OPTIMAL STOPPING OF THE MAXIMUM PROCESS IN A LÉVY MODEL

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ABSTRACT. We study optimal stopping problems whose reward function involves both the state process X and its running maximum S. In this article, the state process is a spectrally negative Lévy process. The problem is in nature a two-dimensional one. We present a method that handles this problem in a systematic way and find, in a general setting, explicit solutions by using excursion theory and scale functions.

Key words: Optimal stopping; excursion theory; spectrally negative Lévy processes; scale functions.
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1. INTRODUCTION

We let $X = (X_t, t \geq 0)$ be a *spectrally negative Lévy process* (i.e., a Lévy process with only negative jumps) and denote by Y the reflected process,

$$Y_t = S_t - X_t$$

where $S_t = \sup_{u \in [0,t]} X_u \vee s$ with $s = S_0$. Hence Y is the excursion of X from its running maximum S. We consider an optimal stopping problem that involves both X and S:

$$\bar{V}(x, s) \quad = \quad \sup_{\tau} \mathbb{E}^{x,s} \left[\int_0^{\tau} e^{-qt} f(X_t, S_t) \mathrm{d}t + e^{-q\tau} g(X_\tau, S_\tau) \right].$$

where the rewards f and g are measurable functions from $\mathbb{R}^2$ to $\mathbb{R}_+$. The rigorous mathematical definition of this problem is presented in Section 2. In this study, we shall solve for optimal strategy and corresponding value function along with optimal stopping region in the (s, x)-plane. In contrast to diffusion case, when the state process X is a spectrally negative Lévy process, no characterization of the value function is yet known even for problems involving X *alone*, while a number of authors have succeeded in extending the classical results by using the technique called *scale functions*. We just name a few here : Baurdoux and Kyprianou [5, 6] for stochastic games, Avram et al. [4], Kyprianou and Palmowski [17], and Loeffen [19] for the optimal dividend problem, Alili and Kyprianou [1] and Avram et al. [3] for American and Russian options. Due to the lack of general characterization, solution techniques presented in these articles are more or less problem-specific. Optimality is usually obtained by so-called "threshold strategy". That is, the player should stop and receive rewards on the first occasion when the state process enters a certain region. Accordingly, in Lévy and other jump models, the authors first qualitatively argue what optimal strategy should be and construct a candidate value function in continuation and stopping region. Then they prove its optimality by verifying the 'quasi-variational inequalities' (see Øksendal and Sulem [20]). Since the problem at hand involves two dimensions; , finding and proving the overall optimal

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strategy may be challenging. There are some papers on two-dimensional optimal stopping problems (involving S and X) that have specific reward functions: for example, Ott [21] and Guo and Zervos [15]. In the former, the author solves problems including a capped version of the Shepp-Shiryaev problem [25], and the latter is another contribution that extends [25] with the reward function $g(x,s) = (x^a s^b - K)^+$, $a, b, K \geq 0$.

The technique used in our study is excursion theory. Excursion theory for spectrally negative Lévy processes has been developed recently. See Bertoin [7] as a general reference. More specifically, an exit problem of the reflected process Y was studied by Avram et al. [3], Pistorius [23] [24] and Doney [11]. In this study, rather than solving for particular reward functions, we shall provide a general framework that tackles this problem, namely two-dimensional involving S and X for spectrally negative Lévy processes. As in the case of diffusions in Egami and Oryu [12], we look at excursions that occur at each level of S, and reduce the problem to an infinite number of one-dimensional optimal stopping problems. This approach is new with Lévy process, but is already utilized with diffusion processes: see, for example, Graversen and Peskir [14] and Alvarez and Matomäki [2]. Having thus far characterized two-dimensional problem as a set of one-dimensional optimal stopping problems, we shall focus on and contribute to obtaining, in a general setting, an explicit form of value function (e.g., (5.4b)) among the class of stopping times $\mathcal{S}' \subsetneq \mathcal{S}$ defined in (2.4), a class of threshold strategies. Our method is general in the sense that it can treat problems in which the conventional method relying on the *smooth-fit principle* does not work (see Section 6). The explicit formula of the value function should be of great help for further analysis.

The key step is to compute $U(s,s)$ in (2.11), or the value of similar kind. This denotes the value if we start with $S = X$. In Section 3, we characterize this value in an integral form by using the excursion theory (see Proposition 3.1). Once this is found, the problem reduces to the one-dimensional problem. In those reduced problems, $U(s,s)$ appears in different situations (see Case (1-L), (2-L), and (3-L)) in Section 4. We shall provide explicit forms of $U(s,s)$ for each of the cases, which is our main contribution (see Propositions 4.1 and 4.2). With $U(s,s)$ at hand, we shall compute $U(x,s)$ for $x \leq s$, the general value and draw a (s,x)-diagram from which one can tell a certain point in $\mathbb{R}^2$ belongs either continuation or stopping region. This is treated in a systematical way in Section 5. Finally in Section 6 we handle lookback options with jumps: $g(x,s) = e^s - ke^x$ as an illustration of how to implement our methods. Note that in Section 2, we discuss sufficient conditions for optimality and the scale function of a spectrally negative Lévy process. Some technical matters are collected in Appendices.

2. Problem with Spectrally Negative Lévy Processes

Let the spectrally negative Levy process $X = \{X_t; t \geq 0\}$ represent the state variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the set of all possible realization of the stochastic economy, and $\mathbb{P}$ is a probability measure defined on $\mathcal{F}$. We denote by $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ the filtration with respect to which X is adapted and with the usual conditions being satisfied. The Laplace exponent ψ of X is given by

$$\psi(\lambda) = \mu\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} (e^{\lambda x} - 1 - \lambda x \mathbb{1}_{(x>-1)})\Pi(\mathrm{d}x),$$

where $\mu \geq 0$, $\sigma \geq 0$, and Π is a measure concentrated on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$. It is well-known that ψ is zero at the origin, convex on $\mathbb{R}_+$ and has a right-continuous inverse:

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}, \quad q \geq 0.$$

The running maximum process $S = \{S_t; t \geq 0\}$ is defined by $S_t = \sup_{u\in[0,t]} X_u \vee s$. In addition, we write Y for the reflected process defined by $Y_t = S_t - X_t$. The payoff is composed of two parts; the running income to be received continuously until stopped and the terminal reward part to be received when the process is stopped. We consider the following optimal stopping problem and the value function $\bar{V}:\mathbb{R}^2\mapsto\mathbb{R}$ associated with initial values $X_0=x$ and $S_0=s$;

$$\bar{V}(x,s) \;=\; \sup_{\tau\in\mathcal{S}}\mathbb{E}^{x,s}\left[\int_0^\tau e^{-qt}f(X_t,S_t)\mathrm{d}t+1\!\!1_{\{\tau<\infty\}}e^{-q\tau}g(X_\tau,S_\tau)\right] \tag{2.1}$$

where $\mathbb{P}^{x,s}(\,\cdot\,):=\mathbb{P}(\,\cdot\,|\,X_0=x,S_0=s)$ and $\mathbb{E}^{x,s}$ is the expectation operator corresponding to $\mathbb{P}^{x,s}$, $q\geq 0$ is the constant discount rate and $\mathcal{S}$ is the set of all $\mathbb{F}$-adapted stopping times. The running income function $f:\mathbb{R}^2\mapsto\mathbb{R}$ is a measurable function that satisfies the regularity condition

$$\mathbb{E}^{x,s}\left[\int_0^\infty e^{-qt}|f(X_t,S_t)|\mathrm{d}t\right]<\infty. \tag{2.2}$$

The reward function $g:\mathbb{R}^2\mapsto\mathbb{R}_+$ and is assumed to be measurable and integrable with respect to the Lévy measure Π. Our main purpose is to calculate $\bar{V}$ and to find the stopping time τ^* which attains the supremum. For each Borel measurable function $l:\mathbb{R}\mapsto\mathbb{R}_+$, we define a stopping time $\tau(l)$ by

$$\tau(l):=\inf\{t\geq 0: S_t-X_t>l(S_t)\}, \tag{2.3}$$

and define a set of stopping times $\mathcal{S}'$ by

$$\mathcal{S}':=\{\tau(l):l:\mathbb{R}\mapsto\mathbb{R}_+\}. \tag{2.4}$$

In other words, $\tau(l)$ is the first time the excursion $S-X$ from level, say $S=s$, becomes greater than some value $l(s)$. When l is constant, for example, $\bar{l}\equiv c$ on $\mathbb{R}$, we write

$$\tau_c:=\inf\{t\geq 0: S_t-X_t>c\}.$$

2.1. **On the Optimal Strategy.** We will reduce the original problem (2.1) to an infinite number of one-dimensional optimal stopping problem and discuss the optimality of the proposed strategy (2.3). Let us denote by $\bar{f}:\mathbb{R}^2\mapsto\mathbb{R}$ the q-potential of f where

$$\bar{f}(x,s):=\mathbb{E}^{x,s}\left[\int_0^\infty e^{-qt}f(X_t,S_t)\mathrm{d}t\right].$$

