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Prudential Capital Controls or Bailouts? The Impact of Different Collateral Constraint Assumptions*

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Abstract

A fast growing literature on small open economy models with pecuniary externalities has provided the theoretical grounds for the policy analysis of macro prudential regulations. Using the framework of Jeanne and Korinek (2010), we investigate whether a *subsidy* on debt during crises as a form of bailout can outperform prudential capital controls. We show that the result depends on the functional form of the collateral constraint faced by households. If households collateralize their assets that they purchase at the same time as their borrowing, subsidizing debt during crises is preferable. If, on the other hand, the maximum borrowing is constrained by the value of their assets that they have purchased before they borrow, a stronger case can be made for prudential capital controls.

JEL Classification: E32, F38, G01, G18

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1 Introduction

Should policymakers in emerging market economies rely on macro prudential policies during normal times or bail out borrowers at the time of a financial crisis? Recent studies have highlighted the importance of prudential controls on cross-border capital flows and macro prudential regulations to prevent inefficient boom-bust cycles. Among others, Jeanne and Korinek (2010, hereafter JK), Bianchi (2011), and Bianchi and Mendoza (2012) emphasize that market-determined asset prices can generate pecuniary externalities that distort financing decisions of economic agents. A key ingredient in their models is an occasionally binding collateral constraint that depends on the market value of the collateral. Based on the framework with this constraint, these early studies advocate that the government should preemptively impose a Pigouvian tax on debt during normal times and internalize the externalities.[1]

This paper extends the JK model to consider policy prescriptions for coping with crises. In particular, we consider two assumptions on the collateral constraint and investigate how the difference in the assumptions affects policy prescriptions for coping with crises. The two collateral constraints differ in the timing in which households' assets are collateralized. To emphasize the timing, we call the collateral constraints either a "beginning-of-period" or an "end-of-period" collateral constraint. The "beginning-of-period" collateral constraint is assumed in JK. Under this constraint, the households collateralize the assets that they have purchased before they borrow. On the other hand, under the "end-of-period" collateral constraints, households collateralize the assets that they purchase at the same time as borrowing. This end-of-period collateral constraint has also been employed by other studies (e.g., Kiyotaki and Moore, 1997 and Bianchi and Mendoza, 2012).

We find that the choice between the two assumptions is not innocuous. We show that, under the beginning-of-period collateral constraint, subsidizing debt during crises cannot achieve better allocation than that under the laissez-faire economy. In other words, the bailout is neutral. By contrast, subsidizing debt during crises can achieve the first-best allocation if households are subject to an "end-of-period" collateral constraint.

Our results suggest that policymakers should have knowledge of the structure of the collateral constraint that the households are faced with. As JK and others suggested, prudential

[1] More recent papers include Benigno, Chen, Otrok, Rebucci, and Young (2012a, 2012b, 2013), Bianchi (2013), Bianchi and Mendoza (2013), Dávilla (2011), Jeanne and Korinek (2012), Korinek and Simsek (2013). Farhi and Werning (2012, 2013) and Schmitt-Grohé and Uribe (2012, 2013) argue for the prudential capital controls under the nominal price or wage rigidities.

capital controls can achieve the second-best allocation. If the beginning-of-period collateral constraint is a more plausible assumption, a policy prescription is that policymakers should rely on the prudential capital controls via the Pigouvian tax on debt. This is because a bailout through subsidizing debt is neutral in terms of welfare. Conversely, under the end-of-period collateral constraint, the bailout can achieve the first-best allocation and thus is more desirable than the prudential capital controls.

In the literature, early studies have noted that the government can restore the first-best allocation in some models with an occasionally binding collateral constraint. For example, using a model similar to Bianchi (2011), Benigno, Chen, Otrok, Rebucci, and Young (2012a, hereafter BCORY) show that, if the government can use additional distortionary policy instruments on top of capital controls (e.g., a price support policy in the form of a subsidy on collateral or collateralized nontradable goods), then the collateral constraint can always be removed and the government can achieve the first-best allocation.[2] Our finding starkly contrasts with BCORY. In this paper, "bailout" refers to the subsidy on foreign debt. To keep the comparison between prudential capital controls and bailouts fair, we assume that the government is given a single policy instrument that affects the cost of borrowing from abroad (i.e., intervention in the credit market). In our model, the government can achieve the first-best allocation without relying on additional distortionary policy measures such as outright purchase of the collateral and subsidy on nontradable good consumption.

If bailouts are theoretically desirable, the next question for policymakers would be whether the bailouts are practically feasible. To answer this question, the present paper also performs numerical experiments and assesses the size and frequency of policy interventions in credit markets. Our assessment suggests that strong expectations for bailouts akin to moral hazard lead to an unrealistically large bailout size and a high frequency of intervention. In fact, on average, a large lump-sum tax equivalent to as much as 31.6 percent of annual household income must be imposed to achieve the first-best allocation. On the frequency, the government needs to intervene almost every year. These results point to a large gap compared with the actual observation because very few – perhaps no – governments in emerging market economies have embarked on such large bailouts with such high frequency. Our experiments

[2]See also Jeanne and Korinek (2012) and Benigno, Chen, Otrok, Rebucci, and Young (2012b) for the case where multiple policy instruments are available for crisis management. Schmitt-Grohé and Uribe (2013) show that the open economy model with downward nominal wage rigidities can give rise to pecuniary externalities. They argue that bailouts by devaluation of the country's currency could restore the first-best allocation. Likewise, using a model of banking, Green (2010) argues that bailouts can lead to a socially efficient outcome.

suggest that further research be called for to explore what can fill the gap.

This paper is organized as follows. Section 2 introduces the JK model with the two collateral constraints used for our comparisons. Section 3 extends the JK model to consider the optimal debt subsidy as a bailout. Section 4 performs numerical experiments. Section 5 concludes.

2 Two collateral constraints

We begin with a small open economy model developed by JK. Suppose that the utility of identical atomistic households is given by $u(c_0) + u(c_1) + c_2$, where $u(c) = c^{1-\sigma}/(1-\sigma)$, $\sigma > 0$, and c_t represents consumption in period t. Domestic households' budget constraints for each period are given by

$$\begin{aligned} c_0 + p_0\theta_1 &= d_1 + p_0 & (1)\\ c_1 + d_1 + p_1\theta_2 &= e + d_2 + p_1\theta_1 & (2)\\ c_2 + d_2 &= \theta_2 y, & (3)\end{aligned}$$

where d_t is the debt to be repaid at the beginning of the period t and θ_t represents the domestic collateral held by the households at the beginning of period t. Here, p_t is the price of collateral traded in a competitive market. Throughout all periods, the world interest rate is set to zero for simplicity. At the beginning of period 0, the households hold one unit of collateral. In this period, they borrow d_1 from abroad and purchase θ_1 as well as the consumption goods.[3] In period 1, the households have three sources of inflow: sales of collateral $p_1\theta_1$, new borrowing d_2, and an endowment e that is not pledgeable to foreign lenders. They use them for consumption c_1, repaying d_1, and purchasing collateral θ_2. In the final period, the households must repay d_2 after receiving returns on collateral y. Following JK, we assume that the return on collateral can be acquired only by domestic agents and the value of collateral in period 2 is lost after the households receive y. We also assume that the supply of collateral assets is inelastic and normalized to one. In this JK model, the linear utility in period 2 implies that, if there is no collateral constraint, consumption in periods 0 and 1 is unity (i.e., the first-best level of consumption).

The model introduces a collateral constraint for d_2. As JK discuss, a low value of e may result in the binding collateral constraint and precipitate a crisis (e.g., sudden stop in

[3] Without loss of generality, the initial value of foreign debt is set to zero.

capital inflows). With the binding collateral constraint in period 1, the desired borrowing is generally impossible and the households must accept a large reduction in c_1. As such, period 1 corresponds to the period of a crisis.

