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A POTENTIAL-THEORETIC APPROACH TO OPTIMAL STOPPING IN A SPECTRALLY LÉVY MODEL

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ABSTRACT. We present a systematic solution method for optimal stopping problem of one-dimensional spectrally negative Lévy processes. Our main tools are based on the potential theory, particularly the Riesz decomposition and the maximum principle. This novel approach allows us to handle a broad class of reward functions. That is, we solve the problem in a general setup without relying on specific form of the reward function. We provide a step-by-step solution procedure, which is applicable to complex solution structures including multiple double-sided continuation regions.

Key words: Optimal stopping; spectrally negative Lévy processes; potential theory; Riesz decomposition; maximum principle.
Mathematics Subject Classification (2020) : Primary: 60G40 Secondary: 60J76

1. INTRODUCTION

1.1. **The problem.** This paper investigates the optimal stopping problem for a one-dimensional spectrally negative Lévy process, a class of real-valued Lévy process with no positive jumps. Let the spectrally negative Lévy process $X = \{X_t; t \geq 0\}$ represent the state variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the set of all possible outcomes, and $\mathbb{P}$ is a probability measure defined on $\mathcal{F}$. We denote by $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ the filtration with respect to which X is adapted, assuming that the usual conditions hold.

For a spectrally negative Lévy process, its Laplace exponent ψ is given by

$$\psi(\theta) = -\gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)} (e^{\theta x} - 1 - \theta x \mathbb{1}_{(-1,0)}(x))\Pi(\mathrm{d}x), \tag{1.1}$$

where $\gamma \in \mathbb{R}$, $\sigma \geq 0$, and Π is a measure concentrated on $(-\infty, 0)$ satisfying $\int_{(-\infty,0)} (1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$. It is well-known that ψ is zero at the origin, convex on $\mathbb{R}_+$ and has a right-continuous inverse:

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}, \quad q \geq 0.$$

Moreover, it is well known that $X_t \to \infty$ as $t \to \infty$ almost surely if and only if $\psi'(0+) > 0$, oscillates if and only if $\psi'(0+) = 0$ and $X_t \to -\infty$ as $t \to \infty$ almost surely if $\psi'(0+) < 0$.

The jumps of the process have a finite mean $\int_{(-\infty,0)} |y|\Pi(\mathrm{d}y) < \infty$ and there is no diffusion component $\sigma = 0$ if and only if the paths have bounded variation. Then we may rewrite (1.1) as

$$\psi(\theta) = \delta\theta + \int_{(-\infty,0)} \left(e^{\theta x} - 1\right)\Pi(\mathrm{d}x),$$

where

$$\delta = -\left(\gamma + \int_{(-1,0)} x\Pi(\mathrm{d}x)\right) \tag{1.2}$$

is the drift coefficient.

Let $\mathcal{L}$ be the infinitesimal generator of X, which is given as

$$\mathcal{L}f(x) = -\gamma f'(x) + \frac{1}{2}\sigma^2 f''(x) + \int_{(-\infty,0)} \left[f(x+y) - f(x) - y\mathbb{1}_{(-1,0)}(y)f'(x)\right] \Pi(\mathrm{d}y). \tag{1.3}$$

Let $q \geq 0$ be a constant and $g(\cdot)$ be a non-negative Borel function such that $\mathbb{E}[e^{-q\tau}g(X_\tau)]$ is well-defined for all $\mathbb{F}$-stopping times τ. We denote by

$$v(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}^x[e^{-q\tau}g(X_\tau)],\ x \in \mathbb{R} \tag{1.4}$$

the value function of the optimal stopping problem with reward function g and discount rate q, where the supremum is taken over the class $\mathcal{T}$ of all $\mathbb{F}$-stopping times.

We define the stopping region Γ and continuation region C as follows:

$$\Gamma = \{x \in \mathbb{R} : v(x) = g(x)\} \quad \text{and} \quad C = \{x \in \mathbb{R} : v(x) > g(x)\}. \tag{1.5}$$

We make the following assumptions. These assumptions are maintained throughout the paper.

Assumption 1. *(i) g' is bounded on $\mathbb{R}$.*

(ii) The Lévy measure Π satisfies

$$\int_{(-\infty,-1)} |x|\Pi(\mathrm{d}x) < \infty.$$

(iii) Π is nonzero measure; that is, there exists a Borel set $A \in (-\infty, 0)$ such that $\Pi(A) > 0$.

(iv) Π does not have atoms.

Assumption (i) and (ii) are adopted to carry out analysis based on the generator $\mathcal{L}$. In fact, (i) and (ii) ensures $\mathcal{L}g$ takes finite values at points that g is differentiable (resp. twice-differentiable) in the case where X has paths of bounded variation (resp. unbounded variation). To see this, we define $I_1(x) := \int_{(-\infty,-1)} [g(x+y) - g(x)]\,\Pi(\mathrm{d}y)$ and $I_2(x) := \int_{(-1,0)} \left[g(x+y) - g(x) - y\mathbb{1}_{(-1,0)}(y)g'(x)\right] \Pi(\mathrm{d}y)$. It is enough to show that I_1 and I_2 is finite due to the fact $\mathcal{L}g(x) = -\gamma g'(x) + (\sigma^2/2)g''(x) + I_1(x) + I_2(x)$. It follows from the mean value theorem that

$$\begin{aligned}
|I_1(x)| &\leq \int_{(-\infty,-1)} |g(x+y) - g(x)|\Pi(\mathrm{d}y) \\
&= \int_{(-\infty,-1)} |g(x+c_y)||y|\Pi(\mathrm{d}y) \quad \Big(\text{for some} \quad c_y \in (-\infty, 0)\Big) \\
&\leq \sup|g'| \int_{(-\infty,-1)} |y|\Pi(\mathrm{d}y) < \infty,
\end{aligned}$$

where the final inequality is obtained from (i) and (ii). As for I_2, it follows from Taylor's theorem that

$$\begin{aligned}
|I_2(x)| &= \Big| \int_{(-1<y<0)} \frac{1}{2} g''(x+c_y)y^2\Pi(\mathrm{d}y)\Big| \quad \Big(\text{for some} \quad c_y \in (-1, 0)\Big) \\
&= \frac{1}{2} \sup_{c\in(-1,0)} |g''(x+c)| \int_{(-1<y<0)} y^2\Pi(\mathrm{d}y) < \infty.
\end{aligned}$$

We conclude that $\mathcal{L}g$ is finite. Note that the assumption (ii) is also imposed in [3], [2], and [6]. The assumption (iii) is introduced to facilitate the application of the maximum principle in Section 2.4. When this assumption fails, X becomes a diffusion process, for which solution methods are already established. Thus, the assumption (iii) is made without loss of generality. The assumption (iv) is adopted to ensure the smoothness of the scale function, which is introduced in Section 2.1.

1.2. **Literature review and research motivation.** Optimal stopping problems for Lévy processes have been extensively studied. However, much of the existing literature has focused on cases where the reward function is explicitly specified. For instance, [25], [1], and [10] investigated the McKean optimal stopping problem (see Section 7.1), while [22] and [26] focused on problems with power-type reward functions. The papers of [16] and [17] consider an optimal alarm problem in the context of capital adequacy management, where the reward function is decreasing and negative in the negative region and identically zero in the positive region.

These works identify the stopping region as a set such that the strategy of stopping (hence accepting reward) upon exiting it would maximize the expected payoff. This approach requires a explicit representation of the expected reward, and thus compels the authors to restrict the form of the reward function for ensuring that the expected payoff is explicitly computable. While this approach is mathematically rigorous and yields novel results, it depends on a delicate guess-and-verify procedure. This procedure tends to be highly problem-specific and hence to be likely to fail to offer general insights into the structure of optimal solutions. In contrast, our work introduces an alternative approach that eliminates the requirement to explicitly specify the reward function. That is, we employ a maximum principle for spectrally negative Lévy processes. This enables us to verify that a candidate solution dominates the reward function without requiring its explicit functional form (see Section 5) and to avoid the guess-and-verify approach. This is particularly helpful in cases such as Section 7.2, where the conventional guess-and-verify method becomes intractable. In our framework, fluctuation identities do not serve as the primary analytical tool, but rather play a secondary role when computing explicit solutions in specific examples.

Our work is also related to [30] and [15], which characterizes more general classes of optimal stopping problems using the Wiener–Hopf factorization and the so-called averaging problem, a term introduced in [30]. What distinguishes our study from these previous works is that we provide a systematic solution method. In [30] and [15], the solution to the optimal stopping problem is characterized via the solution to the averaging problem. However, it is generally difficult to construct such a solution in a tractable or unified manner. In contrast, our approach offers a step-by-step method for solving a broad class of problems (see Section 6). Furthermore, unlike these earlier works that often restrict attention to the one-sided case, our analysis deals with more general stopping regions.

Our method is grounded in a potential-theoretic approach. The use of the potential theory for optimal stopping problems has been explored in some studies. One of the earliest contributions in this direction is [28], who characterized the solution to the optimal stopping problem for one-dimensional diffusion processes using the Martin representation of the value function. More recently, this line of methodology has been extended to multi-dimensional diffusions. For example, [12], and [9] derived integral equations examining the stopping region through the Martin boundary theory.

While those studies focus on representing the value function itself, our work considers an extended Riesz decomposition, allowing us to derive an integral representation not only for the value function but also for general expected reward functions (see Section 3). Thanks to this decomposition we are able to identify necessary conditions that the solution must satisfy even in the presence of negative Lévy jumps (see Section 4).

Additional examples of potential-theoretic approach in the literature include [11], who studied conditions under which a point in the stopping region of optimal stopping problems for diffusion processes with reward function g maximizes g/h for some harmonic function h. Furthermore, [13] extended the results of [30] and [15] to general Hunt processes by using properties of excessive functions.

The rest of the paper is organized as follows. In Section 2, we provide the mathematical preliminaries necessary for this paper. We mainly review the basic properties of spectrally negative Lévy processes, fundamental notion of the potential theory including the Riesz decomposition and the maximum principle, and the smooth Gerber-Shiu function, which gives an explicit representation of harmonic functions. Section 3 constitutes one of the main contributions of this paper, as it establishes a generalized Riesz decomposition for analyzing the properties of expected reward functions. This decomposition plays a key role in deriving a necessary condition for the value function, which is presented in Section 4. We demonstrate that the continuous fit condition (resp. the smooth fit condition) at the left boundary of C — the continuation region — is necessary when X has paths of bounded variation (resp. unbounded variation, respectively), along with the smooth fit condition at the right boundary of C. Section 5 verifies the sufficiency of the necessary conditions derived in Section 4. This section presents another major result of the paper. The proof primarily relies on the maximum principle, which allows for a proof that is independent of the specific form of the reward function. The semi-explicit representation of the smooth Gerber-Shiu function, introduced in [3] (see also (2.7) and (2.8)), plays a crucial role in the verification of the solution via the maximum principle. These representations provide valuable insights into the smoothness and structural properties of the associated harmonic function. In Section 6, we present a systematic procedure for solving general problems, based on the results developed in Section 5. Readers primarily interested in applying the solution of specific optimal stopping problems to practical settings may refer directly to the algorithm presented in this section. Section 7 presents three examples in which we solve the specific problems following the general procedure presented in Section 6.

2. Mathematical tools

2.1. Spectrally negative Lévy process and its scale function. Associated with every spectrally negative Lévy process, there exists a (q-)scale function

$$W^{(q)} : \mathbb{R} \to \mathbb{R}; \quad q \geq 0,$$

that is continuous, strictly increasing on $[0, \infty)$ and 0 on $(-\infty, 0)$. It is uniquely determined by

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) \mathrm{d}x = \frac{1}{\psi(\beta) - q}, \qquad \beta > \Phi(q).$$

For any Borel set A, define the hitting time

$$T_A := \inf\{t \geq 0 : X_t \in A\}.$$

Moreover, for simplicity, we write

$$T_r := T_{(r,\infty)} = \inf\{t \geq 0 : X_t > r\} \quad \text{and} \quad T_\ell^- := T_{(-\infty,\ell)} = \inf\{t \geq 0 : X_t < \ell\}. \tag{2.1}$$

for ℓ and r in $\mathbb{R}$. Then, for fixed $0 < x < a$, we have the following fluctuation identities

$$\mathbb{E}^x \left[e^{-qT_a} 1_{\{T_a < T_0^-, T_a < \infty\}} \right] = \frac{W^{(q)}(x)}{W^{(q)}(a)} \tag{2.2}$$

$$\mathbb{E}^x \left[e^{-qT_0^-} 1_{\{T_a > T_0^-, T_0^- < \infty\}} \right] = Z^{(q)}(x) - Z^{(q)}(a) \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{2.3}$$

where

$$Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(y) \mathrm{d}y, \quad x \in \mathbb{R}.$$

Here we have $Z^{(q)}(x) = 1$ on $(-\infty, 0]$. We also have

$$\mathbb{E}^x \left[e^{-qT_0^-} \right] = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x), \quad x > 0.$$

Fix $q > 0$. The scale function increases exponentially;

$$\lim_{x \to \infty} \frac{W^{(q)}(x)}{e^{\Phi(q)x}} = \frac{1}{\psi'(\Phi(q))} \tag{2.4}$$

Finally, under Assumption 1 (iv), $W^{(q)}$ belongs to $C^1(0, \infty)$ if X has paths of bounded variation, and to $C^2(0, \infty)$ if X has paths of unbounded variation; see [8]. For a comprehensive account of the scale function, we refer the reader to [4, 5, 23, 24]. See also [18, 31] for numerical methods for computing the scale function.

2.2. **The potential theory.** Let $\{H_t\}_{t\geq 0}$ be the Markov kernel of X; for $x \in \mathbb{R}$ and a Borel set A and for a Borel function f,

$$H_t(x, A) = \mathbb{P}^x(X_t \in A) \quad \text{and} \quad H_t f(x) = \mathbb{E}^x[f(X_t)],$$

respectively. Moreover, we extend this notation from deterministic time t to any stopping time τ:

$$H_\tau(x, A) = \mathbb{P}^x(X_\tau \in A).$$

For simplicity, we shall write for hitting times

$$H_{T_A} = H_A.$$

Furthermore, in view of (2.1), we shall use, throughout this paper, the following notation for hitting times of $(-\infty, a)$ and $(-\infty, a) \cup (b, \infty)$:

$$H_{T_a^-} = H_a, \quad \text{and} \quad H_{T_a^- \wedge T_b} = H_{a,b},$$

respectively.

For $q \geq 0$, H_t^q is defined to indicate

$$H_t^q(x, A) = \mathbb{P}^x(X_t \in A; t < \mathbf{e_q}),$$

where $\mathbf{e_q}$ is a random variable independent of X and follows an exponential distribution with rate q.

We define *the resolvent kernel* $\mathbf{G} = \{G_q(x, A)\}_{q\geq 0}$ as

$$G_q(x, A) = \int_0^\infty e^{-qt} H_t(x, A)\mathrm{d}t \quad \text{for } q \geq 0.$$

Similarly, $H_\tau f$, $H_t^q f$, $G_q f$, H_τ^q are defined in the same manner.

A non-negative measurable function u is said to be *q-excessive* if it satisfies the following two conditions:

$$H_t^q u(x) \leq u(x); \quad t \geq 0,\ x \in \mathbb{R},$$
$$\lim_{t\downarrow 0} H_t^q u(x) = u(x); \quad x \in \mathbb{R}.$$

A non-negative measurable function u is said to be *q-superharmonic* (resp. *q-subharmonic*) on open set G if each open subset $A \subset G$ whose closure is compact in G,

$$u \geq H_{A^c}^q u \quad (\text{resp. } u \leq H_{A^c}^q u).$$

A *q-harmonic function* is defined as a function that is both q-superharmonic and q-subharmonic. A function g is said to be superharmonic at a fixed point c if there exists an open neighborhood $U(c)$ of c on which g is superharmonic. The definition of being subharmonic or harmonic at c is given in the same manner. A function is said to be *strictly superharmonic* if it is superharmonic but not harmonic. Similarly, we define a *strictly subharmonic* function as one that is subharmonic but not harmonic. Note that if g is subharmonic at a point c, then for some open neighborhood $U(c)$, we have $g(c) < H_{U(c)^c} g(c) \leq v(c)$; hence, $c \in C$ by the definition of C. This property will be used later when solving the optimal stopping problem.

If u is superharmonic (resp. subharmonic) on G and $\mathcal{L}u$ is defined on G, then $\mathcal{L}u \leq 0$ (resp. $\mathcal{L}u \geq 0$) holds on G. This follows directly from the definition of the generator $\mathcal{L}$. Conversely, if $\mathcal{L}u(x) \leq 0$ (resp. $\mathcal{L}u(x) \geq 0$) for all $x \in G$, then u is superharmonic (resp. subharmonic) on G, as a consequence of Dynkin's formula. For a open set G, $H_{G^c} g$ is harmonic on G. It is followed from $H_{G^c} = H_{A^c} H_{G^c}$ holds for $A \subset G$ (see Theorem 3.4.2 in [14]).

The *co-resolvent kernel* $\hat{\mathbf{G}} = \{\hat{G}_q(x, A)\}_{q\geq 0}$ is defined as

$$\langle f, G_q \rangle = \langle \hat{G}_q, g \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product with respect to Lebesgue measure. It is known that $\hat{\mathbf{G}}$ is the resolvent kernel of $\hat{X} = -X$ ([4]; Section II.1). A q-excessive function for $\hat{\mathbf{G}}$ is called *q-co-excessive*.

Given $q \geq 0$, a jointly measurable function $G_q(x, y)$ is said to be the *q-potential density* if the following conditions are satisfied: (i) $G_q(x, \mathrm{d}y) = G_q(x, y)\mathrm{d}y$; (ii) $\hat{G}_q(y, \mathrm{d}x) = G_q(x, y)\mathrm{d}x$; (iii) $G_q(\cdot, y)$ is q-excessive for each y and $G_q(x, \cdot)$

is q-co-excessive for each x. For a spectrally negative Lévy process, the q-potential density always exists and is written in semi-explicit terms. The q-potential density is given by ([7], Corollary 8.9 in [23])

$$G_q(x,y) = \theta^{(q)}(y-x) = \Phi'(q)e^{-\Phi(q)(y-x)} - W^{(q)}(x-y). \tag{2.5}$$

In case of $q=0$, under the assumption that $\psi(0+)>0$, letting $q \downarrow 0$ in (2.5) yields

$$G(x,y) = \theta^{(0)}(y-x) = \frac{1}{\psi'(0+)} - W(x-y). \tag{2.6}$$

by noting that Φ is a right inverse of ψ. Note that $\theta^{(q)}(\cdot)$ $(q \geq 0)$ is of $C^1(\mathbb{R}/\{0\})$ (resp. $C^2(\mathbb{R}/\{0\})$) when X has paths of bounded variation (resp. unbounded variation). For notational simplicity, we write $\theta^{(0)}(\cdot)$ as $\theta(\cdot)$.

