An approximation method for analysis and valuation of credit correlation derivatives ⋆

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Abstract

This paper presents a model for approximating the value of a basket of default-correlated assets and analyzes subordinate tranches in securitized debt obligations. The model is calibrated to an intensity-based simulation of correlated defaults and represents an alternative computation method to full Monte Carlo simulation. Timing of individual obligor defaults are driven by intensity processes and collateral value is modeled with a jump-diffusion process where the number of jumps corresponds to the total number of defaults in the asset pool. This approach allows decomposition of subordinate obligations in terms of a collection of simpler securities and yields useful risk management information.

Key words: correlated credit risk, structured security, portfolio approximation
JEL Classification: G12, G13

1 Introduction

This article presents a method for valuation and analysis of subordinate tranches of structured securities backed by a pool of defaultable claims. To demonstrate ideas, we focus on collateralized debt obligations (CDOs) which account for a growing share of all asset-backed securities issued. A CDO is a derivative security on a portfolio of bonds, loans, or other credit risky claims. Cash flows from the collateral portfolio are divided into tranches of varying quality and yield which are then sold to investors. Along with other credit correlation products, the CDO has become a

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popular vehicle for investing in and managing credit risk. The asset pool of a CDO is typically a diversified portfolio of debt collateral that may include commercial and real-estate loans, sovereign and corporate bonds, and other asset backed securities. These products are offered in response to interest by insurers, banks, and fund managers who wish to efficiently redistribute, transfer, or speculate on risk. For example, banks issue CDOs to reduce credit exposure and, accordingly, regulatory capital while investors with varying constraints and levels of risk aversion gain access to an asset pool that would not be available otherwise.

The credit correlation market has grown significantly since its beginnings in the late 1980s. Below $5 billion prior to mid-1990s, by 2000 the annual global CDO issuance grew to nearly $100 billion (Flanagan and Sam, 2002). According to a recent survey, correlation products accounted for about a quarter of the market for credit derivatives in 2002, which itself increased by 50% from the previous year to over $2 trillion in notional outstanding value (Patel, 2003). The market demand for CDOs depends on a set of structural trade-offs that determine how the proceeds from the collateral pool are directed into several debt and equity tranches organized by their relative seniority. Default correlation makes evaluation of subordinated products a challenging and computationally intensive exercise. Our aim in this article is to describe a method for efficient valuation and analysis of such securities that offers reduced computational burden but retains the essence of the correlation structure.

Issuers of CDOs face the challenge of matching investor requirements while minimizing funding costs for tranches. The issuer or sponsor often sells the portfolio of debt to a special purpose vehicle (SPV) and takes an equity interest. The SPV uses these debt obligations as its asset pool to issue liabilities in the form of prioritized multi-layer tranches. In a traditional SPV, senior, mezzanine, and subordinate notes are offered along with equity shares. Senior notes appeal to relatively risk averse investors and are paid before mezzanine and lower-subordinated notes, but after management, transaction, and hedging costs are subtracted. Any remaining cash flows go to equity holders. In this paper, we propose a model to approximate the value of collateralized structures by starting with the most subordinate tranche and working our way to the most senior liability piece.

The potential for arbitrage as a result of mispricing is a key practical consideration in constructing debt structures (Goodman and Fabozzi, 2002, see). In most applications, simplifying assumptions allow a quick and rough comparison of the yield offered on a particular tranche with other market opportunities. For example, the rate of return on the equity tranche can be estimated by considering a static portfolio composition and by benchmarking to market rates while adjusting for a fixed annual default rate and recovery. However, due to the great flexibility in configuring deals (subordination structure, quality of collateral and trigger tests, leverage, asset diversity, and delayed cash flow draw features to name a few), a much more detailed analysis is required to structure these securities. For example, increasing the amount of investment grade debt in the collateral pool reduces the need for credit enhancement and requires a smaller equity share; equity capital can be also lowered by increasing diversity in the collateral pool. Equity can account for two to fifteen percent of a typical CDO's capitalization depending on credit quality and asset diversity.

1.1 Modeling default correlation

A straightforward method for building default correlation is to modify or combine single-name default models in a way that allows dependency among entities. Single-name default models provide the marginal default probability distributions that are combined to get a joint default probability distribution. This process contains two key steps: first, selecting or developing a suitable model for single-name default; second, specifying an appropriate structure of dependence that binds together the marginal distributions. Of course, the final result should be consistent with market signals and, ideally, would be efficient and easy to calibrate and implement.

Approximations can be made at each modeling stage to improve efficiency and ease of use. While simpler models offer the benefit of quick analysis, they often rely on coarse assumptions about the correlation structure of collateral assets. Although more efficient in getting order-of-magnitude values, this limits their usefulness in design, valuation, and dynamic risk management. More complex models, those with better ability to capture default correlations, tend to be computationally intensive and less transparent. Here, we provide a brief (and by no means exhaustive) review of default correlation pricing models. [1]

In the *structural* strand of models, pioneered by Merton (1974), default is triggered when a firm's asset value falls below a threshold level. Zhou (2001) proposes a model of joint default in this vein where asset values of two firms are correlated. Overall, parameter estimation can be challenging in these models and they have had mixed success in matching the range of observed term structures of default probabilities needed for pricing credit derivatives.

In response to empirical problems of traditional structural models in reconciling firm value with market rates, a related line of research abstracts from firm value and, instead, focuses on a continuous-time measure of credit quality. Default occurs when credit quality deteriorates to a certain threshold level. These *threshold* models are more versatile and have had greater success in matching implied default probabilities observed from credit spreads or bond prices (see, *e.g.*, Avellaneda and Zhu, 2001; Esteghamat, 2003). Hull and White (2001) introduce a discrete-time version

[1] Many models quantify loss over a fixed horizon to focusing on risk measurement rather than pricing and dynamic hedging (see Crouhy et al. (2000) for a review).

of the threshold model with default dependency by considering pairwise correlation of credit qualities. The default barrier is made time-dependent and is calibrated so that the model is consistent with market information. Overbeck and Schmidt (2003) extend this approach to a model that uses time-scaled correlated Brownian motion to represent uncertainty in credit quality (or firm value). Time scaling is essentially equivalent to introducing time variation in volatility of the credit quality process and adds additional freedom in calibrating the model to marginal default probability distributions. Working in a continuous-time setting, they obtain a quasi-analytic expression for pairwise default distribution to mitigate the high computational burden of threshold models with correlated defaults.

A critical feature of default rates, supported by empirical evidence, is time variation in correlation and the importance of systemic effects. In a recent study Das et al. (2001) find significant systematic variation in default rates over time. Default rates tend to vary with the business cycle, peaking during recessions. These effects have been incorporated in many credit risk measurement models through introduction of common factor dynamics (examples include Wilson (1997) and Pesaran et al. (2003)). However, Adelson (2003) points out that, although it is widely recognized that systematic forces can drive default correlations, most popular models do not capture time-varying credit risk dependencies in structured securities. Adelson cites, as evidence, the failure of these models to account for linkages among credit risks that have led to poor performance in the securitization market in recent years including CDOs and other asset-backed securities. The top three rating agencies, Standard & Poor, Moody, and Fitch, use models with static default correlation to evaluate these securities.

Intensity-based pricing models (*e.g.*, Lando, 1998; Duffie and Singleton, 1999) take hazard rates of default as a primitive and are, by design, well-suited to match credit spreads. Intensity models incorporate the systemic features of default correlation by allowing hazard rates of default to depend on a set of common (*e.g.*, macroeconomic) state factors (see, for example, Duffie and Singleton, 1999; Duffie and Gârleanu, 2001). Giesecke (2003) constructs a joint exponential model of default intensities along with a common shock intensity component. In these models, intensities are often assumed to be independent *conditional* on the state of the economy or factor values.

