta}_c > 0$ and $\overline{\eta}_k > 0$ hold.

[32] Because the slope of the *kk* curve converges to $-\{\delta - d + (1+b)\lambda\} = \rho > 0$, not only *k* but also *c* continues to increase in the limit.

The growth rate of $\dot{k}(t)$ in (22) is lower than the growth rate of $k(t)$ in (23) because $f'(k(t)) < f(k(t))/k(t)$ holds by the concavity of $f(k(t))$. Thus, as time goes to infinity, $\dot{c}(t)/c(t)$ and $\dot{k}(t)/k(t)$ approach zero even though the values of $\dot{c}(t)$ and $\dot{k}(t)$ do not always approach zero in the limit.

We summarize the results above in the next proposition.

Proposition 5 （***Properties of the Optimal Path in the Absence of a Finite Steady State***）

The optimal path derived in Proposition 4 has properties described below.

(1) *The consumption-capital ratio* $c(t)/k(t)$ *asymptotically approaches a positive constant* $-\{\delta - d + (1+b)\lambda\} > 0$ *along the optimal path,*

(2) *The optimal path is a (degenerate) asymptotically BGP, that is, the asymptotic growth rates of per capita consumption, capital and income are zero.*

(3) *The state to which the optimal path asymptotically approaches depends on the conditions for the existence of a steady state:*

(i) *when* $\rho + \delta - d + (1+b)\lambda = 0$ *(*$n = \tilde{n}_S$*) holds, the optimal path asymptotically approaches a steady state where the level of per capita consumption, capital and income are constant over time, i.e.,* $\dot{c}(t) = \dot{k}(t) = \dot{y}(t) = 0$.

(ii) *when* $\rho + \delta - d + (1+b)\lambda < 0$ *(*$n < \tilde{n}_S$*) holds, the optimal path asymptotically approaches a state where the growth rates of per capita consumption, capital and income equal zero, i.e.,* $\lim_{t\to\infty}(\dot{c}(t)/c(t)) = \lim_{t\to\infty}(\dot{k}(t)/k(t)) = \lim_{t\to\infty}(\dot{y}(t)/y(t)) = 0$.

5.3 General Theorem on the Optimal Path

Finally, we identify the optimal paths in the case of $n_R < \tilde{n}_S$ by moving down from the value $n_P < 0$ of population decline. When n lies in the range $\tilde{n}_S < n < n_P$, a finite steady state exists, and the *RIC* curve lies above the *kk* curve (i.e., $n_R < n$). Then, Case A-1 occurs. The optimal path is a saddle-point path converging to the finite steady state ($\dot{c}(t) = \dot{k}(t) = 0$) with $c^* > f(k^*)$ like *AE* and *BE* in Figure 2.

When $n = \tilde{n}_S$ holds, an infinite steady state exists, and the *RIC* curve lies above the *kk* curve. Then, Case B-1 occurs. When n lies in the range $n_R < n < \tilde{n}_S$, no steady state exists, and the *RIC* curve lies above the *kk* curve. Then, Case B-1 occurs. When $n = n_R$ holds, the *RIC* curve coincides with the *kk* curve. Because $n < \tilde{n}_S$ holds, no steady state exists. Then, Case B-2 occurs. In all the three cases ($n_R \le n \le \tilde{n}_S$) of non-existence of a finite steady state, the optimal path is the equilibrium path described in Proposition 4.

We obtain the third theorem on the optimal path for the case of $n_R < \tilde{n}_S$ or $\rho + \delta < 1$.[33] This theorem must be realistically more relevant because $\rho + \delta < 1$ is likely to be satisfied empirically.

Theorem 3:

Consider the Ramsey-Cass-Koopmans model with the discount rate $\rho > 0$ *and child rearing cost* $b \ge 0$. *If* $n_R < \tilde{n}_S$ or $\rho + \delta < 1$ *holds, the optimal path changes depending on the range of exogenous rates* n *of population change, where* $n_P \equiv -\dfrac{\delta + bd}{1+b}$, $\tilde{n}_S \equiv -\dfrac{\rho + \delta + bd}{1+b}$ *and* $n_R \equiv -\dfrac{1 + bd}{1+b}$.

(1) *When the rate of population change is positive (*$n > 0$*) or negative with* $n_P \le n \le 0$*, the optimal path is a saddle-point path converging to the steady state (*$\dot{c}(t) = \dot{k}(t) = 0$*) with* $c^* \le f(k^*)$ *for any initial state* $k_0 > 0$.

[33] The necessary and sufficient condition for the *RIC* curve to lie above the *kk* curve under $\rho + \delta - d + (1+b)\lambda = 0$ is $1 - \delta > -\{\delta - d + (1+b)\lambda\} = \rho$, namely, $1 > \rho + \delta$. This is consistent with the condition for Theorem 3 to hold.

(2) *When the rate of population decline (the absolute value of* $n<0$*) is so small that* $\tilde{n}_S < n < n_P$ *holds, the optimal path is a saddle-point path converging to the steady state (* $\dot{c}(t)=\dot{k}(t)=0$ *) with* $c^* > f(k^*)$ *if the initial state* k_0 *satisfies* $0<k_0\le \bar{k}$*. If the initial state* k_0 *satisfies* $\bar{k}<k_0$*, the optimal path does not exist.*

(3) *When the rate of population decline (the absolute value of* $n<0$*) is so large that* $n_R \le n \le \tilde{n}_S$ *holds,*[34] *the optimal path is the equilibrium path which asymptotically approaches the kk curve from below with the properties described in Proposition 5 for any initial state* $k_0>0$.