An excessive function u is called a *potential* if it is finite almost everywhere and satisfies $\lim_{s\uparrow\infty} H_{(-s,s)^c}u = 0$ almost everywhere. For each y, $G(\cdot,y)$ is a potential since $\hat{G}$ is the co-resolvent kernel of $\hat{X} = -X$ (Proposition 13.1 in [21]). Conversely, every potential can be represented as a composition of the potential densities $G(\cdot,y)$; that is,

Proposition 2.1. *A function u is a potential if and only if there exists a measure μ such that*

$$u = G\mu := \int_{(-\infty,\infty)} G(\cdot,y)\mu(\mathrm{d}y).$$

Moreover, the following decomposition holds for excessive functions that are finite almost everywhere:

Proposition 2.2 (Riesz decomposition)**.** *Every excessive function that is finite almost everywhere can be uniquely decomposed as*

$$u = G\mu + h,$$

where μ is a measure and h is a harmonic function.

For the proofs of these results, see Proposition 7.6 and Theorem 2 in [21], respectively.

2.3. **The smooth Gerber–Shiu function.** Let g be continuous and *left-differentiable* at every point. For $a \in \mathbb{R}$, let h_a be a harmonic function on (a,∞) satisfying the boundary condition $h_a(x) = g(x)$ for $x \leq a$ (and $h'_a(a+) = g'(a)$ when X has paths of unbounded variation). This function is called the *smooth Gerber–Shiu function*, and it is known that it can be represented in terms of W and Z as follows ([3]):

$$h_a^{(q)}(x) = g(a) + g'(a-)(x-a) - \int_0^{x-a} W^{(q)}(x-a-y)J_a(y)\,\mathrm{d}y, \tag{2.7}$$

$$h_a^{(q)\prime}(x) = g'(a-) - \int_{[0,x)} J_a(x-a-y)W^{(q)}(\mathrm{d}y), \tag{2.8}$$

where J_a is given by

$$J_a(x) = g'(a-)\psi'(0+) - q(g'(a-)x + g(a)) + \int_{(x,\infty)} (g(x+a-z) - g(a) + g'(a-)(z-x))\,\Pi(-\mathrm{d}z).$$

When $q=0$, we simply write h_a instead of $h_a^{(q)}$. From (2.7) and (2.8), $h_a^{(q)}$ and $h_a^{(q)\prime}$ are continuous . For each $x \geq a$, the maps $a \mapsto h_a^{(q)}(x)$ and $a \mapsto h_a^{(q)\prime}(x)$ are continuous if g is of C^1-class. Moreover, note that $\lim_{a'\uparrow a} h_{a'}^{(q)} = h_a^{(q)}$ by (2.7).

The asymptotic behavior at infinity is given by (Lemma 5.7 in [3]):

$$\lim_{x\to\infty} \frac{h_a^{(q)}(x)}{W^{(q)}(x-a)} = \kappa(a), \quad \text{where} \quad \kappa(a) = \frac{\sigma^2}{2}g'(a-) + \frac{q}{\Phi(q)}g(a) - L\bar{g}(\Phi(q)), \tag{2.9}$$

with L denoting the Laplace transform and $\bar{g}(x) = \int_{(x,\infty)} (g(x+a-z) - g(a))\,\Pi(-\mathrm{d}z)$. In particular, $\kappa(a)$ is continuous if g is of C^1-class.

Using this formula, one obtains (Proposition 5.4 in [3])

$$H_a^{(q)}g(x) = h_a^{(q)}(x) - W^{(q)}(x-a)\kappa(a), \tag{2.10}$$

$$H_{a,b}^{(q)}g(x) = h_a^{(q)}(x) + W^{(q)}(x-a)\frac{g(b)-h_a^{(q)}(b)}{W^{(q)}(b-a)}. \tag{2.11}$$

When X has paths of bounded variation, $H_a^{(q)}g(x)$ and $H_{a,b}^{(q)}g(x)$ are of $C^1(a,\infty)$ and $C^1(a,b)$, respectively if g is of C^1-class. When X has paths of unbounded variation, they are of $C^2(a,\infty)$ and $C^2(a,b)$ if g is of C^1-class.

We define $h_{a+}^{(q)}$ as follows:

$$h_{a+}^{(q)}(x) = g(a) + g'(a+)(x-a) - \int_0^{x-a} W^{(q)}(x-a-y)J_a(y)\,\mathrm{d}y. \tag{2.12}$$

For later reference, we introduce the smooth Gerber–Shiu function of the exponential. Define the two-variable scale function $Z_q(x,\theta)$ (see [3]) as

$$Z_q(x,\theta) = e^{\theta x} + (q-\psi(\theta))\int_0^x e^{\theta(x-y)}W_q(y)\mathrm{d}y. \tag{2.13}$$

It is known that $Z_q(\cdot\ -a,\theta)$ is the smooth Gerber-Shiu function of $e^{\theta(\cdot\ -a)}$ ([3, Corollary 5.9.]). Moreover, the Laplace transform of $Z_q(x,\theta)$ with respect to x is given by ([2, Remark 5.2.])

$$L[Z_q(x,\theta)](s) = \frac{\psi(s)-\psi(\theta)}{s-\theta}\frac{1}{\psi(s)-q}. \tag{2.14}$$

To simplify notation, we write Z_0 as Z.

2.4. **The maximum principle.** We prove a maximum principle for q-subharmonic and q-superharmonic functions.

Proposition 2.3. *Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that $(\mathcal{L}-q)f(c)$ is defined for a fixed point c. If $f(c)\geq 0$ (resp. $f(c)\leq 0$), $f'(c)=0$ and $f(c)>f(x)$ (resp. $f(c)<f(x)$) for all $x\leq c$, then $(\mathcal{L}-q)f(c)<0$ (resp. $(\mathcal{L}-q)f(c)>0$) holds.*

Proof. Assume that f satisfies $f(c)\geq 0$, $f'(c)=0$ and $f(c)>f(x)$ for all $x\leq c$. We prove $f''(c)\leq 0$. Assume to the contrary that $f''(c)>0$. Then, there exists $\varepsilon>0$ such that $f'(x)<0$ for $x\in(c-\varepsilon,c)$, which is contradict to $f(c)>f(x)$ for all $x\leq c$. Since $\int_{(-\infty,0)}(f(c+y)-f(c))\,\Pi(dy)<0$ from the assumption that Π is nonzero and $f(c)>f(x)$ for all $x\leq c$, we obtain $\mathcal{L}f(c)=(\sigma^2/2)f''(c)+\int_{(-\infty,0)}[f(c+y)-f(c)]\,\Pi(\mathrm{d}y)-qf(c)<0$. The remaining part can be shown in the same manner. □

Proposition 2.4. *Let $f:\mathbb{R}\to\mathbb{R}$ be a q- subharmonic (resp. q- superharmonic) function on an open interval (a,b), where $-\infty\leq a<b\leq\infty$. If f is not constant, then f restricted to $(-\infty,b)$ does not attain its maximum (resp. minimum) at any point in (a,b).*

Proof. We prove the case where f is subharmonic on (a,b). Assume that f is not constant and assume that $f|_{(-\infty,b)}$ attains its maximum at some points. Then, there exists $c\in(a,b)$ such that $f'(c)=0$ and $f(c)>f(x)$ for all $x\leq c$. Hence, Proportion 2.3 leads to $(\mathcal{L}-q)f(c)<0$, which is a contradiction to that f is q- subharmonic. □

Proposition 2.5. *Given a function g on $\mathbb{R}$ and an open interval (a,b), we consider the following boundary value problem:*

$$\begin{cases}(\mathcal{L}-q)h(x)=0 & \text{for } x\in(a,b),\\ h(x)=g(x) & \text{for } x\leq a,\\ h(b)=g(b).\end{cases}$$

If a solution to this boundary value problem exists, then it is unique.

Proof. Since the operator $\mathcal{L}-q$ is linear, it suffices to show that if h satisfies

$$\begin{cases} (\mathcal{L}-q)h(x)=0 & \text{for } x\in(a,b), \\ h(x)=0 & \text{for } x\le a, \\ h(b)=0, \end{cases} \tag{2.15}$$

then $h\equiv 0$. By the continuity of h and the given boundary conditions, h must attain either a maximum or a minimum in (a,b). Thus, by Proposition 2.4, h is constant, and we conclude that $h\equiv 0$. □

3. The Riesz representation of $H^q_{A^c}g$

We shall give an extended version of Riesz representation of expected reward functions $H^q_{A^c}g$ for closed interval $\bar{A}=[a,b]$ and the discount rate $q\ge 0$ and let $\mathcal{L}_q:=\mathcal{L}-q$. From here on, $q\ge 0$ is fixed. Also, when no confusion can arise, we refer an q-excessive function as an excessive function.

The proof makes use of the adjoint operator $\tilde{\mathcal{L}}$ of the infinitesimal operator $\mathcal{L}$, which is given by

$$\tilde{\mathcal{L}}f(x)=-\delta f'(x)+\frac{1}{2}\sigma^2 f''(x)+\int_{(-\infty,0)}\left[f(x+y)-f(x)-y\mathbb{1}_{(-1,0)}(y)f'(x)\right]\tilde{\Pi}(\mathrm{d}y),$$

where $\tilde{\Pi}(A)=\Pi(-A)$. *Keep in mind that Assumption 1 is imposed throughout the paper. Except within the statements of the propositions, we do not explicitly restate this assumption.*

Proposition 3.1. *We assume Assumption 1 holds. Let $A=(a,b)\subset\mathbb{R}$. Assume that $g:\mathbb{R}\to\mathbb{R}_+$ is of $C^1(\mathbb{R}\setminus[a,b])$ (resp. $C^2(\mathbb{R}\setminus[a,b])$) if X has paths of bounded variation (resp. unbounded variation). Then, we obtain the Riesz decomposition*

$$H^q_{A^c}g(x)=\int_{A^c}G_q(x,y)\mu(\mathrm{d}y)+h(x),$$

where h is the q-harmonic function with $h(\infty)=g(\infty)$ and $h(-\infty)=g(-\infty)$ and μ a signed measure. Moreover, μ can be identified as follows: if X has paths of bounded variation,

$$\begin{cases} \mu(\mathrm{d}x)=-\mathcal{L}_q H_{A^c}g(x)\mathrm{d}x\ ;\ x<a, x>b, \\ \mu(\{a\})=-\delta\Delta H_{A^c}g(a), \\ \mu(\{b\})=-\delta\Delta H_{A^c}g(b), \end{cases} \tag{3.1}$$

and if X has paths of unbounded variation,

$$\begin{cases} \mu(\mathrm{d}x)=-\mathcal{L}_q H_{A^c}g(x)\mathrm{d}x\ ;\ x<a, x>b, \\ \mu(\{a\})=-\frac{\sigma^2}{2}\Delta(H_{A^c}g)'(a), \\ \mu(\{b\})=-\frac{\sigma^2}{2}\Delta(H_{A^c}g)'(b)\}, \end{cases} \tag{3.2}$$

where we denote $\Delta f(x):=f(x+)-f(x-)$.

To prove this proposition, we prepare the following two lemmas.

Lemma 3.1. *Let f be of class $C^2_b(\mathbb{R})$. Then there exist q-potentials f_1 and f_2 and a q- harmonic function h such that*

$$f=f_1-f_2+h.$$

Proof. Let F_+ and F_- be the positive part and negative part of $(-\mathcal{L}_q)f$, respectively, and let $f_1:=G_qF_+$ and $f_2:=G_qF_-$. Since G_q is a strongly Feller operator ([19]), f_1 and f_2 is finite everywhere. Now, f_1 and f_2 are potential from Proposition 2.1. Let h be set as $h=f-f_1+f_2=f-G_q(-\mathcal{L}_q)f$. Then we have $\mathcal{L}_qh=\mathcal{L}_qf-\mathcal{L}_qG_q(-\mathcal{L}_q)f=\mathcal{L}_qf-\mathcal{L}_qf=0$. □

Lemma 3.2. *Let f be of class $C^2_b(\mathbb{R})$. The following three statements are equivalent:*

(i) There exists q-potentials f_1 and f_2 such that $f = f_1 - f_2$.
(ii) $\lim_{|x|\to\infty} f(x) = 0$ *holds.*
(iii) $G_q(-\mathcal{L}_q)f = f$ *holds.*

Proof. (i)⇒ (ii): It is trivial from the definition of a potential.
(ii)⇒ (i): We consider the decomposition of $f = f_1 - f_2 + h$ in Lemma 3.1. Since f_1 and f_2 are potential, $\lim_{|x|\to\infty} f_1(x) = \lim_{|x|\to\infty} f_2(x) = 0$. Combining the assumption (ii), we obtain $\lim_{|x|\to\infty} h(x) \equiv 0$, which means that $h = 0$ since h is q-harmonic. Hence, $f = f_1 - f_2$.
(iii)⇒ (i): It is immediate form Lemma 3.1.
(i)⇒ (iii): $f = f_1 - f_2 = G_q(-\mathcal{L}_q)f_1 - G_q(-\mathcal{L}_q)f_2 = G_q(-\mathcal{L}_q)f$. □

Proof of Proposition 3.1. Let $\tilde{\mathcal{L}}_q = \tilde{\mathcal{L}} - q$ and fix $f \in C_0^\infty(\mathbb{R})$ arbitrary. For simplicity, we write in this proof $\mathcal{L}_q$, $\tilde{\mathcal{L}}_q$, G_q, $\hat{G}_q$ and H_{A^c} as $\mathcal{L}$, $\tilde{\mathcal{L}}$, G, $\hat{G}$ and H, respectively. First, we have

$$\langle -\tilde{\mathcal{L}}f, Hg\rangle = \langle -\tilde{\mathcal{L}}_\delta f, Hg\rangle + \langle -\tilde{\mathcal{L}}_\sigma f, Hg\rangle + \langle -\tilde{\mathcal{L}}_j f, Hg\rangle + q\langle f, Hg\rangle,$$

where $\tilde{\mathcal{L}}_\delta f(x) := \delta f'(x)$, $\tilde{\mathcal{L}}_\sigma f(x) := \frac{1}{2}\sigma^2 f''(x)$ and $\tilde{\mathcal{L}}_j f(x) := \int_{\mathbb{R}} [f(x+y) - f(x)]\, \tilde{\Pi}(\mathrm{d}y)$.

Since Hg is harmonic in A and $Hg = g$ in A^c, Hg is of $C^1(\mathbb{R}\backslash\{a,b\})$ in the bounded variation case and of $C^2(\mathbb{R}\backslash\{a,b\})$ in the unbounded variation case. Using integration by parts, we obtain

$$\begin{aligned}
\langle -\tilde{\mathcal{L}}_\delta f, Hg\rangle &= \langle f, \gamma(Hg)'\rangle - \delta f(a)\Delta Hg(a) - \delta f(b)\Delta Hg(b),\\
\langle -\tilde{\mathcal{L}}_\sigma f, Hg\rangle &= \langle f, -\frac{1}{2}\sigma^2(Hg)''\rangle + \frac{1}{2}\sigma^2 f'(a)\Delta Hg(a) + \frac{1}{2}\sigma^2 f'(b)\Delta Hg(b)\\
&\quad - \frac{1}{2}\sigma^2 f(a)\Delta(Hg)'(a) - \frac{1}{2}\sigma^2 f(b)\Delta(Hg)'(b),\\
\langle -\tilde{\mathcal{L}}_j f, Hg\rangle &= \langle f, -\mathcal{L}_j Hg\rangle.
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
\langle -\tilde{\mathcal{L}}f, Hg\rangle =& \langle f, -\mathcal{L}Hg\rangle + f(a)\Big[-\delta\Delta Hg(a) - \frac{1}{2}\sigma^2\Delta(Hg)'(a)\Big]\\
&+ f(b)\Big[-\delta\Delta Hg(b) - \frac{1}{2}\sigma^2\Delta(Hg)'(b)\Big]\\
&+ f'(a)\Big[\frac{1}{2}\sigma^2\Delta Hg(a)\Big] + f'(b)\Big[\frac{1}{2}\sigma^2\Delta Hg(b)\Big].
\end{aligned} \tag{3.3}$$

If X has paths of bounded variation, then $\sigma = 0$ and thus (3.3) is

$$\langle -\tilde{\mathcal{L}}f, Hg\rangle = \langle f, -\mathcal{L}Hg\rangle + f(a)\Big[-\delta\Delta Hg(a)\Big] + f(b)\Big[-\delta\Delta Hg(b)\Big] = \int f(y)\mu(\mathrm{d}y),$$

where μ is given in (5.5). If X has paths of unbounded variation, then we have $\Delta Hg(a) = \Delta Hg(b) = 0$ and thus

$$\langle -\tilde{\mathcal{L}}f, Hg\rangle = \langle f, -\mathcal{L}Hg\rangle + f(a)\Big[-\frac{1}{2}\sigma^2\Delta(Hg)'(a)\Big] + f(b)\Big[-\frac{1}{2}\sigma^2\Delta(Hg)'(b)\Big] = \int f(y)\mu(\mathrm{d}y),$$

where μ is given in (5.6).

Next, it follows that

$$\begin{aligned}\int_{A^c} f(y)\mu(\mathrm{d}y) &= \int_{A^c} \hat{G}(-\tilde{\mathcal{L}})f(y)\mu(\mathrm{d}y) \\ &= \int_{A^c}\left(\int -\tilde{\mathcal{L}}f(x)G(x,y)\mathrm{d}x\right)\mu(\mathrm{d}y) \\ &= \int_{A^c} -\tilde{\mathcal{L}}f(x)\left(\int G(x,y)\mu(\mathrm{d}y)\right)\mathrm{d}x \\ &= \langle -\tilde{\mathcal{L}}f, \int_{A^c} G(\cdot,y)\mu(\mathrm{d}y)\rangle,\end{aligned} \tag{3.4}$$

where the first equality follows from Lemma 3.2 and the second is due to $\hat{G}_q(y,\mathrm{d}x) = G_q(x,y)\mathrm{d}x$. From (3.3) and (3.4), we have for any $f \in C_0^\infty(\mathbb{R})$

$$\langle -\tilde{\mathcal{L}}f, Hg\rangle = \langle -\tilde{\mathcal{L}}f, \int_{A^c} G(\cdot,y)\mu(\mathrm{d}y)\rangle. \tag{3.5}$$

Define $h := Hg - \int_{A^c} G(\cdot,y)\mu(\mathrm{d}y)$. We shall show that h is harmonic to complete the proof. Since the proof for the case where X has paths of bounded variation can be carried out exactly in the same manner, we will provide only for the case of unbounded variation. For this purpose, we show that h is of $C^2(\mathbb{R})$. Note that Hg is of $C^2(\mathbb{R}\backslash\{a,b\})$ as seen above. By the specification of μ,

$$\int_{A^c} G(\cdot,y)\mu(\mathrm{d}y) = -\frac{\sigma^2}{2}\Delta(Hg)'(a)G(x,a) + \int_{A^c\backslash\{a,b\}}(-\mathcal{L}Hg)(y)G(x,y)\mathrm{d}y - \frac{\sigma^2}{2}\Delta(Hg)'(b)G(x,b).$$

Note that in the Lévy case, $G(x,y)$ has a representation $\theta(y-x)$ as in (2.5) and (2.6), from which we see that $\theta(\cdot)$ is of $C^2(\mathbb{R}\backslash\{0\})$ and hence $\int G(\cdot,y)\mu(\mathrm{d}y)$ is of $C^2(\mathbb{R}\backslash\{a,b\})$. Therefore, so is h.