It is important to note that the assumption of conditional independence does not imply defaults are independent. In fact, correlation is orchestrated by time-varying dynamics of the common factors. For instance, hazard rates of default for firms in a particular industry sector may all increase in response to recessionary economic pressures, in turn, increasing the likelihood of default for each firm in the sector. Other firms may be affected more by oil price shocks. Some studies find, however, that the range of correlations achieved with intensity-based models may be limited. There have been three responses to correct this weakness: exploring models of *infectious default*; using *copula* functions; and reexamining factor specifications in intensity models.

In variations of so-called contagion or infectious type models, the assumption of conditional independence is relaxed and default intensities are made to depend on default events of other entities (*e.g.*, Davis and Lo, 2001; Jarrow and Yu, 2001; Yu, 2003b). Contagion models fill an important gap but at the cost of analytic tractability. They can be especially difficult to implement for large portfolios.

In an alternative approach, Li (2000) and Schönbucher and Schubert (2001) show how to apply the Gaussian copula to obtain joint default time distributions for pricing credit correlation securities. A copula is a functional that maps marginal distributions to a joint distribution. Very popular in practice for its simplicity in use, the Gaussian copula imposes arguably unrealistic time dependency on the correlation structure. In particular, dependency among defaults is high initially and rapidly and monotonically decays over time for the t-copula family, including the Gaussian copula. Rogge and Schönbucher (2003) overcome this drawback by considering the Archimedean class of copulae. The copula approach is relatively straightforward to implement, it can match a wide range of default dependencies and, as a result, it holds obvious appeal to practitioners. This flexibility, however, is accompanied with loss of economic intuition about structural connections of default and about choice of copula. Calibration is data intensive and simulations can be very slow, requiring large numbers of simulations to analyze basket securities. Moreover, the copula function and its parameters have to be reestimated as credit events unfold, whether due to idiosyncratic or systematic causes.

In a recent study Yu (2003a) argues that the problem of low correlations in intensity models may be due to insufficient specification of common factors and not to an inherent feature of the approach. Building on empirical results of Duffee (1999) and Driessen (2005), Yu shows high correlations are possible in a simple two-factor model of default intensities.

1.2 Computational issues

Models that do well in capturing individual *and* joint default behavior tend to be computationally intensive and slow. A quick response time is especially important in calculating hedge positions and performing sensitivity analysis in trading environments. Several studies address this issue. Pykhtin and Dev (2003) derive an analytical approximation for tranche loss distributions in a homogenous portfolio of identical loans driven by a single risk factor. Defining default intensities to be conditionally independent functions of a low dimensional vector of latent factors, Gregory and Laurent (2003) illustrate the efficiency gains of calculating default distributions through characteristic functions. Latent factors are endogenously implied and are not connected to macroeconomic variables. The authors give single-factor examples but point out that extension to multiple factors is uncomplicated for the case of Gaussian but not Archimedean copulae. Andersen et al. (2003) consider copulae with conditionally independent default times where default intensities are time-varying but not factor dependent. They obtain computationally advantageous semi-analytic approximations by combining factor reduction techniques on the Gaussian copula correlation matrix with a recursive scheme (rather than characteristic functions or simulations) for evaluating loss distributions. Hull and White (2004) choose a multifactor Gaussian and t copulae under the assumption of independence of default times given the common factors. They employ a recursive relation to evaluate default probabilities with reduced computational requirements over simulation methods.

This paper is in the spirit of these studies. We demonstrate a salient way of computing market prices of subordinate tranches that does not require full simulation analysis and does not impose oversimplifying assumptions. Our approach is based on constructing approximate dynamics for the collateral value with a jump-diffusion process where the diffusion piece reflects gradual changes in portfolio value and the jump component captures shocks due to asset defaults in the portfolio. This process is calibrated to a default model driven by a multifactor conditionally independent vector of intensity processes. Hazard rates of default are linked to state factors, capturing the effect of common factor dynamics but allowing faster computations.

Simulations indicate that for a typical pool, the approximation performs well. The model, of course, would be recalibrated periodically as needed to increase accuracy and for longer maturities. In addition to fast convergence properties, the method provides useful information for structural analysis and design of subordinate claims in correlation derivatives. Based on the approximation scheme, we also present convenient formulae for tranche values which allow decomposition of subordinated claims in terms of call options and ith-to-default derivatives. These results can be used by issuers and potential investors to evaluate, to optimize, and to hedge CDO structures.

2 Setup and assumptions

Assume, in the usual manner, that prices can be determined in an equivalent risk-neutral no-arbitrage environment and there are no market frictions. In this setting, we consider a model of collateral default based on the Cox process approach of Lando (1998) with cross-sectional *conditional* independence of default intensities *given* values of a set of common factors. It should be emphasized that conditional independence does not imply independence of defaults. To the contrary, default intensities are correlated, deriving their dependence from time variation in a set of common latent or observable state variables which may include, for example, interest rates, energy prices, GDP, equity market prices, specific industry sector factors, and inflation. In other words, correlation of default intensities is ascribed to a factor model. The choice of observable factors will depend on the mix of collateral assets in the portfolio. Our assumptions regarding credit events and key features of collateralized debt obligations are detailed below.

There is a complete probability space $(\Omega, \mathcal{G}, \mathbb{P}^*)$ with an increasing family of right-continuous σ-algebras $\mathcal{G} \triangleq \{\mathcal{G}_t\}_{t\geq 0}$ that characterizes uncertainty where $\mathcal{G}_t$ represents all information available at time t. Information arrives from distinct (and independent) sources $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^1 \vee \mathcal{H}_t^2 \vee \cdots \vee \mathcal{H}_t^n$ where $\mathcal{F} \triangleq \{\mathcal{F}_t\}_{t\geq 0}$ and $\mathcal{H}^i \triangleq \{\mathcal{H}_t^i\}_{t\geq 0}$ are filtrations generated by a Brownian motion and default (counting) processes for entities indexed by i. $\mathcal{F}$ is the history of random shocks in the common (*e.g.*, macroeconomic) state processes; $\mathcal{H}^i$ is the default history of entity i. The $\mathcal{F}$-Brownian motion is assumed also to be a Brownian motion in the enlarged filtration $\mathcal{G}$.

For a given default time of obligor i, denoted τ_i, we introduce the associated jump process $1_{\{\tau_i \leq t\}}$ for $t \geq 0$ taking on a value of zero if $\tau_i > t$ and a value of one otherwise. $\mathbb{P}^*$ is the risk-neutral spot martingale measure. Knowledge of state factor history is not sufficient to know whether an entity has defaulted or not. To determine that, either individual default histories or a larger history are needed. This means τ_i is not necessarily a stopping time with respect to $\mathcal{F}$, although it is always a stopping time with respect to the larger $\mathcal{G}$.

The portfolio of assets that represents the collateral pool of a CDO consists of n different obligors or names. Each obligor i in the pool has a random credit event (*e.g.*, default) intensity of λ_t^i of the form $\lambda^i(X_t) : \mathbf{R}^d \to \mathbf{R}_+$ where X_t is a state variable taking values in $\mathbf{R}^d$ representing an aggregate proxy vector for economic, financial, and political variables of consequence. The default time of obligor i, τ_i for $i = 1, \ldots, n$, is governed by the $\mathcal{F}$-hazard process $\Lambda_t^i = \int_0^t \lambda_s^i \, \mathrm{d}s$. Equivalently, the probability that a credit event occurs after time t, given no events have occurred prior to t is $\mathbb{P}^*(\tau_i > t \mid \mathcal{F}_t) = e^{-\Lambda_t^i}$. A default event for obligor i corresponds to the first (and only) jump of its counting process.[2] A counting process $N_t = \sum_{i=1}^n 1_{\{\tau_i \leq t\}}$ keeps track of the total number of default events in the asset portfolio. Default times are assumed conditionally independent with respect to $\mathcal{F}_t$ under $\mathbb{P}^*$, for any $s > 0$,

$$\mathbb{P}^*(\tau_1 > t_1, \ldots, \tau_n > t_n \mid \mathcal{F}_s) = \mathbb{P}^*(\tau_1 > t_1 \mid \mathcal{F}_s) \, \cdots \, \mathbb{P}^*(\tau_n > t_n \mid \mathcal{F}_s).$$

Default correlation arises from dependence of default intensities on state vector X. Other specifications are possible, for example, mapping marginal default distributions through a copula function to obtain the joint distribution. Intensity of a firm could be made dependent of the default history of other firms (introducing infectious defaults). These extensions are not considered in this study.

