Kyoto University,
Graduate School of Economics
Discussion Paper Series

Efficient and (or) fair allocations under market-clearing constraints

Wataru Ishida, Yusuke Iwase, and Taro Kumano

Discussion Paper No. E-24-004

Graduate School of Economics
Kyoto University
Yoshida-Hommachi, Sakyo-ku
Kyoto City, 606-8501, Japan

September, 2024

Efficient and (or) fair allocations under market-clearing constraints*

Wataru Ishida† Yusuke Iwase‡ Taro Kumano§

September 18, 2024

Abstract

Matching markets often encounter admissibility issues due to social concerns and regulations that must be respected. A key situation that has not been thoroughly analyzed in the literature involves the market-clearing requirement, which ensures balance in allocations across multiple matching markets, similar to supply-and-demand dynamics. To address these admissibility issues, we introduce the concept of an admissible set for such problems. We propose two solutions. The first solution is the "fairness-guaranteed stable solution." We identify a requirement on admissible sets that is necessary and sufficient for the non-emptiness of this solution. This requirement ensures that an allocation where no agent is assigned any resources is admissible. We then conduct welfare analysis and comparative statics of this solution. The second solution is called "efficiency-guaranteed stability," which focuses on maximizing efficiency within the constraints of the admissible set. We show that only specific admissible sets allow this solution to be non-empty.

Keyword: Efficiency, Fairness, Matching

*We are grateful to William Thomson for his helpful support and encouragement. We would like to thank Yu Awaya, Keisuke Bando, Kenzo Imamura, Yuichiro Kamada, Onur Kesten, Morimitsu Kurino, James Schummer, Ryo Shirakawa, and seminar and conference participants at the 13th Conference on Economic Design, CTW Summer Camp 2023, RISS joint with KUAS workshop and Keio University for discussion. Ishida is financially supported by KAKENHI Grant Number 24K22605. Iwase is financially supported by KAKENHI Grant Number 22K20162 and 24K16331. Kumano is financially supported from JSPS KAKENHI grant number 20K01558.

†Department of Economics, University of Rochester. Email: wishida@ur.rochester.edu

‡Department of Economics, Kyoto University. Email: iwase.yusuke.4w@kyoto-u.ac.jp

§Department of Economics, Yokohama National University Email: kumano-taro-sp@ynu.ac.jp

1 Introduction

Agents and institutions, which we will call "entities," are usually assigned to each other according to rules based on the preferences of these entities. "Traditional" many-to-one matching theory assumes that each institution has its own individual maximal capacity. For the list of individual capacities, an "admissible set" of allocations can be derived. If there are no other restrictions, we obtain a traditional admissible set.[1]

However, in real-life matching markets, entities often face additional constraints beyond individual capacities. Examples include affirmative action policies or shared capacities that span multiple institutions. The literature has largely treated these issues in isolation, analyzing each constraint separately without considering how they may interact in practice.

In fact, many markets seek to address multiple admissibility issues simultaneously. For instance, an institution may need to meet budget constraints, balance capacity constraints across multiple institutions, and adhere to affirmative action policies. This is further complicated by market complementarity, where one market's admissibility requirements influence, or are influenced by, those of another market. In such cases, an allocation in one market can have ripple effects across others, making the matching process more complex.

Particularly, issues on admissible sets caused by complementarity in multiple markets are governed by balance requirements, which resemble supply-demand dynamics. Institutions often act as intermediaries across multiple independent matching markets. They aim to match agents in one market with those in another, but these matches are interdependent. Admissibility of an allocation depends not only on the characteristics of the entities in a single market but also on the availability of compatible entities in connected markets. Without coordination, there is no guarantee that these allocations will align effectively.

Consider, for example, a nursery school that hires teachers and enrolls children. These processes are typically conducted separately, yet the school must balance the number and specializations of teachers with the needs and number of children to meet budget constraints and legal requirements regarding student-teacher ratios. Similarly, refugee assignments to camps must account for each camp's physical supplies and medical personnel. Balanced assignments prevent overcrowding and related issues, such as supply shortages and health risks. The following example illustrates in more detail how such bal-

[1]For example, some studies on "matching problems with bilateral contracts" (Hatfield and Milgrom, 2005) do not explicitly define the capacity of an institution. In these models, the capacity of an institution is implicitly captured by its choice function, which determines the admissible assignments of the institution.

ance requirements can emerge and why they are essential for efficient market outcomes.

Example 1. *(Nursery schools hiring teachers and enrolling children.)*
Consider the following situation. There are three schools s_1, s_2, and s_3; two teachers t_1 and t_2; and four children c_1, c_2, c_3, and c_4. Each school s has preferences over pairs consisting of a set of teachers and a set of children; and due to its capacity constraint (e.g., size of buildings) it can accommodate at most four children and two teachers. Suppose that the preferences of each school s are "separable." This means that its preferences can be described in terms of a pair consisting of a preference relation over sets of teachers T, P_s^{T}; and a preference relation over sets of children C, P_s^{C}. Each teacher t has preferences P_t over individual schools and being not assigned any school denoted by s_0; and each guardian of child c (for simplicity, just say each child) has preferences P_c over individual schools and s_0.

In the existing literature, this children-teacher-school assignment problem is decomposed into two distinct problems: a teacher-school assignment problem and a children-school assignment problem.

$P^T_{s_1}$	$P^T_{s_2}$	$P^T_{s_3}$	P_{t_1}	P_{t_2}
t_1	t_2	t_1	s_1	s_2
t_2	t_1	t_2	s_2	s_1
				s_3

(Teacher-school assignment problem)

$P^C_{s_1}$	$P^C_{s_2}$	$P^C_{s_3}$	P_{c_1}	P_{c_2}	P_{c_3}	P_{c_4}
c_1	c_1	c_1	s_2	s_2	s_3	s_3
c_2	c_2	c_2	s_3	s_3	s_2	s_2
c_3	c_3	c_3	s_1	s_1		
c_4	c_4	c_4				

(Children-school assignment problem)

However, those two problems are related in several ways: to satisfy legal and budget-balance requirements. Suppose that the following admissibility requrements have to be met:

- There are at most two children per teacher.
- Each nursery school pays each teacher a salary of $3,000.
- Each nursery school earns $2,000 per child for tuition.
- Budget deficit is not allowed.

To obtain an allocation for the two markets, as a starting point, consider the following naive process. First, decide a "tentative" allocation for the teacher-school assignment problem. Remember that for each school, there is a maximal capacity for hiring teachers, namely 2. For this profile of capacities, a traditional admissible set can be derived. Suppose that we apply, for this problem, the teacher-proposing deferred acceptance algorithm. This gives the following stable allocation for this problem:

$$\mu^T = \begin{pmatrix} s_1 & s_2 & s_3 & s_0 \\ t_1 & t_2 & \emptyset & \emptyset \end{pmatrix}.$$

Next, from the allocation for the teacher-school problem and the admissibility requirements, we can deduce the maximal number of children that each school can admit. In this case, the capacities of both schools s_1 and s_2 are 2, and that of school s_3 is 0.[2] We now have a traditional admissible set. Again, suppose that we apply, for this problem, the children-proposing deferred acceptance algorithm. This gives the following stable allocation for this problem:

$$\mu^C = \begin{pmatrix} s_1 & s_2 & s_3 & s_0 \\ \emptyset & c_1, c_2 & \emptyset & c_3, c_4 \end{pmatrix}.$$

By composing the allocations for the two problems, we obtain the following allocation for the entire problem:

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 & s_0 \\ t_1 & t_2, c_1, c_2 & \emptyset & c_3, c_4 \end{pmatrix}.$$

However, at this allocation, given the number of assigned children, s_1 is assigned too many teachers to meet its budget. Thus, this allocation is not admissible. The issue is that, when hiring teachers, a nursery school does not know how many children it will enroll. Therefore, we have to think of a way of making the allocation admissible. One simple approach to help school s_1 meet its budget is by preventing it from confirming the hiring of teacher t_1. Consequently, applying the above sequential process would give the

[2]For example, consider school s_2. Since it hires one teacher at μ^T, by the first requirements above, it can enrol at most two children. Further, to meet the budget balance, it indeed needs to enrol exactly two children. Since here we just naively let the capacity be the maximal number of children that each school can admit, it is 2.

following admissible allocation as final outcome:

$$\mu' = \begin{pmatrix} s_1 & s_2 & s_3 & s_0 \\ \emptyset & t_2, c_1, c_2 & \emptyset & t_1, c_3, c_4 \end{pmatrix}.$$

Consider the following allocation instead:

$$\nu = \begin{pmatrix} s_1 & s_2 & s_3 & s_0 \\ \emptyset & t_1, t_2, c_1, c_2, c_3, c_4 & \emptyset & \emptyset \end{pmatrix}.$$

Allocation ν respects the admissibility requirements. Obviously, each entity finds their assignment at ν at least as desirable as their assignment at μ'; at least one of them (t_1, c_3, and c_4) prefers their assignment at ν.

The difficulty we just described arises from a fundamental economic principle: maintaining balance between supply and demand. On one hand, the supply of available seats in a school is determined by the number of teachers hired, which is influenced by teachers' preferences. On the other hand, the demand for seats is given by children's preferences. These two are independently given from one another. In the procedure previously described, though, the entire allocation is determined in a way that adjusts the demand to fit the allocation obtained for the supply-side. It led to inefficiency or instability in the entire allocation. Similarly, even if we reverse the order in which the markets are cleared, the final allocation may still fail to meet admissibility requirements, efficiency, or stability.[3] To address this issue, we integrate the two markets and clear them simultaneously, akin to a market-clearing mechanism.

Our example here, which integrates multiple markets into a single market, is somewhat related to a previous work by Ostrovsky (2008). However, due to the complex way that admissibility requirements can take, his results are not applicable to the assignment problem described in this example.

In the literature, admissibility requirements have been modeled in two ways. The first approach is to define choice functions (or preferences) so as to combine the requirements. The other way is to directly define the set of admissible allocations (See Section 5.2 for a detailed discussion). Ostrovsky (2008) follows the first approach; and provide a sufficient

[3]Namely, there is a profile of preferences for which first having an allocation for the children-school problem and then for teacher-school problem leads to a violation of admissibility, efficiency, or stability.

condition on choice functions that guarantees the existence of stable allocations.[4] Nevertheless, choice functions of institutions that reflect our requirements do not satisfy the property. Thus, his result does not help solve all problems that integrate multiple markets with complex requirements of the type we are interested in. ♣

Our paper models the admissibility requirements by directly defining the set of admissible allocations. An admissible set is defined as an arbitrary subset of the set of all allocations. This allows us to include any type of requirements or restrictions on allocations such as the one illustrated in the example above.

For traditional admissible sets, regardless of preferences, stable allocations exist.[5] Further, stability implies two important properties: efficiency and a fairness notion, freedom from justified-envy. Unfortunately, for general admissible sets, not only stable allocations may not exist, but efficiency and freedom from justified-envy may not be compatible (Proposition 1). Therefore, we propose two types of stability notions as compromises: *efficiency-guaranteed stability* and *fairness-guaranteed stability*.

We start by examining a solution related to the "core." As commonly understood, a coalition of entities *blocks an allocation* if there is an admissible allocation such that (i) each member of the coalition finds their new assignment at least as desirable as their original ones while some members prefer the new assignment to the original one, and (ii) assignments of members of the coalition are in the coalition.[6] However, this definition may not be appropriate for general admissible sets. Consider a problem with three agents i, j, k and two institutions x, y. At an allocation, agent i is assigned x, agent j is assigned y, and agent k is not assigned any institution. Suppose that agent k prefers x to their assignment; and x prefers additionally accepting agent k to not doing so. Consider the requirement on allocations that the total number of agents assigned the two institutions must be at most 2. According to the definition of a blocking coalition, "the allocation at which agent i is assigned x, agent j is not assigned any institution, and agent k is assigned x," is a candidate of allocations via which $\{i, k, x\}$ blocks the original allocation. However, since

[4]It is actually a conjunction of properties. One of it is called "same-side substitutability." It is easy to verify that the situation here cannot be described by any profile of choice functions that satisfy same-side substitutability.

[5]We assume that each institution has preferences that respect relative and absolute desirability of an individual agent, which is also known as "responsiveness".

[6]There is an alternative definition as follows: a coalition of entities blocks an allocation if there is an admissible allocation *for the coalition* satisfying the same requirements. However, for general admissible sets, we cannot necessarily talk about admissibility of an allocation for coalition.

the "irrelevant" agent-institution pair (j, y) is forced to sever their relation, such a block does not seem reasonable in the context of matching. Accordingly, we propose a natural alternative of blocking notion.

Given an admissible set F, a coalition *F-blocks* an allocation if it blocks the allocation in the usual sense while not changing the assignment of "irrelevant" entities. We define an *efficiency-guaranteed stable allocation* as one in which no coalition can block the allocation in our sense. Although each efficiency-guaranteed stable allocation is efficient, such an allocation may violate freedom from justified-envy. Regarding the issue of existence, we focus on a specific subclass of admissible sets that we call "number-based." An admissible set is *number-based* if, whenever an allocation is admissible, any allocation at which, for each institution, the same number of agents is assigned to the institution is also admissible. We identify a necessary condition on admissible sets in the class that guarantees the existence of such allocations (Theorem 1). Unfortunately, even within the number-based admissible sets, such allocations rarely exist.

We then focus on another stability notion, fairness-guaranteed stability, which achieves fairness while possibly allowing some Pareto improvement. More concretely, we require that an allocation meet the outside option lower bound, be free of justified envy, and be "fairness-constrained non-wasteful", meaning it contains no "fairness-constrained waste." *Fairness-constrained waste*, in contrast to waste in the allocation, permits certain positions in institutions to remain vacant. Specifically, even if these positions could be redistributed to a set of agents who would prefer them to their assignments, doing so would result in justified envy.[7] We identify a necessary and sufficient condition on the admissible set that ensures the non-emptiness of this solution. In contrast to the negative result pertaining to the efficiency-guaranteed stability, that condition can be seen as a minimal requirement on admissible sets. Formally, the fairness-guaranteed stable solution is not empty-valued if and only if the null allocation (i.e., no agent is assigned any institution) is admissible (Theorem 2). Further, we provide an alternative result in the domain of preferences where every entity prefers anyone in the other entity to being unassigned. We identify a sufficient condition on the class of admissible sets for which this solution is not empty-valued (Theorem 3).

In the light of the trade-off between efficiency and fairness that we uncovered in

[7] It is easy to see that fairness-constrained non-wastefulness coincides with non-wastefulness for traditional admissible sets.

our environment, one of these requirements has to be sacrificed. Nonetheless, fairness-guaranteed stability achieves maximal welfare among allocations that meet the outside option lower bound and are free of justified-envy (Proposition 3).

We then conduct a comparative static exercise on the set of fairness-guaranteed stable allocations with respect to enlargement of admissible sets.

The rest of the paper proceeds as follows. In Section 2, we describe the model. In Section 3, we introduce the concept of efficiency-guaranteed stability and present some key results regarding this solution concept. In Section 4, we introduce our main solution concept, fairness-guaranteed stability, and analyze its properties. In Section 4.1, we identify the conditions on admissible sets for which the fairness-guaranteed solution is non-empty. In Section 4.2, we study the comparative statics of the solution with respect to enlargement of admissible sets. In Section 5, we discuss the restriction of the domain of preferences and the relation to the literature. All proofs in the main text are provided in the Appendix A.

