Kyoto University,
Graduate School of Economics
Research Project Center Discussion Paper Series

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Discussion Paper No. E-14-012

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December 2014

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Abstract

Constructing an intra-industry trade model with division of labor within firms, this paper shows that opening up to trade improves firm productivity. Firms choose the number of markets they export. Optimal entry conditions for export markets rule out loss from opening up to trade. Under fixed export costs, opening up to trade makes some firms exit and concentrates labor to surviving firms through recruiting process and induces the division of labor. An increase in the number of markets induces firms to enter more export markets and improves firm productivity in the long run and has the reverse effect on firm productivity in the short run.

Keywords: the division of labor within firms; firm productivity; the optimal number of markets firms enter; fixed export costs

JEL classification numbers : F12

1 Introduction

Adam Smith (1776) indicates an importance of productivity improvement induced by division of labor within a firm using a example of a pin factory; "More labor input, higher marginal labor productivity ". In particular, Adam Smith's famous proposition, *" The division of labor is limited by the extent of the market"* is called *Adam Smith's*

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theorem. Many economists have justified Adam Smith's theorem theoretically and empirically.

These studies treats autarky economy. How can this theorem extend to open economy ? Little is known of this. One of exceptions is Chaney and Ossa (2013). They indicates an increase in the extent of the market induce division of labor within firms theoretically and suggest a new insight for division of labor within firms in the context of intraindustry trade.

Trade costs (transport costs and fixed export costs) seem to be very important for the division of labor induced by trade, though Chaney and Ossa (2013) does not treats these trade costs. In facts, Zadeh (2013) indicates a decrease in trade costs promotes the division of labor within firms by calibrating the model to Mexican firm level data. Furthermore, considering *Adam Smith's theorem* in trade contexts, we must define the extent of markets firms face. When there are many countries (many markets) and trade costs, how many export markets firms enter is non-trivial. This paper shows a simple trade model with trade costs which investigates how trade affect firm's entry and induce the division of labor and change firm productivity.

We construct a model which is very similar to standard intra-industry trade models presented by Krugman (1980) except for the division of labor within firms and fixed export costs. In particularly, we assume that number of countries is $\bar{n}+1$ and each exporting firm must pay fixed trade cost for each destination country. We treat the division of labor within a firm as Chaney and Ossa (2013). Chaney and Ossa (2013) succeeds in formalizing Adam Smith's (1976) pin factory story.

This paper's main results are the following. Opening trade with fixed trade costs makes some firms exit and concentrate labor on surviving firms and hence induces the division of labor and raises firm productivity. This reallocation is achieved through recruiting process. The division of labor induced by trade changes real wage rate. Therefore, gains from opening trade is contributed by both variety expansion and an increase in real wage rate. Optimal entry conditions for exports markets rule out loss from trade. In the long run, an increase in number of the markets promotes entry for export markets, the division of labor and raises firm productivity. In contrast with this, in the short run an increase in number of the markets has reverse effects. Our model indicates a new insight for Adam Smith's theorem in trade model's context. Chaney and Ossa (2013) and Kamei (2014) indicate that the division of labor is limited by the *size* of the market. In contrast with these, our model indicates that the division of labor is limited by the *number* of the markets.

Key factor behind the division of labor is presence of fixed export costs. Most related studies of this paper are Chaney and Ossa (2013) and Kamei (2014). Their mechanism of the division of labor driven are different from this paper. They adopts a model with variable markup as Krugman (1979). This variable markup drives pro-competition effect in opening up to trade and implies that an expansion of labor force and an increase in number of firms make some firms exit and concentrate labor to surviving firms. This labor concentration drives the division of labor. In their model, Adam Smith's (1976) theorem is interpreted as "The division of labor is limited by the *size* of the market ". In contrast to them, that theorem is interpreted as "The division of labor is limited by the *number* of the markets " from the following mechanism. This paper's selection mechanism is subject to the following process. On opening up to trade, some firms try to enter export markets. Theses firms must pay fixed export costs. In order to pay those costs, the firms recruit workers, promote the division of labor, and raise their productivity. This recruiting process raises real wage rate. This makes firms which does not succeed in the recruiting. Fixed export costs play a key role in this recruiting process. Fixed export costs play a key role also in entry for export markets. When the number of markets increases, firm enter more export markets. This is not trivial. At that time, firms face trade off between an increase in total revenue and in total fixed costs. The former effect dominates the latter effect. Therefore , an increase in the number of markets raises optimal entry, concentrates labor on surviving firms, and induces the division of labor.

There are very few papers which analyze international trade explicitly incorporating the division of labor within firms. Kamei (2013) and Francois (1987) are exception. Kamei (2013) also adopts Chaney and Ossa (2013) type's the division of labor in general oligopolistic equilibrium model with variable markup rate and shows a role of regulation for number of firms. Francois adopts Edwards and Starr (1987) type's the division of labor and analyze trade in services and its effect on the division of labor. Zadeh (2013) show a model in which there are two types of workers. Zadeh (2013) focuses on relative specialization and skill premium though trade liberalization. Unlike these papers, this papers focus on relation among the division of labor within firms, optimal entry for export markets and social welfare by constructing a model with many markets and trade costs.

We should distinguish the above results from relatively similar research lines. Firstly, this paper is different from Melitz (2003). He focuses average industry productivity with heterogeneous firms. This paper focuses firm productivity with homogeneous firms in productivity. Secondly, this paper is different from Ethier (1982). In Ethier (1982), on identical production process (task), each intermediate good firm produces horizontally differentiated. An increase in the intermediate goods raises final good firm's productivity. That is, he describes division of labor as love of variety. This paper treats other division of labor. In this paper, a final good firm assigns some interval task to each worker. When the division of labor is induced, each worker engage in narrower task set.

One of research lines related to this paper is the division of labor. Edwards and Starr (1987) presented a model in which division of labor is not sufficient condition for increasing returns to scale. Swanson (1999) presented a very simple model which analyze relationship between human capital investment, the division of labor, and firm productivity. Becker and Murphy (1992) shows explicitly that cost of promoting division of labor is coordination costs. They suggests that the division of labor is not limited by the extent of the market and limited by coordination costs. This property is compatible with this paper's result. Baumgardner (1988) indicates that the more populated counties is, more specialist physician becomes.

The other research lines related to this paper is trade-induced productivity improvement. Grossman and Helpman studies trade-induced R&D. Yeaple (2005) and Bustos (2011) studies technology adoption. McLaren (2000) studies productivity improvement through vertical restructuring. Clerides, Lach and Tybout (1998), Salomon and Shaver (2005), and Martins and Yang (2009) studies empirical analysis. There was no consensus whether an improvement of firm productivity induced by trade. A survey Wagner (2007) indicates that this effect is mixed and unclear. Thereafter, Martins and Yang (2009), however, indicates that many empirical studies recognizes firm productivity improvements induced by trade, considering more than 30 papers. Recently some studies investigate changes in firm organization induced by trade. Caliendo, and Rossi-Hansberg (2012) focuses the number of layers of firms. Davidson, Heyman, Matusz, Sjoholm, and Zhu (2013) focuses skill intensity of firms.

The rest of the paper is constructed in the following way. Section 2 analyzes autarky equilibrium. Section 3 analyzes how opening up to trade induces the division of labor and raises the welfare. Section 4 analyzes how trade liberalization induces the division of labor. and how trade the division of labor is induces in the short run. Conclusion and Appendix follow.

2 The Model

We introduce the division of labor in to the trade model of monopolistic competition with fixed export costs. The setup of the model is based on the idea by Chaney and

Ossa (2013).

2.1 Households

There are L units of household and each household supplies one unit of labor inelastically at wage rate w. Preference of each consumer is given by a C.E.S utility function over a continuum of good indexed by θ : $U = \left[\int_{\theta\in\Theta} c(\theta)^{\rho} d\theta\right]^{1/\rho}$, $0 < \rho < 1$, where the measure of the set Θ represents the mass of available differentiated goods and $c(\theta)$ represents consumption of variety θ. From the standard utility maximization, price index can be obtained as $P = \left[\int_{\theta\in\Theta} (p(\theta))^{1-\sigma} d\theta\right]^{1/(1-\sigma)}$, where $\sigma = 1/(1-\rho) > 1$ is the elasticity of substitution between any two varieties and also represents the price elasticity of demand for each variety.

2.2 Firm's organization

Each firm produces a variety of differentiated final good. As for the production of goods, we modify the model developed by Chaney and Ossa (2013). Many tasks are sequentially distributed over the set $[0, 2]$ in each firm. A firm assigns these tasks to t teams where $t \in R_+$. Since teams are symmetric, identical range of subset of the task set is assigned to each team. One unit of preliminary good for a certain task set $[\underline{\omega}, \overline{\omega}]$ is produced by inputting the following units of labor;

$$l([\underline{\omega}, \overline{\omega}]) = \frac{1}{2}\int_{\underline{\omega}}^{\overline{\omega}} \gamma \left|\frac{\underline{\omega} + \overline{\omega}}{2} - \omega\right| d\omega,\ \ \gamma > 0, \tag{1}$$

where $(\underline{\omega} + \overline{\omega})/2$ denotes this team's core-competency and γ denotes team's burden parameter.[1)] The core-competency is task which this team is the most suited to. As γ is high, certain task set needs more labor force. γ can be interpreted as difficulty of multitask.

This implies that the larger γ is, the less efficient assigning many task sets to one team is: a decrease in γ raises team's performance. Figure 1 illustrates this feature for task set $[0, 4/t]$ when t is a positive integer. Integral term in (1) corresponds to area of two right angled triangles formed in linear symmetry with respect to the vertical direction in Figure 1. [2)]

1) In Chaney and Ossa (2013), the firm assigns each team core-competency. That is, the core-competency is endogenously determined and the optimal core-competency is certainly the mid point in the assigned task set. This is because each task set is symmetric with respect to the core-competency.

2) For assumption of $l(\underline{\omega}, \overline{\omega})$, Chaney and Ossa (2013) adopt a more general form, $l(\underline{\omega}, \overline{\omega}) = \frac{1}{2}\int_{\underline{\omega}}^{\overline{\omega}} \left(\frac{\underline{\omega}+\overline{\omega}}{2} - \omega\right)^{\alpha} d\omega$, where $\alpha >$ is a positive parameter. By formulating $l(\underline{\omega}, \overline{\omega})$ the way as (1),

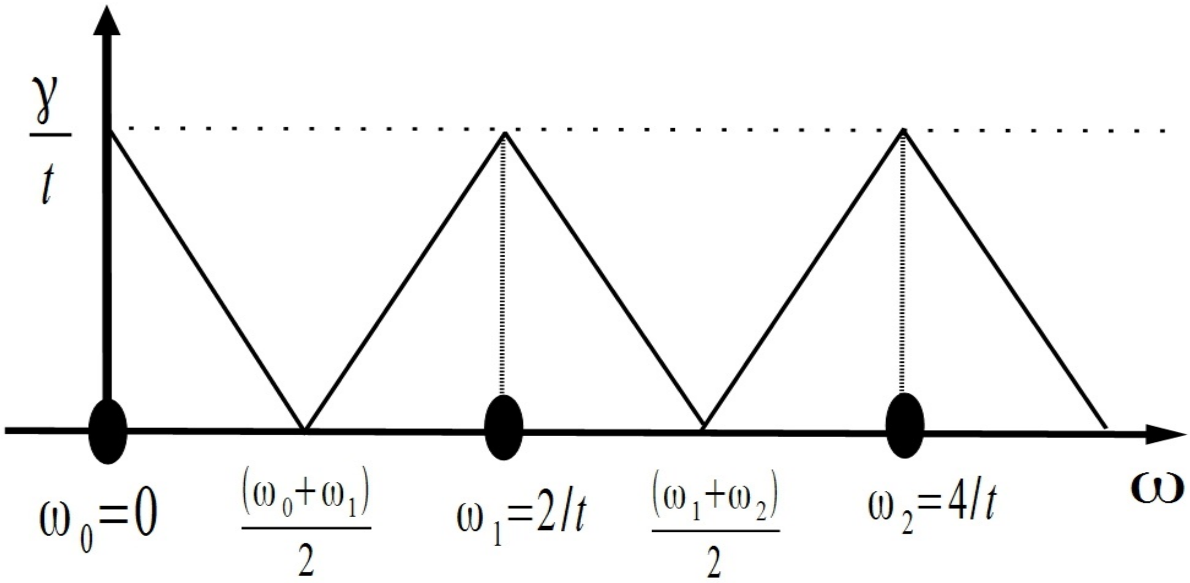

Figure 1: sequential task structure

By combining (1) for each team, one unit of preliminary good for the task set $[0, 2]$ is produced by inputting the following units of labor,[3] $t\left(y\int_0^{2/t}\gamma\frac{\omega}{2}d\omega\right) = t\left(y\int_0^{1/t}\gamma\omega d\omega\right) = \frac{\gamma y}{2t}$.

