OPTIMAL STOPPING PROBLEMS FOR ASSET MANAGEMENT

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Abstract. An asset manager invests the savings of some investors in a portfolio of defaultable bonds. The manager pays the investors coupons at a constant rate and receives management fee proportional to the value of portfolio. She also has the right to walk out of the contract at any time with the net terminal value of the portfolio after the payment of investors' initial funds, but is not responsible for any deficit. To control the principal losses, investors may buy from the manager a limited protection which terminates the agreement as soon as the value of the portfolio drops below a predetermined threshold. We assume that the value of the portfolio is a jump-diffusion process and find optimal termination rule of the manager with and without a protection. We also derive the indifference price of a limited protection. We describe numerical algorithms to calculate expected maximum reward and nearly optimal terminal rules for the asset manager and illustrate them on an example. The motivation comes from the collateralized debt obligations.

1. Introduction

We study two optimal stopping problems of an institutional asset manager hired by ordinary investors who do not have access to certain asset classes. The investors entrust their initial funds in the amount of L to the asset manager. As long as the contract is alive, the investors receive coupon payments from the asset manager on their initial funds at a fixed rate (higher than the risk-free interest rate). In return, the asset manager collects dividend or management fee (at a fixed rate on the market value of the portfolio). At any time, the asset manager has the right to terminate the contract and to walk away with the net terminal value of the portfolio after the payment of the investors' initial funds. However, she is not financially responsible for any amount of shortfall. The asset manager's *first problem* is to find a nonanticipative stopping rule which maximizes her expected discounted total income.

Under the original contract, investors face the risk of losing all or some part of their initial funds. Suppose that the asset manager offers the investors a limited protection against this risk, in the form that the new contract will terminate as soon as the market value of the portfolio goes below a predetermined threshold. The asset manager's *second problem* is to find the fair price for the limited protection and the best time to terminate the contract under this additional clause.

We assume that the market value X of the asset manager's portfolio follows a geometric Brownian motion subject to downward jumps which occur according to an independent Poisson process. As explained in detail in the next section, both the problems and the setting are motivated by those faced by the managers responsible for the portfolios of defaultable bonds, for example, as in

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collateralized debt obligations (CDOs). For a detailed description and the valuation of CDOs, we refer the reader to Duffie and Gârleanu [11], Goodman and Fabozzi [18], Egami and Esteghamat [13] and Hull and White [14]. Briefly, a CDO is a derivative security on a portfolio of bonds, loans, or other credit risky claims. Cash flows from a collateral portfolio are divided into various quality/yield tranches which are then sold to investors. In our setting, for example, the times of the (downward) jumps in the portfolio value process can be thought as the default times of individual bonds in the portfolio.

The difference between the real-world CDOs and our setting is that a CDO has a pre-determined maturity while we assume an infinite time horizon. However, a typical CDO contract has a term of 10-15 years (much longer than, for example, finite-maturity American-type stock options) and is often extendable with the investors' consent. Hence our perpetuality assumption is a reasonable approximation of the reality. We believe that our analysis is also applicable in certain other financial and real-options settings with no fixed maturity, e.g., open-end mutual funds, outsourcing the maintenance of computing, printing or internet facilities in a company or in a university.

To find the solutions of the asset manager's aforementioned problems, we first model them as optimal stopping problems for a suitable jump-diffusion process under a risk-neutral probability measure. By separating the jumps from the diffusion part by means of a suitable dynamic programming operator, similarly to the approach used by Dayanik, Poor, and Sezer [8] and Dayanik and Sezer [9] for the solutions of sequential statistics problems, we solve the the optimal stopping problems by means of successive approximations, which not only lead to accurate and efficient numerical algorithms but also allow us to establish concretely the form of optimal stopping strategies.

Without any protection, the optimal rule of the asset manager turns out to terminate the contract if the market value of the portfolio X becomes too small or too large; i.e., as soon as X exits an interval (a, b) for some suitable constants $0 < a < b < \infty$.