[2] This assumption is temporarily suspended in analyzing *Type 1* claims below, where successive defaults of one bond are considered.

2.1 Collateral tests and seniority

Cash flows from a diversified collateral pool are distributed to two broad classes of investors: equity holders (least subordinate) and note holders (remaining higher priority classes). Periodically, the note holders receive first dollars as pre-determined coupons from the cash flows generated by the portfolio. Equity holders receive dividends as long as note holders receive their share of coupon payments and the market value of the portfolio passes a certain test, for example, requiring that portfolio value divided by capitalization be greater than some pre-determined ratio. At maturity, note holders are entitled to receive principal payments before equity holders. Any residual portfolio value is given to equity holders.

The special purpose vehicle that is setup as a part of a collateralized debt obligation is often equipped with protective clauses to safeguard the investment of senior note holders. For example, the value, quality, or diversity of the asset pool may be periodically measured against benchmark values. A failed test can result in asset liquidation and dissolution of the SPV. Here, we focus on a collateral market value test and, for illustration purposes, consider two opposite scenarios.

> *Type 1 claim* (no collateral test): Some structures incorporate a certain flexibility in dealing with collateral tests. Equity holders can potentially inject additional capital into the structure; senior note holders may allow a grace period for the tests to be met; assets may be sold until the test requirements are met. In order to allow for such contingencies, suppose no matter how the value of the portfolio fluctuates, the CDO under consideration is not liquidated. All tranche disbursements are collected and distributed at maturity. Pricing under this scenario is discussed in Section 3.

> *Type 2 claim* (market value structure): To avoid the risk that note holders do not recover their principal and accrued coupon payments, the contract is terminated, assets are liquidated, and note holders and equity holders are paid out according to the designated prioritization as soon as the market value of the portfolio drops below a threshold level. The threshold level is the par value of a debt tranche or any reference asset value. This more general case is discussed in Section 4.

2.2 Recovery and reinvestment

Carey (2002) shows that average credit loss rates have been significantly higher in the U.S. during recessions when compared to periods of economic expansion. Frye (2003) finds that recovery rates and default rates rise together. Moreover, recovery rates are more sensitive to changes in the default environment (low versus high default years; business cycles) than default rates themselves. Despite these findings, most models focus on default rates and treat loss rates as constant or exogenous variables. This may be partly due to the fact that recovery rates are implied in the corporate bond spreads used to calibrate default probabilities. Moreover, in intensity-based models, hazard rates can be conveniently modified to incorporate correlated recovery rates. In particular, if recovery is defined as a fraction of asset market value just prior to default, the hazard rate can be replaced with a "loss adjusted" hazard rate (see, for example, Duffie and Singleton, 1999).

In this paper, we treat recovery as an independent exogenous variable. An equivalent interpretation is that recovery is a fraction of market value. When a collateral asset defaults, a fraction $\tilde{L}^i$ of its market value is lost. We assume, for simplicity, that $\tilde{L}^i \equiv \tilde{L}$ is constant over all the assets in the portfolio. The defaulted asset is sold immediately at $(1 - \tilde{L})$ fraction of its pre-default value. Sale proceeds are used in one of two ways.

> *Type 1 claim*: Post-default proceeds are invested "in kind." The recovery amount is used to buy a similar type or class of bond governed by the same default intensity process as that of the original asset. The newly purchased asset is again subject to future default. This, in fact, is equivalent to allowing multiple credit events for a single obligor.
>
> *Type 2 claim*: The proceeds from sale of defaulted asset are used to increase shares of the surviving assets in the portfolio or are reinvested in a similar asset. We do not make any assumptions regarding exactly how the recovery amount is allocated except that the new portfolio contains a mix of assets not too drastically different from the original portfolio.

3 Price in the absence of collateral tests

Subordinate tranches of a collateralized obligation resemble call options on more senior tranches. In this section we confirm that, absent collateral tests, indeed the most subordinate tranche is a default-adjusted call option with collateral assets as the underlying. Collateral tests introduce additional complexity which we shall address in the next section. There, the call option analogy is modified to reflect the effect of default losses on the senior notes.

Consider a portfolio of n bonds with default intensities λ_t^i for $i = 1, \ldots, n$. We apply no threshold tests to portfolio value and allow multiple credit events for each bond. In reality, the bond either remains in the portfolio or is replaced with another bond in the same class and of the same credit quality and default propensity as the original. In either case, the portfolio always consists of the same number of bonds with the same default intensity as the original portfolio. Therefore, successive credit events are associated with the same intensity. Thus, the n different default intensities can be combined so that the next credit event occurs at time $\min(\tau_1, \tau_2, \ldots, \tau_n)$

and has intensity $\lambda_t = \sum_{i=1}^{n} \lambda_t^i$. We postulate the following dynamics for the aggregate portfolio value process (the motivation for this assumption is explained in Section 4.2). Fix $(\Omega, \mathcal{G}, \mathbb{P})$ and let V_t be the aggregate portfolio value defined under the empirical probability measure $\mathbb{P}$

$$\frac{\mathrm{d}V_t}{V_{t-}} = \nu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}B_t - L_t(\,\mathrm{d}N_t - \lambda_t \,\mathrm{d}t) \tag{1}$$

where $L_t \geq 0$ is the loss rate of portfolio value ensuing a credit event at time t. N_t is an $\mathcal{F}$-conditional Poisson process with intensity λ_t that keeps count of the number of defaults. ν_t and σ_t are drift and diffusion coefficients corresponding to the aggregate income from the portfolio. We assume they are smooth functions such that the Equation (1) admits a unique and strong solution.

Proposition 1 *Under assumptions of Section 2 for a type 1 claim and dynamics (1), the value of the most subordinate tranche of the basket derivative is given by*

$$V_t^{\text{sub}} = G_t \,\mathbb{E}^*[G_T^{-1}(V_T - D)^+ \mid \mathcal{G}_t] \tag{2}$$

where D is the promised amount to be paid to note holders at time T and

$$G_t = \exp\left(\int_0^t r_u \,\mathrm{d}u\right) \tag{3}$$

with $0 \leq L_t < 1$ and r_t the spot interest rate. The expectation operator $\mathbb{E}^*$ is with respect to an equivalent martingale probability measure $\mathbb{P}^*$ obtained from the empirical probability measure $\mathbb{P}$ via

$$\left.\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}_t} = \exp\left\{\int_0^t \beta_u \,\mathrm{d}B_u - \int_0^t \frac{\beta_u^2}{2} \,\mathrm{d}u + \int_0^t \ln(1+p_u) \,\mathrm{d}N_u - \int_0^t \lambda_u p_u \,\mathrm{d}u\right\}$$

where β_t is the market price of the diffusion risk and p_t is the market price of the jump risk, satisfying

$$1 + p_t > 0 \quad \text{and} \quad \nu_t - r_t + \sigma_t\beta_t - \lambda_t p_t L_t = 0. \tag{4}$$

This proposition states that, under an appropriate change of probability measure, the subordinated basket claim of type 1 can be valued as a call option on collateral value discounted using a short rate r_t. A proof appears in the appendix.

4 Incorporating a market value test

This section presents a more general model (in the framework of type 2 claims introduced in Section 2) for analyzing barrier-type basket options and collateralized debt obligations. In particular, we consider the valuation of subordinated pieces (starting with equity) of a collateralized security that is liquidated when collateral value fails a threshold test. Introducing a collateral test has the effect of modifying Equation (2) so that it reflects the possibility of partial transfer or full liquidation of assets before maturity. Early asset transfer occurs when collateral value falls below a required level and is a guarantee mechanism to safeguard the most senior investors. This guarantee comes at the expense of subordinate investors and its inclusion in pricing adds accounting complications but the approach is similar to that described in the previous section. For the remainder of the paper, we assume all processes are specified under an equivalent martingale measure $\mathbb{P}^*$.