One unit of final good is produced by inputting one unit of preliminary good for task set $[0, 2]$. Organizing one team requires $f(>0)$ units of labor, which is interpreted as coordination costs. [4] Then, y units of final goods is yielded for given number of teams, t, by inputting the following units of labor on *production lines*,

$$l(t, y) = tf + \frac{\gamma y}{2t}. \tag{2}$$

Each firm choose the number of teams, t, so that the above labor input $l(t, y)$ is minimized. In this problem, the firm faces trade off between productivity improvements by increasing the number of teams and an increase in costs of organizing teams.

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we can make the model highly tractable. In Appendix E, we compare both forms in detail and show that the technology which we adopt does not loose generality very much in comparison to the one adopted by Chaney and Ossa (2013).

3) On the right hand side of (1), by dividing the integral term by two, we can get very simple form for the units of labor.

4) f can be interpreted as middle-level management costs. Since each team specializes in certain task set, the firm needs coordinators. Becker and Murphy (1992) emphasizes that coordination costs is the brake for division of labor.

The optimal number of teams t is

$$t = \left(\frac{\gamma y}{2f}\right)^{1/2}. \tag{3}$$

Each firm inputs labor into the production lines and furthermore $f_d(> 0)$ units of labor into the management division, where $f_d(> 0)$ is fixed and wf_d represents overhead production costs. Total labor input is $l + f_d$.

Combining (2) and (3) gives the following total cost function under the optimal organization:

$$TC(y) = wl(y) + wf_d = w(2\gamma f y)^{1/2} + wf_d. \tag{4}$$

This cost function shows that the firm's technology exhibits increasing returns to scale and that marginal cost is decreasing at all levels of output.

From (2), we can obtain the following production function $y = \frac{l^2}{2\gamma f}$. The marginal productivity of labor, MPL, is given by

$$MPL = \frac{dy}{dl} = \frac{l}{\gamma f}. \tag{5}$$

This shows that labor input expansion increases marginal productivity. From (3) and production function, the number of teams is proportional to labor input. That is, an increase in labor input raises the number of teams and labor productivity. Therefore, *the division of labor effect* can be confirmed.

2.3 Equilibrium allocation

We analyze firm's profit maximization problem in a monopolistic competitive market. Each firm faces a residual demand curve with constant elasticity σ and therefore, sets $p = \mu MC(y)$ where $\mu \equiv \sigma/(\sigma - 1)$ and $MC(y) \equiv dTC(y)/dy$. Using (4), this optimal pricing rule is written by PP_A schedule,

$$PP_A : \frac{p}{w} = \frac{\mu}{2}\left(\frac{2\gamma f}{y}\right)^{1/2}. \tag{6}$$

Firms can enter and exit freely. This gives zero profit, $\pi = 0$. This is written by $p = AC(y)$, where $AC(y) \equiv TC(y)/y$. Using (4), this free entry condition is written by FE_A schedule,

$$FE_A : \frac{p}{w} = \left(\frac{2\gamma f}{y}\right)^{1/2} + \frac{f_d}{y}. \tag{7}$$

(6) and (7) characterize $(y, p/w)$ in equilibrium in the following way:

$$y_A = \frac{{f_d}^2}{2\gamma f}B^{-2}, \tag{8}$$

$$\left(\frac{p}{w}\right)_A = B(B+1)\frac{2\gamma f}{f_d}, \tag{9}$$

where $B \equiv \mu/2 - 1$ and subscript "A" represents variables in autarky equilibrium.

Hereafter, we assume the following assumption 1 to ensure the unique internal solution.

Assumption 1.[5] $0 < B < \infty$, *that is,* $2 < \mu < \infty$ $(1 < \sigma < 2)$ *and* $f_d > 0$ *hold.*

We can immediately obtain the next proposition from (8) and (9).

Proposition 1. [6] *Under assumption 1, a unique internal solution in which* $y > 0$ *and* $p/w > 0$ *exists.*

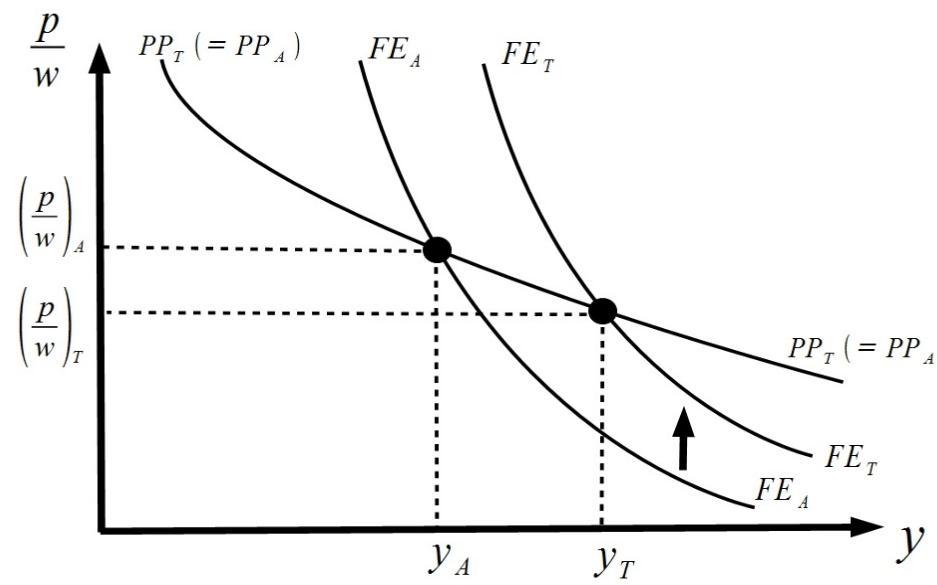

Figure 2: Autarky and Tarding equilibrium in $(y, p/w)$ space.

5) This internal condition makes us reconsider firm's technology represented by (1). See Appendix E for details. However, we will adopt technology in (1) and assumption 1 for analytical simplicity.

6) Why does the the internal solution exist ? In equilibrium, M_A and $\partial MR/\partial y < 0$ hold. Moreover on $y \leq y_A$ and $\partial^2 MR/\partial y^2 \leq \partial^2 MC/\partial y^2 < 0$ hold. This shows that in the equilibrium, if each firm raises the output, the profit decreases or does not change. Therefore, y_A is optimal. Proposition 1 holds even if $L \to \infty$. This is because all the effects of an increase in labor forces are not absorbed into an increase in demand for each variety but into the variety expansion effect.

Note that if $f_d = 0$ holds, the internal solution does not exist.[7] Hence, we need to assume $f_d > 0$ to compare autarky equilibrium allocation and trading equilibrium allocation. Even if $f_d > 0$, under $\sigma \geq 2$, $y \to \infty$. That is, the internal solution requires sufficiently low elasticity of substitution between varieties (strong love of variety).

Figure 2 illustrates the features of autarky equilibrium. The figure has a unique intersection between FE_A curve and PP_A curve : $(y, p/w) = (y_A, (p/w)_A)$. The PP_A curve is cut by the FE_A curve only once. This ensures the unique internal solution. [8]

Substitute (8) into (3) to yield the equilibrium level of t;

$$t_A = \frac{f_d}{2fB}. \tag{10}$$

The equilibrium level of l is obtained by substituting (8) and (10) into (2):

$$l_A = \frac{f_d}{B}. \tag{11}$$

Then, substitute (11) into (5) to yield

$$MPL_A = \frac{f_d}{\gamma f B}. \tag{12}$$

This equation implies that $MPL_A = 2t_A/\gamma = l_A/\gamma f$. Further, $(w/p)_A = t_A/\gamma(B+1) = l_A/2(B+1)\gamma f$ holds. On equilibrium, labor productivity and real wage are proportional to the number of teams and the labor input on production lines.

Now, we can completely characterize the equilibrium allocation by determining the number of varieties. Labor market clearing condition, $L = M(l + f_d)$, gives the following equilibrium number of varieties, M_A by using (11),[9]

$$M_A = \frac{B}{B+1}\frac{L}{f_d}. \tag{13}$$

From (12) and (13), the following proposition is immediately obtained.

Proposition 2.

Under assumption 1, an expansion of aggregate labor force does not induce the division of labor and hence, does not raise firm productivity and only raises the number

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7) When $f_d = 0$ and $B = 0$, equilibrium output y is not determined. When $f_d = 0$ and $B \neq 0$, equilibrium output y is zero or approaches positive infinity.

8) The characteristic of Figure 2 is supported by Appendix. G.

9) To obtain M_A we use labor market clearing condition and do not use budget constraint for each consumer. This approach is justified from Walras' law.

of firms.

Proposition 2 means that the division of labor *is not* limited by the *size* of the market. This result is contrast to Chaney and Ossa (2013) and Kamei (2014) in which pro-competition effect occurs.

The mechanism behind proposition 2 can be explained in the following way. In the short run, number of firms can not be adjust. An expansion of labor force increases employed workers by each firm and hence, improves firm productivity. Then, firms obtain excess profits. However, in the long run with free entry and exit, new firms enter and recruits some workers form incumbent firms. Therefore, the effect of productivity improvements is just outset entirely.

Proposition 2 has an important implication for an trading equilibrium. A firm's allocation in an trading equilibrium without trading costs is accord with one in integrated economy's equilibrium because pro-competition effect does not occur in this model. Therefore, opening up to trade without trade costs does not raise firm productivity. This result is confirmed in Proposition 3. In the next section, we extend the model to the case where there are fixed export costs.

3 Opening Up to International Trade

We extend the model in the previous section to the case of trade among identical $\bar{n}+1$ countries with fixed export costs. The assumption of fixed export costs is essential for division of labor induced by trade. We assume that $\bar{n} \in R_{++}$ for analytical simplicity. Without loss of generality, we focus on the home country's allocation.

3.1 Firm's decision

Firm's decision has two stages. The first stage is market entry process. The second stage is a choice of optimal quantity and price. This problem can be solved using backward induction. We begin with the second problem. In the second stage, the number of export markets firms entry is given.

Each firm faces two types of trade costs. First, firms must export $\tau \in [1, \infty)$ units of product to send one unit of product (iceberg trade cots) to a foreign market. Second, in order to enter export markets, firms must pay fixed costs. Firms entering n export markets must pay fixed cost wnf_x.