In the presence of limited protection (provided to the investors by the asset manager for a fee) at some level $\ell \in (a, L]$, it is optimal for the asset manager to terminate the contract as soon as the value X of the portfolio exits an interval (ℓ, m) for some suitable $m \in [\ell, b)$. Namely, if the protection is binding, i.e., $\ell \in (a, b)$, then the asset manager's optimal continuation region shrinks. *In other words, investors can have limited protection only if they are also willing to give up in part from the upside-potential of their managed portfolio.* "Total protection" (i.e, the case $\ell = L$) wipes out the upside-potential completely since the optimal strategy of the asset manager becomes "stop immediately" in this extreme case (i.e., $\ell = m = L$). Incidentally, a contract with a protection at some level is less valuable than an identical contract without a protection. The difference between these two values gives the fair price of the investors' protection. The investors must pay this difference to the asset manager in order to compensate for the asset manager's lost potential revenues due to "suboptimal" termination of the contract in the presence of the protection. In other words, the asset manager will be willing to provide the protection only if the difference between the expected total revenues with and without the protection is cleared by the investors.

Our model also sheds some light on the *default timing problem* of a single firm. Note that the lower boundary l of the optimal continuation region in the first problem's solution may be interpreted as the "optimal default time" of a CDO. Instead of the value of a portfolio, if X represents the market value of a firm subject to unexpected "bad news" (downward jumps), then the asset manager's first problem and its solution translate into the default and sale timing problem of the firm and its solution. An action (default or sale) is optimal if the value X of the firm leaves the optimal continuation region (a, b). It is optimal to default if X reaches $(0, a]$, and optimal to sell the firm if X reaches $[b, \infty)$. Our solution extends the work of Duffie [10, Chapter 11] who calculates (based on the paper by Leland [17]) the optimal default time for a single firm whose asset value is modeled by a geometric Brownian motion.

Let us also mention that optimal stopping problems (especially, pricing American-type options) for Lévy processes have been extensively studied; see, for example, Chan [3], Pham [21], Mordecki [20, 19], Boyarchenko and Levendroskiĭ [2], Kou and Wang [16] and Asmussen et al. [1].

The problems are formulated in Section 2. The solutions of first and second problems are studied in Sections 3 and 4, respectively. Numerical algorithms are described in Section 5 and illustrated on a numerical example in Section 6.

2. The problem description

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space hosting a Brownian motion $B = \{B_t, t \geq 0\}$ and an independent Poisson process $N = \{N_t, t \geq 0\}$ with the constant arrival rate λ, both adapted to some filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ satisfying usual conditions.

An asset manager borrows L dollars from some investors and invests in some risky asset $X = \{X_t, t \geq 0\}$. The process X has the dynamics

$$\frac{dX_t}{X_{t-}} = (\mu - \delta)\mathrm{d}t + \sigma \mathrm{d}B_t - y_0 \, [\mathrm{d}N_t - \lambda \mathrm{d}t], \quad t \geq 0 \tag{2.1}$$

for some constants $\mu \in \mathbb{R}$, $\sigma > 0$, $\delta > 0$ and $y_0 \in (0, 1)$. We denote by δ the dividend rate or the management fee received by the asset manager. Note that the absolute value of relative jump sizes are equal to y_0, and the jumps are downwards. Therefore, the asset price

$$X_t = X_0 \exp\left\{ \left(\mu - \delta - \frac{1}{2}\sigma^2 + \lambda y_0 \right) t + \sigma B_t \right\} (1 - y_0)^{N_t}, \qquad t \geq 0.$$

is a geometric Brownian motion subject to downward jumps with constant relative jump sizes.

An interesting example of our setting is a portfolio of defaultable bonds as in the *collateralized debt obligations*. Let X_t be the value of a portfolio of k defaultable bonds. After every default, the portfolio loses y_0 percent of its market value. The default times of each bond constitutes a Poisson process with the intensity rate λ_i independent of others. Therefore, defaults occur at the rate $\lambda \triangleq \sum_{i=1}^{k} \lambda_i$ at the level of the portfolio. The loss ratio upon a default is the same constant y_0 across the bonds. The defaulted bond is immediately sold at the market, and a bond with a similar default rate is bought using the sales proceeds. Under this assumption, defaults occur at the fixed rate λ because the number of bonds in the portfolio is maintained at k. Egami and Esteghamat

[13] showed that the dynamics in (2.1) are a good approximation of the dynamics of the aggregate value of individual defaultable bonds when priced in the "intensity-based" modeling framework (see, e.g., Duffie and Singleton [12]). The jump size y_0 on the portfolio level has to be calibrated.