An important finding in Theorem 3 is that the *RCK* model with the discount rate $\rho>0$ induces a new type of optimal path in result (3), which does not appear in the *RCK* model with the discount rate $\rho - n > 0$. While the optimal paths in results (1) and (2) in Theorem 3 are qualitatively the same as the associated paths in results (1) and (2) in Theorem 1 and 2, which are (degenerated) BGPs. In contrast, the optimal paths in result (3) are (degenerate) asymptotically BGPs. Because per capita capital $k(t)$ keeps increasing along this asymptotically BGP, the real wage rate rises while the real rate of interest declines in the long-run.

Let us make two remarks on Theorem 2 and 3. First, the properties of optimal paths in the *RCK* model with the discount rate $\rho>0$ are independent of the child rearing cost (b) because the conditions separating Theorem 2 and 3 ($\rho+\delta<1$ and $\rho+\delta>1$) do not include $b\ge 0$. Second, the *RCK* model with the discount rate $\rho>0$ reveals all the possibilities of optimal paths including an asymptotically BGP in the *RCK* model with declining population. It has thus turned out that the *RCK* model with the

[34] This inequality contains $n=n_R=\tilde{n}_S$ though it does not always contradict $n_R<\tilde{n}_S$. For $n=n_R=\tilde{n}_S$, the steady state $k^*=+\infty$ lies on the *kk* curve. Then, the optimal path is, as described in (3), the equilibrium path which always lies below the *kk* curve with the highest consumption for the same level of capital, and asymptotically approaches the *kk* curve from below.

discount rate $\rho - n > 0$ with the most common formulation of utility function in macroeconomics can only provide a limited view for the possibilities of optimal paths (i.e., can induce only the possibility of a BGP). In this respect, Theorem 3 could be regarded as the general theorem for the *RCK* model with declining population.

5.4 Economic Implications from the Three Theorems

We are now ready to discuss economic implications from the three theorems. First, an optimal path exists under plausible economic situations in the two *RCK* models with declining population.[35] Thus, even after an economy gets into the phase of population decline, the *RCK* model will remain to be useful as the theoretical foundation for deriving the socially optimal path in a perfectly-competitive market economy. In addition, according to results (1) and (2) which are common among the three theorems, the optimal path turns out to be a saddle-point path converging the steady state, which is the same property as in the population-increasing economy. While this seems consistent with our intuitive conjecture,[36] to the best of our knowledge, no existing studies have explicitly derived these results by rigorous theoretical analysis. We also find a new result for the large rates of population decline that per capita consumption comes to be larger than per capita income ($c^* > f(k^*)$) and thus an economy needs to make reversible investment in the long-run. This result could be derived because we explicitly analyzed the *RCK* models with declining population.

Second, taking into account that the solution path in the *RCK* model can be interpreted as the equilibrium path of a decentralized market economy, we could find intrinsic mechanisms that may work

[35] The non-existence of optimal paths (result (2)) in the three theorems occurs only when the initial condition satisfies $\bar{k} < k_0$, which will not actually occur in a population-declining society (recall the explanation just after Theorem 1). The non-existence result (3) in Theorem 2 occurs only in the exceptional situation with $\lambda = 0$ and $d = 1$.

[36] These optimal paths emerge for the ranges of population declining rates $\max\{n_S, n_R\} < n < n_P$ for Theorem 1, $n_R < n < n_P$ for Theorem 2 and $\tilde{n}_S < n < n_P$ for Theorem 3. In section 6 we will show the rates of population decline estimated in *WPP 2019* can be lower than $n_P < 0$.

in the competitive market economy with declining population. In particular, none of the three theorems involve any solution paths with decreasing per capita income and consumption. Thus, the competitive market economy with declining population turns out to have no intrinsic growth mechanism which *decreases* per capita income and the utility of per capita consumption. This finding should be emphasized because there has been a concern in a real society that per capita income and/or welfare may deteriorate if population declines so rapidly. In addition, the previous literature on growth mechanism in the perfectly-competitive market economy seems to share this concern. Ritschl (1985) showed in the Solow-type neoclassical growth model that there exists an unstable steady state under population decline if an exogenous saving rate is negative. Then, per capita capital k and thus per capita income $f(k)$ reduce to zero in the long-run if the initial level of capital k_0 lies below the steady-state value k^*.[37] We have found, however, that this income/welfare-deteriorating result does not apply when we generalize the competitive market economy model by endogenizing the saving rate in the *RCK* model.

