Sequential dictatorship rules in multi-unit object assignment problems with money*

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April 8, 2024

Abstract

We study consistency in multi-unit object allocation problems with money. Objects are identical and each agent has a multi-demand and quasi-linear preferences. We consider the class of weak object monotonic preferences and that of single-peaked preferences. We first show that on those domains, if a rule satisfies consistency, strategy-proofness, individual rationality, no subsidy, non-wasteful tie-breaking, and minimal tradability, then it is a sequential dictatorship rule. Since not all sequential dictatorship rule are strategy-proof and consistent, we then focus on a specific class of sequential dictatorship rules which we call sequential dictatorship rules with lowest tie-breaking. On the weakly object monotonic domain, when the reservation prices are increasing in the number of objects, sequential dictatorship rules with lowest tie-breaking satisfy consistency and independence of unallocated objects if and only if there is a common priority ordering for more than one object and this is an acyclic ordering of the priority ordering for one object. We also show that this condition is a necessary and sufficient condition for a sequential dictatorship rule with lowest tie-breaking to satisfy consistency and independence of unallocated objects on the single-peaked domain.

Keywords. Consistency, Strategy-proofness, sequential dictatorship rule, serial dictatorship rule, weakly object monotonic preferences, single-peaked preferences, acyclicity.

JEL Classification. D44, D71, D61, D82.

*We gratefully acknowledge financial support from the Japan Society for the Promotion of Science (Kazumura, 19K13643, 21H00696; Sakai, 19J10221, 22KJ1785).

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1 Introduction

We investigate consistent allocation rules in the problem of allocating identical objects with money. *Consistency* requires that if an allocation is selected by a rule for an economy and some of the agents leave with their assigned objects, then for the reduced economy, the rule should choose the allocation at which each of the remaining agents receives the same consumption bundle as before. *Consistency* has been studied in many economic models, and its desirability has been discussed from various point of views. For example, *consistency* is considered to be a solidarity principle, a robustness requirement, etc.—The desirability of *consistency* is argued comprehensively in Thomson (2012). Klaus and Nichifor (2020) introduce *consistency* in an object allocation with money. They show that *consistency*, incentive constraints and other mild conditions imply the serial dictatorship: This result is extended to the case of heterogeneous objects (Klaus and Nichifor, 2021). However, they assume that agents are unit-demand, that is, each agent could receive at most one object. Our goal is to investigate the implication of *consistency* in the environment where there are identical objects and each agent can receive multiple objects.

Formally, an *economy* in our model consists of three components: A set of agents, the number of (identical) objects available in the economy, and preferences of the agents. A *(consumption) bundle* consists of a pair of a quantity of objects, and a payment level. We assume that preferences are quasi-linear. A set of preferences is called a *domain*. We consider two specific domains. The first one is the domain of *weakly object monotonic preferences*. Weak object monotonicity requires that at each payment level, receiving more objects should be at least as desirable as receiving less. The other one is the domain of *single-peaked preferences*. If a preference is single-peaked, there is a unique optimal consumption level of the object, and at each payment level, the agent is made at least as well off if the number of objects she receives is closer to the optimal level.

An *(allocation) rule* specifies an allocation for each economy. We follow the same set of properties of rules studied in Klaus and Nichifor (2020, 2021). Aside from *consistency*, there are six properties. A rule is *strategy-proof* if each agent has an incentive to report her true preferences. A rule is *individually rational* if no agent receives a bundle that makes her worse off than she would be if she had received no object and paid nothing. *No subsidy* requires that the payment of each agent should be nonnegative. *Independence of unallocated objects* has a similar spirit as consistency. It requires that the allocation should remain the same even if some of the unassigned objects are removed from the economy. *Non-wasteful tie-breaking* requires that if an agent receives an object, her bundle should not be indifferent to receiving no object and paying nothing. Thus, together with *individual rationality*, *non-wasteful tie-breaking* means that an agent will be made better off by participating in the mechanism as long as she receives an object. *Minimal tractability* requires that given a number of objects, there is an economy with the same number of objects where no object is unassigned.

A rule is *sequential dictatorship* if each number of objects is associated with a reservation price and a priority order over agents, and the allocation for an economy is determined as follows: The agent who has the highest priority with respect to the number of objects existing in the economy chooses the most preferred number of objects at the reservation prices, pays the associated reservation price, and leaves; Next, the agent who has the highest priority among the remaining agents with respect to the number of objects existing in the reduced economy chooses a consumption bundle as in the first agent and leaves; This process continues until all the agent in the economy makes a decision. A *serial dictatorship rule* is a sequential dictatorship rule where the priority order is common for each number of objects.

Our first result shows that on the weakly object monotonic domain and the single-peaked domain, the only rules that satisfy the above mentioned properties are sequential dictatorship rules (Theorem 1). Thus, while this result demonstrates a link between *consistency* and dictatorship as in Klaus and Nichifor (2020, 2021), our result is slightly different from their results in that they show that serial dictatorship rules are the only rules that satisfy the list of properties. Another difference is that while the results in Klaus and Nichifor (2020, 2021) are characterization results, our result is not. Sequential dictatorship rules satisfy all the properties other than *consistency*, *strategy-proofness*, and *independence of unallocated objects*. But some of them violate some of the three properties.

Thus, we then ask when sequential dictatorship rules satisfy *consistency*, *strategy-proofness*, and *independence of unallocated objects*. We first observe that the tie-breaking rule plays an important role for a sequential dictatorship rule to satisfy these three properties (Example 1). Thus, we focus on a specific tie-breaking rule. A rule is *sequential dictatorship rule with lowest tie-breaking* if it is a sequential dictatorship with a tie-breaking rule such that if an agent who is making a decision has multiple optimal consumption bundles, the consumption bundle with the lowest number of objects among them is selected. This tie-breaking rule is one of the natural candidates from the view point of increasing the number of agents who receive objects, which is sometimes a goal in practice. Sequential dictatorship rules with lowest tie-breaking are *strategy-proof*. Hence, we investigate when they satisfy *consistency* and *independence of unallocated objects*.

The result depends on the domain. On the weakly object monotonic domain, we show that a sequential dictatorship rule with lowest tie-breaking satisfies *consistency* and *independence of unallocated objects* if and only if it satisfies the list of conditions we identify (Theorem 2). While the conditions are complicated, the result provides a useful tool for verifying whether a sequential dictatorship rule with lowest tie-breaking satisfies *consistency* and *independence of unallocated objects*. For instance, it is sometimes reasonable to focus on reservation prices that are increasing in the number of objects. Theorem 2 tells that a sequential dictatorship rule with lowest tie-breaking and increasing reservation prices satisfies *consistency* and *independence of unallocated objects* if and only if there is a common priority ordering for more than one object and this is an acyclic ordering of the priority ordering for one object (Corollary 1).

On the single-peaked domain, the condition in Corollary 1 is obtained without assuming the monotonicity of reservation prices. That is, on the single-peaked domain, a sequential dictatorship rule with lowest tie-breaking satisfies *consistency* and *independence of unallocated objects* if and only if there is a common priority ordering for more than one object and this is an acyclic ordering of the priority ordering for one object (Theorem 3).

This paper is organized as follows. The next section reviews the related literature. Section 2 sets up the model and introduces properties of rules. Section 3 defines sequential dictatorship rules. Section 4 provides our results and independence of our axioms. All the proofs appear in Appendix.

1.1 Related literature

The desirability of sequential and serial dictatorship rules has been discussed in various environments. In object allocation problems without money, if each agent can receive at most one object, serial dictatorship rule is the only one that satisfies efficiency, strategy-proofness, non-bossiness, and neutrality (Svensson, 1999). When an agent can receive multiple objects but monetary transfers are not allowed, there is a connection between sequential dictatorship rules and solidarity principles. For example, serial dictatorship rules are the only rules that satisfy *efficiency*, *strategy-proofness*, and either *population monotonicity* or *consistency* (Klaus and Miyagawa, 2002).

In object allocation problems without money, *non-bossiness* is also a key to induce sequential and serial dictatorship rules. Indeed, when an agent can receive multiple objects, sequential dictatorship rules are the only rules that satisfy *efficiency*, *strategy-proofness*, and *non-bossiness* (Pápai, 2001; Ehlers and Klaus, 2003), and serial dictatorship rules are the only rules that satisfy *resource monotonicity* together with the above mentioned three properties (Ehlers and Klaus, 2003).[1]

[1] A similar result is obtained in the environment where each agent is restricted to receive always the same amount of objects (Hatfield, 2009).

Klaus and Nichifor (2020) is the first paper that introduces serial dictatorship rules in object allocation problems with money, and their result is extended to the case of heterogeneous objects (Klaus and Nichifor, 2021). Their results and ours demonstrate that there is a link between solidarity principles and sequential dictatorship rules even in the environment with money. For *non-bossiness*, when there is a single object, sequential dictatorship rules are the only rules that satisfy *strategy-proofness*, *non-bossiness*, *individual rationality*, and *no subsidy* (Shinozaki, 2022).[2]

Acyclicity of priorities has been considered as a condition that guarantees the desirability of certain rules. For example, in priority-based object allocation problems without money, acyclicity is known as a necessary and sufficient condition for the deferred acceptance rule to be *efficient* (Ergin, 2002). In the same model, the top trading cycle rule is *stable* if and only if the priorities are acyclic (Kesten, 2006). It is also true that the immediate acceptance rule is *strategy-proof* and *stable* if and only if priorities are acyclic (Kumano, 2013). The acyclicity conditions introduced in those papers are all different. The acyclicity conditions imposed in Kesten (2006) and Kumano (2013) are stronger than that in Ergin (2002). But the acyclicity conditions in Kesten (2006) and Kumano (2013) are independent. Our acyclicity condition is the one introduced by Ergin (2002).

Acyclicity conditions are related to *consistency*. Indeed, the deferred acceptance rule is *consistent* if and only if priorities are acyclic (Ergin, 2002), and the immediate acceptance rule is *consistent* if and only if priorities are acyclic (Kumano, 2013). Our paper is related to these papers in the sense that our results demonstrate that *consistency* (and other conditions) lead to acyclicity of priorities even in the model with money if preferences are single-peaked or reservation prices are increasing.

In object allocation problems with money, the Vickrey rule and the minimum price Walrasian rule have played central roles.[3] The Vickrey rule is the unique rule that satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* if preferences are quasi-linear (Holmström, 1979). When preferences are not necessarily quasi-linear and agents are unit-demand, the minimum price Walrasian rule is the only rule for those properties (Saitoh and Serizawa, 2008; Sakai, 2008; Morimoto and Serizawa, 2015; Zhou and Serizawa, 2018). When agents are multi-demand like our model, a Walrasian equilibrium may not exits. When preferences are quasi-linear, the existence of a Walrasian equilibrium is guaranteed if preferences satisfy gross sub-

[2] To be precise, Shinozaki (2022) characterizes sequential dictatorship using *pairwise strategy-proofness*, and *non-imposition*. *Non-bossiness* implies *pairwise strategy-proofness*, and *non-imposition* is equivalent to *individual rationality* and *no subsidy* under *strategy-proofness*.

[3] These rules are equivalent when agents are unit-demand and either objects are identical or preferences are quasi-linear.

stitutability (Kelso and Crawford, 1982).[4] When preferences may not be quasi-linear, a Walrasian equilibrium exists if and only if a Walrasian equilibrium exists in the corresponding quasi-linear economies (Baldwin et al., 2023). Our paper is different from those papers in that we focus on sequential dictatorship rules. Miyagawa (2001) considers the house allocation problem with money and shows that if a rule satisfies *strategy-proofness*, *non-bossiness*, *individual rationality*, and *ontoness*, then it is a fixed-price core mechanism. This paper is similar to ours in that both papers show that a list of axioms leads to fixed prices.

2 Preliminaries

We consider an economy where multiple units of an object are to be allocated to a set of agents, and each agent pays some amount of money. We allow an agent to receive multiple units of the object. Thus, a **(consumption) bundle** of an agent is a pair $\mathbb{Z}_+ \times \mathbb{R}$.[5] The set of potential agents is $\mathbb{N}$. Let $\mathcal{N}$ be the family of subsets of $\mathbb{N}$ such that for each $N \in \mathcal{N}$, $0 < |N| < \infty$.

2.1 Preferences

Each agent i has a complete and transitive preference relation R_i over $\mathbb{Z}_+ \times \mathbb{R}$. Throughout the paper, we assume that each preference relation R_i satisfies the following properties.

Quasi-linearity: There is a **valuation function** $v_i : \mathbb{Z}_+ \to \mathbb{R}$ such that (i) $v_i(0) = 0$, and (ii) for each pair $(x, t), (y, s) \in \mathbb{Z}_+ \times \mathbb{R}$, $(x, t)\ R_i\ (y, s)$ if and only if $v_i(x) - t \geq v_i(y) - s$.

Desirability of the object: For each $x \in \mathbb{Z}_+ \setminus \{0\}$, $(x, 0)\ R_i\ (0, 0)$.

We consider two classes of preferences.

Definition 1 *A preference relation R_i is* **weakly object monotonic** *if for each pair $x, y \in \mathbb{Z}_+$ with $x \geq y$, and each $t \in \mathbb{R}$, $(x, t)\ R_i\ (y, t)$.*

Let $\mathcal{R}^{WO}$ be the class of weakly object monotonic preferences.

[4] The existence result is extended to the cases where certain complementarities are allowed (Sun and Yang, 2006; Teytelboym, 2014; Baldwin and Klemperer, 2019).

[5] $\mathbb{Z}_+$ denotes the set of nonnegative integers.

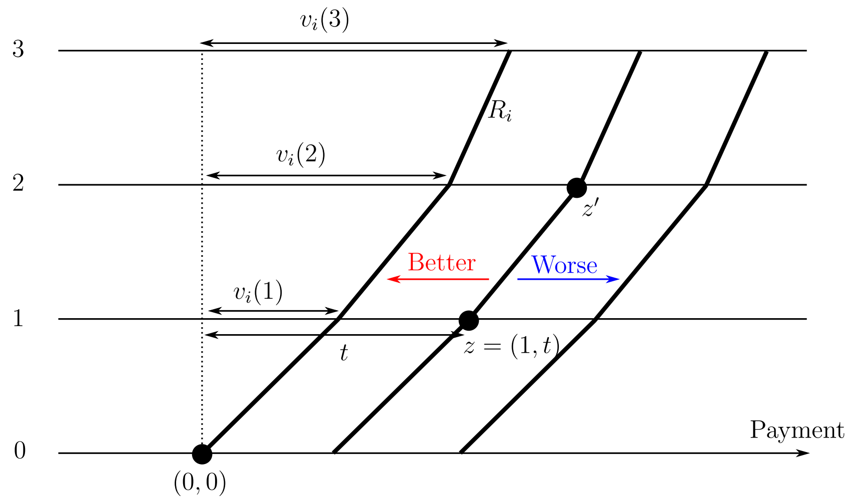

Figure 1: Consumption set and indifference curves.