2 Model

There are a finite set of agents A and a finite set of institutions I. We call the components of $A \cup I$ *entities*. An agent is assigned at most one institution, while an institution can be assigned to multiple agents. Each agent $a \in A$ has strict preferences R_a over $I \cup \{\emptyset\}$, where $\emptyset$ means being assigned the *outside option*.[8] Let $\mathcal{R}_a$ be the set of all preferences of agent $a \in A$. Each institution $i \in I$ has strict preferences R_i over 2^A. Let $\mathcal{R}_i$ be the set of all preferences of institution $i \in I$. Given $h \in A \cup I$, P_h is the asymmetric part of R_h. Let $\mathcal{R} \equiv \prod_{h \in A \cup I} \mathcal{R}_h$. Let $R \in \mathcal{R}$ be our generic notation for a profile of preferences. If there is no danger of confusion, we write $a \; R_i \; a'$ instead of $\{a\} \; R_i \; \{a'\}$, and $A \cup a \; R_i \; A \cup a'$ instead of $A \cup \{a\} \; R_i \; A \cup \{a'\}$.

Institution i's preferences R_i are *separable* if both of the following conditions hold:[9]

(i) for each $A' \subseteq A$ and each pair $a, a' \in A \setminus A'$,

$$A' \cup a \; R_i \; A' \cup a' \iff a \; R_i \; a', \text{ and}$$

[8] A binary relation is strict if it is complete, transitive, and antisymmetric.

[9] Separability is closely related to what the literature calls responsiveness with respect to a capacity. In the definition of responsiveness, the second requirement of separability is modified as follows: for a set of agents, adding a "desirable" agent improves the set "if the capacity allows it." Since explicit capacities for each institution are not present in our model, the property cannot be property defined. Consequently, we instead directly deem such an allocation as inadmissible.

(ii) for each $A' \subseteq A$ and each $a \in A \setminus A'$,

$$A' \cup a \; R_i \; A' \iff a \; R_i \; \emptyset.$$

Throughout the paper, we assume that for each $i \in I$, R_i are separable.

An *allocation* is a function $\mu : A \cup I \to 2^A \cup I \cup \{\varnothing\}$ satisfying the following conditions:

(i) for each $a \in A$, $\mu(a) \in I \cup \{\varnothing\}$,

(ii) for each $i \in I$, $\mu(i) \in 2^A$, and

(iii) for each $a \in A$ and each $i \in I$, $\mu(a) = i$ if and only if $a \in \mu(i)$.

Let $\boldsymbol{\varnothing}$ be the *null allocation*; that is, for each $a \in A$, $\boldsymbol{\varnothing}(a) = \varnothing$. Let M be the set of all allocations.

Each problem has its own *admissible set* F, which is defined as a non-empty subset of all allocations, $F \subseteq M$ with $F \neq \emptyset$. For each $\mu \in M$, if $\mu \in F$, then μ is *F-admissible*. For simplicity, we say that μ is admissible. A *problem* is a list (A, I, R, F). In what follows, unless otherwise mentioned, we fix (A, I, F). Thus, a problem is simply described as a profile R.

An allocation $\mu \in F$ meets *the outside option lower bound for $R \in \mathcal{R}$* if

(i) for each $a \in A$, $\mu(a) \; R_a \; \varnothing$, and

(ii) for each $i \in I$ and each $a \in \mu(i)$, $a \; P_i \; \emptyset$.

Suppose that there is no institution that any agent prefers to their outside option. Then, if an allocation that meets the outside option lower bound exists for the problem, it is unique— it is the null allocation. Namely, if for each problem, such an allocation exists for that problem, the null allocation is admissible. Conversely, as long as the null allocation is in the admissible set, since for each problem the null allocation meets the outside option lower bound, there is an allocation that meets the outside option lower bound regardless of preferences. Hence, we have the following observation:

Observation 1. *Let $F \subseteq M$ be an admissible set. For each $R \in \mathcal{R}$, there is an allocation that meets the outside option lower bound for R if and only if $\boldsymbol{\varnothing} \in F$.*

The following two properties of allocations are essential. For each pair $\mu, \nu \in F$, ν *Pareto dominates* μ *for* $R \in \mathcal{R}$ if (i) for each $h \in A \cup I$, $\nu(h)\ R_h\ \mu(h)$ and (ii) there is $h \in A \cup I$ such that $\nu(h)\ P_h\ \mu(h)$. Allocation $\mu \in F$ is *efficient for* $R \in \mathcal{R}$ if no other admissible allocation Pareto dominates μ for R. Allocation $\mu \in F$ is *free of justified envy for* $R \in \mathcal{R}$ if for each pair $(a, i) \in A \times I$, if $i\ P_a\ \mu(a)$, then for each $a' \in \mu(i)$, $a'\ P_i\ a$.

Let P be the correspondence that associates with each problem the set of allocations that meet the outside option lower bound and are efficient. Similarly, let E be the correspondence that associates with each problem the set of allocations that meet the outside option lower bound and are free of justified envy.

3 Preliminary observation

In traditional matching problems, F is derived only by a profile of institution by institution maximal capacities. Formally, an admissible set F is *traditional* if there is $c \in \mathbb{N}_+^I$ such that

$$F = \{\mu \in M \mid \text{ for each } i \in I, |\mu(i)| \leq c_i\}.$$

In traditional admissible sets, efficiency and fairness are compatible. However, in general, efficiency and fairness are incompatible:

Proposition 1. *There are problems (A, I, F, R) such that $P(R) \cap E(R) = \emptyset$.*[10] [11]

Accordingly, we first consider a solution that places more importance on efficiency than on fairness. Our first attempt extends the notion of the "core". To begin with, we formally define the core in our environment.[12] Consider a pair of admissible allocations $\mu, \nu \in F$. A set of entities, or *coalition*, $C \subseteq A \cup I$ *blocks* μ *for* $R \in \mathcal{R}$ *via* ν if the following holds:

(i) for each $a \in C \cap A$, $\nu(a)\ R_a\ \mu(a)$ and $\nu(a) \in C \cup \{\varnothing\}$,

(ii) for each $i \in C \cap I$, $\nu(i) \subseteq C$ and $\nu(i)\ R_i\ \mu(i)$, and

(iii) there is $h \in C$ such that $\nu(h)\ P_h\ \mu(h)$.

[10] All proofs are included in the Appendix A.

[11] In the literature, some classes of admissible sets are uncovered for which efficient and free of justified-envy allocations exist for each problem. For example, see Kamada and Kojima (2017).

[12] More formally, since we distinguish preferences from admissible allocations, our notion of core is referred to as "α-core" in the literature.

An allocation $\mu \in F$ is in the *core of* $R \in \mathcal{R}$ if it meets the outside option lower bound for R and no coalition blocks μ for R.

When F is not traditional, this core notion seems too strong to be appropriate. This is because there may be too much freedom for the choices of allocations via which a coalition may block a given allocation (ν in the above definition). In essence, in the definition above, the assignment for entities outside the coalition C at the proposed allocation ν is not necessarily equal to the original assignment at μ. Put differently, to ensure the admissibility of ν, there might exist an agent-institution pair outside of C that may need to be unmatched. What is more concerning is the possibility that an agent outside of C could be reassigned an institution to which that agent prefers the outside option. To prevent this, we uniquely identify the allocation via which a coalition can block in a manner that does not alter the assignments of those not involved in C.

Consider a pair of admissible allocations $\mu, \nu \in F$. A pair consisting of a set of agents and a set of institutions $C \subseteq A \cup I$ *F-blocks* μ *for* $R \in \mathcal{R}$ *via* ν if the following holds:

(i) C blocks μ for R via ν,

(ii) for each $a \in A \backslash C$ with $\mu(a) \in C$, $\nu(a) = \emptyset$, and

(iii) for each $a \in A \backslash C$ with $\mu(a) \in I \backslash C$, $\nu(a) = \mu(a)$.

Definition 1. An allocation $\mu \in F$ is *efficiency-guaranteed stable for* $R \in \mathcal{R}$ if it meets the outside option lower bound for R and no coalition F-blocks μ for R.

Note that even if there is no F-blocking coalition at an allocation, the allocation may violate the outside option lower bound (See Example 4 in Appendix B).

Proposition 2. *For each $R \in \mathcal{R}$, any efficiency-guaranteed stable allocation for R is efficient for R.*

We ask whether efficiency-guaranteed stable allocations exist irrespective of what (A, I, R) is. We begin by focusing on a class of admissible sets for which admissibility only depends on the number of assigned agents in each institution, which we call the *distribution* of agents across institutions. For each $\mu \in M$, let $w(\mu) \equiv (|\mu(i)|)_{i \in I}$ be the *distribution of* μ. Namely, for each $i \in I$, $w_i(\mu)$ represents the number of agents to which institution i is assigned at μ. For each $w \in \mathbb{Z}_+^{|I|}$, the L^1-norm of w, $||w|| \equiv \Sigma_{i \in I} w_i$, is the number of agents to which an institution is assigned at μ. An admissible set $F \subseteq M$ is *number-based* if for each pair $\mu, \mu' \in M$, whenever $w(\mu) = w(\mu')$ and $\mu \in F$, then $\mu' \in F$. For each number-based admissible set $F \subseteq M$, there is an indicator function $f : \mathbb{Z}_+^{|I|} \to \{0, 1\}$ such that for each $w \in \mathbb{Z}_+^{|I|}$, $f(w) = 1$ if and only if (i) $||w|| \leq |A|$ and (ii) for each $\mu \in M$ with $w(\mu) = w$, $\mu \in F$. Given admissible set F and corresponding indicator function f, a *distribution* $w \in \mathbb{Z}_+^{|I|}$ *is admissible* if $f(w) = 1$.

We impose a condition on a number-based admissible set F. For each pair of vectors $w^1, w^2 \in \mathbb{Z}_+^{|I|}$ with $||w^1|| = ||w^2||$, let $W(w^1, w^2) \subseteq \mathbb{Z}_+^{|I|}$ be the following set of non-negative $|I|$-dimensional vectors:

$$W(w^1, w^2) \equiv \{w \in \mathbb{Z}_+^{|I|} \mid \text{ (i) } w \leq w^1 \vee w^2 \text{ and (ii) } ||w|| > ||w^1|| = ||w^2||\}.$$

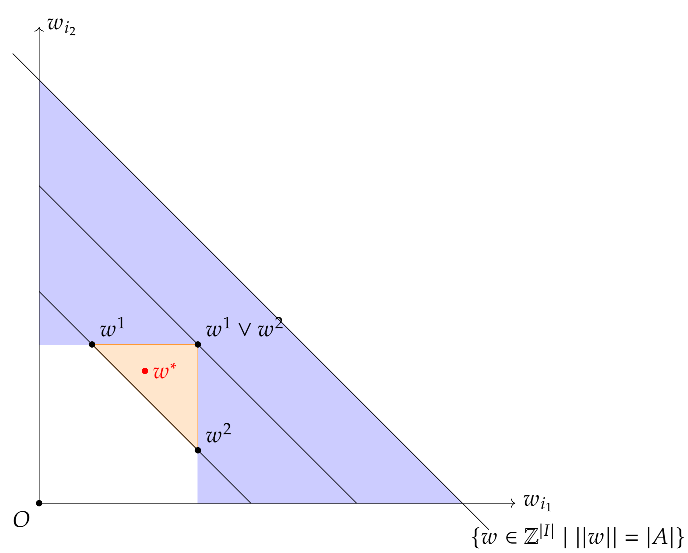

Figure 1: The distributions in the orange area corresponds to $W(w^1, w^2)$ with the case of $|I| = 2$, where the boundary passing through w^1 and w^2 is not included. In-betweeness requires that if there is no admissible distribution in blue area, then an admissible distribution w^* exists in the orange area.

Let a pair of $w^1 \in \mathbb{Z}_+^{|I|}$ and $w^2 \in \mathbb{Z}_+^{|I|}$ have the same L^1-norm. Then, the condition on the number-based F, defined formally below, indicates that there is an allocation in F whose distribution is in $W(w^1, w^2)$.

Definition 2. A number-based admissible set $F \subseteq M$ satisfies *in-betweenness* if its corresponding indicator function f satisfies the following condition:
For each pair of admissible distributions $w^1, w^2 \in \mathbb{Z}_+^{|I|}$ satisfying

(i) $||w^1|| = ||w^2||$,

(ii) $||w^1 \vee w^2|| \leq |A|$, and

(iii) $\{w \in \mathbb{Z}_+^{|I|} \mid f(w) = 1\} \subseteq \{w \in \mathbb{Z}_+^{|I|} \mid w \leq w^1 \vee w^2\}$,

there is an admissible distribution $w^* \in W(w^1, w^2)$.

Let ES be the correspondence that associates with each problem the set of efficiency-guaranteed stable allocations for the problem. Theorem 1 states that, for number-based admissible sets, in-betweenness plays a crucial role for ES to be well-defined:

Theorem 1. *Let $F \subseteq M$ be a number-based admissible set. For each $R \in \mathcal{R}$, $ES(R) \neq \emptyset$ only if (1) $\varnothing \in F$ and (2) F satisfies in-betweenness.*[13]

From Theorem 1, unfortunately, even for a certain domain of admissible sets, very stringent condition on the admissible set is required for the efficiency-guaranteed stable solution to be non–empty-valued[14]

Since an efficiency-guaranteed stable allocation places efficiency above fairness, even if such an allocation exists, it generally violates freedom from justified-envy:

Example 2. *(ES allows justified-envy.)*
Let $A = \{a, b\}$ and $I = \{i, j\}$. Let $F \subseteq M$ and $R \in \mathcal{R}$ be the following:

$$F = \left\{ \begin{pmatrix} i & j \\ b & a \end{pmatrix}, \begin{pmatrix} i & j \\ a & b \end{pmatrix}, \varnothing \right\}, \quad \begin{array}{c|c} R_a & R_b \\ \hline i & j \\ j & i \\ \varnothing & \varnothing \end{array}, \quad \begin{array}{c|c} R_i & R_j \\ \hline a & a \\ b & b \\ \emptyset & \emptyset \end{array}.$$

[13] When we study number-based admissible sets, we induce the indicator function f that corresponds to F. On the contrary, the literature studies a problem where the indicator function $f : \mathbb{Z}_+^{|I|} \to \{0, 1\}$ is given as a primitive of the problem (e.g. Kamada and Kojima (2017); Aziz et al. (2022)). To adapt such models, we modify *in-betweenness* by simply dropping the requirement (ii) on the pair w^1 and w^2. This condition, *in-betweeness** is a necessary condition on f that guarantees the non-emptiness of ES regardless of the set of agents. Formally, we have the following statement that is a counterpart result about an implication of (1) and (2) in Theorem 1 of Kamada and Kojima (2017):
Theorem 1*: *For each set of agents A and each profile $R \in \mathcal{R}$, $ES(R) \neq \emptyset$ only if* (1) $f(\mathbf{0}) = 1$ *and* (2) f *satisfies in-betweeness**. Note that its proof is analogous to the proof of Theorem 1 except the part specifying a particular set A.

[14] See Example 3 in Appendix B for the non-existence of efficiency-guaranteed stable allocations.

Let the first allocation in F be μ, and the second allocation in F be μ'. Consider μ. Then, a prefers i to their assignment at μ, namely j. Further, i is assigned b, to which a is preferred according to R_i. Then, (a, i) has justified-envy at μ.