Instead, we find that per capita income and consumption may be constant in the long-run along the market equilibrium path because result (2) of the three theorems shows that the solution path converges to the steady state with $\dot{c}(t) = \dot{k}(t) = 0$. This is qualitatively the same property as in the *RCK* model with increasing population. Not only that, under more rapid population decline, we find a new type of market equilibrium path in result (3) of Theorem 3: per capita income and consumption *increase* along the asymptotically BGP in the long-run. Thus, even if technological progress is absent, the competitive market economy turns out to have the intrinsic growth mechanism which makes per capita income and consumption constant or increasing in the long-run. In the terminology by Jones (2022), the competitive

[37] Ritschl (1985) proceeds to show that a steady state exists under population decline when the classical saving function is introduced. Felderer (1988) shows the existence of a steady state for any signs of population change by replacing the (ad-hoc) classical saving function with the life-cycle hypothesis, which has a sound microeconomic foundation. They focused on the existence of a steady state and did not always emphasize the concern about decreasing per capita income and welfare.

market economy does not have any mechanisms which induces the "Empty Planet" result but it involves the intrinsic mechanism which can induce the "Expanding Cosmos" result.

Third, we need to deviate from the assumption of perfect competition if we should take into consideration the possibility of deteriorating per capita income and welfare in a decentralized market economy with declining population. It must be important to incorporate missing factors including technological progress, factor-input externalities, increasing returns or imperfect market structure, and investigate the consequences and implications by explicitly analyzing the associated growth theories. Then, the present analyses of the *RCK* models may provide the benchmark that enables us to clarify how such factors can work through what mechanism in deciding behaviors of per capita income and welfare in the market equilibrium and optimal paths under population decline.

6. Empirical Relevance of Growth Paths Specific to Population Decline

In this section we will show that the optimal paths specific to population decline can emerge in empirically more relevant situations than economists have ever thought by introducing the child rearing cost $b > 0$ in the *RCK* model with the discount rate $\rho > 0$.

6.1 Empirical Strategy and Methodology

We first derive the critical rates $(n_P, \tilde{n}_S, n_P)$ of population decline numerically and then examine whether the rates of population change estimated in the United Nation's *WPP 2019* can be lower than these negative critical rates. We take a strategy with the following two points for this empirical investigation.

First, we use the "low(-fertility) variant" among the three estimations for 2020-2100 provided in *WPP 2019.* Taking into consideration that Jones (2022) has recently emphasized global population decline as a distinct possibility,[38] it will be of substantial significance to explore a future possibility of realizing a growth path specific to population decline under global population decline, which is consistent with the "low variant" estimation in many regions of the world.[39]

Second, we make use of the data of child rearing costs in Japan provided in Cabinet Office (2005). This comprehensive and detailed report is based on the Japanese government's newest investigation of child rearing costs in the Japanese society as a whole. To the best of our knowledge, it is the only source of aggregate child rearing costs in Japan,[40] where substantial decline in fertility has been observed.

Let us next explain the methodology for deriving the critical rates $(n_P, \tilde{n}_S, n_R)$, which depend mainly on ρ, δ and b. First, we set the rate of time preference at ρ=0.01, taking into consideration that Jones (2022) sets it at $\rho = 0.011$. Second, we suppose the depreciation rate of capital $\delta = 0.03$ based on the statement by Jones (2022) "Empirically, rates of population decline are perhaps 1 percent or smaller, whereas depreciation rates are 3 percent or 5 percent or more." (p.3492). Third, we estimate the values of b in a real term using $b = TCRC(t)/\lambda(t)K(t)$. This formula is derived by substituting $\eta(t) = b(t)\{K(t)/L(t)\}$ into total child rearing costs $TCRC(t) = \eta(t)\lambda(t)L(t)$ in the entire economy at a point in time *t*. Then, we follow three steps for the estimation of b.

First, Cabinet Office (2005) reports nominal child rearing costs in the entire economy (including the costs for age 18-21) and labor costs for domestic child care (i.e., the opportunity cost of parental leave) for 2002. We divide the sum of these costs by the consumer price index in 2002 (evaluated with 2000 price) and thus obtain the real *TCRC*.

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38 He mentions "The fact that so many rich countries already have fertility below replacement indicates that a future with negative population growth is a possibility that deserves further consideration" on p.3490.

39 The rates of population change in the "low variant" estimation in *WPP 2019* are all negative for Asia, Europe, North America and Latin America and Caribbean after 2050. Even if the estimated rates for African countries are included, global population decline is predicted after 2055-60 in the "low variant" estimation.

40 "Report on Investigation of Child Rearing Costs via Internet" (2009) is also available. However, this report provides highly disaggregated data separating too many items, so that it is rather difficult to obtain the data of aggregate child rearing costs.

Second, we find the data of physical capital (K) in a real term for 2000-2009 (evaluated with the average price in 2000) in GROSS FIXED CAPITAL FORMATION (installation-base) of the National Accounts of Japan. Third, the fertility and death rates (λ and d) can be calculated simply by dividing the numbers of births and deaths by the total population for each year, taking up the data for 2000-2009 corresponding to the period of available data on real physical capital. We use the data in Chapter 2 of *Statistical Handbook of Japan 2022* for the numbers of births and deaths and Long-term Time Series Data in *Population Estimates* (2020) for the total population (both are provided by Statistics Bureau in Ministry of Internal Affairs and Communications of Japan). Then, we can derive the critical rates $(n_P, \tilde{n}_S, n_R)$ for each year during 2000-2009 by assuming the real *TCRC* in 2002 for all years during 2000-2009 (thus, the critical rates $(n_P, \tilde{n}_S, n_R)$ for 2002 should be the most reliable).