DEFINITION **2** *A preference relation* R_i *is* **single-peaked** *if there is* $p(R_i) \in \mathbb{Z}_+$ *such that for each pair* $x, y \in \mathbb{Z}_+$ *and each* $t \in \mathbb{R}$, *if* $y < x < p(R_i)$ *or* $y > x > p(R_i)$, $(p(R_i), t)\ P_i\ (x, t)\ R_i\ (y, t)$.

Let $\mathcal{R}^{SP}$ be the class of single-peaked preferences.

Figure 1 is an illustration of the consumption set $\mathbb{Z}_+ \times \mathbb{R}$. In this diagram, each horizontal line represents the set of real numbers, and each point on the lines represents a payment for the amount of the object specified on the left side of the line. The vertical dotted line in this diagram connects the points where the payment is zero. For example, the point z corresponds to the consumption bundle $(1, t)$.

The kinked lines are "indifference curves" of a preference relation R_i. That is, if bundles are on the same indifference curve, the bundles are indifferent for the preference relation. For example, z and z' in Figure 1 are on the same indifference curve, and hence, $z\ I_i\ z'$. Bundles to the left (resp. right) of an indifference curve are better (resp. worse) than the bundles on the indifference curve. Indifference curves of a quasi-linear preference relation are parallel to each other as shown in Figure 1. Thus, we can illustrate a quasi-linear preference relation by drawing just a single indifference curve. The valuation function of a quasi-linear preference relation corresponds to the payment levels at the bundles indifferent to $(0, 0)$.

The preference relation in Figure 1 is weakly object monotonic since the valuation function is nondecreasing. On the other hand, the preference relation in Figure 2 is single-peaked. It has a unique peak, which is 1, and when the payment is fixed, the agent finds an amount $x \in \mathbb{Z}_+$ of the object at least as well as another one $y \in \mathbb{Z}_+$ if x is closer to 1 than y.

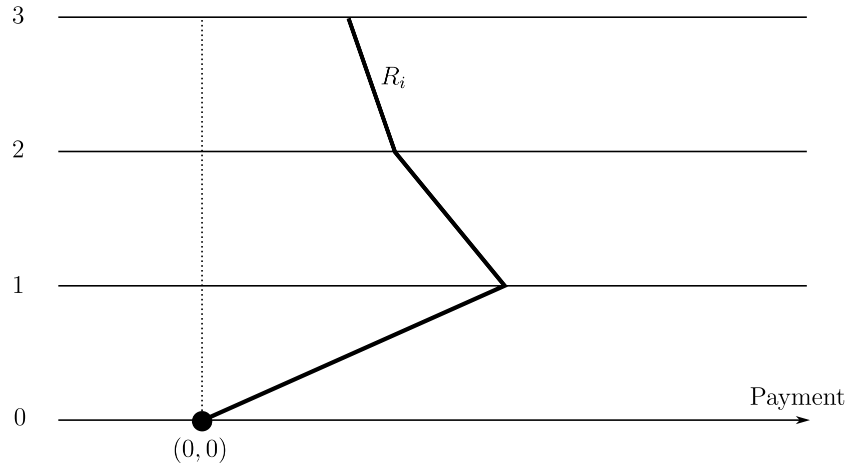

Figure 2: An indifference curve of a single-peaked preference relation.

2.2 Economies, rules, and properties of rules

An **economy** is a tuple $e := (N, m, R) \in \mathcal{N} \times \mathbb{Z}_+ \times \mathcal{R}^{|N|}$, where m is the number of (identical) objects available in the economy and $R := (R_1, \ldots, R_{|N|}) \in \mathcal{R}^{|N|}$ is a preference profile for N. Given $N \in \mathcal{N}$, $R \in \mathcal{R}^{|N|}$, and $N' \subseteq N$, denote $R_{N'} = (R_i)_{i \in N'}$. Let $\mathcal{E}$ be a generic notation for the set of economies and call it a **domain**. We use the notation $\mathcal{R}$ to denote the class of preferences associated with $\mathcal{E}$. Let $\mathcal{E}^{WO}$ be the set of economies such that for each $(N, m, R) \in \mathcal{E}^{WO}$, $R \in (\mathcal{R}^{WO})^{|N|}$, and call it the **weakly object monotonic domain**. Similarly, let $\mathcal{E}^{SP}$ be the set of economies such that for each $(N, m, R) \in \mathcal{E}^{SP}$, $R \in (\mathcal{R}^{SP})^{|N|}$, and call it the **single-peaked domain**.

Given $(N, m) \in \mathcal{N} \times \mathbb{Z}_+$, a **(feasible) allocation** for (N, m) is a tuple $((x_i, t_i))_{i \in N} \in (\mathbb{Z}_+ \times \mathbb{R})^{|N|}$ such that $\sum_{i \in N} x_i \leq m$. Denote the set of allocations for (N, m) by $A(N, m)$.

An **(allocation) rule** is a mapping $f : \mathcal{E} \to \cup_{(N,m) \in \mathcal{N} \times \mathbb{Z}_+} A(N, m)$ such that for each $e := (N, m, R) \in \mathcal{E}$, $f(e) \in A(N, m)$. Given $e := (N, m, R) \in \mathcal{E}$ and $i \in N$, let $f_i(e)$ be the consumption bundle assigned to agent i at e, and we write $f_i(e) = (x_i(e), t_i(e))$ where $x_i(e) \in \{1, \ldots, m\}$ is the number of objects assigned to i and $t_i(e)$ is her payment. Given $e := (N, m, R) \in \mathcal{E}$ and $N' \subseteq N$, let $f_{N'}(e) = (f_i(e))_{i \in N'}$.

Now we introduce properties of rules. The first property requires that the rule should be consistent in the sense that if some agents leave an economy with their assigned bundles, the allocation for the remaining agents remain the same.

Consistency: For each $(N, m, R) \in \mathcal{E}$ and each $N' \subseteq N$,

$$f_{N'}(N, m, R) = f(N', m - \sum_{i \in N \setminus N'} x_i(N, m, R), R_{N'}).$$

The following property requires that each agent should always have an incentive to report her true preference relation.

Strategy-proofness: For each $(N, m, R) \in \mathcal{E}$, each $i \in N$, and each $R'_i \in \mathcal{R}$,

$$f_i(N, m, R) \; R_i \; f_i(N, m, (R'_i, R_{-i})).$$

The following property requires that an agent should not be assigned a bundle that makes her worse off than she would be if she had received no object and paid nothing.

Individual rationality: For each $e := (N, m, R) \in \mathcal{E}$ and each $i \in N$, $f_i(e) \; R_i \; (0, 0)$.

The next property requires that the payment of each agent should be nonnegative.

No subsidy: For each $e := (N, m, R) \in \mathcal{E}$ and each $i \in N$, $t_i(e) \geq 0$.

The following property has a similar spirit as *consistency*. It requires that when the allocation should remain the same even if some of the unassigned objects are removed from the economy.

Independence of unallocated objects: For each $(N, m, R) \in \mathcal{E}$ and each $m' \in \mathbb{Z}_+$ with $m > m' \geq \sum_{i \in N} x_i(N, m, R)$,

$$f(N, m, R) = f(N, m', R).$$

The following property requires that if an agent receives objects, her assigned bundle is not indifferent to receiving no object and paying nothing.

Non-wasteful tie-breaking: For each $e := (N, m, R) \in \mathcal{E}$, there is no agent $i \in N$ such that $x_i(e) \neq 0$ and $f_i(e) \; I_i \; (0, 0)$.

The final property requires that there is always an economy where no object is unallocated.

Minimal tradability: For each $(N, m) \in \mathcal{N} \times \mathbb{Z}_+$, there is $R \in \mathcal{R}^{|N|}$ such that $\sum_{i \in N} x_i(N, m, R) = m$.

3 Sequential dictatorship rules

Given $m, m' \in \mathbb{Z}_+$ with $m \leq m'$, let $[m, m']_{\mathbb{Z}} := \{m, m+1, \ldots, m'\}$. We call it an **(integer) interval**. In particular, for each $m \in \mathbb{Z}_+$, let $[m] := [0, m]_{\mathbb{Z}}$. A **reservation price** is a mapping $r : \mathbb{Z}_+ \to \mathbb{R}_+$ such that $r(0) = 0$. Given a preference relation R_i, a reservation price r, and $m \in \mathbb{Z}_+$, let

$$B(R_i, m, r) := \{x \in [m] : \text{for each } y \in [m],\ (x, r(x))\ R_i\ (y, r(y))\}.$$

A **priority ordering** is a complete, antisymmetric, and transitive binary relation $\succ$ over $\mathbb{N}$. Given a priority ordering $\succ$ and $N \subseteq \mathbb{N}$, let $top(\succ, N) \in N$ be the agent who has the highest priority in N. That is, for each $i \in N$ with $i \neq top(\succ, N)$, $top(\succ, N) \succ i$.

Definition 3 *A rule f on $\mathcal{E}$ is a* **sequential dictatorship** *if there are a profile of reservation prices $(r_i)_{i \in \mathbb{N}}$ and a profile of priority orderings $(\succ^m)_{m \in \mathbb{Z}_+}$ such that for each $e := (N, m, R) \in \mathcal{E}$, $f(e)$ is determined as follows:*

- *Let $i_1 := top(\succ^m, N)$. If $0 \notin B(R_{i_1}, m, r_{i_1})$, then $x_{i_1}(e) \in B(R_{i_1}, m, r_{i_1})$ and $t_{i_1}(e) = r_{i_1}(x_{i_1}(e))$. Otherwise, $f_{i_1}(e) = (0, 0)$.*
- *Let $k \in [2, |N|]_{\mathbb{Z}}$ and denote $i_k = top(\succ^{m - \sum_{k' < k} x_{i_{k'}}(e)}, N \setminus \{i_1, \ldots, i_{k-1}\})$. If $0 \notin B(R_{i_k}, m - \sum_{k' < k} x_{i_{k'}}(e), r_{i_k})$, then $x_{i_k}(e) \in B(R_{i_k}, m - \sum_{k' < k} x_{i_{k'}}(e), r_{i_k})$ and $t_{i_k}(e) = r_{i_k}(x_{i_k}(e))$. Otherwise, $f_{i_k}(e) = (0, 0)$.*

A rule f is a **serial dictatorship** *if it is a sequential dictatorship and for each pair $m, m' \in \mathbb{Z}_+$, $\succ^m = \succ^{m'}$.*

4 Results

Our first result states that if the domain is either the weakly object monotonic or the single-peaked domain, then a rule that satisfies the list of properties we consider must be a sequential dictatorship rule.

Theorem 1 *Let $\mathcal{E} \in \{\mathcal{E}^{WO}, \mathcal{E}^{SP}\}$. Let f be a rule on $\mathcal{E}$ that satisfies consistency, strategy-proofness, individual rationality, no subsidy, independence of unallocated objects, non-wasteful tie-breaking, and minimal tradability. Then f is a sequential dictatorship rule.*

It is easy to see that sequential dictatorship rules satisfy, *individual rationality*, *no subsidy*, *non-wasteful tie-breaking*, and *minimal tradability*. However some sequential dictatorship rules or even some serial dictatorship rules violate *consistency*, *strategy-proofness*, or *independence of unallocated objects*. The following is an example of a serial dictatorship that violates *consistency*, *strategy-proofness*, and *independence of unallocated objects*.

EXAMPLE **1** *Let* $\mathcal{E} = \{\mathcal{E}^{WO}, \mathcal{E}^{SP}\}$.*Let* $r \in \mathbb{R}_{++}$. *For each* $i \in \mathbb{N}$, *let* r_i *be such that for each* $m \in \mathbb{Z}_+$, $r_i(m) = m \cdot r$. *Let* $\delta \in \mathbb{R}_{++}$. *Let* $i, j, k \in \mathbb{N}$ *be distinct agents. Let* $R_i^* \in \mathcal{R}$ *be such that for each* $x \in [4]$ *with* $x > 0$,

$$v_i^*(x) = x \cdot r + \delta.$$

Let $R_j^* := R_i^*$. *Note that for each* $m \in [4]$ *with* $m > 0$, $B(R_i^*, m, r_i) = B(R_j^*, m, r_j) = [1, m]_{\mathbb{Z}}$.

Let $R_k^* \in \mathcal{R}$ *be such that for each* $x \in [4]$,

$$v_k^*(x) = \begin{cases} 0 & \text{if } x \leq 1, \\ x \cdot r + \delta & \text{otherwise.} \end{cases}$$

Note that in both $\mathcal{R}^{WO}$ *and* $\mathcal{R}^{SP}$ *there are preferences that satisfy the condition of* R_i^* *and preferences that satisfy the condition of* R_k^*. *Denote* $R^* := (R_i^*, R_j^*, R_k^*)$.

Let

$$\mathcal{E}^* := \{(N, m, R) : N \subseteq \{i, j, k\}, m \leq 4, R_i = R_i^*, R_j = R_j^*, R_k \in \mathcal{R}\}.$$

Let f *be a serial dictatorship rule such that the associated reservation price vector is* $(r_i)_{i \in \mathbb{N}}$, *the corresponding priority ordering* $\succ$ *satisfies* $i \succ j \succ k$, *and for each* $e := (N, m, R) \in \mathcal{E}^*$,

$$x_i(e) = \begin{cases} 2 & \text{if } e = (\{i, j, k\}, 4, R^*), \\ \min B(R_i, m, r_i) & \text{otherwise,} \end{cases}$$

and

$$x_j(e) = \min B(R_j, m - x_i(e), r_j).$$

By the definition of f, $x_i(\{i, j, k\}, 4, R^*) = 2$, $x_j(\{i, j, k\}, 4, R^*) = 1$, *and thus* $f_k(\{i, j, k\}, 4, R^*) = (0, 0)$. *On the other hand,* $x_i(\{i, j, k\}, 3, R^*) = x_i(\{i, k\}, 3, R_{-j}^*) = 1$. *Hence,* f *violates* consistency *and* independence of unallocated objects.