Suppose that the asset manager pays the investors a coupon of c percent on the face value of the initial borrowing L on a continuously compounded basis. We assume $c < \delta$. The asset manager has the right to terminate the contract at any time $\tau \in \mathbb{R}_+$ and receive $(X_\tau - L)^+$. Dividend and coupon payments to the parties cease upon the termination of the contract. Let $0 < r < c$ be the risk-free interest rate, and $\mathcal{S}$ be the collection of all $\mathbb{F}$-stopping times. The *asset manager's first problem* is to find her maximum expected discounted total income

$$U(x) \triangleq \sup_{\tau \in \mathcal{S}} \mathbb{E}^\gamma_x \left[e^{-r\tau}(X_\tau - L)^+ + \int_0^\tau e^{-rt}(\delta X_t - cL)\mathrm{d}t \right], \quad x \in \mathbb{R}_+ \tag{2.2}$$

and a stopping time $\tau^* \in \mathcal{S}$ which attains the supremum (if such τ^* exists) under the condition

$$0 < r < c < \delta.$$

In (2.2), the expectation $\mathbb{E}^\gamma$ is taken under the equivalent martingale measure $\mathbb{P}^\gamma$ for a specified market price γ of the jump risk.

In the real CDOs, the dividend payment is often subordinated to the coupon payment. But since we allow the possibility that the asset manager's net running cash flow $\delta X_t - cL$ becomes negative, our formulation has more stringent requirement on the asset manager than a simple subordination.

In the *asset manager's second problem*, the investors' assets have limited protection. In the presence of the limited *protection at level* $\ell > 0$, the contract terminates at time $\widetilde{\tau}_{(\ell,\infty)} \triangleq \inf\{t \geq 0 : X_t \notin (\ell, \infty)\}$ automatically. The asset manager wants to maximize her expected total discounted earnings as in (2.2), but now the supremum has to be taken over all $\mathbb{F}$-adapted stopping times $\tau \in \mathcal{S}$ which are less than or equal to $\widetilde{\tau}_{(\ell,\infty)}$ almost surely.

3. The solution of the asset manager's first problem

In the no-arbitrage pricing framework, the value of a contract contingent on the asset X is the expectation of the total discounted payoff of the contract under some equivalent martingale measure. Since the dynamics of X in (2.1) contain jumps, there are more than one equivalent martingale measure. The restriction to $\mathcal{F}_t$ of every equivalent martingale measure $\mathbb{P}^\gamma$ in a large class admits a Radon-Nikodym derivative in the form of

$$\left.\frac{d\mathbb{P}^\gamma}{d\mathbb{P}}\right|_{\mathcal{F}_t} \triangleq \eta_t \quad \text{and} \quad \frac{d\eta_t}{\eta_{t-}} = \beta \mathrm{d}B_t + (\gamma - 1)[\mathrm{d}N_t - \lambda \mathrm{d}t], \quad t \geq 0 \quad (\eta_0 = 1), \tag{3.1}$$

which has the solution $\eta_t = \exp\left\{\beta B_t - \frac{1}{2}\beta^2 t + N_t \log\gamma - (\gamma-1)\lambda t\right\}$, $t \geq 0$ for some constants $\beta \in \mathbb{R}$ and $\gamma > 0$. The constants β and γ are known as the market price of the diffusion risk and the market price of the jump risk, respectively, and satisfy the drift condition

$$\gamma > 0 \quad \text{and} \quad \mu - r + \sigma\beta - \lambda y_0(\gamma - 1) = 0. \tag{3.2}$$

Then the discounted value process $\{e^{-(r-\delta)t}X_t : t \geq 0\}$ before the dividends are paid is a $(\mathbb{P}^\gamma, \mathbb{F})$-martingale; see, e.g., Pham [21], Colwell and Elliott [4], Cont and Tankov [5]. Girsanov theorem implies that $B_t^\gamma \triangleq B_t - \beta t$, $t \geq 0$ is a standard Brownian motion, and N_t, $t \geq 0$ is a homogeneous Poisson process with intensity $\lambda\gamma$ independent of B^γ under the new measure $\mathbb{P}^\gamma$. The dynamics of X can be rewritten as

$$\begin{aligned}\frac{dX_t}{X_{t-}} &= [\mu - \delta + \beta\sigma - \lambda y_0(\gamma-1)]\mathrm{d}t + \sigma \mathrm{d}B_t^\gamma - y_0\,[\mathrm{d}N_t - \lambda\gamma \mathrm{d}t], \\ &= (r-\delta)\mathrm{d}t + \sigma \mathrm{d}B_t^\gamma - y_0\,[\mathrm{d}N_t - \lambda\gamma\mathrm{d}t], \quad t \geq 0, \end{aligned} \tag{3.3}$$

where the equality $\mu - \delta + \beta\sigma - \lambda y_0(\gamma - 1) = r - \delta$ follows from the drift condition in (3.2). Using Itô's rule, one can also easily verify that

$$X_t = X_0 \exp\left\{\left(r - \delta - \frac{1}{2}\sigma^2 + \lambda\gamma y_0\right)t + \sigma B_t^\gamma\right\}(1-y_0)^{N_t}, \qquad t \geq 0. \tag{3.4}$$