We focus on firms entering n export markets. These firm's output for home market is denoted by y_d and the one for one foreign market is denoted by y_x. Then, we can define total output of firms as $y_t = y_d + ny_x$.

Firm's production function is given by $y_t = \frac{l^2}{2\gamma f}$, where l represents labor inputs in production lines to sell for $n+1$ markets. This firm's total labor inputs is $l + f_d + nf_x$.

This gives the marginal product of labor; $MPL_n = \frac{l}{\gamma f}$. Total cost function are given by

$$TC(y_t) = w\left[(2\gamma f y_t)^{1/2} + f_d + nf_x\right]. \tag{14}$$

Note that under the cost function, the following relationship holds;

$$TC(y_t) < TC(y_d) + nTC(y_x). \tag{15}$$

This implies that each firm's total profits can not be decomposed into profits from the home market and those from the export markets; $\pi_t \neq \pi_d + n\pi_x$.

Price for home market is denoted by p_d and price for export market as p_x. Mill price in export market is $p_x = \tau p_d$ from the assumption.

Home consumers buy goods from n foreign countries as the trade balanced condition is satisfied. Home consumers face all countries' s brands and $(n/\bar{n})M$ brands on the average per one foreign country. Hence, the price index is given by $P_T = \left[\int_{\theta\in\Theta} (p_d(\theta))^{1-\sigma}\, d\theta + \bar{n}\int_{\theta^*\in\Theta^*} [\tau p_d(\theta^*)]^{1-\sigma}\, d\theta^*\right]^{1/(1-\sigma)}$, where an asterisk represents foreign brands.

Accounting for final good market clear condition, firm profit maximization is characterized by the following optimal price setting;[10] $PP: p_d = \mu MC(y_t)$.

3.2 Trading equilibrium and the division of labor induced by trade

We define trading equilibrium in almost the same way as autarky equilibrium. We, however, need to account furthermore firm's decision of export market entry. Subscript T represents variables in trading equilibrium. Then, we define trading equilibrium in the following way.

Definition 1. *We define trading equilibrium as a equilibrium which satisfies the following conditions.*

(I) Optimal price setting rules, free entry conditions, goods market clearing conditions, labor market clearing conditions, and trade balanced conditions are satisfied.

10) See Appendix G for the details.

(II) No firms have incentive to deviate from the equilibrium.

We consider firm's decision of the second stage (optimal price and output), treating the positive number of markets firms enter, $n > 0$ as given. Firm's equilibrium allocation is characterized by optimal pricing rule (PP_T) and free entry condition (FE_T). These conditions are derived by using $PP : p_d = \mu MC(y_t)$ and Eq. (14) in Appendix. A. Figure 2 illustrates the features of trading equilibrium.

In Figure 2, positive fixed export costs shift FE curve upward. Note that the free entry condition holds for the world market as a whole and that the only difference between aurtaky and trading equilibrium condition is the fixed cost term. This implies that we can obtain $y_{t,T}$ and $(w/p_d)_T$ by replacing f_d with $f_d + nf_x$ in y_A and $(w/p_d)_A$. Furthermore, we can obtain t_T, l_T, MPL_T and M_T by such a operation, which are derived in details in Appendix A.

From trading equilibrium allocation, We obtain $y_{t,T} > y_{t,A}$, $(w/p_d)_T > (w/p_d)_A$, $t_T > t_A$, $l_T > l_A$, $MPL_T > MPL_A$ and $M_T < M_A$, which are shown in Appendix. A.

$M_T < M_A$ means that some firms exit. $t_T > t_A$ and $MPL_T > MPL_A$ mean that the division of labor is induced by opening trade. These are driven by an increase in l (remember $l_T > l_A$).

Those results are summarized in the following proposition.

Proposition 3. *Under assumption 1 and given $n > 0$, opening trade with positive fixed export costs induces the division of labor.*

We can explain this results the following way. On opening up to trade, some firms try to enter export markets. Theses firms must pay fixed export costs. In order to pay those costs, the firms recruit workers. Why do the firms recruit workers ? The firms recruit workers to promote the division of labor and raise their productivity. This recruiting process raise real wage rate. This makes firms which does not succeed in the recruiting.

We should note that this selection mechanism is different from that of Chaney and Ossa (2013), Kamei (2014). Their selection mechanism is driven by pro-competition effect.

An allocation in an trading equilibrium without trading costs is accord with one in integrated economy's equilibrium. Therefore, Proposition 3 immediately implies the following.

Corollary 1. *Under assumption 1, opening trade without trade costs does not raise firm productivity.*

Proof. See Eq (A.8) in Appendix A. Q.E.D.

This result is parallel to the result in Melitz (2003). [11] Therefore, positive fixed export costs are essential for Proposition 3.

3.3 Optimal entry

The next, we consider firm's decision of the first stage ; firm's entry process. Firms decide the number of markets the firms enter while keeping the number of markets the other firms enter.

The number of export markets firms should enter depends on parameter set. The optimal number is uniquely determined under certain assumptions as shown in proposition 4. In order to clarify those assumptions, we introduce a function $G(n)$.

Definition 2. *We define function $G(n)$, which is a function of $n \in R$ in the following way:*

$$G(n) = \frac{1 + \tau^{1-\sigma} n}{\left(1 + \frac{f_x}{f_d} n\right)^{2-\sigma}}. \tag{16}$$

In addition, We define a value n_c which satisfy the following condition:

$$n_c \in R - \{0\} \ \wedge \ G(n_c) = 1$$

For analytical simplicity, we focus on an equilibrium in which all firms enter the same number of markets. In order to focus such a equilibrium, we impose the following assumption.

Assumption 2. *We assume the following condition:*

$$f_d < (2 - \sigma)\tau^{\sigma-1} f_x <\rightarrow n_c < \bar{n}. \tag{17}$$

Can the number of export markets each firm enters distribute ? The following lemma 1 indicates that such a distribution is degenerate distribution in a certain condition.

Lemma 1. *All th other firms also enter n export markets, when assumption 1 holds and some firms enter $n \in (0, \bar{n}]$ export markets which satisfies the following conditions*

11) Melitz's (2003) footnote 24 says "*In the absence of such costs (...), opening to trade will not induce any distributional changes among firms, and heterogeneity will not play an important role.*"

(18) or (19);

$$G(n) > 1, \tag{18}$$

$$n = \frac{1}{\sigma - 1}\left[(2-\sigma)\tau^{\sigma-1} - \frac{f_d}{f_x}\right]. \tag{19}$$

Proof. See Appendix I. Q.E.D.

$G(n) > 1$ holds if n is sufficiently high relative to trade costs, τ and f_x/f_d.

If all firms enter n export markets, when an equilibrium condition (II) in definition 1 is satisfied ? The following lemma 2 answer this question.

Lemma 2. *When assumption 1 holds and the number of export markets each firm enters is identical, that a equilibrium in which all firms enter n exports markets uniquely exists is equivalent to the following condition:*

$$\forall \hat{n} \in [0, \bar{n}],\ \hat{n} \neq n \ \wedge \ G(n) > G(\hat{n}). \tag{20}$$

Proof. See Appendix K. Q.E.D.

Lemma 1 and Lemma 2 imply derive the following proposition 4.

Proposition 4.

When Assumption 1 and 2 hold, there is the unique equilibrium in which all firms entry $\bar{n}+1$ markets. That is, $n_T = \bar{n}$.

Proof. See Appendix L. Q.E.D.

Proposition 4 indicates that if assumption 2 holds, only $\bar{n}$ satisfies (18) of lemma 1 and (20) of lemma 2 simultaneously. That is, $G(\bar{n}) > 1$ holds and $\bar{n}$ maximize G. On the other hand, n of (19) is not equilibrium since the n minimize G.

If a condition $f_d < (2-\sigma)\tau^{\sigma-1}f_x$ in (17) of Assumption 2 does not bind, $G(\bar{n}) > 1$ holds and $\bar{n}$ maximize G without constraint $n_c < \bar{n}$. We can interpret this in the following way. As explained in Appendix D, the numerator of $G(n)$, $(1+n\tau)$, and the denominator of $G(n)$, $(1+n\frac{f_x}{f_d})^{2-\sigma}$ can be interpreted as respectively entry gain and entry loss. That a condition $f_d < (2-\sigma)\tau^{\sigma-1}f_x$ does not bind means trade costs, τ and f_x, are sufficiently low. This implies high entry gain and low entry loss. Therefore, firms have incentive to enter as many market as possible.

By using $n_T = \bar{n}$, we can completely characterize the equilibrium.

3.4 Gains from trade

In trading equilibrium, the real wage rates are identical in all countries and hence the indirect utility function is given by $V_T = (w/P)_T$. Since countries are symmetric, the following proposition is obtained;

Proposition 5. *Under assumption 1 and 2, $V_T > V_A$.*

Proof. With some calculation, we can obtain $V_T > V_A \leftrightarrow G(n) > 1$ under assumption 1 (see Appendix B for proof). Assumption 2 certifies $G(n) > 1$ (see Appendix L for proof). Q.E.D.

The necessary condition (18), $G(n) > 1$, for optimal entry is equivalent to a condition $V_T > V_A$. That is, optimal entry conditions rule out loss from opening up to trade.

We next decompose gains form trade. M_W denotes the equilibrium number of varieties which are consumed in each countries. Since countries are symmetry, $M_W = (1 + n_T)M_T$ holds.

From V_T of (B.3), $V_T = (w/p_d)_T \hat{M}^{1/(\sigma-1)}$, we can decompose gains form trade into the change in real wage (*productivity effect*) and the change in the effective number of varieties (*effective variety effect*), where the effective number of varieties is defined as $\hat{M}_W \equiv (1 + n_T \tau^{1-\sigma})M_T$.

$\hat{M}_W$ shows that if each consumer consumes each brand in units of $(w/p_d)_T$, the effective number of varieties the consumer can consume is less than M_W, accounting for variable trade cost τ. [12)]

Productivity effect is positive from proposition 3. In the contrast with this, when $\hat{M}_W$ rises ? $\hat{M}_W > M_A$ is equivalent to the following condition:

$$f_d > \tau^{\sigma-1} f_x. \tag{21}$$

In these settings we can obtain the following proposition.

Proposition 6. *Under assumption 1 and 2, gains from opening trade is decomposed to productivity effect and effective variety effect.*

(I) When (21) holds, effective variety effect is positive and then, gains from opening trade always exists.

(II) When (21) does not hold, under assumption 2, effective variety effect is nega-

12) We define $\hat{M}_W$ in this manner in order to measure firm productivity with firm's output at "factory gate".

tive and then, positive productivity effect dominates this effect and gains from opening trade exists.

Proof. *(I) Note that $\hat{M}_W > M_A$ is equivalent to (21). (21) $\Rightarrow$ (18) holds. Therefore, $\hat{M}_W > M_A \Rightarrow$ (18). (II) Intersection of set of (18) and complement set of (21) is not empty. See Appendix H in details. Q.E.D.*

Note that condition (21) is independent of n in contrast with (18). (21) demands trade costs (combination of τ and f_x) is sufficiently low relative to f_d. That is, when entry gain is sufficiently high and entry loss is sufficiently low, the effective variety effect is positive.

4 Trade liberalization

4.1 Trade liberalization in the long run

We define trade liberalization as a decrease in variable trade cost τ or a decrease in fixed export cost, f_x or an increase in the number of trading partners, $\bar{n}$. Note that change in τ and in f_x are worldwide since all countries are identical.

From MPL_T of (A.7), M_T of (A.10), and V_T of (B.4), we can implement comparative statics analysis for trade liberalization in the following way.