Let $R_k' \in \mathcal{R}$ *be such that for each* $x \in [4]$,

$$v_k'(x) = \begin{cases} 0 & \text{if } x \leq 1, \\ x \cdot r + 2\delta & \text{otherwise.} \end{cases}$$

By the definition of f, $x_i(\{i, j, k\}, 4, (R_i^*, R_j^*, R_k')) = x_j(\{i, j, k\}, 4, (R_i^*, R_j^*, R_k')) = 1$, *and thus,* $f_k(\{i, j, k\}, 4, (R_i^*, R_j^*, R_k')) = (2, 2r)$. *Thus,* $f_k(\{i, j, k\}, 4, (R_i^*, R_j^*, R_k'))\ P_k^*\ f_k(\{i, j, k\}, 4, R^*)$. *Hence,* f *is not strategy-proof.*

The above example shows that tie-breaking rules play an important role to determine whether a sequential dictatorship rule satisfies *consistency*, *strategy-proofness*, and *independence of unallocated objects*. Indeed, tie-breaking rules are the only factor that makes a sequential dictatorship rule manipulable.[6] Given this observation, in the next section we focus on sequential dictatorship rules with specific tie-breaking rules, and investigate when those rules satisfies the properties we impose.

4.1 Sequential dictatorship with lowest tie-breaking

We focus on sequential dictatorship rules where ties are broken in such a way that each agent is assigned the lowest number of objects among her most preferred numbers of objects. This tie-breaking rule is reasonable when the planner wants to increase the number of agents who receive an object.

Definition 4 *A rule f on $\mathcal{E}$ is a* **sequential dictatorship rule with lowest tie-breaking** *if there are a profile of reservation prices $(r_i)_{i\in\mathbb{N}}$ and a profile of priority orderings $(\succ^m)_{m\in\mathbb{Z}_+}$ such that for each $e := (N, m, R)$, $f(e)$ is determined as follows:*

- *Let $i_1 := top(\succ^m, N)$. Then, $x_{i_1}(e) = \min B(R_{i_1}, m, r_{i_1})$ and $t_{i_1}(e) = r_{i_1}(x_{i_1}(e))$.*
- *Let $k \in [2, |N|]_{\mathbb{Z}}$ and denote $i_k = top(\succ^{m-\sum_{k'<k} x_{i_{k'}}(e)}, N \setminus \{i_1, \dots, i_{k-1}\})$. Then, $x_{i_k}(e) = \min B(R_{i_k}, m - \sum_{k'<k} x_{i_{k'}}(e), r_{i_k})$ and $t_{i_k}(e) = r_{i_k}(x_{i_k}(e))$.*

Note that sequential dictatorship rules with lowest tie-breaking are *strategy-proof*. On the other hand, sequential dictatorship rules with lowest tie-breaking may not satisfy *consistency* and *independence of unallocated objects*. Therefore, we investigate when a sequential dictatorship rules with lowest tie-breaking satisfy the two properties.

Before stating our results in this section, we introduce the notion of acyclicity. Given a priority ordering $\succ$, $N \subseteq \mathbb{N}$, and $i \in N$, let $rank_i(\succ, N)$ be the rank of agent i among N with respect to $\succ$.

Definition 5 *Let $\succ$ and $\succ'$ be priority orderings over $\mathbb{N}$. Then, $\succ$ is an* **acyclic ordering** *of $\succ'$ if for all $i \in \mathbb{N}$, $|rank_i(\succ', \mathbb{N}) - rank_i(\succ, \mathbb{N})| \leq 1$.*

Remark 1 *Let $\succ$ and $\succ'$ be priority orderings over $\mathbb{N}$. Then, $\succ$ is an acyclic ordering of $\succ'$ if and only if there are no $i, j, k \in \mathbb{N}$ such that $i \succ j \succ k$ and $k \succ' i$.*

[6] To be precise, take a sequential dictatorship rule with reservation prices $(r_i)_{i\in\mathbb{N}}$ and an economy $e := (N, m, R)$. If each agent $i \in N$ has strict preferences over the set of bundles that the rule may assign to her, that is, $\{(x, r_i(x)) : x \leq m\}$, then no agent benefits from misreporting her preferences at the economy.

4.1.1 Weakly object monotonic domain

We first introduce some notations. For each pair $i, j \in \mathbb{N}$, let $M_{i,j} := \{m \in \mathbb{Z}_+ : i \succ^m j\}$. Note that for each pair $i, j \in \mathbb{N}$, $M_{i,j}$ can be described as a collection of disjoint intervals. That is, there is a collection $\mathcal{I}_{i,j}$ of intervals such that $M_{i,j} = \cup_{I \in \mathcal{I}_{i,j}} I$ and for each distinct pair $I, I' \in \mathcal{I}_{i,j}$, $I \cap I' = \emptyset$. For each $m \in M_{i,j}$, let $I_{i,j}(m)$ be the interval in $\mathcal{I}_{i,j}$ that contains m. For each $m \in M_{i,j}$ let $\underline{m}_{i,j} := \min I_{i,j}(m)$. Note that if $\underline{m}_{i,j} > 1$, $j \succ^{\underline{m}_{i,j}-1} i$. Finally, let $M^{\min}_{i,j} := \{\underline{m}_{i,j} : m \in M_{i,j}\}$.

To state our result on the weakly object monotonic domain, we introduce the notion of feasible path. Let f be a sequential dictatorship and denote the corresponding reservation prices and priority orderings by $(r_i)_{i \in \mathbb{N}}$ and $(\succ^m)_{m \in \mathbb{Z}_+}$, respectively. Given $m \in \mathbb{Z}_+$ and $K \in [2, |N|]_{\mathbb{Z}}$, a pair $(\{i_k\}_{k=1}^K, \{x_k\}_{k=1}^K)$ of sequences of distinct agents and positive integers is a **feasible path at** m if the following hold:

- For each $k \in [1, K-1]_{\mathbb{Z}}$, $i_k = top(\succ^{m - \sum_{k' \in [1,k-1]_{\mathbb{Z}}} x_{k'}}, \{i_k\}_{k'=k}^K)$.
- For each $k \in [1, K-1]_{\mathbb{Z}}$ and each $x \in [x_k + 1, m - \sum_{k' \in [1,k-1]_{\mathbb{Z}}} x_{k'}]_{\mathbb{Z}}$, $r_{i_k}(x_k) \leq r_{i_k}(x)$.
- $m - \sum_{k \in [1,K]_{\mathbb{Z}}} x_k \geq 0$.

The feasible path guarantees that there is an economy with $\{i_1, \ldots, i_K\}$ and m objects such that each agent i_k receives x_k objects under sequential dictatorship rules with lowest tie-breaking. The first condition states that under sequential dictatorship rules, agents make a decision in the order of $i_1, i_2, \ldots, i_K$ as long as each agent i_k chooses x_k in her tern. The second condition states that for each $k \in [1, K]_{\mathbb{Z}}$, there is indeed a preference relation such that i_k selects x_k in her turn. To see this point, take any $k \in [1, K]_{\mathbb{Z}}$. Let $R_{i_k} \in \mathcal{R}$ be such that for each $x \in \mathbb{Z}_+$,

$$v_{i_k}(x) \begin{cases} = 0 & \text{if } x < x_k, \\ > r_{i_k}(x_k) & \text{if } x = x_k, \\ = v_{i_k}(x_k) & \text{otherwise.} \end{cases}$$

Then, $\min B(R_{i_k}, m - \sum_{k' \in [1,k-1]_{\mathbb{Z}}} x_{k'}, r_{i_k}) = x_k$, and hence, i_k is assigned x_k under sequential dictatorship rules with lowest tie-breaking. Finally, the third condition states that it is feasible that each agent i_k receives x_k when there are m objects in the economy.

The following result provides a necessary and sufficient condition for a sequential dictatorship rule with lowest tie-breaking to satisfy *consistency* and *independence of unallocated objects* when the domain is the weakly object monotonic domain.

Theorem **2** *Let $\mathcal{E} = \mathcal{E}^{WO}$. Let f be a sequential dictatorship with lowest tie-breaking on $\mathcal{E}$, and $(r_i)_{i \in \mathbb{N}}$ and $(\succ^m)_{m \in \mathbb{Z}_+}$ be the corresponding reservation prices and priority orderings, respectively. Then, the following conditions are equivalent.*

- *f satisfies consistency and independence of unallocated objects.*
- *For each $i, j \in \mathbb{N}$ with $i \neq j$, f satisfies the following three conditions.*
  1. *For each $m \in M_{i,j}^{\min}$ and each $x \in [2, m-1]_{\mathbb{Z}}$, $r_i(m) < r_i(x)$.*
  2. *Suppose $j \succ^1 i$. Let $m \in M_{i,j}$. If $r_i(1) \leq \min_{x \in [\underline{m}_{i,j}, m]_{\mathbb{Z}}} r_i(x)$, then there is no $k \in \mathbb{N}$ such that $i \succ^m k$ and $k \succ^{m-1} j$.*
  3. *Suppose $i \succ^1 j$. Let $m \in M_{i,j}$ be such that $\underline{m}_{i,j} > 1$. If $r_i(1) \leq \min_{x \in [\underline{m}_{i,j}, m]_{\mathbb{Z}}} r_i(x)$, then there is no feasible path $(\{i_k\}_{k=1}^K, \{x_k\}_{k=1}^K)$ at m such that*
     - *$(i_1, x_1) = (i, 1)$,*
     - *$i_K = j$,*
     - *for some $L \subseteq [2, K-1]_{\mathbb{Z}}$, $j = top(\succ^{m - \sum_{k \in L} x_k}, \{i_k\}_{k \in [1,K]_{\mathbb{Z}} \setminus L})$.*

Theorem 2 gives a useful tool to verify whether a sequential dictatorship rule with lowest tie-breaking satisfies *consistency* and *independence of unallocated objects*. For example, it is sometimes reasonable to consider reservation prices that are increasing in the number of objects. Theorem 2 provides conditions on priority orderings that are compatible with *consistency* and *independence of unallocated objects* when reservation prices are increasing.

Formally, let f be a sequential dictatorship rule with lowest tie-breaking that satisfies *consistency* and *independence of unallocated objects*, and $(r_i)_{i \in \mathbb{N}}$ and $(\succ^m)_{m \in \mathbb{Z}_+}$ be the corresponding reservation prices and priority orderings, respectively. Assume that for each $i \in \mathbb{N}$, r_i is *increasing*, i.e., for each pair $x, x' \in \mathbb{Z}_+$ with $x > x'$, $r_i(x) > r_i(x')$. Suppose that for some $i, j \in \mathbb{N}$ and $m, m' \in \mathbb{Z}_+$ with $m > m' > 1$, $i \succ^m j$ and $j \succ^{m'} i$. Condition 1 of Theorem 2 implies that $r_i(m') > r_i(\underline{m}_{i,j})$. By $m' < \underline{m}_{i,j}$, this contradicts the fact that r_i is increasing. Hence, Condition 1 implies that for each pair $m, m' \in \mathbb{Z}_+$ with $m > m' > 1$, $\succ^m = \succ^{m'}$.

Next, suppose that for some $i, j \in \mathbb{N}$, $i \succ^2 j$ and $j \succ^1 i$. By Condition 2 of Theorem 2, there is no $k \in \mathbb{N}$ such that $i \succ^2 k \succ^1 j$. That is, $\succ^2$ is an acyclic ordering of $\succ^1$. Therefore, we obtain the following corollary.

Corollary **1** *Let $\mathcal{E} = \mathcal{E}^{WO}$. Let f be a sequential dictatorship with lowest tie-breaking on $\mathcal{E}$, and $(r_i)_{i \in \mathbb{N}}$ and $(\succ^m)_{m \in \mathbb{Z}_+}$ be the corresponding reservation prices and priority orderings, respectively. Assume that for each $i \in \mathbb{N}$, r_i is increasing. Then, the following conditions are equivalent.*

- *f satisfies consistency and independence of unallocated objects.*

- $(\succ^m)_{m\in\mathbb{Z}_+}$ *satisfies the following two conditions.*
  1. *For each* $k, k' \in \mathbb{Z}_+$ *with* $k \geq 2$ *and* $k' \geq 2$, $\succ^k = \succ^{k'}$.
  2. $\succ^1$ *is an acyclic ordering of* $\succ^k$, *where* $k \geq 2$.

4.1.2 Single-peaked domain

When preferences are single-peaked, the condition of Corollary 1 is a necessary and sufficient condition for a sequential dictatorship with lowest tie-breaking to satisfy *consistency* and *independence of unallocated objects*.

THEOREM **3** *Let* $\mathcal{E} = \mathcal{E}^{SP}$. *Let* f *be a sequential dictatorship with lowest tie-breaking on* $\mathcal{E}$, *and* $(r_i)_{i\in\mathbb{N}}$ *and* $(\succ^m)_{m\in\mathbb{Z}_+}$ *be the corresponding reservation prices and priority orderings, respectively. Then, the following conditions are equivalent.*

- f *satisfies consistency and independence of unallocated objects.*
- $(\succ^m)_{m\in\mathbb{Z}_+}$ *satisfies the following two conditions.*
  1. *For each* $k, k' \in \mathbb{Z}_+$ *with* $k \geq 2$ *and* $k' \geq 2$, $\succ^k = \succ^{k'}$.
  2. $\succ^1$ *is an acyclic ordering of* $\succ^k$, *where* $k \geq 2$.

4.2 Independence of axioms

The conclusion of Theorem 1 does not hold if we drop any of the properties, as shown by the following examples. Throughout this section, we assume that for each $m, m' \in \mathbb{Z}_+$, $\succ^m = \succ^{m'} = \succ$, and for each $i \in \mathbb{N}$, $r_i : \mathbb{Z}_+ \to \mathbb{R}_+$ is increasing. Also, for each $i \in \mathbb{N}$, let r_i^0 be such that for each $m \in \mathbb{Z}_+$, $r_i^0(m) = 0$.

EXAMPLE **2 (Consistency)** *Let* f *be such that for each* $e := (N, m, R) \in \mathcal{E}$, *and each* $i \in N$,

- *if* $i = top(\succ, N)$,

$$x_i(e) = \min B(R_i, m, r_i) \text{ and } t_i(e) = r_i(x_i(e)),$$

- *otherwise,* $f_i(e) = (0, 0)$.

This rule satisfies all axioms in Theorem 1 but consistency.

EXAMPLE **3 (Strategy-proofness)** *Let* f *be such that for each* $e := (N, m, R) \in \mathcal{E}$ *and each* $i \in N$,

$$x_i(e) = \min B(R_i, m - \textstyle\sum_{j\succ i} x_j(e), r_i^0) \text{ and } t_i(e) = v_i(x_i(e)).$$

This rule satisfies all axioms in Theorem 1 but strategy-proofness.