The infinitesimal generator of the process X under the probability measure $\mathbb{P}^\gamma$ coincides with the second order differential-difference operator

$$(\mathcal{A}^\gamma f)(x) \triangleq (r - \delta + \lambda\gamma y_0)\,x\,f'(x) + \frac{1}{2}\,\sigma^2\,x^2\,f''(x) + \lambda\gamma[f(x(1-y_0)) - f(x)] \tag{3.5}$$

on the collection of twice-continuously differentiable functions $f(\cdot)$.

Because $\{e^{-(r-\delta)t}X_t, t \geq 0\}$ is a martingale under $\mathbb{P}^\gamma$, we have $\mathbb{E}_x^\gamma[\int_0^\infty \delta X_t e^{-rt}\mathrm{d}t] = \int_0^\infty \delta x e^{-\delta t}\mathrm{d}t = x$, and for every stopping time $\tau \in \mathcal{S}$, the strong Markov property implies that $\mathbb{E}_x^\gamma[\int_0^\tau \delta X_t e^{-rt}\mathrm{d}t] =$

$$\begin{aligned}\mathbb{E}_x^\gamma\left[\int_0^\infty \delta X_t e^{-rt}\mathrm{d}t\right] - \mathbb{E}_x^\gamma\left[\int_\tau^\infty \delta X_t e^{-rt}\mathrm{d}t\right] &= x - \mathbb{E}_x^\gamma\left[e^{-r\tau}\int_0^\infty \delta X_{\tau+s}\,e^{-rs}\mathrm{d}s\right] \\ &= x - \mathbb{E}_x^\gamma\left[e^{-r\tau}\,\mathbb{E}_{X_\tau}^\gamma\left(\int_0^\infty \delta X_s\,e^{-rs}\mathrm{d}s\right)\right] = x - \mathbb{E}_x^\gamma\left[e^{-r\tau}X_\tau\right], \quad x \in \mathbb{R}_+.\end{aligned}$$

Because $\mathbb{E}_x^\gamma[\int_0^\tau cLe^{-rt}\mathrm{d}t] = \frac{cL}{r} - \mathbb{E}_x^\gamma[\frac{cL}{r}\,e^{-r\tau}]$ for every $\tau \in \mathcal{S}$ and $x \in \mathbb{R}_+$, we can rewrite the asset manager's first problem in (2.2) as

$$U(x) = V(x) + x - \frac{cL}{r}, \quad x \in \mathbb{R}_+, \tag{3.6}$$

where

$$V(x) \triangleq \sup_{\tau\in\mathcal{S}} \mathbb{E}_x^\gamma\left[e^{-r\tau}\left((X_\tau - L)^+ - X_\tau + \frac{cL}{r}\right)\right], \quad x \in \mathbb{R}_+. \tag{3.7}$$

is a discounted optimal stopping problem with the terminal reward function

$$h(x) \triangleq (x-L)^+ - x + \frac{cL}{r}, \qquad x \in \mathbb{R}_+. \tag{3.8}$$

We fix the market price γ of jump risk, and the market price β is determined by the drift condition in (3.2). In the remainder, we shall describe the solution of the optimal stopping problem (3.7).

Let $T_1, T_2, \ldots$ be the arrival times of process N. Observe that $X_{T_{n+1}} = (1-y_0)X_{T_{n+1}-}$ and

$$\frac{X_{T_n+t}}{X_{T_n}} = \exp\left\{\left(r - \delta + \lambda\gamma y_0 - \frac{\sigma^2}{2}\right)t + \sigma(B_{T_n+t}^\gamma - B_{T_n}^\gamma)\right\}, \quad 0 \leq t < T_{n+1} - T_n,\ n \geq 1.$$