Proposition 7. *Under assumption 1 and 2, trade liberalization has impacts on equilibrium allocation and social welfare in the following way.*

(I) A decrease in τ does not change $y_{t,T}$, l_T, n_T, MPL_T, M_T, and $\hat{M}_W$ and, raises V_T.

(II) A decrease in f_x does not change n_T, reduces $y_{t,T}$, l_T, MPL_T and, raises $M_T, \hat{M}_W$, and V_T.

(III) An increase in $\bar{n}$ raises $y_{t,T}$, l_T, n_T, MPL_T and V_T and, reduces M_T. Then, $\hat{M}_W$ is raised (is not raised) if (21) holds (does not hold).

Proof. *see Appendix C. Q.E.D.*

The mechanism behind the above results is in the following way.

To begin with, we consider effects on n_T. A decrease in τ and f_x does not n_T. These changes implies higher entry gain and lower entry loss. This induce firms to enter more export markets but n_T is bind at $n_T = \bar{n}$. Therefore, n_T does not changes.

An increase in $\bar{n}$ raise n_T. This is not trivial. When firms raise n_T, firms face trade off between an increase in total revenue $r_t = r_d + nr_x$ and in total fixed costs $f_d + nf_x$. The former effect dominates the latter effect. Therefore , an increase in $\bar{n}$ raises n_T, concentrates labor on surviving firms, and induces the division of labor.

The next, we consider effects on M_T, V_T. A decrease in τ raises $(1+\bar{n}\tau^{1-\sigma})^{\frac{1}{\sigma-1}}$ and hence causes the positive effective variety effect. The other hand, it does not changes $\frac{w}{p_d}$ and hence, raises social welfare.

A decrease in f_x has the negative productivity effect as Proposition 3. It raises M_T and does not change $(1+\bar{n}\tau^{1-\sigma})^{1/(\sigma-1)}$. Hence, $\hat{M}$ does not change. The positive effective variety effect dominates the negative productivity effect and hence, social welfare improves.

An increase in $\bar{n}$ raises w/p_d through raising $n_T f_x$ as Proposition 3. Further more, it raises $(1+n_T\tau^{1-\sigma})^{\frac{1}{\sigma-1}}$ and reduces M_T from M_T of (A.10). Therefore, whether the effective number of varieties rises or not depends on whether (21) holds or not. Then, welfare is raised even if effective variety effect is negative because the effect is dominated by positive productivity effect.

This effect of an increase in $\bar{n}$ and proposition 2 indicate a new insight for Adam Smith's theorem. That is, the division of labor is *not* limited by the *size* of the market but is limited by the *number* of the markets.

Remember our model and Melitz (2003) have common theoretical foundations and however, Proposition 7 shows that the change in productivity by trade liberalization is very different from Melitz (2003) model. 13) 14)

4.2 Trade liberalization in the short run

Up to the previous section, we have studied equilibria with free entry and exit by imposing zero profit condition except for section 3.3. That is, these equilibria have time span in which the number of firms can be adjusted. We call such a time span long run. In this section, we study short run trading equilibrium in which the number of firms can not be adjusted and zero profit condition is not imposed.

Proposition 7 indicates that a change in variable trade costs has no effects on the division of labor. We, then focus on changes in f_x and $\bar{n}$.

In the short run, from labor market clearing condition $Ml + M(f_d + \bar{n}f_x) = L$, we

13) In Melitz (2003), all trade liberalization policies raise aggregate productivity.

14) We should notice that we can directly compare our model's "industry average productivity" to that Melitz (2003) because measurement is different. In order to apply our model's measurement, we should measure aggregate productivity of Melitz (2003) with Φ defined in (D.2) in Melitz (2003)'s appendix D.3. Aggregate productivity Φ is measured with output at "factory gates". Footnote (20)'s aggregate productivity is measured by Φ.

can obtain labor input on production lines, l in the following way

$$l_S = \frac{L}{M} - (f_d + \bar{n} f_x) \tag{20}$$

where subscript "S" represents variables in the short run trading equilibrium. (21), production function $y = l^2/2\gamma f$, and optimal team numbers $t(y)$ in (3) give t in the short run equilibrium:

$$t_S = \frac{l_S}{2f} = \frac{1}{2f}\left[\frac{L}{M} - (f_d + \bar{n} f_x)\right]. \tag{21}$$

From (21), comparative statics in the short run is obtained immediately in the following proposition 8.

Proposition 8. *In the short run equilibrium, comparative statics for* t_S *is obtained in the following way.*

(I) A decrease in τ *does not change* t_S.

(II) A decrease in f_x *raises* t_S.

(III) An increase in $\bar{n}$ *decreases* t_S.

Note that A decrease in f_x and an increase in n does not induce labor reallocation across firm. The results (II) and (III) are explained as reorganization process. (not labor concentration on surviving firms).

In the short run, a decrease in f_x induce firms to increase labor input in production lines and to promotes the division of labor. In the contrast with this, an increase in $\bar{n}$ induce firms to increase labor input in headquarter lines and to refrain the division of labor.

4.3 A decomposition of trade liberalization in the long run

We furthermore decompose the effect of trade liberalization in the long run. From Appendix A, the effect of opening trade on firms is the same as an effect of an increase in n on the ones. Hereafter, we focus on the division of labor induced by an increase in n.

The change in MPL can be decomposed into the technology effect and the division of labor effect in the following way:

$$\frac{dMPL_T}{d\bar{n}} = \underbrace{\frac{dMPL_T}{dl_T}\frac{\partial l_T}{\partial n_T}\frac{dn_T}{d\bar{n}}}_{\textbf{reallocation within firms } (-)} + \underbrace{\frac{dMPL_T}{dl_T}\frac{\partial l_T}{\partial M_T}\frac{dM_T}{dn_T}\frac{dn_T}{d\bar{n}}}_{\textbf{reallocation across firms } (+)} > 0. \tag{22}$$

This shows that the effect of labor reallocation *across* firms (positive effect) dominates the effect of that *within* firms (negative effect).

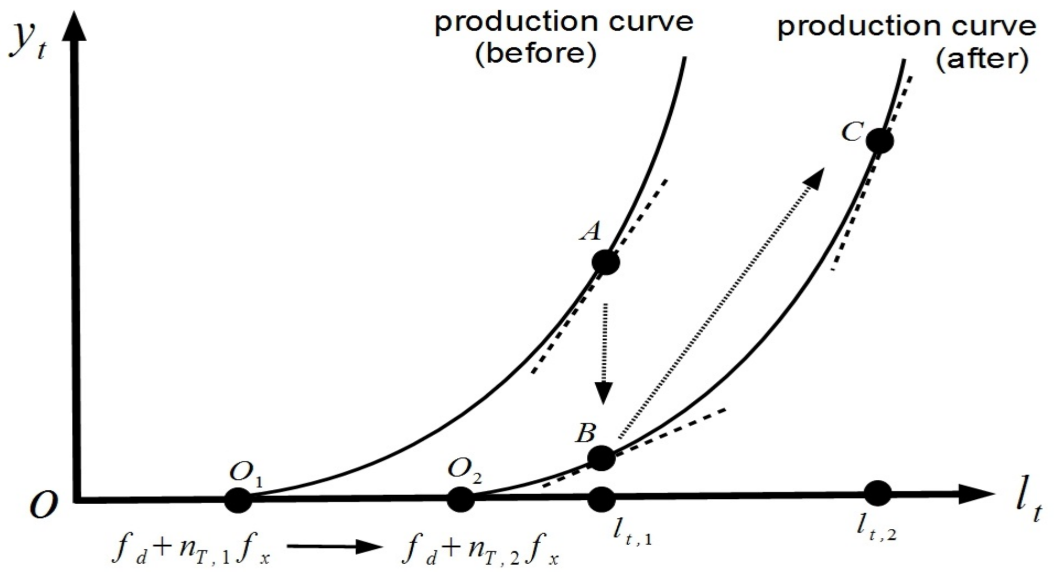

Figure 3: production curve and the change in MPL.

This property seems to be novel. We can get a graphical intuition by Figure 3. Figure 3 illustrates two production curves (curves before and after a change in n) in $(l_t - y)$ space. l_t is firm's total labor inputs. That is, l_t is defined as $l_t = l + f_d + n_T f_x$. Before the change in n, we let each firm's employment and production be at point A. The change in n effect on the firm's actions can be decomposed to two effects.

The first effect is a transition from point A to point B. In this transition, l decrease by interval O_1O_2. This indicates that firms reassign labor of interval O_1O_2 from the production division to the management division while keeping l_t units of total labor. This reassignment effect on productivity is negative as shown in figure 3 where the slope of the tangent decreases.

The second effect is a transition from point B to point C. Just after n increased, all incumbent firms earn negative profit. This make some firms exit and concentrate workers on survived firms. In this transition, l_t increases by interval $l_{t,1}l_{t,2}$. This indicates that these firms succeed in recruiting new workers and assign them jobs of production division. This concentration effect on productivity is positive as shown in figure 3 where the slope of the tangent increases.

An increase in l in the long run (as shown by interval $l_{t,1}l_{t,2}$) is greater than a decrease in l in the short run (as shown by interval O_1O_2). Therefore, the former effect dominates the later effect as shown in figure 3 where the slope of the tangent at point C is greater than that at point A.

5 Conclusion

This paper analyzes how trade induces entry for export markets, the division of labor and changes firm productivity.

Under positive fixed exported cots, opening up to trade makes some firms exit and concentrates labor to surviving firms through recruiting process and induces the division of labor. This new result has different mechanism from Chaney and Ossa (2013) and Kamei (2014), in which the division of labor is induced through pro-competition effect. Furthermore, optimal entry conditions for export markets rule out loss from opening up to trade.

In the long run, an increase in the number of the markets promotes the division of labor and raises firm productivity. In contrast with this, in the short run an increase in the number of the markets has reverse effects. This result is related with Adam Smith's theorem, " The division of labor is limited by the extent of the market". This paper's model indicates a new insight in trade model's context. Chaney and Ossa (2013) and Kamei (2014) indicate that the division of labor is limited by the *size* of the market. In contrast with these, our model indicates that the division of labor is limited by the *number* of the markets.

Problem and extension will be left. First, it seem to be strange that a reduction in variable trade cost has no effect on firm productivity. Second, it is natural that team coordinators are skilled workers and product workers are unskilled workers as Borghans and Weel (2006) indicates .

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Appendix.

Appendix A : Proof of proposition 3 and Corollary 1

The equilibrium allocation is obtained as with autarky equilibrium. We treat n as given. From total cost function eq (14) and optimal price setting $p_d = \mu MC(y_t)$, the following equilibrium conditions are given

$$PP_T : \frac{p_d}{w} = \frac{\mu}{2}\left(\frac{2\gamma f}{y}\right)^{1/2}, \tag{A.1}$$

$$FE_T : \frac{p_d}{w} = \left(\frac{2\gamma f}{y_t}\right)^{1/2} + \frac{f_d + nf_x}{y_t}. \tag{A.2}$$

Note that free entry condition can not be defined for each market but can be defined for the only whole world markets.