6.2 Empirical Relevance of the Critical Rates of Population Decline

In Table 6.1, we show the critical rates $(n_P, \tilde{n}_S, n_R)$ of population decline derived from the *RCK* model with the discount rate $\rho > 0$.

Year	$n_P = -\dfrac{\delta + bd}{1+b}$ (%)	$\tilde{n}_S = -\dfrac{\rho + \delta + bd}{1+b}$ (%)	$n_R = -\dfrac{1+bd}{1+b}$ (%)
2000	−1.07	−1.21	−14.66
2001	−1.08	−1.22	−14.65
2002	−1.08	−1.22	−14.61
2003	−1.10	−1.23	−14.41
2004	−1.11	−1.25	−14.54
2005	−1.14	−1.28	−14.28
2006	−1.15	−1.29	−14.80

2007	−1.17	−1.32	−15.10
2008	−1.20	−1.34	−15.34
2009	−1.19	−1.34	−15.098

Table 6.1 Critical Rates of Population Decline under Depreciation Rate $\delta = 0.03$

(Source: Authors' calculation)

Let us compare the critical rates for 2002 ($n_P = -1.08$%, $\tilde{n}_S = -1.22$% and $n_R = -14.61$%) with the rates of population change in the "low variant" estimation in *WPP* 2019, which we present in Table 6.2.

Region	Country	n(%)	period	n(%)	Period
Asia	Japan	−1.18	2055−60	−1.33	2060−65
	South Korea	−1.20	2050−55	−1.41	2055−60
	Thailand	−1.21	2055−60	−1.32	2060−65
	Taiwan	−1.12	2055−60	−1.24	2060−65
	China	−1.13	2060−65	−1.27	2070−75
Southern Europe	Greece	−1.20	2055−60	−1.32	2060−65
	Italy	−1.20	2055−60	−1.31	2060−65
	Portugal	−1.11	2055−60	−1.24	2065−70
	Spain	−1.09	2055−60	−1.26	2060−65
Eastern Europe	Bulgaria	−1.15	2030−35	−1.25	2045−50
	Romania	−1.11	2050−55	−1.23	2055−60
	Moldova	−1.13	2040−45	−1.24	2045−50
	Ukraine	−1.11	2040−45	−1.28	2050−55

Table 6.2 The Estimated Rates of Population Decline Exceeding the Critical Rates

(Source: *World Population Prospects 2019*)

It is found that not a few countries in many regions of the world will experience the negative rates of population change lower than the critical rates n_P and $\tilde{n}_S$ in a few or several decades (though they will still be higher than $n_R < 0$). For example, Japan will experience the rates of population decline n= -1.18 (<-1.08=n_P)% for 2055-60 and n=-1.33 (<-1.22=n_S)% for 2060-65. Then, the Japanese economy will get into the phases of $n_S < n < n_P$ and $n_R < n < n_S$ in these periods, respectively (for reference, n=-1.06% for 2050-55). While you can find the other countries with similar estimated rates of population decline in Table 6.2, there are more countries in the "low variant" estimation in *WPP 2019* (the corresponding data have not been shown here), which may get into the phases of $\tilde{n}_S < n < n_P$ and $n_R < n < \tilde{n}_S$ by 2100.

To sum up, by introducing child rearing costs in the *RCK* model, the optimal paths specific to population decline may emerge in empirically more relevant situations than has ever been thought in the future covered by *WPP 2019*.

7. Concluding Remarks

In this paper we investigate how population decline may affect the optimal path in two types of Ramsey-Cass-Koopmans (*RCK*) model with child rearing costs. One is the model with the discount rate $\rho - n > 0$ and the other is the model with the discount rate ρ. An optimal path exists in both models under economically plausible conditions, that is, the possibilities of non-existence of an optimal path can be realized in a theoretically extreme case or should not occur in a real economy which enters into the phase of population decline from that of population increase. Therefore, even under population decline, the *RCK* model remains to be a reliable fundamental theory that can identifies the socially optimal path in a competitive market economy.

Furthermore, it is found that a new type of optimal path exists in the *RCK* model with the discount rate ρ, which is also worth analyzing under population decline. The existence and properties of an optimal path depend on the range of the rates of population decline, regardless of the child rearing costs. First, when population decline is mild, the optimal path is a saddle-point path converging to a finite steady state, as in the standard *RCK* model with increasing population. Second, when population decline is faster, the optimal path is a saddle-point path converging, by *reversible investment*, to a finite steady state (i.e., a balanced growth path (*BGP*)), at which per capita consumption is *larger* than per capita income. Third, when population decline is even faster, a new type of optimal path emerges in the *RCK* model with the discount rate ρ: the optimal path can be an *asymptotically BGP*, along which both per capita consumption and income keep *increasing* permanently even without technological progress. We have also show empirical relevance of these optimal paths by Japanese data: the negative rates of population change estimated in *World Population Prospects 2019* can be lower than the theoretical threshold values below which an optimal path has different properties from those under population increase.

Finally, as an interesting suggestion form this paper, if we should pay attention to the widely shared concern that per capita income and consumption *may decrease* due to population decline, we need to deviate from the assumption of perfect competition. It is of substantial importance to explicitly analyze equilibrium and optimal paths in growth theories with technological progress and/or some market failures like factor-input externalities, increasing returns, or imperfect market structure. These directions should be further explored in future research on growth theory under population decline.