From the strong Markov property of (X,S) and the regularity condition (2.2), we have

$$\mathbb{E}^{x,s}\left[\int_0^\tau e^{-qt}f(X_t,S_t)\mathrm{d}t\right]=\bar{f}(x,s)-\mathbb{E}^{x,s}\left[1\!\!1_{\{\tau<\infty\}}e^{-q\tau}\bar{f}(X_\tau,S_\tau)\right], \tag{2.5}$$

whose derivation is standard. Hence the value function $\bar{V}$ can be written as

$$\bar{V}(x,s) \;=\; \bar{f}(x,s)+V(x,s),$$

where

$$V(x,s):=\sup_{\tau\in\mathcal{S}}\mathbb{E}^{x,s}\left[1\!\!1_{\{\tau<\infty\}}e^{-q\tau}(g-\bar{f})(X_\tau,S_\tau)\right]. \tag{2.6}$$

We shall concentrate on $V(x,s)$. By the dynamic programming principle, we can write $V(x,s)$ as

$$V(x,s)=\sup_{\tau\in\mathcal{S}}\mathbb{E}^{x,s}\left[1\!\!1_{\{\tau<\theta\}}e^{-q\tau}(g-\bar{f})(X_\tau,S_\tau)+1\!\!1_{\{\theta<\tau\}}e^{-q\theta}V(X_\theta,S_\theta)\right], \tag{2.7}$$

for any stopping time $\theta \in \mathcal{S}$. See, for example, Pham [22] page 97. Now we set $\theta = T_s$ in (2.7). For each level $S = s$ from which an excursion occurs, the value S does not change during the excursion. Now let us set stopping times (first passage times) T_m as

$$T_m = \inf\{t \geq 0 : X_t > m\}.$$

Hence, during the first excursion interval from $S_0 = s$ and $S_t = s$ for any $t \leq T_s$, and (2.7) can be written as the following one-dimensional problem for the state process X;

$$V(x,s) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^{x,s}\left[1\!\!1_{\{\tau < T_s\}} e^{-q\tau}(g - \bar{f})(X_\tau, s) + 1\!\!1_{\{T_s < \tau\}} e^{-qT_s} V(s,s)\right]. \tag{2.8}$$

Now we can look at *only* the process X and find $\tau^* \in \mathcal{S}$. In relation to (2.8), we consider the following one-dimensional optimal stopping problem as for X and its value function $\widehat{V} : \mathbb{R}^2 \mapsto \mathbb{R}$;

$$\widehat{V}(x,s) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^{x,s}\left[1\!\!1_{\{\tau < T_s\}} e^{-q\tau}(g - \bar{f})(X_\tau, s) + 1\!\!1_{\{T_s < \tau\}} e^{-qT_s} K\right], \tag{2.9}$$

where $K \geq 0$ is a constant. Now the following lemma provides a sufficient condition for threshold strategies (i.e., stopping times from $\mathcal{S}'$) to be optimal for (2.9). See also Theorem 2.2 in Øksendal and Sulem [20] for this type of *verification theorem.*

Lemma 2.1. Fix some $s \in \mathbb{R}$, and let a differential operator $\mathcal{A}$ defined by

$$\mathcal{A}w(x) := \mu w'(x) + \frac{\sigma^2}{2} w''(x) + \int_0^\infty \Pi(\mathrm{d}y)[w(x+y) - w(x) - yw'(x) 1\!\!1_{\{-1<y\}}].$$

If there exist $z^* \in (s-b, s)$ and a function $w \in \mathcal{C}^1((-\infty, s]) \cap \mathcal{C}^2((-\infty, s)\backslash\{z^*\})$ such that

(i) $w(s) = K$,
(ii) $\mathcal{A}w(x) - qw(x) = 0$ and $w(x) > (g - \bar{f})(x,s)$ on $x \in (z^*, s)$,
(iii) $\mathcal{A}w(x) - qw(x) < 0$ and $w(x) = (g - \bar{f})(x,s)$ on $x \in (-\infty, z^*]$,

then $w(x) = \widehat{V}(x,s)$ for every $x \in (-\infty, s]$ and the $\mathbb{F}$-stopping time $\tau^* = \inf\{t \geq 0; X_t < z^*\}$ gives supremum in (2.9).

Proof. We postpone the proof to Appendix A. □

To use Lemma 2.1, one usually constructs a candidate $w(\cdot)$ and proves the required inequalities (ii) and (iii). However, for the optimal stopping problems in spectrally negative Lévy models, this procedure tends to be problem-specific, depending on various data such as functions f, g and process X. It is because no general results about the optimality of threshold strategy have been proved. In this study, having thus far characterized our two-dimensional problem (2.1) as a set of one-dimensional optimal stopping problems (2.8), we shall focus on and contribute to obtaining, in a general setting, an explicit form of solution (5.4b) among the class of stopping times $\mathcal{S}' \subsetneq \mathcal{S}$. Accordingly, we consider the value function $\bar{U}$ (instead of $\bar{V}$) which is defined by

$$\bar{U}(x,s) := \bar{f}(x,s) + U(x,s), \tag{2.10}$$

where

$$U(x,s) := \sup_{\tau \in \mathcal{S}'} \mathbb{E}^{x,s}\left[1\!\!1_{\{\tau < \infty\}} e^{-q\tau}(g - \bar{f})(X_\tau, S_\tau)\right]. \tag{2.11}$$

Note that, for our problem, the optimality of (5.4b) for (2.8) (and hence for (2.1)) is given by verifying the conditions in Lemma 2.1 with $K = U(s,s)$.

2.2. **Scale functions.** We review some mathematically important facts before solving the problem. Associated with every spectrally negative Lévy process, there exists a (q-)scale function

$$W^{(q)} : \mathbb{R} \mapsto \mathbb{R}; \quad q \geq 0,$$

that is continuous, strictly increasing on $[0, \infty)$ and 0 on $(-\infty, 0)$. It is uniquely determined by

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) \mathrm{d}x = \frac{1}{\psi(\beta) - q}, \qquad \beta > \Phi(q).$$

Fix $a > x > 0$ and define

$$T_a := \inf\{t \geq 0 : X_t > a\} \quad \text{and} \quad T_0^- := \inf\{t \geq 0 : X_t < 0\}.$$

then we have

$$\mathbb{E}^x \left[e^{-qT_a} 1_{\{T_a < T_0^-, T_a < \infty\}} \right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}$$

and

$$\mathbb{E}^x \left[e^{-qT_0^-} 1_{\{T_a > T_0^-, T_0^- < \infty\}} \right] = Z^{(q)}(x) - Z^{(q)}(a) \frac{W^{(q)}(x)}{W^{(q)}(a)},$$

where

$$Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(y) \mathrm{d}y, \quad x \in \mathbb{R}.$$

Here we have $Z^{(q)}(x) = 1$ on $(-\infty, 0]$. We also have

$$\mathbb{E}^x \left[e^{-qT_0^-} \right] = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x), \quad x > 0.$$

In particular, $W^{(q)}$ is continuously differentiable on $(0, \infty)$ if Π does not have atoms and $W^{(q)}$ is twice-differentiable on $(0, \infty)$ if $\sigma > 0$; see, e.g., Chan et al.[9]. Throughout this paper, we assume the former:

Assumption 1. *We assume that* Π *does not have atoms.*

Fix $q > 0$. The scale function increases exponentially; $W^{(q)}(x) \sim \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))}$ as $x \uparrow \infty$. There exists a (scaled) version of the scale function $W_{\Phi(q)} = \{W_{\Phi(q)}(x); x \in \mathbb{R}\}$ that satisfies

$$W_{\Phi(q)}(x) = e^{-\Phi(q)x} W^{(q)}(x), \quad x \in \mathbb{R}$$

and

$$\int_0^\infty e^{-\beta x} W_{\Phi(q)}(x) \mathrm{d}x \quad = \frac{1}{\psi(\beta + \Phi(q)) - q}, \quad \beta > 0.$$

Moreover $W_{\Phi(q)}(x)$ is increasing, and as is clear from the exponential increase of $W^{(q)}$,

$$W_{\Phi(q)}(x) \uparrow \frac{1}{\psi'(\Phi(q))} \quad \text{as } x \uparrow \infty.$$

Regarding its behavior in the neighborhood of zero, it is known that

(2.12)

$$W^{(q)}(0) = \left\{ \begin{array}{ll} 0, & \text{unbounded variation} \\ \frac{1}{d}, & \text{bounded variation} \end{array} \right\} \quad \text{and} \quad W_+^{(q)'}(0) = \left\{ \begin{array}{ll} \frac{2}{\sigma^2}, & \sigma > 0 \\ \infty, & \sigma = 0 \text{ and } \Pi(0,\infty) = \infty \\ \frac{q+\Pi(0,\infty)}{d^2}, & \text{compound Poisson} \end{array} \right\}$$

where $d := \mu - \int_{(-1,0)} x\Pi(\mathrm{d}x)$. See Lemmas 4.3-4.4 of Kyprianou and Surya [18]. Moreover, we have (see e.g., Pistorius [24])

$$\lim_{x \to \infty} \frac{W_+^{(q)'}(x)}{W^{(q)}(x)} = \Phi(q). \tag{2.13}$$

For a comprehensive account of the scale function, we refer the reader to [7, 8, 16, 18]. See also [13, 26] for numerical methods for computing the scale function.