Similarly to (3.3), we have, for any $f \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned}0 = \langle -\tilde{\mathcal{L}}f, h\rangle =& \langle f, -\mathcal{L}h\rangle + f(a)\Big[-\delta\Delta h(a) - \frac{1}{2}\sigma^2\Delta h'(a)\Big] \\ &+ f(b)\Big[-\delta\Delta h(b) - \frac{1}{2}\sigma^2\Delta h'(b)\Big] \\ &+ f'(a)\Big[\frac{1}{2}\sigma^2\Delta h(a)\Big] + f'(b)\Big[\frac{1}{2}\sigma^2\Delta h(b)\Big]\end{aligned} \tag{3.6}$$

where the first equality is due to (3.5). This formula implies that $\Delta h(a) = \Delta h(b) = \Delta h'(a) = \Delta h'(b) = 0$. Thus, h is of $C^1(\mathbb{R})$. Moreover, observing the first term of the right-hand side of (3.6), since h is of $C^2(\mathbb{R}\backslash\{a,b\})$, we have $\mathcal{L}h = 0$ on $\mathbb{R}\backslash\{a,b\}$. Hence we have for $x < a$,

$$0 = \mathcal{L}h(x) = -\gamma h'(x) + \frac{1}{2}\sigma^2 h''(x) + \int_{(-\infty,0)}\left[h(x+y) - h(x) - y\mathbb{1}_{(-1,0)}(y)h'(x)\right]\Pi(\mathrm{d}y).$$

To apply the dominated convergence theorem, we establish the following estimate. Note that h' is bounded in view of Assumption 1 (i) because $h-g$ tends to zero as $x \to \pm\infty$. The latter statement is in turn follows from the definition of h and the fact that $\int_{A^c} G(\cdot,y)\mu(\mathrm{d}y)$ is a potential. For each $x < a$, we have

$$\begin{aligned}&\Big|\int_{(-\infty,0)}\left[h(x+y) - h(x) - y\mathbb{1}_{(-1,0)}(y)h'(x)\right]\Pi(\mathrm{d}y)\Big| \\ &\le \int_{(-\infty,-1)}|h(x+y)-h(x)|\,\Pi(\mathrm{d}y) + \int_{(-1,0)}\left|h(x+y)-h(x)-y\mathbb{1}_{(-1,0)}(y)h'(x)\right|\Pi(\mathrm{d}y) \\ &\le \int_{(-\infty,-1)}|h'(x+c_1(y))|\,|y|\,\Pi(\mathrm{d}y) + \int_{(-1,0)}|h''(x+c_2(y))|\,|y|^2\,\Pi(\mathrm{d}y) \quad (\text{for some } c_1(y) < -1 \text{ and } -1 < c_2(y) < 0) \\ &\le \sup|h'|\int_{(-\infty,-1)}|y|\,\Pi(\mathrm{d}y) + \sup_{x\in(-1,0)}|h''(x)|\int_{(-1,0)}|y|^2\,\Pi(\mathrm{d}y) < \infty,\end{aligned}$$

where the second inequality is obtained from the mean value theorem and Taylor's theorem. Letting $x \uparrow a$, we obtain from the dominated convergence theorem,

$$0 = -\gamma h'(a) + \frac{1}{2}\sigma^2 h''(a-) + \int_{(-\infty,0)} \left[h(a+y) - h(a) - y\mathbb{1}_{(-1,0)}(y)h'(x)\right] \Pi(\mathrm{d}y)$$

by the fact that $h \in C^1(\mathbb{R})$ shown above. By applying the same argument to the case $x > a$, we obtain

$$0 = -\gamma h'(a) + \frac{1}{2}\sigma^2 h''(a+) + \int_{(-\infty,0)} \left[h(a+y) - h(a) - y\mathbb{1}_{(-1,0)}(y)h'(x)\right] \Pi(\mathrm{d}y).$$

It follows that $h''(a-) = h''(a+)$. Similarly, we obtain $h''(b-) = h''(b+)$. We have shown that h belongs to $C^2(\mathbb{R})$. Hence since $\mathcal{L}h = 0$ holds on the entire real line, we have shown that h is harmonic. □

We prepare the following corollary for use later.

Corollary 3.1. *Let $A = (-\infty, c) \cup (a, b) \subset \mathbb{R}$ and assume $c < a$. Assume that $g : \mathbb{R} \to \mathbb{R}_+$ is of $C^1(\mathbb{R} \setminus ((-\infty, c] \cup [a, b])) \cap C(\mathbb{R})$ (resp. $C^2(\mathbb{R} \setminus ((-\infty, c] \cup [a, b])) \cap C(\mathbb{R})$) if X has paths of bounded variation (unbounded variation). Moreover, we assume that g is differentiable at c. Then, we obtain the Riesz decomposition*

$$H^q_{A^c} g(x) = \int_{A^c} G_q(x, y)\mu(\mathrm{d}y) + h(x),$$

where h is the q-harmonic function with $h(\infty) = g(\infty)$ and $h(-\infty) = g(-\infty)$ and μ a signed measure. Moreover, μ can be identified as follows: if X has paths of bounded variation,

$$\begin{cases} \mu(\mathrm{d}x) = -\mathcal{L}_q H^q_{A^c} g(x)\mathrm{d}x\ ;\ x < c, c < x < a, x > b, \\ \mu(\{c\}) = -\delta\Delta H^q_{A^c} g(c) \\ \mu(\{a\}) = -\delta\Delta H^q_{A^c} g(a), \\ \mu(\{b\}) = -\delta\Delta H^q_{A^c} g(b), \end{cases}$$

and if X has paths of unbounded variation,

$$\begin{cases} \mu(\mathrm{d}x) = -\mathcal{L}_q H^q_{A^c} g(x)\mathrm{d}x\ ;\ x < c, c < x < a, x > b, \\ \mu(\{c\}) = -\frac{\sigma^2}{2}\Delta(H^q_{A^c} g)'(c), \\ \mu(\{a\}) = -\frac{\sigma^2}{2}\Delta(H^q_{A^c} g)'(a), \\ \mu(\{b\}) = -\frac{\sigma^2}{2}\Delta(H^q_{A^c} g)'(b)\}. \end{cases}$$

In particular, if $q = 0$ and g attains its maximum at c, then $\mu(\{c\}) = 0$ holds in the both cases.

Proof. The proof in the former part follows the same argument as in Proposition 3.1. To prove the last statement, assume that $q = 0$ and g attains its maximum at c. Then, we have $H_{A^c} g(x) = g(c)$ for $x \in (-\infty, c]$ and $H_{A^c} g(x) = g(x)$ for $x \in [c, a)$. By the continuity of g, we have $\Delta H_{A^c} g(c) = 0$. Moreover, since $g'(c) = 0$ from the assumption that g attains its maximum at c, we obtain $\Delta(H_{A^c} g)'(c) = 0$. Therefore, we obtain $\mu(\{c\}) = 0$ in both bounded and unbounded cases.

□

4. NECESSARY CONDITIONS

We present a necessary condition for the optimal stopping problem by using the Riesz representation given in Section.3. When X has paths of bounded variation, the continuous fit condition is a necessary condition, while for unbounded variation, the smooth fit condition is required. This result is consistent with, and can be seen as a generalization of, previous findings such as those by [1, 17, 25].

From here on, we consider only the case where

$$q = 0 \text{ and } \lim_{t\to\infty} X_t = \infty \text{ almost surely under } \mathbb{P}.$$

This assumption does not affect generality. If $q = 0$ and X oscillates, then the problem is trivial. Since, for any point x, the probability of reaching x in finite time is 1 regardless of the initial distribution, the optimal stopping time τ^* is given as $\tau^* = \inf\{t \geq 0 : X_t = \max g\}$ if g attains its maximum. There is no optimal stopping time if g does not attain its maximum. In cases where $q > 0$ or [$q = 0$ and $\lim_{t\to\infty} X_t = -\infty$], the problem can be reduced to $q = 0$ and $\lim_{t\to\infty} X_t = \infty$ by the exponential change of measure:

$$\frac{\mathrm{d}\tilde{\mathbb{P}}^x}{\mathrm{d}\mathbb{P}^x}\bigg|_{\mathcal{F}_t} := e^{\Phi(q)(X_t - x) - qt}. \tag{4.1}$$

Then, we have

$$\begin{aligned}
\mathbb{E}^x[e^{-qt}g(X_t)] &= e^{\Phi(q)x}\mathbb{E}^x[e^{-\Phi(q)X_t}g(X_t)e^{\Phi(q)(X_t-x)-qt}] \\
&= e^{\Phi(q)x}\tilde{\mathbb{E}}^x[e^{-\Phi(q)X_t}g(X_t)] \\
&= e^{\Phi(q)x}\tilde{\mathbb{E}}^x[\tilde{g}(X_t)],
\end{aligned}$$

where $\tilde{g}(x) = e^{-\Phi(q)x}g(x)$ and $\tilde{\mathbb{E}}$ denotes the expectation operator under $\tilde{\mathbb{P}}$. Note that a straightforward calculation yields

$$\tilde{\psi}(\theta) = \psi(\theta + \Phi(q)) - q, \tag{4.2}$$

$$\tilde{\mathcal{L}}\tilde{g}(x) = e^{-\Phi(q)x}(\mathcal{L} - q)g(x), \tag{4.3}$$

where $\tilde{\psi}$ is the Laplace exponent and $\tilde{\mathcal{L}}$ is the generator of X under $\tilde{\mathbb{P}}$. This identity is used in practical applications; see Section 7.1. Moreover, since $\tilde{\psi}'(0+) = \psi'(\Phi(q)) > 0$ on the account of the strict convexity of ψ and $\Phi(q) > 0$, $(X, \tilde{\mathbb{P}})$ always drifts to infinity. Hence this case can be reduced to $q = 0$ and $\lim_{t\to\infty} X_t = \infty$.

In the following, we use the notation

$$H_{T_a^-} = H_a, \quad \text{and} \quad H_{T_a^- \wedge T_b} = H_{a,b}.$$

Proposition 4.1. *We assume Assumption 1 holds. Suppose that X has paths of bounded variation (resp. unbounded variation). Let $-\infty < a < b < \infty$. Suppose that g satisfies the same assumptions as in Proposition 3.1. If $v(x) = H_a g(x)$, then we have*

$$H_a g(a) = g(a) \quad \Big(\text{resp. } (H_a g)'(a+) = g'(a)\Big).$$

If $v(x) = H_{a,b}g(x)$, then we have

$$\begin{cases} H_{a,b}g(a) &= g(a) \quad \Big(\text{resp. } (H_{a,b}g)'(a+) = g'(a)\Big), \\ H_{a,b}g(b) &= g(b) \quad \Big(\text{resp. } (H_{a,b}g)'(b-) = g'(b)\Big) \end{cases}$$

Remark 4.1. (i) If X has paths of unbounded variation, then $H_a g(a) = g(a)$and $H_a g(b) = g(b)$ holds automatically, since any point a and b are regular in this case.
(ii) In the case of $v(x) = H_{a,b}g(x)$, even when X has paths of bounded variation, the identity $(H_{a,b}g)'(b-) = g'(b)$ holds, as is implied by Proposition 5.3.

Proof of Proposition 4.1. We use the fact that the value function is excessive majorant of g (Theorem 1 in [29]). First, we consider the case that X has paths of bounded variation $\Big($resp. unbounded variation$\Big)$ and $v = H_a g$. Since $v = H_a g$ is excessive, the corresponding signed measure μ in Proposition 3.1 must be a measure from Proposition 2.2. Thus, it is necessary that $\mu(\{a\}) \geq 0$, which is equivalent to $\Delta H_a g(a) \leq 0$ because we have $\delta = \psi'(0+) > 0$ from the assumption

$X_t \to \infty$ as $t \to \infty$ almost surely$\Big($resp. $\Delta(H_a g)'(a) \leq 0$ because of $\sigma^2 \geq 0\Big)$. On the other hand, since $v = H_a g$ is majorant of g and $g(x) = H_a(x)$ for $x \leq a$, $\Delta H_a g(a) \geq 0$ $\Big($resp. $\Delta(H_a g)'(a) \geq 0$ $\Big)$is necessary. Thus, $\Delta H_a g(a) = 0$, so that $H_a g(a) = g(a)$ $\Big($resp. $(H_a g)'(a+) = g'(a)$ $\Big)$ is obtained.

The argument in the latter case proceeds in exactly the same manner as in the first part. □

5. SUFFICIENT CONDITIONS

In this section, we present sufficient conditions under which the necessary condition derived in Section 4 becomes sufficient. We verify these conditions separately in the cases where the stopping region is one-sided (Section 5.1) and two-sided (Section 5.2), respectively. *Again keep in mind that Assumption 1 is imposed throughout the paper. Except within the statements of the propositions, we do not explicitly restate this assumption.*

The key idea in the proofs of these propositions is to apply the maximum principle (Proposition 2.3) in order to show that the candidate value function is a majorant of g. Since the use of this maximum principle as a verification tool for the optimal stopping problem is methodologically novel, we include the full proofs here in the main text rather than relegating them to an appendix.

These sufficient conditions also serve as powerful tools for solving a wide range of optimal stopping problems for spectrally negative Lévy process *in a systematic way*. Their general usage is described in Section 6. We also provide concrete examples in Section 7. These subsequent sections will make it clear that the verification presented here is comprehensive.

If g attains a maximum, then let $\beta := \max\{x : g(x) \geq g(y) \text{ for all } \ y\}$ and let $\hat{g} := H_{-\infty,\beta} g$. Note that we have

$$\hat{g}(x) := \begin{cases} g(\beta) & \textit{if} \quad x \leq \beta \\ g(x) & \textit{if} \quad x \geq \beta \end{cases} \tag{5.1}$$

Assumption 2 (Assumptions on g)**.** *We present some assumptions for the reward function g. If g attains its maximum, we interpret the following conditions as applying to $\hat{g}$ instead of g. If g does not attain its maximum, we formally set $\beta = -\infty$:*

- *(a) g is a non-negative, continuous function such that $g \not\equiv 0$ and $\lim_{x\to\infty} g(x) = 0$.*
- *(b) There exists $\ell_0 \in \mathbb{R}$ such that $\mathcal{L}g(x) > 0$ for $x \in (\ell_0, \infty)$ and $g'(\ell_0+) \geq g'(\ell_0-)$.*
- *(c) There exists $a \in \mathbb{R}$ such that $\mathcal{L}g(x) \leq 0$ for $x \in (-\infty, a)$.*
- *(d) There exists finitely many points $\{p_1, \dots, p_n\} \in (-\infty, \ell_0)$ such that g belongs to $C^1(\mathbb{R} \setminus \{p_1, \dots, p_n, \ell_0\})$ and $g'(p_i+) \geq g'(p_i-)$ for $i = 1, \dots, n$ if X has paths of bounded variation, and to $C^2(\mathbb{R} \setminus \{\beta, p_1, \dots, p_n, \ell_0\}) \cap C^1(\mathbb{R} \setminus \{\ell_0\})$ if X has paths of unbounded variation.*

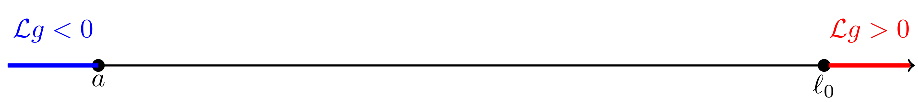

FIGURE 1. The real line with points $a < \ell_0$; the red segment represents the region (ℓ_0, ∞) where $\mathcal{L}g > 0$, and the blue segment represents $(-\infty, a)$ where $\mathcal{L}g < 0$.

Remark 5.1. At first glance, the reader may suspect that a and ℓ_0 are arbitrarily assumed to be optimal boundaries. However, this is not the case. These points do not actually serve as optimal boundaries.

The relative positions of a and ℓ_0 are illustrated in Figure 1. The assumptions (a), (b) and (c) are in place throughout the remainder of the paper. The reader may regard these assumptions as ad hoc and expedient. However, they are in fact *not*. We shall now discuss the motivation and justification of each assumption.

(a) The assumption $\lim_{x\to\infty} g(x) = 0$ does not restrict the generality of the problem. To see this, we consider the case where the assumption is not satisfied. If $\lim_{x\to\infty} g(x) = M$ for some $0 < M < \infty$, then we can solve the problem by taking the reward function as $g - M$. If $\lim_{x\to\infty} g(x) = \infty$ or the limit does not exists, then, for any fixed x, there exists $y > x$ such that $g(y) > g(x)$. Hence, combining this with the assumptions that $q = 0$ and $X_t \to \infty$ as $t \to \infty$, we obtain $H_{y,\infty} g(x) = g(y) > g(x)$. Thus $x \in C$, and thus; $C = \mathbb{R}$. The rest of the conditions in the assumption (a) are standard.

(b) Since $\lim_{x\to\infty} g(x) = 0$ from the assumption (a), $\lim_{x\to\infty} g'(x) = \lim_{x\to\infty} g''(x) = 0$ if the limits exist. On the other hand, $\int_{(y<0)}[g(x+y) - g(x)]\Pi(\mathrm{d}y) > 0$ for sufficiently large x from the assumption (a). Hence, we obtain $\mathcal{L}g(x) = \delta g'(x) + \frac{\sigma^2}{2} g''(x) + \int_{(y<0)}[g(x+y) - g(x)]\Pi(\mathrm{d}y) > 0$ for sufficiently large x. Thus, there exists $\ell_0 \in \mathbb{R}$ such that $\mathcal{L}g(x) > 0$ for $x \in (\ell_0, \infty)$; that is, under the assumption (a), the first assumption of (b) is always satisfied. However, the second assumption might not be satisfied. That is, there may exist $\ell_0 \in \mathbb{R}$ such that $\mathcal{L}g(x) > 0$ for $x \in (\ell_0, \infty)$, while $g'(\ell_0+) < g'(\ell_0-)$.

(c) Here we assume that $g(x)$ is eventually *not subharmonic* as x goes negatively large. Let us point out that it is unnecessary to consider this case where there exists a such that $\mathcal{L}g(x) > 0$ for $x \in (-\infty, a)$. If g attains its maximum at β, then $\mathcal{L}\hat{g}(x) = 0$ for $x < \beta$ holds; and thus, the case of $\mathcal{L}g(x) > 0$ does not arise. If g does not attain its maximum, then it follows from the proof of Proposition 5.1 that there exists no optimal stopping time. Thus, this case is of no significance. As a result, without loss of generality, we may assume that there is no a such that $\mathcal{L}g(x) > 0$ for $x \in (-\infty, a)$. However, it cannot be said that assumption (c) holds in general: we cannot rule out the possibility that the set $\{x : \mathcal{L}g(x) > 0\}$ may have infinitely many connected components, although this is rather a restricted case.