2.1 The beginning-of-period collateral constraint

Each household faces a collateral constraint of the form

$$d_2 \leq \phi\theta_1 p_1. \tag{4}$$

Note that, following JK, the borrowing capacity is constrained by the market value of collateral at the *beginning* of period 1. The parameter $\phi \in (0, 1]$ represents the ceiling on the leverage in the collateral constraint.[4] In a symmetric equilibrium, θ_1 must be unity because the supply of collateral asset is one. Not surprisingly, pecuniary externalities arise from the feedback loop between the collateral price and borrowing. When a sufficiently low e takes place, the collateral constraint binds. The households try to prevent consumption reduction by decreasing net demand for their collateral. The households' deleveraging results in declines in collateral prices, and the decline in p_1 further tightens their collateral constraints. While each atomistic household takes p_1 as given, the households' decision as a whole has the general equilibrium effect on p_1. As a result, the general equilibrium effect cannot be internalized by price-taking households and the laissez-faire equilibrium is not generally Pareto efficient. This result calls for the "macro prudential policies" that have been widely discussed in the literature.[5]

JK show that, in their stochastic model where e is random, a Pigouvian tax τ can replicate the *second-best* allocation solved by the constrained social planner.[6] Replace the period-1 budget constraint (2) by

$$c_1 + (1+\tau)\, d_1 - T + p_1\theta_2 = e + d_2 + p_1\theta_1, \tag{5}$$

where the government runs a balanced budget: the lump-sum transfers T equal tax on debt

[4] We consider a slightly more general constraint because JK assume $\phi = 1$.

[5] Examples of this research include Bianchi (2011), Jeanne and Korinek (2011), Korinek (2011), Bianchi and Mendoza (2012) and Benigno et al. (2012a, 2012b, 2013).

[6] See JK for details on the constrained social planner's problem in maximizing the households' utility subject to the resource constraints and the same collateral constraint as that of households.

τd_1. In JK, the following macro prudential tax is proposed:

$$\tau = \frac{\phi \mathbb{E}_0 \left[\lambda_{sp} p' \left(m_1\right)\right]}{\mathbb{E}_0 \left[u' \left(c_1\right)\right]}, \tag{6}$$

where $\mathbb{E}_0$ denotes the expectations operator conditional on the information at $t = 0$.[7] In (6), λ_{sp} is the Lagrange multiplier for the collateral constraint that the social planner faces, $u'(c_1)$ denotes the marginal utility of consumption, and $p'(m_1) > 0$ is the derivative of p_1 with respect to the level of the liquid net worth $m_1 \equiv e - d_1$. The shadow price of holding debt λ_{sp} and the asset pricing function $p(m_1)$ are obtained from the constrained social planner's problem in which she internalizes the general equilibrium effect of m_1 on p_1.[8] This prudential tax on debt reduces borrowing in the pre-crisis period and can mitigate reductions in asset prices in the crisis period.

2.2 The end-of-period collateral constraint

As a variant of the collateral constraint (4), consider

$$d_2 \leq \phi \theta_2 p_1, \tag{7}$$

where the value of collateral is evaluated by the *end*-of-period holding of collateral θ_2, rather than θ_1. As discussed in JK, their key results on prudential taxes remain the same across the two collateral constraints:[9] (i) the same collateral constraint in equilibrium (i.e., $d_2 \leq \phi p_1$); (ii) the same feedback loop between the collateral price and borrowing; and (iii) the same form of the Pigouvian tax. These results remain essentially unaltered even under the infinite-horizon setting.[10]

[7] More specifically, the information set does not include the realization of e.

[8] The asset pricing function is given by $p(m_1) = y/u'(c_1) = y/u'(d_2 + m_1)$ and differs from the asset pricing function under the laissez-faire economy. In other words, JK follow the "constrained efficiency" definition of Kehoe and Levine (1993) in their social planner's problem.

[9] See footnote 4 in JK.

[10] Jeanne and Korinek (2011) numerically reconfirm the robustness to the assumptions on collateral constraints in Appendix A.2.

3 The optimal debt subsidy as a bailout

We consider how changes in the assumption regarding the collateral constraint affect bailouts. For simplicity, we assume in this section that e is deterministic rather than stochastic. This simplification allows us to obtain an explicit solution for the optimal debt subsidy, but has no effect on our argument.

To introduce bailouts, we replace (2) by

$$c_1 + d_1 + p_1\theta_2 = e + (1+s)\,d_2 - S + p_1\theta_1, \tag{8}$$

where $s \geq 0$ is a subsidy on debt and S is the lump-sum tax, satisfying $S = sd_2$.[11] A balanced government budget ensures that household resources are kept unchanged both intra- and inter-temporally. Thus, the only distinction between the prudential capital controls and bailouts is the question of whether to raise the cost of debt before a crisis or to reduce it during one.

In the following two subsections, we will present propositions on the optimal subsidy under the two differing collateral constraints, (4) and (7). Then, we will interpret the two propositions in the context of policy implications.

3.1 The beginning-of-period collateral constraint

The households maximize their utility $u(c_0) + u(c_1) + c_2$, subject to the budget constraints (1), (8), (3) and the beginning-of-period collateral constraint (4). The first-order conditions are

$$u'(c_0) = u'(c_1) \tag{9}$$

$$(1+s)\,u'(c_1) = 1 + \lambda_m \tag{10}$$

$$p_0 = p_1\left[\frac{u'(c_1) + \lambda_m\phi}{u'(c_0)}\right] \tag{11}$$

$$p_1 = \frac{y}{u'(c_1)}. \tag{12}$$

Here λ_m represents the Lagrange multiplier for (4). In (10), the households choose d_2 by comparing the marginal cost $1+\lambda_m$ on the right-hand side with the marginal benefit $(1+s)\,u'(c_1)$

[11] Following Jeanne and Korinek (2012), our policy analysis rules out the possibility that the government uses the non-distortionary inter-temporal lump-sum taxes and transfers to fully relax the collateral constraint.

on the left-hand side. Other things being equal, a higher subsidy on debt encourages a household to hold more debt during a crisis. The asset pricing equations in each period are given by (11) and (12). In (11), $\lambda_m \phi p_1$ represents the extra benefit of holding more collateral under the beginning-of-period collateral constraint. This extra benefit increases only p_0 but has no effect on p_1.

The first proposition establishes that, under the beginning-of-period collateral constraint, subsidizing debt during a crisis does not improve the welfare, compared to the laissez-faire economy.

Proposition 1 *Suppose that the household maximizes the utility of $u(c_0)+u(c_1)+c_2$ subject to (1), (8), (3) and the beginning-of-period collateral constraint (4). Then, the optimal subsidy on debt s^* is zero if $(2-e)/y \leq \phi \leq 1$. If $0 < \phi < (2-e)/y$, on the other hand, the allocation is fully independent of s and is equivalent to the allocation under the laissez-faire economy.*

Proof. It is straightforward to obtain the unconstrained first-best allocation: $c_{FB,0} = c_{FB,1} = d_{FB,1} = 1$, $c_{FB,2} = y - 2 + e$, and $d_{FB,2} = 2 - e$. Likewise, it can be easily shown that the price of collateral under the first-best allocation is $p_{FB,0} = p_{FB,1} = y$. For $(2-e)/y \leq \phi \leq 1$, the collateral constraint does not bind under the laissez-faire economy ($d_{FB,2} < \phi p_{FB,1}$). Hence, the optimal subsidy s^* is trivially zero. We next consider the case of $0 < \phi < (2-e)/y$, which means that the collateral constraint binds under the laissez-faire economy. In this case, together with (12), (4) implies (i) $d_2 = \phi y / u'(c_1)$. Because of the market clearing conditions for collateral (i.e., $\theta_1 = \theta_2 = 1$) and the government budget constraint, (1), (8), and (3) can be simplified to (ii) $c_0 = d_1$, (iii) $c_1 + d_1 = e + d_2$, and (iv) $c_2 + d_2 = y$. Furthermore, (9) implies that (v) $c_0 = c_1$. The allocation in this decentralized economy, $\{c_0, c_1, c_2, d_1, d_2\}$, can be fully determined by (i) - (v) if the unique equilibrium exists. Because (i) - (v) do not include s, the resulting allocation is fully independent of s. Therefore, the allocation must be equivalent to that under the laissez-faire economy. ■