EXAMPLE 4 **(Individual rationality)** *Let f be such that for each $e := (N, m, R) \in \mathcal{E}$, and each $i \in N$,*

- *if $i = top(\succ, N)$, $f_i(e) = (m, P)$, where $P \in \mathbb{R}$ be such that $P > 0$,*
- *otherwise, $f_i(e) = (0, 0)$.*

This rule satisfies all axioms in Theorem 1 but individual rationality*.*

EXAMPLE 5 **(No subsidy)** *Let f be such that for each $e := (N, m, R) \in \mathcal{E}$, and each $i \in N$,*

- *if $i = top(\succ, N)$, $f_i(e) = (m, P)$, where $P \in \mathbb{R}$ be such that $P < 0$,*
- *otherwise, $f_i(e) = (0, 0)$.*

This rule satisfies all axioms in Theorem 1 but no subsidy*.*

EXAMPLE 6 **(Independence of unallocated objects)** *Let f be such that for each $e := (N, m, R) \in \mathcal{E}$, and each $i \in N$,*

- *if $\{j \in N : (m, r_j(m)) \, P_j \, (0, 0))\} = \emptyset$, $f_i(e) = (0, 0)$ and*
- *if $\{j \in N : (m, r_j(m)) \, P_j \, (0, 0))\} \neq \emptyset$,*

$$f_i(e) = \begin{cases} (m, r_i(m)) & \text{if } i = top(\succ, \{j \in N : (m, r_j(m)) \, P_j \, (0, 0))\}), \\ (0, 0) & \text{otherwise.} \end{cases}$$

This rule satisfies all axioms in Theorem 1 but independence of unallocated objects*.*

EXAMPLE 7 **(Non-wasteful tie-breaking)** *Let f be such that for each $e := (N, m, R) \in \mathcal{E}$, and each $i \in N$,*

$$x_i(e) = \max B(R_i, m - \textstyle\sum_{j \succ i} x_j(e), r_i) \text{ and } t_i(e) = r_i(x_i(e)).$$

This rule satisfies all axioms in Theorem 1 but non-wasteful tie-breaking*.*

EXAMPLE 8 **(Minimal tradability)** *The no-trade rule satisfies all axioms in Theorem 1 but not* minimal tradability*.*[7]

A Proofs

We introduce a notation for sequential dictatorship rules, which will be used in . We will use this notation in the proofs of Theorems 2 and 3. Let f be a sequential dictatorship and denote the corresponding priority orderings $(\succ^m)_{m \in \mathbb{Z}_+}$. For each $e := (N, m, R) \in \mathcal{E}$, let $i_1(e) = top(\succ^m, N)$, and for each $k \in [2, |N|]_{\mathbb{Z}}$,

$$i_k(e) = top(\succ^{m - \sum_{k' < k} x_{i_{k'}(e)}(e)}, N \setminus \{i_1(e), \ldots, i_{k-1}(e)\}).$$

[7]The no trade rule is a rule such that for each economy, each agent is assigned $(0, 0)$.

A.1 Proof of Theorem 1

The proof consists of six steps.

Step **1** *For each $i \in \mathbb{N}$, there is a reservation price $r_i : \mathbb{Z}_+ \to \mathbb{R}_+$ such that for each $e := (N, m, R) \in \mathcal{E}$ with $i \in N$, $t_i(e) = r_i(x_i(e))$.*

Proof: Let $i \in \mathbb{N}$. We first show the following claim.

Claim **1** *Let $m \in \mathbb{Z}_+$. There is $r_i(m) \in \mathbb{R}_+$ such that for each $R_i \in \mathcal{R}$ with $x_i(\{i\}, m, R_i) = m$, $t_i(\{i\}, m, R_i) = r_i(m)$.*

Proof: By *minimal tradability*, there is $R_i \in \mathcal{R}$ such that $x_i(\{i\}, m, R_i) = m$. Let $r_i(m) := t_i(\{i\}, m, R_i)$. By *no subsidy*, $r_i(m) \geq 0$.

Suppose that there is $R_i' \in \mathcal{R}$ such that $x_i(\{i\}, m, R_i') = m$ and $t_i(\{i\}, m, R_i') \neq r_i(m)$. Without loss of generality, assume $t_i(\{i\}, m, R_i') > r_i(m)$. Then,

$$f_i(\{i\}, m, R_i) = (m, r_i(m))\, P_i'\, (m, t_i(\{i\}, m, R_i')) = f_i(\{i\}, m, R_i'),$$

which contradicts *strategy-proofness*. Hence, $t_i(\{i\}, m, R_i') = r_i(m)$. □

Let $e := (N, m, R) \in \mathcal{E}$ be such that $i \in N$. By *consistency* and *independence of unallocated objects*,

$$f_i(e) = f_i(\{i\}, m - \sum_{j \in N \setminus \{i\}} x_j(e), R_i) = f_i(\{i\}, x_i(e), R_i).$$

Hence, by Claim 1, $t_i(e) = t_i(\{i\}, x_i(e), R_i) = r_i(x_i(e))$. ■

To simplify the notation, for each $i \in \mathbb{N}$, each $m \in \mathbb{Z}_+$, and each $R_i \in \mathcal{R}$, we write $B_i(R_i, m)$ instead of $B(R_i, m, r_i)$ throughout the proof of Theorem 1.

Step **2** *Let $e = (N, m, R) \in \mathcal{E}$ and $i \in N$. Then, $x_i(e) \in B_i(R_i, m - \sum_{j \in N \setminus \{i\}} x_j(e))$.*

Proof: Denote $m' := m - \sum_{j \in N \setminus \{i\}} x_j(e)$. By *consistency*, $x_i(e) = x_i(\{i\}, m', R_i)$. Thus, to complete the proof, it is sufficient to show to show $x_i(\{i\}, m', R_i) \in B_i(R_i, m')$. Let $m^* := \max B_i(R_i, m')$. If $m^* = 0$, then *individual rationality* implies $x_i(\{i\}, m', R_i) \in B_i(R_i, m')$. Thus, assume $m^* > 0$.

Claim **2** *There is $R_i' \in \mathcal{R}$ such that $x_i(\{i\}, m', R_i') = m^*$.*

Proof: If $m^* = m'$, *minimal tradability* guarantees the existence of such a preference relation. Thus, assume $m^* < m'$. The proof depends on the domain we consider.

Case 1. $\mathcal{E} = \mathcal{E}^{WO}$. Since R_i is weakly object monotonic, for each $x \in [m^* + 1, m']_{\mathbb{Z}}$,

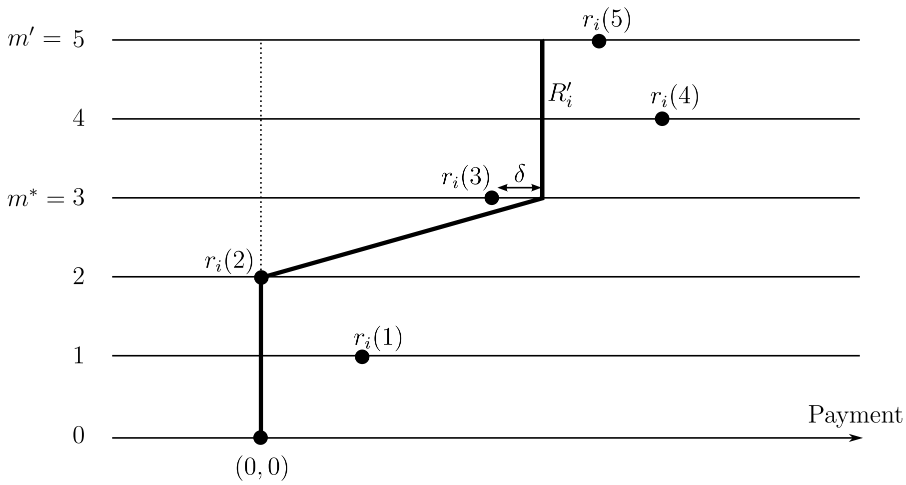

Figure 3: An illustration of R'_i when $m' = 5$, $m^* = 3$, and $\mathcal{R} = \mathcal{R}^{WO}$.

$r_i(m^*) < r_i(x)$. Let $\delta \in \mathbb{R}_{++}$ be such that $\delta < \min_{x \in [m^*+1, m']_{\mathbb{Z}}} r_i(x) - r_i(m^*)$. Let $R'_i \in \mathcal{R}$ be such that for each $x \in \mathbb{Z}_+$,

$$v'_i(x) = \begin{cases} 0 & \text{if } x < m^*, \\ r_i(m^*) + \delta & \text{otherwise.} \end{cases}$$

Figure 3 is an illustration of R'_i for the case where $m' = 5$ and $m^* = 3$. By *individual rationality* and *non-wasteful tie-breaking*, $x_i(\{i\}, m', R'_i) \geq m^*$. Further, for each $x \in [m^*+1, m']_{\mathbb{Z}}$, $v_i(x) - r_i(x) = r_i(m^*) + \delta - r_i(x) < 0$, which implies $(0,0)\ P'_i\ (x, r_i(x))$. Thus, $x_i(\{i\}, m', R'_i) \leq m^*$. Hence, $x_i(\{i\}, m', R'_i) = m^*$.

Case 2. $\mathcal{E} = \mathcal{E}^{SP}$. Let $\delta \in \mathbb{R}_+$ and $R' \in \mathcal{R}$ be such that for each $x \in \mathbb{Z}_+$,

$$v'_i(x) = \begin{cases} r_i(m^*) + \delta & \text{if } x = m^*, \\ 0 & \text{otherwise.} \end{cases}$$

Figure 4 is an illustration of R'_i for the case where $m' = 5$ and $m^* = 3$. Since $r_i(x) \geq 0$ for each $x \in \mathbb{Z}_+$, *individual rationality* and *non-wasteful tie-breaking* imply that $x_i(\{i\}, m', R'_i) = m^*$. □

By Claim 2, there is $R'_i \in \mathcal{R}$ such that $f_i(\{i\}, m', R'_i) = (m^*, r_i(m^*))$. By *strategy-proofness*, $f_i(\{i\}, m', R_i)\ R_i\ f_i(\{i\}, m', R'_i) = (m^*, r_i(m^*))$. Hence, by $m^* \in B_i(R_i, m')$, $x_i(\{i\}, m', R_i) \in B_i(R_i, m')$. ■

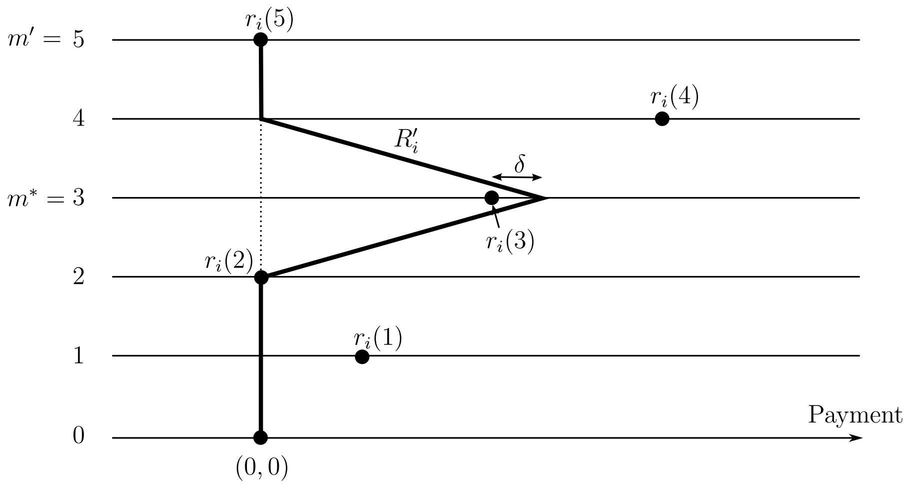

Figure 4: An illustration of R'_i when $m' = 5$, $m^* = 3$, and $\mathcal{R} = \mathcal{R}^{SP}$.

Given $i \in \mathbb{N}$ and $m \in \mathbb{Z}_+$, we introduce the following notations.

$$\mathcal{R}_i(m) := \{R_i \in \mathcal{R} : \text{for each } x \in [1, m-1]_{\mathbb{Z}}, (m, r_i(m))\ P_i\ (0,0)\ R_i\ (x, r_i(x))\}, \text{ and}$$
$$\mathcal{R}_i^+(m) := \{R_i \in \mathcal{R} : (m, r_i(m))\ P_i\ (m-1, r_i(m))\ P_i\ \cdots\ P_i\ (1, r_i(1))\ P_i\ (0,0)\}.$$

Step **3** *Let $i \in \mathbb{N}$ and $m \in \mathbb{Z}_+$. Then, $\mathcal{R}_i(m) \neq \emptyset$ and $\mathcal{R}_i^+(m) \neq \emptyset$.*

Proof: Let $\delta \in \mathbb{R}_{++}$ and $R_i \in \mathcal{R}$ be such that for each $x \in [m]$,

$$v_i(x) = \begin{cases} 0 & \text{if } x < m, \\ r_i(m) + \delta & \text{if } x = m. \end{cases}$$

Note that in both $\mathcal{R}^{WO}$ and $\mathcal{R}^{SP}$, there is a preference relation that satisfies the above condition. Clearly, $R_i \in \mathcal{R}_i(m)$, and thus, $\mathcal{R}_i(m) \neq \emptyset$.

Next, we show $\mathcal{R}_i^+(m) \neq \emptyset$. Let $\delta \in \mathbb{R}_{++}$. We define $R_i \in \mathcal{R}$ as follows: Let $v_i(1) = r_i(1) + \delta$, and for each $x \in [1, m]_{\mathbb{Z}}$, inductively define

$$v_i(x) = v_i(x-1) + \max\{r_i(x) - r_i(x-1), 0\} + \delta.$$

Figure 5 is an illustration of R_i. Note that in both $\mathcal{R}^{WO}$ and $\mathcal{R}^{SP}$, there is a preference relation that satisfies the above condition.

We show $R_i \in \mathcal{R}_i^+(m)$. To show this claim, it is sufficient to show that for each

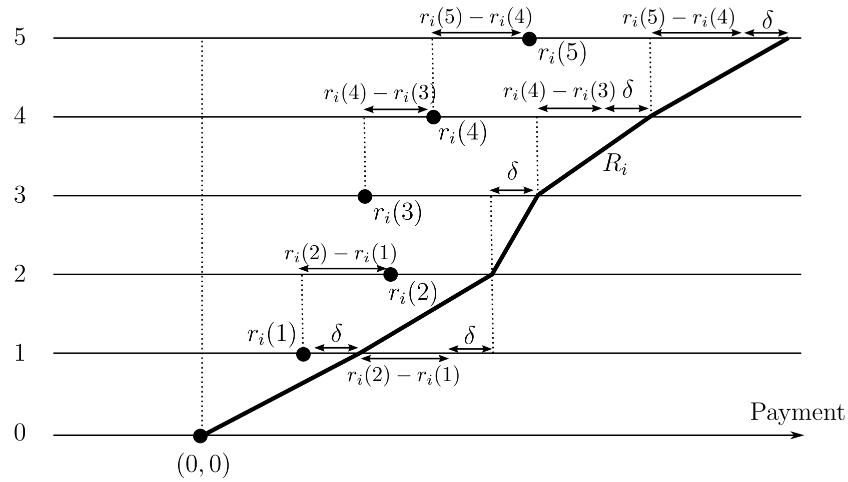

Figure 5: An illustration of R_i in Step 3.