From the above discussion, we conclude that these assumptions generally hold, except in the following two exceptional cases: when there exists $\ell_0 \in \mathbb{R}$ such that $\mathcal{L}g(x) > 0$ for $x \in (\ell_0, \infty)$ and $g'(\ell_0+) < g'(\ell_0-)$; or when the set $\{x : \mathcal{L}g(x) > 0\}$ has infinitely many connected components.

As for the assumption (d), as we shall explain in Section 6, situations that satisfy (d) naturally arise in the process of solving a broad class of problems . Let us emphasize, through the setup in the assumption (d), that we can take care of a reward function which fails to be differentiable at a point. A function of this kind includes payoff of typical options contract (see Section 7).

We refer to Assumption 2(a) simply as the assumption (a), whenever there is no risk of confusion. The same convention applies to Assumptions 2(b), (c), and (d).

Remark 5.2. We provide some information that may be helpful in solving actual problems.

(i) A tractable sufficient condition for the assumption (b) is as follows:

(b') There exists x_0 such that $g(x) = 0$ for all $x \geq x_0$.

(ii) If g attains its maximum, then the assumption (c) is automatically satisfied by choosing $a = \beta$.

We shall show in Propositions 5.1 that the following is a necessary condition for the existence of the optimal stopping time:

$$\text{(A)} \qquad \text{There exists a negatively large } a \text{ such that } \mathcal{L}g \leq 0 \text{ on } (-\infty, a) \text{ and } h_a(x) > g(x); x > a,$$

where h_a is the smooth Gerber-Shiu function for $q = 0$ defined in (2.7). In particular, h_a is harmonic on (a, ∞) and satisfies the boundary condition $h_a(x) = g(x)$ for $x \leq a$. In addition, if X has paths of unbounded variation , it satisfies $h'_a(a+) = g'(a)$. We refer this condition as Condition (A) hereafter. Figure 2 illustrates the relative positions of g and its smooth Gerber–Shiu function when g satisfies Condition (A).

We investigate the behavior of the smooth Gerber–Shiu function h_a depending on the sign of $\mathcal{L}g$ in Lemmas 5.1, 5.2, and 5.3 below.

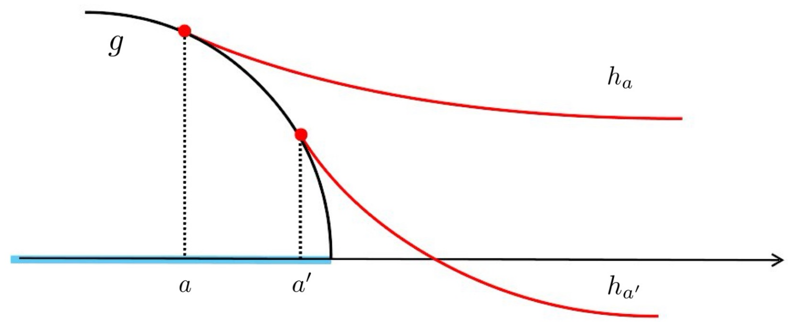

FIGURE 2. Relative position of h_a and g is presented. Though $h_{a'}$ is not majorant of g, h_a is majorant of g if we take a sufficiently large in the negative direction. Hence, g in this figure satisfies Condition (A).

Lemma 5.1. *If g satisfies $\mathcal{L}g < 0$ on (a, b), then for $a < b$, $f := h_a - g$ is strictly increasing on (a, b). In particular, $h_a > g$ on (a, b).*

Proof. We only provide a proof for the case where X has paths of unbounded variation to avoid repeating analogous reasoning[1]. We have $f = 0$ on $(-\infty, a]$, $\mathcal{L}f > 0$ on (a, b) and $f'(a+) = 0$ by the definition of h_a. We prove that there exists $\varepsilon > 0$ such that $f > 0$ on $(a, a + \varepsilon)$. It follows that for $x \in (a, a + (b - a)/2) =: E$,

$$\begin{aligned}
0 < c &:= \min_{x\in \bar{E}} -\mathcal{L}g(x) = \min_{x\in \bar{E}} \mathcal{L}f(x) \\
&\le \mathcal{L}f(x) = \delta f'(x) + \frac{1}{2}\sigma^2 f''(x) + \int_{(-\infty,0)} [f(x + y) - f(y) - y\mathbb{1}_{(-1,0)}(y)f'(x)]\Pi(dy),
\end{aligned}$$

where the first inequality is obtain from $\mathcal{L}g < 0$ on $(-\infty, b)$. Letting $x \downarrow a$, we obtain, from the dominated convergence theorem as in the proof of Proposition 3.1 and the fact that $f'(a+) = 0$,

$$0 < c \le \delta f'(a) + \frac{1}{2}\sigma^2 f''(a+) + \int_{(-\infty,0)} [f(a + y) - f(y) - y\mathbb{1}_{(-1,0)}(y)f'(x)]\Pi(dy) = \frac{1}{2}\sigma^2 f''(a+),$$

and thus $f''(a+) > 0$. This and $f'(a+) = 0$ imply that $f > 0$ on $(a, a + \varepsilon)$ for some $\varepsilon > 0$. Therefore, if f is not strictly increasing at some points on (a, b), there exists $c \in (a, b)$ such that $f'(c) = 0$ and $f(c) > f(x)$ for $x < c$. However, it would follow from Proposition 2.3 that $-\mathcal{L}g(c) = \mathcal{L}f(c) < 0$, which contradicts to $\mathcal{L}g(c) < 0$ because of $c \in (a, b)$. Therefore, f is strictly increasing on (a, b) and $f > 0$ on (a, b). □

Lemma 5.2. *If g satisfies $\mathcal{L}g < 0$ on (a_2, a_1) for $a_2 < a_1$, then we have $h_{a_2} > h_{a_1}$ on (a_1, ∞).*

Proof. Choose a_1 and a_2 such that $a_2 < a_1$ and we show that $h_{a_1} < h_{a_2}$ on (a_1, ∞). Let $f = h_{a_2} - g$. From the previous lemma, f is strictly increasing and $f > 0$ on (a_2, a_1). Define $h := h_{a_2} - h_{a_1}$, which is harmonic on (a_1, ∞). Since $h = f$ on $(-\infty, a_1]$, we have $h = 0$ on $(-\infty, a_2]$, $h > 0$ on $[a_2, a_1]$ and $h'(a_1) = f'(a_1) > 0$. Now, assume to the contrary that h is not strictly increasing at some points on (a_1, ∞). Then, there exist $c > a_1$ such that $h'(c) = 0$ and $h(c) > h(x)$ for all $x < c$. Hence, $\mathcal{L}h(c) < 0$ is obtained from Proposition 2.3, which is a contradiction to the harmonicity of h at $c \in (a_1, \infty)$ and we conclude h is strictly increasing on (a_1, ∞). Since $h(a_1) > 0$ holds, we obtain $h > 0$ on (a_1, ∞), which is equivalent to $h_{a_1} < h_{a_2}$ on (a_1, ∞). □

Lemma 5.3. *If g satisfies $\mathcal{L}g > 0$ for an open interval (ℓ, r), then $h_{\ell+}(x) < g(x)$ for $x \in (\ell, r)$, where $h_{\ell+}$ is defined in* (2.12).

[1]In the bounded variation case, $f(a) = 0$ holds by the definition of f and we can derive $f'(a+) > 0$ in the similar manner.

Proof. Define $s := g - h_{\ell+}$. Similar to the argument for f in Lemma 5.1, since s is subharmonic on (ℓ, r) and $s = 0$ on $(-\infty, \ell]$, s is strictly increasing on (ℓ, r). In particular, $s > 0$ on (ℓ, r). □

Recall the definition of the stopping and continuous region Γ and C in (1.5).

Proposition 5.1. *Assumptions 1 and 2 (a), (b) and (c) are in place.*

- *(I) Assume that g does not attain the maximum. If Condition (A) is violated, then there exists no optimal stopping time; that is, $v(x) > H_\Gamma g(x)$ for some x.*
- *(II) Assume that g attains its maximum. Then, $\hat{g}$ satisfies Condition (A).*

Proof. (I) g is superharmonic on $(-\infty, \ell)$ for some ℓ from the assumption (c). Take an arbitrary $a \in (-\infty, \ell)$. We prove that $a \in C$. Let $\tilde{a} < a$. Since Condition (A) is violated, we have $h_{\tilde{a}}(x) \le g(x)$ for some $x > \tilde{a}$. On the other hand, $h_{\tilde{a}}(a) \ge g(a)$ from Lemma 5.1 and the assumption (c). Since g and $h_{\tilde{a}}$ is continuous, there exists $\tilde{b}$ such that $h_{\tilde{a}}(\tilde{b}) = g(\tilde{b})$ holds. It follows from (2.11) and Lemma 5.2 that we obtain

$$v(a) \ge H_{\tilde{a},\tilde{b}}(a) = h_{\tilde{a}}(a) > h_a g(a) = g(a).$$

Hence we obtain $a \in C$, and thus $(-\infty, \ell) \in C$ since a is arbitrary.

From the above result, C has a connected component $(-\infty, b)$ for some $b \ge \ell$. It follows that there must exist a region $[b, m]$ for some $m \le \infty$ that belongs to Γ. We show that T_Γ is not the optimal stopping time. Assume to the contrary that $v(x) = H_\Gamma g(x)$. Since the process starting from $x \in (-\infty, b)$ exits C from b at finite time from the assumption $X_t \to \infty$ as $t \to \infty$ and the support of $\mathbb{P}^x[X_{T_{(-\infty,b)^c}} \in \mathrm{d}y]$ is $\{b\}$ (recall X is spectrally negative), we obtain, for $x \in (-\infty, b)$,

$$g(x) \le v(x) = H_\Gamma g(x) = H_{(-\infty,b)^c} g(x) = g(b).$$

It follows from this result and the assumption (a) that g attains the maximum, which is a contradiction to the assumption that g does not have the maximum. We conclude that there exists no optimal stopping time.

(II) Note that point β defined above is well-defined due to the assumption (a). $\hat{g}$ is (super)hamonic on $(-\infty, \beta)$ and we have, for $x > \beta$, $\hat{h}_\beta(x) = \hat{g}(\beta) > \hat{g}(x)$, where $\hat{h}_\beta$ is the smooth Gerber-Shiu function for $\hat{g}$ at β. Therefore, Condition (A) holds.

□

5.1. **One-sided stopping region.** First, we consider the case where g does not attain the maximum.

Proposition 5.2 (One-sided case)**.** *Assumptions 1 and 2- (a), (b), (c), and (d) are in force. Suppose that X has the bounded variation (resp. unbounded variation). Suppose that g does not attain the maximum. Consider the equation:*

$$H_a g(a) = g(a) \quad \Big(\textit{resp. } (H_a g)'(a+) = g'(a)\Big) \tag{5.2}$$

The following (I) and (II) hold:

- *(I) Suppose that Condition (A) holds. Then, there exists a solution a^* of (5.2) such that $H_{a^*} g$ is a majorant of g. If $\mathcal{L}g \le 0$ on $(-\infty, a^*)$, $v(x) = H_{a^*} g(x)$. Moreover, $T_\Gamma = T_{(-\infty,a^*]}$ is the optimal stopping time.*
- *(II) Suppose that Condition (A) does not hold. Then, (5.2) has no solution and there is no optimal stopping time.*

Remark 5.3. It follows from (2.10) that (5.2) is equivalent to $\kappa(a) = 0$, which is in turn equivalent to $\lim_{x\to\infty} h_a(x) = 0$.

To prove this proposition, we prepare the following lemmas.

Lemma 5.4. *Suppose that g satisfies the assumption (d). ℓ_0 is as defined in the assumption (b). If X has paths of bounded variation, then discontinuities of $\kappa(\cdot)$ for $a < \ell_0$ are downward jumps only, and discontinuities of $a \mapsto H_a g(a)$ are upward jumps only. If X is paths of unbounded variation, then $\kappa(\cdot)$ and $a \mapsto (H_a g)'(a+)$ for $a < \ell_0$ are continuous.*

Proof. If X has paths of bounded variation, it follows from the definition of κ in (2.9) and $g'(x+) \geq g'(x-)$ for all x in the assumption (d) that the discontinuities of $\kappa(\cdot)$ are downward jumps only. Moreover, it follows from the result above and (2.10) that discontinuities of $a \mapsto H_a g(a)$ are upward jumps only. The remaining statement directly follows from (2.7), (2.8), (2.9) and (2.10). $\square$

Lemma 5.5. *We assume that the assumptions stated in Proposition 5.2 are satisfied. Define*

$$S := \{a \in \mathbb{R} \mid H_a g(a) = g(a)\} \quad \Big(\textit{resp. } S = \{a \in \mathbb{R} \mid (H_a g)'(a+) = g'(a)\}\Big)$$

and

$$M := \{a \in \mathbb{R} \mid H_a g \textit{ is a majorant of } g\}.$$

If Condition (A) is satisfied, then $S \neq \varnothing$.

Proof. It follows from Condition (A) that there exists a_1 such that $h_{a_1}(x) > g(x)$ for all $x > a_1$. Hence, from (2.9) and assumption (a), we obtain $\kappa(a_1) = h_{a_1}(\infty)/W(\infty) > g(\infty)/W(\infty) = 0$.

Since we have $h_{\ell_0+}(x) < g(x)$ for $x \in (\ell_0, \infty)$ from Lemma 5.3 and the assumption (b), it follows, from $g'(\ell_0+) - g'(\ell_0-) \geq 0$ also in the assumption (b), that $h_{\ell_0}(x) \leq h_{\ell_0+}(x) < g(x)$ for $x \in (\ell_0, \infty)$. Recall (2.7) for the relation between $h_a(\cdot)$ and $g'(a-)$. It follows from $h_{\ell_0}(x) = \lim_{\varepsilon \downarrow 0} h_{\ell_0 - \varepsilon}(x)$ for $x \geq \ell_0$ by the continuity of $g'(a)$ for $a < \ell_0$ that $h_{\ell_0-\varepsilon}(x) < g(x)$ for some $x \in (\ell_0, \infty)$ and sufficiently small $\varepsilon > 0$.

Define $a_2 := \ell_0 - \varepsilon$. We prove $h_{a_2}(\infty) < 0$; that is, $\kappa(a_2) < 0$. Assume to the contrary that $h_{a_2}(\infty) = (h_{a_2} - g)(\infty) \geq 0$, where the equality is obtained from the assumption (a). Combining this with the fact that $(h_{a_2} - g)$ falls below zero somewhere on (ℓ_0, ∞), there exists $c > \ell_0$ such that $(h_{a_2} - g)(c) \leq 0$, $(h_{a_2} - g)'(c) = 0$ and $(h_{a_2} - g)(c) < (h_{a_2} - g)(x)$ for all $x \leq c$. Hence, from Proposition 2.3, we obtain $\mathcal{L}(h_{a_2} - g)(c) > 0$, which is a contradiction to the fact that $(h_{a_2} - g)$ is superharmonic on (ℓ_0, ∞). Therefore, it follows from $\kappa(a_1) > 0$, $\kappa(a_2) < 0$ and Lemma 5.4 that there exists $a \in (a_1, a_2)$ such that $\kappa(a) = 0$, which is equivalent to $a \in S$ as checked in Remark 5.3 (i).

$\square$

Lemma 5.6. *We assume that the assumptions stated in Proposition 5.2 are satisfied. The sets S and M are defined as in Lemma 5.5. If $a \in S \cap M^c$, then there exists $a' \in S$ such that $a' > a$.*

Proof. The proof differs depending on whether X has paths of bounded variation or unbounded variation. We will prove each case in turn.

X has paths of bounded variation:

Let $a \in S \cap M^c$ and then $H_a g$ is not majorant of g. We consider the following two cases depending on the regions where $H_a g$ is below g.