3.2 The end-of-period collateral constraint

We next consider the same bailout under the end-of-period collateral constraint (7). While the first-order conditions (9) and (10) remain the same as before, (11) and (12) must be replaced by

$$p_0 = p_1 \tag{13}$$

$$p_1 = \frac{y}{u'(c_1) - \lambda_m \phi}, \tag{14}$$

respectively. In contrast to (11) and (12), the extra benefit of holding the collateral $(\lambda_m \phi p_1)$ affects both p_0 and p_1 under the end-of-period collateral constraint.

The next proposition states that the optimal s can replicate the unconstrained first-best allocation. Proposition 2 summarizes our second main result.

Proposition 2 *Suppose that the household maximizes the utility of $u(c_0)+u(c_1)+c_2$ subject to (1), (8), (3), and the end-of-period collateral constraint (7). Then, there exists time-consistent s^* with which the decentralized equilibrium achieves the unconstrained first-best allocation, and s^* is equal to the equilibrium shadow price of holding debt in the decentralized economy:*

$$s^* = \lambda_m (s^*) \geq 0. \tag{15}$$

If $0 < \phi < (2-e)/y$, s^* is given by

$$s^* = \frac{1}{\phi} - \frac{y}{2-e} > 0. \tag{16}$$

Otherwise, $s^* = 0$.

Proof. As in the proof of Proposition 1, the optimal subsidy for $(2-e)/y \leq \phi \leq 1$ is trivial: $s^* = \lambda_m = 0$ and the collateral constraint does not bind under s^*. For $0 < \phi < (2-e)/y$, $d_{FB,2} > \phi p_{FB,1}$. If a positive s^* exists, it must be the case that $\lambda_m(s^*) > 0$ and $d_{FB,2} = \phi p_1$. Here, from (10), we note that λ_m depends on s. Substituting the unconstrained first-best allocation into (10) and (14) yields $s = \lambda_m(s)$ and $p_1 = y/(1-s\phi)$, respectively. Therefore, s^* in (16) is obtained by eliminating p_1 from $p_1 = y/(1-s\phi)$ and $d_{FB,2} = \phi p_1$. Because $d_{FB,2}$ is feasible under p_1 satisfying $p_1 = y/(1-s^*\phi)$, s^* achieves the first-best allocation $\{c_{FB,0}, c_{FB,1}, c_{FB,2}, d_{FB,1}, d_{FB,2}\}$. The solution s^* is a function of exogenous variables and hence time-consistent.[12] ■

[12] In the appendix that is available upon request, we show that Proposition 2 can be extended to the continuous time version of the model.

3.3 Interpretation

Proposition 1 indicates that, if the collateral constraint is given by (4) as in JK, subsidizing debt during crises does not improve the welfare. This result implies that the prudential capital controls that can achieve the second-best allocation are strictly preferred to the bailout. By contrast, however, Proposition 2 shows that, if the collateral constraint is replaced by (7), the optimal subsidy s^* can prevent fire sales of collateral and achieve the first-best allocation. In terms of policy prescription, the bailout outperforms the prudential capital controls under the assumption of the end-of-period collateral constraint.

The key to understanding our result is the price of collateral p_1 at a time of crisis. Suppose that the collateral constraint binds in period 1 (i.e., $0 < \phi < (2-e)/y$). In this case, the government may wish to intervene in the credit market with $s > 0$ if it can inflate the price of collateral. Moreover, the households can enjoy even the first-best level of consumption if the asset price is inflated to ensure that $d_{FB,2} = \phi p_1$. Therefore, $d_{FB,2}$ uniquely determines the target level of the asset price for the government:

$$p_1 = \frac{d_{FB,2}}{\phi} = \frac{2-e}{\phi} > y, \tag{17}$$

where the strict inequality is ensured by the assumption of $\phi < (2-e)/y$.

The question is whether the government can inflate asset prices. Under (4), the subsidy on new borrowing has no effect on households' decisions because their borrowing capacity is predetermined by $\phi\theta_1 p_1$. Households' demand for collateral (θ_2) is not stimulated by the subsidy. As a result, p_1 remains unaffected by the subsidy and the only option for the households is to demand less collateral (θ_2) to compensate for consumption. After all, the allocation turns out to be the same as that under the laissez-faire economy with a low price of collateral. By contrast, under (7), households know that if they buy more collateral (θ_2), then they can borrow more, because θ_2 affects their borrowing capacity. The lower cost of new debt owing to the subsidy provides them with financing for the purchase of new collateral (θ_2). This financing for the new collateral purchase stimulates the demand, which results in higher p_1.

From the viewpoint of the government, p_1 under (14) can be seen as a function of the policy instrument s. In particular, along with (10), the asset pricing equation (14) can be expressed as

$$p_1 = \frac{y}{[1-(1+s)\phi]\,u'(c_1) + \phi}.$$

This equation indicates that the government can control the price of collateral by choosing s. If the government sets $s = s^* = \lambda_m(s^*)$,

$$p_1 = \frac{y}{1 - s^* \phi}. \tag{18}$$

This s^* is consistent with the target price given by (17), while satisfying all the first-order conditions and the constraints. Hence, under the end-of-period collateral constraint, the government can achieve the first-best allocations.

In the literature, BCORY propose a different policy prescription that can restore the first-best allocation. In this regard, Proposition 2 in our paper is related to their findings. The optimal subsidy in our model, however, starkly differs from the BCORY's prescription in three respects. First, the policy instrument in this paper differs from BCORY's. The government in BCORY subsidizes the household collateral purchase, while that in our model subsidizes household borrowing. Intervening in collateral markets may require additional capacity (i.e., additional policy instruments) of the government.[13] Our argument is applicable to any government that has the capacity to implement capital controls (i.e., the capacity to affect the cost of borrowing). The second difference is slightly technical. Under the BCORY's prescription, the collateral constraint never binds. In the context of our model, this means that $d_{FB,2} < \phi p_1$ and $\lambda_m = 0$. To ensure the effectiveness of the optimal subsidy in our model, λ_m needs to take strictly positive values, because, as indicated by (14), the non-zero Lagrange multiplier enables asset price inflation. Third, the BCORY's prescription can also be interpreted as crisis prevention rather than bailouts, because crises never take place in BCORY. By contrast, the optimal subsidy in our model requires actual fiscal expenditure and, in fact, needs to be activated as a tangible intervention.

4 Quantitative evaluation of the optimal subsidy

Proposition 2 in the previous section indicates that theoretically there exists an "optimal bailout" in the form of subsidy on debt during a crisis. However, a natural question for policymakers would be whether the optimal subsidy is *practically* feasible. To answer this question, we extend the three-period model to a stochastic infinite-horizon model that can

[13] See Propositions 3 and 4 in BCORY. They employ the collateral constraint in which the income from tradable and non tradable endowments can be pledged as collateral as in Bianchi (2011). Using this setup of the model, the government commits to supporting the relative price of non tradables to tradables during crises by either a subsidy on nontradable good consumption or a tax on tradable good consumption.

be calibrated to the data. Using the extended model, we examine (i) whether the policy intervention for implementing the "optimal bailout" is realistic in terms of the size; and (ii) how frequently the policy intervention needs to be made to restore the first-best allocation. As a preparatory step, we first show that the result of Proposition 2 continues to hold even in the case of a stochastic infinite-horizon model and that the optimal subsidy on debt becomes state-contingent. We then explore how s^* fluctuates over time to assess the size and frequency of bailouts in a calibrated model.