These conditions give

$$y_{t,T} = \frac{(f_d + nf_x)^2}{2\gamma f} B^{-2}, \tag{A.3}$$

$$\left(\frac{p_d}{w}\right)_T = B(B+1)\frac{2\gamma f}{f_d + nf_x}, \tag{A.4}$$

(A.3) gives the rest of firm's behaviors;

$$t_T = \frac{f_d + nf_x}{2fB}, \tag{A.5}$$

$$l_T = \frac{f_d + nf_x}{B}. \tag{A.6}$$

From (A.6), MPL_T is given by

$$MPL_T = \frac{f_d + nf_x}{B\gamma f}. \tag{A.7}$$

A difference between MPL_T and MPL_A given by

$$MPL_T - MPL_A = \frac{nf_x}{B\gamma f}. \tag{A.8}$$

Labor market clearing condition in open economy is given by

$$M = \frac{L}{(2\gamma f y_t)^{1/2} + f_d + nf_x}, \tag{A.9}$$

where M represents the number of home country's firms which pay overhead production costs in the home country, f_d.

By substituting (A.3) into (A.9), we obtain the equilibrium number of varieties:[15)]

$$M_T = \frac{B}{B+1}\frac{L}{f_d + nf_x}. \tag{A.10}$$

Note that when f_x is positive, $y_{t,T} > y_A$, $(w/p_d)_T > (w/p_d)_A$, $t_T > t_A$, $l_T > l_A$, and $M_T < M_A$ hold. The other hand, when f_x is zero, $y_{t,T} = y_A$, $(w/p_d)_T = (w/p_d)_A$,

15) We check equilibrium conditions. For simplicity, we treat n as a positive integer. We have already imposed final good market clear conditions for $n+1$ countries to obtain equation $p_d = \mu MC(y_t)$. Equilibrium M_T can be obtained by imposing labor market clear conditions for n countries. This is because final good market clear conditions for $n+1$ countries and labor market clear conditions for n countries derives trade balance conditions for $n+1$countries from Warlas law.

$t_T = t_A$, and $M_T = M_A$ hold.

Appendix B : Proof of Proposition 5.

Social Welfare in closed economy

We treat representative household's utility as a measure of social welfare. Under the utility maximization, indirect utility function of each household is $V_A = (w/P)_A$. On equilibrium, firms set identical price, p and from the definition of P, the following relation is given,

$$V_A = \left(\frac{w}{p}\right)_A M_A^{\frac{1}{\sigma-1}}. \tag{B.1}$$

Note that the indirect utility can be decomposed to real wage rate and the number of varieties. We substitute (9) and (13) into (B.1) and consequently, obtain equilibrium social welfare,

$$V_A = (2\gamma f)^{-1} L^{\frac{1}{\sigma-1}} (B+1)^{\frac{-\sigma}{\sigma-1}} B^{\frac{2-\sigma}{\sigma-1}} f_d{}^{\frac{\sigma-2}{\sigma-1}}. \tag{B.2}$$

Social Welfare in open economy economy

In trading equilibrium, the real wage rates are identical in the all countries and hence, the indirect function is given by $V_T = (w/P)_T$.

Consumers of home country face $(n/\bar{n})M$ brands on the average per one foreign country. Then, P_T can be rewritten in the following way:

$$\begin{aligned} P_T^{1-\sigma} &= \int_{\theta\in\Theta} (p(\theta))^{1-\sigma}\, d\theta + \bar{n} \int_{\theta^*\in\Theta^*} [\tau p(\theta^*)]^{1-\sigma}\, d\theta^* \\ &= M_T p_{d,T}^{1-\sigma} + \bar{n} \left(\tfrac{n}{\bar{n}}\right) M_T \tau^{1-\sigma} p_{d,T}^{1-\sigma} \\ &= M_T p_{d,T}^{1-\sigma} (1 + n\tau^{1-\sigma}). \end{aligned}$$

Since countries are symmetric, the social welfare is obtained by

$$V_T = \left(\frac{w}{p_d}\right)_T \left[(1 + n_T \tau^{1-\sigma}) M_T\right]^{\frac{1}{\sigma-1}}. \tag{B.3}$$

By substituting (A.4) and (A.10)) into (B.3), we can obtain equilibrium social welfare;

$$V_T = (2\gamma f)^{-1} L^{\frac{1}{\sigma-1}} (B+1)^{\frac{-\sigma}{\sigma-1}} B^{\frac{2-\sigma}{\sigma-1}} (f_d + n_T f_x)^{\frac{\sigma-2}{\sigma-1}} (1 + n_T \tau^{1-\sigma})^{\frac{1}{\sigma-1}}. \tag{B.4}$$

This expression is rewritten as

$$V_T = \frac{V_A}{f_d^{\frac{\sigma-2}{\sigma-1}}}(f_d + n_T f_x)^{\frac{\sigma-2}{\sigma-1}}(1 + n_T\tau^{1-\sigma})^{\frac{1}{\sigma-1}}.$$

Comparing V_T of (B.4) to V_A of (B.2), the following relationship is obtained:

$$V_T > V_A \leftrightarrow 1 + n_T\tau^{1-\sigma} > \left(1 + n_T\frac{f_x}{f_d}\right)^{2-\sigma} \leftrightarrow G(n_T) > 1. \tag{B.5}$$

Appendix C : Proof of Proposition 7.

Properties except for properties of V_T and $\hat{M}_T$ in (III) are straightforward accounting for Appendix.

Property $d\hat{M}_T/d\bar{n}$

From $(F.)$ in Appendix F, we can obtain the following condition:

$$\frac{d\hat{M}_W}{d\bar{n}} = \frac{d(1+\bar{n}\tau^{1-\sigma})}{d\bar{n}}M_T + (1+\bar{n}\tau^{1-\sigma})\frac{dM_T}{d\bar{n}} = \frac{\tau^{1-\sigma}f_d - f_x}{f_d + \bar{n}f_x}M_T.$$

This condition implies :

$$\frac{d\hat{M}_W}{d\bar{n}} > 0 \leftrightarrow \tau^{1-\sigma}f_d - f_x > 0. \tag{C.1}$$

Property $dV_T/d\bar{n} > 0$

From $(B.4)$ in Appendix B, we can treat V_T as a function of n and we can obtain $V_T = const \times (f_d + \bar{n}f_x)^{\frac{\sigma-2}{\sigma-1}}(1+\bar{n}\tau^{1-\sigma})^{\frac{1}{\sigma-1}}$, where constant term is positive. From this equation, we can obtain the following condition for sign of $dV_T/d\bar{n}$:

$$sign\left(\frac{dV_T}{d\bar{n}}\right) = sign\left(\frac{d[(f_d+\bar{n}f_x)^{\frac{\sigma-2}{\sigma-1}}(1+\bar{n}\tau^{1-\sigma})^{\frac{1}{\sigma-1}}]}{d\bar{n}}\right).$$

This condition implies :

$$sign\left(\frac{dV_T}{d\bar{n}}\right) = sign\left((\sigma-1)\bar{n} + \frac{f_d}{f_x} - (2-\sigma)\tau^{\sigma-1}\right).$$

Then, we can obtain

$$\frac{dV_T}{d\bar{n}} > 0 \leftrightarrow \bar{n} > \frac{\tau^{\sigma-1}(2-\sigma) - \frac{f_d}{f_x}}{\sigma - 1}. \tag{C.2}$$

When $\frac{\tau^{\sigma-1}(2-\sigma)-\frac{f_d}{f_x}}{\sigma-1} < 0$ holds, $\bar{n} > 0 > \frac{\tau^{\sigma-1}(2-\sigma)-\frac{f_d}{f_x}}{\sigma-1}$. On the other hand, when $\frac{\tau^{\sigma-1}(2-\sigma)-\frac{f_d}{f_x}}{\sigma-1} > 0$ holds, we can obtain $\bar{n} > n_c > \frac{\tau^{\sigma-1}(2-\sigma)-\frac{f_d}{f_x}}{\sigma-1}$ from Appendix. Therefore,

$$\bar{n} > \frac{\tau^{\sigma-1}(2-\sigma) - \frac{f_d}{f_x}}{\sigma - 1} \tag{C.3}$$

Proof of relation; (21) $\Rightarrow$ $(C.3)$ **holds and the converse relation does not hold.**

(21) $\Rightarrow$ $(C.3)$ holds because the following relation holds;.

$$\forall \sigma \in (1,2),\ \ (\sigma-1)n + \frac{f_d}{f_x} > (\sigma-1)n + \tau^{\sigma-1} > (2-\sigma)\tau^{\sigma-1}.$$

We assume $(\sigma, \tau, f_x/f_d) = (3/2, 4, 2)$. The, for any $n > 1$, (21) holds and (C.3) does not hold. Hence, the converse relation does not hold. Q.E.D.

Appendix D : Deviation condition and interpretation of G

We can interpret G, from deviation condition shown in the following way. In this appendix, we use subscript n, $\hat{n}$ which represents the number of export markets firms enter.

We call *deviation condition* for a condition that a firm has incentive to enter $\hat{n} \neq n$ export markets when all the other firms enter n export markets. We shows deviation condition is equivalent to $G(\hat{n}) > G(n)$. That is, we show the following proposition: When $P_T^{1-\sigma} = Mp_{d,n}^{1-\sigma}(1 + n\tau^{1-\sigma})$ holds, the following condition holds:

$$\text{For given } n,\ \forall M, [\pi_{\hat{n}} > \pi_n] \rightarrow [G(\hat{n}) > G(n)]$$

As with the manner in proof of non-deviation condition of Appendix K, from Appendix J, we can obtain profits of each type of firm in the following way

$$\pi_n = \frac{B}{B+1}\frac{LI}{M} - w(f_d + nf_x),$$

$$\pi_{\hat{n}} = \frac{B}{B+1}\frac{LI}{M}\frac{1+\hat{n}\tau^{1-\sigma}}{1+n\tau^{1-\sigma}}\left(\frac{f_d + \hat{n}f_x}{f_d + nf_x}\right)^{\sigma-1} - w(f_d + \hat{n}f_x).$$

From some M, $\pi_{\hat{n}} > 0 = \pi_n$. This implies $G(\hat{n}) > G(n)$.

Now, we can interpret in the following way. Though lemma 2 demands $G(n)$ must be maximized *on equilibrium*, the above result demands $G(n)$ must be maximized *off equilibrium*. Therefore, The numerator of $G(n)$, $(1 + n\tau)$, can be interpreted as entry gain. The denominator of $G(n)$, $(1 + n\frac{f_x}{f_d})^{2-\sigma}$, can be interpreted as entry loss.

For On Line Appendix.

Appendix E : Generality and validity of the technology in comparison to the one adopted by Chaney and Ossa (2013)

In this section, We examine that how general and valid the technology which we adopt in equation (1) is in comparison to the one adopted by Chaney and Ossa (2013).

Generality of the technology in (1)

The technology we adopted is different from the one adopted by Chaney and Ossa (2013), in two points. Equation (1) in this paper corresponds to the following equation in Chaney and Ossa (2013) ;

$$l(\underline{\omega}, \overline{\omega}) = \frac{1}{2} \int_{\underline{\omega}}^{\overline{\omega}} \left(\frac{\underline{\omega} + \overline{\omega}}{2} - \omega \right)^{\alpha} d\omega. \tag{E.1}$$

Equation (E.1) and (1) are equal, when $\alpha = 1$ in (E.1) and $\gamma = 1$ in (1).

We examine a characteristic of parameter, α by seeing shape of $l(\underline{\omega}, \overline{\omega})$. For simplicity, we assume $\gamma = 1$ and $t = 1$. When $\alpha = 1$, integral term of the right hand side in (E.1) is the area formed by "Benchmark Line" in Figure 4. When $\alpha > 1$, the one is the area formed by "Curve H" in Figure 4. When $0 < \alpha < 1$, the one is the area formed by "Curve L" in Figure 4. Figure 4 implies that the effect of an increase in α is parallel to the effect of a decrease in γ.