3. Characterization of $U(s,s)$

Now we look to an explicit solution of $\bar{U}$ for $\tau \in \mathcal{S}'$. Let us introduce the probability measure $\widetilde{\mathbb{P}}^{x,s}$ such that the Radon-Nikodym derivative between $\widetilde{\mathbb{P}}^{x,s}$ and $\mathbb{P}^{x,s}$ is defined by

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{P}}^{x,s}}{\mathrm{d}\mathbb{P}^{x,s}}\right|_{\mathcal{F}_t} = e^{-qt+\Phi(q)(X_t - x)}.$$

Under $\widetilde{\mathbb{P}}^{x,s}$, X has the Laplace exponent $\widetilde{\psi}$ defined by

$$\begin{aligned} \widetilde{\psi}(\lambda) &= \psi(\lambda + \Phi(q)) - \psi(\Phi(q)) \\ &= \left(\sigma^2 \Phi(q) + \mu + \int_{(-\infty,0)} x(e^{\Phi(q)x} - 1)1\!\!1_{\{x>-1\}}\Pi(\mathrm{d}x) \right) \lambda \\ &\quad + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} (e^{\lambda x} - 1 - \lambda x 1\!\!1_{\{x>-1\}}) e^{\Phi(q)x}\Pi(\mathrm{d}x). \end{aligned}$$

Note that since $\widetilde{\psi}'(0+) = \psi'(\Phi(q)+) > 0$, X drifts to ∞ for $q \geq 0$.

Let $W_{\Phi(q)} : \mathbb{R} \mapsto \mathbb{R}$ be the scale function of X under $\widetilde{\mathbb{P}}^{x,s}$, that is, $W_{\Phi(q)}$ has the Laplace transform

$$\int_0^\infty e^{-\lambda x} W_{\Phi(q)}(x)dx = \frac{1}{\widetilde{\psi}(\lambda)}.$$

In addition, we define the process $\eta = \{\eta_t; t \geq 0\}$ of the height of the excursion as

$$\eta_u := \sup\{(S-X)_{T_{u-}+w} : 0 \leq w \leq T_u - T_{u-}\}, \text{ if } T_u > T_{u-},$$

and $\eta_u = 0$ otherwise, where $T_{u-} := \inf\{t \geq 0 : X_t \geq u\} = \lim_{m \to u-} T_m$. Then η is a Poisson point process, and we denote its characteristic measure under $\widetilde{\mathbb{P}}^{x,s}$ by $\tilde{\nu}$. It is known that there is a relation between $W_{\Phi(q)}$ and $\tilde{\nu}$:

$$W_{\Phi(q)}(x) = c \exp\left(-\int_x^\infty \tilde{\nu}[u,\infty)\mathrm{d}u \right), \tag{3.1}$$

where c is some positive constant. See Bertoin [7] (page 195) for an explanation of this identity.

Recall that if $\tau \in \mathcal{S}'$, equation (2.5) can be written as

$$\mathbb{E}^{x,s}\left[\int_0^\tau e^{-qt} f(X_t, S_t)\mathrm{d}t\right] = \bar{f}(x,s) - \mathbb{E}^{x,s}\left[\mathbb{1}_{\{\tau<\infty\}} e^{-q\tau}\bar{f}(X_\tau, S_\tau)\right].$$

Accordingly, the function U in (2.11) is

$$U(x,s) = \sup_{\tau\in\mathcal{S}'} \mathbb{E}^{x,s}\left[\mathbb{1}_{\{\tau<\infty\}} e^{-q\tau}(g-\bar{f})(X_\tau, S_\tau)\right]. \tag{3.2}$$

As a first step, we consider the case $X_0 = S_0$. By using the stopping times $T_m = \inf\{t \geq 0 : X_t > m\}$ and the strong Markov property of (X, S), when $\tau(l) \in \mathcal{S}'$ and $S_0 = X_0 = s$, the first term of the expectation in (3.2) can be written as follows:

$$\begin{aligned}
&\mathbb{E}^{s,s}\left[\mathbb{1}_{\{\tau(l)<\infty\}} e^{-q\tau(l)}(g-\bar{f})(X_{\tau(l)}, S_{\tau(l)})\right] \\
=& \int_s^\infty \mathbb{E}^{s,s}\left[\mathbb{1}_{\{\tau(l)<\infty, S_{\tau(l)}\in\mathrm{d}m\}} e^{-q\tau(l)}(g-\bar{f})(X_{\tau(l)}, S_{\tau(l)})\right] \\
=& \int_s^\infty \mathbb{E}^{s,s}\left[\mathbb{1}_{\{T_m\leq\tau(l)\}} e^{-qT_m}\mathbb{E}^{m,m}\left[e^{-q\tau_{l(m)}}(g-\bar{f})(X_{\tau_{l(m)}}, S_{\tau_{l(m)}})\mathbb{1}_{\{S_{\tau_{l(m)}}\in\mathrm{d}m\}}\right]\right] \\
=& \int_s^\infty \mathbb{E}^{s,s}\left[\mathbb{1}_{\{S_{\tau(l)}\geq m\}} e^{-qT_m}\right]\Big((g-\bar{f})(m-l(m),m) \\
&\times\mathbb{E}^{m,m}\left[e^{-q\tau_{l(m)}}\mathbb{1}_{\{Y_{\tau_{l(m)}-}=l(m), S_{\tau_{l(m)}}\in\mathrm{d}m\}}\right] \\
&+\iint_A (g-\bar{f})(m-y+h,m) \\
&\times\mathbb{E}^{m,m}\left[e^{-q\tau_{l(m)}}\mathbb{1}_{\{X_{\tau_{l(m)}}-X_{\tau_{l(m)}-}\in\mathrm{d}h, S_{\tau_{l(m)}}\in\mathrm{d}m, Y_{\tau_{l(m)}-}\in\mathrm{d}y\}}\right]\Big),
\end{aligned} \tag{3.3}$$

where

$$A = \{(y,h)\in\mathbb{R}^2; y-h\in[l(m),\infty), h<0, y\in[0,l(m)]\}.$$

Now we examine each term on the right-hand side of the last (third) equality of (3.3). First, since X is a spectrally negative process and S is its running maximum process, by (3.1) we have, for $m \geq s$,

$$\begin{aligned}
\mathbb{E}^{s,s}\left[\mathbb{1}_{\{S_{\tau(l)}\geq m\}} e^{-qT_m}\right] &= \widetilde{\mathbb{E}}^{s,s}\left[e^{-(m-s)\Phi(q)}\mathbb{1}_{\{S_{\tau(l)}\geq m\}}\right] \\
&= e^{-(m-s)\Phi(q)}\widetilde{\mathbb{P}}^{s,s}(S_{\tau(l)}\geq m) \\
&= \exp\left(-\int_s^m\left(\frac{W'_{\Phi(q)}(l(u)+)}{W_{\Phi(q)}(l(u))}+\Phi(q)\right)\mathrm{d}u\right) \\
&= \exp\left(-\int_s^m \frac{W_+^{(q)'}(l(u))}{W^{(q)}(l(u))}\mathrm{d}u\right).
\end{aligned} \tag{3.4}$$