However, no coalition F-blocks μ for R. Suppose by contradiction that there is $C \subseteq A \cup I$ that F-blocks μ for R via $\nu \in F$. First, suppose that $\nu = \emptyset$. Since each agent is assigned an institution that they prefer to their outside option at μ, no agent would benefit from belonging to the coalition. Thus, $\nu \neq \emptyset$, so that $\nu = \mu'$. By the second and third requirements of the definition of F-blocking, $b \in C \cap A$. However, this is contrary to the first requirement of the definition of F-blocking. Therefore, no coalition blocks μ for R via some admissible allocation, so that $\mu \in ES(R)$. ♣

4 Main Results

In the previous section, we showed that our first approach raises an existence issue; efficiency-guaranteed stable allocations rarely exist. Consequently, in contrast to prioritizing efficiency over fairness, this section delves into an alternative solution concept that places a greater emphasis on fairness.

In this section, we introduce our principal solution, "fairness-guaranteed stability." We establish its non-emptiness under a minimal condition on admissible sets; and we examine its comparative statics with respect to admissible sets.

4.1 Fairness-guaranteed stability

For traditional admissible sets, the core coincides with what we refer to as "the stable correspondence." It is characterized by three key properties: meeting the outside option lower bound, freedom from justified-envy, and "non-wastefulness." Given an allocation μ, an agent $a \in A$, and an institution $i \in I$, μ is *wasteful for i from the perspective of a* if i would enable a to occupy its vacant position and both would benefit from doing so.[15] Importantly, given an allocation and an institution, there may be multiple agents of whose perspective an allocation is wasteful for the institution. In such cases, determining who should occupy the vacant position first may be important. A natural candidate is the agent who is most preferred by the institution among such agents. We concern waste from the perspective of

[15]For detail, see Appendix C.2.

such an agent, which we informally call "fairness-constrained waste."

The following is a group-version of such a "non-wastefulness" requirement, also taking admissibility into consideration. An allocation $\mu \in F$ is *fairness-constrained non-wasteful for* R if no set of distinct agents $A' \subseteq A$ and no sequence of institutions $(i_a)_{a \in A'} \in I^{A'}$ indexed by agents in A' satisfy the following three requirements:

(i) for each $a \in A'$, $i_a \; P_a \; \mu(a)$ and $a \; P_{i_a} \; \emptyset$,

(ii) for each pair $a \in A'$ and $b \in A \setminus \{c \in A' | i_c = i_a\}$,

$$[b \in A' \text{ and } i_a \; P_b \; i_b] \Rightarrow a \; P_{i_a} \; b$$
$$[b \notin A' \text{ and } i_a \; P_b \; \mu(b)] \Rightarrow a \; P_{i_a} \; b$$

(iii) the following allocation ν is admissible: for each $b \in A$,

$$\nu(b) = \begin{cases} i_b & \text{if } b \in A' \\ \mu(b) & \text{otherwise.} \end{cases}$$

Condition (ii) corresponds to what we previously described as the distinction between "fairness-constrained waste" and "waste." In other words, we do not consider "waste" as an actual waste if it gives rise to an additional occurrence of justified-envy. Importantly, for traditional admissible sets, fairness-constrained non-wastefulness is equal to non-wastefulness. Remember that the property is a component of the core's characterization explained above.

Now we are ready to define our main solution concept and result.

Definition 3. An allocation $\mu \in F$ is *fairness-guaranteed stable for* $R \in \mathcal{R}$ if it meets the outside option lower bound for R, is free of justified envy for R, and is fairness-constrained non-wasteful for R.

Let FS be the correspondence that associates with each problem the set of fairness-guaranteed stable allocations for the problem.

Theorem 2. *Let $F \subseteq M$ be an admissible set. For each $R \in \mathcal{R}$, $FS(R) \neq \emptyset$ if and only if $\varnothing \in F$.*

One may wonder why we consider a group-version of waste as opposed to a pairwise version of waste. The reason is that, for general admissible sets, the pairwise notion seems inappropriate. To see this, we first formally define the pairwise version.

An allocation $\mu \in F$ is *pairwise fairness-constrained non-wasteful for* $R \in \mathcal{R}$ if no pair of agent $a \in A$ and institution $i \in I$ satisfies

(i) $i \; P_a \; \mu(a)$ and $a \; P_i \; \emptyset$,

(ii) for each $b \in A \setminus a$, $i \; P_b \; \mu(b)$ implies $a \; P_i \; b$, and

(iii) the following allocation ν is admissible: for each $b \in A$,

$$\nu(b) = \begin{cases} i & \text{if } b = a \\ \mu(b) & \text{otherwise.} \end{cases}$$

We also define the notion of pairwise fairness-guaranteed stability.

Definition 4. An allocation $\mu \in F$ is *pairwise fairness-guaranteed stable for* $R \in \mathcal{R}$ if it meets the outside option lower bound for R, is free of justified envy for R, and is pairwise fairness-constrained non-wasteful for R.

Let PFS be the correspondence that associates with each problem the set of pairwise fairness-guaranteed stable allocations for the problem. The existence of pairwise fairness-guaranteed stable allocations follows immediately from Theorem 2.

Corollary 1. *Let $F \subseteq M$ be an admissible set. For each $R \in \mathcal{R}$, a pairwise fairness-guaranteed stable allocation for R exists if and only if $\varnothing \in F$.*

Both solutions reduce to stability for traditional admissible sets. However, the pairwise notion seems inappropriate for general admissible sets. To see this, recall Example 2. Both $\varnothing$ and μ' are pairwise fairness-guaranteed stable for the problem, and clearly $\varnothing$ is Pareto dominated by μ'. That is, there is an admissible set for which a pairwise fairness-guaranteed stable allocation may be Pareto dominated by another pairwise fairness-guaranteed stable allocation.

In contrast, although fairness-guaranteed stable allocations achieve fairness while leaving the possibility of some Pareto improvement, it is "fairness-constrained efficient"; namely there is no improvement among allocations that meet the outside option lower bound and are free of justified envy. Formally, an allocation is *fairness-constrained efficient* for $R \in \mathcal{R}$ if it is in $E(R)$ and no allocation in $E(R)$ Pareto dominates it. Let EE be the correspondence that associates with each problem the set of fairness-constrained efficient allocations. We have the following relations between solutions:

Proposition 3. *For each $R \in \mathcal{R}$, $FS(R) \subseteq EE(R) \subseteq E(R)$.*

To conclude this subsection, we formally illustrate our leading example, Example 1 in Introduction.

Example 1 (revisited). Let $A = \{t_1, t_2, c_1, c_2, c_3, c_4\}$ and $I = \{s_1^t, s_1^c, s_2^t, s_2^c, s_3^t, s_3^c\}$. For convenience, let $T = \{t_1, t_2\}$ and $C = \{c_1, c_2, c_3, c_4\}$. The profile of preferences is as follows:

$$\begin{array}{ccc} P_{s_1^t} & P_{s_2^t} & P_{s_3^t} \\ \hline t_1 & t_2 & t_1 \\ t_2 & t_1 & t_2 \\ \emptyset & \emptyset & \emptyset \\ & & \\ & & \end{array}, \quad \begin{array}{cc} P_{t_1} & P_{t_2} \\ \hline s_1^t & s_2^t \\ s_2^t & s_1^t \\ \emptyset & s_3^t \\ & \emptyset \\ & \end{array}, \quad \begin{array}{ccc} P_{s_1^c} & P_{s_2^c} & P_{s_3^c} \\ \hline c_1 & c_1 & c_1 \\ c_2 & c_2 & c_2 \\ c_3 & c_3 & c_3 \\ c_4 & c_4 & c_4 \\ \emptyset & \emptyset & \emptyset \end{array}, \quad \begin{array}{cccc} P_{c_1} & P_{c_2} & P_{c_3} & P_{c_4} \\ \hline s_2^c & s_2^c & s_3^c & s_3^c \\ s_3^c & s_3^c & s_2^c & s_2^c \\ s_1^c & s_1^c & \emptyset & \emptyset \\ \emptyset & \emptyset & & \\ & & & \end{array}$$

Let $\tilde{M} \subseteq M$ be the set of allocations such that for each $\mu \in \tilde{M}$ and each $i \in \{1,2,3\}$, $\mu(s_i^t) \subseteq T$ and $\mu(s_i^c) \subseteq C$. The set of all allocations that meet the requirements illustrated in Example 1 is the following:

$$F \equiv \left\{ \mu \in \tilde{M} \mid \forall i \in \{1,2,3\}, |\mu(s_i^t)| = 2|\mu(s_i^c)| \right\}.$$

For this problem (A, I, R, F),

$$FS(R) = \left\{ \begin{pmatrix} s_1^t & s_1^c & s_2^t & s_2^c & s_3^t & s_3^c \\ \emptyset & \emptyset & t_1, t_2 & c_1, c_2, c_3, c_4 & \emptyset & \emptyset \end{pmatrix} \right\}.$$

As noted in the Introduction, the following allocation μ is not fairness-guaranteed stable:

$$\mu = \begin{pmatrix} s_1^t & s_1^c & s_2^t & s_2^c & s_3^t & s_3^c \\ \emptyset & \emptyset & t_2 & c_1, c_2 & \emptyset & \emptyset \end{pmatrix}.$$

Let $A' \equiv \{c_3, c_4, t_1\}$ and $(i_{c_3}, i_{c_4}, i_{t_1}) \equiv (s_2^c, s_2^c, s_2^t)$. Then, it is easy to see that μ is not fairness-guaranteed stable for R due to $(A', (i_a)_{a \in A'})$.

However, μ is a pairwise fairness-guaranteed stable allocation. As noted in the Introduction, μ is Pareto dominated by $FS(R)$. Hence, we should care about the general group version of a fairness-guaranteed stable allocation rather than about its pairwise version.[16]

4.2 Comparative statics

The specification of an admissible set reflects some policy or restrictions in matching markets. Therefore, a market designer will want to know the welfare effect of changes in admissible sets. Our next task is to analyze comparative statics with respect to admissible sets. Throughout this subsection, we fix (A, I, R) and vary $F \subseteq M$. Accordingly, our generic notation for a problem is an admissible set F instead of a preference profile R. For each admissible set $F \subseteq M$, let $FS(F)$ be the set of fairness-guaranteed stable allocations for F.

Proposition 4 indicates that, within fairness-guaranteed stable allocations, expanding the admissible set never hurts all entities.[17]

Proposition 4. *Let $F, F' \subseteq M$ be a pair of problems such that $F \subseteq F'$. Then, for each $\mu \in FS(F')$, there is no $\nu \in FS(F)$ that Pareto dominates μ for F'.*

5 Discussion

5.1 Restriction on the domain of preferences

In some applications, it is natural to assume that (i) each agent prefers any institution to the outside option and (ii) each institution prefers filling its position to leaving it vacant. In this subsection, for such a class of preferences, we investigate how the results change.

[16]Note that other than μ and $FS(R)$, there are three more pairwise fairness-guaranteed stable allocations for the problem: $\begin{pmatrix} s_1^t & s_1^c & s_2^t & s_2^c & s_3^t & s_3^c \\ t_1 & c_1, c_2 & \emptyset & \emptyset & \emptyset & \emptyset \end{pmatrix}$, $\begin{pmatrix} s_1^t & s_1^c & s_2^t & s_2^c & s_3^t & s_3^c \\ \emptyset & \emptyset & \emptyset & \emptyset & t_2 & c_1, c_2 \end{pmatrix}$, and $\emptyset$.

[17]As a special case of the relation $F \subseteq F'$, we can consider the effect of institution-capacities incremental on agents' welfare in a traditional matching problem. Assume that each institution $i \in I$ has a *capacity* that represents the maximum number of agents it can accommodate. Adding capacities is a good example of expanding admissible sets. We can regard adding one capacity of institution $i \in I$ as introducing one institution (with the same preference as institution i) into the problem. By the classical result such as Crawford (1991), as long as we focus on agent-optimal stable allocations (formally defined in Appendix B in the Online Appendix), introducing one institution makes all agents weakly better off, while making all institutions weakly worse off. Since fairness-guaranteed stability corresponds to agent-optimal stability in the traditional matching problems (see Proposition 6 in Appendix C.2), adding capacities makes all agents weakly better off, while making institutions weakly worse off in terms of fairness-guaranteed stability. This result is consistent with Proposition 4.

For each $a \in A$, let $\overline{\mathcal{R}}_a \subseteq \mathcal{R}_a$ be the *class of all-acceptable preferences for agents*:

$$\overline{\mathcal{R}}_a \equiv \{R_a \in \mathcal{R}_a \mid \text{ for each } i \in I, i\ R_a\ \varnothing\}.$$

Similarly, for each $i \in I$, let $\overline{\mathcal{R}}_i \subseteq \mathcal{R}_i$ be the *class of all-acceptable preferences of agents*:

$$\overline{\mathcal{R}}_i \equiv \{R_i \in \mathcal{R}_i \mid \text{ for each } a \in A, a\ R_i\ \emptyset\}.$$

Let $\overline{\mathcal{R}} \equiv \prod_{h \in A \cup I} \overline{\mathcal{R}}_h$ be the *all-acceptable class*.

An admissible set $F \subseteq M$ is *weakly number-based* if there is $\mu \in F$ such that for each $\nu \in M$ with $w(\nu) = w(\mu)$, $\nu \in F$. Namely, for any weakly number-based admissible set, there is at least one list of numbers such that each allocation whose number-distribution corresponds to the list is admissible.[18] Obviously, each number-based admissible set is also weakly number-based.

Theorem 3 states that as long as there is at least one "admissible number distribution", the existence of fairness-guaranteed stable allocations is guaranteed.

Theorem 3. *Let $F \subseteq M$ be a weakly number-based admissible set. For each $R \in \overline{\mathcal{R}}$, $FS(R) \neq \emptyset$.*

For each admissible set $F \subseteq M$, if $\varnothing \in F$, then F is weakly number-based, but the converse does not hold. Since for "the full domain of preferences", $\varnothing \in F$ is necessary and sufficient for FS to be well-defined, Theorem 3 strengthens the existence result of fairness-guaranteed stable allocations by restricting the domain of preferences.

5.2 Related literature

In this section, we discuss how our paper relates to the previous literature. There are two main approaches to study matching problems with complex requirements on them.

One approach consists in defining a choice function by combining social requirements and institution's preferences: for example, in school choice problems (Echenique and Yenmez, 2015; Ehlers et al., 2014; Erdil and Kumano, 2019; Sönmez and Yenmez, 2022), and in army branch assignment problems (Sönmez and Switzer, 2013; Kominers and

[18] An interesting subclass of weakly number-based admissible sets describes a *floor constraint*: an admissible set $F \subseteq M$ represents a *floor constraint* if for each $\mu, \nu \in M$ with $w(\mu) \leq w(\nu)$, $\mu \in F$ implies $\nu \in F$. In the class of floor constraints, Akin (2021) defines *floor-respecting stability* and shows its existence. Our efficiency-guaranteed stability corresponds to floor-respecting stability in this class. Hence, ES is well-defined for floor constraints.

Sönmez, 2016). Stability in the usual sense can be defined with such a profile of choice functions without any modification. Further, in the literature, a combination of properties on choice functions has been uncovered that guarantees the existence of stable allocations.[19] However, some requirements on matching markets cannot be represented by means of a single choice function that represents a single institution's preferences. For example, a single choice function cannot express the requirement that a particular number of agents should be assigned some set of institutions.

The other approach consists in explicitly defining admissible allocations in addition to institution's preferences. This approach reflects constraints such as social concerns or market restrictions more directly, as done in this paper. On one hand, it can describe situations more freely than the former approach; on the other hand, a solution concept such as stability must be modified to fit the model. There are a number of studies of matching problems with constraints that follow this approach. We introduce some of them by categorizing them into specific classes of admissible sets.