If we adopts the technology in (E.1), the equilibrium allocation are rewritten by;

$$l_{p,A} = \frac{2(\alpha+1) - \mu}{\mu - (\alpha+1)} f_d,$$

$$y_A = \left(\frac{\alpha+1}{\mu - (\alpha+1)} f_d \right)^{\alpha+1} \left(\frac{\alpha}{\alpha+1} \frac{1}{f} \right)^{\alpha},$$

$$MPL_A = (\alpha+1) \left[\left(\frac{\alpha}{\alpha+1} \right) \left(\frac{\alpha+1}{\mu - (\alpha+1)} \right) \frac{f_d}{f} \right]^{\alpha},$$

$$t_A = \left(\frac{\alpha}{\alpha+1} \right) \left(\frac{\alpha+1}{\mu - (\alpha+1)} \right) \frac{f_d}{f}.$$

The next table shows that the effect of an increase in α is parallel to the effect of a decrease in γ on certain conditions.

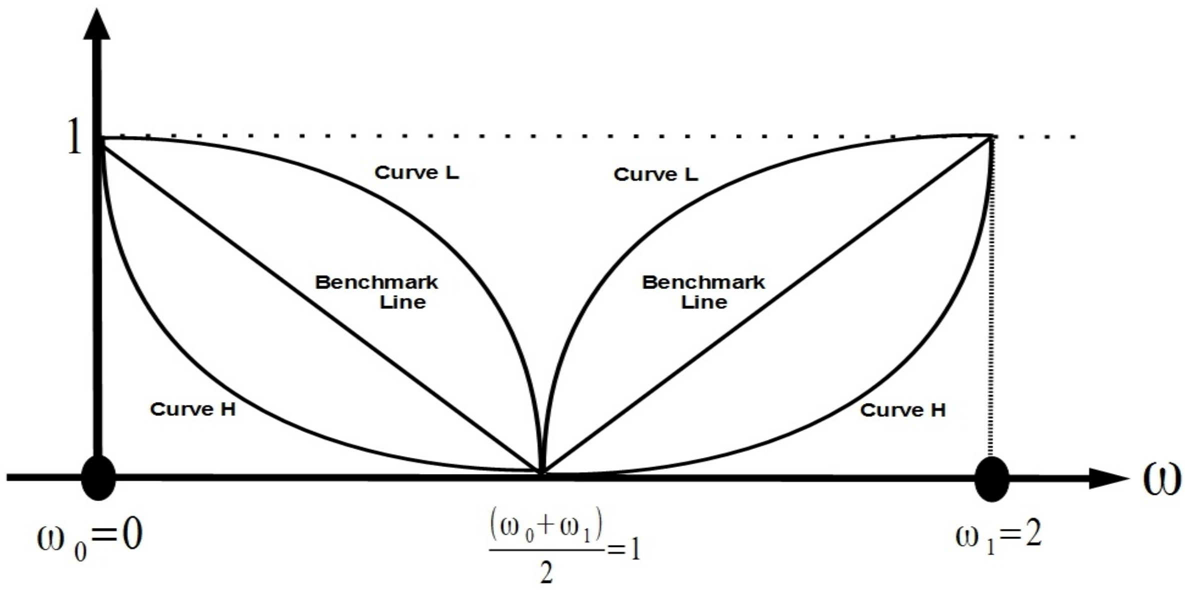

Figure 4: comparison between sequential task structures

Table 5

	l_A	y_A	MPL_A
$\alpha \uparrow$	0	+ only if $t_A > 1$	+ only if $t_A > 1$

α's amplification effect also occurs on certain conditions. Moreover, effect of f does not change. Therefore, this suggests that the technology which we adopt does not loose generality very much in comparison to the one adopted by Chaney and Ossa (2013) .

Validity of the technology in (1)

Martins, Scarpetta and Pilat (1996) shows that almost all industries in OECD have markup rate which belongs to set $(1, 2)$. Therefore, the internal solution condition $2 < \mu$ does not seems to have reality. This property highly depends on organization parameter α. If we adopts the technology in (E.1), internal solution condition is

$$\mu > \alpha + 1.$$

Therefore, by assuming organization parameter α to be in (0,1), model's mark-up rate μ can be consistent with the empirical studies.

However, assuming α to be in (0,1) makes tractability of the model decrease. For analytical simplicity, we assume α to be 1.

Appendix F : Shape of PP_A curve and FE_A curve in Figure 2

In this section, we examine shape of PP_A curve and FE_A curve in Figure 2.

We define $Z(y)$ as difference between right hand side of (7), PP_A relation and of (8), FE_A relation ;

$$Z(y) \equiv \frac{\mu}{2}\left(\frac{2\gamma f}{y}\right)^{1/2} - \left[\left(\frac{2\gamma f}{y}\right)^{1/2} + \frac{f_d}{y}\right] = B(2\gamma f)^{1/2}y^{-1/2} - f_d y^{-1}.$$

Certainly, $Z(y_A) = 0$ holds.

The derivative of function $Z(y)$ is given by

$$Z'(y) = -2^{-1}B(2\gamma f)^{1/2}y^{-3/2} + f_d y^{-2}.$$

When $y = y_A^*$, $Z'(y_A^*) = 0$ holds, where y_A^* is given by

$$y_A^* = 2\frac{f_d}{B^2\gamma f} = 4\frac{f_d}{B^2 2\gamma f} = 4y_A.$$

From $B > 0$, when $y < 4y_A$, $Z'(y) > 0$ holds and when $y > 4y_A$, $Z'(y) < 0$ holds. Furthermore, for the second order derivative of function $Z(y)$, $Z''(64y_A/9) = 0$ holds.

The limits of function $Z(y)$ are given by

$$\lim_{y\to\infty} Z(y) = 0,$$

$$\lim_{y\to 0} Z(y) = -\infty.$$

The above relations are proved in the following way.

Proof.

$$\lim_{y\to\infty} Z(y) = \lim_{y\to\infty} \frac{B(2\gamma f)^{1/2}y^{1/2} - f_d}{y} = \frac{0 - f_d}{\infty} \to 0,$$

$$\lim_{y\to 0} Z(y) = \lim_{y\to 0} \frac{B(2\gamma f)^{1/2}y^{1/2} - f_d}{y} = \frac{-f_d}{0} \to -\infty.$$

Q.E.D.

According to the above results, the shape of $Z(y)$ is the one as shown in Figure 5.

Figure 5 is consistent to Figure 2 and hence, Figure 2 is supported.

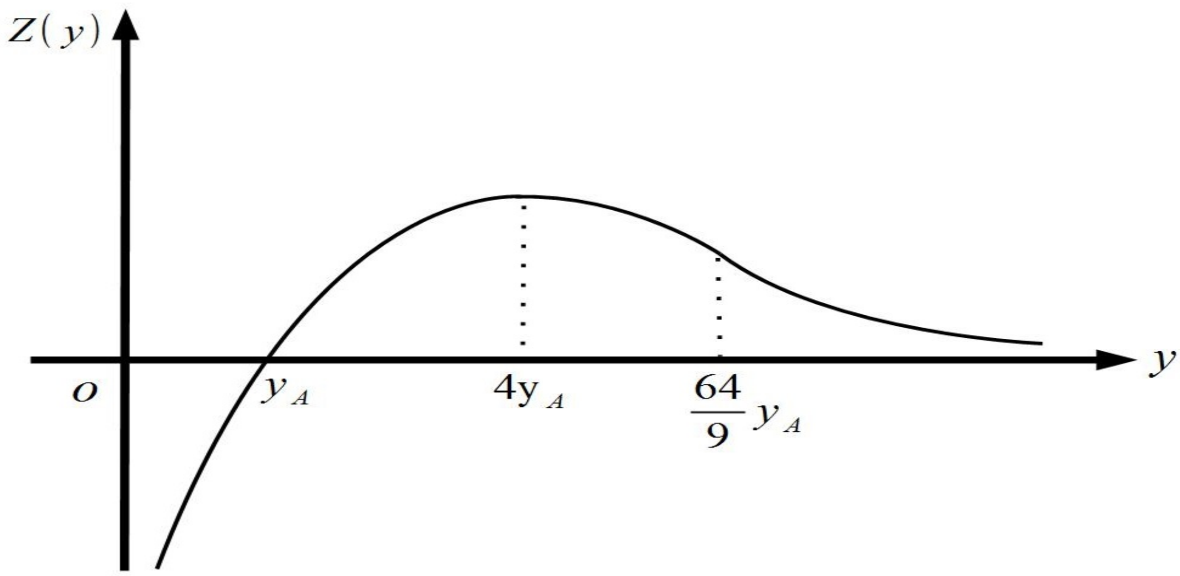

Figure 5: the shape of Z(y)

Appendix G : Derivation of optimal quantity-price rule on trading equilibrium

In open economy, home country's house holds have preference represented by utility function $U = \left[\int_{\theta\in\Theta} c(\theta)^{\rho} d\theta + n\int_{\theta^*\in\Theta^*} c(\theta^*)^{\rho} d\theta\right]^{1/\rho}$, $0<\rho<1$. Utility maximization drives price index $P_T = \left[\int_{\theta\in\Theta}(p(\theta))^{1-\sigma}d\theta + \bar{n}\int_{\theta^*\in\Theta^*}[\tau p(\theta^*)]^{1-\sigma}d\theta^*\right]^{1/(1-\sigma)}$.

On trading equilibrium, all firms' profit are zero and then, each household's income contains the only wage income. Consumption of each household in each country for domestic brand is respectively;

$$c = p_d^{-\sigma}(P_T)^{\sigma-1}w, \;\; c^* = p_d^{*-\sigma}(P_T^*)^{\sigma-1}w^*. \tag{G.1}$$

Consumption of the one for foreign brand (import brand) is respectively;

$$c' = (\tau p_d^*)^{-\sigma}(P_T)^{\sigma-1}w, \;\; c^{*\prime} = (\tau p_d)^{-\sigma}(P_T^*)^{\sigma-1}w^*. \tag{G.2}$$

Prime represents consumption for import brand. The above equations show that the elasticity of demand for price is σ regardless of source countries.

From definition of iceberg cost τ, export revenue is defined as $r_x \equiv p_x\dfrac{y_x}{\tau}$. Since mill price in export market is $p_x = \tau p_d$, export revenue can be rewritten as $r_x = \tau p_d\dfrac{y_x}{\tau} = p_d y_x$. Total revenue from all markets $r_t = r_d + nr_x$ can be rewritten as

$r_t = p_d y_d + p_d n y_x = p_d y_t$. Total profit from all markets π_t is;

$$\pi_t = p_d y_n - TC(y_n). \tag{G.3}$$

Market clear condition for home country's brand is;

$$y_t = Lc + n\tau L^* c^{*\prime}. \tag{G.4}$$

(G.1), (G.2), and (G.4) derive

$$y_t = L[p_d^{-\sigma} P_T^{\sigma-1} w] + n\tau L^*[(\tau p_d)^{-\sigma} P_T^{*\sigma-1} w^*]. \tag{G.5}$$

This shows that each firm faces individual demand curve whose elasticity of demand for price is σ. From (G.3), and (G.5), profit maximization problem derives

$$PP : p_d = \mu MC(y_t).$$

Appendix H : Complements for Proof of Proposition 6

Proof of property that (21) is equivalent to $\hat{M}_W > M_A$.

From (13) and (D.10), we can get

$$\hat{M}_W - M_A = \frac{B}{B+1} \frac{n(f_d \tau^{1-\sigma} - f_x)}{f_d(f_d + n f_x)}.$$

From this relation, $f_d \tau^{1-\sigma} - f_x > 0$ is equivalent to $\hat{M}_W > M_A$. Hence (21) is equivalent to $\hat{M}_W > M_A$. Q.E.D.