Le us define for every $n \geq 0$ the standard Brownian motion $B^{\gamma,n}_t := B^{\gamma}_{T_n+t} - B^{\gamma}_{T_n}$, $t \geq 0$ and Poisson process $T^{(n)}_k := T_{n+k} - T_n$, $k \geq 0$, respectively, under $\mathbb{P}^{\gamma}$ and the one-dimensional diffusion process

$$Y^{y,n}_t \triangleq y \exp\left\{\left(r - \delta + \lambda\gamma y_0 - \frac{\sigma^2}{2}\right) t + \sigma B^{\gamma,n}_t\right\},\ t \geq 0, \tag{3.9}$$

which has dynamics

$$Y^{y,n}_0 = y \quad \text{and} \quad \mathrm{d}Y^{y,n}_t = Y^{y,n}_t \left[(r - \delta + \lambda\gamma y_0)\mathrm{d}t + \sigma \mathrm{d}B^{\gamma,n}_t\right], \quad t \geq 0 \tag{3.10}$$

and infinitesimal generator (under $\mathbb{P}^{\gamma}_x$)

$$(\mathcal{A}^{\gamma}_0 f)(y) = \frac{\sigma^2 y^2}{2} f''(y) + (r - \delta + \lambda\gamma y_0) y f'(y) \tag{3.11}$$

acting on twice-continuously differentiable functions $f : \mathbb{R}_+ \mapsto \mathbb{R}$. Then X coincides with $Y^{X_{T_n},n}$ on $[T_n, T_{n+1})$ and jumps to $(1-y_0)Y^{X_{T_n},n}_{T_{n+1}-T_n}$ at time T_{n+1} for every $n \geq 0$; namely,

$$X_{T_n+t} = \begin{cases} Y^{X_{T_n},n}_t, & 0 \leq t < T_{n+1} - T_n, \\ (1-y_0)Y^{X_{T_n},n}_{T_{n+1}-T_n}, & t = T_{n+1} - T_n. \end{cases}$$

For $n = 0$, we shall write $Y^{y,0} \equiv Y^y = y \exp\left\{(r - \delta - \lambda\gamma y_0 - \sigma^2/2)t + \sigma B^{\gamma}_t\right\}$ and $Y^{X_0,0} \equiv Y^{X_0}$.

3.1. **A dynamic programming operator.** Let $\mathcal{S}_B$ denote the collection of all stopping times of the diffusion process Y^{X_0}, or equivalently, Brownian motion B. Let us take any arbitrary but fixed stopping time $\tau \in \mathcal{S}_B$ and consider the following stopping strategy toward the solution of (3.7):

(i) on $\{\tau < T_1\}$ stop at time τ,

(ii) on $\{\tau \geq T_1\}$, update X at time T_1 to $X_{T_1} = (1-y_0)Y^{X_0}_{T_1}$ and continue optimally thereon.

The value of this new strategy is $\mathbb{E}^{\gamma}_x[e^{-r\tau}h(X_\tau)1_{\{\tau<T_1\}} + e^{-rT_1}V(X_{T_1})1_{\{\tau \geq T_1\}}]$ and equals

$$\begin{aligned}\mathbb{E}^{\gamma}_x &\left[e^{-r\tau}h(Y^{X_0}_\tau)1_{\{\tau<T_1\}} + e^{-rT_1}V((1-y_0)Y^{X_0}_{T_1})1_{\{\tau\geq T_1\}}\right] \\ &= \mathbb{E}^{\gamma}_x\left[e^{-(r+\lambda\gamma)\tau}h(Y^{X_0}_\tau) + \int_0^\tau \lambda\gamma e^{-(r+\lambda\gamma)t}V((1-y_0)Y^{X_0}_t)\mathrm{d}t\right].\end{aligned}$$

If for every bounded function $w : \mathbb{R}_+ \mapsto \mathbb{R}_+$ we introduce the operator

$$(Jw)(x) \triangleq \sup_{\tau\in\mathcal{S}_B} \mathbb{E}^{\gamma}_x\left[e^{-(r+\lambda\gamma)\tau}h(Y^{X_0}_\tau) + \int_0^\tau \lambda\gamma e^{-(r+\lambda\gamma)t}w((1-y_0)Y^{X_0}_t)\mathrm{d}t\right], \quad x \geq 0, \tag{3.12}$$

then we expect that the value function $V(\cdot)$ of (3.7) to be the unique fixed point of operator J; namely, $V(\cdot) = (JV)(\cdot)$, and that $V(\cdot)$ is the pointwise limit of the successive approximations

$$\begin{aligned} v_0(x) &\triangleq h(x) = (x-L)^+ - x + \frac{cL}{r}, && x \geq 0, \\ v_n(x) &\triangleq (Jv_{n-1})(x), && x \geq 0,\ n \geq 1. \end{aligned}$$