4.1 CDO default

We refer to the asset transfer or liquidation event as structural or *CDO default* to distinguish it from individual collateral defaults or *credit events* of obligors represented in the asset basket. To specify the trigger level that precipitates asset liquidation, consider a reference asset with time t price

$$Y_t = D\,\mathbb{E}^* \left[\exp\left(- \int_t^T r_s \, \mathrm{d}s \right) \middle| \mathcal{G}_t \right]$$

such that at maturity T, it grows to $Y_T = D$. The threshold level D can be flexibly chosen on a contract by contract basis. For example, D may be the notional debt amount. Note holders and equity holders may negotiate on the level D and the corresponding reference zero coupon bond. In practice, if portfolio value is below a certain percentage of capitalization, senior tranche holders may be repaid to the extent that the value of the portfolio meets the test. This arrangement corresponds to shifting both portfolio value, V, and reference value, Y, curves downward. To give equity holders more ‘cushion,’ the magnitude of the shift for the Y curve can be made greater than that of V curve. Note holders claim the assets if $\tau < T$ where

$$\tau \triangleq \inf \{ t \geq 0 : V_t \leq Y_t \} .$$

is a stopping time of the portfolio value process. If $V_\tau = Y_\tau$, they recover D almost surely at maturity T.

4.2 Asset pool dynamics

As individual asset values change either independently or in response to common economic or sectoral shocks, the value of the collateral portfolio fluctuates. In our setup, the value of the asset portfolio is affected by random fluctuations in the state process and by the number of default events in the obligor pool. These variables can be simulated to generate portfolio value scenarios and, in turn, to evaluate CDO claims. This procedure can be computationally intensive and does not provide a transparent view of the impact of various design parameters on the value of a particular tranche. In this section, we propose an approximation to the dynamics of the collateral pool that allows faster valuations and a salient way of performing sensitivity analysis on subordinated structures.

The motivation for our approximating process is as follows. Since defaults in the asset portfolio are guided by the default intensities of individual obligations, there is reason to believe that, especially for diversified portfolios, the remaining uncertainty in portfolio value is mainly due to systemic risk factors (*e.g.*, interest rate movements, equity market valuations). Thus, on the aggregate, portfolio value may tend to vary in accordance with general economic conditions *adjusted* by default events. From this point of view, a diffusion process with a jump component linked to the number of credit events in the portfolio is a natural choice. Thus, we assume evolution of collateral value can be approximated by the $\mathbb{P}^*$-process

$$V_t = V_0 \, \exp\left(\int_0^t (\mu_s - \frac{1}{2}\sigma_s^2)\, \mathrm{d}s + \int_0^t \sigma_s \, \mathrm{d}B_s\right)(1-L)^{N_t} \tag{5}$$

with corresponding stochastic differential equation

$$\mathrm{d}V_t = V_{t-}(\mu_t \, \mathrm{d}t - L_t \, \mathrm{d}N_t + \sigma_t \, \mathrm{d}B_t) \tag{6}$$

where μ_t, σ_t, N_t, and B_t are defined similarly as in Equation (1).[3] Note that the running number of defaults affects portfolio value through the counting process that drives N_t. In other words, individual default events in the asset pool have a direct bearing on the approximated value of the portfolio. Preliminary tests, outlined in Section 5.1, provide support for this assertion, increasing with effectiveness as for diversified asset pools.

The value of the equity piece of the CDO is, then,

$$V_t^{\text{sub}} = \mathbb{E}^*\left[e^{-\int_t^T r_s \, \mathrm{d}s}(V_T - D)\, 1_{\{\tau > T\}} \,|\, \mathcal{G}_t\right] \tag{7}$$

where D represents the aggregate claim of note holders at maturity.

Equation (7) can be easily evaluated by Monte Carlo simulation of the underlying default and portfolio value processes, thus bypassing the need for a full simulation of individual asset dynamics in the collateral pool. Portfolio value, as defined

[3] An intuitive mathematical argument is as follows: If each defaultable bond has an intensity affine in the state vector X where each component of X follows an independent square root diffusion process, then it is well known that bond prices are exponential affine. A first-order approximation, via the central limit theorem, for the value of a diversified portfolio is a lognormal process if the number of the bonds is sufficiently large.

above, can itself be considered an aggregate credit score or state process (following a geometric Brownian motion with jumps or GBMJ, for short) indicating the overall credit quality of the CDO where threshold crossings signal credit events as described, for example, in Avellaneda and Zhu (2001), Bélanger et al. (2001), and Esteghamat (2003). The following section presents more explicit variations of the above valuation formula under different scenarios of response to impeding failure of the collateral to meet its market value test. The variations rely on the conditional independence of default events under the risk neutral measure $\mathbb{P}^*$ and require fewer simulations to evaluate.

4.3 Valuation

Section 3 developed an intuitive valuation formula based on the well-understood call option-like feature of subordinated debt. In this section, we extend this idea to a more general case by showing that a subordinate tranche can be treated as a collection of ith-to-default basket derivatives plus a default-adjusted portfolio of call options. This decomposes subordinate structures into simpler securities with obvious pricing and hedging benefits.

Assume, momentarily, that the default time τ of the CDO coincides with default of a collateral asset. This restriction is revisited shortly. It can be equated, roughly, with affording greater flexibility to pool managers in dealing with impending and predictable shortages or failures of collateral tests. Additionally, to reduce the cost of default, managers are given a grace period to meet collateral tests before note holders assume ownership of the assets. Another situation, compatible with the coincident default restriction, is that of a collateral pool consisting of high-yield debt, where individual default events most likely will dominate general economic uncertainty. Therefore, CDO defaults will tend to be caused by large shifts in credit yield spreads and not by gradual degradation of portfolio value. If the portfolio consists of floating rate debt, the effect of price fluctuations due to interest rate movements are mitigated, supporting the coincident-default scenario.

Denote the i-th default time by $\tau_{(i)}$, its hazard process by $\Lambda_t^{(i)}$, and its default intensity by $\lambda_t^{(i)}$. Our assumption implies that, for example, $\{\tau_i = \tau, \tau_{i+1} = \tau\}$ is a null event. Using the conditional independence of default times yields a more explicit formulation for Equation (7) by conditioning on the event that the CDO default time τ corresponds to the i-th default in the portfolio,

$$V_0^{\text{sub}} = \sum_{i=1}^{n} \mathbb{E}^*_{(i)}[\cdot]\, \mathbb{P}^*(\tau_{(i)} = \tau) + \mathbb{E}^* \left[e^{-\int_0^T r_s\, \mathrm{d}s} (V_T - D)^+ \right] \mathbb{P}^*(\tau_{(n)} < \tau) \tag{8}$$

with $\mathbb{E}^*_{(i)}[\cdot] = \mathbb{E}^*[e^{-\int_0^T r_s\, \mathrm{d}s}(V_T - D)^+ \sum_{j=1}^{n} \int_T^\infty P_j(u)\, \lambda_u^j\, e^{-\Lambda_u^j}\, \mathrm{d}u]$ and $P_j(u) := \sum_{\mathbf{j}_-, \mathbf{j}_+} \prod_{k \in \mathbf{j}_-} (1 - e^{-\Lambda_u^k}) \prod_{l \in \mathbf{j}_+} e^{-\Lambda_u^l}$ where $\mathbf{j}_-$, $\mathbf{j}_+$ are sequences of obligors defaulting before and after default of the j-th entity, respectively. The summation is taken with respect to all such permutations. Expectation $\mathbb{E}^*_{(i)}[\cdot]$ denotes value of a contract promising to pay $(V_T - D)^+$ at maturity T only if the i-th default occurs after T (following the convention of Bielecki and Rutkowski (2002)). The second term in (8) corresponds to the event that collateral defaults do not cause CDO default. Now, consider the probability part of Equation (8), that is, $\mathbb{P}^*(\tau = \tau_{(i)}) = \mathbb{P}^*(\tau \leq \tau_{(i)}) - \mathbb{P}^*(\tau \leq \tau_{(i-1)})$. We claim the probability can be rewritten as

$$\mathbb{P}^*(\tau \leq \tau_{(i)}) = \int_0^\infty \mathbb{P}^* \left(\frac{V_{\tau_{(i)}-}}{Y_{\tau_{(i)}}} \leq (1-L)^{-i} \mid \tau_{(i)} = t \right) \mathrm{d}\mathbb{P}^*(\tau_{(i)} \leq t). \tag{9}$$