4.1 The infinite-horizon model

We consider the stochastic infinite-horizon model, similar to Jeanne and Korinek (2011) and Bianchi and Mendoza (2012). The households choose d_{t+1} and θ_{t+1} to maximize

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u(c_t)\right], \tag{19}$$

where β is the discount factor satisfying $\beta \in (0,1)$. Each household faces the period-by-period budget constraint:

$$c_t + d_t + p_t\theta_{t+1} = \theta_t e_t + (1+s_t)\frac{d_{t+1}}{R} - S_t + p_t\bar{\theta}, \tag{20}$$

and the end-of-period occasionally binding collateral constraint:

$$\frac{d_{t+1}}{R} \leq \phi p_t \theta_{t+1}. \tag{21}$$

In this maximization problem, d_{t+1} is non-state-contingent one-period debt. The real interest rate on the non-state-contingent debt is $R > 1$ rather than unity. Every period, each household receives the exogenous endowment of collateral $\bar{\theta}$, which is normalized to one. He receives a stochastic income e_t based on the predetermined share of collateral assets θ_t (i.e., dividends). As before, the value of collateral is lost after receiving the return on collateral. The budget constraint here is basically the same as (8), but is also similar to (3) in terms of the returns on collateral. Finally, the collateral constraint is the same as (7) except for R.

The first-order conditions are standard:

$$
\begin{aligned}
(1+s_t)\,u'(c_t) &= \beta R\mathbb{E}_t u'(c_{t+1}) + \lambda_{m,t} && (22)\\
p_t &= \beta\frac{\mathbb{E}_t\left[u'(c_{t+1})\,e_{t+1}\right]}{u'(c_t)-\lambda_{m,t}\phi} && (23)\\
0 &= \left[\phi p_t\theta_{t+1}-\frac{d_{t+1}}{R}\right]\lambda_{m,t},\ \lambda_{m,t}\geq 0,\ \text{and}\ \phi p_t\theta_{t+1}-\frac{d_{t+1}}{R}\geq 0. && (24)
\end{aligned}
$$

In equilibrium, the markets for collateral and consumption goods clear: $\theta_{t+1}=\bar{\theta}$ and $c_t+d_t = e_t+d_{t+1}/R$ for all t, respectively. As before, we assume a balanced budget of the government: $S_t = s_t d_{t+1}/R$.

The first-best allocation must satisfy the following first-order conditions:

$$
\begin{aligned}
u'(c_{FB,t}) &= \beta R\mathbb{E}_t\left[u'(c_{FB,t+1})\right] && (25)\\
p_{FB,t} &= \beta\frac{\mathbb{E}_t\left[u'(c_{FB,t+1})\,e_{t+1}\right]}{u'(c_{FB,t})}, && (26)
\end{aligned}
$$

which yield the policy functions $c_{FB,t}=c_{FB}(d_t,e_t)$ and $d_{FB,t+1}=d_{FB}(d_t,e_t)$ as functions of state variables. Likewise, the asset pricing function is obtained from the model without the collateral constraint: $p_{FB,t}=p_{FB}(d_t,e_t)$. The following proposition shows that there exists a state-contingent optimal subsidy $s_t = s(d_t,e_t)$ consistent with the first-best allocation in the stochastic infinite-horizon model.

Proposition 3 *Suppose that the household maximizes (19) subject to (20) and the end-of-period collateral constraint (21). Then, there exists a price function $p_s(d_t,e_t)$ and a time-consistent subsidy $s^*(d_t,e_t)$, with which the decentralized equilibrium characterized by (20)-(24) achieves the unconstrained first-best allocation $\{c_{FB,t},d_{FB,t+1}\}_{t=0}^{\infty}$. Furthermore, the subsidy $s_t^*=s^*(d_t,e_t)$ is proportional to the Lagrange multiplier for (21):*

$$s^*(d_t,e_t)=\frac{\lambda_m(d_t,e_t;s_t^*)}{u'\left[c_{FB}(d_t,e_t)\right]}\geq 0, \tag{27}$$

where $\lambda_m(d_t,e_t;s_t)$ represents the Lagrange multiplier for the collateral constraint, given s_t. This subsidy s_t^* is time-consistent. Furthermore, if $d_{FB}(d_t,e_t)/R>\phi p_{FB}(d_t,e_t)$, $s^*(d_t,e_t)$ is given by

$$s^*(d_t,e_t)=\frac{1}{\phi}-\frac{p_{FB}(d_t,e_t)}{d_{FB}(d_t,e_t)/R}>0. \tag{28}$$

Otherwise, $s^*(d_t,e_t)=0$.

Proof. We consider two cases for the states of the economy (d_t, e_t): (i) $d_{FB}(d_t, e_t)/R \leq \phi p_{FB}(d_t, e_t)$ and (ii) $d_{FB}(d_t, e_t)/R > \phi p_{FB}(d_t, e_t)$. Here, the conditions distinguish whether or not the first-best level of debt is feasible in the laissez-faire economy. For each case, we will confirm that the first-order conditions (22)–(24) are satisfied under $s^*(d_t, e_t)$ when they are evaluated at the first-best allocation $c_{FB}(d_t, e_t)$ and $d_{FB}(d_t, e_t)$.

Consider the states satisfying (i). If $s^*(d_t, e_t) = 0$, $\lambda_m[d_t, e_t, s^*(d_t, e_t)] = 0$. Then, the first-order conditions of (22) and (23) are the same as (25) and (26), and the allocation is the first-best. Therefore, $p_s(d_t, e_t) = p_{FB}(d_t, e_t)$ characterized by (26). The optimality of $s^*(d_t, e_t) = 0$ is confirmed for the states satisfying (i).

Next, for the states satisfying (ii), consider the price of collateral that achieves $d_{FB}(d_t, e_t)$ with the binding collateral constraint. If

$$p_s(d_t, e_t) = \frac{d_{FB}(d_t, e_t)}{\phi R}, \tag{29}$$

then (24) is satisfied together with $\lambda_m(d_t, e_t; s^*(d_t, e_t)) \geq 0$. Combining (27), (23), and (29) yields

$$\frac{d_{FB}(d_t, e_t)}{\phi R} = \beta \frac{\mathbb{E}_t\left(u'\left\{c_{FB}\left[d_{FB}(d_t, e_t), e_{t+1}\right], e_{t+1}\right\} e_{t+1}\right)}{u'\left[c_{FB}(d_t, e_t)\right](1 - s_t^*\phi)}, \tag{30}$$

for all states (d_t, e_t) satisfying (ii). Using the above equation and (26), we can solve for $s_t^* = s^*(d_t, e_t)$ and the solution turns out to be (28). Because s_t^* is chosen to satisfy (30), (23) and (24) are obviously satisfied at the first-best allocation. Finally, using (27), (25) can be rewritten as

$$[1 + s^*(d_t, e_t)]\, u'\left[c_{FB}(d_t, e_t)\right] = \beta R \mathbb{E}_t\left(u'\left\{c_{FB}\left[d_{FB}(d_t, e_t), e_{t+1}\right]\right\}\right) + \lambda_m\left[d_t, e_t; s^*(d_t, e_t)\right],$$

which is exactly (22) under $s_t = s^*(d_t, e_t)$. Therefore, all the first-order conditions (22)–(24) are satisfied under $s^*(d_t, e_t)$. This s^* is completely determined by the current state of the economy and hence is time-consistent. ■

The proposition confirms that, as in the case of the three-period model, the optimal subsidy $s^*(d_t, e_t)$ can achieve the first-best allocation. The mechanism behind the result is the same as that in the three-period model. The government can fully control the asset price to avoid fire sales by choosing s_t.