Proof of property that (21) $\Rightarrow$ (18) holds.

We show that (21) $\Rightarrow$ (18) holds. (21) is equivalent to $\tau^{1-\sigma} > f_x/f_d$. From $2 - \sigma > 0$, this relation implies;

$$\left(1 + n\tau^{1-\sigma}\right)^{2-\sigma} > \left(1 + n\frac{f_x}{f_d}\right)^{2-\sigma}.$$

The other hand, from $0 < 2 - \sigma < 1$, the following relation holds;

$$1 + n\tau^{1-\sigma} > \left(1 + n\tau^{1-\sigma}\right)^{2-\sigma}.$$

Therefore, (21) $\Rightarrow$ (18) holds.

We show that the converse relation does not hold. We assume $\sigma = 3/2$, $\tau = 4$, and $f_x/f_d = 2$. Then, $1 < \tau^{\sigma-1} f_x/f_d$ holds from $1 < 4^{1/2}2 = 4$. That is, $f_d < \tau^{\sigma-1} f_x$ holds. If $n > 4$ holds, the following relation holds;

$$1 + 4^{-1/2}n > (1 + 2n)^{1/2}.$$

That is, there is a pair of $(\sigma, \tau, f_x, f_d, n)$ such that satisfies both (18) and $f_d < \tau^{\sigma-1} f_x$. Therefore, (18) $\Rightarrow$ (21) does not hold. Q.E.D.

Appendix I : Proof of Lemma 1.

In this appendix, we examine weather n distributes or not on trading equilibrium.

Consider two firm (firm a and firm b). Firm a and firm b enter the domestic firm and enter n_a, n_b export markets respectively.

Firm i's good market clearing condition is given by

$$y_{t,i} = Lc_d + n_i L\tau c^{*\prime} \tag{I.1}$$

where i denotes a or b.

Firms' optimal pricing rules and zero profit conditions give

$$\frac{y_{t,a}}{y_{t,b}} = \frac{(f_d + n_a f_x)^2}{(f_d + n_b f_x)^2}. \tag{I.2}$$

Optimal consumption conditions, (G.1), (G.2) and optimal pricing rules give

$$\frac{Lc_d + n_a L\tau c^{*\prime}}{Lc_d + n_b L\tau c^{*\prime}} = \frac{P_T^{\sigma-1}(1 + n_a\tau^{1-\sigma})Lwp_{d,a}^{-\sigma}}{P_T^{\sigma-1}(1 + n_b\tau^{1-\sigma})Lwp_{d,b}^{-\sigma}} = \frac{(1 + n_a\tau^{1-\sigma})}{(1 + n_b\tau^{1-\sigma})}\frac{(f_d + n_a f_x)^\sigma}{(f_d + n_b f_x)^\sigma} \tag{I.3}$$

From equation (I.1), (I.2) and (I.3), we can get

$$\frac{1 + n_a\tau^{1-\sigma}}{1 + n_b\tau^{1-\sigma}} = \left(\frac{f_d + n_a f_x}{f_d + n_b f_x}\right)^{2-\sigma}. \tag{I.4}$$

We define $\delta \in [0, \frac{\bar{n}}{n_b}]$ as $n_a = \delta n_b$. Using this δ, we can rewrite $(I.4)$ in the following way:

$$\frac{1 + n_b\tau^{1-\sigma}\delta}{1 + n_b\tau^{1-\sigma}} = \left(\frac{f_d + n_b f_x\delta}{f_d + n_b f_x}\right)^{2-\sigma}. \tag{I.5}$$

Of course, $\delta = 1$ satisfies $(I.5)$. Does the other values of δ satisfies $(I.5)$?

Left hand side of (I.5) is linear for δ and this is denoted by $J(\delta)$. Right hand side of (I.5) is nonlinear for δ and this is denoted by $K(\delta)$. A line represented by $J(\delta)$ has the following intersection in (δ, J) space

$$J(0) = \frac{1}{1 + n_b \tau^{1-\sigma}}. \tag{I.6}$$

A curve represented by $K(\delta)$ has the following intersection in (δ, K) space

$$K(0) = \left(\frac{f_d}{f_d + n_b f_x}\right)^{2-\sigma}. \tag{I.7}$$

When $K(0) > J(0)$ holds, these line and curve have a unique cross point , as is shown in Figure 6. The unique point is a point whose δ is one. Note $K(0) > J(0)$ is equivalent to $G(n_b) > 1$.

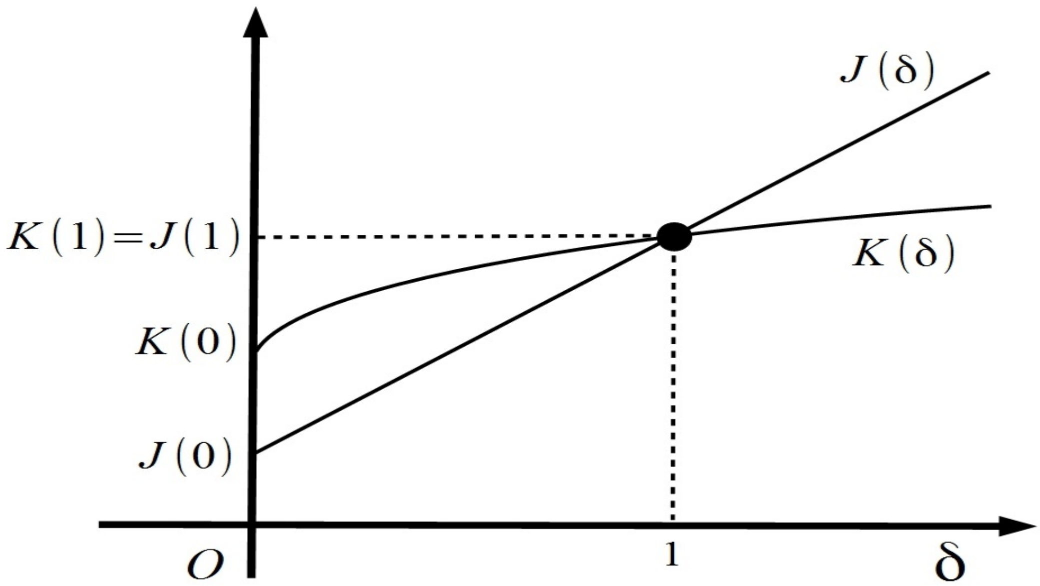

Figure 6: n's degeneration 1.

When $K(0) < J(0)$ holds, δ has multiple solutions except for a case in which $J'(1) = K'(1)$. This case is described in Figure 7. In this case, line J and curve K come in contact with each other at $\delta = 1$.

$J'(1) = K'(1)$ is equivalent to

$$n = (2 - \sigma)\tau^{\sigma-1} - \frac{f_d}{f_x}.$$

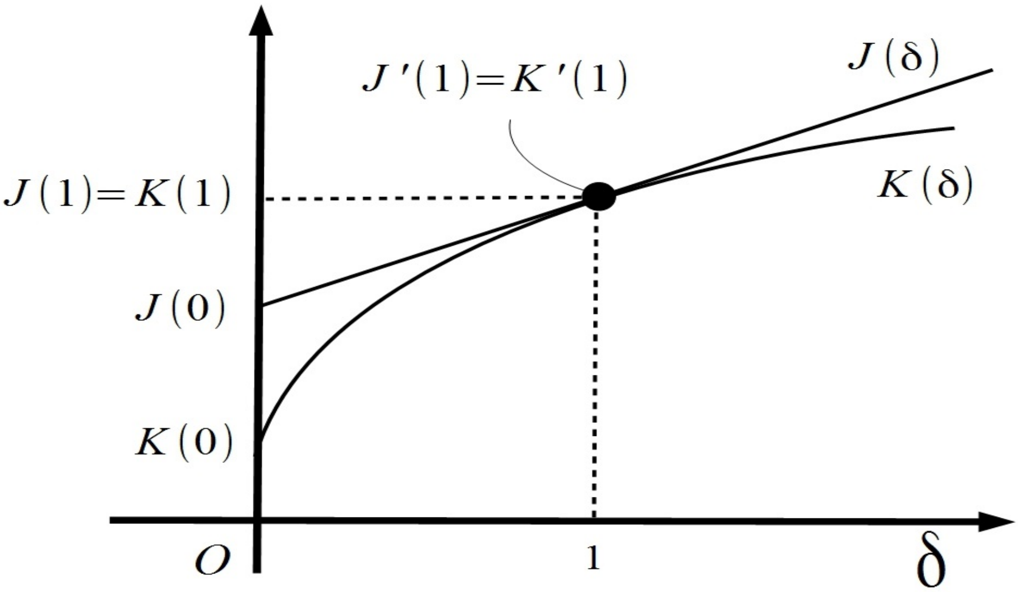

Figure 7: n's degeneration 2.

Appendix J : Short run equilibrium

We use this Appendix to prove Lemma 2 and to decompose effects of trade liberalization into short run and long run effect.

This equation derives demands of home consumers for home country:

$$\begin{aligned} y_t = & \quad Lc + n\tau Lc^{*\prime} \\ = & \quad LIp_d P_T^{\sigma-1} + n\tau LI(\tau p_{d,n})^{-\sigma} P_T^{\sigma-1} \\ = & \quad LIp_d^{-\sigma} P_T^{\sigma-1}(1 + n\tau^{1-\sigma}) \\ = & \quad LIp_d^{-1} M^{-1}. \end{aligned}$$

This equation derives total revenue of home country:

$$r_t = p_d y_t = LIM^{-1}.$$

Firms input $(2\gamma f y_t)^{1/2} + (f_d + nf_x)$ units of labor and this derives total cost function:

$$TC = w\left[(2\gamma f y_t)^{1/2} + (f_d + nf_x)\right].$$

This equation and optimal pricing rule gives:

$$p_d = w(B+1)(2\gamma f)^{1/2} y_t^{-1/2}.$$

From these conditions we can obtain optimal total cost function of short run:

$$TC = \frac{r_t}{B+1} + w(f_d + nf_x).$$

We substitute r_t and TC into π_d and obtain

$$\begin{aligned}\pi = & \quad r_t - TC_t \\ = & \quad r_t - \left[\tfrac{r_t}{B+1} + w(f_d + nf_x)\right] \\ = & \quad \tfrac{B}{B+1} r_t - w(f_d + nf_x) \\ = & \quad \tfrac{B}{B+1}\tfrac{LI}{M} - w(f_d + nf_x).\end{aligned}$$

This condition and consumer's income $I = w + M\pi_t/L$ give

$$\frac{I_t}{w} = \mu - \frac{M(f_d + nf_x)}{L},$$

$$\frac{\pi_t}{w} = \frac{\mu L}{\sigma M} - \mu\frac{(f_d + nf_x)}{L}.$$

These equations derives

$$l_t = \frac{L}{M},$$

$$y_t = \frac{\left[\frac{L}{M} - (f_d + nf_x)\right]^2}{2\gamma f},$$

$$\frac{p_d}{w} = \frac{(B+1)(2\gamma f)}{L/M - (f_d + nf_x)}.$$

Appendix K : Proof of Lemma 2

In this appendix, we use subscript n, $\hat{n}$ which represents the number of export markets firms enter.