To see this, consider two shadow processes: One to keep track of a hypothetical portfolio value which experiences no loss as a result of credit events; another to accumulate a loss percentage. Each time there is a default event, compute post-loss asset value by multiplying the shadow price by the accumulated loss and check whether it is above or below the value of the reference bond. If post-loss value is greater than Y_t, the reference price, let the value process continue. Otherwise, declare CDO default. If the i-th default causes structural default, we combine all the losses at that time by multiplying the shadow price by $(1-L)^i$. Since the probability distribution of i-th asset default is known, we can calculate

$$\mathbb{P}^*(\tau \leq \tau_{(i)}) = \sum_{j=1}^{n} \int_0^\infty \mathbb{P}^* \left(\frac{V_u}{Y_u} \leq (1-L)^{-i} \right) P_j(u)\, \lambda_u^j\, e^{-\Lambda_u^j}\, \mathrm{d}u. \tag{10}$$

The probability just inside the integral has a well-known analytic solution when drift and diffusion parameters are constants. Namely,

$$\mathbb{P}^* \left(\frac{V_{t-}}{Y_t} \leq (1-L)^{-i} \right) = \Phi \left(\frac{-\gamma_0 - \gamma_1 t}{\sqrt{t}} \right) + e^{-2\gamma_0\gamma_1} \Phi \left(\frac{-\gamma_0 + \gamma_1 t}{\sqrt{t}} \right)$$

where Φ is the cumulative standard normal distribution, $\gamma_0 = \frac{\ln(V_0/Y_0) + \ln(1-L)^i}{\sqrt{\sigma_V^2 + \sigma_Y^2}}$, $\gamma_1 = \frac{\nu_V - \nu_Y}{\sqrt{\sigma_V^2 + \sigma_Y^2}}$, $\nu_V = \mu_V - \sigma_V^2/2$, and $\nu_Y = \mu_Y - \sigma_Y^2/2$.

Proposition 2 *Under assumptions of Section 2 for a type 2 claim with the additional assumption that the CDO default necessarily coincide with defaults of the collateral pool, the value of the least senior tranche of CDO at time zero is*

$$\begin{aligned} V_0^{\text{sub}} = \sum_{i=1}^{n} \mathbb{P}^*(\tau_{(i)} = \tau)\, \mathbb{E}^* &\left[e^{-\int_0^T r_s\, \mathrm{d}s} (V_T - D)^+ \sum_j \int_T^\infty P_j(u)\, \lambda_u^j\, e^{-\Lambda_u^j}\, \mathrm{d}u \right] \\ &+ \mathbb{P}^*(\tau_{(n)} < \tau) \mathbb{E}^* \left[e^{-\int_0^T r_s\, \mathrm{d}s} (V_T - D)^+ \right] \end{aligned} \tag{11}$$

where $P_j(u) \triangleq \sum_{\mathbf{j}_-, \mathbf{j}_+} \prod_{k \in \mathbf{j}_-} (1 - e^{-\Lambda_u^k}) \cdot \prod_{l \in \mathbf{j}_+} e^{-\Lambda_u^l}$. $\mathbb{P}^*(\tau_{(i)} = \tau)$ *is given by Equation (10) and* $\mathbb{P}^*(\tau_{(n)} < \tau) = 1 - \sum_{i=1}^n \mathbb{P}^*(\tau_{(i)} = \tau)$.

Thus, with the collateral test in place, the most subordinate tranche can be decomposed into a default-adjusted call option plus a series of ith-to-default securities.

Remark 1 Since the sum value of all tranches must equal the value of collateral (net of fees), the above pricing formula can be used successively to value each tranche of a CDO by working up from the least to the most senior claim. For example, to price a mezzanine tranche just above equity, introduce a new reference curve, Z_t such that $Z_T < Y_T = D$ almost surely. If asset value V_t crosses curve Z_t at a time $\tau_M \in (0, T]$, mezzanine and equity are both in default. This new asset value process yields the value of the combined mezzanine and equity pieces at time zero, V_0^{M+E}. Hence we have $V_0^M = V_0^{M+E} - V_0^E$. Finally, the value of the remaining senior piece at time zero, V_0^S is $V_0^S = V_0 - V_0^{M+E}$.

Remark 2 Removing the restriction that structural CDO default coincide with a collateral default has the effect of modifying Equation (11) to account for the possibility of predictable defaults (formally, accessible default times, *i.e.*, those caused by gradual decline of portfolio value). Note that the event that CDO default is caused by the diffusion component of portfolio value and the event that it is caused by obligor defaults are mutually exclusive. The probability of CDO default in the interval between two collateral defaults is, by the strong Markov property of V, on the set $\{\tau_{(i)} < \tau\}$,

$$\mathbb{P}^*(\tau_{(i)} < \tau < \tau_{(i+1)}) = \mathbb{E}^*[\mathbb{E}^*_{V_{\tau_{(i)}}}[1_{\{0<\tau<\tau_{(1)}\}}]]. \tag{12}$$

This says that, conditioning upon the post-ith default value of V, the probability that the CDO default occurs before the next collateral default can be calculated by the inner expectation. If we denote the value of the least senior piece from Proposition 2 by V_0^*,

$$V_0^{\text{sub}} = V_0^* - \mathbb{E}^*\left[e^{-\int_0^T r_s\,\mathrm{d}s}(V_T - D)^+\right]\sum_{i=0}^{n}\mathbb{P}^*(\tau_{(i)} < \tau < \tau_{(i+1)}, \tau_{(i)} < \tau)$$

with the convention $\tau_0 \triangleq 0$ and $\tau_{n+1} \triangleq T$. However, evaluation of Equation (12) necessarily involves simulating V just after the default of the ith asset. Hence Monte Carlo simulation of sample paths in (7) is more efficient. However, the adjustment factor is typically small so that Proposition 2 can be used to bypass extra simulations.

One can identify simple hedge strategies based on Proposition 2. Of course, the equity investor can reduce exposure to the most likely losses by hedging out few of the most default-prone names. Alternatively, one can insure the equity tranche against first default by purchasing a 1st-to-default swap on the collateral or subset thereof. The value of this strategy can be deduced from the pricing formula.

5 Performance tests

5.1 *Asset dynamics*

How well does our approximation scheme work? First, to test the performance of our assumption regarding portfolio value given in Equation (5), we set up a hypothetical collateral pool of $n = 20$ zero coupon bonds and simulate the value of each bond in the portfolio using an intensity based (IB) model of correlated default for both type 1 and 2 claims. Recall for a type 1 claim sales proceeds of defaulted bonds are reinvested in a similar bond that is subject to default again. For a type 2 claim, we specify how the sales proceeds of defaulted bonds are distributed in the formula for portfolio value as follows

$$V_t = \sum_{i=1}^{n} \left(1_{\{\tau_i > t\}} \mathbb{E}^* \left[e^{-\int_t^T (\tilde{L}\lambda_u^i + r_u)\, \mathrm{d}u} \middle| \mathcal{G}_t \right] + 1_{\{\tau_i \le t\}} (1 - \tilde{L}) P_{\tau_{i-}} e^{\int_{\tau_i}^t r_u \, \mathrm{d}u} \right) \quad (13)$$

where P_{τ_i-} is the value of the i-th bond just prior to its default. In other words, once a bond defaults (*i.e.*, $\tau_i \le t$), it is sold at the price of $(1 - \tilde{L})$ times its value just prior to default and the revenue is invested in a default-free savings account until maturity.