4.2 The size and frequency of bailouts

4.2.1 Calibration

The parameters are mainly taken from Bianchi and Mendoza (2012). For the household's preference, $u(c_t) = c_t^{1-\sigma}/(1-\sigma)$, with $\sigma = 2.0$. The discount factor β is set to 0.96, calibrating the model to the annual frequency. The ceiling on the household borrowing per collateral asset ϕ is set at 0.36. We assume that the total factor productivity in Bianchi and Mendoza (2012) can be translated into the stochastic process of the dividend e_t in our model. The stochastic process of the total factor productivity in Bianchi and Mendoza (2012) follows a log-normal AR(1) process. We thus employ the same stochastic process for e_t as theirs: $\log(e_{t+1}) = \rho \log(e_t) + \eta_{t+1}$, where $\eta_t \sim N(0, \sigma_\varepsilon)$ for all t. Here ρ and σ_ε are set to 0.53 and 0.014, respectively. To ensure stationary $d_{FB}(d_t, e_t)$, we assume that households face a small risk premium on their foreign debt.[14] Specifically, we replace the budget constraint (20) with a slightly different form:

$$c_t + d_t + p_t\theta_{t+1} = \theta_t e_t + (1+s_t)\frac{d_{t+1}}{R_{t+1}} - S_t + p_t\bar{\theta},$$

where $R_{t+1} = R + \psi[\exp(d_{t+1}) - 1]$, R =1.028 and ψ =0.01.[15]

To confirm that parameters are reasonably calibrated, we simulate the laissez-faire economy with the occasionally binding collateral constraint. We interpret GDP in our model as $e_t\bar{\theta}$. In the simulation, the mean of the debt-to-GDP ratio is 33.5 percent. This ratio is in close proximity to the calibration target in Benigno et al. (2013), who calibrate their model to the Mexican economy based on the updated dataset of Lane and Milesi-Ferretti (2007). We also compute the probability of binding collateral constraints in the laissez-faire economy as a proxy of the crisis probability. The probability is 6.57 percent, which is broadly in line with the literature.[16]

[14] See Schmitt-Grohé and Uribe (2003).

[15] It is straightforward to show that Proposition 3 can be extended to the case with a risk premium on foreign debt.

[16] For reasonable values of the crisis probability, Benigno et al. (2013) target the crisis probability at 8 percent per year. In the empirical studies, Basel Committee on Banking Supervision (2010) reports two empirical crisis probabilities for 11 advanced economies over 1985-2009, based on the datasets of Reinhart and Rogoff (2008) and Laeven and Valencia (2008). The estimated crisis probabilities are 5.2 percent in Reinhart and Rogoff (2008) and 4.1 percent in Laeven and Valencia (2008).

4.2.2 Simulation results

Table 1 compares the moments generated by the model without intervention (i.e., the laissez-faire economy) with those generated under the optimal subsidy. Overall, the optimal subsidy on debt takes extremely large values, meaning that massive bailouts are required to restore the first-best allocation. On average, s_t^* is 67.0 percent. To finance this subsidy, the government needs to impose a large lump-sum tax equivalent to as much as 31.6 percent of annual household income ($e\bar{\theta}$). Figure 1 plots the simulated path of the optimal subsidy over 300 periods. The figure indicates that the government activates bailouts very frequently. In the figure, s_t^* takes a value of zero only in period 13. This means that, over 300 periods, bailouts are activated 299 times.

The reason for the massive and frequent bailouts is straightforward. Expectations of bailouts strongly incentivize households to hold a large amount of debt. As long as the debt in the first-best allocation is sufficiently large compared to that in the laissez-faire economy, the collateral constraint is likely to bind. Whenever the collateral constraint binds, the government bails out households by encouraging new borrowing that is used to repay the large amount of existing debt. Given their expectations that they will be bailed out, it is optimal for households to roll over the large amount of new borrowing. This new borrowing substantially increases the probability that the collateral constraint will continue to bind in the next period. As a result, the government almost always needs to bail out households.

We argue that there is a gap between theory and data. For comparison, we take an empirical estimate reported by Frydl (1999). Based on Frydl (1999), an empirically realistic size of a bailout would amount to somewhere between 1 to 3 percent of GDP for a single year in the aftermath of a crisis.[17] Comparisons of the sizes of intervention between the model and the practice suggest that the optimal subsidy would be difficult to implement, perhaps because of political conflicts which are not taken into account in our model.

[17] Frydl (1999) discusses two empirical works estimating (i) fiscal costs (resolution cost) and (ii) the length of financial crises as his baseline estimates (Caprio and Klingebiel, 1996 and Lindgren, Garcia, and Saal, 1996). In the two empirical papers, the average fiscal costs are 13.6 percent of GDP in the former and 7.2 percent in the latter, whereas the average duration of crises is 4.5 years and 6.2 years, respectively. We compute the average fiscal cost for a single year by dividing average total fiscal costs by the average length of financial crises, suggesting that the average fiscal cost for a single year ranges from 1.16 (7.2/6.2) to 3.02 (13.6/4.5) percent of GDP.

4.2.3 Sensitivity of simulation results

Although unrealistically large and frequent bailouts seem difficult to implement, the size and frequency depend on parameters in the model. Therefore, one could argue that it is possible to attain a realistic size and frequency of bailouts by changing parameters in the model. To consider this argument, this subsection discusses whether the difficulty of the optimal subsidy is robust to changes in the model parameters.

Proposition 3 suggests that s_t^* can take lower values under looser collateral constraint (i.e., a larger ϕ). Under a larger ϕ, the government may activate bailouts less often, because the collateral constraint is less likely to bind. Hence, in the economy with a large ϕ, the optimal subsidy can be practically feasible. In Figure 2, we plot the optimal subsidy in the economy with $\phi = 0.60$. With this parameterization, the maximum value of s_t^* over 300 periods is 2.90 percent and the size of policy intervention measured by the lump-sum tax relative to GDP is 1.71 percent. On the frequency, the incidence of the policy activation is two out of 300 periods. Thus, bailouts appear to be easier to implement in an economy with $\phi = 0.60$. However, the model fails to explain crisis probabilities under the laissez-faire economy: in our simulations, the probability of binding collateral constraints in the laissez-faire economy is only 0.25 percent, much lower than that reported in the previous studies. Therefore, it is sensible to conclude that, although an "optimal bailout" may be implementable in an economy resilient to negative shocks, it would not be feasible in most economies that are fragile to shocks.

The optimal subsidy s_t^* can be also affected by the risk-premium parameter ψ. In our benchmark simulation, ψ is set to a somewhat large value of 0.01. In the literature on the small open economy real business cycle model, this parameter is usually set to a very small number to calibrate the model without the collateral constraint to match the volatility of the debt-to-GDP ratio. For example, Schmitt-Grohé and Uribe (2003) set this parameter to 0.0007. We could have used their parameterization, but the level of the debt in this case is much higher than what we obtained in the benchmark simulation. Because (28) indicates that a larger amount of foreign debt *increases* the size and frequency of bailouts, this low ψ makes the optimal subsidy more unrealistic.