Other than the class of number-based admissible sets, which we denote by $\mathcal{F}^{\#}$ introduced in Section 3, we introduce two more special classes: the general upper-bound class $\overline{\mathcal{F}}$ and the institution-by-institution class $\mathcal{F}^{\prod}$.[20] Figure 2 illustrates the entire class of admissible sets and each subclass:

[19]More precisely, the defined choice functions are required to satisfy a sort of substitutability. See Hatfield and Kojima (2010) for example. Another example is a model with supply chain networks (Ostrovsky, 2008). The model can illustrate the situation in Example 1 with choice functions that encompass the requirements on the market. However, the induced choice function does not satisfy "same-side substitutability" that is a sufficient condition on choice functions to guarantee existence of stable allocations. However, in the example, as we saw, stable allocations exist.

[20]Formal definitions of $\overline{\mathcal{F}}$ and $\mathcal{F}^{\prod}$ are in Appendix C.1 and C.2, respectively.

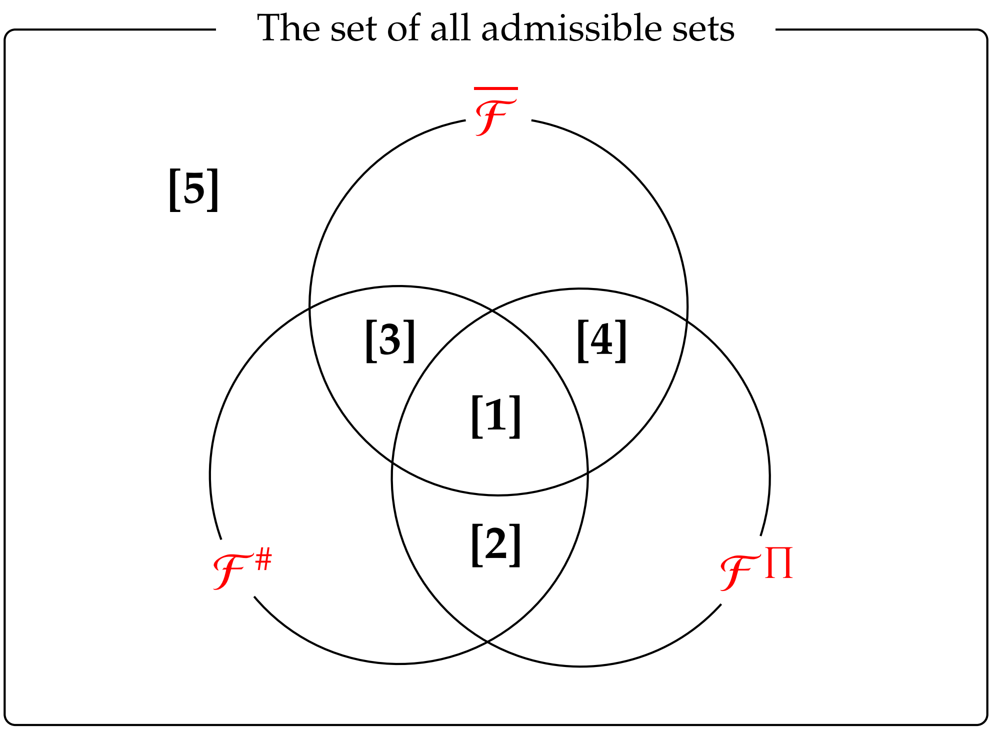

Figure 2: Classification of admissible sets.

In class [1], many-to-one matching problems have been developed with an admissible set associated with only the list of capacities (or ceiling constraints) (Gale and Shapley, 1962). Other than ceiling constraints or capacities, an admissible set is also subjected to *floor-constraints* (where each institution has a minimum number of agents it must accept). This allows us to study both [1] and [2] (e.g. (Akin, 2021)). Besides the list of institution-wise capacities, there would be a type of ceiling constraints restricting the number of agents that a set of institutions can jointly accommodate. Problems with such admissible sets, namely, both [1] and [3], is investigated (Kamada and Kojima, 2015, 2017, 2018).[21] In both [1] and [4], a fairness-guaranteed solution, named "the student-optimal fair solution," is investigated in Kamada and Kojima (2023).[22] The solution coincides with one of our solutions, the fairness-guaranteed stable solution.[23] A nursery school assignment problem under a "balance" requirement that differs from the one illustrated in Example 1 is studied in Kamada and Kojima (2022). The admissible set investigated in their model allows us to

[21] Also, see Aziz et al. (2022) and Cho et al. (2022).

[22] While they relax the assumption of general upper-bound at the model level, they retain it for their main results.

[23] See also Root (2019) and Delacrétaz et al. (2023).

study not only the entire class of [1] but also a part of [5], which balances the number of "inter-district transfer of children." Even though the admissible set illustrated in Example 1 also belongs to [5], it is not in the class studied in Kamada and Kojima (2022). Note that our model allows us to accommodate any admissible set in the above figure. More importantly, in any admissible set, as long as the null allocation is admissible, fairness-guaranteed stable allocations exist.

Appendix

A Proofs

Proof of Proposition 1

The proof is by means of an example. Let $A = \{a, b\}$ and $I = \{i\}$. Let $\mu \in M$ be such that $\mu(a) = i$ and $\mu(b) = \emptyset$. Let $F = \{\boldsymbol{\varnothing}, \mu\}$.

Let $R \in \mathcal{R}$ be such that (i) each agent prefers institution i to being unmatched and (ii) institution i prefers agent b to agent a and to being unmatched. Then, $P(R) = \{\mu\}$ and $E(R) = \{\boldsymbol{\varnothing}\}$. □

Proof of Proposition 2

Let $R \in \mathcal{R}$ and $\mu \in F$. Suppose that an allocation $\nu \in F$ Pareto dominates μ for R. Let $A' \equiv A$ and $I' \equiv I$. We claim that $C = A' \cup I'$ F-blocks μ for R via ν. Since ν Pareto dominates μ for R, the first requirement in the definition of F-blocking coalitions holds. Further, since $A \setminus A' = I \setminus I' = \emptyset$, all the other requirements trivially hold. Hence, $\mu \notin ES(R)$. □

Proof of Theorem 1

Let $F \subseteq M$ be a number-based admissible set and let f be the corresponding indicator function. If $\boldsymbol{\varnothing} \notin F$, then Observation 1 implies that there is a problem for which no allocation meets the outside option lower bound. Hence, let us assume that $\boldsymbol{\varnothing} \in F$.

Suppose by contradiction that F violates in-betweenness. Then, there is a pair of admissible distributions $w^1, w^2 \in \mathbb{Z}_+^{|I|}$ such that

(i) $||w^1|| = ||w^2||$,

(ii) $||w^1 \vee w^2|| \leq |A|$,

(iii) $\{w \in \mathbb{Z}_+^{|I|} \mid f(w) = 1\} \subseteq \{w \in \mathbb{Z}_+^{|I|} \mid w \leq w^1 \vee w^2\}$, and

(iv) for each $w^* \in W(w^1, w^2)$, $f(w^*) = 0$.

We claim that there is a problem for which any admissible allocation that meets the outside option lower bound has an F-blocking coalition.

Step 1: Construction of the problem.

We first define the following three sets:

$$I^1 \equiv \{i \in I \mid w_i^1 > w_i^2\} = \left\{i_1^1, i_2^1, \dots, i_{|I^1|}^1\right\},$$
$$I^2 \equiv \{i \in I \mid w_i^2 > w_i^1\} = \left\{i_1^2, i_2^2, \dots, i_{|I^2|}^2\right\}, \text{ and}$$
$$I^= \equiv \{i \in I \mid w_i^1 = w_i^2\} = \left\{i_1^=, i_2^=, \dots, i_{|I^=|}^=\right\}.$$

Then, $\{I^1, I^2, I^=\}$ is a partition of I.

For each $i \in I$, let $m_i \equiv \min\{w_i^1, w_i^2\}$ and $w_i \equiv \max\{w_i^1, w_i^2\}$. Let $m \equiv (m_i)_{i\in I}$ and $w \equiv (w_i)_{i\in I}$.

By Assumption (ii), $||w|| \leq |A|$. Then, there is $A^* \subseteq A$ such that $|A^*| = ||w||$. For convenience, let $\{A^0, A^1, A^2\}$ be a partition of A^* such that (i) $|A^0| = ||m||$, (ii) $|A^1| = ||w^1|| - ||m||$, and (iii) $|A^2| = ||w^2|| - ||m||$. By Assumption (i), $|A^1| = |A^2|$. Let $\{A(i)\}_{i\in I}$ be a partition of A^0 such that for each $i \in I$, $|A(i)| = m_i$. For each $i \in I$, let $A(i) = \{a_1^i, a_2^i, \dots, a_{m_i}^i\}$.

For convenience, we define an allocation that may not be admissible. Let $\bar{\mu} \in M$ be such that (i) for each $i \in I^=$, $\bar{\mu}(i) = A(i)$, (ii) for each $i \in I^1$, $A(i) \subseteq \bar{\mu}(i) \subseteq A(i) \cup A^2$, (iii) for each $i \in I^2$, $A(i) \subseteq \bar{\mu}(i) \subseteq A(i) \cup A^1$, and (iv) $w(\bar{\mu}) = w$.[24]

Given $\bar{\mu}$, we introduce two more notations. For each $k \in \{1, 2, \dots, |I^2|\}$, let $A_k^1 \equiv \bar{\mu}(i_k^2) \backslash A(i)$. Similarly, for each $k \in \{1, 2, \dots, |I^1|\}$, let $A_k^2 \equiv \bar{\mu}(i_k^1) \backslash A(i)$.

Let $R \in \mathcal{R}$ be the following:

- Preference for $a \in A \setminus A^*$

  For each $a \in A \backslash A^*$ and each $i \in I$,

$$R_a : \emptyset,\ i.$$

- Preference for $a \in A^0 = \bigcup_{i\in I} A(i)$

[24] By $w(\bar{\mu}) = w$ and the construction of A, for each $a \in A$, $\bar{\mu}(a) \neq \emptyset$.

For each $i \in I$, each $a \in A(i)$, and each $i' \in I \backslash \{i\}$,

$$R_a : i, \varnothing, i'.$$

- For each $k \in \{1, 2, ..., |I^2|\}$, preference for $a \in A^1_k$
  For each $i \in I^=$, and each $i' \in I^1$,

$$R_a: ..., i, ..., i', ..., \underbrace{i^2_{k-1}, i^2_{k-2}, ..., i^2_1, i^2_{|I^2|}, ..., i^2_{k+1}, i^2_k}_{I^2}, \varnothing$$

- For each $k \in \{1, 2, ..., |I^1|\}$, preference for $a \in A^2_k$
  For each $i \in I^=$, and each $i' \in I^2$,

$$R_a: ..., i, ..., i', ..., \underbrace{i^1_{k-1}, i^1_{k-2}, ..., i^1_1, i^1_{|I^1|}, ..., i^1_{k+1}, i^1_k}_{I^1}, \varnothing$$

- For each $k \in \{1, 2, ..., |I^1|\}$, preference for $i^1_k \in I^1$

1. For each $a \in A(i^1_k)$, each $a' \in A^1$, each $a'' \in A^2$, and each $a''' \in A \backslash (A(i^1_k) \cup A^1 \cup A^2)$,

$$R_{i^1_k} : ..., a, ..., a'', ..., a', ..., \emptyset, ..., a''', ...$$

2. For each $(a_1, ..., a_{|I^1|}) \in \prod_{\ell=1}^{|I^1|} A^2_\ell$,

$$R_{i^1_k} : ..., a_k, ..., a_{k-1}, ..., a_1, ..., a_{|I^1|}, ..., a_{|I^1|-1}, ..., a_{k+1}, ...$$

- For each $k \in \{1, 2, ..., |I^2|\}$, preference for $i^2_k \in I^2$

1. For each $a \in A(i^2_k)$, each $a' \in A^1$, each $a'' \in A^2$, and each $a''' \in A \backslash (A(i^2_k) \cup A^1 \cup A^2)$,

$$R_{i^2_k} : ..., a, ..., a', ..., a'', ..., \emptyset, ..., a''', ...$$

2. For each $(a_1, ..., a_{|I^2|}) \in \prod_{\ell=1}^{|I^2|} A^1_\ell$,

$$R_{i^2_k} : ..., a_k, ..., a_{k-1}, ..., a_1, ..., a_{|I^2|}, ..., a_{|I^2|-1}, ..., a_{k+1}, ...$$

- Preference for $i \in I^{=}$
  For each $a \in A(i)$, each $a' \in A^2$, each $a'' \in A^1$, and each $a''' \in A \backslash (A(i) \cup A^1 \cup A^2)$,

$$R_i : \ldots, a, \ldots, a', \ldots, a'', \ldots, \emptyset, \ldots, a''', \ldots$$

Then, we have a problem (A, I, R, F). Again, we simply refer to the problem as R.

(End of Step 1.)

Suppose by contradiction that $ES(R) \neq \emptyset$. Let $\mu \in ES(R)$. Let $\bar{I} \equiv \{i \in I^1 \cup I^2 \mid A(i) \not\subseteq \mu(i)\}$. Namely, $\bar{I}$ is the set of institutions such that each institution $i \in \bar{I}$ is not assigned a student in $A(i)$ at μ. The following step illustrates the requirements on μ.

Step 2: There is $\{\tilde{A}(i)\}_{i \in I \backslash (\bar{I} \cup I^{=})}$ such that (i) for each $i \in I \backslash (\bar{I} \cup I^{=})$, $\tilde{A}(i) \subseteq A^1 \cup A^2$ and (ii) μ is the following:

$$\text{for each } i \in I, \quad \mu(i) \begin{cases} \subsetneq A(i) & \text{if } i \in \bar{I} \\ \subseteq A(i) & \text{if } i \in I^{=} \\ = A(i) \cup \tilde{A}(i) & \text{otherwise.} \end{cases} \qquad \cdots\cdots (*)$$

To prove the statement, we provide a lemma. The lemma states that no institution i in $\bar{I} \cup I^{=}$ is assigned any agent who does not belong to $A(i)$.

Lemma 1. *For each $i \in \bar{I} \cup I^{=}$, $\mu(i) \subseteq A(i)$.*

Proof of Lemma 1. Suppose by contradiction that there is $i \in \bar{I} \cup I^{=}$ such that $\mu(i) \not\subseteq A(i)$; namely, there is an agent-institution pair $(a, i) \in A \times I$ such that (i) $i \in \bar{I} \cup I^{=}$, (ii) $a \in A \backslash A(i)$, and (iii) $\mu(a) = i$.

First, we claim that there is $a' \in A(i)$ such that $\mu(a') \neq i$. Suppose $i \in \bar{I}$. Then, by the definition of $\bar{I}$, the claim automatically holds. Suppose $i \in I^{=}$. Suppose by contradiction that for each $a' \in A(i)$, $\mu(a') = i$. Then, $m_i + 1 = |A(i)| + |\{a\}| \leq w_i(\mu)$ together with $w_i = m_i$ imply $w_i < w_i(\mu)$; namely $w(\mu) \not\leq w$. Then, Assumption (iii) implies $f(w(\mu)) = 0$. This means that μ is not admissible, a contradiction. Hence, there is $a' \in A(i)$ such that $\mu(a') \neq i$ for the case of $i \in I^{=}$, too.