We call non-deviation condition for a condition that a firm does not have incentive to enter $\hat{n} \neq n$ export markets when all the other firms enter n export markets. This condition certifies existence of trading equilibrium. We shows non-deviation condition is equivalent to $G(n) > G(\hat{n})$. That is, we show the following proposition:

$$\text{For given } n, \forall M, \forall \hat{n}(\neq n), [\pi_n > \pi_{\hat{n}}] \rightarrow [G(n) > G(\hat{n})].$$

From Appendix J and $P_T^{1-\sigma} = Mp_{d,n}^{1-\sigma}(1 + n\tau^{1-\sigma})$, profit of firms entering n export markets is

$$\pi_n = \frac{B}{B+1}\frac{LI}{M} - w(f_d + nf_x).$$

On the other hand, profit of firms entering $\hat{n}(\neq n)$ export markets is derived in the following way.

Demands for this firms is

$$\begin{aligned}
y_{\hat{n}} = &\quad Lc + \hat{n}\tau Lc^{*\prime} \\
= &\quad LIp_{d,\hat{n}}P_T^{\sigma-1} + \hat{n}\tau LI(\tau p_{d,\hat{n}})^{-\sigma}P_T^{\sigma-1} \\
= &\quad LIp_{d,\hat{n}}^{-\sigma}P_T^{\sigma-1}(1+\hat{n}\tau^{1-\sigma}) \\
= &\quad p_{d,\hat{n}}^{-1}LIM^{-1}\tfrac{1+\hat{n}\tau^{1-\sigma}}{1+n\tau^{1-\sigma}}\left(\tfrac{p_{d,n}}{p_{d,\hat{n}}}\right)^{\sigma-1} \\
= &\quad p_{d,\hat{n}}^{-1}LIM^{-1}\tfrac{1+\hat{n}\tau^{1-\sigma}}{1+n\tau^{1-\sigma}}\left(\tfrac{f_d+\hat{n}f_x}{f_d+nf_x}\right)^{\sigma-1}.
\end{aligned}$$

This gives

$$r_{\hat{n}} = p_{d,\hat{n}}y_{\hat{n}} = LIM^{-1}\frac{1+\hat{n}\tau^{1-\sigma}}{1+n\tau^{1-\sigma}}\left(\frac{f_d+\hat{n}f_x}{f_d+nf_x}\right)^{\sigma-1}.$$

Optimal pricing rule gives $TC_{\hat{n}} = \frac{r_{\hat{n}}}{B+1} + w(f_d + \hat{n}f_x)$ and these conditions give

$$\pi_{\hat{n}} = \frac{B}{B+1}\frac{LI}{M}\frac{1+\hat{n}\tau^{1-\sigma}}{1+n\tau^{1-\sigma}}\left(\frac{f_d+\hat{n}f_x}{f_d+nf_x}\right)^{\sigma-1} - w(f_d+\hat{n}f_x).$$

$\forall M, \forall \hat{n}(\neq n), [\pi_n > \pi_{\hat{n}}]$ hold for M which $\pi_n = 0$. That is, for some M, $\pi_n = 0 > \pi_{\hat{n}}$ hold. This condition is equivalent to

$$f_d + nf_x = \frac{B}{B+1}\frac{LI}{wM} < \frac{1+n\tau^{1-\sigma}}{1+\hat{n}\tau^{1-\sigma}}\left(\frac{f_d+nf_x}{f_d+\hat{n}f_x}\right)^{\sigma-1}(f_d+\hat{n}f_x).$$

This condition is equivalent to $G(n) > G(\hat{n})$.

Appendix L : Proof of Proposition 3

To characterize of n_T, we begin with clarifying property of $G(\cdot)$.

Property of $G(\cdot)$

From definition of $G'(n)$, we can obtain the following condition:

$$G'(n) = \quad \left[\tau^{1-\sigma}\left(1+\tfrac{f_x}{f_d}n\right)^{2-\sigma} - (1+\tau^{1-\sigma}n)(2-\sigma)\left(1+\tfrac{f_x}{f_d}n\right)^{1-\sigma}\tfrac{f_x}{f_d}\right]\left(1+\tfrac{f_x}{f_d}n\right)^{-2(2-\sigma)}$$

$$
\begin{aligned}
&= \left(1+\tfrac{f_x}{f_d}n\right)^{2-\sigma}\left[\tau^{1-\sigma}-(1+\tau^{1-\sigma}n)(2-\sigma)\left(1+\tfrac{f_x}{f_d}n\right)^{-1}\tfrac{f_x}{f_d}\right]\left(1+\tfrac{f_x}{f_d}n\right)^{-2(2-\sigma)} \\
&= \left[\tau^{1-\sigma}-(1+\tau^{1-\sigma}n)(2-\sigma)\left(n+\tfrac{f_d}{f_x}\right)^{-1}\right]\left(1+\tfrac{f_x}{f_d}n\right)^{-(2-\sigma)} \\
&= \left[\tau^{1-\sigma}\left(x+\tfrac{f_d}{f_x}\right)-(1+\tau^{1-\sigma}n)(2-\sigma)\right]\left(1+\tfrac{f_x}{f_d}n\right)^{-(2-\sigma)}\left(n+\tfrac{f_d}{f_x}\right)^{-1} \\
&= \left[(\sigma-1)\tau^{1-\sigma}n+\tau^{1-\sigma}\tfrac{f_d}{f_x}-(2-\sigma)\right]\underbrace{\left(1+\frac{f_x}{f_d}n\right)^{-(2-\sigma)}\left(n+\frac{f_d}{f_x}\right)^{-1}}_{+}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
& G'(n)\gtreqless 0 \\
\leftrightarrow \quad & (\sigma-1)\tau^{1-\sigma}n+\tau^{1-\sigma}\tfrac{f_d}{f_x}-(2-\sigma)\gtreqless 0 \\
\leftrightarrow \quad & n \gtreqless \underbrace{\frac{1}{\sigma-1}}_{+}\left[\tau^{2(\sigma-1)}(2-\sigma)-\tfrac{f_d}{f_x}\right].
\end{aligned}
$$

We define n_m as $G'(n_m)=0$. That is, $n_{min}=[\tau^{2(\sigma-1)}(2-\sigma)-\frac{f_d}{f_x}]/(\sigma-1)$.

Characterization of n_T

When $n_{min}<0$ holds, $n_c<0$ holds because $G(0)=1$ holds and G is increasing for $n\geq n_{min}$. We can interpret this in Figure 7.

In this case, n_T is characterized in the following way. From lemma 2, n_T maximizes G and from lemma $G(n_T)>1$. Since G is increasing for $n\geq 0$ and $G(0)=1$, G is maximized at $n=\bar{n}$ and $G(\bar{n})>1$. Therefore, $n_T=\bar{n}$.

When $n_{min}>0$ holds, that is, when $f_d<(2-\sigma)\tau^{\sigma-1}f_x$, $n_c>0$ holds because $G(0)=1$ holds and G is increasing for $n\geq n_{min}$. We can interpret this in Figure 7.

In this case, n_T is characterized in the following way. When $\bar{n}<n_c$ holds, G is maximized at $n=0$ for $n\in[0,\bar{n}]$. $G(0)>1$ does not holds since $G(0)=1$. Therefore, $n_T\neq 0$ holds. When $\bar{n}<n_c$ holds, n_T does not exists. On the other hand, when $\bar{n}>n_c$ holds, $n_T=\bar{n}$ since G is increasing for $n\geq n_c$ and $G(\bar{n})>1$.

For n of (19) in lemma 1, $n=n_{min}$ holds. Even though $n_{min}>0$, n_m do not maximize G. Therefore, n of (19) is not n_T.

The above analysis certifies n_T as the unique solution.

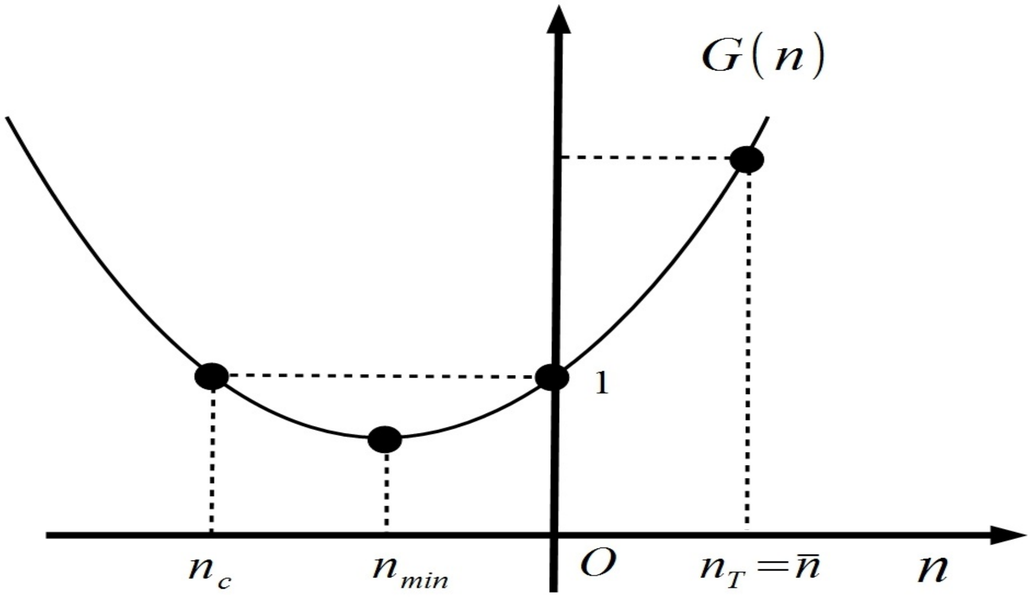

Figure 8: Function $G(n)$: A case of $n_c < 0$

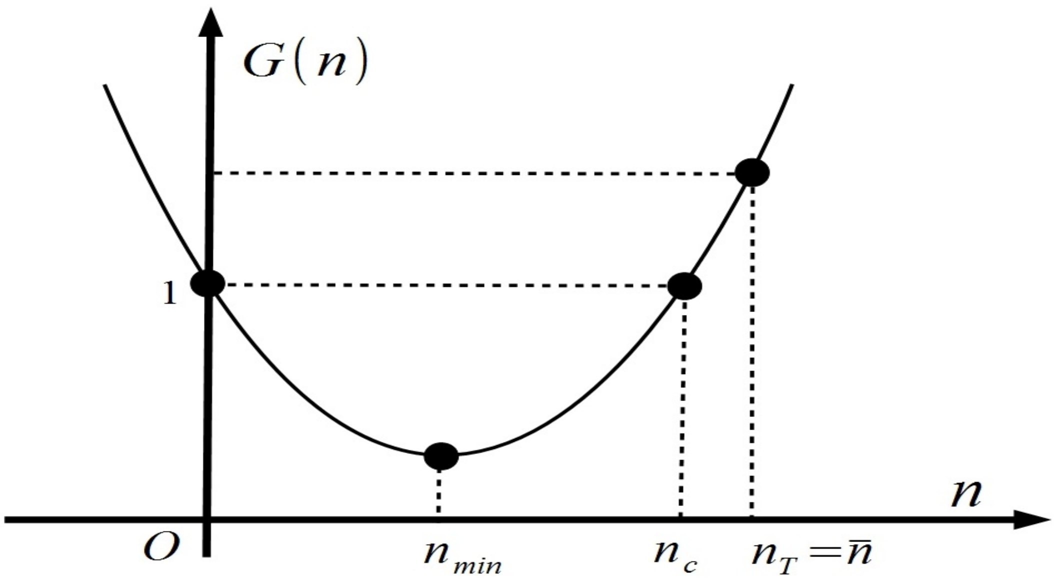

Figure 9: Function $G(n)$: A case of $n_c > 0$