5 Concluding remarks

This paper analyzed the roles of bailouts in managing financial crises. Using the simple framework employed by Jeanne and Korinek (2010), we showed that the timing assumption of the collateral constraint has a non-negligible impact on the debate regarding macro prudential policies and bailouts. If we employ Jeanne and Korinek's (2010) original functional form of the collateral constraint, the policy that subsidizes debt during crises does not increase the welfare, and thus this bailout fails to outperform the prudential capital controls. If households can collateralize their assets that they purchase at the same time as their borrowing, the same policy can improve the welfare much more than the prudential capital controls. Our analytical results in the deterministic three-period model can be extended into a more general stochastic infinite-horizon framework. Using the extended model, we examined the practical feasibility of the optimal subsidy on debt. In terms of the practical feasibility, however, the optimal subsidy would be massive in size and require very frequent activation. We conclude that, while an "optimal bailout" is theoretically feasible, in practice it would be difficult to implement.

We presented our results in the simplified models to obtain the results analytically. As in the seminal works by Jeanne and Korinek (2010) and Bianchi (2011), we considered an endowment economy without an endogenous labor supply. To extend the analysis into a more general framework would be an important step for future research.

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A Appendix

This appendix describes a continuous-time version of the model in Section 3.

A.1 Setup

Suppose that the households live only between 0 and 1, and maximize their life-time utility. The households' maximization problem is

$$\max_{\{c_t, x_t, \theta_t, d_t\}_{t=0}^{1}} \int_0^1 u(c_t)dt + c_1^+, \tag{31}$$

$$\begin{aligned}
\text{s.t. } \dot{d}_t &= p_t x_t + c_t - \varepsilon - s_t\left(d_t + \dot{d}_t\right) + S_t & (32)\\
\dot{\theta}_t &= x_t & (33)\\
\dot{d}_t &\leq \phi p_t(\theta_t + x_t) - d_t \text{ for } t \in [0,1] & (34)\\
c_1^+ + d_1 &= \theta_1 y, & (35)
\end{aligned}$$

where $u(c) = c^{1-\sigma}/(1-\sigma)$ and $\sigma > 0$. To be consistent with the setup in the discrete time model, we introduce the linear (instantaneous) utility at an instant after $t = 1$. This consumption is denoted by c_1^+. If there were no collateral constraint, the marginal utility is one and all resources remaining at $t = 1$ are consumed at this instant. Because the households obtain returns on collateral y at $t = 1$, the remaining resources are $\theta_1 y - d_1$. This is represented by (35). Regarding other constraints, (32) is the budget constraint, where ε is an endowment that is not pledgeable to foreign lenders. In (33), we introduce x_t to represent changes in the demand for collateral. The households are also faced with the end-of-period collateral constraint (34).[18] In this problem, we assume that the initial conditions are $d_0 = 0$ and $\theta_0 = 1$.

Before deriving the first-order conditions, we rewrite the constraints. First, (32) can be written as

$$\dot{d}_t = \frac{1}{1+s_t}(p_t x_t + c_t - \varepsilon - s_t d_t + S_t). \tag{36}$$

[18] The beginning-of-period collateral constraint can be written as $\dot{d}_t \leq \phi p_t \theta_t - d_t$

Using (36), we eliminate $\dot{d}_t$ from (34) and rewrite the collateral constraint as

$$\phi p_t(\theta_t + x_t) - \frac{1}{1+s_t}(p_t x_t + c_t - \varepsilon + d_t + S_t) \geq 0 \tag{37}$$

The Lagrangian is given by

$$\begin{aligned}
\mathbb{L} &= \int_0^1 u(c_t)dt + (\theta_1 y - d_1) \\
&\quad + \int_0^1 \gamma_t \left[\frac{1}{1+s_t}(p_t x_t + c_t - \varepsilon - s_t d_t + S_t) - \dot{d}_t\right] dt + \int_0^1 \mu_t \left[x_t - \dot{\theta}_t\right] dt \\
&\quad + \int_0^1 \lambda_t \left[\phi p_t(\theta_t + x_t) - \frac{1}{1+s_t}(p_t x_t + c_t - \varepsilon + d_t + S_t)\right] dt \\
&= \int_0^1 \left\{H\left(c_t, x_t, d_t, \gamma_t, \mu_t\right) + \lambda_t \left[\phi p_t(\theta_t + x_t) - \frac{1}{1+s_t}(p_t x_t + c_t - \varepsilon + d_t + S_t)\right]\right\} dt \\
&\quad + (\theta_1 y - d_1) - \int_0^1 \left(\gamma_t \dot{d}_t + \mu_t \dot{\theta}_t\right) dt,
\end{aligned} \tag{38}$$

where $H\left(c_t, x_t, d_t, \gamma_t, \mu_t\right)$ is given by

$$H\left(c_t, x_t, d_t, \gamma_t, \mu_t\right) = u(c_t) + \gamma_t \left[\frac{1}{1+s_t}(p_t x_t + c_t - \varepsilon - s_t d_t + S_t)\right] + \mu_t x_t.$$

Further, applying the integral by parts to $\int_0^1 \left(\gamma_t \dot{d}_t + \mu_t \dot{\theta}_t\right) dt$ in (38), we have

$$\begin{aligned}
\mathbb{L} &= \int_0^1 \left\{H\left(c_t, x_t, d_t, \gamma_t, \mu_t\right) + d_t \dot{\gamma}_t + \theta_t \dot{\mu}_t\right. \\
&\quad \left. + \lambda_t \left[\phi p_t(\theta_t + x_t) - \frac{1}{1+s_t}(p_t x_t + c_t - \varepsilon + d_t + S_t)\right]\right\} dt \\
&\quad + (\theta_1 y - d_1) - \gamma_1 d_1 - \mu_1 \theta_1 + \mu_0.
\end{aligned} \tag{39}$$

The first-order conditions for c_t, x_t, d_t, and θ_t are

$$\begin{aligned} u'(c_t) &= \frac{\lambda_t - \gamma_t}{1+s_t} && (40)\\ \frac{(\gamma_t - \lambda_t)\, p_t}{1+s_t} + \mu_t &= -\lambda_t \phi p_t && (41)\\ \dot{\gamma}_t &= \frac{\lambda_t + s_t \gamma_t}{1+s_t} && (42)\\ \dot{\mu}_t &= -\lambda_t \phi p_t, && (43) \end{aligned}$$

respectively. The terminal conditions for γ_t and μ_t are obtained from the first-order condition for d_1 and θ_1:

$$\begin{aligned} \gamma_1 &= -1 && (44)\\ \mu_1 &= y && (45) \end{aligned}$$

The government budget is assumed to be balanced at every t: $s_t\left(d_t + \dot{d}_t\right) = S_t$ and the supply of collateral is constant (i.e., $\theta_t = 1$ for $t \in [0,1]$). Using the assumptions of the balanced government budget and the constant supply of collateral (i.e., $\dot{\theta}_t = x_t = 0$), the laws of motion for d_t and the complementary slackness condition can be simplified. In particular, the equilibrium law of motion for d_t is

$$\dot{d}_t = c_t - \varepsilon, \tag{46}$$

and the complementary slackness condition is

$$(\phi p_t - d_t - c_t + \varepsilon)\,\lambda_t = 0, \lambda_t \geq 0,\ \phi p_t - d_t - c_t + \varepsilon \geq 0, \tag{47}$$

where the last inequality can be obtained from (37) as follows:

$$\begin{aligned} 0 &\leq \phi p_t(\theta_t + x_t) - \frac{1}{1+s_t}(p_t x_t + c_t - \varepsilon + d_t + S_t)\\ &= \phi p_t - \frac{1}{1+s_t}\left[c_t - \varepsilon + d_t + s_t\left(\dot{d}_t + d_t\right)\right]\\ &= \phi p_t - \frac{1}{1+s_t}\left[c_t - \varepsilon + s_t\,(c_t - \varepsilon) + (1+s_t)\,d_t\right]\\ &= \phi p_t - d_t - c_t + \varepsilon. \end{aligned}$$

Here the second equality uses the government budget constraint and the third equality comes from (46).