For μ to meet the outside option lower bound for R, by construction of R, $\mu(a') = \emptyset$. Let $A' \equiv (\mu(i) \cup \{a'\}) \setminus \{a\}$, $I' \equiv \{i\}$, and $\nu \in M$ be the allocation such that

$$\text{for each } \bar{a} \in A, \quad \nu(\bar{a}) = \begin{cases} i & \text{if } \bar{a} = a' \\ \emptyset & \text{if } \bar{a} = a \\ \mu(\bar{a}) & \text{otherwise.} \end{cases}$$

Since $w(\nu) = w(\mu)$ and $f(w(\mu)) = 1$, $\nu \in F$. Agent a and institution i prefer ν to μ, and any other member of the coalition match the same ones at ν as the ones at μ. Hence, (A', I') F-blocks μ for R via ν. □

By Lemma 1 and construction of $\bar{I}$, there is a family of (possibly empty) disjoint sets $\{\tilde{A}(i)\}_{i \in I \setminus (\bar{I} \cup I^=)}$ such that (i) for each $i \in I \setminus (\bar{I} \cup I^=)$, $\tilde{A}(i) \subseteq A^1 \cup A^2$ and (ii) μ is the following:

$$\text{for each } i \in I, \quad \mu(i) \begin{cases} \subsetneq A(i) & \text{if } i \in \bar{I} \\ \subseteq A(i) & \text{if } i \in I^= \\ = A(i) \cup \tilde{A}(i) & \text{otherwise.} \end{cases} \qquad \cdots\cdots (*)$$

(End of Step 2.)

In the next step, we check the distribution of μ.

Step 3: $w(\mu) \in W(w^1, w^2)$**.**

Suppose not, one of the following holds:

1. $||w(\mu)|| \leq ||w^1|| = ||w^2||$ or
2. $w(\mu) \not\lesssim w$.

If $w(\mu) \not\lesssim w$, then by Assumption (iii), μ is not admissible. Hence, it is not the case that $w(\mu) \not\lesssim w$. Suppose that $||w(\mu)|| \leq ||w^1|| = ||w^2||$.

Step 3-1: For each $a' \in A^1 \cup A^2$**,** $\mu(a') \in I^1 \cup I^2$**.**

We claim that for each $a' \in A^1$, $\mu(a') \in I^1 \cup I^2$.[25] Suppose by contradiction that there is

[25] The statement such that for each $a' \in A^2$, $\mu(a') \in I^1 \cup I^2$ can be shown analogously.

$a' \in A^1$ such that $\mu(a') \notin I^1 \cup I^2$. Let $a' \in A^1$ be such that $\mu(a') \notin I^1 \cup I^2$.

The next lemma shows that no institution in I^1 (I^2, respectively) is assigned an agent in A^1 (A^2, respectively) at μ.

Lemma 2. *(1) For each $i \in I^1 \backslash \bar{I}$, $\tilde{A}(i) \subseteq A^2$ and (2) for each $i \in I^2 \backslash \bar{I}$, $\tilde{A}(i) \subseteq A^1$.*

Proof of Lemma 2. First, we claim that for each $a \in A^2$, $\mu(a) \in I^1 \cup \{\varnothing\}$. By (∗) obtained in Step 2, $\mu(a') = \varnothing$. Then, by an analogous argument to one made in the proof of Lemma 1, it is easy to see that for each $a \in A^2$, $\mu(a) \notin I^2$. Again by (∗), for each $a \in A^2$, $\mu(a) \in I^1 \cup \{\varnothing\}$, which implies (1).

Suppose that there is $a'' \in A^2$ such that $\mu(a'') = \varnothing$. By construction of R, an analogous argument to one made in Lemma 1 leads to the condition that for each $a \in A^1$, $\mu(a) \in I^2 \cup \{\varnothing\}$. Hence, (ii) holds. On the contrary, suppose that for each $a'' \in A^2$, $\mu(a'') \neq \varnothing$. Then, for each $a \in A^2$, $\mu(a) \in I^1$. By (∗) and by Assumption (iii), for each $a \in A^1$, $\mu(a) \in I^2 \cup \{\varnothing\}$. Again, (2) holds in this case. □

Before moving to the next lemma, let $\tilde{I} \equiv \{i \in I^2 \mid w_i(\mu) > m_i\}$.

Lemma 3. *For each $i \in \bar{I}$ and each $a \in \bar{\mu}(i) \backslash A(i)$, $\mu(a) = \varnothing$.*

Proof of Lemma 3. Suppose by contradiction that there is a pair $(a, i) \in A \times \bar{I}$ such that (i) $a \in \bar{\mu}(i) \backslash A(i)$ and (ii) $\mu(a) \neq \varnothing$. Without loss of generality, suppose that $i \in I^2$; and let $i^2_{k^0} \equiv i$. By construction of $\bar{\mu}$, $a \in A^1$. By Lemma 2 and by (∗), $\mu(a) \in \tilde{I}$. Let $i^2_{k^1} \equiv \mu(a)$. Since $i^2_{k^0} \in \bar{I}$ and $a \notin A(i^2_{k^0})$, $\mu(a) \neq i^2_{k^0}$; namely $i^2_{k^1} \neq i^2_{k^0}$. Then, Assumption (iii) implies that there is $a^1 \in A^1_{k^1}$ such that $\mu(a^1) \neq i^2_{k^1}$. By Lemma 2, $\mu(a^1) \in \tilde{I} \cup \{\varnothing\}$. If $\mu(a^1) = \varnothing$, then $((\mu(i^2_{k^1}) \cup \{a^1\}) \backslash \{a\}, \{i^2_{k^1}\})$ F-blocks μ for R via an admissible allocation that is obtained from μ by replacing a with a^1 by $i^2_{k^1}$. This violates $\mu \in ES(R)$. Hence, $\mu(a^1) \in \tilde{I} \backslash \{i^2_{k^1}\}$. Let $i^2_{k^2} \equiv \mu(a^1)$. An analogous argument shows that there is $a^2 \in A^1_{k^2}$ such that $\mu(a^2) \neq i^2_{k^2}$. By repeating this argument, and since $i^2_{k^0} \in \bar{I}$, Assumption (iii) and Lemma 2 imply that there are $i^2_{k^*} \in \tilde{I}$, $a^* \in A^1_{k^*}$, and $a \in \mu(i^2_{k^*})$ such that $\mu(a^*) = \varnothing$ and $a \notin A^1_{k^*}$.

Let $a \in \mu(i^2_{k^*})$ with $a \notin A^1_{k^*}$. Then, by an analogous argument to one made in Lemma 1, $((\mu(i^2_{k^*}) \cup \{a^*\}) \backslash \{a\}, \{i^2_{k^*}\})$ F-blocks μ for R via an admissible allocation, a contradiction. □

Lemma 4. *Suppose $|I^2| \geq 3$. Let a distinct triple $k^1, k^2, k^3 \in \{1, 2, ..., |I^2|\}$, $a_1 \in A^1_{k^1}$, and $a_3 \in A^1_{k^3}$. Then, we have*

$$a_1 \; P_{i^2_{k^2}} \; a_3 \;\; \Rightarrow \;\; i^2_{k^3} \; P_{a_1} \; i^2_{k^2}.$$

Proof of Lemma 4. Suppose that $a_1 \; P_{i^2_{k^2}} \; a_3$.

Case 1: $k^1, k^3 \in [1, k^2)$**.** By construction of $R_{i^2_{k^2}}$, $k^1 > k^3$. Then, $k^2 > k^1 > k^3$ holds. By construction of $R_{i^2_{k^2}}$, we have $i^2_{k^3} \; P_{a_1} \; i^2_{k^2}$. Similarly, if $k^1, k^3 \in (k^2, |I^2|]$, then $k^1 > k^3 > k^2$, so that $i^2_{k^3} \; P_{a_1} \; i^2_{k^2}$.

Case 2: $k^1 \in [1, k^2)$ **and** $k^3 \in (k^2, |I^2|]$. Since $1 \leq k^1 < k^2 < k^3 \leq |I^2|$, by construction of R_{a_1}, $i^2_{k^3} \; P_{a_1} \; i^2_{k^2} \; P_{a_1} \; i^2_{k^1}$.

Case 3: $k^3 \in [1, k^2)$ **and** $k^1 \in (k^2, |I^2|]$. By construction of $R_{i^2_{k^2}}$, $a_3 \; P_{i^2_{k^2}} \; a_1$, so that this is not the case. □

The next lemma states that each agent in A^1 who is assigned an institution prefers any institution in $(I^2 \cap \bar{I}) \cup I^1 \cup I^=$ to their assignment.

Lemma 5. *For each $a \in A^1$ with $\mu(a) \in I^2$; and each $i \in (I^2 \cap \bar{I}) \cup I^1 \cup I^=$, $i \; P_a \; \mu(a)$.*

Proof of Lemma 5. Let $a \in A^1$ with $\mu(a) \in I^2$. By construction of preferences, it is obvious that for each $i \in I^1 \cup I^=$, $i \; P_a \; \mu(a)$. In what follows, we claim that for each $i \in I^2 \cap \bar{I}$, $i \; P_a \; \mu(a)$. Suppose that $I^2 \cap \bar{I} \neq \emptyset$; otherwise, the statement obviously holds. Without loss of generality, suppose that $a \in A^1_k$; namely $a \in \bar{\mu}(i^2_k) \backslash A(i^2_k)$. For clarity of the proof, rename a as a_k.

First, suppose that $\mu(a_k) = \bar{\mu}(a_k)$, namely $\mu(a_k) = i^2_k$. By $(*)$, since $a_k \notin A(i^2_k)$ and $\mu(a_k) = i^2_k$, $i^2_k \notin \bar{I}$. Then, by construction of preferences, for each $i \in I^2 \cap \bar{I}$, $i \; P_{a_k} \; i^2_k = \mu(a_k)$.

Second, suppose that $\mu(a_k) \neq \bar{\mu}(a_k)$. Without loss of generality, let $i^2_{k'} \equiv \mu(a_k)$. Note that $i^2_k \neq i^2_{k'}$; and by $(*)$ and Lemma 3, $\{i^2_k, i^2_{k'}\} \cap \bar{I} = \emptyset$. Let $k'' \in \{1, 2, \ldots, |I^2|\}$ be such that $i^2_{k''} \in I^2 \cap \bar{I}$. Then, k, k', and k'' are all distinct. Let $a_{k''} \in A^1_{k''}$. By Lemma 3, $\mu(a_{k''}) = \varnothing$. Then, if $a_{k''} \; P_{i^2_{k'}} \; a_k$, by a similar argument to one made in Lemma 1, μ is F-blocked for R via an admissible allocation. Hence, $a_k \; P_{i^2_{k'}} \; a_{k''}$. By Lemma 4, $i^2_{k''} \; P_{a_k} \; i^2_{k'} = \mu(a_k)$. Since $i^2_{k''}$ is arbitrarily chosen from $I^2 \cap \bar{I}$, for each $i \in I^2 \cap \bar{I}$, $i \; P_{a_k} \; \mu(a_k)$. □

We conclude Step 3-1 by constructing an allocation ν from μ such that $w(\nu) = w^1$, and showing that a coalition F-blocks μ for R via ν. To do this, we need additional notations. For each $i \in I^=$, let $e_i \equiv m_i - |\mu(i)|$ and let $e \equiv \sum_{i \in I^=} e_i$. By $(*)$, for each $i \in I^=$, $e_i \geq 0$.

First, we claim that each agent in $\bigcup_{i \in I^2 \backslash \bar{I}} (\mu(i) \backslash A(i))$ can be reassigned to some institution outside of $I^2 \setminus \bar{I}$ so as to be its distribution is less than w^1. To do this, we show the following equation:

$$\sum_{i\in I^2\setminus\bar{I}}(|\mu(i)|-m_i) \le \sum_{i\in I^2\cap\bar{I}}(m_i-|\mu(i)|)+\sum_{i\in I^1}(w_i-|\mu(i)|)+e\cdots(**)$$

Note that by assumption, we have $||w(\mu)|| \le ||w^1||$; so that

$$\begin{aligned}
& ||w(\mu)|| \le ||w^1|| \\
\iff & \sum_{i\in I^=}|\mu(i)|+\sum_{i\in I^1}|\mu(i)|+\sum_{i\in I^2}|\mu(i)| \le \sum_{i\in I^=}m_i+\sum_{i\in I^1}m_i+\sum_{i\in I^2}m_i+|A^1| \\
\iff & \sum_{i\in I^2}(|\mu(i)|-m_i) \le \sum_{i\in I^1}(m_i-|\mu(i)|)+|A^1|+\sum_{i\in I^=}(m_i-|\mu(i)|) \\
\iff & \sum_{i\in I^2}(|\mu(i)|-m_i) \le \sum_{i\in I^1}(w_i-|\mu(i)|)+\sum_{i\in I^=}(m_i-|\mu(i)|) \quad \because\ |A^1|=\sum_{i\in I^1}(w_i-m_i) \\
\iff & \sum_{i\in I^2\setminus\bar{I}}(|\mu(i)|-m_i)+\sum_{i\in I^2\cap\bar{I}}(|\mu(i)|-m_i) \le \sum_{i\in I^1}(w_i-|\mu(i)|)+\sum_{i\in I^=}(m_i-|\mu(i)|) \\
\iff & \sum_{i\in I^2\setminus\bar{I}}(|\mu(i)|-m_i) \le \sum_{i\in I^2\cap\bar{I}}(m_i-|\mu(i)|)+\sum_{i\in I^1}(w_i-|\mu(i)|)+\sum_{i\in I^=}(m_i-|\mu(i)|) \\
\iff & \sum_{i\in I^2\setminus\bar{I}}(|\mu(i)|-m_i) \le \sum_{i\in I^2\cap\bar{I}}(m_i-|\mu(i)|)+\sum_{i\in I^1}(w_i-|\mu(i)|)+e.
\end{aligned}$$

Then, we construct, from μ, an admissible allocation ν whose distribution is exactly w^1 in the following manner:

Case 1: $\sum_{i\in I^2\setminus\bar{I}}(|\mu(i)|-m_i) \le \sum_{i\in I^2\cap\bar{I}}(m_i-|\mu(i)|)$.

Let each agent in $\bigcup_{i\in I^2\setminus\bar{I}}(\mu(i)\setminus A(i))$ be reassigned to an institution i in $I^2\cap\bar{I}$ so that each $i\in I^2\cap\bar{I}$ is assigned at most m_i agents. Let this allocation be denoted by ν^1. Since $\sum_{i\in I^2\setminus\bar{I}}(|\mu(i)|-m_i) \le \sum_{i\in I^2\cap\bar{I}}(m_i-|\mu(i)|)$, the allocation ν^1 indeed exists.

We construct ν from ν^1 as follows: For each $i\in I^2\cap\bar{I}$ that is assigned fewer than m_i agents at ν^1, let it be reassigned some agents in $A(i)\setminus\mu(i)$ so as to be assigned exactly m_i agents. Then, for each $i\in I^=$, all $A(i)\setminus\mu(i)$ agents are assigned to i. Then, for each each $i\in I^1$, let each agent in $A^2\cup A(i)$ who are assigned $\varnothing$ at μ be assigned to i so as to be assigned exactly w_i agents; let the resulting allocation be ν. By $(*)$ and Lemma 2, $w(\nu)=w^1$.

Case 2: $\sum_{i\in I^2\setminus\bar{I}}(|\mu(i)|-m_i) > \sum_{i\in I^2\cap\bar{I}}(m_i-|\mu(i)|)$.