Next, from Theorems 1 and 2 in Pistorius [24], we have

$$\begin{aligned}
&\mathbb{E}^{m,m}\left[e^{-q\tau_{l(m)}}\mathbb{1}_{\{X_{\tau_{l(m)}}-X_{\tau_{l(m)}-}\in\mathrm{d}h, S_{\tau_{l(m)}}\in\mathrm{d}m, Y_{\tau_{l(m)}-}\in\mathrm{d}y\}}\right] \\
&\quad= \mathbb{1}_{\{y-h>l(m)\}}\Pi(\mathrm{d}h)\left(W_+^{(q)'}(y) - \frac{W_+^{(q)'}(l(m))}{W^{(q)}(l(m))}W^{(q)}(y)\right)\mathrm{d}y\mathrm{d}m,
\end{aligned}$$

and

$$\mathbb{E}^{m,m}\left[e^{-q\tau_{l(m)}}1\!\!1_{\{Y_{\tau_{l(m)}-}=l(m),S_{\tau_{l(m)}}\in \mathrm{d}m\}}\right]=\frac{\sigma^2}{2}\left(\frac{W_+^{(q)\prime}(l(m))^2}{W^{(q)}(l(m))}-W_+^{(q)\prime\prime}(l(m))\right)\mathrm{d}m.$$

Putting together, if $\tau(l)\in\mathcal{S}'$, (3.3) becomes

$$\begin{aligned}
&\mathbb{E}^{s,s}\left[1\!\!1_{\{\tau(l)<\infty\}}e^{-q\tau(l)}(g-\bar{f})(X_{\tau(l)},S_{\tau(l)})\right]\\
&=\int_s^\infty \mathbb{E}^{s,s}\left[1\!\!1_{\{\tau(l)<\infty,S_{\tau(l)}\in\mathrm{d}m\}}e^{-q\tau(l)}(g-\bar{f})(X_{\tau(l)},S_{\tau(l)})\right]\\
&=\int_s^\infty \exp\left(-\int_s^m\frac{W_+^{(q)\prime}(l(u))}{W^{(q)}(l(u))}\mathrm{d}u\right)\left(\frac{\sigma^2}{2}\left(\frac{W_+^{(q)\prime}(l(m))^2}{W^{(q)}(l(m))}-W_+^{(q)\prime\prime}(l(m))\right)\right.\\
&\times(g-\bar{f})(m-l(m),m)+\int_0^{l(m)}\mathrm{d}y\int_{-\infty}^{y-l(m)}\Pi(\mathrm{d}h)(g-\bar{f})(m-y+h,m)\\
&\left.\times\left(W_+^{(q)\prime}(y)-\frac{W_+^{(q)\prime}(l(m))}{W^{(q)}(l(m))}W^{(q)}(y)\right)\right)\mathrm{d}m.
\end{aligned}$$

Hence we have, up to this point, proved the following:

Proposition 3.1. *When $X_0=S_0=s$, the function U can be represented by*

$$U(s,s)=\sup_l\int_s^\infty\exp\left(-\int_s^m\frac{W_+^{(q)\prime}(l(u))}{W^{(q)}(l(u))}\mathrm{d}u\right)\Psi_m(l(m))\mathrm{d}m \tag{3.5}$$

where $\Psi_m(z):(0,\infty)\mapsto\mathbb{R}$ is defined by

$$\begin{aligned}
\Psi_m(z) \;:=\;& \frac{\sigma^2}{2}\left(\frac{W_+^{(q)\prime}(z)^2}{W^{(q)}(z)}-W_+^{(q)\prime\prime}(z)\right)(g-\bar{f})(m-z,m)\\
&+\int_0^z\mathrm{d}y\int_{-\infty}^{y-z}\Pi(\mathrm{d}h)(g-\bar{f})(m-y+h,m)\left(W_+^{(q)\prime}(y)-\frac{W_+^{(q)\prime}(z)}{W^{(q)}(z)}W^{(q)}(y)\right),
\end{aligned}\tag{3.6}$$

provided that the integral is finite.

Recall that $l(s)$ denotes the height of the excursion $Y=S-X$ when $S=s$. The representation in Proposition 3.1 applies to general cases.

4. Computing $U(s,s)$

We examine how to compute $U(s,s)$. In solving an optimal stopping problem involving S and X, one of the aims is to draw a diagram like Figure 1. Note that we draw the diagram with s on the horizontal axis since it is better understood than otherwise. For distinct points in the (s,x)-diagram, we need to determine whether a point in $\mathbb{R}^2$ is in the continuation region (C) or stopping region (Γ). The task in this section[1] is to compute the value $U(s,s)$ at a point (s,s) on the diagonal and to determine whether it belongs to C or Γ.

As stressed before, once we fix $S=s$, the problem reduces to one-dimensional problems in X and hence the way to find $U(s,s)$ for a fixed s is similar to the one described in Section 2.1. Let us denote by $\Sigma_s\subseteq\mathbb{R}$ (resp.

[1]Once this is done, then the next task is to examine the points (s,x) by moving downwards to $x=0$ from the diagonal $x=s$. We take this in Section 5.

$\mathcal{C}_s$) the stopping region (resp. continuation region) with respect to the reward $(g-\bar{f})(X_\tau, s)$ with s fixed. That is, continuation and stopping region for the problem, $\sup_{\tau\in\mathcal{S}'}\mathbb{E}^{x,s}[e^{-q\tau}(g-\bar{f})(X_\tau,s)]$.

Note that this Σ_s (resp. $\mathcal{C}_s$) should be distinguished from the stopping region $\Gamma\subseteq\mathbb{R}^2$ (resp. C) of the problem (2.1), the final object to figure out. Due to the dependence of the reward on s, there are certain situations which we need to be careful about. To discuss further, we shall hereafter assume the following:

Assumption 2. Denote by $x^*(s)$ the threshold point that separates $\mathcal{C}_s$ and Γ_s with respect to the problem

$$\sup_{\tau\in\mathcal{S}'}\mathbb{E}^{x,s}[e^{-q\tau}(g-\bar{f})(X_\tau,s)]$$

associated with this s.

(i) $(g-\bar{f})(x,s)$ is increasing in s, and
(ii) the continuation region $\mathcal{C}_s$ corresponding to $(g-\bar{f})(X_\tau,s)$ is in the form of $A_s:=(-\infty,x^*(s)]$ or $B_s:=[x^*(s),\infty)$.

The first assumption is merely to restrict our problems to practical ones because we are solving maximization problems. The second is to make our argument concrete and simplistic. More complicated structure can be handled by some combinations of the cases prescribed below.

4.1. **Case (1-L):** $s\in\Sigma_s$**.** If $s\in\Sigma_s$ for the problem $\sup_{\tau\in\mathcal{S}'}\mathbb{E}^{x,s}[e^{-q\tau}(g-\bar{f})(X_\tau,s)]$, we need to consider the possibility that a greater value can be obtained, instead of stopping immediately, if one stops X during an excursion from some upper level $s'>s$. This value is represented as $U(s,s)$ in Proposition 3.1. Once $U(s,s)$ is obtained, we can compute, for this s,

$$U(x,s)=\sup_{\tau\in\mathcal{S}'}\mathbb{E}^{x,s}\left[1\!\!1_{\{\tau<T_s\}}e^{-q\tau}(g-\bar{f})(X_\tau,s)+1\!\!1_{\{T_s<\tau\}}e^{-qT_s}U(X_s,s)\right], \tag{4.1}$$

which is the equation with V replaced by U in (2.8) since we are looking to $\tau\in\mathcal{S}'$. This part shall be treated in the next section. In this section, we shall compute $U(s,s)$ for $s\in\Sigma_s$.

Proposition 4.1. *Let s be fixed in $\mathbb{R}$. Under $q\ge 0$ and $\sigma>0$ instead of Assumption 1, suppose further that (1) $\Psi_s:\mathbb{R}_+\mapsto\mathbb{R}$ is continuous and (2) the* net *reward function $(g-\bar{f})(x,s)$ is nondecreasing in the second argument. Then we have*

$$U(s,s)=\frac{\Psi_s(l^*(s))W^{(q)}(l^*(s))}{W_+^{(q)'}(l^*(s))-W^{(q)}(l^*(s))}, \tag{4.2}$$

and $l^*(s)$ is the maximizer of the map $z\mapsto\frac{\Psi_s(z)W^{(q)}(z)}{W_+^{(q)'}(z)-W^{(q)}(z)}$ on $[0,\infty)$.