To simulate the value of the portfolio using Equation (5) we estimated the model with constant parameters μ, σ and L. Recall L is the fractional loss of portfolio given a default event; it is not $\tilde{L}$, the individual collateral loss fraction. The three parameters were estimated by calibrating to an intensity based model. A summary of the procedure is described below.

(1) We use a multifactor intensity process for each of twenty bonds $i = 1, ..., 20$. Driessen (2005) finds that a risk-neutral default intensity of the form

$$\lambda_{i,t} = \alpha_i + \beta_{i,r} r_t + \beta_{i,v} v_t + \gamma_{i,1} F_{1,t} + \gamma_{i,2} F_{2,t} + G_{i,t} \quad (14)$$

captures corporate bond spread behavior well, where r_t and v_t are short-rate factors in an affine term structure model, $F_{1,t}$ and $F_{2,t}$ are common factors to every firm, and $G_{i,t}$ is a firm-specific factor. Both the common and firm-specific factors are assumed to follow independent square root diffusions and hence specified by a triple (κ, θ, σ) where $dX_t = \kappa(\theta - X_t) + \sigma\sqrt{X_t}\, \mathrm{d}B_t^*$ for $X = F_1$, F_2 or G. This model is capable of producing high default correlations.

We follow the specification and calibration result of Yu (2003a) for the above parameters where the short-rate factors are ignored (*i.e.*, β_r and β_v are set to zero). Table 1 summarizes our chosen parameter values. Credit quality deteriorates as index i increases. Consistent with observations from Table 3 of Yu (2003a), values of (κ, θ, σ) for firm specific factors do not change when

Table 1
Parameter values for a sample basket of 20 bonds with intensity processes of Equation (14).

	κ	θ	σ	α	γ_1	γ_2
F_1	.030	.0050	.0160			
F_2	.049	.0050	.0160			
1	.017	.0065	.0170	.0054	5.80	0.78
2	.018	.0065	.0157	.0056	6.40	1.00
3	.019	.0065	.0137	.0058	7.20	0.44
4	.020	.0065	.0137	.0060	8.00	0.70
5	.021	.0065	.0147	.0062	8.60	0.20
6	.022	.0065	.0157	.0064	9.20	0.56
7	.023	.0065	.0137	.0066	9.80	1.00
8	.024	.0065	.0170	.0068	10.40	0.70
9	.025	.0065	.0157	.0070	11.00	0.90
10	.026	.0065	.0167	.0072	11.60	1.50
11	.027	.0065	.0157	.0074	12.20	1.60
12	.017	.0065	.0177	.0076	12.80	1.70
13	.018	.0065	.0147	.0078	13.40	1.90
14	.019	.0065	.0137	.0080	14.00	2.10
15	.020	.0065	.0147	.0082	14.60	3.60
16	.021	.0065	.0137	.0084	15.20	3.10
17	.022	.0065	.0157	.0086	15.80	4.50
18	.023	.0065	.0167	.0088	16.40	4.90
19	.024	.0065	.0157	.0090	17.00	5.20
20	.025	.0065	.0167	.0092	17.60	4.70

credit quality varies but correlation parameters γ_1 and γ_2 increase as credit quality deteriorates. The short rate process is modeled as a square root diffusion

$$\mathrm{d}r_t = (0.0774 - 0.86 r_t)\ \mathrm{d}t + 0.0348\sqrt{r_t}\ \mathrm{d}B_t^*. \tag{15}$$

(2) For each simulation path in Step 1, compute maximum likelihood estimates for the three parameters of Equation (5), namely, μ, σ, and L.
(3) Take the average of the estimates for μ, σ, and L obtained above and denote them $\hat{\mu}$, $\hat{\sigma}$, and $\hat{L}$, respectively. The resulting annualized estimates, based on 1000 scenarios, are listed in Table 2. Create paths based on Equation (5) using $\hat{\mu}$, $\hat{\sigma}$, and $\hat{L}$.
(4) Compare the two sets of paths generated in Steps 1 and 3. Figure 1 shows sample average portfolio value paths. For both type 1 and type 2 claim, the estimation for each t is reasonably accurate. Snapshots of portfolio value distributions for the two collections (Figure 2) show that the approximating process provides not only average sample paths but also a reasonable distribution of portfolio value. This information is useful in managing risk exposure. The jump-diffusion approximation may be used for quick sensitivity analysis and risk measurement (*e.g.*, computing value-at-risk), eliminating the necessity for

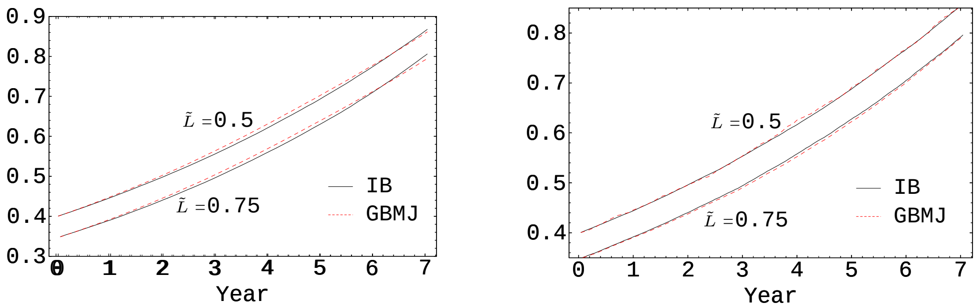

Fig. 1. *Average values of hypothetical collateral pools of zero coupon bonds maturing in seven years. Price paths are shown for the intensity based model (IB) of Equation (14) and the jump diffusion model (GMBJ) calibrated to intensity based simulations. The left plot is for a type 1 claim; the plot on the right is for a type 2 claim.*

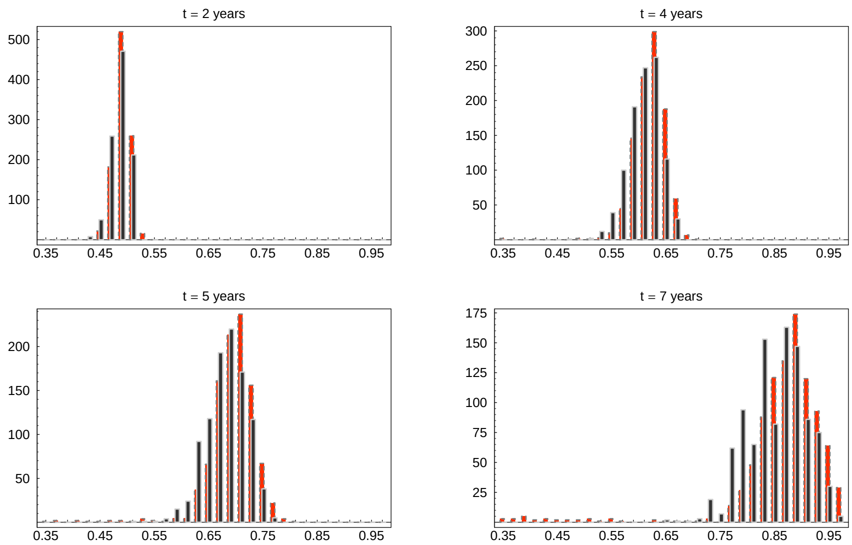

Fig. 2. *Distribution of portfolio value according to an intensity based model (darker bars) and the maximum-likelihood estimate of the GBMJ model (lighter bars). Loss fraction is set to 0.5.*

Table 2
Parameter estimates for a sample basket of 20 bonds with intensities given by (14).