Case I: First, we assume that there exists $\varepsilon > 0$ such that $H_a g(c) < g(c)$ for all $c \in (a, a+\varepsilon)$. That is, $H_a g(x)$ fails to be a majorant of $g(x)$ immediately $x > a$. Then, since $H_a g(a) = g(a)$ holds, there exists $a_1 > a$ such that $\frac{d}{dy}(H_a g(y) - g(y)) \leq 0$ for $y \in [a, a_1]$. We consider the path where the process X starting from a_1 at time zero, and define the following three events:

$$\begin{aligned}
A_1 &= \Big\{\omega \in \Omega : T^-_{a_1} = \infty\Big\}, \\
A_2 &= \Big\{\omega \in \Omega : T^-_{a_1} < \infty \quad \text{and} \quad X_{T^-_{a_1}} \in (-\infty, a)\Big\}, \\
A_3 &= \Big\{\omega \in \Omega : T^-_{a_1} < \infty \quad \text{and} \quad X_{T^-_{a_1}} \in [a, a_1)\Big\}.
\end{aligned}$$

Note that they are disjoint each other and $A_1 \cup A_2 \cup A_3 = \Omega$. Let $p_i = \mathbb{P}^{a_1}[A_i]$ for $i \in \{1, 2, 3\}$. It follows that

$$H_{a_1} g(a_1) = p_1 \cdot 0 + p_2 \int_{(-\infty, a)} g(z) \mathbb{P}^{a_1}\Big[X_{T^-_{a_1}} \in \mathrm{d}z \Big| A_2\Big] + p_3 \int_{[a, a_1)} g(z) \mathbb{P}^{a_1}\Big[X_{T^-_{a_1}} \in \mathrm{d}z \Big| A_3\Big], \quad \text{and}$$

$$\begin{aligned} H_a g(a_1) &= p_1 \cdot 0 + p_2 \int_{(-\infty,a)} g(z)\mathbb{P}^{a_1}\Big[X_{T_a^-} \in \mathrm{d}z \Big| A_2\Big] + p_3 \int_{(-\infty,a)} g(z)\mathbb{P}^{a_1}\Big[X_{T_a^-} \in \mathrm{d}z, T_a^- < \infty \Big| A_3\Big] \\ &= p_1 \cdot 0 + p_2 \int_{(-\infty,a)} g(z)\mathbb{P}^{a_1}\Big[X_{T_{a_1}^-} \in \mathrm{d}z \Big| A_2\Big] + p_3 \underbrace{\int_{(-\infty,a)} g(z)\mathbb{P}^{a_1}\Big[X_{T_a^-} \in \mathrm{d}z, T_a^- < \infty \Big| A_3\Big]}_{=(\bigstar)}, \end{aligned}$$

where the last equality is due to $T_{a_1}^- = T_a^-$ on A_2. Since T_a^- is a hitting time, we can write $T_a^- = T_{a_1}^- + T_a^- \circ \theta_{T_{a_1}^-}$ for $a_1 > a$ (see [27], Section III.7). Then by using the strong Markov property at time T_a^- and Fubini's theorem, we have

$$\begin{aligned} (\bigstar) &= \int_{(-\infty,a)} g(z) \int_{[a,a_1)} \mathbb{P}^y\Big[X_{T_a^-} \in \mathrm{d}z, T_a^- < \infty\Big]\mathbb{P}^{a_1}\Big[X_{T_{a_1}^-} \in \mathrm{d}y \Big| A_3\Big] \\ &= \int_{[a,a_1)} \left(\int_{(-\infty,a)} g(z) \int_{[a,a_1)} \mathbb{P}^y\Big[X_{T_a^-} \in \mathrm{d}z, T_a^- < \infty\Big]\right)\mathbb{P}^{a_1}\Big[X_{T_{a_1}^-} \in \mathrm{d}y \Big| A_3\Big] \\ &= \int_{[a,a_1)} H_a g(y)\mathbb{P}^{a_1}\Big[X_{T_{a_1}^-} \in \mathrm{d}y \Big| A_3\Big]. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} H_{a_1} g(a_1) - H_a g(a_1) &= p_3 \int_{[a,a_1)} \Big(g(y) - H_a g(y)\Big)\mathbb{P}^{a_1}\Big[X_{T_{a_1}^-} \in \mathrm{d}y \Big| A_3\Big] \\ &\le p_3\Big(g(a_1) - H_a g(a_1)\Big) \int_{[a,a_1)} \mathbb{P}^{a_1}\Big[X_{T_{a_1}^-} \in \mathrm{d}y \Big| A_3\Big] \\ &= p_3\Big(g(a_1) - H_a g(a_1)\Big) \\ &\le g(a_1) - H_a g(a_1), \end{aligned}$$

where the first inequality is followed directly from the choice of a_1. Thus, $H_{a_1} g(a_1) \le g(a_1)$ is shown. On the other hand, define a_2 as in the proof of Lemma 5.5, in which we found $\kappa(a_2) < 0$. Since we have $H_{a_2} g(x) = h_{a_2} g(x) - W(x-a_2)\kappa(a_2)$ from (2.10) and $h_{a_2}(a_2) = g(a_2)$ from the definition of h_{a_2} and the assumption that X has paths of bounded variation, it follows that

$$H_{a_2} g(a_2) = g(a_2) - W(0)\kappa(a_2) \ge g(a_2)$$

Hence by Lemma 5.4, there exists $a_2 \ge a' \ge a_1 > a$ such that $H_{a'} g(a'+) = g(a')$ so that $a' \in S$

Case II: Secondly, we assume that there exists *no* $\varepsilon > 0$ such that $H_a g(c) < g(c)$ for all $c \in (a, a+\varepsilon)$. Then $a_1 := \inf\{x > a : H_a g(x) < g(x)\}$ such that $a_1 \in (a,\infty)$ exists because of our assumption $a \in S \cap M^{\mathrm{c}}$. By the similar argument to Case I, we have

$$H_{a_1} g(a_1) - H_a g(a_1) = p_3 \int_{[a,a_1)} \Big(g(y) - H_a g(y)\Big)\mathbb{P}^{a_1}\Big[X_{T_{a_1}^-} \in \mathrm{d}y \Big| A_3\Big] \le 0,$$

where the last inequality is obtained from the choice of a_1. Thus, it follows that $H_{a_1} g(a_1) \le H_a g(a_1) = g(a_1)$, where the last equality is obtained from the choice of a_1. The remaining argument is exactly same as in Case I.

X has paths of unbounded variation:

Let us set $a \in S \cap M^{\mathrm{c}}$. It then follows that $H_a g$ is not majorant of g. As in the bounded variation case, we divide into the two cases.

Case I: First, we assume that there exists $\varepsilon > 0$ such that $(H_a g)'(c) < g'(c)$ for all $c \in (a, a+\varepsilon)$. Then, since $(H_a g)'(a+) = g'(a)$ holds, there exists $a_1 > a$ such that $g'(x) \geq (H_a g)'(x)$ for all $x \in [a, a_1]$ and define $h(x) := H_{a_1} g(x) - H_a g(x)$ for $x \in \mathbb{R}$. This function solves the following boundary value problem:

$$\begin{cases} \mathcal{L}h = 0; \quad x \in (a_1, \infty) \\ \lim_{x\to\infty} h(x) = 0, \\ h(x) = f(x); \quad x \leq a_1, \end{cases} \tag{5.3}$$

where f is defined as $f(x) := g(x) - H_a g(x)$ for $x \leq a_1$ and satisfies $f(x) = 0$ for $x \leq a$, $f(x) \geq 0$ for $x \in [a, a_1]$ and $f'(x) \geq 0$ for $x \in [a, a_1]$ by the choice of a_1. Note that the harmonicity of $H_a g(x)$ and $H_{a_1} g(x)$ on $x \in (a_1, \infty)$ are again due to Theorem 3.4.2 in [14], and $\lim_{x\to\infty} h(x) = 0$ is due to the assumption that $g(x)$ tapers off as $x \to \infty$. Hence by the maximum principle (Proposition 2.4), $h(x) \leq h(a_1)$ for all $x \geq a_1$. In particular, we have $h'(a_1+) \leq 0$, which is equivalent to $(H_{a_1} g)'(a_1+) \leq (H_a g)'(a_1+)$. Recalling, by the choice of a_1, that we have $g'(a_1) \geq (H_a g)'(a_1)$, we obtain $(H_{a_1} g)'(a_1+) \leq g'(a_1)$. On the other hand, define a_2 as in the proof of Lemma 5.5. Then, we have $\kappa(a_2) < 0$, in which we found $H_{a_2} g(x) = h_{a_2} g(x) - W(x - a_2)\kappa(a_2)$ from (2.10) and $h'_{a_2}(a_2+) = g'(a_2)$ from the definition of h_{a_2} and the assumption that X has paths of unbounded variation, it follows that

$$(H_{a_2} g)'(a_2) = g'(a_2) - W'(0)\kappa(a_2) \geq g'(a_2)$$

Thus, by the continuity of $x \mapsto (H_x g)'(x+)$ (Lemma 5.4), there exists $a_2 \geq a' \geq a_1 > a$ such that $(H_{a'} g)'(a'+) = g'(a')$, which implies $a' \in S$ and $a' > a$.

Case II: Secondly, we assume that there exists *no* $\varepsilon > 0$ such that $(H_a g)'(c) < g'(c)$ for all $c \in (a, a+\varepsilon)$. Let $a_1 := \inf\{x > a : H_a g(x) < g(x)\}$ and h and f be defined as in Case I. The existence of $a_1 < \infty$ is guaranteed again by the assumption $a \in S \cap M^c$. Then h solves the boundary value problem (5.3) and f satisfies $f(a_1) = 0$, $f(x) = 0$ for $x \in (-\infty, a]$, and $f(x) \leq 0$ for $x \in [a, a_1]$. By the maximum principle (Proposition 2.4), $h(x) \leq 0$ for $x \geq a_1$. In particular, $h'(a_1+) = 0$ holds, which is equivalent to $(H_{a_1} g)'(a_1+) = (H_a g)'(a_1) \leq g'(a_1)$. The remaining argument is exactly same as in Case I.

□

Lemma 5.7. *We assume that the assumptions stated in Proposition 5.2 are satisfied. The sets S and M are defined as in Lemma 5.5. If $a \in S \cap M$, then $\mathcal{L}g(a) \leq 0$ holds.*

Proof. Let $a \in S \cap M$.

X has paths of bounded variation: For $x > a$,

$$0 = \mathcal{L}H_a g(x) = \delta(H_a g)'(x) + \int_{(y<0)} \Big[H_a g(x+y) - H_a g(x)\Big]\Pi(\mathrm{d}y). \tag{5.4}$$

Since $H_a g(x) = g(x)$ for $x \leq a$ holds by $a \in S$, letting $x \downarrow a$ in the right hand side of (5.4) gives along with the dominated convergence theorem as in the proof of Proposition 3.1,

$$\text{(RSH of (5.4))} \to \delta(H_a g)'(a+) + \int_{(y<0)} \Big[g(a+y) - g(a)\Big]\Pi(\mathrm{d}y), \quad \text{and}$$

$$0 = \delta(H_a g)'(a+) + \int_{(y<0)} \Big[g(a+y) - g(a)\Big]\Pi(\mathrm{d}y).$$

Since $H_a g$ is majorant of g and $H_a g(a) = g(a)$ holds, $(H_a g)'(a+) \geq g'(a)$. Therefore,

$$\mathcal{L}g(a) = \delta g'(a) + \int_{(y<0)} \Big[g(a+y) - g(a)\Big]\Pi(\mathrm{d}y) \leq 0.$$

X **has paths of unbounded variation:** Similarly, it follows that from $H_a g(a) = g(a)$ and $(H_a g)'(a+) = g'(a)$ that

$$0 = \lim_{x \downarrow a} \mathcal{L} H_a g(x) = -\gamma g'(a) + \frac{1}{2}\sigma^2 (H_a g)''(a+) + \int_{\mathbb{R}} \left[g(a+y) - g(a) - y \mathbb{1}_{(-1,0)}(y) g'(a) \right] \Pi(\mathrm{d}y).$$

Since $H_a g$ is majorant of g and $(H_a g)'(a+) = g'(a)$ holds, $(H_a g)''(a+) \geq g''(a)$. Therefore, we obtain $\mathcal{L} g(a) \leq 0$.

□

Now we are ready for proving the proposition:

Proof of Proposition 5.2. (I) First we show $S \cap M \neq \varnothing$. From Lemma 5.5, $S \neq \varnothing$. By the assumption (b), ℓ_0 is an upper bound of S. Indeed, for any $\ell' > \ell_0$, we have $h_{\ell'}(x) < g(x)$ for $x > \ell'$ from Lemma 5.3. Therefore, as in the proof of Lemma 5.5, using the maximum principle (Proposition 2.3), we obtain $h_{\ell'}(\infty) < g(\infty) = 0$, and thus; $\kappa(\ell') < 0$. Hence, $\ell' \notin S$ from Remark 5.3. This implies that S is bounded from above. Since $a \mapsto H_a g(a)$ and g is continuous (see Lemma 5.4), S is a closed set. Thus, there exists the maximum element $c \in S$. It follows from Lemma 5.6 that $c \in M$. Hence $S \cap M$ is not empty.

Let $a^* \in S \cap M$. It suffices to show that $H_{a^*} g$ is a majorant of g and excessive (Lemma 11.1 in [23]). By the definition of M, $H_{a^*} g$ is majorant of g. We prove that $H_{a^*} g$ is an excessive function. It follows from Proposition 3.1 that, for $x \in \mathbb{R}$, We prove that $H_{a^*} g$ is an excessive function. It follows from Proposition 3.1 that there exist a signed measure μ such that, for $x \in \mathbb{R}$,

$$\begin{aligned} H_{a^*} g(x) &= \int_{(-\infty, a^*)} G(x,y) \mu(\mathrm{d}y) \\ &= \int_{(-\infty, a)} G(x,y) (-\mathcal{L} H_{a^*} g(y)) \mathrm{d}y + \mu(\{a^*\}) G(x, a^*) + h(x) \\ &= \int_{(-\infty, a^*)} G(x,y) (-\mathcal{L} H_a{}^* g(y)) \mathrm{d}y + h(x), \end{aligned}$$

where h is some harmonic function and the third equality holds since $\Delta H_{a^*} g(a) = 0$ (resp. $\Delta (H_{a^*} g)'(a) = 0$) when X has paths of bounded variation (resp. unbounded variation). Recall that $a^* \in S$. Now, from the assumption that $\mathcal{L} g(x) \leq 0$ holds for all $x < a^*$, we find $-\mathcal{L} H_{a^*} g(y) = -\mathcal{L} g(y) \geq 0$ for each $y < a^*$. Hence, since μ is a measure, $H_{a^*} g$ is excessive.

(II) This follows immediately from Proposition 5.1 (I).

□

Next, we consider the case where g attains the maximum.

Corollary 5.1. *Suppose that X has the bounded variation (resp. unbounded variation). Assumptions 2- (a), (b), (c), and (d) are in force. Assume that g attains the maximum. Then, there exists a solution a^* of (5.2) such that $H_{a^*} \hat{g}$ is a majorant of g. If $\mathcal{L} \hat{g} \leq 0$ on $(-\infty, a^*) \setminus \{\beta\}$, then $v(x) = H_{\beta, a^*} g(x)$. Moreover, $T_\Gamma = T_{[\beta, a^*]}$ is the optimal stopping time.*

Proof. Noting that Condition (A) always holds (Proposition 5.1 (II)), the proof—except for the excessivity of H_{a^*} under the assumption $\mathcal{L} \hat{g} \leq 0$ on $(-\infty, a^*) \setminus \{\beta\}$—is exactly the same as in Proposition 5.2. The proof of the remaining part is given below. It follows from Corollary 3.1 that there exist a signed measure μ such that, for $x \in \mathbb{R}$,

$$H_{a^*} g(x) = \int_{(-\infty, a^*]} G(x,y) \mu(\mathrm{d}y),$$

and μ is given by if X has paths of bounded variation,

$$\begin{cases} \mu(\mathrm{d}x) = -\mathcal{L}_q H_{a^*} g(x)\mathrm{d}x \; ; \; x \in (-\infty, a^*) \setminus \{\beta\}, \\ \mu(\{\beta\}) = 0, \\ \mu(\{a\}) = -\delta \Delta H_{a^*} g(a), \end{cases} \tag{5.5}$$

and if X has paths of unbounded variation,

$$\begin{cases} \mu(\mathrm{d}x) = -\mathcal{L}_q H_{a^*} g(x)\mathrm{d}x \; ; \; x \in (-\infty, a^*) \setminus \{\beta\}, \\ \mu(\{\beta\}) = 0 \\ \mu(\{a\}) = -\frac{\sigma^2}{2} \Delta (H_{a^*} g)'(a), \end{cases} \tag{5.6}$$

We obtain $\mu\{a^*\} = 0$ from $a^* \in S$ and obtain $\mu(\mathrm{d}x) = -\mathcal{L} H_{a^*} g(x) = -\mathcal{L} g(x) \geq 0$ for $x \in (-\infty, a^*) \setminus \{\beta\}$ by the assumption. Therefore, μ is a measure, and thus; $H_{a^*} g$ is excessive. □

5.2. **Two-sided stopping region.** In two-sided stopping region case, we impose the following assumption (e) instead of (d). If g attains its maximum, we interpret the following conditions as applying to $\hat{g}$ instead of g. If g does not attain its maximum, we formally set $\beta = -\infty$:

- (e) There exists $p_1 < \cdots < p_n < \ell \leq r < \ell_0$ satisfying the following conditions:
  - (i) g belongs to $C^1((-\infty, r) \setminus \{p_1, \ldots, p_n, \ell\})$ if X has paths of bounded variation, and g belongs to $C^2((-\infty, r) \setminus \{p_1, \ldots, p_n, \ell, \beta\}) \cap C^1((-\infty, r) \setminus \{\ell\})$ if X has paths of unbounded variation.
  - (ii) $\mathcal{L} g(x) > 0$ for $x \in (\ell, r)$ and $\mathcal{L} g(x) \leq 0$ for $x \in (-\infty, \ell) \setminus \{p_1, \ldots, p_n\}$.
  - (iii) $g'(\ell+) \geq g'(\ell-)$
  - (iv) $\ell < r$ if X has paths of bounded variation, and $g'(\ell+) > g'(\ell-)$ or $\ell < r$ if X has paths of unbounded variation.

We refer to Assumption 2(e) simply as the assumption (e), whenever there is no risk of confusion. Although the assumption (e) may appear ad hoc on the surface, it is in fact *not*. As explained in Section 6, situations satisfying this assumption naturally arise in the process of systematically solving a broad class of problems. Let us emphasize that we are capable of handling a reward function which fails to be differentiable at a point. A function of this kind includes payoff of typical options contract (see Section 7).

In this case, consider the following system for (a, b)

$$\begin{cases} (H_{a,b} g)(a) = g(a) \quad \Big(\text{resp. } (H_{a,b} g)'(a+) = g'(a)\Big) \\ (H_{a,b} g)'(b-) = g'(b) \end{cases} \tag{5.7}$$

We define *the maximum solution* (a', b') of (5.7) as (i) (a, b) is not a solution to (5.7) for any $a < a'$ and b, and (ii) (a', b) is not a solution to (5.7) for any $b > b'$. By the definition, the maximum solution is unique if it exists.

Proposition 5.3 (Two-sided case)**.** *Assumptions 1 and 2 (a), (b), (c) and (e) are in force. Suppose that X has paths of bounded variation (resp. unbounded variation). Suppose that g does not attain its maximum. Define*

$$a' := \sup A, \text{ where } A := \{a : a < \ell\} \cap \{a : h_a(x) > g(x) \text{ for all } x > a\} \tag{5.8}$$

and assume that $a' > p_n$.

- *(I) Suppose that Condition (A) is satisfied. If there exists $b > a'$ such that $h_{a'}(b) = g(b)$, then (5.7) has the maximum solution (a', b') such that $a' < \ell < r < b'$. $H_{a', b'} g$ is majorant of g, and*

$$E := (a', b') \setminus \{b \in (a', b') : h_{a'}(b) = g(b)\}. \tag{5.9}$$

is contained in the continuation region C. Moreover, $\mathcal{L} g(b') \leq 0$ holds if $\mathcal{L} g$ is defined at b'.

(II) Suppose that Condition (A) *is satisfied. If there does not exist* $b > a'$ *such that* $h_{a'}(b) = g(b)$*, then* $v(x) = H_{a'}g(x)$*. Moreover,* $T_\Gamma = T_{(-\infty,a']}$ *is the optimal stopping time.*

(III) Suppose that Condition (A) *does not holds. Then,* (5.7) *has no solution and there is no optimal stopping time.*

Remark 5.4. Recall Section 2.3 and note that (a, b) satisfies the first equation of (5.7) if and only if it satisfies $h_a(b) = g(b)$. To show this, suppose that $h_a(b) = g(b)$. We know that h_a is the harmonic function on (a,∞) with the boundary condition $h_a(x) = g(x)$ for $x \leq a$ is continuous (resp. smooth) at a when X has paths of bounded variation (resp. unbounded variation). Hence, with the assumption $h_a(b) = g(b)$, the restriction of h_a on the set $(-\infty, b]$, that is, $h_a|_{(-\infty,b]}$ is the harmonic function on (a, b) with the boundary condition $h_a|_{(-\infty,b]}(x) = g(x)$ for $x \in (-\infty, a] \cup \{b\}$. On the other hand, $H_{a,b}g$ is also the harmonic function on (a, b) with the boundary condition $H_{a,b}g(x)(x) = g(x)$ for $x \in (-\infty, a] \cup \{b\}$; and thus, by the uniqueness of the boundary value problem (Proposition 2.5), $h_a(x) = H_{a,b}g(x)$ for $x \in (-\infty, b]$. Therefore, from this and the definition of h_a, we obtain the first equation of (5.7). The converse can be shown in a similar manner.