Let $\bar{B} \subseteq \bigcup_{i\in I^2\setminus\bar{I}}(\mu(i)\setminus A(i))$ be such that $|\bar{B}| = \sum_{i\in I^2\cap\bar{I}}(m_i-|\mu(i)|)$; and $B \equiv (\bigcup_{i\in I^2\setminus\bar{I}}(\mu(i)\setminus A(i)))\setminus\bar{B}$.

Just let each agent in $\bar{B}$ be reassigned to an institution i in $I^2\cap\bar{I}$ in a way that makes each $i\in I^2\cap\bar{I}$ be assigned exactly m_i agents; and let it ν^1.

Case 2-1: $|B|\le e$.

Let each agent in B be reassigned to an institution in $I^=$ so that each $i\in I^=$ is assigned at most m_i agents. Let this allocation be denoted by ν^2. From ν^2, perform the operation as we did in Case 1; we obtain a corresponding ν.

Case 2-2: $|B|>e$.

Let $\bar{C}\subseteq B$ be such that $|\bar{C}|=e$; and $C\equiv B\setminus\bar{C}$. Let each agent in $\bar{C}$ be reassigned to an institution i in $I^=$ so that each $i\in I^=$ is assigned at most m_i agents. Let this allocation be denoted by ν^2. Then, from ν^2, let each agent in C be reassigned to an institution in I^1. Perform the operation as in Case 1; we obtain a corresponding ν.

Let $I'\equiv\{i\in I\mid \mu(i)\neq\nu(i)\}\setminus\tilde{I}$ and let $A'\equiv\bigcup_{i\in I'}\nu(i)$. By construction of preferences and Lemma 5, for each $a\in A'$, $\nu(a)\ R_a\ \mu(a)$. By separability of institutions' preferences, for each $i\in I'$, $\nu(i)\ P_i\ \mu(i)$. Hence, $A'\cup I'$ F-blocks μ for R via ν, a contradiction.

Step 3-2: $||w(\mu)|| = ||w|| - e$.

By Step 3-1, every agent $a'\in A^1\cup A^2$ is assigned an institution $i\in I^1\cup I^2$ at μ; that is, $\bigcup_{i\in I^1\cup I^2}\tilde{A}(i) = A^1\cup A^2$. By Assumption (iii), $\bar{I}=\emptyset$ holds. Then, we have

$$\begin{aligned}\sum_{i\in I^1\cup I^2} w_i(\mu) &= \sum_{i\in I^1\cup I^2}(|A(i)|+|\tilde{A}(i)|)\\ &= \sum_{i\in I^1\cup I^2}(m_i+|\tilde{A}(i)|)\\ &= \sum_{i\in I^1\cup I^2} m_i + |A^1|+|A^2| \quad \because \bigcup_{i\in I^1\cup I^2}\tilde{A}(i) = A^1\cup A^2\\ &= \sum_{i\in I^1\cup I^2} w_i \quad \because w = w^1\vee w^2. \end{aligned} \tag{1}$$

We claim that for each $i \in I^1 \cup I^2$, $w_i(\mu) = w_i$. Since $\mu \in F$, Assumption (iii) implies that for each $i \in I^1 \cup I^2$, $w_i(\mu) \leq w_i$. Then, if there is $i \in I^1 \cup I^2$ such that $w_i(\mu) \neq w_i$, $\sum_{i \in I^1 \cup I^2} w_i(\mu) < \sum_{i \in I^1 \cup I^2} w_i$. This violates equation (1). Hence, for each $i \in I^1 \cup I^2$, $w_i(\mu) = w_i$. Then,

$$
\begin{aligned}
||w(\mu)|| &= \sum_{i \in I^1 \cup I^2} w_i(\mu) + \sum_{i \in I^=} w_i(\mu) \\
&= \sum_{i \in I^1 \cup I^2} w_i + \sum_{i \in I^=} w_i(\mu) && \because \forall i \in I^1 \cup I^2, w_i(\mu) = w_i \\
&= \sum_{i \in I^1 \cup I^2} w_i + \sum_{i \in I^=} (m_i - e_i) && \because \forall i \in I^=, e_i = m_i - w_i(\mu) \\
&= \sum_{i \in I^1 \cup I^2} w_i + \sum_{i \in I^=} w_i - \sum_{i \in I^=} e_i && \because \forall i \in I^=, w_i^1 = w_i^2 \\
&= ||w|| - e.
\end{aligned}
$$

Step 3-3: Conclusion of Step 3.

First, we claim that $e < \sum_{i \in I^1} (w_i(\mu) - m_i)$. Suppose by contradiction that $\sum_{i \in I^1} (w_i(\mu) - m_i) \leq e$. Then, by a similar argument to one made in Case 2-1 in Step 3-1, there is an admissible allocation η via which a coalition F-blocks μ for R, a contradiction. Hence,

$$\begin{aligned}
||w(\mu)|| &= ||w|| - e && \because \text{Step 3-2} \\
&> ||w|| - \sum_{i\in I^1}(w_i(\mu) - m_i) && \because e < \sum_{i\in I^1}(w_i(\mu) - m_i) \\
&= \sum_{i\in I} w_i - \sum_{i\in I^1}(w_i(\mu) - m_i) \\
&= \sum_{i\in I^2} w_i + \sum_{i\in I^=} w_i + \sum_{i\in I^1} m_i && \because \forall i \in I^1, w_i(\mu) = w_i \\
&= \sum_{i\in I^2} w_i + \sum_{i\in I^=} m_i + \sum_{i\in I^1} m_i && \because \forall i \in I^=, w_i^1 = w_i^2 \\
&= \sum_{i\in I^2}(m_i + |\tilde{A}(i)|) + \sum_{i\in I^=} m_i + \sum_{i\in I^1} m_i && \because \forall i \in I^2, w_i(\mu) = w_i \\
&= \sum_{i\in I^2} m_i + |A^1| + \sum_{i\in I^=} m_i + \sum_{i\in I^1} m_i \\
&= \sum_{i\in I} m_i + |A^1| \\
&= ||w^1||.
\end{aligned}$$

This contradicts to the assumption that $||w(\mu)|| \leq ||w^1||$. In conclusion, $w(\mu) \in W(w^1, w^2)$.
(End of Step 3.)

The conclusion of Step 3 violates Assumption (iv). □

Proof of Theorem 2

If $\emptyset \notin F$, then for a problem, no allocation meets the outside option lower bound for the problem (Observation 1).

To the contrary, suppose that $\emptyset \in F$. Let $R \in \mathcal{R}$. To prove existence, we construct a fairness-guaranteed stable allocation step by step. We say that allocation $\mu \in F$ is *fair for R* if it meets the outside option lower bound and is free of justified envy for R.

Step 0: Find a fair allocation for R.
Note that $\emptyset$ is fair for R. Hence, there is a fair allocation for R. Let μ^0 be a fair allocation for R.

Step $t(\geq 1)$: Eliminate waste.

Let μ^{t-1} be a fair allocation for R, which is obtained in Step $t-1$ of the proof. If μ^{t-1} is fairness-constrained non-wasteful, then $\mu^{t-1} \in FS(R)$. Thus, suppose that μ^{t-1} is fairness-constrained wasteful. Let $A' \subseteq A$ and $(i_a)_{a \in A'} \in I^{A'}$ be such that

(i) for each $a \in A'$, $i_a \ P_a \ \mu^{t-1}(a)$ and $a \ P_{i_a} \ \emptyset$,

(ii) for each $a \in A'$ and each $b \in A \backslash \{c \in A' \mid i_c = i_a\}$,

$$[b \in A' \text{ and } i_a \ P_b \ i_b] \implies a \ P_{i_a} \ b$$
$$[b \notin A' \text{ and } i_a \ P_b \ \mu^{t-1}(b)] \implies a \ P_{i_a} \ b,$$

(iii) the following allocation ν is admissible: for each $b \in A$,

$$\nu(b) = \begin{cases} i_b & \text{if } b \in A' \\ \mu^{t-1}(b) & \text{otherwise} \end{cases} .$$

Let $I' \equiv \bigcup_{a \in A'} \{i_a\}$. Let $\mu^t \equiv \nu$ be the allocation defined in the above. First, we claim that μ^t meets the outside option lower bound for R. Let $a \in A'$. By (i), $\mu^t(a) \ P_a \ \mu^{t-1}(a)$. Since μ^{t-1} meets the outside option lower bound for R, $\mu^{t-1}(a) \ R_a \ \varnothing$. By transitivity of preferences, $\mu^t(a) \ R_a \ \varnothing$. Let $a \in A \backslash A'$. By (iii), $\mu^t(a) = \mu^{t-1}(a)$, so that $\mu^t(a) \ R_a \ \varnothing$.

Let $i \in I'$. By (i), for each $a \in \mu^t(i) \backslash \mu^{t-1}(i)$, $a \ P_i \ \emptyset$. Since μ^{t-1} meets the outside option lower bound for R, for each $a \in \mu^t(i)$, $a \ R_i \ \emptyset$. Let $i \in I \backslash I'$. Then $\mu^t(i) \subseteq \mu^{t-1}(i)$. Since μ^{t-1} meets the outside option lower bound for R, for each $a \in \mu^t(i)$, $a \ P_i \ \emptyset$.

Second, we claim that μ^t is free of justified envy for R. Suppose by contradiction that there is a pair $(a, i) \in A \times I$ such that $i \ P_a \ \mu^t(a)$ and there is $a' \in \mu^t(i)$ such that $a \ P_i \ a'$.

We claim that $i \ P_a \ \mu^{t-1}(a)$. Suppose that $a \in A'$. By (i), $\mu^t(a) \ P_a \ \mu^{t-1}(a)$. By transitivity of preferences, since $i \ P_a \ \mu^t(a)$, $i \ P_a \ \mu^{t-1}(a)$. Suppose that $a \in A \backslash A'$. By (iii), $\mu^t(a) = \mu^{t-1}(a)$, so that $i \ P_a \ \mu^{t-1}(a)$.

Then, since μ^{t-1} is free of justified envy for R, for each $a'' \in \mu^{t-1}(i)$, $a'' \ P_i \ a$. If $i \notin I'$, then since $\mu^t(i) = \mu^{t-1}(i) \backslash A'$, $\mu^t(i) \subseteq \mu^{t-1}(i)$. This violates the supposition that there is $a' \in \mu^t(i)$ such that $a \ P_i \ a'$. Suppose that $i \in I'$. If $a' \in \mu^{t-1}(i)$, then since μ^{t-1} is free of justified envy for R, $a' \ P_i \ a$, a contradiction. Suppose that $a' \notin \mu^{t-1}(i)$. Then, $\mu^t(a') = i \neq \mu^{t-1}(a')$, so that by (iii), $a' \in A'$. Since $a \in A \setminus \{c \in A' \mid i_c = i\}$ and $i \ P_a \ \mu^t(a)$, by (ii), $a' \ P_i \ a$ must hold regardless of whether $a \in A'$ or not. This is a contradiction to the fact that μ^t is not free of justified envy for R.

By construction of μ^t, for each $a \in A$, $\mu^t(a)\ R_a\ \mu^{t-1}(a)$. Hence, μ^t Pareto dominates for the agents μ^{t-1}.

Since the set of admissible allocations is finite, so that it is bounded above according to the agents' preferences, there is $k \in \mathbb{N}_+$ such that $\mu^k \in FS(R)$. □

Proof of Proposition 3

Let $R \in \mathcal{R}$ and $\mu \in FS(R)$. Suppose by contradiction that $\mu \notin EE(R)$. Since μ meets the outside option lower bound and is free of justified envy for R, there is $\nu \in E(R)$ such that ν Pareto dominates μ for R. Let $A' \equiv \{a \in A \mid \nu(a) \neq \mu(a)\}$ and for each $a \in A'$, let $i_a \equiv \nu(a)$.

We claim that the pair consisting of A' and $(i_a)_{a\in A'}$ satisfies (i)–(iii) in the definition of fairness-constrained non-wastefulness. Since ν Pareto dominates μ, for each $a \in A'$, $i_a = \nu(a)\ P_a\ \mu(a)$. Since ν meets the outside option lower bound for R, for each $a \in A'$, $a\ P_{i_a}\ \emptyset$. Thus, the pair consisting of A' and $(i_a)_{a\in A'}$ satisfies (i). Moreover, for each $a \in A \backslash A'$, $\nu(a) = \mu(a)$. Hence, the pair also satisfies (iii).

Lastly, we check that the pair also satisfies (ii). Let $a \in A'$ and $b \in A \setminus \{c \in A' \mid i_c = i_a\}$. Suppose that $b \in A'$ and $i_a\ P_b\ i_b$. Since $i_b = \nu(b)$, $i_a = \nu(a)$, and ν is free of justified envy for R, $a\ P_{i_a}\ b$ holds. Suppose that $b \notin A'$ and $i_a\ P_b\ \mu(b)$. Since $\mu(b) = \nu(b)$, $i_a = \nu(a)$, and ν is free of justified envy for R, $a\ P_{i_a}\ b$ holds. Therefore, the pair consisting of A' and $(i_a)_{a\in A'}$ satisfies (ii).

In conclusion, μ is fairness-constrained wasteful for R, so that $\mu \notin FS(R)$, a contradiction. □

Proof of Proposition 4

Let $F, F' \subseteq M$ be a pair of problems with $F \subseteq F'$. Let a pair $(\mu, \nu) \in FS(F) \times FS(F')$. Then, by Proposition 3, $(\mu, \nu) \in EE(F) \times EE(F')$. Since $EE(F) \subseteq F \subseteq F'$, $\mu \in F'$. Since $\nu \in EE(F')$, μ does not Pareto dominate ν; otherwise $\nu \notin EE(F')$. □

Proof of Theorem 3

Let $F \subseteq M$ be a weakly number-based admissible set. Then, there is $w \in \mathbb{Z}_+^I$ such that for each $\mu \in M$ with $w(\mu) = w$, $\mu \in F$. Let $R \in \overline{\mathcal{R}}$. For each $i \in I$, set w_i as a capacity of institution i and then apply the DA algorithm. Since the preference profile is all-acceptable, DA produces an allocation, denoted by μ^0, whose distribution is w. Since DA is free of justified envy, μ^0 is free of justified envy for R. Moreover, since the preference profile is all-acceptable, μ^0 meets the outside option lower bound for R. Then, by the same argument in the proof of Theorem 2, there is $\mu^* \in F$ such that $\mu^* \in FS(R)$. □

B Examples

Example 3. *(ES may be empty-valued.)*
Let $A = \{a, b\}$ and $I = \{i, j\}$. Let $R \in \mathcal{R}$ be the following:

R_a	R_b	R_i	R_j
i	j	$\{b\}$	$\{a\}$
j	i	$\{a\}$	$\{b\}$
$\varnothing$	$\varnothing$	$\emptyset$	$\emptyset$

Let $F \subseteq M$ be number-based and let f be the corresponding indicator function such that for each $w \in \mathbb{Z}_+^{|I|}$, $f(w) = 1$ if and only if $||w|| \leq 1$.

We claim that any allocation that meets the outside option lower bound for R has an F-blocking coalition for R via an admissible allocation. Let μ be an allocation that meets the outside option lower bound for R. Note that, according to f, at most one agent can be matched with an institution at μ. Further, since $(\{a\}, \{i\})$ F-blocks $\varnothing$ via an admissible allocation at which agent a is assigned institution i and no other agent is assigned any institution. Hence, at μ, one and only one agent is assigned an institution.