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Table 1. Excerpt from *World Population Prospects 2019* (with medium fertility rates)

Medium Variant

Medium variant	Average annual rate of population change (percentage)															
Region, country or area *	**2020-2025**	**2025-2030**	**2030-2035**	**2035-2040**	**2040-2045**	**2045-2050**	**2050-2055**	**2055-2060**	**2060-2065**	**2065-2070**	**2070-2075**	**2075-2080**	**2080-2085**	**2085-2090**	**2090-2095**	**2095-2100**
Asia	0.77	0.62	0.49	0.36	0.25	0.14	0.04	-0.05	-0.12	-0.19	-0.25	-0.29	-0.33	-0.35	-0.37	-0.39
Europe	-0.05	-0.12	-0.17	-0.20	-0.23	-0.26	-0.29	-0.33	-0.34	-0.32	-0.28	-0.24	-0.19	-0.16	-0.14	-0.14
Latin America and the Caribbean	0.84	0.70	0.56	0.43	0.32	0.22	0.11	0.02	-0.07	-0.16	-0.24	-0.30	-0.36	-0.40	-0.44	-0.46
Northern America	0.59	0.56	0.53	0.45	0.38	0.34	0.32	0.33	0.34	0.33	0.30	0.27	0.25	0.24	0.24	0.25
Eastern Asia	0.21	0.05	-0.08	-0.20	-0.30	-0.40	-0.49	-0.56	-0.59	-0.60	-0.60	-0.60	-0.59	-0.55	-0.51	-0.50
China	0.26	0.09	-0.05	-0.17	-0.27	-0.38	-0.47	-0.54	-0.57	-0.58	-0.59	-0.59	-0.58	-0.55	-0.51	-0.50
China, Hong Kong SAR	0.68	0.67	0.25	0.05	-0.09	-0.16	-0.17	-0.15	-0.15	-0.17	-0.21	-0.21	-0.15	-0.04	0.08	0.17
Japan	-0.40	-0.53	-0.60	-0.66	-0.69	-0.69	-0.71	-0.76	-0.82	-0.84	-0.80	-0.70	-0.63	-0.57	-0.54	-0.52
Republic of Korea	0.03	-0.07	-0.18	-0.36	-0.53	-0.69	-0.86	-0.99	-1.03	-1.02	-1.00	-0.97	-0.93	-0.88	-0.83	-0.72
South-Eastern Asia	0.91	0.77	0.63	0.49	0.37	0.26	0.16	0.07	-0.01	-0.08	-0.13	-0.18	-0.23	-0.26	-0.30	-0.33
Cambodia	1.26	1.07	0.94	0.84	0.70	0.56	0.41	0.29	0.17	0.05	-0.06	-0.15	-0.21	-0.26	-0.32	-0.38
Indonesia	0.97	0.83	0.69	0.57	0.44	0.32	0.21	0.12	0.05	-0.01	-0.05	-0.10	-0.14	-0.19	-0.24	-0.28
Lao People's Democratic Re	1.33	1.13	0.95	0.79	0.63	0.47	0.31	0.16	0.02	-0.11	-0.24	-0.34	-0.43	-0.51	-0.58	-0.65
Malaysia	1.19	0.99	0.80	0.62	0.50	0.41	0.33	0.24	0.13	0.02	-0.07	-0.14	-0.17	-0.19	-0.19	-0.21
Myanmar	0.76	0.68	0.54	0.38	0.23	0.11	0.02	-0.05	-0.12	-0.18	-0.25	-0.31	-0.35	-0.37	-0.38	-0.38
Philippines	1.28	1.14	1.00	0.84	0.70	0.57	0.44	0.33	0.23	0.14	0.04	-0.05	-0.13	-0.20	-0.26	-0.30
Singapore	0.77	0.60	0.39	0.18	0.01	-0.12	-0.20	-0.25	-0.28	-0.30	-0.30	-0.28	-0.24	-0.19	-0.13	-0.06
Thailand	0.15	0.01	-0.13	-0.26	-0.39	-0.52	-0.63	-0.71	-0.74	-0.75	-0.74	-0.73	-0.73	-0.74	-0.74	-0.71
Viet Nam	0.76	0.60	0.41	0.28	0.20	0.13	0.03	-0.08	-0.18	-0.25	-0.29	-0.31	-0.31	-0.32	-0.32	-0.34
Southern Asia	1.07	0.92	0.77	0.62	0.48	0.36	0.25	0.14	0.03	-0.08	-0.17	-0.24	-0.30	-0.35	-0.39	-0.42
India	0.92	0.80	0.66	0.50	0.35	0.23	0.13	0.03	-0.08	-0.18	-0.27	-0.34	-0.39	-0.42	-0.45	-0.47
Central & South America																