$x \in [1, m]_{\mathbb{Z}}$, $(x, r_i(x))\ P_i\ (x-1, r_i(x-1))$. Let $x \in [1, m]_{\mathbb{Z}}$. Then,

$$\begin{aligned} v_i(x) - r_i(x) &= v_i(x-1) + \max\{r_i(x) - r_i(x-1), 0\} + \delta - r_i(x) \\ &> v_i(x-1) + r_i(x) - r_i(x-1) - r_i(x) \\ &= v_i(x-1) - r_i(x-1), \end{aligned}$$

which implies $(x, r_i(x))\ P_i\ (x-1, r_i(x-1))$. Thus, $R_i \in \mathcal{R}_i^+(m)$, and hence, $\mathcal{R}_i^+(m) \neq \emptyset$. ■

Remark **2** *By individual rationality and non-wasteful tie-breaking, for each $e := (N, m, R)$ and each $i \in N$, if $R_i \in \mathcal{R}_i(m)$, then $x_i(e) \in \{0, m\}$.*

Step **4** *Construction of priority orderings.*

To construct priority orderings, we show the following claims.

Claim **3** *Let $e := (N, m, R) \in \mathcal{E}$ be such that $R \in \prod_{i \in N} \mathcal{R}_i(m)$. There is $i \in N$ such that $x_i(e) = m$.*

Proof: Suppose by contradiction that for each $i \in N$, $x_i(e) \neq m$. By $R \in \prod_{i \in N} \mathcal{R}_i(m)$ and Remark 2, for each $i \in N$, $x_i(N, m, R) = 0$. Take any $i \in N$. By $R_i \in \mathcal{R}_i(m)$, $B_i(R_i, m) = \{m\}$. Thus, by Step 2, $x_i(\{i\}, m, R_i) = m$. Therefore, $x_i(\{i\}, m, R_i) \neq x_i(N, m, R)$, contradicting *consistency*. □

Claim **4** *Let $i, j \in \mathbb{N}$ be such that $i \neq j$, and $m \in \mathbb{Z}_+$. There is $k \in \{i, j\}$ such that for each $R \in \mathcal{R}_i(m) \times \mathcal{R}_j(m)$, $x_k(\{i, j\}, m, R) = m$.*

Proof: Take any $R \in \mathcal{R}_i(m) \times \mathcal{R}_j(m)$. By Claim 3, either $x_i(\{i,j\}, m, R) = m$ or $x_j(\{i,j\}, m, R) = m$. Without loss of generality, assume $x_i(\{i,j\}, m, R) = m$.

Let $R' \in \mathcal{R}_i(m) \times \mathcal{R}_j(m)$. We show $x_i(\{i,j\}, m, R') = m$. By $x_i(\{i,j\}, m, R) = m$, $R'_i \in \mathcal{R}_i(m)$, and *strategy-proofness*, $x_i(\{i,j\}, m, (R'_i, R_j)) = m$. Thus, $x_j(\{i,j\}, m, (R'_i, R_j)) = 0$.

By *strategy-proofness*, $(0,0) = f_j(\{i,j\}, m, (R'_i, R_j))\ R_j\ f_j(\{i,j\}, m, R')$. Thus, by $R'_j \in \mathcal{R}_j(m)$ and Remark 2, $x_j(\{i,j\}, m, R') = 0$. Hence, by Claim 3, $x_i(\{i,j\}, m, R') = m$. □

Given $m \in \mathbb{Z}_+$, let $\succ^m$ be a binary relation over $\mathbb{N}$ such that for each distinct pair $i, j \in \mathbb{N}$,

$$i \succ^m j \Leftrightarrow [\forall R \in \mathcal{R}_i(m) \times \mathcal{R}_j(m), x_i(\{i,j\}, m, R) = m].$$

By Claim 4, this binary relation is complete and antisymmetric.

Finally, we show that for each $m \in \mathbb{Z}_+$, $\succ^m$ is transitive. Let $m \in \mathbb{Z}_+$ and $i, j, k \in \mathbb{N}$ be such that $i \succ^m j$ and $j \succ^m k$. Let $R \in \prod_{\ell \in \{i,j,k\}} \mathcal{R}_\ell(m)$. By Claim 3, there is an agent who receives m objects at $f(\{i,j,k\}, m, R)$.

If $x_j(\{i,j,k\}, m, R) = m$, then by *consistency*, $x_j(\{i,j\}, m, R_{-k}) = m$, contradicting $i \succ^m j$. If $x_k(\{i,j,k\}, m, R) = m$, then by *consistency*, $x_k(\{j,k\}, m, R_{-i}) = m$, contradicting $j \succ^m k$. Thus, $x_i(\{i,j,k\}, m, R) = m$. By *consistency*, $x_i(\{i,k\}, m, R_{-j}) = x_i(\{i,j,k\}, m, R) = m$. Thus, by the definition of $\succ^m$, $i \succ^m k$. Hence, $\succ^m$ is transitive.

Step **5** *Let $(N, m) \in \mathcal{N} \times \mathbb{Z}_+$ and $i \in N$ be such that $i = top(\succ^m, N)$. Let $R \in \prod_{j \in N} \mathcal{R}^+_j(m)$. Then $x_i(N, m, R) = m$.*

Proof: We begin with a claim which states that there is exactly one agent who receives the object.

Claim **5** $|\{j \in N : x_j(N, m, R) \neq 0\}| = 1$.

Proof: First suppose $\{j \in N : x_j(N, m, R) \neq 0\} = \emptyset$. Then, $x_j(N, m, R) = 0$ for each $j \in N$ and thus $B_i(R_i, m - \sum_{j \in N \setminus \{i\}} x_j(N, m, R)) = B_i(R_i, m)$. This and Step 2 imply $x_i(N, m, R) \in B_i(R_i, m)$. By $R_i \in \mathcal{R}^+_i(m)$, $B_i(R_i, m) = \{m\}$. This contradicts $x_i(N, m, R) = 0$.

Now, suppose $|\{j \in N : x_j(N, m, R) \neq 0\}| > 1$. Let $j, k \in \{\ell \in N : x_\ell(N, m, R) \neq 0\}$ and denote $m' := x_j(N, m, R) + x_k(N, m, R)$. Without loss of generality, assume $j \succ^{m'} k$. By *consistency* and *independence of unallocated objects*,

$$x_j(N, m, R) = x_j(\{j,k\}, m - \sum_{\ell \in N \setminus \{j,k\}} x_\ell(N, m, R), (R_j, R_k)) = x_j(\{j,k\}, m', (R_j, R_k)).$$

Thus, by $x_j(N,m,R) < m'$, $x_j(\{j,k\},m',(R_j,R_k)) < m'$. By $R_j \in \mathcal{R}_j^+(m)$,

$$(m', r_j(m'))\ P_j\ f_j(\{j,k\},m',(R_j,R_k)).$$

Let $R' \in \mathcal{R}_j(m') \times \mathcal{R}_k(m')$. By Remark 2, $f_j(\{j,k\},m',(R'_j,R_k)) \in \{(0,0),(m',r_j(m'))\}$. Thus, by $(m',r_j(m'))\ P_j\ f_j(\{j,k\},m',(R_j,R_k))$ and *strategy-proofness*, $x_j(\{j,k\},m',(R'_j,R_k)) = 0$. By Step 2 and $R_k \in \mathcal{R}_k^+(m')$, $x_k(\{j,k\},m',(R'_j,R_k)) \in B_k(R_k,m') = \{m'\}$. Therefore, by *strategy-proofness* and $R'_k \in \mathcal{R}_k(m')$, $f_k(\{j,k\},m',R') = (m',r_k(m'))$. However, this contradicts $j \succ^{m'} k$. □

Let $j \in \{k \in N : x_k(N,m,R) \neq 0\}$. We show that $j = i$. Suppose by contradiction that $j \neq i$. By Step 2 and $R_j \in \mathcal{R}_j^+(m)$, $x_j(N,m,R) \in B_j(R_j,m) = \{m\}$, and so $x_j(N,m,R) = m$. Thus, by *consistency*, $x_j(\{i,j\},m,R_{\{i,j\}}) = m$. However, this contradicts $i \succ^m j$. ■

STEP **6** *Let $(N,m) \in \mathcal{N} \times \mathbb{Z}_+$ and $i \in N$ be such that $i = top(\succ^m, N)$. Let $R \in \mathcal{R}^{|N|}$. Then, $x_i(N,m,R) \in B_i(R_i,m)$.*

Proof: Let $R' \in \prod_{j\in N} \mathcal{R}_j^+(m)$. By Step 5 and $i = top(\succ^m, N)$, $x_i(N,m,R') = m$.

CLAIM **6** $x_i(N,m,(R'_i,R_{-i})) = m$.

Proof: We show that for each $N' \subseteq N \setminus \{i\}$, if $x_i(N,m,(R_{N'\setminus\{j\}},R'_{-N'\setminus\{j\}})) = m$ for each $j \in N'$, then $x_i(N,m,(R_{N'},R'_{-N'})) = m$. Note that by $x_i(N,m,R') = m$, this completes the proof.

Let $N' \subseteq N \setminus \{i\}$. Assume that for each $j \in N'$, $x_i(N,m,(R_{N'\setminus\{j\}},R'_{-N'\setminus\{j\}})) = m$. For each $j \in N'$, by *strategy-proofness*,

$$(0,0) = f_j(N,m,(R_{N'\setminus\{j\}},R'_{-N'\setminus\{j\}}))\, R'_j\, f_j(N,m,(R_{N'},R'_{-N'})).$$

Thus, by *individual rationality* and *non-wasteful tie-breaking*, for each $j \in N'$,

$$x_j(N,m,(R_{N'},R'_{-N'})) = 0.$$

By *consistency*, $x_i(N,m,(R_{N'},R'_{-N'})) = x_i(N\backslash N',m,R'_{-N'})$. By $R'_{-N'} \in \prod_{j\in N\backslash N'} \mathcal{R}_j^+(m)$, $i = top(\succ^m, N\backslash N')$, and Step 5, $x_i(N\backslash N',m,R'_{-N'}) = m$. Hence, $x_i(N,m,(R_{N'},R'_{-N'})) = m$. □

Let $m^* := \max B_i(R_i,m)$. Suppose $m^* = m$. By *strategy-proofness* and Claim 6, $f_i(N,m,R)\, R_i\, f_i(N,m,(R'_i,R_{-i})) = (m,r_i(m))$. Hence $x_i(N,m,R) \in B_i(R_i,m)$. Next, suppose $m^* = 0$. Then, *individual rationality* implies $x_i(N,m,R) \in B_i(R_i,m)$.

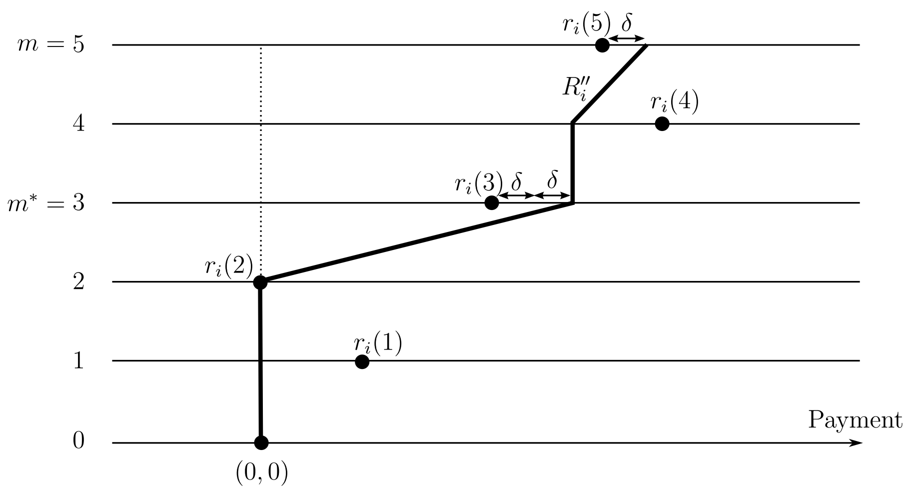

Figure 6: An illustration of R'_i in Step 6 when $m = 5$, $m^* = 3$, and $\mathcal{R} = \mathcal{R}^{WO}$.

Finally, suppose $0 < m^* < m$. There are two cases.

Case 1. $\mathcal{R} = \mathcal{R}^{WO}$. By $m^* = \max B_i(R_i, m)$, for each $x \in [m^*+1, m]_{\mathbb{Z}}$, $r_i(m^*) < r_i(x)$. Take $\delta \in \mathbb{R}_{++}$ such that $2\delta < \min_{x \in [m^*+1,m]_{\mathbb{Z}}} r_i(x) - r_i(m^*)$. Let $R''_i \in \mathcal{R}$ be such that for each $x \in \mathbb{Z}_+$,

$$v''_i(x) = \begin{cases} 0 & \text{if } x < m^* \\ r_i(m^*) + 2\delta & \text{if } m^* \leq x < m, \\ r_i(m) + \delta & \text{otherwise.} \end{cases}$$

Figure 6 is an illustration of R_i for the case where $m = 5$ and $m^* = 3$. It is clear that R''_i is weakly object monotonic. Note that by $2\delta < \min_{x \in [m^*+1,m]_{\mathbb{Z}}} r_i(x) - r_i(m^*)$, $\{x \in [m] : (x, r_i(x)) \ P''_i \ (0, 0)\} = \{m^*, m\}$. By Claim 6 and *strategy-proofness*, $f_i(N, m, (R''_i, R_{-i})) \ R''_i \ f_i(N, m, (R'_i, R_{-i})) = (m, r_i(m))$. Thus, $x_i(N, m, (R''_i, R_{-i})) \in \{m^*, m\}$.

Suppose $x_i(N, m, (R''_i, R_{-i})) = m$. Then for each $j \in N \setminus \{i\}$, $x_j(N, m, (R''_i, R_{-i})) = 0$. Thus, by Step 2, $m = x_i(N, m, (R''_i, R_{-i})) \in B_i(R''_i, m)$, However, $v''_i(m^*) - r_i(m^*) = 2\delta > \delta + r_i(m) - r_i(m) = v''_i(m) - r_i(m)$. This implies $m \notin B_i(R''_i, m)$, a contradiction. Thus, $x_i(N, m, (R''_i, R_{-i})) = m^*$.

By *strategy-proofness*, $f_i(N, m, R) \, R_i \, f_i(N, m, (R''_i, R_{-i})) = (m^*, r_i(m^*))$. Hence, by $m^* \in B_i(R_i, m)$, $x_i(N, m, R) \in B_i(R_i, m)$.