By symmetry of the preference profile, without loss of generality, we check the case in which agent a is matched with an institution. If agent a is matched with institution i at μ, then $(\{b\}, \{i\})$ F-blocks μ via an admissible allocation at which agent b is assigned institution i and no other agent is assigned any institution. Moreover, if agent a is matched with institution j at μ, then $(\{a\}, \{i\})$ F-blocks μ via an admissible allocation in which agent a is assigned institution i and no other agent is assigned any institution. Therefore, no efficient-guaranteed stable allocation exists for the problem. ♣

Example 4. *(Outside option lower bound and non-existence of F-blocking coalition are independent.)*
Let $A = \{a, b, c\}$ and $I = \{i, j\}$. Let $R \in \mathcal{R}$ be the following:

R_a	R_b	R_c
i	j	$\varnothing$
$\varnothing$	$\varnothing$	

R_i	R_j
$\{a,c\}$	$\{b\}$
$\{a\}$	$\emptyset$
$\{c\}$	
$\emptyset$	

Let $F \subseteq M$ be number-based and let f be the corresponding indicator function such that for each $w \in \mathbb{Z}_+^{|I|}$, $f(w) = 1$ if and only if $w \in \{(0,0),(0,1),(2,1)\}$.

Consider the following two admissible allocations μ and μ':

$$\mu = \begin{pmatrix} i & j \\ a,c & b \end{pmatrix} \quad \text{and} \quad \mu' = \begin{pmatrix} i & j \\ \emptyset & b \end{pmatrix}.$$

Since $\varnothing \; P_c \; \mu(c) = i$, μ does not meet the outside option lower bound for R. We claim that no coalition F-blocks μ for R. Suppose by contradiction that a coalition $A' \cup I' \subseteq A \cup I$ F-blocks μ for R via an admissible allocation ν. It is easy to verify that agent c is the only agent who can be potentially better off at μ. Then, $c \in A'$ and thus $\nu(c) = \varnothing$. Since $\nu \in F$, $w(\nu) \in \{(0,0),(0,1)\}$. In either case, $\nu(a) = \varnothing$, implying $\mu(a) = i \; P_a \; \varnothing = \nu(a)$. By the first requirement in the definition of an F-blocking coalition, $a \notin A'$. Then, by the second and third requirements in the definition of an F-blocking coalition, $\mu(a) = i \in I'$. However, $w(\nu) \in \{(0,0),(0,1)\}$ implies $\nu(i) = \emptyset$, which violates the first requirement of F-blocking coalitions. Therefore, μ has no F-blocking coalition.

In contrast, $\boldsymbol{\varnothing}$ meets the outside option lower bound for R but $(\{b\},\{j\})$ F-blocks $\boldsymbol{\varnothing}$ for R via μ'. ♣

C Relation to the literature

In Appendix C, we relate our paper to the literature. In Appendix C.1, we formally introduce "general upper-bound" class of admissible sets. Then, we compare our solutions to solutions proposed in the literature. In Appendix C.2, we formally define "institution-by-institution" class of admissible sets. Then, we compare our solutions to solutions proposed in the literature.

C.1 Number-based and general upper-bound class

First, we formally define *general upper-bound class* denoted by $\overline{\mathcal{F}}$.[26] To do this, first, we define a suballocation of an allocation. For each pair of allocations $\mu, \nu \in M$, ν is a *suballocation of* μ if for each $a \in A$, $\nu(a) \in \{\varnothing, \mu(a)\}$. For each $\mu \in M$, let $SB(\mu)$ be the set of all suballocations of μ. A general upper-bound class is, formally,

$$\overline{\mathcal{F}} \equiv \{F \subseteq M \mid \text{for each pair } \mu, \nu \in M, \text{ if } \nu \in SB(\mu) \text{ and } \mu \in F, \text{ then } \nu \in F\}.$$

Namely, constraints that belong to the class allow admissible allocations to remain admissible when some agents unmatch their assignment.

Kamada and Kojima (2017) consider the number-based class that satisfies the general upper-bound, that is, $\overline{\mathcal{F}} \cap \mathcal{F}^{\#}$. Note that for the number-based class, an admissible set $F \subseteq M$ is general upper-bound if and only if for each $w, w' \in \mathbb{Z}_+^{|I|}$, $w' \leq w$ and $f(w) = 1$ imply $f(w') = 1$, where f is the indicator function that corresponds to F.[27]

We introduce three pairwise stability notions proposed in the literature for admissible sets in $\overline{\mathcal{F}} \cap \mathcal{F}^{\#}$. Each of them is a combination of three properties: meeting the outside option lower bound, being freedom from justified envy, and being their own non-wasteful. We begin with a few notations. For each $i \in I$, let $e^i \in \mathbb{Z}_+^{|I|}$ be the i-th unit vector, and let $\mathbf{0} \in \mathbb{Z}_+^{|I|}$ be the zero-vector. In addition, let $e^{\varnothing} \equiv \mathbf{0}$. First, we introduce three non-wastefulness notions:

An allocation $\mu \in F$ is *KK non-wasteful for* $R \in \mathcal{R}$ if no agent-institution pair $(a, i) \in A \times I$ satisfies

(i) $i \; P_a \; \mu(a)$ and $a \; P_i \; \emptyset$, and

(ii) $f(w(\mu) + e^i - e^{\mu(a)}) = 1$.

An allocation $\mu \in F$ is *KK strongly non-wasteful for* $R \in \mathcal{R}$ if no agent-institution pair $(a, i) \in A \times I$ satisfies

(i) $i \; P_a \; \mu(a)$ and $a \; P_i \; \emptyset$, and

(ii) $f(w(\mu) + e^i) = 1$.

[26] The term is introduced in Kamada and Kojima (2023) for slightly different meanings.

[27] The literature has analyzed the general upper-bound class as a whole and includes research on efficient allocations (Imamura and Kawase, 2022). There is another subclass of constraints, called the matroid class. It is a subclass of $\overline{\mathcal{F}}$ and has some intersection with both $\mathcal{F}^{\#}$ and $\mathcal{F}^{\prod}$.

An allocation $\mu \in F$ is *ABB non-wasteful for* $R \in \mathcal{R}$ if no agent-institution pair $(a, i) \in A \times I$ satisfies

(i) $i\ P_a\ \mu(a)$ and $a\ P_i\ \emptyset$,

(ii) for each $c \in \{b \in A \mid i\ P_b\ \mu(b) \text{ and } b\ P_i\ a\}$, $f(w(\mu) + e^i - e^{\mu(c)}) = 1$, and

(iii) $f(w(\mu) + e^i - e^{\mu(a)}) = 1$.

Then, we define three pairwise stability notions for each non-wastefulness notion: An allocation $\mu \in F$ is *KK pairwise (KK pairwise weakly; ABB pairwise, resp.) stable for* $R \in \mathcal{R}$ if it meets the outside option lower bound, is free of justified envy, and KK non-wasteful (KK strongly non-wasteful; ABB non-wasteful, resp.) for R.[28]

We define four correspondences. Let S^{KK} be the correspondence that associates each problem with the set of KK-pairwise stable allocations for the problem. Similarly, WS^{KK} and S^{ABB} associate each problem with the set of KK pairwise weakly stable allocations and ABB pairwise stable allocations, respectively. An allocation $\mu \in F$ is *agent-optimal KK pairwise weakly stable for* $R \in \mathcal{R}$ if it is in $WS^{KK}(R)$ and no allocation in $WS^{KK}(R)$ dominates μ for the agents for R. Let $\overline{WS}^{KK}$ be the correspondence that associates each problem with the set of all agent-optimal KK pairwise weakly stable allocations for the problem.

We have the following relations between those solutions and solutions presented in our paper:

Proposition 5. *Let $F \in \overline{\mathcal{F}} \cap \mathcal{F}^{\#}$. We have the following:*

(1) For each $R \in \mathcal{R}$,

$$FS(R) = \overline{WS}^{KK}(R) \subseteq PFS(R) = S^{ABB}(R) \subseteq WS^{KK}(R).$$

(2) For each $R \in \mathcal{R}$,

$$ES(R) \subseteq S^{KK}(R) \subseteq PFS(R).$$

(3) There is $F \in \overline{\mathcal{F}} \cap \mathcal{F}^{\#}$ and $R \in \mathcal{R}$ such that

[28] KK pairwise stability is originally called strong stability in Kamada and Kojima (2017). However, in the current paper, in order to distinguish "pairwise" notions and "group-wise" notions, we gave it a different name. ABB pairwise stability is originally called cut-off stability in Aziz et al. (2022).

$$S^{KK}(R)\backslash FS(R) \neq \emptyset \text{ and } FS(R)\backslash S^{KK}(R) \neq \emptyset.$$

Proof of Proposition 5. Let $F \in \overline{\mathcal{F}} \cap \mathcal{F}^{\#}$ and f be the corresponding indicator function.

(1) Let $R \in \mathcal{R}$.

$\underline{PFS(R) \subseteq S^{ABB}(R)}$

Let $\mu \in PFS(R)$. Suppose that $\mu \notin S^{ABB}(R)$. Since μ meets the outside option lower bound and is free of justified envy for R, there exists an agent-institution pair $(a, i) \in A \times I$ that satisfies the three requirements in the definition of ABB non-wastefulness. We claim that μ is pairwise fairness-constrained wasteful for R.

First, let $A' \equiv \{b \in A \mid i \ P_b \ \mu(b)\}$ and let $a' \in A'$ be the agent such that for each $c \in A'$, $a' \ R_i \ c$. We check that $f(w(\mu) + e^i - e^{\mu(a')}) = 1$. If $a' = a$ holds, then by (iii) in the definition of ABB non-wastefulness, $f(w(\mu) + e^i - e^{\mu(a')}) = 1$. If $a' \neq a$ holds, then by $a' \ P_i \ a$ and (ii) in the definition of ABB non-wastefulness, $f(w(\mu) + e^i - e^{\mu(a')}) = 1$. Let $\nu \in M$ be the following: for each $b \in A$,

$$\nu(b) = \begin{cases} i & \text{if } b = a' \\ \mu(b) & \text{otherwise.} \end{cases}$$

It is easy to check that $w(\nu) = w(\mu) + e^i - e^{\mu(a')}$. By $f(w(\mu) + e^i - e^{\mu(a')}) = 1$, $f(w(\nu)) = 1$, so that ν is admissible. Hence, (iii) in the definition of pairwise fairness-constrained non-wastefulness holds.

Second, to check (ii) in the definition of pairwise fairness-constrained non-wastefulness, let $b \in A \setminus \{a'\}$ with $i \ P_b \ \mu(b)$. Then, by $b \in A'$ and the definition of a', $a' \ P_i \ b$. So, (ii) in the definition of pairwise fairness-constrained non-wastefulness holds.

Lastly, note that by $a' \in A'$, $i \ P_{a'} \ \mu(a')$, and by transitivity of P_i and the definition of a', $a' \ R_i \ a \ P_i \ \emptyset$. Hence, (i) in the definition of pairwise fairness-constrained non-wastefulness holds.

Therefore, due to (a', i), μ is not pairwise fairness-constrained wasteful, a contradiction to $\mu \in PFS(R)$.

$\underline{S^{ABB}(R) \subseteq PFS(R)}$

Let $\mu \in S^{ABB}(R)$. Suppose that $\mu \notin PFS(R)$. Since μ meets the outside option lower bound and is free of justified envy for R, there exists an agent-institution pair $(a, i) \in A \times I$ that satisfies the three requirements in the definition of pairwise fairness-constrained non-wastefulness. Let $\nu \in F$ be the admissible allocation described by (iii) in pairwise fairness-constrained non-wastefulness. We claim that μ is ABB wasteful for R.

First, it is easy to check that $w(\mu) + e^i - e^{\mu(a)} = w(\nu)$. Since ν is admissible—that is, $f(w(\nu)) = 1$—we have $f(w(\mu) + e^i - e^{\mu(a)}) = 1$. So, (iii) in the definition of ABB non-wastefulness holds.

Second, to check (ii) in the definition of ABB non-wastefulness, let $b \in A \setminus a$ with $i \; P_b \; \mu(b)$. By (ii) in the definition of pairwise fairness-constrained non-wastefulness, $a \; P_i \; b$. Hence, $\{c \in A \mid i \; P_c \; \mu(c) \text{ and } c \; P_i \; a\} = \emptyset$. So, (ii) in the definition of ABB non-wastefulness trivially holds.

Lastly, note that by the definition of pairwise fairness-constrained non-wastefulness, $i \; P_a \; \mu(a)$ and $a \; P_i \; \emptyset$. Thus, (i) in the definition of ABB non-wastefulness trivially holds.

Therefore, due to (a, i), μ is ABB wasteful, a contradiction to $\mu \in S^{ABB}(R)$.

$\underline{S^{ABB}(R) \subseteq WS^{KK}(R)}$

This relation has been shown in Proposition 4 in Aziz et al. (2022).

$\underline{FS(R) \subseteq PFS(R)}$:

By definition of pairwise fairness-constrained non-wastefulness and fairness-constrained non-wastefulness, we are done.

$\underline{FS(R) \subseteq \overline{WS}^{KK}(R)}$

Let $\mu \in FS(R)$. Suppose that $\mu \notin \overline{WS}^{KK}(R)$. Then, $(*)$ μ is not KK pairwise weakly stable for R, or $(**)$ μ is KK pairwise weakly stable for R; and there is $\nu \in WS^{KK}(R)$ that Pareto dominates for the agents μ for R.

Suppose that $(*)$ holds. For the number-based and general upper-bound class, we understand that any pairwise fairness-guaranteed stable allocation is KK pairwise weakly stable. Hence, μ is not pairwise fairness-guaranteed stable for R, a contradiction to $\mu \in FS(R)$.

Suppose that $(**)$ holds. Then, there exists $\nu \in WS^{KK}(R)$ such that

$$\forall a \in A, \; \nu(a) \; R_a \; \mu(a),$$
$$\exists a \in A, \; \nu(a) \; P_a \; \mu(a).$$

Let $A' \equiv \{a \in A \mid \nu(a) \; P_a \; \mu(a)\}$ and for each $a \in A'$, let $i_a \equiv \nu(a)$. Note that since μ meets the outside option lower bound for R, $i_a \in I$. Then, (i) for each $a \in A'$, $i_a = \nu(a) \; P_a \; \mu(a)$ and $a \; P_{i_a} \; \emptyset$, where the latter holds by being outside option lower bound of ν for R. Moreover, since ν is free of justified envy for R, (ii) for each $a \in A'$ and $b \in A \setminus \{c \in A' | i_c = i_a\}$, if $b \in A'$ and $i_a \; P_b \; \nu(b) = i_b$, then $a \; P_{i_a} \; b$ holds; if $b \notin A'$ and $i_a \; P_b \; \nu(b) = \mu(b)$, then $a \; P_{i_a} \; b$ holds. In addition, it is easy to check that (iii) for each $b \in A$, if $b \in A'$, then $\nu(b) = i_b$; if $b \notin A'$, then $\nu(b) = \mu(b)$. Hence, μ satisfies (i)–(iii) in the definition of fairness-constrained non-wastefulness, a contradiction to $\mu \in FS(R)$.