Lemma 1. *Let $w_1, w_2 : \mathbb{R}_+ \mapsto \mathbb{R}$ be bounded. If $w_1(\cdot) \leq w_2(\cdot)$, then $(Jw_1)(\cdot) \leq (Jw_2)(\cdot)$. If $w(\cdot)$ is nonincreasing convex function such that $h(\cdot) \leq w(\cdot) \leq cL/r$, then $(Jw)(\cdot)$ has the same properties.*

The proof easily follows from the linearity of $y \mapsto Y_t^y$ for every fixed $t \geq 0$ and the definition of the operator J. The next proposition guarantees the existence of unique fixed point of J.

Proposition 2. *For every bounded $w_1, w_2 : \mathbb{R}_+ \mapsto \mathbb{R}$, we have $\|Jw_1 - Jw_2\| \leq \frac{\lambda\gamma}{r+\lambda\gamma}\|w_1 - w_2\|$, where $\|w\| = \sup_{x\in\mathbb{R}_+} |w(x)|$; namely, J acts as a contraction mapping on the bounded functions.*

Proof. Because $w_1(\cdot), w_2(\cdot)$ are bounded, $(Jw_1)(\cdot)$ and $(Jw_2)(\cdot)$ are finite, and for every ε and $x > 0$, there are ε-optimal stopping times $\tau_1(\varepsilon, x)$ and $\tau_2(\varepsilon, x)$, which may depend on ε and x, such that

$$(Jw_i)(x) - \varepsilon \leq \mathbb{E}_x^\gamma\Big[e^{-(r+\lambda\gamma)\tau_i(\varepsilon,x)} h(Y^{X_0}_{\tau_i(\varepsilon,x)}) + \int_0^{\tau_i(\varepsilon,x)} \lambda\gamma e^{-(r+\lambda\gamma)t} w_i((1-y_0)Y_t^{X_0})\mathrm{d}t\Big], \quad i = 1, 2.$$

Therefore, $(Jw_1)(x) - (Jw_2)(x) \leq \varepsilon + \|w_1 - w_2\| \int_0^\infty \lambda\gamma e^{-(r+\lambda\gamma)t}\mathrm{d}t = \varepsilon + \|w_1 - w_2\|\frac{\lambda\gamma}{r+\lambda\gamma}$. Interchanging the roles of $w_1(\cdot)$ and $w_2(\cdot)$ gives $|(Jw_1)(x) - (Jw_2)(x)| \leq \varepsilon + \|w_1 - w_2\|\frac{\lambda\gamma}{r+\lambda\gamma}$ for every $x > 0$ and $\varepsilon > 0$. Taking the supremum of both sides over $x \geq 0$ completes the proof. □

Lemma 3. *The sequence $(v_n)_{n\geq 0}$ of successive approximations is nondecreasing. Therefore, the pointwise limit $v_\infty(x) \triangleq \lim_{n\to\infty} v_n(x)$, $x \geq 0$ exists. Every $v_n(\cdot)$, $n \geq 0$ and $v_\infty(\cdot)$ are nonincreasing, convex, and bounded between $h(\cdot)$ and cL/r.*

Lemma 3 follows from repeated applications of Lemma 1. Proposition 4 below shows that the unique fixed point of J is the uniform limit of successive approximations.

Proposition 4. *The limit $v_\infty(\cdot) = \lim_{n\to\infty} v_n(\cdot) = \sup_{n\geq 0} v_n(\cdot)$ is the unique bounded fixed point of operator J. Moreover, $0 \leq v_\infty(x) - v_n(x) \leq \frac{cL}{r}(\frac{\lambda\gamma}{r+\lambda\gamma})^n$ for every $x \geq 0$.*