Case 2. $\mathcal{R} = \mathcal{R}^{SP}$. Let $v^* := \max_{x \in [m^*,m]_{\mathbb{Z}}} r_i(x)$ and $\delta \in \mathbb{R}_{++}$. Let $R''_i \in \mathcal{R}$ be such

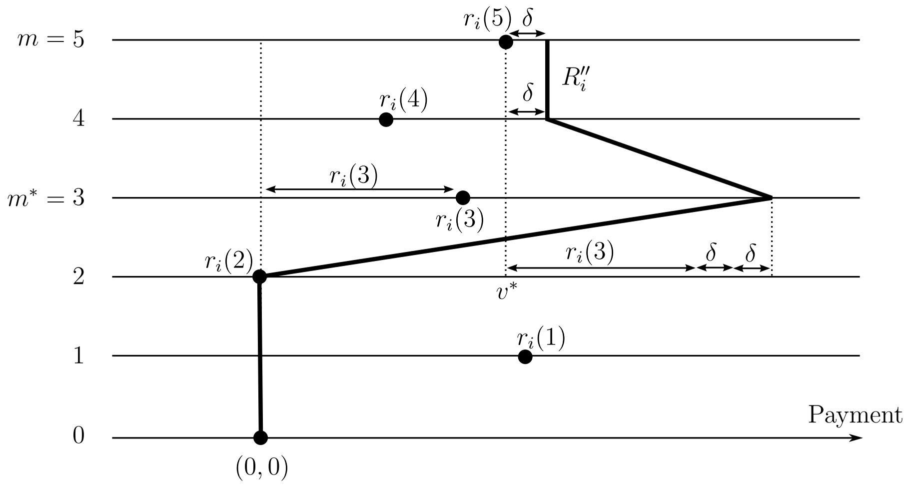

Figure 7: An illustration of R'_i in Step 6 when $m=5$, $m^*=3$, and $\mathcal{R}=\mathcal{R}^{SP}$.

that for each $x \in \mathbb{Z}_+$,

$$v''_i(x) = \begin{cases} 0 & \text{if } x < m^*, \\ v^* + r_i(m^*) + 2\delta & \text{if } x = m^*, \\ v^* + \delta & \text{if } x > m^*. \end{cases}$$

Figure 7 is an illustration of R_i for the case where $m=5$ and $m^*=3$. Note that R_i is single-peaked. Note also that $\{x \in [m] : (x, r_i(x)) \; P''_i \; (0,0)\} = [m^*, m]_{\mathbb{Z}}$. By Claim 6 and *strategy-proofness*, $f_i(N, m, (R''_i, R_{-i})) \; R''_i \; f_i(N, m, (R'_i, R_{-i})) = (m, r_i(m))$. Thus, $x_i(N, m, (R''_i, R_{-i})) \in [m^*, m]_{\mathbb{Z}}$.

Suppose that there is $x \in [m^*+1, m]_{\mathbb{Z}}$ such that $x_i(N, m, (R''_i, R_{-i})) = x$. By *consistency* and *independence of unallocated objects*, $x_i(\{i\}, x, R''_i) = x_i(N, m, (R''_i, R_{-i})) = x$. By Step 2, $x = x_i(\{i\}, x, R''_i) \in B_i(R''_i, x)$. However, by the definition of R''_i, $v_i(m^*) - r_i(m^*) = v^* + 2\delta > v_i(x) - r_i(x)$, a contradiction. Hence, $x_i(N, m, (R''_i, R_{-i})) = m^*$.

By *strategy-proofness*, $f_i(N, m, R) \, R_i \, f_i(N, m, (R''_i, R_{-i})) = (m^*, r_i(m^*))$. Hence, by $m^* \in B_i(R_i, m)$, $x_i(N, m, R) \in B_i(R_i, m)$. ■

Step 7 *Completing the proof.*

Let $e := (N, m, R) \in \mathcal{E}$. Let $i_1 := top(\succ^m, N)$. By Step 6, $x_{i_1}(e) \in B_{i_1}(R_{i_1}, m)$. Next, let $i_2 := top(\succ^{m - x_{i_1}(e)}, N \setminus \{i_1\})$. By *consistency*, $x_{i_2}(e) = x_{i_2}(N \setminus \{i_1\}, m - x_{i_1}(e), R_{-i_1})$. By Step 6, $x_{i_2}(N \setminus \{i_1\}, m - x_{i_1}(e), R_{-i_1}) \in B_{i_2}(R_{i_2}, m - x_{i_1}(e))$. Thus, $x_{i_2}(e) \in B_{i_2}(R_{i_2}, m - x_{i_1}(e))$. By continuing this procedure, we obtain the desired result. ■

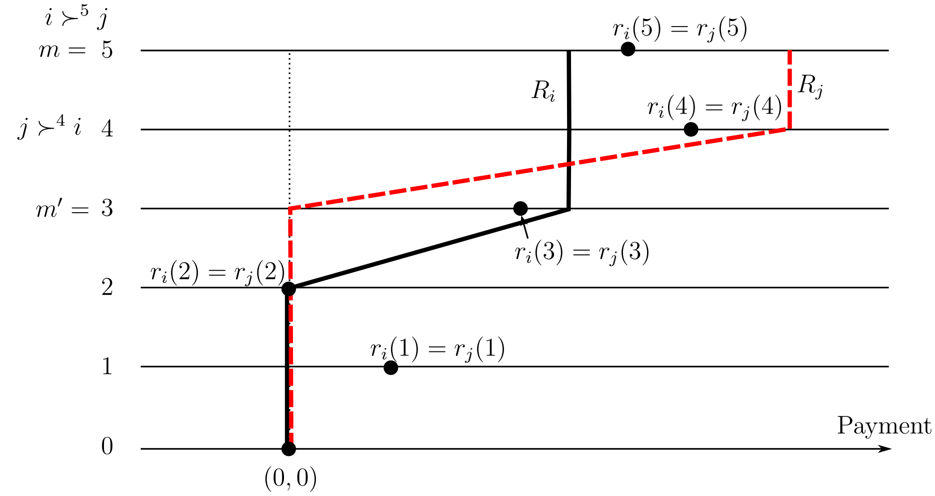

Figure 8: An illustration of R_i and R_j in the proof of Condition 1 when $m = 5$, $m' = 3$, and $r_i = r_j$.

A.2 Proof of Theorem 2

Our proof consists of two parts. In Part I, we show that if f satisfies *consistency* and independence of unallocated objects, then it satisfies Conditions 1, 2, and 3. In Part II, we show the other direction.

Part I. Suppose f satisfies *consistency* and *independence of unallocated objects*. Take any pair of agents $i, j \in \mathbb{N}$. Let $m \in \mathbb{Z}_+$.

Condition 1: Suppose by contradiction that there exist $m \in M_{i,j}^{\min}$ and $m' \in [2, m-1]_{\mathbb{Z}}$ such that $r_i(m) \geq r_i(m')$. Without loss of generality, we can assume that for each $m'' \in [m'+1, m-1]_{\mathbb{Z}}$, $r_i(m'') \geq r_i(m)$.

Let $R_i \in \mathcal{R}$ be such that for each $x \in \mathbb{Z}_+$,

$$v_i(x) \begin{cases} = 0 & \text{if } x < m', \\ > r_i(m') & \text{if } x = m', \\ = v_i(m') & \text{otherwise.} \end{cases}$$

Let $R_j \in \mathcal{R}$ be such that for each $x \in \mathbb{Z}_+$,

$$v_j(x) \begin{cases} = 0 & \text{if } x < m-1, \\ > r_j(m-1) & \text{otherwise.} \end{cases}$$

Denote $R := (R_i, R_j)$ and $e := (\{i, j\}, m, R)$. Figure 9 illustrates R_i and R_j for the case where $m = 5$, $m' = 3$, and $r_i = r_j$.

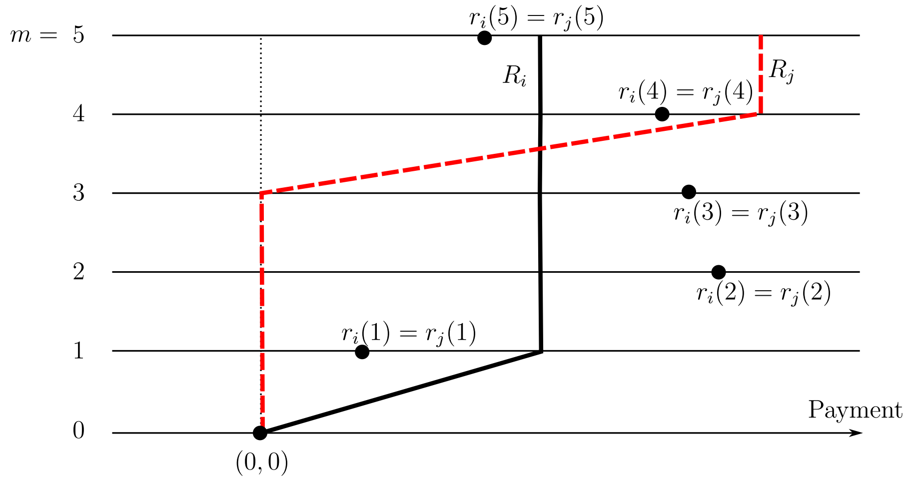

Figure 9: An illustration of R_i and R_k in the proof of Condition 2 when $m = 5$ and $r_i = r_k$.

For each $x \in [m'-1]$, $r_i(x) \geq 0$, and thus, $(m', r_i(m'))\ P_i\ (x, 0)\ R_i\ (x, r_i(x))$. Thus, by $r_i(m') \leq \min_{x \in [m'+1, m]_{\mathbb{Z}}} r_i(x)$, $m' = \min B(R_i, m, r_i)$. Therefore, by $i \succ^m j$ and the definition of f, $x_i(e) = \min B(R_i, m, r_i) = m'$. By $m' > 1$, $m - m' < m' - 1$. Thus, by the definition of f and the definition of R_j, $x_j(e) = \min B(R_j, m - m', r_j) = 0$.

Let $e' := (\{i, j\}, m-1, R)$. By $m' < m$ and $x_i(e)+x_j(e) = m'$, $x_i(e)+x_j(e) \leq m-1$. Thus, by *independence of unallocated objects*, $x_j(e') = x_j(e) = 0$.

However, by $m \in M_{i,j}^{\min}$, $j \succ^{m-1} i$. Thus, $x_j(e') = \min B(R_j, m-1, r_j)$. By the definition of R_j, $\min B(R_j, m-1, r_j) = m - 1 \neq 0$, a contradiction.

Condition 2: Suppose by contradiction that $r_i(1) \leq \min_{x \in [\underline{m}_{i,j}, m]_{\mathbb{Z}}} r_i(x)$ and there is $k \in \mathbb{N}$ such that $i \succ^m k$ and $k \succ^{m-1} j$.[8] Since f satisfies Condition 1, we can assume that for each $x \in [2, \underline{m}_{i,j} - 1]_{\mathbb{Z}}$, $r_i(x) > r_i(\underline{m}_{i,j}) \geq r_i(1)$.

Let $R_i \in \mathcal{R}$ be such that for each $x \in \mathbb{Z}_+$,

$$v_i(x) \begin{cases} > r_i(1) & \text{if } x = 1, \\ = v_i(1) & \text{otherwise.} \end{cases}$$

Let $R_j \in \mathcal{R}$ be such that $v_j(1) > r_j(1)$. Let $R_k \in \mathcal{R}$ be such that for each $x \in \mathbb{Z}_+$,

$$v_k(x) \begin{cases} = 0 & \text{if } x < m - 1, \\ > r_k(m-1) & \text{if } x = m - 1, \\ = v_k(m-1) & \text{otherwise.} \end{cases}$$

[8]By $m \in M_{i,j}$ and $j \succ^1 i$, $m \geq \underline{m}_{i,j} > 1$.

Let $R := (R_i, R_j, R_k)$ and $e := (\{i,j,k\}, m, R)$. By the definition of f,

$$x_i(e) = 1,\ x_k(e) = m-1, \text{ and } x_j(e) = 0.$$

Let $e' := (\{i,j\}, 1, (R_i, R_j))$. By $j \succ^1 i$ and $v_j(1) > r_j(1)$, $x_j(e') = 1$. However, by $m - x_k(e) = 1$ and $x_j(e) = 0$, this contradicts *consistency*.

Condition 3: Suppose by contradiction that $r_i(1) \le \min_{x \in [\underline{m}_{i,j}, m]_{\mathbb{Z}}} r_i(x)$ and there is a feasible path $(\{i_k\}_{k=1}^K, \{x_k\}_{k=1}^K)$ at m such that

- $(i_1, x_1) = (i, 1)$,
- $i_K = j$,
- for some $L \subseteq \{i_2, \ldots, i_{K-1}\}$, $j = \mathit{top}(\succ^{m-\sum_{i_k \in L} x_k}, \{i_0, i_1, \ldots, i_K\} \setminus L)$.

Since f satisfies Condition 1, we can assume that for each $x \in [2, \underline{m}_{i,j} - 1]_{\mathbb{Z}}$, $r_i(x) > r_i(\underline{m}_{i,j}) \ge r_i(1)$.

Let $R_i \in \mathcal{R}$ be such that for each $x \in \mathbb{Z}_+$,

$$v_i(x) \begin{cases} > r_i(1) & \text{if } x = 1, \\ = v_i(1) & \text{otherwise.} \end{cases}$$

For each $k \in [2, K-1]_{\mathbb{Z}}$, let $R_{i_k} \in \mathcal{R}$ be such that for each $x \in \mathbb{Z}_+$,

$$v_{i_k}(x) \begin{cases} = 0 & \text{if } x < x_k, \\ > r_{i_k}(x_k) & \text{if } x = x_k, \\ = v_{i_k}(x_k) & \text{otherwise.} \end{cases}$$

Let $R_j \in \mathcal{R}$ be such that $v_j(1) = r_j(1) + \delta$ and for each $x \in \mathbb{Z}_+$ with $x > 1$,

$$v_j(x) = v_j(x-1) + \max\{r_j(x) - r_j(x-1), 0\} + \delta.$$

Note that for each $x \in \mathbb{Z}_+$, $B(R_j, x, r_j) = \{x\}$.

Let $N := \{i_1, \ldots, i_K\}$, $R := (R_{i_1}, \ldots, R_{i_K})$, and $e := (N, m, R)$. By the definitions of f and feasible path, $x_i(e) = 1$, for each $k \in [2, K-1]_{\mathbb{Z}}$, $x_{i_k}(e) = x_k$, and $x_j(e) = m - 1 - \sum_{k \in [2,K-1]_{\mathbb{Z}}} x_k$.