From the above argument, the system (5.7) is equivalent to the following:

$$\begin{cases} h_a(b) &= g(b) \\ h_a'(b) &= g'(b). \end{cases} \tag{5.10}$$

This representation is employed in the proof of Proposition 5.3 and practical applications (see Section 7.2).

Proof of Proposition 5.3. Since this proposition can be proved using the same tools as in the proof of Proposition 5.2, the proof is deferred to Appendix A. □

As Corollary 5.1 in the one-sided case, the following Corollary is obtained by using Proposition 5.1 (ii).

Corollary 5.2. *Suppose that* X *has the bounded variation (resp. unbounded variation). Assumptions* (a), (b), (c), *and* (e) *are in force. Define define* β *and* $\hat{g}$ *as in* (5.1)*. Let* $\hat{h}_a$ *be the smooth Gerber-Shiu function for* $\hat{g}$ *at* a*. Then,*

(I) If there exists $b > a'$ *such that* $\hat{h}_{a'}(b) = g(b)$*, then* (5.7) *has the maximum solution* (a', b') *such that* $a' < \ell < r < b'$*.* $H_{a',b'}g$ *is majorant of* g*, and*

$$E := (a', b') \setminus \{b \in (a', b') : \hat{h}_{a'}(b) = g(b)\}. \tag{5.11}$$

and $(-\infty, \beta)$ *are contained in the continuation region* C*. Moreover,* $\mathcal{L}g(b') \leq 0$ *holds if* $\mathcal{L}g$ *is defined at* b'*.*

(II) If there does not exist $b > a'$ *such that* $\hat{h}_{a'}(b) = g(b)$*, then* $v(x) = H_{a'}\hat{g}(x) = H_{[\beta,a']}H$*. Moreover,* $T_\Gamma = T_{[\beta,a']}$ *is the optimal stopping time.*

Proof of Corollary 5.2. See Appendix A. □

6. General procedure to solve optimal stopping

We give a general procedure to solve the optimal stopping problem (1.4) for a reward function g satisfying (a), (b), and (c) in Assumption 2. Our strategy is to successively "eliminate" the regions where $\mathcal{L}g > 0$, starting from $-\infty$. Recall that we discuss the justification of these assumptions in detail at the beginning of Section 5. By following the procedure, the reader will see that Assumptions (d) and (e) are satisfied in due course.

Assume that there exists finitely many points $\ell_1 \leq r_i < \ell_2 \leq \cdots < \ell_n \leq r_n$ such that $g'(\ell_+) \geq g'(\ell_-)$, $g \in C^1(\mathbb{R} \setminus \{\ell_1, \ldots, \ell_n\})$ (resp. $g \in C^2(\mathbb{R} \setminus \{\ell_1, \ldots, \ell_n\})$) if X has paths of bounded variation (resp. unbounded variation) and $\mathcal{L}g > 0$ holds on (ℓ_i, r_i) for each i. Moreover, we assume that $\ell_i < r_i$ if X has paths of bounded variation and that $\ell_i < r_i$ or $g'(\ell_+) > g'(\ell_-)$ if X has paths of unbounded variation. We refer to each interval (ℓ_i, r_i) (when $\ell_i < r_i$) and (ℓ_0, ∞), and each singleton ℓ_i (when $\ell_i = r_i$), as a *subharmonic component*. We denote by D the union of all subharmonic components. Note that the number of subharmonic components is at least 1 since (ℓ_0, ∞) is a subharmonic component. The relative positions of the points, for example in the case $n = 4$, are illustrated in Figure 3. The method of solution proceeds through the following steps (Step 1 to Step 6).This solution process is summarized in the flowchart in Figure 5.

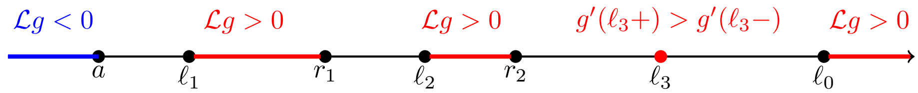

FIGURE 3. The real line with points $a < \ell_1 < r_1 < \ell_2 < r_2 < \ell_3 < \ell_0$ when $n = 4$; the red segment represents the region $D = (\ell_1, r_1) \cup (\ell_2, r_2) \cup \{\ell_3\} \cup (\ell_0, \infty)$, and the blue segment represents $(-\infty, a)$ where $\mathcal{L}g < 0$.

Step 1 (Exponential change of measure): We reduce the problem to the case $q = 0$ and $\lim_{t\to\infty} X_t = \infty$ by the exponential change of measure in (4.1). The objective function after the change of measure is again denoted by g.

Step 2 (Left-side flattening): We check whether g attains its maximum. If it does, replace g with $\hat{g}$ and proceed to Step 4, where $\hat{g}$ is defined in Proposition 5.1 (ii).[2] If not, proceed to Step 3.

Step 3 (Verifying Condition (A)): We now verify Condition (A). If the condition is not satisfied, then there exists no optimal stopping time. If it is satisfied, we proceed to Step 4.

Step 4 (Examining the number of subharmonic components n): We examine the number of subharmonic components n. If $n = 1$, then we proceed to Step 5. If $n \geq 2$, we proceed to Step 6.

Step 5 (Applying Proposition 5.2): Since $n = 1$, we know that g satisfies the assumptions in Corollary 6.1 including the assumption (d). We can apply Corollary 6.1 stated later. If g does not attain its maximum, then Corollary 6.1 (II) is applied and we obtain $v(x) = H_{a^*}g(x)$ and $\Gamma = (-\infty, a^*]$, where a^* is a solution of (5.2). If g attains the maximum, then Corollary 6.1 (III) is applied and we obtain $v(x) = H_{\beta,a^*}g(x)$ and $\Gamma = [\beta, a^*]$.

Step 6 (Applying Proposition 5.3): (1) First, we apply Proposition 5.3 to $(\ell, r) = (\ell_1, r_1)$. Define a'_1 as in (5.8). Note that g and (ℓ_1, r_1) satisfy the assumption (e) by the construction of D.

(2) If Proposition 5.3 (II) can be applied, then we obtain $(a'_1, \infty) \in C$ and the problem has been solved. The solution is $v(x) = H_{a'_1}g(x)$ and $\Gamma = (-\infty, a'_1]$ $\big($resp. $v(x) = H_{\beta,a'_1}g(x)$ and $\Gamma = [\beta, a'_1]$ $\big)$ if g does not attain its maximum (resp. g attains its maximum).

(3) If Proposition 5.3 (I) can be applied, then let (a'_1, b'_1) be the maximal solution of (5.7). Define $g_1 := H_{a'_1,b'_1}g$. Note that g_1 and (ℓ_2, r_2) satisfy the assumptions in Proposition 5.3 as noted in Remark 6.1 (i). Replace g and (ℓ_1, r_1) by g_1 and (ℓ_2, r_2), respectively, and return to Step 6 (1). Now either of the following two cases arises: If Proposition 5.3 (II) is applicable for this g_1, then we obtain $v(x) = H_{a'_2,\infty}H_{a'_1,b'_1}g$ and $\Gamma = \mathbb{R} \setminus ((a'_1, b'_1) \cup (a'_2, \infty))$: we are done. If Proposition 5.3 (I) can be applied to g_1, we set $g_2 := H_{a'_2,b'_2}g_1$.

(4) We repeat this procedure, say, $m \leq n - 1$ times until either g_m becomes superharmonic on $(-\infty, \ell_0)$ or Proposition 5.3 (II) is applicable. Finally, in the case that g_m is superharmonic on $(-\infty, \ell_0)$, we apply Corollary 6.1 to g_m.

Remark 6.1. To ensure that Step 6 can be carried out, we make the following two observations.

(i) We show that g_1 satisfies the assumptions in Proposition 5.3 . First, since (a'_1, b'_1) solves (5.7), note that g_1 satisfies the assumption (e) for $(p_1, p_2) = (a'_1, b'_1)$. Hence, it is sufficient to show that $a'_2 > b'_1$. Since we have $g_1 = H_{a'_1.b'_1}g = h_{a'_1}$ on $(-\infty, b'_1]$ and $g'_1(b'_1) = h'_{a'_1}(b'_1)$ from the fact that (a'_1, b'_1) solves (5.7), it follows that both $h^1_{b'_1}$ and $h_{a'_1}$ solve the same boundary value problem: $\mathcal{L}h = 0$ on (b'_1, ∞), $h = g_1$ on $(-\infty, b'_1]$ and $h'(b'_1) = g'(b'_1)$, where $h^1_{b'_1}$ is the smooth Gerber-Shiu function for g_1 at b'_1. Hence, by the uniqueness of the boundary value problem, we obtain $h^1_{b'_1} = h_{a'_1}$ on $[b'_1, \infty)$. Hence, since $h_{a'_1} > g = g_1$ on (b'_1, ∞) from the definition of b'_1, it follows that $h^1_{b'_1} > g$ on (b'_1, ∞). Therefore, $a'_2 > b'_1$. Refer to Figure 4 for the spatial relationship among g, g_1, $h_{a'_1}$, and $h^1_{b'_1}$.

[2]It would be more precise to continue using the notation $\hat{g}$, but to avoid unnecessary complexity, we instead write g. However, the original notation $\hat{g}$ is retained in Figure 5.

(ii) We show that $m \leq n-1$. Since $a_i' < b_i' < b_{i+1}'$ holds as shown in (i), each (a_i', b_i') for $i = 1, \ldots, m$ is a connected component of C. Thus, C has $m+1$ connected components, including the connected component obtained by applying Corollary 6.1 at the final application. However, since D has n connected components, the number of connected components of C is at most n. Hence, we obtain $m+1 \leq n$.

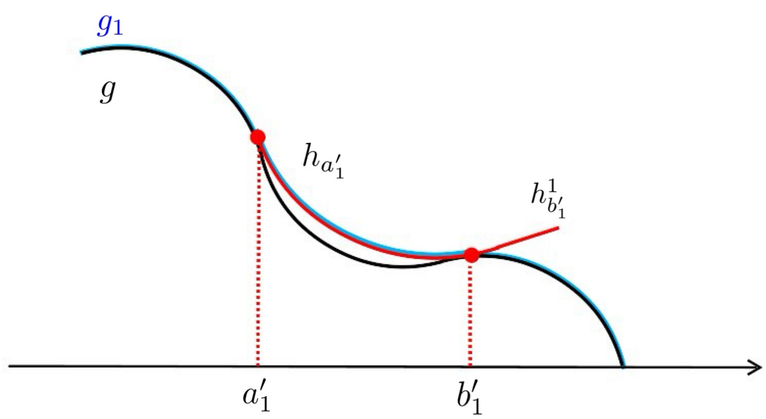

FIGURE 4. g is shown as a black curve, and g_1 is shown as a blue curve. The figure illustrates that the smooth Gerber–Shiu function $h_{a_1'}$ for g at a_1' coincides with the smooth Gerber–Shiu function $h^1_{b_1'}$ for g_1 at b_1' on the interval (b_1', ∞).

Corollary 6.1 (Corollary to Proposition 5.2)**.** *Suppose that X has the bounded variation (resp. unbounded variation). Assumptions (a), (b), (c) and (d) are in force. Suppose that g (resp. $\hat{g}$) is superhamonic on $(-\infty, \ell_0)$ if g does not attain (resp. attains) the maximum. The following (I), (II) and (III) hold:*

(I) Assume that g does not attain its maximum and Condition (A) *is satisfied. Then, there exists a solution a^* of* (5.2) *such that $H_{a^*}g$ is a majorant of g and $v(x) = H_{a^*}g(x)$. Moreover, $T_\Gamma = T_{(-\infty, a^*]}$ is the optimal stopping time.*

(II) Assume that g does not attain its maximum and Condition (A) *is violated. Then,* (5.2) *has no solution and there is no optimal stopping time.*

(III) Assume that g attains its maximum. Then, there exists a solution a^* of (5.2) *such that $H_{a^*}\hat{g}$ is a majorant of g and $v(x) = H_{\beta, a^*}g(x)$. Moreover, $T_\Gamma = T_{[\beta, a^*]}$ is the optimal stopping time.*

Proof. This statement follows immediately from Proposition 5.2 and Lemma 5.7. □

7. EXAMPLES

7.1. **One-sided stopping region case.** We consider the Mckean optimal stopping problem with respect to spectrally negative Lévy process ([25], [1], and [10]). The problem is given by

$$v_M(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x\Big(e^{-q\tau} g(x)\Big), \text{ where } g_M(x) = (K - e^x)_+.$$

We exclude the case where $q = 0$ but $\lim_{t\to\infty} X_t \nrightarrow \infty$ with probability 1 since there is no optimal stopping time in this case. If $q = 0$ and $\lim_{t\to\infty} X_t \to -\infty$ with probability 1, then the optimal strategy is to never stop due to $g(-\infty) > g(x)$ for all x. If $q = 0$ and X oscillates, then the process is recurrent and there exists no optimal stopping time. (*Step 1*) By the exponential change of measure (4.1), this problem is equivalent to the following problem:

$$v_M(x) = \sup_{\tau \in \mathcal{T}} \tilde{\mathbb{E}}_x \tilde{g}_M(x), \text{ where } \tilde{g}_M(x) = e^{-\Phi(q)x}(K - e^x)_+.$$

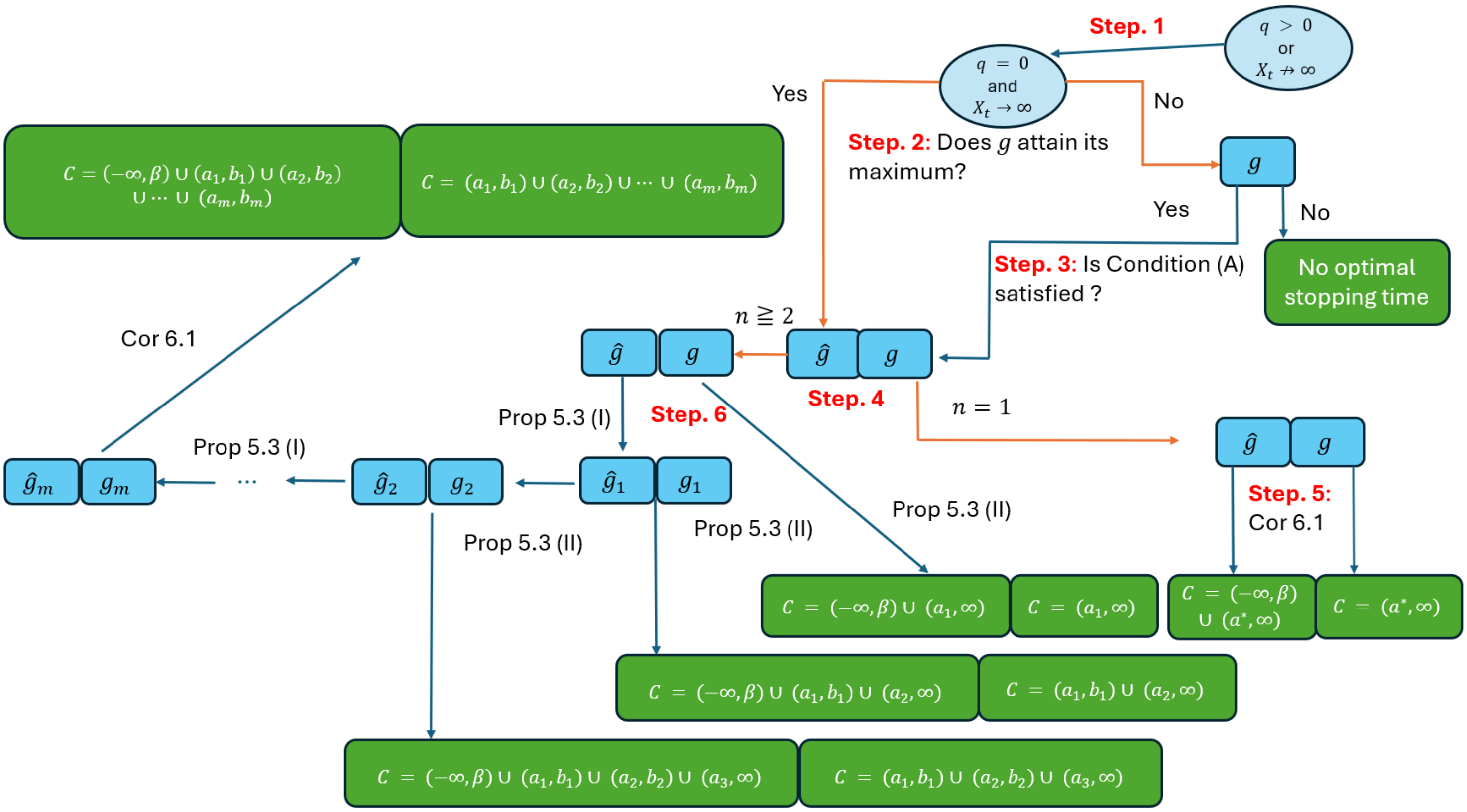

FIGURE 5. The flowchart above illustrates the procedure from *Step 1* through *Step 6*. In this figure, we keep the original notation $\hat{g}$. The function $\hat{g}_1$ is defined as the result of applying Proposition 5.3 to $\hat{g}$. However, in the main text, we do not distinguish g_1 from $\hat{g}_1$, and instead write g_1 for simplicity. The functions $\hat{g}_2, \dots, \hat{g}_m$ are defined in the same way.

(*Step 2*) Since $\lim_{x\to-\infty} g(x) = K$ and $g(x) < K$ for all x, g does not attain its maximum. For now, we assume that Condition (A) holds. We will confirm this fact later.

(*Step 4*) We show that this problem satisfies the assumption Corollary 6.1 . Let $\tilde{\mathcal{L}}$ be the generator of X under $\tilde{\mathbb{P}}$. It follows from (4.3) and a straightforward calculation that for $x < \log K$

$$\begin{aligned}\tilde{\mathcal{L}}\tilde{g}_M(x) &= e^{-\Phi(q)x}(\mathcal{L}-q)g_M(x)\\ &= e^{-\Phi(q)x}\Big(-\mathcal{L}e^x - q(K-e^x)\Big)\\ &= e^{-\Phi(q)x}\Big(-\psi(1)e^x - q(K-e^x)\Big) < 0.\end{aligned}$$

On the other hand, it follows from (4.3) and a straightforward calculation that for $x > \log K$

$$\begin{aligned}\tilde{\mathcal{L}}\tilde{g}_M(x) &= e^{-\Phi(q)x}(\mathcal{L}-q)g_M(x)\\ &= e^{-\Phi(q)x}\int_{(y<\log K - x)}(K-e^{x+y})\Pi(\mathrm{d}y) > 0.\end{aligned}$$

Hence, the assumption in Corollary 6.1 is satisfies.