Proof. From the equation (3.1), we have for any $\epsilon > 0$,

$$\begin{aligned} U(s,s) &= \sup_l \Bigg[\exp\left(-\int_s^{s+\epsilon} \frac{W_+^{(q)'}(l(u))}{W^{(q)}(l(u))} \mathrm{d}u\right) \\ &\quad \times \int_{s+\epsilon}^{\infty} \exp\left(-\int_{s+\epsilon}^{m} \frac{W_+^{(q)'}(l(u))}{W^{(q)}(l(u))} \mathrm{d}u\right) \Psi_m(l(m)) \mathrm{d}m \\ &\quad + \int_s^{s+\epsilon} \exp\left(-\int_s^{m} \frac{W_+^{(q)'}(l(u))}{W^{(q)}(l(u))} \mathrm{d}u\right) \Psi_m(l(m)) \mathrm{d}m \Bigg] \\ &= \sup_l \Bigg[\exp\left(-\int_s^{s+\epsilon} \frac{W_+^{(q)'}(l(u))}{W^{(q)}(l(u))} \mathrm{d}u\right) U^0(s+\epsilon, s+\epsilon) \\ &\quad + \int_s^{s+\epsilon} \exp\left(-\int_s^{m} \frac{W_+^{(q)'}(l(u))}{W^{(q)}(l(u))} \mathrm{d}u\right) \Psi_m(l(m)) \mathrm{d}m \Bigg] \end{aligned}$$

This expression motivates us to set $U_\epsilon : \mathbb{R} \mapsto \mathbb{R}$ as

$$U_\epsilon(s) := \sup_{l(s)} \left[\exp\left(-\frac{\epsilon W_+^{(q)'}(l(s))}{W^{(q)}(l(s))}\right) U(s+\epsilon, s+\epsilon) + \epsilon \Psi_s(l(s)) \right]. \tag{4.3}$$

Then we have $\lim_{\epsilon \downarrow 0} U_\epsilon(s) = U(s,s)$. Since $\lim_{\epsilon \downarrow 0} U(s+\epsilon, s+\epsilon) = U(s,s)$, the optimal threshold $l^*(s)$ should satisfy

$$\lim_{\epsilon \downarrow 0} U_\epsilon(s) = \lim_{\epsilon \downarrow 0} \left[\exp\left(-\frac{\epsilon W_+^{(q)'}(l^*(s))}{W^{(q)}(l^*(s))}\right) U(s+\epsilon, s+\epsilon) + \epsilon \Psi_s(l^*(s)) \right].$$

For taking limits of $\epsilon \downarrow 0$, we need the following lemma whose proof is postponed to Appendix B:

Lemma 4.1. *Under the assumptions of Proposition 4.1, we have* $U_\epsilon(s) = \frac{1}{1+\epsilon} U(s+\epsilon, s+\epsilon)$ *for* $\epsilon > 0$ *sufficiently small.*

Suppose that we have proved the lemma. From (4.3) with Lemma 4.1, we obtain

$$\begin{aligned} U(s,s) &= \lim_{\epsilon \downarrow 0} \frac{U_\epsilon(s) - \exp\left(-\frac{\epsilon W_+^{(q)'}(l^*(s))}{W^{(q)}(l^*(s))}\right) U(s+\epsilon, s+\epsilon)}{1 - (1+\epsilon) \exp\left(-\frac{\epsilon W_+^{(q)'}(l^*(s))}{W^{(q)}(l^*(s))}\right)} \\ &= \lim_{\epsilon \downarrow 0} \frac{\epsilon \Psi_s(l^*(s))}{1 - (1+\epsilon) \exp\left(-\frac{\epsilon W_+^{(q)'}(l^*(s))}{W^{(q)}(l^*(s))}\right)} = \frac{\Psi_s(l^*(s)) W^{(q)}(l^*(s))}{W_+^{(q)'}(l^*(s)) - W^{(q)}(l^*(s))} \end{aligned}$$

where the last equality is obtained by L'Hôpital's rule. Finally, by using (2.12) and (3.6)

$$\lim_{z \downarrow 0} \frac{\Psi_s(z) W^{(q)}(z)}{W_+^{(q)'}(z) - W^{(q)}(z)} = (g - \bar{f})(s,s)$$

which is desired and hence $l^*(s)$ is the value which gives supremum to $\frac{\Psi_s(z) W^{(q)}(z)}{W_+^{(q)'}(z) - W^{(q)}(z)}$ on $[0, \infty)$. ☐

Remark 4.1. (i) If $q > 0$, a sufficient condition for the continuity of Ψ_s is the continuity of f and g. This is a consequence of the continuity of $x \mapsto \mathbb{E}^{x,s}[f(X_t, S_t)]$ for all $t \geq 0$ and $s \in \mathbb{R}_+$, $W^{(q)} \in C^2$ and

$$|\bar{f}(x,s) - \bar{f}(y,s)| \leq q^{-1}|f(x,s) - f(y,s)| \tag{4.4}$$

for all $x, y \in \mathbb{R}$. If $q = 0$, sufficient conditions for the continuity of Ψ_s are more restrictive, for example, the boundedness of f.
(ii) $\frac{\Psi_s(z)W^{(q)}(z)}{W_+^{(q)\prime}(z) - W^{(q)}(z)}$ is the value for the strategy l with $l(s) = z$ and $l = l^*$ for every $m > s$; that is, this amount is obtained if we stop when X goes below $s - z$ in the excursion at level $S = s$ and, if not, use optimal strategy for the higher levels $S > s$. □

4.2. **Case (2-L):** $s \in A_s := [x^*(s), \infty)$**.** A_s is defined in Assumption 2. In this case, similar to (1-L), a positive $l^*(s)$ may lead to improvement of the value of $U(s,s)$, so that we use Proposition 4.1.

4.3. **Case (3-L):** $s \in B_s := (-\infty, x^*(s)]$**.** B_s is defined in Assumption 2. For this case, the typical situation is that $x^*(s)$ is monotonically decreasing in s. See Figure 1. The curve separating the region Γ and C_2 corresponds to the function $x^*(s)$. Then define the point $\hat{s}$ such that

$$s = x^*(s) \tag{4.5}$$

holds.

Proposition 4.2. *In Case (3-L) with (4.5), the value function $U(s,s)$ is represented by*

$$U(s,s) = e^{-\Phi(q)(\hat{s}-s)}(g - \bar{f})(\hat{s}, \hat{s}) \tag{4.6}$$

for $s \leq \hat{s}$.

Proof. Recall that there are no upward jumps. Hence one receives $(g-\bar{f})(x^*(\hat{s}), \hat{s}) = (g-\bar{f})(\hat{s}, \hat{s})$ when stopping there. Recall also that from (3.4) we have

$$\mathbb{E}^{s,s}\left[1\!\!1_{\{S_{\tau(l)} \geq \hat{s}\}} e^{-qT_{\hat{s}}}\right] = \exp\left(-\int_s^{\hat{s}} \frac{W_+^{(q)\prime}(l(u))}{W^{(q)}(l(u))} \mathrm{d}u\right).$$

Since we are not stopping until $s = \hat{s}$ in this case, the excursion characteristic measure ν is obtained by letting $l \to \infty$. Then we have $\lim_{l\to\infty} \frac{W_+^{(q)\prime}(l(u))}{W^{(q)}(l(u))} = \Phi(q)$ for all u due to (2.13). Combing these facts, we have (4.6). □

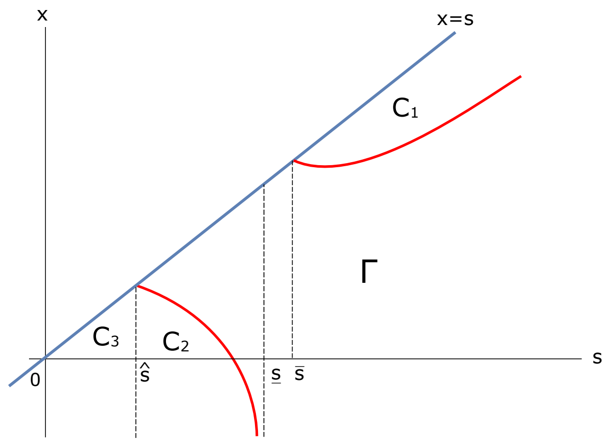


FIGURE 1. The solution (s, x)-diagram in a schematic presentation.