El Salvador	0.48	0.40	0.26	0.18	0.06	-0.05	-0.16	-0.28	-0.41	-0.54	-0.68	-0.83	-0.98	-1.11	-1.22	-1.31
Mexico	0.96	0.81	0.68	0.54	0.41	0.29	0.18	0.08	-0.01	-0.10	-0.17	-0.24	-0.31	-0.38	-0.43	-0.46
Brazil	0.60	0.44	0.30	0.16	0.05	-0.05	-0.15	-0.25	-0.34	-0.43	-0.51	-0.57	-0.61	-0.62	-0.63	-0.62
Chile	0.13	0.22	0.43	0.28	0.14	0.02	-0.08	-0.17	-0.24	-0.29	-0.34	-0.39	-0.41	-0.43	-0.43	-0.41
Eastern Europe	-0.24	-0.35	-0.41	-0.43	-0.41	-0.40	-0.41	-0.44	-0.47	-0.47	-0.43	-0.36	-0.28	-0.23	-0.22	-0.23
Belarus	-0.14	-0.26	-0.35	-0.37	-0.35	-0.34	-0.34	-0.37	-0.40	-0.41	-0.38	-0.32	-0.24	-0.18	-0.17	-0.21
Bulgaria	-0.77	-0.82	-0.88	-0.89	-0.87	-0.86	-0.88	-0.92	-0.94	-0.93	-0.88	-0.80	-0.72	-0.67	-0.68	-0.72
Czechia	0.09	-0.02	-0.10	-0.12	-0.09	-0.06	-0.08	-0.14	-0.20	-0.21	-0.15	-0.06	0.03	0.09	0.11	0.10
Hungary	-0.31	-0.37	-0.45	-0.50	-0.51	-0.49	-0.48	-0.49	-0.52	-0.53	-0.50	-0.44	-0.38	-0.34	-0.30	-0.26
Poland	-0.18	-0.31	-0.42	-0.50	-0.56	-0.60	-0.64	-0.69	-0.75	-0.83	-0.87	-0.86	-0.79	-0.71	-0.64	-0.60
Republic of Moldova	-0.30	-0.45	-0.60	-0.71	-0.78	-0.83	-0.90	-0.98	-1.09	-1.18	-1.21	-1.16	-1.06	-0.95	-0.88	-0.86
Romania	-0.49	-0.50	-0.54	-0.58	-0.61	-0.64	-0.67	-0.71	-0.73	-0.71	-0.66	-0.61	-0.57	-0.55	-0.53	-0.54
Russian Federation	-0.11	-0.25	-0.31	-0.30	-0.25	-0.22	-0.22	-0.25	-0.27	-0.26	-0.20	-0.12	-0.05	-0.02	-0.03	-0.06
Slovakia	-0.04	-0.17	-0.30	-0.40	-0.45	-0.47	-0.49	-0.54	-0.61	-0.67	-0.67	-0.61	-0.52	-0.43	-0.38	-0.37
Ukraine	-0.65	-0.70	-0.73	-0.74	-0.75	-0.77	-0.81	-0.85	-0.87	-0.86	-0.82	-0.74	-0.65	-0.59	-0.57	-0.58
Southern Europe	-0.22	-0.29	-0.32	-0.36	-0.44	-0.54	-0.65	-0.75	-0.79	-0.77	-0.71	-0.62	-0.54	-0.50	-0.49	-0.49
Greece	-0.52	-0.47	-0.42	-0.42	-0.47	-0.56	-0.68	-0.77	-0.81	-0.80	-0.72	-0.62	-0.53	-0.47	-0.46	-0.46
Italy	-0.20	-0.28	-0.31	-0.34	-0.43	-0.57	-0.68	-0.77	-0.80	-0.74	-0.64	-0.56	-0.51	-0.49	-0.48	-0.47
Portugal	-0.27	-0.30	-0.34	-0.39	-0.47	-0.55	-0.62	-0.67	-0.68	-0.65	-0.57	-0.49	-0.42	-0.40	-0.39	-0.36
Spain	-0.08	-0.15	-0.20	-0.24	-0.31	-0.41	-0.55	-0.68	-0.76	-0.77	-0.70	-0.56	-0.42	-0.35	-0.34	-0.36
Western Europe	0.13	0.09	0.05	-0.01	-0.07	-0.12	-0.15	-0.15	-0.13	-0.09	-0.08	-0.07	-0.06	-0.04	-0.02	-0.00
Austria	0.22	0.16	0.08	0	-0.06	-0.12	-0.16	-0.18	-0.16	-0.14	-0.13	-0.13	-0.11	-0.06	0.00	0.05
Belgium	0.29	0.25	0.21	0.16	0.11	0.06	0.01	0.00	0.01	0.03	0.04	0.05	0.06	0.07	0.08	0.08
France	0.24	0.19	0.16	0.10	0.03	-0.03	-0.07	-0.09	-0.08	-0.06	-0.05	-0.04	-0.05	-0.06	-0.06	-0.07
Germany	-0.06	-0.09	-0.12	-0.16	-0.21	-0.26	-0.28	-0.26	-0.22	-0.17	-0.15	-0.14	-0.11	-0.06	-0.02	0.01
Netherlands	0.21	0.15	0.06	-0.05	-0.14	-0.20	-0.23	-0.22	-0.19	-0.15	-0.13	-0.15	-0.17	-0.17	-0.16	-0.14

Table 2. Excerpt from *World Population Prospects 2019* (with low fertility rates)