We proceed to (*Step 5*). From Remark 5.3, $a \in S$, where S is defined in Proposition 5.2, is equivalent to $\lim_{x\to\infty} h_a(x) = 0$. To calculate this, we identify h_a.

Let $\tilde{Z}(x,\theta)$ be the two-variable scale function for $q=0$ under $\tilde{\mathbb{P}}$. First, we see that

$$\tilde{Z}(x,\theta) = e^{-\Phi(q)x} Z_q(x, \theta+\Phi(q)), \tag{7.1}$$

where Z_q is the two-variable scale function under $\mathbb{P}$, introduced in (2.13). It follows from (2.14) and (4.2) that

$$\begin{aligned} L(\tilde{Z}(x,\theta))[s] &= \frac{\tilde{\psi}(s)-\tilde{\psi}(\theta)}{s-\theta}\frac{1}{\tilde{\psi}(s)} \\ &= \frac{\psi(s+\Phi(q))-\psi(\theta+\Phi(q))}{s-\theta}\frac{1}{\psi(s+\Phi(q))-q} \\ &= L(e^{-\Phi(q)x}\tilde{Z}_q(x,\theta+\Phi(q)))[s]. \end{aligned}$$

Hence, we obtain (7.1).

By the similar discussion in Section 2.3, we know that $\tilde{Z}(\cdot\; -a,\theta)$ is the smooth Gerber-Shiu function of $e^{\theta(\cdot\; -a)}$ at a. Hence, since we have, for $a<\log K$,

$$g(x) = Ke^{-\Phi(q)a}e^{-\Phi(q)(x-a)} - e^{(-\Phi(q)+1)a}e^{(-\Phi(q)+1)(x-a)},$$

we obtain the smooth Gerber-Shiu function of g at a as

$$h_a(x) = Ke^{-\Phi(q)a}\tilde{Z}(x-a,-\Phi(q)) - e^{(-\Phi(q)+1)a}\tilde{Z}(x-a,-\Phi(q)+1).$$

Therefore, it follows from (7.1) that, for $a<\log K$, $h_a(x) = e^{-\Phi(q)x}\Big(KZ_q(x-a,0)-e^aZ_q(x-a,1)\Big)$.

Note that we have, for $q>0$ or $\theta\neq 0$,

$$\begin{aligned} \lim_{x\to\infty} Z_q(x,\theta)e^{-\Phi(q)x} &= \lim_{x\to\infty}\frac{1+(q-\psi(\theta))\int_0^x e^{-\theta y}W_q(y)\mathrm{d}y}{e^{(\Phi(q)-\theta)x}}, \\ &= \lim_{x\to\infty}\frac{(q-\psi(\theta))e^{-\theta x}W_q(x)}{(\Phi(q)-\theta)e^{(\Phi(q)-\theta)x}}, \\ &= \frac{q-\psi(\theta)}{\psi'(\Phi(q))(\Phi(q)-\theta)}, \end{aligned} \tag{7.2}$$

where the first equality is obtained from the definition of $Z(x,\theta)$, the second is followed from l'Hôpital's rule, and the final is followed from (2.4). In the case where $q=0$ and $\theta=0$, we have $Z_0(x,0)=1$. Hence, it follows that $\lim_{x\to\infty}h_a(x)=0$ is equivalent to

$$a = \left\{ \begin{array}{ll} \log\left(K\frac{q}{\Phi(q)}\frac{\Phi(q)-1}{q-\psi(1)}\right) & \text{if} \quad q>0 \\ \log\left(K\psi'(0+)\frac{\Phi(q)-1}{q-\psi(1)}\right) & \text{if} \quad q=0 \end{array} \right\} =: a^*. \tag{7.3}$$

As noted at the beginning of this section, we may assume that X has positive drift when $q=0$, so that $\psi'(0+)>0$. Hence, the argument of the logarithm in (7.3) is positive in the case $q=0$, and the expression is well-defined. Hence, this yields $S=\{a^*\}$. Therefore, we obtain from Corollary 6.1 that $\Gamma=(-\infty,a^*]$ and $v_M(x)=H_{a^*}g(x)=KZ_q(x-a^*,0)-e^{a^*}Z_q(x-a^*,1)$. The corresponding graph of the solution is shown in Figure 6.[3]

Finally, we confirm Condition (A). First, we consider the case where $q>0$. For any fixed x, it follows that

$$\begin{aligned} \lim_{a\to-\infty}h_a(x) &= \lim_{a\to-\infty}e^{-\Phi(q)x}\Big(KZ_q(x-a,0)-e^aZ_q(x-a,1)\Big) \\ &= \lim_{a\to-\infty}e^{-\Phi(q)a}\Big(KZ_q(x-a,0)e^{-\Phi(q)(x-a)}-e^aZ_q(x-a,1)e^{-\Phi(q)(x-a)}\Big). \end{aligned}$$

Since we have $\lim_{a\to-\infty}e^{-\Phi(q)a}=\infty$ and $\lim_{a\to-\infty}\Big(KZ_q(x-a,0)e^{-\Phi(q)(x-a)}-e^aZ_q(x-a,1)e^{-\Phi(q)(x-a)}\Big)=K\frac{q}{\psi'(0)\Phi(q)}\in(0,\infty)$ from (7.2), we obtain $\lim_{a\to-\infty}h_a(x)=\infty>g(x)$. Hence, Condition (A) is satisfied for $q>0$.

[3] For the numerical analysis, we take $\psi(\theta)=c\theta-\lambda(1-\mu(\mu+\theta)^{-1})$ with $\mu=1.5$, $\lambda=1$, and $c=1.2$.

Next, we assume that $q = 0$. Then, noting that $Z(x, 0) = 1$, by the same calculation, we have $\lim_{a\to-\infty} h_a(x) = K > g(x)$. Hence, Condition (A) is satisfied.

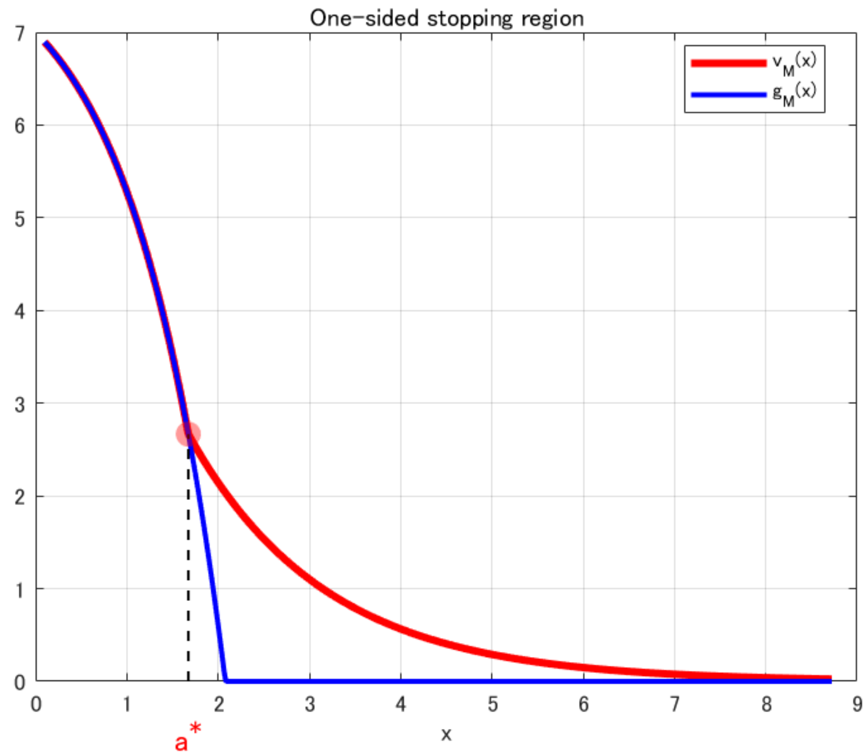

FIGURE 6. One-sided stopping region case

7.2. **Two-sided stopping region case.** We define the reward function g as follows:

$$g(x) = \begin{cases} K - e^x, & \text{if } x \le d, \\ \max\left\{(K - e^d) - l(x-d),\ 0\right\}, & \text{if } x \ge d, \end{cases}$$

where $K > 0$, $l > 0$, $d < \log K$ are constants. To simplify the discussion, we assume $q = 0$. If $\lim_{t\to\infty} X_t = -\infty$ holds with probability 1 or X oscillates, then there is no optimal stopping time as in Section 7.1. Hence, we can assume $\lim_{t\to\infty} X_t = \infty$ holds with probability 1 without loss of generality; and thus, we can skip *Step 1*. Let v be the value function. To exclude trivial cases, we additionally assume that the functions g and v_M intersect for $x > d$ and $d \in (a^*, \log K)$, where v_M and a^* are defined in Section 7.1. Let $p > d$ be the smallest intersection point of g and v_M. Note that as $d \to \log K$, the problem reduces to the McKean optimal stopping problem in Section 7.1.

We can show that g does not attain its maximum and satisfies Condition (A) by the same argument in Section 7.1 (*Step 2 and Step 3*).

(*Step 4*) We show that $g'(d+) > g'(d-)$ to carry out *Step 4*. Under the assumption $\psi'(0) > 0$, the scale function W is increasing and concave (see [2] and [8]), which implies that $Z(\cdot, 1)$ is also increasing and concave. Hence, since we have $h_a(x) = K - e^a Z(x - a, 1)$ as seen in Section 7.1, it follows that $h_a(x)$ is convex on (a, ∞). Recall that the functions g and $v_M|_{(a^*,\infty)} = h_{a^*}$ intersect for $x > d$ and $d \in (a^*, \log K)$. Assume to the contrary that $g'(d+) \le g'(d-)$. It follows that for any $x > d$

$$\begin{aligned}
v_M(x) = h_{a^*}(x) &= h_{a^*}(d) + \int_d^x h'_{a^*}(y)\mathrm{d}y \\
&\geq h_{a^*}(d) + \int_d^x g'_M(a^*)\mathrm{d}y \\
&\geq g_M(d) + \int_d^x g'_M(a^*)\mathrm{d}y = g(d) + \int_d^x g'(a^*)\mathrm{d}y \\
&> g(d) + \int_d^x g'(d-)\mathrm{d}y \\
&\geq g(d) + \int_d^x g'(d+)\mathrm{d}y = g(x),
\end{aligned}$$

where the first inequality is obtained from the fact that h_{a^*} is convex and $h'_{a^*}(a^*) = g_M((a^*)$, the second inequality is obtained from $h_{a^*}(d) = v_M(d) \geq g_M(d)$, the third inequality is obtained from the strictly concavity of g on $(-\infty, d)$, and the final inequality is obtained from the assumption that $g'(d+) \leq g'(d-)$. This contradicts to the assumption that g and $v_M|_{(a^*,\infty)} = h_{a^*}$ intersect for $x > d$. Therefore, since we have $g'(d+) > g'(d-)$, the number of subharmonic components $n \geq 2$ when X has paths of unbounded variation. Moreover, we can prove $n \geq 2$ holds in the case where X has paths of bounded variation. Since the proof is technical, we defer it to Appendix. We proceed to (*Step 6*).

(*Step 6*) Consider the system in (5.10). We have $h_a(x) = K - e^a Z(x-a, 1)$ as seen in Section 7.1. For $a \leq d$, since g is superharmonic, $h_a(x)$ is decreasing in a for each x. Moreover, h_a is convex by the same argument as above. Therefore, the system in (5.10) has at most one solution. Since $Z(\cdot, 1)$ is bounded, $h_a > g$ for sufficiently small a. On the other hand, h_a^* and g intersects Hence, (5.10) has the unique solution (a, b) such that $a < a^* < d < b$. Since $(-\infty, a)$ is superharmonic from $a < d$, it follows from Proposition 5.3, that (a, b) is contained in the continuation region.

Finally, define $g_1 := H_{a,b}g$ and we solve the problem for the reward function g_1 by applying Corollary 6.1. Note that it satisfies the assumption in Corollary 6.1 due to that g_1 is superharmonic on $(-\infty, d + (K - e^d)/l)$ and is subharmonic on $(d + (K - e^d)/l, \infty)$. The latter is clear from the fact $g_1 \equiv 0$ on $(d + (K - e^d)/l, \infty)$. To see the former, we define $s(x) = h_a(x) - g_1(x)$ for $x \in (-\infty, d + (K - e^d)/l)$. Then, $s = 0$ on $(-\infty, b)$, and s is increasing and concave at $(b, d + (K - e^d)/l)$. Thus, s is subharmonic on $(-\infty, d + (K - e^d)/l)$. Hence, g_1 is superharmonic on $(-\infty, d + (K - e^d)/l)$. Therefore, we can apply Corollary 6.1.

Define S and M as in Proposition 5.2. By Remark 5.3, S can be characterized as

$$S = \left\{ a \in \mathbb{R} : -\frac{\sigma^2}{2}l + \psi'(0)\Big((K - e^d) - l(a-d)\Big) - \int_0^\infty \mathrm{d}x \int_{(x,\infty)} \Pi(\mathrm{d}z)\,(g_1(x+a-z) - g_1(a)) = 0 \right\}.$$

From Corollary 6.1, if S is nonempty, then there exists $\tilde{a} \in S \cap M$. In this case, the stopping region Γ and the continuation region C are given by $\Gamma = (-\infty, a] \cup [b, \tilde{a}]$ and $C = (a, b) \cup (\tilde{a}, \infty)$, respectively. If $S = \varnothing$, then $\Gamma = \varnothing$ and $C = \mathbb{R}$.

To obtain a numerical solution, we specify X as the process whose Laplace exponent is

$$\psi(\theta) = c\theta - \lambda(1 - \mu(\mu + \theta)^{-1}), \tag{7.4}$$

where c is the drift rate, $\lambda > 0$ is the rate of arrival rate, and $\mu > 0$ is the parameter associated with the exponentially distributed jumps. We assume that $c - \lambda/\mu > 0$ so that $\psi'(0) > 0$. Then, the scale function W, as given in [20], and the function $Z(\cdot, 1)$, obtained from Equation (5.17) in [3], are expressed as follows:

$$W(x) = \frac{1}{c}\left(1 + \frac{\lambda}{c\mu - \lambda}\left(1 - e^{-(\mu - c^{-1}\lambda)x}\right)\right),\ Z(x, 1) = \frac{-\lambda}{(c\mu - \lambda)(\mu + 1)} e^{-(\mu - c^{-1}\lambda)x} + \frac{\psi(1)}{\psi'(0)}.$$

Moreover, S is identified by

$$S = \left\{ a \in \mathbb{R} : \left(c - \frac{\lambda}{\mu}\right)\left(l(d-a) + K - e^{d}\right) - \frac{1}{\mu}\int_{(0,\infty)} (g_1(a-y) - g_1(a))\lambda\mu e^{-\mu y} dy \right\}.$$

The parameters are set as $\mu = 1.5$, $\lambda = 1$, $c = 1.2$, $K = 8$, $l = 0.4$, and $d = 1.8$. Under these values, the computed solutions are $a = 1.5986$, $b = 3.7229$, and $\tilde{a} = 5.8136$. The corresponding graph of the solution is shown in Figure 7.

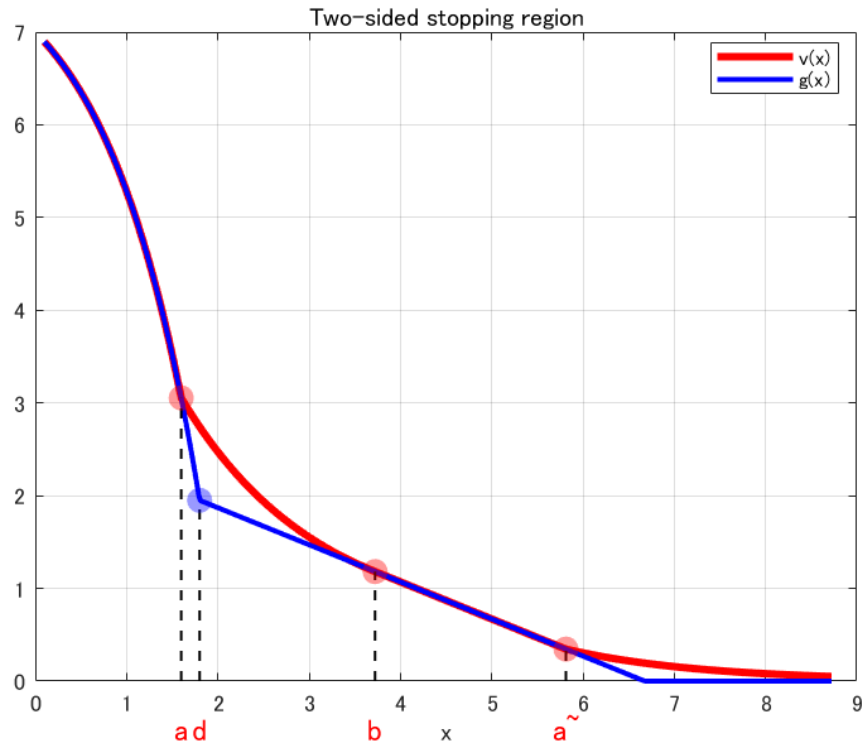

FIGURE 7. Two-sided stopping region case

7.3. **Hump-shaped case.** We consider the following hump-shaped reward function:

$$g(x) = \max\{1 - x^2, 0\}.$$

We assume that $q = 0$, $\lim_{t\to\infty} X_t = \infty$ $\mathbb{P}$-a.s., and $\int_{(x<0)} |x|\Pi(\mathrm{d}x) < \infty$ to simplify the discussion. Since we have $q = 0$ and $\lim_{t\to\infty} X_t = \infty$ $\mathbb{P}$-a.s., (*Step 1*) can be skipped. (*Step 2*) g attains its maximum at 0; and thus $\hat{g}$ is given by

$$\hat{g}(x) = \begin{cases} 1 & \text{if} \quad x \le 0, \\ 1 - x^2 & \text{if} \quad 0 \le x \le 1, \\ 0 & \text{if} \quad x \ge 1. \end{cases}$$

We denote $\hat{g}$ as g again, and we proceed to (*Step 4*).