$\overline{WS}^{KK}(R) \subseteq FS(R)$

Let $\mu \in \overline{WS}^{KK}(R)$. Suppose $\mu \notin FS(R)$. Since μ is admissible, free of justified envy, and meets the outside option lower bound for R, there exist a set of agents $A' \subseteq A$ and a sequence of institutions $(i_a)_{a \in A'} \in I^{A'}$ satisfying (i)–(iii) in fairness-constrained non-wastefulness. Let $\nu \in F$ be the admissible allocation described by (iii) in fairness-constrained non-wastefulness. Then, we understand that

$$\forall a \in A \setminus A', \; \nu(a) = \mu(a) \; R_a \; \mu(a),$$
$$\forall a \in A', \; \nu(a) = i_a \; P_a \; \mu(a).$$

Hence, ν Pareto dominates for the agents μ for R. Since μ is an agent-optimal KK pairwise weakly stable for R, ν is not KK pairwise weakly stable for R. Note that since μ meets the outside option lower bound and is free of justified envy for R, (ii) and (iii) in fairness-constrained non-wastefulness imply that ν also meets the outside option lower bound and is free of justified envy for R. Thus, there is a pair $(a, i_1) \in A \times I$ such that $i_1 \; P_a \; \nu(a)$, $a \; P_{i_1} \; \emptyset$, and $f(w(\nu) + e^{i_1}) = 1$.

Let $a_1 \in A$ be the agent such that for each $a' \in A$ with $i_1 \; P_{a'} \; \nu(a')$ and $a' \; P_{i_1} \; \emptyset$, $a_1 \; R_{i_1} \; a'$. Consider the following allocation: for each $a' \in A$,

$$\nu^1(a') = \begin{cases} i_1 & \text{if } a' = a_1 \\ \nu(a') & \text{otherwise.} \end{cases}$$

That is, the allocation ν^1 just assigns agent a_1 with institution i_1, while assigning anyone else with the same institution (possibly $\varnothing$) at ν. Since $w(\nu) + e^{i_1} \geq w(\nu^1)$ and $f(w(\nu) + e^{i_1}) = 1$, general upper-bound implies $f(w(\nu^1)) = 1$; that is, ν^1 is admissible. It is easy to check that ν^1 meets the outside option lower bound for R. Moreover, by the definition of a_1, ν^1 is free of justified envy for R. Since ν^1 Pareto dominates for the agents ν for R, ν^1 also Pareto dominates for the agents μ for R Thus, ν^1 cannot be KK pairwise weakly stable for R. Since ν^1 is admissible, free of justified envy, and meets the outside option lower bound for R, there is a pair $(a', i_2) \in A \times I$ such that $i_2 \ P_{a'} \ \nu^1(a'), a' \ P_{i_2} \ \emptyset$, and $f(w(\nu^1) + e^{i_2}) = 1$.

Let $a_2 \in A$ be the agent such that for each $a'' \in A$ with $i_2 \ P_{a''} \ \nu^1(a'')$ and $a'' \ P_{i_2} \ \emptyset$, $a_2 \ R_{i_2} \ a''$. Consider the following allocation: for each $a'' \in A$,

$$\nu^2(a'') = \begin{cases} i_2 & \text{if } a'' = a_2 \\ \nu^1(a'') & \text{otherwise.} \end{cases}$$

That is, the allocation ν^2 just assigns agent a_2 with institution i_2, while assigning anyone else with the same institution (possibly $\varnothing$) at ν^1. Since $w(\nu^1)+e^{i_2} \geq w(\nu^2)$ and $f(w(\nu^1)+e^{i_2}) = 1$, general upper-bound implies $f(w(\nu^2)) = 1$; that is, ν^1 is admissible. It is easy to check that ν^2 meets the outside option lower bound for R. Moreover, by the definition of a_2, ν^2 is free of justified envy for R. Since ν^2 Pareto dominates ν^1 in terms of agents' welfare, ν^2 also Pareto dominates μ in terms of agents' welfare. Thus, ν^2 cannot be KK pairwise weakly stable for R. Since ν^2 is admissible, free of justified envy, and meets the outside option lower bound for R, there is a pair $(a'', i_3) \in A \times I$ such that $i_3 \ P_{a''} \ \nu^2(a''), a'' \ P_{i_3} \ \emptyset$, and $f(w(\nu^2) + e^{i_3}) = 1$.

By repeating the same procedure, due to finiteness of allocations, we obtain admissible allocation ν^t at which any agent is assigned with the most preferred institution. The allocation ν^t is obviously KK pairwise weakly stable for R and Pareto dominates for the agents μ for R, a contradiction to $\mu \in \overline{WS}^{KK}(R)$.

(2) Let $R \in \mathcal{R}$.

<u>$ES(R) \subseteq S^{KK}(R)$</u>

Suppose that there exists an allocation $\mu \in ES(R) \setminus S^{KK}(R)$. Since μ is not KK pairwise stable but efficiency-guaranteed stable for R, $(*)$ μ is not free of justified envy for R or $(**)$ μ is KK wasteful for R.

Suppose $(*)$. Then, there exists an agent-institution pair $(a, i) \in A \times I$ such that $i \ P_a \ \mu(a)$ and there is $a' \in \mu(i)$ with $a \ P_i \ a'$. Consider a set of agents $A' \equiv (\mu(i) \setminus \{a'\}) \cup \{a\}$, a set of institutions $I' \equiv \{i\}$, and an allocation $\nu \in M$ such that $\nu(a) = i$, $\nu(a') = \varnothing$, and for all $a'' \in A \setminus \{a, a'\}$, $\nu(a'') = \mu(a'')$. Since $w(\nu) = w(\mu) - e^{\mu(a)}$ and $f(w(\mu)) = 1$, the general upper-bound implies $f(w(\nu)) = 1$; that is, ν is admissible. Then, it is easy to check that $C = A' \cup I'$ F-blocks μ for R via the admissible allocation ν. Therefore, $\mu \notin ES(R)$, a contradiction.

Suppose that (**) holds. Then, there exists an agent-institution pair $(a, i) \in A \times I$ such that $i \; P_a \; \mu(a)$, $a \; P_i \; \emptyset$, and $f(w(\mu) + e^i - e^{\mu(a)}) = 1$. Consider a set of agents $A' \equiv \mu(i) \cup \{a\}$, a set of institutions $I' \equiv \{i\}$, and an allocation $\nu \in M$ such that $\nu(a) = i$ and for all $a' \in A \setminus \{a\}$, $\nu(a') = \mu(a')$. Note that $w(\nu) = w(\mu) + e^i - e^{\mu(a)}$. Hence, by the supposition, ν is admissible. Then, it is easy to check that $C = A' \cup I'$ F-blocks μ for R via an admissible allocation ν. Therefore, $\mu \notin ES(R)$, a contradiction.

<u>$S^{KK}(R) \subseteq PFS(R)$</u>

By (1) in Proposition 5, we understand $PFS(R) = S^{ABB}(R)$. Also, by Proposition 4 in Aziz et al. (2022), $S^{KK}(R) \subseteq S^{ABB}(R)$. Hence, we are done.

(3) We prove the statement by means of an example. Let $A = \{a, b, c\}$ and $I = \{i, j, k\}$. Let $R \in \mathcal{R}$ be the following:

R_a	R_b	R_c
i	j	k
j	i	$\varnothing$
$\varnothing$	$\varnothing$	

R_i	R_j	R_k
$\{a, b\}$	$\{a, b\}$	{c}
$\{b\}$	$\{a\}$	$\emptyset$
$\{a\}$	$\{b\}$	
$\emptyset$	$\emptyset$	

Let $F \in \overline{\mathcal{F}} \cap \mathcal{F}^{\#}$ and let f be the corresponding indicator function such that for each $w \in \mathbb{Z}_+^{|I|}$, $f(w) = 1$ if and only if for some $w' \in \{(1, 0, 1), (1, 1, 0), (0, 1, 1)\}$, $w \leq w'$.

Consider two allocations, μ and ν:

$$\mu = \begin{pmatrix} i & j & k \\ b & \emptyset & c \end{pmatrix} \text{ and } \nu = \begin{pmatrix} i & j & k \\ b & a & \emptyset \end{pmatrix}.$$

Then, it is easy to check that $\mu \in FS(R) \setminus S^{KK}(R)$ and $\nu \in S^{KK}(R) \setminus FS(R)$. □

C.2 Traditional admissible sets

First, we formally define *institution-by-institution class* denoted by $\mathcal{F}^{\Pi}$. It is defined as follows:

$$\mathcal{F}^{\Pi} \equiv \left\{F \subseteq M \mid \text{there is } (F_i)_{i\in I} \in (2^{2^A})^I \text{ such that } F = \{\mu \in M \mid \text{for each } i \in I, \mu(i) \in F_i\}\right\}.$$

Note that for each $F \in \overline{\mathcal{F}} \cap \mathcal{F}^{\#} \cap \mathcal{F}^{\Pi}$, there is $c \in \mathbb{Z}_+^{|I|}$ such that F is a traditional admissible set associated with c.

An allocation $\mu \in F$ is *non-wasteful for* $R \in \mathcal{R}$ if there is no pair of agent $a \in A$ and institution $i \in I$ such that

(i) $i \; P_a \; \mu(a)$ and $a \; P_i \; \emptyset$, and

(ii) $|\mu(i)| < c_i$.

Definition 5. An allocation $\mu \in F$ is *stable for* $R \in \mathcal{R}$ if it meets the outside option lower bound, is free of justified envy, and is non-wasteful for R.

An allocation $\mu \in F$ is *agent-optimal stable for* $R \in \mathcal{R}$ if it is stable and no stable allocation Pareto dominates for the agents it. Let S be the correspondence that associates each problem with the set of stable allocations for the problem. Analogously, let AS be the correspondence that associates each problem with the set of agent-optimal stable allocations for the problem.

Proposition 6. *Let $F \subseteq M$ be a traditional admissible set. For each $R \in \mathcal{R}$,*

$$FS(R) = AS(R) \subseteq ES(R) = PFS(R) = S(R).$$

Proof of Proposition 6. Let $F \subseteq M$ be a traditional admissible set and let $R \in \mathcal{R}$.

$\underline{ES(R) = S(R)}$

It is easy to see that for the traditional $F \subseteq M$, any efficiency-guaranteed stable allocation is in the core, and vice versa. Since the core is equal to the set of stable allocations, we have $ES(R) = S(R)$.

$\underline{PFS(R) \subseteq S(R)}$

It suffices to show that pairwise fairness-constrained non-wastefulness implies non-wastefulness. Let $\mu \in F$ be an allocation that is wasteful for R. Then, there is a pair of agent $a \in A$ and institution $i \in I$ that satisfy $i \; P_a \; \mu(a)$, $a \; P_i \; \emptyset$, and $|\mu(i)| < c_i$. Consider the following allocation ν: for each $b \in A$,

$$\nu(b) = \begin{cases} i & \text{if } \; b = a \\ \mu(b) & \text{otherwise} \end{cases}.$$

Since $|\mu(i)| < c_i$ and F is traditional, $\nu \in F$; that is, ν is admissible.

Let $A' \equiv \{a' \in A \mid i \; P_{a'} \; \mu(a')\}$ and let $a^* \in A'$ be such that for all $a'' \in A' \setminus \{a^*\}$, $a^* \; P_i \; a''$. Then, due to (a^*, i), μ is pairwise fairness-constrained wasteful for R. Hence, pairwise fairness-constrained non-wastefulness implies non-wastefulness.

<u>$S(R) \subseteq PFS(R)$</u>

It suffices to show that non-wastefulness implies pairwise fairness-constrained non-wastefulness. Let $\mu \in F$ be an allocation that is pairwise fairness-constrained wasteful for R. Then, there is a pair of agent $a \in A$ and institution $i \in I$ that satisfy the three requirements in pairwise fairness-constrained non-wastefulness. Let $\nu \in F$ be the admissible allocation described by (iii) in pairwise fairness-constrained non-wastefulness. Since ν is admissible and F is traditional, $|\nu(i)| \leq c_i$. By the construction of ν and $a \in \nu(i) \setminus \mu(i)$, $|\mu(i)| < c_i$ holds. Then, due to (a, i), μ is wasteful for R. Hence, non-wastefulness implies pairwise fairness-constrained non-wastefulness.

<u>$FS(R) \subseteq PFS(R)$</u>

By definition of pairwise fairness-constrained non-wastefulness and fairness-constrained non-wastefulness, we are done.

<u>$FS(R) = AS(R)$</u>

Since F is traditional, the set of stable allocations corresponds to the set of KK pairwise weakly stable allocations. Hence, $AS(R) = \overline{WS}^{KK}(R)$ holds. By Proposition 5, $FS(R) = \overline{WS}^{KK}(R)$. Since any traditional $F \subseteq M$ falls into the number-based class, we have $FS(R) = AS(R)$. □

References

Akin, S. (2021). Matching with floor constraints. *Theoretical Economics*, 16:911–942.

Aziz, H., Baychkov, A., and Biro, P. (2022). Cutoff stability under distributional constraints with an application to summer internship matching. *Mathematical Programming*.

Cho, S. H., Koshimura, M., Mandal, P., Yahiro, K., and Yokoo, M. (2022). Impossibility of weakly stable and strategy-proof mechanism. *Economics Letters*, 217.

Crawford, V. P. (1991). Comparative statics in matching markets. *Journal of Economic Theory*, 54:389–400.

Delacrétaz, D., Kominers, S. D., and Teytelboym, A. (2023). Matching mechanisms for refugee resettlement. *American Economic Review*, 113(10):2689–2717.

Echenique, F. and Yenmez, B. (2015). How to control controlled school choice. *Amerian Economic Review*, 105:2679–94.

Ehlers, L., Hafalir, I. E., Yenmez, M. B., and Yildirim, M. A. (2014). School choice with controlled choice constraints: Hard bounds versus soft bounds. *Journal of Economic Theory*, 153:648–683.

Erdil, A. and Kumano, T. (2019). Efficiency and stability under substitutable priorities with ties. *Journal of Economic Theory*, 184.

Gale, D. and Shapley, L. S. (1962). College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15.

Hatfield, J. W. and Kojima, F. (2010). Substitutes and stability for matching with contracts. *Journal of Economic Theory*, 145(5):1704–1723.

Hatfield, J. W. and Milgrom, P. R. (2005). Matching with contracts. *American Economic Review*, 95(4):913–935.

Imamura, K. and Kawase, Y. (2022). Efficient matching under general constraints. *Games and Economic Behabior*, Forthoming.

Kamada, Y. and Kojima, F. (2015). Efficient matching under distributional constraints: Theory and applications. *American Economic Review*, 105:67–99.

Kamada, Y. and Kojima, F. (2017). Stability concepts in matching under distributional constraints. *Journal of Economic Theory*, 168:107–142.

Kamada, Y. and Kojima, F. (2018). Stability and strategy-proofness for matching with constraints: A necessary and sufficient condition. *Theoretical Economics*, 13:761–793.

Kamada, Y. and Kojima, F. (2022). Ekkyo matching: How to integrate fragmented matching markets for welfare improvement. *Working paper*.

Kamada, Y. and Kojima, F. (2023). Fair matching under constraints: Theory and applications. *Review of Economic Studies*.

Kominers, S. and Sönmez, T. (2016). Matching with slot-specific priorities: Theory. *Theoretical Economics*, 11:683–710.

Ostrovsky, M. (2008). Stability in supply chain networks. *American Economic Review*, 98(3):897–923.

Root, J. (2019). Stable matching under general constraints. *Working paper*.

Sönmez, T. and Switzer, T. (2013). Matching with (branch-of-choice) contracts at the united states military academy. *Econometrica*, 81(2):451–488.

Sönmez, T. and Yenmez, B. (2022). Affirmative action in india via vertical, horizontal, and overlapping reservations. *Econometrica*, pages 1143–1176.