Let $e' := (N \setminus L, m - \sum_{i_k \in L} x_k, R_{N \setminus L})$. By $j = \mathit{top}(N \setminus L, m - \sum_{i_k \in L} x_k)$ and the definition of R_j,

$$x_j(e') = \min B(R_j, m - \sum_{i_k \in L} x_k, r_j) = m - \sum_{i_k \in L} x_k > m - 1 - \sum_{k \in [2,K-1]_{\mathbb{Z}}} x_k = x_j(e).$$

This contradicts *consistency*.

Part II. Suppose f satisfies Conditions 1, 2, and 3. We show that f satisfies *consistency* and *independence of unallocated objects*.

Claim 7 *Let* $e := (N, m, R) \in \mathcal{E}$*,* $N' \subseteq N$*, and* $m' \in [m]$ *be such that* $\sum_{i \in N'} x_i(e) \leq m'$*. Denote* $e' = (N', m', R_{N'})$*. Let* $i \in N'$*. Denote* $j := i_1(e')$*. If* $m_i(e) > m_j(e)$ *and* $m_i(e) \geq m'$*, then* $x_i(e) \leq 1$*.*

Proof: Suppose by contradiction that $m_i(e) > m_j(e)$, $m_i(e) \geq m'$, and $x_i(e) > 1$. Let $m^* = m_i(e)$. By $m^* > m_j(e)$, $i \succ^{m^*} j$. By $j = i_1(e')$, $j \succ^{m'} i$. Thus, by $m^* \geq m'$, $m' < \underline{m^*}_{i,j}$. By $\sum_{k \in N'} x_k(e) \leq m'$, $x_i(e) \leq m'$. Therefore,

$$1 < x_i(e) < \underline{m^*}_{i,j}.$$

By Condition 1, $r_i(x_i(e)) > r_i(\underline{m^*}_{i,j})$. This implies that $(\underline{m^*}_{i,j}, r_i(\underline{m^*}_{i,j}))\ P_i\ (x_i(e), r_i(x_i(e)))$. Thus, by $m^* \geq \underline{m^*}_{i,j}$, $x_i(e) \notin B(R_i, m^*, r_i)$, a contradiction. □

Consistency. Suppose by contradiction that there are $e := (N, m, R) \in \mathcal{E}$ and $N' \subseteq N$ such that $f_{N'}(e) \neq f(N', m - \sum_{i \in N \setminus N'} x_i(e), R_{N'})$. Let $m' := m - \sum_{i \in N \setminus N'} x_i(e)$ and $e' := (N', m - \sum_{i \in N \setminus N'} x_i(e), R_{N'})$. We assume $m' < m$ without loss of generality. For each $k \in [1, |N|]_{\mathbb{Z}}$, denote $i_k := i_k(e)$. Similarly, for each $k \in [1, |N'|]_{\mathbb{Z}}$, denote $j_k := i_k(e')$.

Step 1 *There are* $e^* := (N^*, m^*, R^*)$ *and* $N^{**} \subseteq N^*$ *such that*

(a) $i_1(e^*) \neq i_1(e^{**})$,

(b) $x_{i_1(e^{**})}(e^*) \neq x_{i_1(e^{**})}(e^{**})$,

(c) $i_1(e^*) \in N^{**}$ *and* $x_{i_1(e^*)}(e^*) \neq 0$,

where $e^{**} = (N^{**}, m^* - \sum_{i \in N^* \setminus N^{**}} x_i(e^*), R^*_{N^{**}})$.

Proof: By $f_{N'}(e) \neq f(e')$, there is $K' \in [1, |N'|]_{\mathbb{Z}}$ such that $f_{j_{K'}}(e) \neq f_{j_{K'}}(e')$ and for each $k \in [1, K' - 1]_{\mathbb{Z}}$, $f_{j_k}(e) = f_{j_k}(e')$. Let

$$\hat{N}' := \{j_1, \ldots, j_{K'-1}\}.$$

Let $K \in [1, |N|]_{\mathbb{Z}}$ be such that $i_K \in N' \setminus \hat{N}'$, $x_{i_K}(e) \neq 0$, and for each $k \in [1, K-1]_{\mathbb{Z}}$, $i_k \notin N' \setminus \hat{N}'$ or $x_{i_k}(e) = 0$. Let

$$\hat{N} := \{i_1, \ldots, i_{K-1}\}.$$

Let $e^* := (N^*, m^*, R^*) \in \mathcal{E}$ be such that

$$N^* = N \setminus \hat{N},\ m^* = m - \sum_{i \in \hat{N}} x_i(e),\ \text{and } R^* := R_{N^*}.$$

Let $N^{**} := N' \setminus (\hat{N} \cup \hat{N}')$ and $e^{**} := (N^{**}, m^* - \sum_{i \in N^* \setminus N^{**}} x_i(e^*), R^*_{N^{**}})$. It is clear that $N^{**} \subseteq N^*$. Note that $m^* = m - \sum_{i \in \hat{N}} x_i(e) = m - \sum_{k \in [1, K-1]} x_{i_k}(e) = m_{i_K}(e)$.

Claim 8 $m^* - \sum_{i \in N^* \setminus N^{**}} x_i(e^*) = m_{j_{K'}}(e')$

Proof: By $N^{**} \subseteq N^*$ and the definitions of N^* and N^{**},

$$\begin{aligned}
\sum_{i \in N^* \setminus N^{**}} x_i(e^*) &= \sum_{i \in N^*} x_i(e) - \sum_{i \in N^{**}} x_i(e) \\
&= \sum_{i \in N \setminus \hat{N}} x_i(e) - \sum_{i \in N' \setminus (\hat{N} \cup \hat{N}')} x_i(e) \\
&= \sum_{i \in N} x_i(e) - \sum_{i \in \hat{N}} x_i(e) - (\sum_{i \in N'} x_i(e) - \sum_{i \in \hat{N}'} x_i(e) - \sum_{i \in (N' \setminus \hat{N}') \cap \hat{N}} x_i(e)) \\
&= \sum_{i \in N \setminus N'} x_i(e) - \sum_{i \in \hat{N}} x_i(e) + \sum_{i \in \hat{N}'} x_i(e) + \sum_{i \in (N' \setminus \hat{N}') \cap \hat{N}} x_i(e) \\
&= \sum_{i \in N \setminus N'} x_i(e) - \sum_{i \in \hat{N}} x_i(e) + \sum_{i \in \hat{N}'} x_i(e),
\end{aligned}$$

where the last equality follows since for each $i \in (N' \setminus \hat{N}') \cap \hat{N}$, $x_i(e) = 0$. Therefore, by $m' = m - \sum_{i \in N \setminus N'} x_i(e)$ and $m^* = m - \sum_{i \in \hat{N}} x_i(e)$,

$$\begin{aligned}
m^* - \sum_{i \in N^* \setminus N^{**}} x_i(e^*) &= m - \sum_{i \in \hat{N}} x_i(e) - (\sum_{i \in N \setminus N'} x_i(e) - \sum_{i \in \hat{N}} x_i(e) + \sum_{i \in \hat{N}'} x_i(e)) \\
&= m' - \sum_{i \in \hat{N}'} x_i(e) \\
&= m' - \sum_{i \in \hat{N}'} x_i(e') \\
&= m_{j_{K'}}(e'),
\end{aligned}$$

where the third equality follows since for each $i \in \hat{N}'$, $x_i(e) = x_i(e')$. □

Claim 9 *Let $K'' \in [1, |N|]_{\mathbb{Z}}$ be such that $i_{K''} = j_{K'}$. Then, $K'' > K$*

Proof: Suppose by contradiction that $K'' \leq K$. Then,

$$m_{i_{K''}}(e) = m - \sum_{k \in [1, K''-1]_{\mathbb{Z}}} x_{i_k}(e) \geq m - \sum_{k \in [1, K-1]_{\mathbb{Z}}} x_{i_k}(e) = m_{i_K}(e) = m^*.$$

Thus, by Claim 8,

$$m_{j_{K'}}(e) = m_{i_{K''}}(e) \geq m^* - \sum_{i \in N^* \setminus N^{**}} x_i(e^*) = m_{j_{K'}}(e').$$

Note that $x_{j_{K'}}(e) \leq m_{i_{k''}}(e')$. Therefore, by $m_{j_{K'}}(e) \geq m_{j_{K'}}(e')$ and $x_{j_{K'}}(e) = \min B(R_{j_{K'}}, m_{j_{K'}}(e), r_{j_{K'}})$, $x_{j_{K'}}(e) = \min B(R_{j_{K'}}, m_{j_{K'}}(e'), r_{j_{K'}})$. By the definition of f, this implies $x_{j_{K'}}(e) = x_{j_{K'}}(e')$. However, this contradicts the definition of $j_{K'}$. □

It is clear from the definition of i_K that $i_K \in N^*$. By $N^* = N \setminus \hat{N} = \{i_k : k \in [K, |N|]_{\mathbb{Z}}\}$ and the definition of i_K, for each $i \in N^*$, $i_K \succ^{m_{i_K}(e)} i$. Thus,

$$i_1(e^*) = i_K.$$

By the definition of $j_{K'}$, $j_{K'} \in N' \setminus \hat{N}'$. By Claim 9, $j_{K'} \notin \hat{N}$. Thus, $j_{K'} \in N^{**}$. By $N^{**} = N' \setminus (\hat{N} \cup \hat{N}') \subseteq \{j_k : k \in [K', |N'|]_{\mathbb{Z}}\}$ and the definition of $j_{K'}$, for each $i \in N^{**}$, $j_{K'} \succ^{m_{j_{K'}}(e')} i$. Thus, by Claim 8,

$$i_1(e^{**}) = j_{K'}.$$

Now we show that Conditions (a), (b), and (c) are satisfied. First, by Claim 9,

$$i_1(e^*) = i_K \neq i_{K''} = j_{K'} = i_1(e^{**}).$$

Note that by $N^* = \{i_k; k \in [K, |N|]_{\mathbb{Z}}\}$ and $m^* = m_{i_K}(e)$, for each $i \in N^*$, $x_i(e^*) = x_i(e)$. Note also that by $m_{i_1(e^{**})}(e^{**}) = m_{j_{K'}}(e')$, $f_{i_1(e^{**})}(e^{**}) = f_{j_{K'}}(e')$. Thus, by the definition of $j_{K'}$,

$$x_{i_1(e^{**})}(e^{**}) = x_{j_{K'}}(e') \neq x_{j_{K'}}(e) = x_{j_{K'}}(e^*) = x_{i_1(e^{**})}(e^*).$$

Finally, by the definition of i_K, $x_{i_1(e^*)}(e^*) = x_{i_K}(e^*) = x_{i_K}(e) \neq 0$. Further, by the definition of i_K, $i_K \in N' \setminus \hat{N}'$ and $i_K \notin \hat{N}$. Hence, $i_1(e^*) = i_K \in N^{**}$. □

By Step 1, without loss of generality, we can assume that

- $i_1 \neq j_1$.
- $x_{j_1}(e) \neq x_{j_1}(e')$,
- $i_1 \in N'$ and $x_{i_1}(e) \neq 0$,

We also assume without loss of generality that for each $i \in N$, $x_i(e) \neq 0$. Denote $i = i_1$ and $j = j_1$. Note that by $m > m'$ and Claim 7, $x_i(e) = 1$.

STEP **2** $m' > m_j(e)$.

Proof: Suppose by contradiction that $m_j(e) \geq m'$. By the definition of f, $x_j(e) = \min B(R_j, m_j(e), r_j)$. Then, by $x_j(e) \leq m' \leq m_j(e)$, this implies $x_j(e) = \min B(R_j, m', r_j)$. Thus, by the definition of f, $x_j(e') = x_j(e)$, contradicting the definition of j. □

STEP **3** $j \neq i_2(e)$.

Proof: Suppose by contradiction that $j = i_2(e)$. By $i = i_1(e)$ and $x_i(e) = 1$, $m_j(e) = m - 1$. However, by Step 2, $m' > m_j(e)$. This implies $m' \geq m$, a contradiction. □

STEP **4** $i \succ^1 j$.

Proof: By $i = i_1$, $i \succ^m j$. By $j = j_1$ and $i \in N'$, $j \succ^{m'} i$. Thus, by $m > m'$, $\underline{m}_{i,j} > m'$. Note that if there is $x \in [\underline{m}_{i,j}, m]_{\mathbb{Z}}$ such that $r_i(x) < r_i(1)$, then $(x, r_i(x))\ P_i\ (1, r_i(1))$, contradicting the fact that $1 = x_i(e) \in B(R_i, m, r_i)$. Therefore,

$$r_i(1) \leq \min_{x \in [\underline{m}_{i,j}, m]_{\mathbb{Z}}} r_i(x).$$

By Step 3 and $i = i_1$, $i_2 \notin \{i, j\}$. By $i = i_1$, $i \succ^m i_2$. Further, by $x_i(e) = 1$, $m_{i_2}(e) = m - 1$. Thus, $i_2 \succ^{m-1} j$. Hence, by Condition 2, $i \succ^1 j$. □

Let $K \in [1, |N|]_{\mathbb{Z}}$ be such that $j = i_K$. By Step 3, $K > 2$. Let

$$N^* := \{i_1, \ldots, i_K\}.$$

We assume, without loss of generality, that for each $k \in [2, K-1]_{\mathbb{Z}}$, $x_{i_k}(e) \neq 0$.[9] By Step 2, there is $K' \in [1, K-1]_{\mathbb{Z}}$ such that $m_{i_{K'}}(e) \geq m'$ and $m_{i_{K'+1}}(e) < m'$. Let

$$N^{**} := \{i_1, \cdots i_{K'}\}.$$

STEP **5** *The pair* $(\{i_k\}_{k=1}^K, \{x_{i_k}(e)\}_{k=1}^K)$ *is a feasible path at* m.

Proof: Let $k \in [1, K-1]_{\mathbb{Z}}$. Note that $m - \sum_{k' \in [1,k-1]_{\mathbb{Z}}} x_{i_{k'}}(e) = m_k(e)$. Thus, for each $k' \in [k+1, K]$, $i_k \succ^{m - \sum_{k'' \in [1,k-1]_{\mathbb{Z}}} x_{i_{k''}}(e)} i_{k'}$.

Let $x \in [x_{i_k}(e) + 1, m - \sum_{k' \in [1,k-1]_{\mathbb{Z}}} x_{i_{k'}}(e)]_{\mathbb{Z}}$. Suppose by contradiction that $r_{i_k}(x_{i_k}(e)) > r_{i_k}(x)$. Then, $(x, r_{i_k}(x))\ P_{i_k}\ (x_{i_k}(e), r_{i_k}(x_{i_k}(e)))$. Since $m_{i_k}(e) = m - \sum_{k' \in [1,k-1]_{\mathbb{Z}}} x_{i_{k'}}(e)$, $x \leq m_{i_k}(e)$. Thus, $x_{i_k}(e) \notin B(R_{i_k}, m_{i_k}(e), R_{i_k})$, a contradiction.