5. GENERAL SOLUTION $U(x, s)$

Finally, let us consider the case of $S_0 > X_0$. Given $U(s, s)$, we shall find the general solution for

$$U(x,s) = \sup_{\tau \in \mathcal{S}'} \mathbb{E}^{x,s}[e^{-q\tau}(g - \bar{f})(X_\tau, S_\tau)], \quad x \le s.$$

While it is next to impossible to exhaust all the patterns of diagrams like this, it should be instructive to consider a case like Figure 1 since it seems typical for optimal stopping with $(g - \bar{f})(x, s)$ that satisfies Assumption 2 and it contains a variety of situations (see (A)∼ (C) below) that may occur in real problems. As we mentioned in Section 2.1, this is a problem only about X. But conventionally, one needs to apply conventional solution methods as cited in Section 1 and to use a guesswork about the form of candidate value functions in the continuation region. In contrast, we use the excursion approach to find explicit forms:

(A) Let us study the region $\underline{s > \bar{s}}$ in Figure 1. This corresponds to what we examined in Case (1-L) for finding $U(s, s)$ for s in this region. More precisely, $(-\infty, s] = \Gamma \cup \mathrm{C}_1$ with $\Gamma = (-\infty, s - l^*(s)]$ and $\mathrm{C}_1 = (s - l^*(s), s]$ being stopping and continuation region, respectively. The curve separating C_1 and Γ is the trace of the points $l^*(s)$. By (2.8), $\bar{U}(x, s)$ can be represented in terms of $\bar{U}(s, s)$ as follows:

$$\bar{U}(x,s) \quad = \quad \bar{f}(x,s) + \sup_{\tau \in \mathcal{S}'} \mathbb{E}^{x,s}\left[\mathbb{1}_{\{T_s < \tau\}} e^{-qT_s}(\bar{U} - \bar{f})(s,s) + \mathbb{1}_{\{\tau < T_s\}} e^{-q\tau}(g - \bar{f})(X_\tau, s)\right]. \tag{5.1}$$

Note that $(\bar{U} - \bar{f}) = U$ and hence this equation is the same as (4.1). Set $\tau = \tau(l)$. Then, from (2.12), the first term in (5.1) can be written by

$$\mathbb{E}^{x,s}\left[\mathbb{1}_{\{T_s < \tau\}} e^{-qT_s}(\bar{U} - \bar{f})(s,s)\right] = \frac{W^{(q)}(l(s) + x - s)}{W^{(q)}(l(s))}(\bar{U} - \bar{f})(s,s). \tag{5.2}$$

For the second term, For $x \in [s-l(s), s]$, by splitting into two cases and using Theorem 1 and 2 in Pistorius [24] one more time,

$$\begin{aligned}
&\mathbb{E}^{x,s}\left[\mathbb{1}_{\{\tau<T_s\}}e^{-q\tau}(g-\bar{f})(X_\tau, s)\right] \\
&= \mathbb{E}^{x,s}\left[e^{-q\tau_{l(s)}}\mathbb{1}_{\{Y_{\tau_{l(s)}-}=l(s), S_{\tau_{l(s)}}=s\}}\right](g-\bar{f})(s-l(s), s) \\
&+ \iint_A (g-\bar{f})(s-y+h, s)\mathbb{E}^{x,s}\left[e^{-q\tau_{l(s)}}\mathbb{1}_{\{X_{\tau_{l(s)}}-X_{\tau_{l(s)}-}\in \mathrm{d}h, S_{\tau_{l(s)}}=s, Y_{\tau_{l(s)}-}\in \mathrm{d}y\}}\right] \\
&= \frac{\sigma^2}{2}\left(W_+^{(q)\prime}(l(s)+x-s) - \frac{W_+^{(q)\prime}(l(s))}{W^{(q)}(l(s))}W^{(q)}(l(s)+x-s)\right) \\
&\times (g-\bar{f})(s-l(s), s) + \int_0^{l(s)} \mathrm{d}y \int_{y-b}^{y-l(s)} \Pi(\mathrm{d}h)(g-\bar{f})(s-y+h, s) \\
&\times \left(\frac{W^{(q)}(l(s)+x-s)}{W^{(q)}(l(s))}W^{(q)}(y) - W^{(q)}(y+x-s)\mathbb{1}_{\{y>s-x\}}\right).
\end{aligned} \tag{5.3}$$

where A is defined in (3.3). Now we can write down $\bar{U}(x, s)$. First, for the stopping regions, it becomes from (5.1)

$$\bar{U}(x, s) = g(x, s), \quad x \in (-\infty, s-l^*(s)] \tag{5.4a}$$

Next, for the continuation region, by combining these terms (5.2) and (5.3), and denoting the optimal deviation from a given level s by $l^*(s)$, we can write $\bar{U}(x, s)$ for $x \in (s-l^*(s), s]$,

$$\begin{aligned}
\bar{U}(x, s) &= \bar{f}(x, s) + \frac{W^{(q)}(l^*(s)+x-s)}{W^{(q)}(l^*(s))}(\bar{U}-\bar{f})(s, s) \\
&+ \frac{\sigma^2}{2}(g-\bar{f})(s-l^*(s), s) \\
&\times \left(W_+^{(q)\prime}(l^*(s)+x-s) - \frac{W_+^{(q)\prime}(l^*(s))}{W^{(q)}(l^*(s))}W^{(q)}(l^*(s)+x-s)\right) \\
&+ \int_0^{l^*(s)} \mathrm{d}y \int_{-\infty}^{y-l^*(s)} \Pi(\mathrm{d}h)(g-\bar{f})(s-y+h, s) \\
&\times \left(\frac{W^{(q)}(l^*(s)+x-s)}{W^{(q)}(l^*(s))}W^{(q)}(y) - W^{(q)}(y+x-s)\mathbb{1}_{\{y>s-x\}}\right).
\end{aligned} \tag{5.4b}$$

Note that $\bar{U}-\bar{f}=U$ and $U(s, s)$ is given by (4.2) and that $l^*(s)+x-s>0$ for $x \in (s-l^*(s), s]$.

(B) For the region $s \in [\hat{s}, \underline{s})$, we have again $s \in \Sigma_s$ (Case 1-L) but when applying Proposition 4.1 at this s, it turns out that $U(s, s) = (g-\bar{f})(s, s)$ with $l^*(s) = 0$. That is, the maximum of (4.2) is attained by $l^*(s) = 0$. Hence the structure of stopping rule is that $\mathrm{C}_2 \cup \Gamma = (-\infty, x^*(s)] \cup [x^*(s), s]$ (see Figure 1). The curve separating C_2 and Γ is the trace of points $x^*(s)$. Then the task is to find the value $x^*(s)$ for

$$U(x, s) = \sup_{\tau\in\mathcal{S}'}\mathbb{E}^{x,s}[e^{-q\tau}(g-\bar{f})(X_\tau, S_\tau)] = \sup_{x(s)\in(-\infty,s]}\mathbb{E}^{x,s}[e^{-qT_{x(s)}}(g-\bar{f})(x(s), s)], \quad x \le s$$

because X has no upward jumps. It is well known (and can be also derived by the argument in Section 4.3) that

$$\mathbb{E}^{x,s}[e^{-qT_{x(s)}}(g-\bar{f})(x(s),s)] = e^{-\Phi(q)(x(s)-x)}(g-\bar{f})(x(s),s).$$

Then the value function in the continuation region can be found by finding $x^*(s)$ that maximizes this quantity. In summary,

$$U(x,s) = \begin{cases} e^{-\Phi(q)(x^*(s)-x)}(g-\bar{f})(x^*(s),s), & x \in (-\infty, x^*(s)], \\ (g-\bar{f})(x,s), & x \in [x^*(s), s]. \end{cases}$$

(C) Finally, let us proceed to the region $\underline{(-\infty,\hat{s}]}$. This is what we study in Case (3-L). If $x^*(s)$ is monotonically decreasing in s as in Figure 1, we have

$$U(x,s) = e^{-\Phi(q)(\hat{s}-x)}(g-\bar{f})(\hat{s},\hat{s}), \quad x \in (-\infty, s]$$

by the argument that derives Proposition 4.2. In this range of s, there is no stopping region.

6. Lookback Option in a Jump Model

To further illustrate the method we have presented in a concrete example, we study the lookback option with exponential jumps. In (2.1), the data are

$$f(x,s) \equiv 0, \quad g(x,s) = e^s - ke^x, \quad k \in [0,1]$$

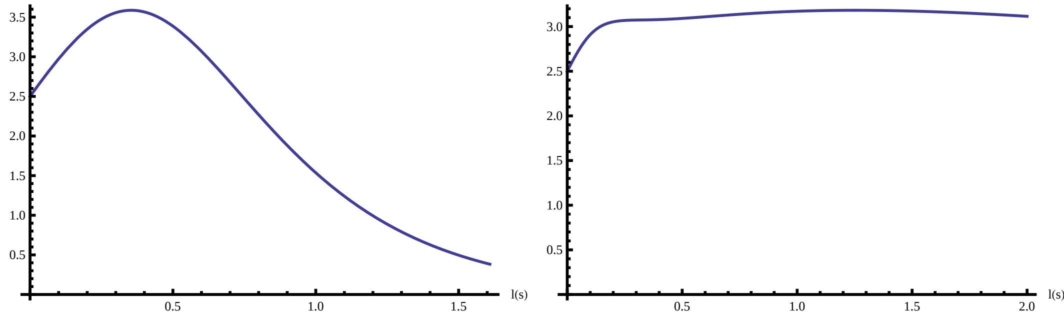
with the process

$$X_t = x + \mu t + \sigma B_t + \sum_{i=1}^{N(t)} \xi_i,$$

where B is a standard Brownian motion, N is a Poisson process with intensity θ, and ξ_i $(i = 1,2,\ldots)$ are independent identically distributed random variables whose distributions are exponential with parameter η under $\mathbb{P}$. The Laplace exponent ψ of X is $\psi(\gamma) = \mu\gamma + \frac{\sigma^2\gamma^2}{2} - \frac{\theta\gamma}{\eta+\gamma}$ and $\psi(\gamma) = q$ has three solutions $\Phi(q) > 0$, $-\gamma_1 < 0$, and $-\gamma_2 < 0$ (with $-\gamma_2 < -\eta < -\gamma_1 < 0$) and q-scale function $W^{(q)}$ of X is represented with these values;

$$W^{(q)}(x) = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} + \frac{e^{-\gamma_1 x}}{\psi'(-\gamma_1)} + \frac{e^{-\gamma_2 x}}{\psi'(-\gamma_2)}.$$