(*Step 4*) We verify that g satisfies the assumption in Corollary 6.1. We have

$$\mathcal{L}g(x) = -2x\delta - \sigma^2 + I(x), \quad \text{with} \quad I(x) = \int_{(-\infty,0)} [g(x+y) - g(x)]\Pi(\mathrm{d}y), \tag{7.5}$$

where δ is defined in (1.2). Note that $I(0) = 0$ by the definition of g. Furthermore, since g is concave, I is increasing and convex. First, we show that I is increasing. For $x \in \mathbb{R}$ and $p > 0$, we have $I(x+p) = \int_{(-\infty,0)}[g(x+p+y)-g(x+p)]\Pi(\mathrm{d}y) \geq \int_{(-\infty,0)}[g(x+y) - g(x)]\Pi(\mathrm{d}y) = I(x)$, where the concavity of g is used to obtain the second inequality. Second, we show that I is convex. For $\alpha \in (0,1)$ and $x, x' \in \mathbb{R}$, we have

$$\begin{aligned}
I(\alpha x + (1-\alpha)x') &= \int_{(-\infty,0)} [g(\alpha(x+y) + (1-\alpha)(x'+y)) - g(\alpha x + (1-\alpha)x')]\Pi(\mathrm{d}y) \\
&\leq \int_{(-\infty,0)} \alpha g(x+y) + (1-\alpha)g(x'+y) - \alpha g(x) - (1-\alpha)g(x')\Pi(\mathrm{d}y) \\
&= \alpha I(x) + (1-\alpha)I(x'),
\end{aligned}$$

where the concavity of g is used to obtain the second inequality.

It follows from the fact that I is increasing and convex and $I(0) = 0$ that $\mathcal{L}g(0) \leq 0$ and the sign of $\mathcal{L}g$ changes once at most in $(0,1)$. Moreover, g is (super)harmonic on $(-\infty, 0)$ and subharmonic on $(1,\infty)$. Therefore, g satisfied the assumption in Corollary 6.1.

(*Step 6*) The remaining steps are the same as in Section 7.2. To obtain a numerical solution, the Laplace exponent of X is specified as (7.4). Then, we obtain from a straightforward calculation

$$S = \left\{ a \in (0,1) : (c - \lambda/\mu)(1-a^2) - \frac{2\lambda}{\mu^3}\left[e^{-\mu a} + (\mu a - 1)\right] \right\}.$$

The parameters are set as $\mu = 1.5$, $\lambda = 1$ and $c = 1.2$. Under these values, the computed solutions are $S = \{a^*\}$ with $a^* = 0.7260$. Hence, the stopping region Γ and continuation region C is given by $\Gamma = [0, a^*]$ and $C = (-\infty, 0) \cup (a^*, \infty)$. The corresponding graph of the solution is shown in Figure 8.

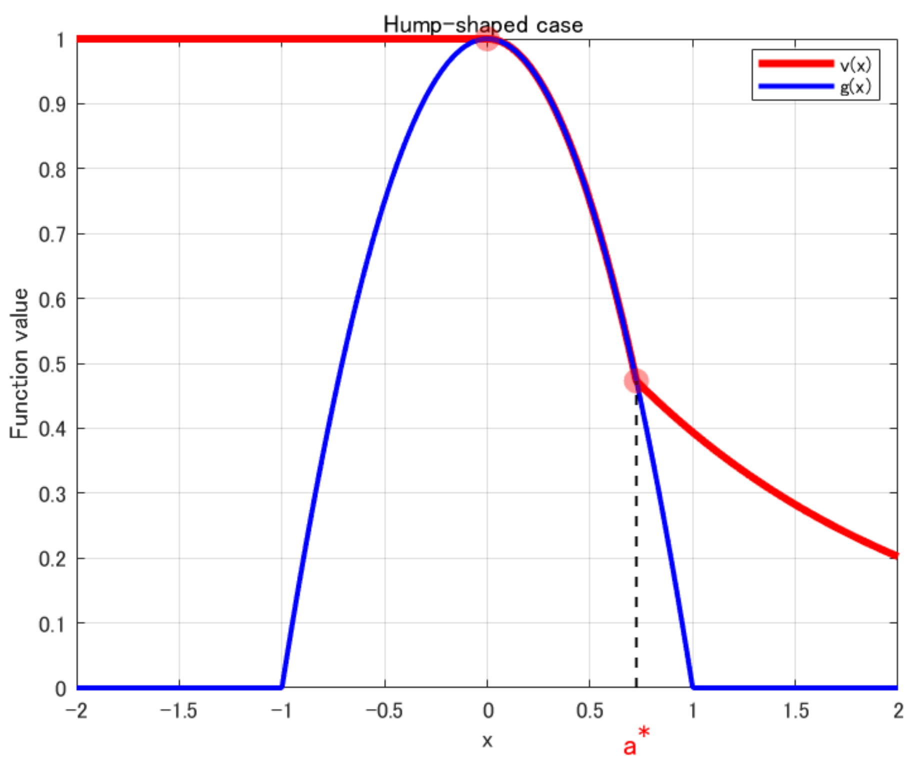

FIGURE 8. Hump-shaped case

Appendix A. Proofs

We collect in this appendix the proofs of Proposition 5.3, Corollary 5.2, and $n \geq 2$ in *Step 4* for the bounded variation case in Section 7.2.

Proof of Proposition 5.3. Before we start, the reader should be reminded of (2.7), (2.12), and the observations that follow as well as the κ function defined in (2.9). First, we show that $a' < \ell$. Consider the case where $\ell < r$. Since we have $h_{\ell+}(x) < g(x)$ for $x \in (\ell, r)$ from Lemma 5.3, it follows from the assumption (e), $g'(\ell+) - g'(\ell-) \geq 0$ that $h_\ell(x) \leq h_{\ell+}(x) < g(x)$ for $x \in (\ell, r)$. It follows, from $h_\ell(x) = \lim_{\varepsilon \downarrow 0} h_{\ell-\varepsilon}(x)$ for $x \geq \ell$ by the continuity of $g'(a)$ for $a < \ell$, that $h_{\ell-\varepsilon}(x) \leq g(x)$ for some $x \in (\ell, r)$, and thus; $a' < \ell$. The remaining case to consider is $\ell = r$ when X has paths of unbounded variation. Then, from the assumption (e), we have $g'(\ell+) - g'(\ell-) > 0$; and thus, $h'_\ell(\ell+) = g'(\ell-) < g'(\ell+)$ by (2.8) for the equality. Fix sufficiently small $\delta > 0$. From the continuity of g' on $(\ell, \ell+\delta)$, $h'_\ell(x+) < g'(x+)$ for $x \in (\ell, \ell+\delta)$. Hence, we obtain $g(\ell+\delta) = g(\ell) + \int_\ell^{\ell+\delta} g'(x+)\mathrm{d}x = h_\ell(\ell) + \int_\ell^{\ell+\delta} g'(x+)\mathrm{d}x > h_\ell(\ell) + \int_\ell^{\ell+\delta} h'_\ell(x+)\mathrm{d}x = h_\ell(\ell+\delta)$. Therefore, it follows from $h_\ell = \lim_{\varepsilon \downarrow 0} h_{\ell-\varepsilon}$ and the same discussion for the case where $\ell < r$ that $a' < \ell$ holds.

Next, we prove

$$h_{a'}(x) \geq g(x) \text{ for all } x. \tag{A.1}$$

If $h_{a'}(x) < g(x)$ for some x, then it follows from the continuity of $a \mapsto h_a(x)$ and Lemma 5.2 that $h_{a'-\varepsilon}(x) < g(x)$ for sufficiently small $\varepsilon > 0$, which is a contradiction to the definition of a'. Here, note that the continuity of the mapping $a \mapsto h_a(x)$ and the assumptions in Lemma 5.2 follow from the fact that g is of class C^1 (resp. class C^2) at an neighborhood of a' if X has paths of bounded variation (resp. unbounded variation). This regularity of g is in turn a consequence of the assumption (e) and $a' > p_n$.

(I) We start the proof of (I). First, by the assumption, we have

$$h_{a'}(x) = g(x) \text{ for some } x > a'. \tag{A.2}$$

From (A.1) and (A.2), there exists a closed set $E_0 \subset (a', \infty)$ such that E_0^{c} is not empty and

$$\begin{cases} h_{a'}(x) & = g(x);\ x \in E_0^{\mathrm{c}}, \\ h_{a'}(x) & > g(x);\ x \in E_0. \end{cases} \tag{A.3}$$

We show that E_0^{c} is compact. Assume to the contrary that E_0^{c} is not compact. Then, $(c, \infty) \subset E_0^{\mathrm{c}}$ for sufficiently large c. Thus, $h_{a'}(x) = g(x)$ for all $x \in (c, \infty)$. Letting $x \to \infty$, we obtain, from assumption (b), $h_{a'}(\infty) = 0$. Hence, we can apply Proposition 5.2 and the optimal stopping problem has a one-sided stopping region, which is a contradiction to the assumption of this proposition. Therefore, E_0^{c} and let b' be the maximum element of E_0^{c}. In particular,

$$h_{a'}(b') = g(b'). \tag{A.4}$$

Next, we prove that

$$h'_{a'}(b') = g'(b'). \tag{A.5}$$

Note that $h'_{a'}(b') \leq g'(b')$ holds: assume to the contrary that $h'_{a'}(b') > g'(b')$ holds, it follows from (A.4) that $h_{a'}(b'-\varepsilon) < g(b'-\varepsilon)$ for a sufficiently small $\varepsilon > 0$, which contradicts (A.3). Similarly, we can show the converse direction $h'_{a'}(b') \geq g'(b')$. To prove this, it suffices to consider $b'+\varepsilon$ instead of $b'-\varepsilon$. Therefore, in view of Remark 5.4, it follows from (A.4) and (A.5) that (a', b') solves (5.7). The fact that this solution (a', b') is maximal and the fact that $H_{a',b'}g = h_{a'}$ is majorant of g are clear from the construction of (a', b'). Thus, $E = (a', b') \cap E_0$ holds, where E is defined in (5.11). Moreover, since $v(x) \geq H_{a',b'}g(x) = h_{a'}(x) > g(x)$ for $x \in E = (a', b') \cap E_0$ from (A.3), E is contained in C.

Finally, we prove that $\mathcal{L}g(b') \le 0$. We only provide a proof for the case where X has paths of unbounded variation to avoid repeating analogous reasoning[4]. Noting that $h_{a'}^{''}(b') \ge g''(b')$ from (A.3), (A.4) and (A.5), we obtain

$$\begin{aligned}\mathcal{L}g(b') &= \delta g(b') + \frac{\sigma^2}{2} g''(b') + \int_{(y<0)} [g(b'+y) - g(b)]\Pi(\mathrm{d}y),\\ &= \delta h_{a'}^{'}(b') + \frac{\sigma^2}{2} g''(b') + \int_{(y<0)} [h_{a'}(b'+y) - h_{a'}(b')]\Pi(\mathrm{d}y),\\ &\le \delta h_{a'}^{'}(b') + \frac{\sigma^2}{2} h_{a'}^{''}(b') + \int_{(y<0)} [h_{a'}(b'+y) - h_{a'}(b')]\Pi(\mathrm{d}y) = \mathcal{L}h_{a'}(b') = 0,\end{aligned}$$

where the second equality is obtained from (A.4) and (A.5) and the final equality is obtained from the harmonicity of $h_{a'}(b')$.

(II) We now proceed to the proof of (II). By the assumption, $h_{a'}(x) > g(x)$ for all $x > a'$ Let $d > a'$ be arbitrary. By the assumption for contradiction, $h_{a'}(x) > g(x)$ for $x \in [a', d]$. Take any $x \in [a', d]$. It follows from the continuity of $a \mapsto h_a(x)$ and Lemma 5.2 that $h_{a'+\varepsilon(x)}(x) > g(x)$ for sufficiently small $\varepsilon(x) > 0$.

Since $h_{a'+\varepsilon(x)}(\cdot)$ and g are continuous, there exists a open neighborhood $U(x)$ of x such that $h_{a'+\varepsilon(x)} > g$ on $U(x)$. Hence, by the compactness of $[a', d]$, there exists finite many points $x_1, x_2 \dots, x_N$ $(N < \infty)$ such that

$$[a', d] \subset \bigcup_{i=1}^{N} U(x_i).$$

Define $\varepsilon_d := \min\{\varepsilon(x_i) : i = 1, \dots, N\} > 0$. Then $h_{a'+\varepsilon} > g$ on $[a', d]$ for any $\varepsilon \in (0, \varepsilon_d)$. Thus, by the definition of a', we obtain

(A.6) For any $0 < \varepsilon < \varepsilon_d$, there exists $x \in (d, \infty)$ such that $h_{a'+\varepsilon}(x) \le g(x)$.

Take $d > x_0$, the latter defined in the assumption (b), and $\varepsilon \in (0, \varepsilon_d)$. Note that $h_{a'+\varepsilon}(y) \le 0$ for some $y \in (d, \infty)$ from (A.6) and $h_{a'+\varepsilon} = g \ge 0$ on $(-\infty, a'+\varepsilon]$. Then $h_{a'+\varepsilon}(x) \le 0$ for all $x \ge y$. Indeed, if $h_{a'+\varepsilon_d}(x) > 0$ for some $x \ge y$, then there exists $c > x_0$ such that $h_{a'+\varepsilon}(c) \le 0$, $h'_{a'+\varepsilon}(c) = 0$, and $h_{a'+\varepsilon}(c) < h_{a'+\varepsilon}(x)$ for all $x \le c$, and thus; $\mathcal{L}h_{a'+\varepsilon}(c) > 0$ from Proposition 2.3, which is a contradiction to the fact that $h_{a'+\varepsilon}$ is harmonic on $(a'+\varepsilon, \infty)$.

Now we have obtained $h_{a'+\varepsilon}(x) \le 0$ for all $x \ge y$. In particular, $\kappa(a'+\varepsilon) = \lim_{x\to\infty} \frac{h_{a'+\varepsilon}(x)}{W(x-a'-\varepsilon)} \le 0$, where the first equality is obtained from (2.9). Taking the limit as $\varepsilon \to 0$ and using the continuity of κ at a', we obtain $\kappa(a') = \lim_{x\to\infty} \frac{h_{a'}(x)}{W(x-a')} \le 0$. On the other hand, recall that we started with the assumption that $h_{a'}(x) > g(x)$ for all $x > a'$, we have $\kappa(a') = \lim_{x\to\infty} \frac{h_{a'}(x)}{W(x-a')} > \lim_{x\to\infty} \frac{g(x)}{W(x-a')} \ge 0$. Hence, $\kappa(a') = 0$. Therefore, from (2.10), $H_{a'}g(x) = h_{a'}(x) > g(x)$ for $x > a'$. Combining this with $H_{a'}g(x) = g(x)$ for $x \le a'$, it follows that $H_{a'}g$ is majorant of g. Moreover, since g is superharmonic on $(-\infty, a')$ from the assumption (e), we could apply Proposition 5.2 to conclude that the optimal stopping problem has a one-sided stopping region.

(III) Finally, (III) follows from Lemma 5.1.

□

Proof of Corollary 5.2. In the proof of Proposition 5.3, we should note that Lemma 5.3 can be applied in a neighborhood of a' because g is of class C^2 near a'. Therefore, in the case where g attains its maximum at β, it is necessary to show that $\beta < a'$ in order to ensure that $\hat{g}$ is C^2 in a neighborhood of a': recall that $\hat{g}$ is not assumed to be in C^2 at β. The inequality $\beta \le a'$ follows immediately from the definitions of β and a'. We now show that $\beta \ne a'$. Suppose, to the contrary, that $\beta = a'$. Then, by the proof of Proposition 5.3, (A.2) holds, i.e., there exists $x > \beta$ such that $\hat{h}_\beta(x) = \hat{g}(x)$. However, since we have $\hat{g}'(\beta) = 0$ and $\hat{g}(x) = \hat{g}(\beta)$ for all $x \le \beta$ by the definition of β, we have $\hat{h}_\beta(x) = \hat{g}(\beta) > \hat{g}(x)$ from (2.7), which

[4] In the bounded variation case, $h_{a'}(b') = g(b')$ holds and we can derive $h_{a'}^{'}(b') \ge g'(b')$ in the similar manner.

is a contradiction. Therefore, $\beta < a'$ holds. The rest of the argument proceeds in exactly the same manner as in the proof of Proposition 5.3.

□

Finally, we prove $n \geq 2$ holds in the bounded variation case in *Step 4* in Section 7.2.

Proof. Assume to the contrary that $n = 1$. Let $f(x) = \mathbb{1}_{\{x \leq p\}}(v_M - g)$. We have $f \equiv 0$ on $(a^*, p)^{\mathrm{c}}$ from the fact that $v_M = g$ on $(-\infty, a^*]$. By the construction of p, $v_M \geq g$ on (a^*, p); and thus, $f \geq 0$. Since $f'(a^*) = v_M'(a^*) - g'(a^*) = 0$ holds, $v_M|_{(a^*,\infty)} = h_{a^*}|_{(a^*,\infty)}$ is convex on (a^*, ∞) and g is concave on (a^*, d), it follows that $f(d) > f(x)$ for $x \in [a^*, d)$. Moreover, since g is linear on (d, p), v_M is convex on (d, p) and $g(p) = v_M(p)$ holds, it follows that $f(d) > f(x)$ for $x \in (d, p]$. Hence, f attains its maximum at d.

Consider the Riesz decomposition of $H_{(p,\infty)^{\mathrm{c}}} f$. Note that $H_{(p,\infty)^{\mathrm{c}}} f$ also attains its unique maximum at d. As in the proof of Proposition 3.1, we obtain the following decomposition:

$$H_{(p,\infty)^{\mathrm{c}}} f(x) = \int_{(-\infty,p)} G(x, y)\mu(\mathrm{d}y) + h(x),$$

where h is a harmonic function and μ is a signed measure such that

$$\begin{cases} \mu(\mathrm{d}y) & = -\mathcal{L}f(y)\ ;\ y \in (-\infty, p) \setminus \{d\}, \\ \mu(\{d\}) & = 0, \\ \mu(\{p\}) & = -\delta(H_{(p,\infty)^{\mathrm{c}}} f(p+) - H_{(p,\infty)^{\mathrm{c}}} f(p-)). \end{cases}$$

Note that $\mu(\{p\}) = -\delta(H_{(p,\infty)^{\mathrm{c}}} f(p+) - H_{(p,\infty)^{\mathrm{c}}} f(p-)) = -\delta H_{(p,\infty)^{\mathrm{c}}} f(p+) \leq 0$. Moreover, since $-\mathcal{L}f(y) = \mathcal{L}g(y) \leq 0$ for $y \in (-\infty, p) \setminus \{d\}$ from the assumption of contradiction, $-\mu$ is a measure; and thus, $-H_{(p,\infty)^{\mathrm{c}}} f$ is excessive. Hence, it follows that $\{H_{(p,\infty)^{\mathrm{c}}} f(X_t)\}_{t \geq 0}$ is submartingale ([14, Proposition 2.1.1]). Therefore, letting $B(d)$ be a bounded open neighborhood of d and letting τ_d be the first exit time from $B(d)$, it follows that $H_{(p,\infty)^{\mathrm{c}}} f(d) \leq \mathbb{E}^d[H_{(p,\infty)^{\mathrm{c}}} f(X_{t \wedge \tau_d})] = \int H_{(p,\infty)^{\mathrm{c}}} f(y)\mathbb{P}^d(X_{t \wedge \tau_d})$. However, this contradicts the fact that $H_{(p,\infty)^{\mathrm{c}}} f$ attains its unique maximum at d. Therefore, we conclude that $n \geq 2$.

□

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