Finally, it is clear that $m - \sum_{k \in [1,K]_{\mathbb{Z}}} x_{i_k}(e) \geq 0$. Hence, $(\{i_k\}_{k=1}^K, \{x_{i_k}(e)\}_{k=1}^K)$ is a feasible path at m. □

STEP **6** *Completing the proof.*

[9] If $\{i_k : k \in [2, K-1]_{\mathbb{Z}}$ and $x_{i_k}(e) = 0\} \neq \emptyset$, let $N^* := \{i_1, \ldots, i_K\} \setminus \{i_k : k \in [2, K-1]_{\mathbb{Z}}$ and $x_{i_k}(e) = 0\}$. Then, the rest of the proof works.

We derive a violation of Condition 3. Note that

$$\begin{aligned}
\sum_{k\in N^{**}\cap N'} x_k(e) &= \sum_{k\in N^{**}\cap N'} x_k(e) + \sum_{k\in N^{**}\setminus N'} x_k(e) - \sum_{k\in N^{**}\setminus N'} x_k(e) \\
&= m - m_{i_{K'+1}}(e) - \sum_{k\in N^{**}\setminus N'} x_k(e) \\
&> m - m' - \sum_{k\in N^{*}\setminus N'} x_k(e) \\
&= \sum_{k\in N\setminus N'} x_k(e) - \sum_{k\in N^{*}\setminus N'} x_k(e) \\
&= \sum_{k\in N^{*}\setminus N'} x_k(e) + \sum_{k\in N\setminus(N^{*}\cup N')} x_k(e) - \sum_{k\in N^{*}\setminus N'} x_k(e) \\
&= \sum_{k\in N^{*}\setminus N'} x_k(e),
\end{aligned}$$

where the inequality follows from $m_{i_{K'+1}}(e) < m'$ and $N^{**} \subseteq N^*$, and the third equality follows from $m' = m - \sum_{k\in N\setminus N'} x_k(e)$. By Claim 7, for each $k \in N^{**} \cap N'$, $x_k(e) \leq 1$. Therefore, there is $\hat{N} \subseteq N^{**} \cap N'$ such that $i \notin \hat{N}$ and

$$\sum_{k\in\hat{N}} x_k(e) = \sum_{k\in N\setminus(N'\cup N^*)} x_k(e).$$

Let $L := \hat{N} \cup (N^* \setminus N')$. Note that $\hat{N} \cap (N^* \setminus N') = \emptyset$. Then,

$$m - \sum_{k\in L} x_k(e) = m - (\sum_{k\in\hat{N}} x_k(e) + \sum_{k\in N^*\setminus N'} x_k(e)) = m - (\sum_{k\in N\setminus(N'\cup N^*)} x_k(e) + \sum_{k\in N^*\setminus N'} x_k(e)) = m'.$$

Further, note that for each $k \in N^* \setminus L$, $k \in N'$. Therefore, by $j = j_1$, $j = top(\succ^{m'}, N^* \setminus L)$. This contradicts Condition 3.

Independence of unallocated objects. Suppose by contradiction that there are $e := (N, m, R) \in \mathcal{E}$ and $m' \in \mathbb{Z}_+$ such that $\sum_{i\in N} x_i(e) \leq m' < m$ and $f(e) \neq f(N, m', R)$. Denote $e' := (N, m', R)$. For each $k \in [1, |N|]_{\mathbb{Z}}$, denote $i_k := i_k(e)$ and $i'_k := i_k(e')$, respectively.

By $f(e) \neq f(e')$, there is $K \in [1, |N|]_{\mathbb{Z}}$ such that $x_{i'_K}(e') \neq x_{i'_K}(e)$ and for each $k \in [1, K-1]_{\mathbb{Z}}$, $x_{i'_k}(e') = x_{i_k}(e)$. Let $N^* := \{i'_1, \ldots, i'_{K-1}\}$. Since f satisfies *consistency* as we have shown,

$$\begin{aligned}
&f(N \setminus N^*, m - \sum_{i\in N^*} x_i(e), R_{N\setminus N^*}) = f_{N\setminus N^*}(e), \text{ and} \\
&f(N \setminus N^*, m' - \sum_{i\in N^*} x_i(e'), R_{N\setminus N^*}) = f_{N\setminus N^*}(e').
\end{aligned}$$

Note that $i_1(N \setminus N^*, m' - \sum_{i \in N^*} x_i(e'), R_{N \setminus N^*}) = i'_K$. Thus, without loss of generality, we can assume that

$$f_{i'_1}(e) \neq f_{i'_1}(e').$$

Claim **10** $m_{i'_1}(e) < m'$.

Proof: Suppose by contradiction that $m_{i'_1}(e) \geq m'$. By $x_{i'_1}(e) \leq m'$ and $x_{i'_1}(e) = \min B(R_{i'_1}, m_{i'_1}(e), r_{i'_1})$, $x_{i'_1}(e) = \min B(R_{i'_1}, m', r_{i'_1})$. By the definition of f, this implies $x_{i'_1}(e') = x_{i'_1}(e)$, a contradiction. □

Let $K' \in [1, |N|]_{\mathbb{Z}}$ be such that $i'_1 = i_{K'}$. By Claim 10 and $m > m'$, $K' > 1$. By $m' > m_{i'_1}(e)$ and Claim 7, there is $k \in [1, K'-1]_{\mathbb{Z}}$ such that $m_{i_k}(e) = m'$. This implies that $i_k \succ^{m'} i'_1$. However, this contradicts $i'_1 = top(N, \succ^{m'})$. ■

A.3 Proof of Theorem 3

Our proof consists of two parts. In Part I, we show that if f satisfies *consistency* and *independence of unallocated objects*, then it satisfies Conditions 1 and 2. In Part II, we show the other direction.

Part I. Suppose that f satisfies *consistency* and *independence of unallocated objects*.

Condition 1. To show Condition 1, it is sufficient to show that for each $m \in \mathbb{Z}_+$ with $m > 1$, and each pair $i, j \in \mathbb{N}$, if $i \succ^m j$, then $i \succ^{m+1} j$. Suppose by contradiction that there are $m \in \mathbb{Z}_+$ and $i, j \in \mathbb{N}$ such that $m > 1$, $i \succ^m j$ and $j \succ^{m+1} i$.

Let $R_i \in \mathcal{R}$ be such that for each $x \in \mathbb{Z}_+$,

$$v_i(x) \begin{cases} > \max\{r_i(m), r_j(m)\} & \text{if } x = m, \\ = 0 & \text{otherwise.} \end{cases}$$

Let $R_j := R_i$ and $R = (R_i, R_j)$. Note that for each $k \in \{i, j\}$, $B(R_k, m+1, r_k) = B(R_k, m, r_k) = \{m\}$. Let $e := (\{i, j\}, m+1, R)$ and $e' := (\{i, j\}, m, R)$.

By $j \succ^{m+1} i$ and the definition of f, $x_j(e) = m$ and $x_i(e) = 0$. Then, by *independence of unallocated objects*, $x_i(e') = x_i(e) = 0$. However, by $i \succ^m j$ and the definition of f, $x_i(e') = m$, a contradiction.

Condition 2. By Condition 1, it is sufficient to show that $\succ^2$ is an acyclic ordering of $\succ^1$. Suppose by contradiction that there are $i, j, k \in \mathbb{N}$ such that $i \succ^2 j \succ^2 k$ and $k \succ^1 i$.

Let $R_i \in \mathcal{R}$ be such that for each $x \in \mathbb{Z}_+$,

$$v_i(x) \begin{cases} > \max\{r_i(1), r_k(1)\} & \text{if } x = 1, \\ = 0 & \text{otherwise.} \end{cases}$$

Let $R_j \in \mathcal{R}$ be such that for each $x \in \mathbb{Z}_+$,

$$v_j(x) = \begin{cases} > r_j(2) & \text{if } x = 2, \\ = 0 & \text{otherwise.} \end{cases}$$

Let $R_k := R_i$. Note that for each $\ell \in \{i, k\}$ and each $m \in \mathbb{Z}_+ \setminus \{0\}$, $B(R_\ell, m, r_\ell) = \{1\}$. Note also that $B(R_j, 2, r_j) = \{2\}$. Let $R := (R_i, R_j, R_k)$. Let $e := (\{i, j, k\}, 3, R)$ and $e' := (\{i, k\}, 1, R_{-j})$.

By $\succ^3 = \succ^2$ and the definition of f, $x_i(e) = 1$, $x_j(e) = 2$, and $x_k(e) = 0$. By *consistency*, $x_k(e') = x_k(e) = 0$. However, by $k \succ^1 i$ and $B(R_k, m, r_k) = \{1\}$, $x_k(e') = 1$, a contradiction.

Part II. Suppose f satisfies Conditions 1 and 2. To prove that f satisfies *consistency* and *independence of unallocated objects*, it is sufficient to show the following: For each $e := (N, m, R) \in \mathcal{E}$, each $N' \subseteq N$, and each $m' \in \mathbb{Z}_+$ with $\sum_{i \in N'} x_i(e) \leq m' \leq m$, $f_{N'}(e) = f(N', m', R_{N'})$.

Let $e := (N, m, R) \in \mathcal{E}$, $N' \subseteq N$, and $m' \in \mathbb{Z}_+$ be such that $\sum_{i \in N'} x_i(e) \leq m' \leq m$. Denote $e' := (N', m', R_{N'})$. For simplicity, for each $k \in [1, |N|]_{\mathbb{Z}}$, denote $i_k := i_k(e)$. Also, for each $k \in [1, |N'|]_{\mathbb{Z}}$, denote $j_k := i_k(e')$.

Suppose by contradiction that $f_{N'}(e) \neq f(e')$. Then, there $K \in [1, |N'|]_{\mathbb{Z}}$ such that $x_{j_K}(e') \neq x_{j_K}(e)$ and for each $k \in [1, K-1]_{\mathbb{Z}}$, $x_{j_k}(e') = x_{j_k}(e)$. Note that

$$\sum_{k \in [K, |N'|]_{\mathbb{Z}}} x_{j_k}(e) \leq m' - \sum_{k \in [1, K-1]_{\mathbb{Z}}} x_{j_k}(e) = m' - \sum_{k \in [1, K-1]_{\mathbb{Z}}} x_{j_k}(e') = m_{j_K}(e').$$

Claim **11** $m_{i_K(e')}(e') > 1$.

Proof: Suppose by contradiction that $m_{j_{K(e')}}(e') \leq 1$. If $m_{j_{K(e')}}(e') = 0$, then $x_{j_K}(e) = x_{j_K}(e') = 0$, a contradiction. Thus, $m_{j_{K(e')}}(e') = 1$. Note that $x_{j_K}(e) \leq m_{j_K}(e') = 1$. There are two cases.

Case 1. $x_{j_K}(e') = 0$. By $m_{j_K}(e')$ and the definition of f, $(0, 0)\ R_{j_K}\ (1, r_{j_K}(1))$. By this and $x_{j_K}(e) \leq 1$, $x_{j_K}(e) = 0 = x_{j_K}(e')$, a contradiction.

Case 2. $x_{j_K}(e') = 1$. By $x_{j_K}(e') \neq x_{j_K}(e)$, $x_{j_K}(e) = 0$. Also by $x_{j_K}(e') = 1$, $(1, r_{j_K}(1))\ P_{j_K}\ (0,0)$. Thus, by $x_{j_K}(e) = 0$, $m_{j_K}(e) = 0$.

Note that

$$m - \sum_{j \in N \setminus N'} x_j(e) - \sum_{k \in [1,K-1]_{\mathbb{Z}}} x_{j_k}(e) \geq \sum_{j \in N'} x_j(e) - \sum_{k \in [1,K-1]_{\mathbb{Z}}} x_{j_k}(e) = \sum_{k \in [K,|N'|]_{\mathbb{Z}}} x_{j_k}(e) = 1,$$

where the inequality follows from $\sum_{j \in N} x_j(e) \leq m$. Thus, by $m_{j_K}(e) = 0$, there is $j \in \{j_k : k \in [K, |N'|]_{\mathbb{Z}}\}$ such that $x_j(e) \geq 1$. By $x_j(e) \leq m_{j_K}(e') = 1$, $x_j(e) = 1$.

By $j \in \{j_k : k \in [K, |N'|]_{\mathbb{Z}}\}$ and $m_{j_K}(e') = 1$, $j_K \succ^1 j$. Further, by $x_j(e) = 1$, $m_j(e) \geq 1$. If $m_j(e) = 1$, then by $m_{j_K}(e) = 0$, $j \succ^1 j_K$, a contradiction. Thus, $m_j(e) > 1$.

By $m_j(e) > 1$ and Condition 1, $j \succ^2 j_K$. By $m_j(e) > 1$ and $m_{j_K}(e) = 0$, there is $k \in N \setminus \{j_K, j\}$ such that $m_j(e) > m_k(e) > m_{j_K}(e)$.

If $m_k(e) = 1$, then $k \succ^1 j_K \succ^1 j$ and $j \succ^2 j_K$, contradicting the fact that $\succ^2$ is an acyclic ordering of $\succ^1$. If $m_k(e) > 1$, then $j \succ^2 k \succ^2 j_K$ and $j_K \succ^1 j$, contradicting the fact that $\succ^2$ is an acyclic ordering of $\succ^1$. □

By Claim 11, $m_{j_K}(e') \geq 2$. We show $m_{j_K}(e) \geq m_{j_K}(e')$. Suppose by contradiction that $m_{j_K}(e) < m_{j_K}(e')$. Note that by $m - \sum_{i \in N \setminus N'} x_i(e) - \sum_{k \in [1,K-1]_{\mathbb{Z}}} x_{j_k}(e) \geq m_{j_K}(e')$, there is $k \in [K+1, |N'|]_{\mathbb{Z}}$ such that $m_{j_k}(e) > m_{j_K}(e)$. By Condition 1, this implies $j_k \succ^2 j_K$. However, by $m_{j_K}(e') \geq 2$, $K < k$, and Condition 1, $j_K \succ^2 j_k$, a contradiction. Hence, $m_{j_K}(e) \geq m_{j_K}(e')$.

Since $m_{j_K}(e) \geq m_{j_K}(e')$, $B(R_{j_K}, m_{j_K}(e'), r_{j_K}) \subseteq B(R_{j_K}, m_{j_K}(e), r_{j_K})$. Thus, by $x_{j_K}(e) = \min B(R_{j_K}, m_{j_K}(e), r_{j_K})$ and $x_{j_K}(e) \leq m_{j_K}(e')$, $x_{j_K}(e) = \min B(R_{j_K}, m_{j_K}(e'), r_{j_K})$. Hence, by the definition of f, $x_i(e') = x_i(e)$, a contradiction. ■

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