	Type 1 Claim		Type 2 Claim	
Parameter	$\tilde{L} = 0.5$	$\tilde{L} = 0.75$	$\tilde{L} = 0.5$	$\tilde{L} = 0.75$
$\hat{\mu}$	0.130113	0.148487	0.127336	0.147311
$\hat{\sigma}$	0.010970	0.011989	0.010462	0.011646
$\hat{L}$	0.018750	0.027002	0.026595	0.040311

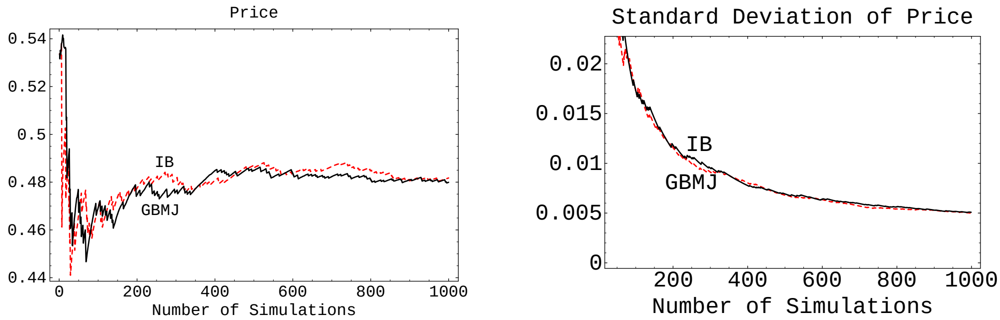

Fig. 3. *Running average of prices generated by the pure intensity based and GBMJ simulations for a collateral pool consisting of twenty bonds (left plot). Standard errors of price estimates are shown on the right.*

full simulations.

It is important to note that the relative good fit of the above experiment can be adversely affected by extreme concentration of default intensities. For example, in the extreme case of 20 identical default intensities, average portfolio value can decline sharply after a certain point. This implies that a GBMJ process with a constant growth factor is not sufficient to describe the behavior of portfolio value. The constant-parameter model works well for standard collateral structures.

5.2 Derivative price

In this section, we use Driessen's multifactor model to price an equity tranche. Again, the collateral basket consists of the twenty defaultable bonds defined in Table 1 each maturing in seven years. If there are no defaults, the total value of the basket reaches one dollar. The contract holder is paid one dollar if the market value of the basket does not ever go below a reference curve. The reference curve Y is arbitrarily (but appropriately) specified as a zero coupon bond whose value at time 0 is $\$0.287$ and grows in seven years to $\$0.70$. The loss rate of each bond is $\tilde{L} = 0.75$ and the sales proceeds of defaulted bonds are used to purchase a similar bond with the same default intensity. Figure 3 shows the running average of time zero prices for this contract generated by intensity based and by GBMJ simulations with parameters as estimated above. Running values of standard errors of prices are shown on the right. Prices from the two methods (pure simulation versus approximation) converge to almost the same value of \$0.48 as the number of simulations increase. The GBMJ approximation, however, requires significantly less computation time. Once jump-diffusion parameters are estimated there is no need to run an IB-based simulation to find, for example, the present value of a contract under different reference curve scenarios or with different payoffs.

Continuing with the example, suppose either (a) interest rates shifts up by 100 basis

Table 3
Sensitivity analysis for model of Equation (14). Percentage changes (rounded) from benchmark values are given in parentheses.

	Price	$\hat{\mu}$	$\hat{\sigma}$	$\hat{L}$
Benchmark values	0.480	0.148	0.0120	0.0270
Interest rate $+100$ bp	0.438 (-9%)	0.145 (-2%)	0.0119 (-1%)	0.0275 (2%)
Bond exchange	0.469 (-3%)	0.148 (-0%)	0.0119(-1%)	0.0297 (10%)

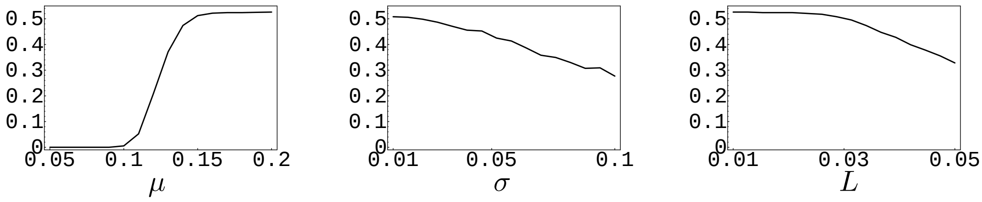

Fig. 4. *Equity price sensitivity based on 1000 simulations.*

point, or (b) the least risky bond (the first bond) is replaced by a bond with identical default intensity to the riskiest bond (twentieth bond). Prices shift in response and new parameters are estimated for the GBMJ approximation as shown in Table 3. The price declines in both cases as expected. Whether a shift in credit spreads or a recomposition of the collateral basket, these changes are translated into a change in GBMJ parameters. Many factors that would influence the pricing are summarized by the three parameters given the default intensity parameters. The shift in price, as each of the GBMJ parameters changes from its benchmark value, is shown in Figure 4. Likelihood of CDO default is chosen to be relatively high in this example. As a result, equity price is especially sensitive to shifts in drift (driven mainly by changes in individual default intensities).

5.3 Performance of the pricing formula

In the previous section we tested the fit of the GBMJ approximation to full Monte Carlo simulation. Here, we compare the convergence rate of the pricing Equation (7) via GMBJ simulation with that from formula (11) of Section 4. We consider a simple experiment and choose parameters of λ_i large enough to cause defaults in the portfolio rather frequently and assume that CDO default is caused only by jumps. In order to reduce computational time, we select a portfolio of five bonds ($n = 5$) and a constant discount rate $r = 0.06$. The procedure is as follows.

(1) Given intensity parameters $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ which we describe below and the parameters of the portfolio value and the reference bond process

$$(L, \mu_V, \mu_Y, \sigma_V, \sigma_Y) = (0.4/n, 0.095, 0.062, 0.2, 0.15),$$

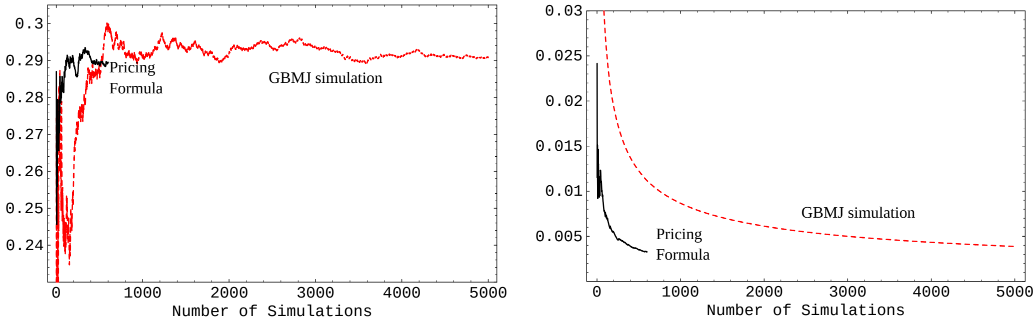

Fig. 5. *Running average of prices generated by the pricing formula and by direct simulation assuming a GBM process with jumps (left plot). The collateral pool consists of five bonds. Standard errors of price estimates are shown on the right.*

and initial values of $V_0 = 0.85$ and $Y_0 = 0.54$, generate scenarios of the underlying state variable X_t and calculate the value (at time 0) of receiving 1 dollar if portfolio value does not fall below the reference bond price for $T = 10$ years. The terminal payoff ($0 or $1) is discounted back to time 0. Initial values of V and Y are chosen so that the two curves are sufficiently apart from each other at time 0. The state process X_t, representing general economic activity or equity market index, is geometric Brownian motion with drift and diffusion parameters 0.1 and 0.25, respectively, and initial value $X_0 = 1$. The intensity processes are $\lambda_{i,t}$= $\alpha_i + \beta_i X_t$ where β_1=0.07, β_2=0.05, β_3=0.04, β_4=0.035, β_5=0.03, α_i = 0 for all i. On average, these values cause around 60% of the bonds (*i.e.*, 3 bonds) to default.