Low Variant

Low Variant	Average annual rate of population change (percentage)															
Region, country or area *	**2020-2025**	**2025-2030**	**2030-2035**	**2035-2040**	**2040-2045**	**2045-2050**	**2050-2055**	**2055-2060**	**2060-2065**	**2065-2070**	**2070-2075**	**2075-2080**	**2080-2085**	**2085-2090**	**2090-2095**	**2095-2100**
Asia	0.59	0.35	0.16	0.04	-0.09	-0.24	-0.40	-0.55	-0.69	-0.80	-0.91	-1.01	-1.12	-1.21	-1.30	-1.38
Europe	-0.20	-0.35	-0.45	-0.50	-0.54	-0.60	-0.69	-0.79	-0.86	-0.89	-0.89	-0.88	-0.87	-0.87	-0.89	-0.93
Latin America and the Caribbean	0.65	0.41	0.22	0.09	-0.04	-0.18	-0.33	-0.48	-0.62	-0.75	-0.88	-1.01	-1.14	-1.27	-1.38	-1.48
Northern America	0.42	0.30	0.21	0.14	0.07	0.01	-0.05	-0.09	-0.12	-0.15	-0.19	-0.24	-0.30	-0.34	-0.37	-0.37
Eastern Asia	0.05	-0.18	-0.36	-0.49	-0.61	-0.76	-0.90	-1.04	-1.14	-1.22	-1.28	-1.34	-1.40	-1.44	-1.46	-1.52
China	0.10	-0.14	-0.32	-0.45	-0.58	-0.74	-0.89	-1.03	-1.13	-1.20	-1.27	-1.34	-1.41	-1.45	-1.48	-1.54
China, Hong Kong SAR	0.49	0.41	-0.03	-0.19	-0.33	-0.44	-0.51	-0.57	-0.63	-0.67	-0.72	-0.74	-0.71	-0.63	-0.54	-0.44
China, Taiwan Province of C	-0.05	-0.22	-0.38	-0.54	-0.72	-0.87	-1.00	-1.12	-1.24	-1.34	-1.41	-1.45	-1.42	-1.37	-1.32	-1.25
Japan	-0.53	-0.73	-0.87	-0.94	-0.98	-1.01	-1.06	-1.18	-1.33	-1.43	-1.44	-1.39	-1.35	-1.34	-1.37	-1.43
Republic of Korea	-0.13	-0.32	-0.48	-0.64	-0.81	-0.99	-1.20	-1.41	-1.56	-1.64	-1.68	-1.70	-1.71	-1.74	-1.79	-1.77
South-Eastern Asia	0.73	0.49	0.29	0.16	0.02	-0.14	-0.29	-0.44	-0.58	-0.69	-0.79	-0.89	-1.00	-1.11	-1.21	-1.30
Cambodia	1.06	0.76	0.57	0.46	0.30	0.09	-0.11	-0.29	-0.46	-0.64	-0.82	-0.98	-1.12	-1.25	-1.39	-1.54
Indonesia	0.79	0.55	0.35	0.22	0.08	-0.09	-0.25	-0.40	-0.52	-0.62	-0.71	-0.80	-0.91	-1.02	-1.14	-1.25
Lao People's Democratic Re	1.12	0.82	0.57	0.41	0.23	0.02	-0.20	-0.40	-0.60	-0.79	-0.99	-1.18	-1.36	-1.55	-1.74	-1.93
Malaysia	0.99	0.69	0.45	0.30	0.17	0.07	-0.06	-0.21	-0.37	-0.51	-0.64	-0.75	-0.84	-0.91	-0.96	-1.01
Myanmar	0.57	0.38	0.16	0.01	-0.15	-0.31	-0.47	-0.61	-0.74	-0.86	-0.98	-1.11	-1.23	-1.33	-1.41	-1.46
Philippines	1.09	0.85	0.64	0.49	0.33	0.16	-0.00	-0.16	-0.31	-0.45	-0.59	-0.73	-0.87	-1.00	-1.13	-1.24
Singapore	0.60	0.36	0.13	-0.06	-0.24	-0.40	-0.53	-0.63	-0.70	-0.74	-0.76	-0.77	-0.76	-0.74	-0.70	-0.61
Thailand	-0.01	-0.25	-0.44	-0.57	-0.72	-0.89	-1.06	-1.21	-1.32	-1.38	-1.43	-1.48	-1.57	-1.68	-1.78	-1.84
Viet Nam	0.58	0.33	0.10	-0.02	-0.11	-0.23	-0.38	-0.56	-0.72	-0.85	-0.94	-1.02	-1.10	-1.17	-1.24	-1.32
Southern Asia	0.88	0.63	0.42	0.28	0.13	-0.03	-0.21	-0.38	-0.54	-0.69	-0.83	-0.97	-1.11	-1.23	-1.35	-1.44
India	0.73	0.50	0.30	0.15	-0.00	-0.16	-0.33	-0.50	-0.65	-0.80	-0.94	-1.07	-1.20	-1.32	-1.43	-1.50
Central & South America																