Example 6.1 (Classical Lookback Option)**.** Note that if we put $\theta = 0$, then X is a Brownian motion with drift and $\psi(\gamma) = q$ has only two solutions ($\Phi(q)$ and one negative solution). Before we move on, it shall be beneficial to take up the case with no jumps, which is the classical lookback option in [25]. We shall confirm that Proposition 4.1 provides the same solution in the literature. The process is $X = \mu t + \sigma B_t$. From Proposition 4.1, $U(s,s)$ can be computed as follows: First, $W^{(q)}(x) = \frac{e^{\alpha_2 x} - e^{\alpha_1 x}}{\sqrt{\mu^2 + 2\sigma^2 q}}$ where α_1 and α_2 are a negative and a positive solution of $\mu\alpha + \frac{1}{2}\sigma^2\alpha^2 = q$, respectively. Since we assume that there are no jumps, $\Psi_m(s)$ reduces to only the first term of (3.6). It can be confirmed by a simple computation that (4.2) provides an optimal excursion height $l^*(s)$ and $U(s,s)$. See Figure 2-(a) for a numerical example. $\bar{U}(x,s) = U(x,s)$ is then computed. See the literature (e.g. Section 3.2.1 in [12]). The optimal strategy, for given s, the stopping region is $(-\infty, s-l^*]$ and the continuation region is $(s-l^*, s]$. □



(a) Plain Lookback Option (b) With Negative Jumps

FIGURE 2. Lookback Options: (a) The original problem is written in terms of geometric Brownian motion $\mathrm{d}X_t = \nu X_t \mathrm{d}t + \sigma X_t \mathrm{d}B_t$ and $g(x', s') = s' - kx'$, i.e., $s' = e^s$ and $x' = e^x$. With the parameter set $(\nu, \sigma, q, k) = (0.05, 0.25, 0.15, 0.5)$ and at $s' = 5$, the solution is $\beta = 0.701636$ such that $l^*(s') = (1-\beta)s'$. In terms of Browinian motion, it is $g(x, s) = e^s - ke^x$ and $\mathrm{d}X_t = \mu \mathrm{d}t + \sigma \mathrm{d}B_t$ where $\mu = \nu - \frac{1}{2}\sigma^2$. When $s = \log 5$, our solution is provided at $l^*(s) = l^* = 0.35434$ that maximizes (4.2). See the graph. This has the relationship $\beta = e^{-l^*}$ as desired. The value of $U(s, s)$ is 3.58648 which is the same as the one in the original problem. (b) With $\theta = 1.5$ and $\eta = 1$, the optimal deviation $l^*(s) = 1.2497$ and the value $U(s, s) = 3.1822$ both decrease.

Now we resume the case with exponential jumps. The first step corresponds to the analysis in Section 4. We compute $U(s, s)$ by Proposition 4.1. We set $\theta = 1.5$ and $\eta = 1$. In Figure 2-(b), we plot the map $z \mapsto \frac{\Psi_s(z) W^{(q)}(z)}{W_+^{(q)\prime}(z) - W^{(q)}(z)}$ (see (4.2)) and observe that the maximizer is $l^*(s) = 1.2497$, independent of s. The value $\frac{\Psi_s(z) W^{(q)}(z)}{W_+^{(q)\prime}(z) - W^{(q)}(z)}$ seems relatively unaffected by the levels of $l(s)$. A possible explanation is as follows: the mean jump size $\frac{1}{\eta}$ is 1 in this example, much larger than the drift μ. Hence no matter where $l(s)$ is set, it should be jumps that may bring the process into the stopping region. This fact may lead to the flat curve in Figure 2-(b).

For the next step, we refer to Section 5. With the information of $l^*(s)$ and the corresponding $U(s, s) = 3.1822$, we solve the resultant one-dimensional optimal stopping

$$\bar{U}(x, s) = U(x, s) = \sup_{\tau \in \mathcal{S}'} \mathbb{E}^{x,s}[1\!\!1_{\{\tau < T_s\}} e^{-q\tau}(s - kX_\tau) + 1\!\!1_{\{T_s < \tau\}} e^{-qT_s} U(s, s)].$$

Analogous to the plain lookback option case in Example 6.1, optimal strategy is in the form of $\Gamma = (-\infty, s - l^*]$ and $\mathrm{C} = (s - l^*, s]$. Hence the case (A) in Section 5 applies and thereby use (5.4b) to compute $U(x, s)$, the value function whose stopping time is in the set of $\mathcal{S}'$. Figure 3 shows the two graphs of $U(x, s)$ for $s = \log 5$. The blue curve shows the value function without jumps. The solid line is for the continuation region and the dashed line is for the stopping region. Similarly, the red curve is the value function with jumps. While the smooth-fit principle works at $s - l^*(s) = 1.2551$ without jumps, it does not at $s - l^*(s) = 0.3597$ when there are jumps. We see that the two value functions coincide on $[0, 0.3597]$. Finally, note that the optimal threshold excursion level $l^*(s)$ in fact does *not* depend on s. This is reasonable considering the form of the reward function and is the same structure as the classical problem. Hence the (s, x)-diagram has, for all $s \geq 0$, the shape in the region explained in (A) of

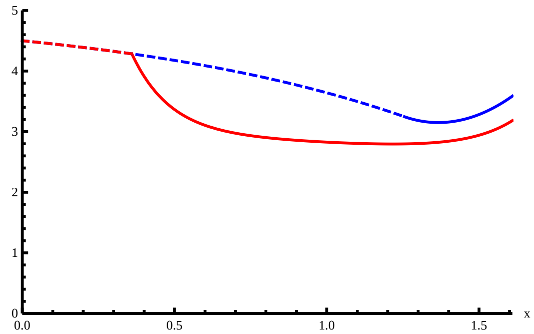


FIGURE 3. comparison of two cases: Lookback Options $U(x,s)$ on the continuation region: No jumps (Blue) and with jumps (Red) with $\theta = 1.5$ and $\eta = 1$. In the respective stopping regions, the curves are dashed. In the jump case, $s - l^*(s) = \log(5) - 1.2497 = 0.3597$ and at that point the smooth-fit principle does not work, while in the classical case, the smooth-fit works at $s - l^*(s) = 1.2551$. The two value functions coincide on $[0, 0.3597]$.

Section 5. Specifically, we conclude

$$U(x,s) = \begin{cases} \frac{W^{(q)}(l^*+x-s)}{W^{(q)}(l^*)}U(s,s) + \frac{\sigma^2}{2}g(s-l^*,s) \times \left(W_+^{(q)'}(l^*+x-s) - \frac{W_+^{(q)'}(l^*)}{W^{(q)}(l^*)}W^{(q)}(l^*+x-s) \right) \\ \qquad + \int_0^{l^*} \mathrm{d}y \int_{-\infty}^{y-l^*} \Pi(\mathrm{d}h) g(s-y+h,s) \\ \qquad \times \left(\frac{W^{(q)}(l^*+x-s)}{W^{(q)}(l^*)} W^{(q)}(y) - W^{(q)}(y+x-s) 1\!\!1_{\{y>s-x\}} \right), & x \in (s-l^*, s], \\ e^s - ke^x, & x \in (-\infty, s-l^*], \end{cases}$$

where the value in the continuation region is thanks to (5.4b).