It is straightforward to obtain the unconstrained first-best allocation in this model. Suppose that $\lambda_t = 0$ and $s_t = 0$. Let variables with subscript *FB* be the variables holding under the unconstrained economy. Then (42) reduces to $\dot{\gamma}_{FB,t} = 0$. This implies that, together with (44), $\gamma_{FB,t} = -1$ for $t \in [0,1]$ since $\gamma_{FB,t}$ is constant. Under the assumption of the CRRA utility, it immediately follows from (40) and (46) that $c_{FB,t} = 1$ and $\dot{d}_{FB,t} = 1 - \varepsilon$ for $t \in [0,1]$. The differential equation for $d_{FB,t}$ yields $d_{FB,t} = (1-\varepsilon)\,t$. To summarize, the first-best allocation is

$$d_{FB,t} = (1-\varepsilon)\,t \tag{48}$$

$$c_{FB,t} = 1 \tag{49}$$

for $t \in [0,1]$ and $c^{+}_{FB,1} = y - (1-\varepsilon)$. To obtain the price of collateral in the unconstrained economy, (41) and (43) become $p_{FB,t} = \mu_{FB,t}$ and $\dot{\mu}_{FB,t} = 0$, respectively. Because $\mu_1 = y$ from (45), it immediately follows

$$p_{FB,t} = y \text{ for } t \in [0,1]\,. \tag{50}$$

A.2 The bailouts under the end-of-period collateral constraint

In the continuous time model, we have the following proposition similar to Proposition 2 in the paper.

Proposition A.1. *Suppose that the household maximizes the utility of $\int_0^1 u(c_t)dt + c_1^{+}$ subject to (32), (33), (35) and the end-of-period collateral constraint (34). Then, there exists time-consistent subsidy s_t^* with which the decentralized equilibrium achieves the unconstrained first-best allocation and s_t^* is equal to the equilibrium shadow price of holding debt in the decentralized economy:*

$$s_t^* = \lambda_t \geq 0.$$

Depending on the tightness of the collateral constraint, the optimal subsidy s_t^* is characterized in the following three cases.

Case (i): *If* $0 < \phi < (1-\varepsilon)\left[3\exp(-1)-1\right]/\left[y\exp(-1)\right]$*,* $s_t^* > 0$ *for* $t \in [0,1]$ *and given by*

$$s_t^* = \frac{3\exp(t-1)-1}{(t+1)\phi} - \frac{y\exp(t-1)}{(t+1)(1-\varepsilon)} \tag{51}$$

Case (ii): *If* $(1-\varepsilon)\left[3\exp(-1)-1\right]/\left[y\exp(-1)\right] \leq \phi < 2(1-\varepsilon)/y$*,* $s_t^* = 0$ *for* $t \in [0,\tau]$ *but (51) holds for* $t \in (\tau, 1]$*, where* $\tau = 1 - \log\left[3 - y\phi/(1-\varepsilon)\right]$*.*
Case (iii): *If* $2(1-\varepsilon)/y \leq \phi$*,* $s_t^* = 0$ *for* $t \in [0,1]$*.*

To prove Proposition A.1, we need to show that, for all cases, we can construct a set of continuously differentiable price, Lagrange multiplier, and costate variables $\{p_t, \lambda_t, \mu_t, \gamma_t\}_{t\in[0,1]}$, where $\{c_{FB,t}, d_{FB,t}\}_{t\in[0,1]}$ and $\{p_t, \lambda_t, \mu_t, \gamma_t\}_{t\in[0,1]}$ satisfy the first-order conditions (40)-(43) and the complementary slackness condition (47) with the optimal subsidy $\{s_t^*\}_{t\in[0,1]}$.

In the proof, we first consider Case (iii). In this case, the collateral constraint does not bind and thus there is no need to activate the bailouts (i.e., $s_t^* = 0$ for all $t \in [0,1]$) to achieve the first-best allocation. Next, we turn to Cases (i) and (ii). Case (i) represents that the collateral constraint always binds for all $t \in [0,1]$ while Case (ii) represents that it binds only for $t \in [\tau, 1]$. In both cases, we show that all first-order conditions are satisfied under the first-best allocation and the optimal subsidy.

The first step is to show that, under the condition for ϕ in Case (iii), the first-best allocation is achieved under $s_t^* = \lambda_t = 0$ for $t \in [0,1]$. The condition $\phi \geq 2(1-\varepsilon)/y$ is equivalent to

$$\begin{aligned} 0 &\leq \phi y - 2(1-\varepsilon) \\ &\leq \phi p_{FB,t} - (1-\varepsilon)t - (1-\varepsilon) \\ &= \phi p_{FB,t} - d_{FB,t} - c_{FB,t} + \varepsilon, \end{aligned}$$

for all $t \in [0,1]$. Thus, $d_{FB,t} + c_{FB,t} - \varepsilon \leq \phi p_{FB,t}$ for all $t \in [0,1]$ if $\phi \geq 2(1-\varepsilon)/y$. From (47), $\lambda_t = 0$ for all $t \in [0,1]$, and $\mu_{FB,t} = p_{FB,t} = y$ trivially satisfy the first order conditions.

Next, we consider Cases (i) and (ii), where s_t^* can be positive for some t. Under $s_t = \lambda_t$, (40) can be written as

$$\frac{\lambda_t - \gamma_t}{1+\lambda_t} = u'(c_{FB,t}) = 1 \Rightarrow \gamma_t = -1 \text{ for } t \in [0,1]. \tag{52}$$

This implies $\dot{\gamma}_t = 0$ from (42) and is consistent with (44). Note that $c_{FB,t} = 1$ and (48) are consistent with $c^{+}_{FB,1} = y - (1 - \varepsilon)$ due to (46). Also, (52) simplifies (41) to

$$\mu_t \;=\; p_t - \lambda_t \phi p_t. \tag{53}$$

For Case (i), we suppose that $\lambda_t > 0$ for all $t \in [0, 1]$ and construct prices under the supposition. We then confirm that the supposition of $\lambda_t > 0$ is consistent with all first-best allocation and the constructed μ_t and p_t. To compute μ_t, we solve the differential equation for μ_t under $s_t = \lambda_t > 0$:

$$\dot{\mu}_t - \mu_t + \frac{d_{FB,t} + c_{FB,t} - \varepsilon}{\phi} = 0,$$

which can be obtained from (43), (53), and the complementary slackness condition implying $p_t = (d_{FB,t} + c_{FB,t} - \varepsilon)/\phi$. The above equation immediately follows from (48) and (49) that

$$\dot{\mu}_t - \mu_t + \frac{(t+1)(1-\varepsilon)}{\phi} = 0. \tag{54}$$

The general solution to this differential equation is

$$\mu_t = \frac{(t+2)(1-\varepsilon)}{\phi} + C\exp(t), \tag{55}$$

where C denotes an arbitrary constant. By the terminal condition (45), (55) is

$$\mu_t = \frac{(t+2)(1-\varepsilon)}{\phi} + \left[y - \frac{3(1-\varepsilon)}{\phi}\right]\exp(t-1). \tag{56}$$

Taking derivative of this equation with respect to t yields

$$\dot{\mu}_t = \frac{1-\varepsilon}{\phi} + \left[y - \frac{3(1-\varepsilon)}{\phi}\right]\exp(t-1). \tag{57}$$

Now, we can find the path of λ_t and the optimal subsidy shown in the proposition. Substituting (57) and $p_t = (d_{FB,t} + c_{FB,t} - \varepsilon)/\phi = (t+1)(1-\varepsilon)/\phi$ into (43) yields

$$\frac{1-\varepsilon}{\phi} + \left[y - \frac{3(1-\varepsilon)}{\phi}\right]\exp(t-1) = -\lambda_t (t+1)(1-\varepsilon).$$

Solving for λ_t, we have

$$\begin{aligned}\lambda_t &= -\frac{1}{\phi(t+1)} - \left[\frac{y}{(1-\varepsilon)(t+1)} - \frac{3}{\phi(t+1)}\right]\exp(t-1)\\ &= \frac{3\exp(t-1)-1}{(t+1)\phi} - \frac{y\exp(t-1)}{(t+1)(1-\varepsilon)}, \qquad (58)\end{aligned}$$

Note that $s_t = \lambda_t$ leads to (51) in the proposition.