El Salvador	0.26	0.08	-0.13	-0.21	-0.35	-0.51	-0.68	-0.87	-1.06	-1.27	-1.51	-1.79	-2.13	-2.49	-2.87	-3.30
Mexico	0.77	0.51	0.32	0.19	0.04	-0.11	-0.28	-0.43	-0.57	-0.70	-0.83	-0.96	-1.10	-1.25	-1.39	-1.51
Brazil	0.41	0.15	-0.05	-0.17	-0.30	-0.43	-0.58	-0.73	-0.89	-1.04	-1.18	-1.31	-1.42	-1.53	-1.63	-1.71
Chile	-0.05	-0.05	0.11	-0.04	-0.19	-0.33	-0.47	-0.61	-0.74	-0.85	-0.95	-1.04	-1.14	-1.22	-1.28	-1.33
Eastern Europe	-0.39	-0.58	-0.70	-0.74	-0.75	-0.78	-0.85	-0.95	-1.04	-1.11	-1.12	-1.09	-1.05	-1.03	-1.05	-1.12
Belarus	-0.29	-0.49	-0.62	-0.67	-0.69	-0.72	-0.78	-0.87	-0.96	-1.02	-1.05	-1.04	-1.00	-0.97	-0.99	-1.07
Bulgaria	-0.91	-1.04	-1.16	-1.20	-1.21	-1.25	-1.33	-1.45	-1.57	-1.65	-1.68	-1.67	-1.65	-1.68	-1.79	-1.97
Czechia	-0.06	-0.24	-0.37	-0.40	-0.39	-0.39	-0.45	-0.57	-0.69	-0.75	-0.73	-0.65	-0.58	-0.54	-0.54	-0.57
Hungary	-0.46	-0.61	-0.74	-0.80	-0.84	-0.85	-0.89	-0.97	-1.08	-1.16	-1.17	-1.15	-1.14	-1.14	-1.14	-1.15
Poland	-0.33	-0.54	-0.70	-0.79	-0.87	-0.95	-1.05	-1.17	-1.31	-1.46	-1.59	-1.66	-1.67	-1.65	-1.65	-1.68
Republic of Moldova	-0.48	-0.71	-0.90	-1.03	-1.13	-1.24	-1.38	-1.55	-1.76	-1.96	-2.10	-2.17	-2.16	-2.14	-2.16	-2.24
Romania	-0.63	-0.73	-0.83	-0.89	-0.95	-1.01	-1.11	-1.23	-1.33	-1.39	-1.40	-1.41	-1.43	-1.48	-1.56	-1.66
Russian Federation	-0.27	-0.48	-0.60	-0.61	-0.60	-0.61	-0.66	-0.75	-0.83	-0.87	-0.86	-0.82	-0.77	-0.75	-0.79	-0.87
Slovakia	-0.20	-0.40	-0.58	-0.69	-0.77	-0.82	-0.90	-1.02	-1.17	-1.30	-1.37	-1.37	-1.33	-1.29	-1.29	-1.33
Ukraine	-0.80	-0.94	-1.03	-1.06	-1.11	-1.18	-1.28	-1.40	-1.51	-1.59	-1.62	-1.60	-1.57	-1.56	-1.61	-1.70
Southern Europe	-0.36	-0.51	-0.60	-0.65	-0.74	-0.86	-1.02	-1.18	-1.31	-1.37	-1.35	-1.30	-1.26	-1.27	-1.32	-1.39
Greece	-0.66	-0.69	-0.70	-0.72	-0.78	-0.89	-1.04	-1.20	-1.32	-1.38	-1.37	-1.30	-1.24	-1.21	-1.27	-1.34
Italy	-0.33	-0.50	-0.58	-0.62	-0.73	-0.88	-1.04	-1.20	-1.31	-1.31	-1.26	-1.21	-1.19	-1.22	-1.28	-1.34
Portugal	-0.41	-0.52	-0.62	-0.68	-0.76	-0.87	-0.99	-1.11	-1.21	-1.24	-1.21	-1.14	-1.10	-1.12	-1.18	-1.22
Spain	-0.22	-0.36	-0.46	-0.52	-0.60	-0.72	-0.90	-1.09	-1.26	-1.35	-1.33	-1.22	-1.09	-1.05	-1.09	-1.18
Western Europe	-0.02	-0.14	-0.24	-0.29	-0.36	-0.44	-0.52	-0.59	-0.62	-0.62	-0.62	-0.64	-0.67	-0.68	-0.70	-0.72
Austria	0.06	-0.08	-0.21	-0.27	-0.34	-0.43	-0.53	-0.61	-0.64	-0.65	-0.67	-0.70	-0.72	-0.70	-0.66	-0.63
Belgium	0.14	0.01	-0.08	-0.14	-0.19	-0.27	-0.36	-0.43	-0.48	-0.48	-0.49	-0.51	-0.53	-0.55	-0.57	-0.60
France	0.09	-0.04	-0.13	-0.20	-0.28	-0.36	-0.45	-0.53	-0.58	-0.60	-0.61	-0.63	-0.68	-0.73	-0.79	-0.85
Germany	-0.21	-0.32	-0.40	-0.44	-0.50	-0.58	-0.65	-0.70	-0.71	-0.69	-0.69	-0.70	-0.71	-0.69	-0.68	-0.68
Netherlands	0.06	-0.09	-0.24	-0.34	-0.44	-0.53	-0.61	-0.68	-0.70	-0.69	-0.70	-0.74	-0.82	-0.88	-0.93	-0.95