APPENDIX A. PROOF OF LEMMA 2.1

Proof. First we prove $w(x) \geq \widehat{V}(x,s)$ for every $x \in \mathbb{R}_+$. The Itô's rule (see e.g. Cont and Tankov[10] page 277) gives us

$$\begin{aligned} &e^{-q(t\wedge T_s)} w(X_{t\wedge T_s}) \\ &\qquad = w(X_0) - \int_0^{t\wedge T_s} q e^{-qu} w(X_u) \mathrm{d}u \\ &\qquad + \mu \int_0^{t\wedge T_s} e^{-qu} w'(X_u) \mathrm{d}u + \sigma \int_0^{t\wedge T_s} e^{-qu} w'(X_u) \mathrm{d}B_u \\ &\qquad + \frac{\sigma^2}{2} \int_0^{t\wedge T_s} e^{-qu} w''(X_u) \mathrm{d}u \\ &\qquad + \int_0^{t\wedge T_s} \int_0^{\infty} \mathrm{d}u \Pi(\mathrm{d}y) e^{-qu} [w(X_u+y) - w(X_{s-}) - yw'(X_{s-}) 1\!\!1_{\{-1<y\}}] \\ &\qquad + \int_0^{t\wedge T_s} \int_0^{\infty} (M(\mathrm{d}u,\mathrm{d}y) - \mathrm{d}u\Pi(\mathrm{d}y)) e^{-qu} [w(X_u+y) - w(X_{u-})] \end{aligned}$$

where we denote by M the Poisson random measure associated with X. By collecting terms, we have

$$
\begin{aligned}
&= w(X_0) - \int_0^{t\wedge T_s} q e^{-qu} w(X_u)\mathrm{d}u, \\
&+ \int_0^{t\wedge T_s} e^{-qu} w'(X_u)\mathrm{d}u + \sigma \int_0^{t\wedge T_s} e^{-qu} w'(X_u)\mathrm{d}B_u \\
&+ \int_0^{t\wedge T_s} e^{-qu} (\mathcal{A}w(X_u) - qw(X_u))\mathrm{d}u \\
&+ \int_0^{t\wedge T_s} \int_0^{\infty} (M(\mathrm{d}u,\mathrm{d}y) - \mathrm{d}u\Pi(\mathrm{d}y)) e^{-qu} [w(X_u + y) - w(X_{u-})].
\end{aligned}
$$

Since the process $\{X_{t\wedge T_s} : t \geq 0\}$ does not leave the interval $[s-b, s]$, the integrals with respect to the Brownian motion B and the compensated jump measure (i.e., the last term) are martingales. Therefore, by taking expectations, we have

$$
w(x) = \mathbb{E}^{x,s}\left[e^{-q(t\wedge T_s)} w(X_{t\wedge T_s})\right] - \mathbb{E}^{x,s}\left[\int_0^{t\wedge T_s} e^{-qu}(\mathcal{A}w(X_u) - qw(X_u))\mathrm{d}u\right]. \tag{A.1}
$$

By the assumption

$$
\mathcal{A}w(x) - qw(x) \leq 0, \quad x \in (-\infty, s],
$$

we have

$$
w(x) \geq \mathbb{E}^{x,s}\left[e^{-q(\tau\wedge T_s)} w(X_{\tau\wedge T_s})\right], \; x \in (-\infty, s], \tau \in \mathcal{S}. \tag{A.2}
$$

Now, for any stopping time $\tau \in \mathcal{S}$, by using (i)-(iv),

$$
\begin{aligned}
w(x) &\geq \mathbb{E}^{x,s}\left[e^{-q(\tau\wedge T_s)} w(X_{\tau\wedge T_s})\right] \\
&\geq \mathbb{E}^{x,s}\left[1\!\!1_{\{\tau < T_s\}} e^{-q\tau}(g - \bar{f})(X_\tau, s) + 1\!\!1_{\{T_s < \tau\}} e^{-qT_s} K\right].
\end{aligned} \tag{A.3}
$$

Taking the supremum over the set $\mathcal{S}$, we have $w(x) \geq \widehat{V}(x,s)$

On the other hand, if we substitute $\tau^* = \inf\{t \geq 0; X_t < z^*\}$ for τ in (A.1)-(A.3), then all the inequalities are satisfied as equalities thanks to the assumptions (i)-(iii). Therefore, $w(x) = \widehat{V}(x,s)$ for every $x \leq s$. □

APPENDIX B. PROOF OF LEMMA 4.1

Lemma 4.1 is the following claim: *Under the assumptions of Proposition 4.1, we have* $U_\epsilon(s) = \frac{1}{1+\epsilon} U(s+\epsilon, s+\epsilon)$ *for* $\epsilon > 0$ *sufficiently small.*

Proof. In view of (3.4) and (3.6), the probabilistic meaning of (4.3) is that $U_\epsilon(s)$ is attained when one chooses the excursion height $l(s)$ optimally in the following optimal stopping:

$$
U_\epsilon(s) = \sup_{l_D(s)} \mathbb{E}^{s,s}[e^{-qT_{s+\epsilon}} 1\!\!1_{\{T_{s+\epsilon} \leq T^-_{s-l(s)}\}} U(s+\epsilon, s+\epsilon) + e^{-qT^-_{s-l(s)}} 1\!\!1_{\{T^-_{s-l(s)} < T_{s+\epsilon}\}} (g - \bar{f})(s - l(s), s)], \tag{B.1}
$$

that is, if the excursion from s does not reach the level of $l(s)$ before X reaches $s+\epsilon$, one shall receive $U(s+\epsilon, s+\epsilon)$ and otherwise, one shall receive the reward. Note that without loss of generality, we have set $k(x,s) \equiv 0$. Since $(g - \bar{f})(x,s)$ is assumed to be nondecreasing in s, (B.1) implies that $U_\epsilon(s) \leq U(s+\epsilon, s+\epsilon)$. As $\epsilon \downarrow 0$, it is clear that $U(s+\epsilon, s+\epsilon) \downarrow U(s,s)$ and $U_\epsilon(s) \downarrow U(s,s)$. Let us set $\alpha(\epsilon) := \frac{1}{1+\epsilon} \in (0,1)$ for $\epsilon > 0$. Suppose, for a contradiction, that we have

$$\alpha(\epsilon) \cdot U(s+\epsilon, s+\epsilon) < U_\epsilon(s) < U(s+\epsilon, s+\epsilon), \tag{B.2}$$

for all $\epsilon > 0$. While the second inequality always hold, the first inequality leads to a contradiction to the fact that the function $\epsilon \mapsto (1-\alpha(\epsilon))U(s+\epsilon, s+\epsilon)$ is continuous for all s. Indeed, due to the monotonicity of $\alpha(\epsilon) \cdot U(s+\epsilon, s+\epsilon)$ in ϵ, we would have $U_\epsilon(s) > U(s,s) > \alpha(\epsilon) \cdot U(s+\epsilon, s+\epsilon)$ for all $\epsilon > 0$. Hence one cannot make the distance between $U(s+\epsilon, s+\epsilon)$ and $\alpha(\epsilon) \cdot U(s+\epsilon, s+\epsilon)$ arbitrarily small without violating (B.2). This shows that there exists an $\epsilon' = \epsilon'(s)$ such that $\epsilon < \epsilon'$ implies that $U_\epsilon(s) \leq \alpha(\epsilon) \cdot U(s+\epsilon, s+\epsilon)$.

For the converse direction, in (B.1), one could choose a stopping time that visits the left boundary $-\infty$, then by reading (3.4) with $l(u) = +\infty$ and $m = s+\epsilon$, (B.1) becomes

$$U_\epsilon(s) \geq \lim_{a \uparrow \infty} \exp\left(-\int_s^{s+\epsilon} \frac{W_+^{(q)'}(a)}{W^{(q)}(a)} \mathrm{d}u\right) U(s+\epsilon, s+\epsilon) = e^{-\epsilon\Phi(q)} U(s+\epsilon, s+\epsilon)$$

for *any* $\epsilon > 0$. The equal sign is due to (2.13). Now suppose that there were no ϵ's such that $U_\epsilon(s) \geq U(s+\epsilon, s+\epsilon)$. It follows that for any ϵ, we would have

$$U_\epsilon(s) > e^{-\epsilon\Phi(q)} U(s+\epsilon, s+\epsilon) > e^{-\epsilon\Phi(q)} U_\epsilon(s).$$

Then by letting $\epsilon \downarrow 0$, it would be $U(s,s) > U(s,s)$ for all $s \in \mathcal{I}$, which is absurd. Since the convergence of $e^{-\epsilon\Phi(q)} \uparrow 1$ is monotone in ϵ, there exists an $\epsilon'' = \epsilon''(s) > 0$ such that $\epsilon < \epsilon''$ implies that $U_\epsilon(s) \geq U(s+\epsilon, s+\epsilon)$. Hence for any s, we have $U_\epsilon(s) \geq \alpha(\epsilon) U(s+\epsilon, s+\epsilon)$ for $\epsilon < \epsilon''$. This completes the proof of Lemma 4.1. □

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