Proof. Since $v_n(\cdot) \nearrow v_\infty(\cdot)$ as $n \to \infty$, and every $v_n(\cdot)$ is bounded from below by $\frac{c-r}{r}L$, and $\mathbb{E}^\gamma_\cdot\big[\int_0^\tau e^{-(r+\lambda\gamma)t}\frac{c-r}{r}L\mathrm{d}t\big] < \infty$ for every $\tau \in \mathcal{S}_B$, the monotone convergence theorem implies that

$$\begin{aligned} v_\infty(x) = \sup_{n\geq 0} v_n(x) &= \sup_{\tau\in\mathcal{S}_B} \lim_{n\to\infty} \mathbb{E}_x^\gamma\Big[e^{-(r+\lambda\gamma)\tau}h(Y_\tau^{X_0}) + \int_0^\tau \lambda\gamma e^{-(r+\lambda\gamma)t} v_n((1-y_0)Y_t^{X_0})\mathrm{d}t\Big] \\ &= \sup_{\tau\in\mathcal{S}_B} \mathbb{E}_x^\gamma\Big[e^{-(r+\lambda\gamma)\tau}h(Y_\tau^{X_0}) + \int_0^\tau \lambda\gamma e^{-(r+\lambda\gamma)t} v_\infty((1-y_0)Y_t^{X_0})\mathrm{d}t\Big] = (Jv_\infty)(x). \end{aligned}$$

Thus, $v_\infty(\cdot)$ is the bounded fixed point of contraction mapping J. Lemma 3 implies $0 \leq v_\infty(\cdot) - v_n(\cdot)$, and $\|v_\infty - v_n\| = \|Jv_\infty - Jv_{n-1}\| \leq \frac{\lambda\gamma}{r+\lambda\gamma}\|v_\infty - v_{n-1}\| \leq \ldots \leq (\frac{\lambda\gamma}{r+\lambda\gamma})^n \frac{cL}{r}$ for every $n \geq 1$. □

3.2. **The solution of the optimal stopping problem in (3.12).** We shall next solve the optimal stopping problem Jw in (3.12) for every fixed $w : \mathbb{R}_+ \mapsto \mathbb{R}$ which satisfies the following assumption:

Assumption 5. *Let $w : \mathbb{R}_+ \mapsto \mathbb{R}$ be nonincreasing, convex, bounded between $h(\cdot)$ and cL/r, and $w(+\infty) = \frac{c-r}{r}L$ and $w(0+) = \frac{c}{r}L$.*

We shall calculate the value function $(Jw)(\cdot)$ and explicitly identify an optimal stopping rule. Because $w(\cdot)$ is bounded, we have

$$\mathbb{E}_x^\gamma\left[\int_0^\infty e^{-(r+\lambda\gamma)t}|w((1-y_0)Y_t^{X_0})|\mathrm{d}t\right] \leq \|w\| \int_0^\infty e^{-(r+\lambda\gamma)t}\mathrm{d}t = \frac{\|w\|}{r+\lambda\gamma} < \infty \quad x \geq 0,$$

and for every stopping time $\tau \in \mathcal{S}_B$, the strong Markov property of Y^{X_0} at time τ implies that

$$(3.13)\qquad \begin{aligned}(Hw)(x) &\triangleq \mathbb{E}_x^\gamma\left[\int_0^\infty e^{-(r+\lambda\gamma)t} w((1-y_0)Y_t^{X_0})\mathrm{d}t\right]\\ &= \mathbb{E}_x^\gamma\left[\int_0^\tau e^{-(r+\lambda\gamma)t} w((1-y_0)Y_t^{X_0})\mathrm{d}t\right] + \mathbb{E}_x^\gamma\left[e^{-(r+\lambda\gamma)\tau}(Hw)(Y_\tau^{X_0})\right].\end{aligned}$$

Therefore, $\mathbb{E}_x^\gamma[\int_0^\tau e^{-(r+\lambda\gamma)t} w((1-y_0)Y_t^{X_0})\mathrm{d}t] = (Hw)(x) - \mathbb{E}_x^\gamma[e^{-(r+\lambda\gamma)\tau}(Hw)(Y_\tau^{X_0})]$, and we can write the expected payoff $\mathbb{E}_x^\gamma[e^{-(r+\lambda\gamma)\tau}h(Y_\tau^{X_0}) + \int_0^\tau \lambda\gamma e^{-(r+\lambda\gamma)t} w((1-y_0)Y_t^{X_0})\mathrm{d}t]$ in (3.12) as $\lambda\gamma(Hw)(x) + \mathbb{E}_x^\gamma\left[e^{-(r+\lambda\gamma)\tau}\left\{h - \lambda\gamma(Hw)\right\}(Y_\tau^{X_0})\right]$ for every $\tau \in \mathcal{S}_B$ and $x > 0$. If we define