(2) Estimate the value of the equity tranche for each of the two methods by using the same parameter values to simulate asset value dynamics of Equation (5).
(3) Compare the running averages of the prices from the two methods.

Results of this test procedure are as follows.

- The pricing formula clearly converges faster than direct simulation of the GMBJ value process. This implies that formula-based pricing has the potential to significantly reduce computation times compared with direct intensity based simulation. Figure 5 shows that 500 scenarios are enough for formula-based pricing to achieve a standard error of about one percent, while convergence in case of GMBJ simulation pricing is much slower.
- Since the pricing formulae involve integration to infinity, choice of a practical upper bound can be an important technical issue for accuracy (see Equation (8)). A larger upper bound has two opposite effects on price. It has a positive effect on price by increasing expected payoff; it has a negative effect because the likelihood that no collateral default causes a structural default declines. The two effects tend to offset each other, thus, diminishing sensitivity of price convergence on the choice of upper bound.
- The pricing formula also performed well under a linear two-factor process $\lambda_i =$

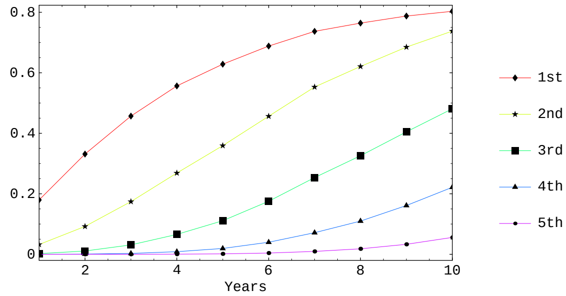

Fig. 6. *The risk-neutral probability of the* ith *entity defaulting.*

$\beta_i X_1 + \gamma_i X_2$ where X_1 and X_2 are described by

$$\begin{pmatrix} \mathrm{d}X_1 \\ \mathrm{d}X_2 \end{pmatrix} = \begin{pmatrix} .095X_1 \\ .0774 - .026X_2 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} .2X_1 & 0 \\ .088\sqrt{X_2} & .055\sqrt{X_2} \end{pmatrix} \begin{pmatrix} \mathrm{d}B_1^* \\ \mathrm{d}B_2^* \end{pmatrix} \tag{16}$$

where β_1=0.07, β_2=0.05, β_3=0.04, β_4=0.035, β_5=0.03, γ_1=0.05, γ_2=0.02, γ_3=0.04, γ_4=0.035, and γ_5=0.03.

Figure 6 plots the (risk-neutral) probability of default of the ith collateral. There is an 80% chance that the first default occurs and a 5% chance that the fifth default occurs by year ten. As a byproduct, the likelihood of CDO defaulting on and prior to the i-th collateral default is tabulated in Table 4. The risk-neutral probability that the first collateral default will lead to CDO default is only about 2%; there is a 79% chance that by the fifth default, the CDO has also defaulted.

Table 4
Likelihood of CDO default prior to i-th collateral default, denoted (i).

(1)	(2)	(3)	(4)	(5)
2.11%	13.68%	37.86%	64.09%	78.96%

6 Conclusion

This paper has touched on three ways of analyzing CDO structures. These are (1) pure intensity-based simulations of defaults, (2) simulations based on the asset dynamics of Section 4.2, and (3) valuation formulae derived from the same dynamics by assuming conditional independence of default times given state factors. The collateral value process of Section 4.2 approximates dynamics of the asset pool and yields a natural way of decomposing subordinated claims into portfolios of options and default derivatives on the underlying baskets of assets. In turn, this identifies possible risk management strategies. The approach also promises computational advantages over pure intensity-based simulation in analyzing and designing CDOs. Outputs and intermediate calculations of the pricing formula can also be used as control variates in analyzing other structures.

Other correlation derivatives can be priced in this framework. Although we have focused on valuation of equity, it is clear that the framework presented here, including the market value test specification of Section 4.1, allows a salient recursive valuation of tranches above equity. For example, with equity priced, to find the value of a mezzanine tranche just above equity, one shifts the reference (market test) bond curve downwards so that its value at maturity appeals to senior note holders. The relative value of the mezzanine note is, then, found relative to equity. This procedure is repeated until the value of the most senior tranche is obtained as the difference between total initial collateral value and the value of all the subordinate portions (including all the intermediate tranches and equity). In this fashion, one computes all relative prices of tranches, enabling evaluation of each tranche relative to others.

Section 5 compared a direct simulation of the GBMJ approximating process against the pricing formulae of Section 4. The standard error of calculations decreases rapidly with each added simulation run when using the pricing formula, although each run is slower.[4] Moreover, as a byproduct, the pricing formula provides: (a) value of ith-to-default securities, and (b) probabilities $\mathbb{P}^*(\tau_{(i)} = \tau)$ that the ith collateral default precipitates CDO default, leading to asset liquidation. Both items are valuable sources of information in risk management. Consider, for example, a fund manager who has invested in a number of notes and equity tranches of different CDOs. In order to reduce the prospect of concentrated defaults which can result in sudden and sustained reductions in fund value, the manager may wish to diversity the fund's assets in order to spread potential losses over the investment horizon. The above information can be used to adjust portfolio allocations (as well as tune CDO structures) to reduce the chance of large simultaneous losses. This information is not readily obtained from direct simulations.

A Proof of Proposition 1

By Girsanov's theorem, process $B_t^* = B_t - \int_0^t \beta_u \, \mathrm{d}u$ follows a Brownian motion for $t \in [0, T]$ under $\mathbb{P}^*$. Note that the jump direction is negative. With this preparation, we deflate the portfolio value process V_t with the price of the money market security

[4] Run times were about the same order of magnitude for the 600 versus 5000 simulations shown using naïve coding.

G from Equation (3) to obtain $\widetilde{V}_t \triangleq V_t G_t^{-1}$ which has dynamics

$$
\begin{aligned}
\mathrm{d}\widetilde{V}_t &= \widetilde{V}_t \left((\nu_t - r_t)\, \mathrm{d}t + \sigma_t \, \mathrm{d}B_t - L_t \, \mathrm{d}N_t + L_t \lambda_t \, \mathrm{d}t \right) \\
&= \widetilde{V}_t \left((\nu_t - r_t)\, \mathrm{d}t + \sigma_t \, \mathrm{d}B_t^* + \sigma_t \beta_t \, \mathrm{d}t - L_t \, \mathrm{d}N_t + (1+p_t) L_t \lambda_t \, \mathrm{d}t - p_t L_t \lambda_t \right) \\
&= \widetilde{V}_t \left(\sigma_t \, \mathrm{d}B_t^* - L_t \, \mathrm{d}N_t + (1+p_t) L_t \lambda_t \, \mathrm{d}t \right) \\
&= \widetilde{V}_t \left(\sigma_t \, \mathrm{d}B_t^* - L_t (\, \mathrm{d}N_t - \lambda_t^* \, \mathrm{d}t) \right). \qquad \text{(A.1)}
\end{aligned}
$$

The third equality is due to (4) and in the last equality $\lambda_t^* \triangleq (1+p_t)\lambda_t$. Since the intensity of N_t under the measure $\mathbb{P}^*$ is λ_t^* and the compensated Poisson process $N_t - \int_0^t \lambda_u^* \, \mathrm{d}u$ is $\mathbb{P}^*$-martingale, Equation (A.1) shows that $\widetilde{V}$ is a $\mathbb{P}^*$-martingale with respect to $\mathcal{G}$. Hence a pricing formula for the equity portion of the claim is

$$V_t^{\text{sub}} = G_t \, \mathbb{E}^*[G_T^{-1}(V_T - D)^+ \mid \mathcal{G}_t]$$

where D is the note holder's promised principal.

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