To complete the proof for Case (i), we must confirm the supposition $\lambda_t > 0$ as well as $\mu_t > 0$ for all $t \in [0, 1]$, consistent with the interpretation as the Lagrange multiplier. Here, if we eliminate y from (58) using the condition $\phi < (1-\varepsilon)[3\exp(-1) - 1]/[y\exp(-1)]$ in Case (i), we can confirm that $\lambda_t > 0$ for all $t \in [0, 1]$. Moreover, (43) implies that $\dot{\mu}_t < 0$ for all $t \in [0, 1]$ because $\lambda_t > 0$ and $p_t = (d_{FB,t} + c_{FB,t} - \varepsilon)/\phi > 0$ for all $t \in [0, 1]$. Note that evaluating (56) at $t = 0$, we have

$$\mu_0 = \frac{2(1-\varepsilon)}{\phi} + \left[y - \frac{3(1-\varepsilon)}{\phi}\right]\exp(-1) > 0$$

because $2 - 3\exp(-1) > 0$. From the terminal condition (15), $\mu_1 = y > 0$. Therefore, given that $\mu_0 > 0$, $\mu_1 > 0$, and $\dot{\mu}_t < 0$ for all $t \in [0, 1]$, μ_t must be positive for $t \in [0, 1]$, consistent with the interpretation as the Lagrange multiplier for $\dot{\theta}_t = x_t$.

For Case (ii), we need to consider the time periods $t \in [0, \tau]$ and $t \in (\tau, 1]$ separately. In particular, we suppose that $\lambda_t = 0$ for $t \in [0, \tau)$ and $\lambda_t > 0$ for $t \in [\tau, 1]$, and show that the constructed λ_t and μ_t are consistent with all first-order conditions and the supposition.

For $t \in (\tau, 1]$, we construct μ_t by solving the differential equation for μ_t under $\lambda_t > 0$ as in Case (i). From (57), $\ddot{\mu}_t = [y - 3(1-\varepsilon)/\phi]\exp(t-1)$, and $\ddot{\mu}_t < 0$ because Case (ii) satisfies $\phi < 3(1-\varepsilon)/y$. Since $\lambda_t = 0$ at $t = \tau$ where $\tau = 1 - \log[3 - y\phi/(1-\varepsilon)]$, we have $\dot{\mu}_t = 0$ at $t = \tau$. Therefore, $\ddot{\mu}_t < 0$ implies that $\dot{\mu} < 0$ for all $t \in (\tau, 1]$, confirming the supposition of $\lambda_t > 0$. Furthermore, the terminal condition for $\mu_1 = y$ and $\dot{\mu}_t < 0$ for all $t \in (\tau, 1]$ ensure that μ_t must be positive for all $t \in (\tau, 1]$.

For $t \in [0, \tau)$, the supposition of $s_t = \lambda_t = 0$ implies that $\mu_t = p_t$ from (53) and $\dot{\mu}_t = 0$ from (43). Hence, p_t and μ_t are constant at $p_t = \mu_t = \mu_0$ for $t \in [0, \tau]$. This μ_0 is obtained from (56) at $t \to \tau$. That is,

$$\begin{aligned}\mu_0 &= \mu_\tau\\ &= \frac{(\tau+1)(1-\varepsilon)}{\phi} = \frac{d_{FB,\tau} + c_{FB,\tau} - \varepsilon}{\phi} = p_\tau > 0,\end{aligned}$$

because μ_t must be continuously differentiable for all $t \in [0,1]$ including $t = \tau$. This positive μ_t is consistent with the interpretation as the Lagrange multiplier. Finally, we confirm the supposition of $\lambda_t = 0$ for $t \in [0, \tau)$. The obtained μ_τ implies

$$\frac{d_{FB,t} + c_{FB,t} - \varepsilon}{\phi} < \frac{d_{FB,\tau} + c_{FB,\tau} - \varepsilon}{\phi} = p_\tau = p_t,$$

for any $t \in [0, \tau)$, leading to $\phi p_t > d_{FB,t} + c_{FB,t} - \varepsilon$. This is consistent with the complementary slackness condition under $\lambda_t = 0$ and thus confirms the supposition of $\lambda_t = 0$.

Table 1: Average and standard deviations of the variables

	Laissez-faire economy	Bailouts
Debt	0.335	0.471
	(0.016)	(0.045)
Asset price	0.962	1.273
	(0.018)	(0.121)
Consumption	0.990	0.985
	(0.015)	(0.009)
Subsidy (percent)	-	67.0
		(22.4)
Fiscal cost (percentage of GDP)	-	31.6
		(13.2)

Note: The numbers in each entry report the average and the standard deviations of the debt position, asset price, and consumption in the laissez-faire economy and the economy with bailouts. The numbers without parentheses refer to the average of variables, and the numbers in parentheses below the average are the standard deviations of the corresponding variables. These moments are from the numerically approximated ergodic distributions with 1,000 periods and 1,000 simulations. The second column also reports the average and standard deviation of the optimal subsidy and the fiscal cost of bailouts measured by the lump-sum tax relative to the GDP.

Figure 1: Simulated path of s_t^*: Benchmark calibration

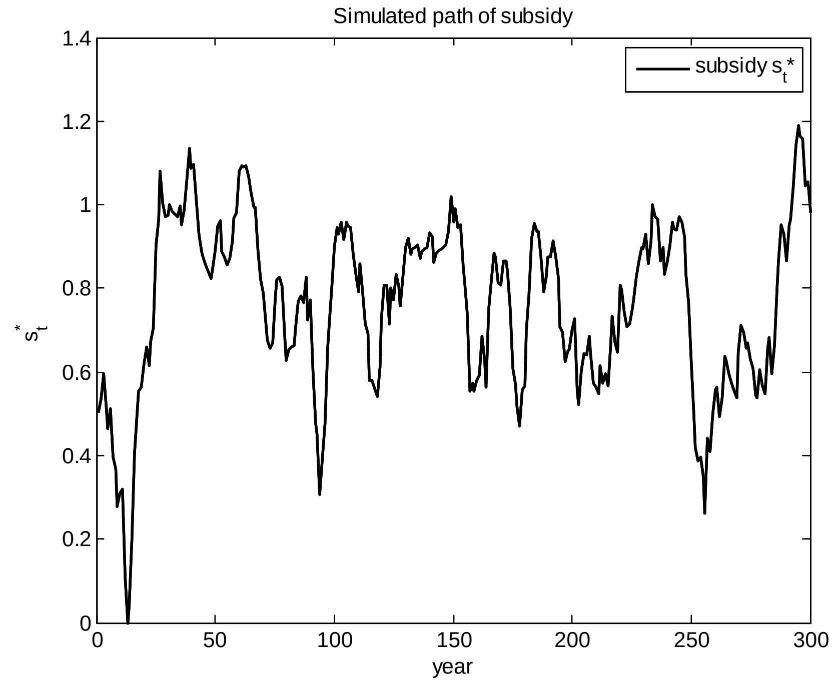

Figure 2: Simulated path of s_t^*: Economy with $\phi = 0.60$