$$(3.14)\qquad (Gw)(x) \triangleq \sup_{\tau\in\mathcal{S}_B} \mathbb{E}_x^\gamma\left[e^{-(r+\lambda\gamma)\tau}\left\{h - \lambda\gamma(Hw)\right\}(Y_\tau^{X_0})\right], \quad x > 0,$$

then the value function in (3.12) can be calculated by

$$(3.15)\qquad (Jw)(x) = \lambda\gamma(Hw)(x) + (Gw)(x), \quad x > 0.$$

Let us first calculate $(Hw)(\cdot)$. Let $\psi(\cdot)$ and $\varphi(\cdot)$ be, respectively, the increasing and decreasing solutions of the second order ordinary differential equation $(\mathcal{A}_0 f)(y) - (r+\lambda\gamma)f(y) = 0$, $y > 0$ with boundary conditions, respectively, $\psi(0+) = 0$ and $\varphi(+\infty) = 0$, where $\mathcal{A}_0$ is the infinitesimal generator in (3.11) of diffusion process $Y^{X_0} \equiv Y^{X_0,0}$. One can easily check that

$$(3.16)\qquad \psi(y) = y^{\alpha_1} \quad \text{and} \quad \varphi(y) = y^{\alpha_0} \quad \text{for every } y > 0,$$

with the Wronskian

$$(3.17)\qquad W(y) = \psi'(y)\varphi(y) - \psi(y)\varphi'(y) = (\alpha_0+\alpha_1)y^{\alpha_0+\alpha_1-1}, \quad y > 0,$$

where $\alpha_0 < \alpha_1$ are the roots of the characteristic function $g(\alpha) = \frac{\sigma^2}{2}\alpha(\alpha-1) + (r - \delta + \lambda\gamma y_0)\alpha - (r+\lambda\gamma)$ of the above ordinary differential equation. Because both $g(0) < 0$ and $g(1) < 0$, we have

$$\alpha_0 < 0 < 1 < \alpha_1.$$

Let us denote the hitting and exit times of diffusion process Y^{X_0}, respectively, by

$$\begin{aligned}\tau_a &\triangleq \inf\{t \geq 0;\ Y_t^{X_0} = a\}, && a > 0,\\ \tau_{ab} &\triangleq \inf\{t \geq 0;\ Y_t^{X_0} \notin (a,b)\}, && 0 < a < b < \infty,\end{aligned}$$

and define operator

$$(3.18)\qquad (H_{ab}w)(x) \triangleq \mathbb{E}_x^\gamma\left[\int_0^{\tau_{ab}} e^{-(r+\lambda\gamma)t} w((1-y_0)Y_t^{X_0})\mathrm{d}t + 1_{\{\tau_{ab}<\infty\}} e^{-(r+\lambda\gamma)\tau_{ab}} h(Y_{\tau_{ab}}^{X_0})\right]$$

and

$$\psi_a(y) \triangleq \psi(y) - \frac{\psi(a)}{\varphi(a)}\varphi(y) \quad \text{and} \quad \varphi_b(y) \triangleq \varphi(y) - \frac{\varphi(b)}{\psi(b)}\psi(y) \quad \text{for every } y > 0,$$

which are, respectively, the increasing and decreasing solutions of $(\mathcal{A}_0 f)(y) - (r+\lambda\gamma)f(y) = 0$, $a < y < b$ with boundary conditions, respectively, $f(a) = 0$ and $f(b) = 0$. The Wronskian of $\psi_a(\cdot)$ and $\varphi_a(\cdot)$ becomes

$$W_{ab}(y) = \psi_a'(y)\varphi_b(y) - \psi_a(y)\varphi_b'(y) = \left[1 - \frac{\psi(a)}{\varphi(a)}\frac{\varphi(b)}{\varphi(b)}\right] W(y), \quad y > 0 \tag{3.19}$$

in terms of the Wronskian $W(\cdot)$ in (3.17) of $\psi(\cdot)$ and $\varphi(\cdot)$.

Lemma 6. *We have*

(i) $\mathbb{E}_x^\gamma \left[e^{-(r+\lambda\gamma)\tau_a} 1_{\{\tau_a < \tau_b\}}\right] = \frac{\varphi_b(x)}{\varphi_b(a)}$ *for every* $0 < a \le x \le b < \infty$.

(ii) $\mathbb{E}_x^\gamma \left[e^{-(r+\lambda\gamma)\tau_b} 1_{\{\tau_a > \tau_b\}}\right] = \frac{\psi_b(x)}{\psi_b(a)}$ *for every* $0 < a \le x \le b < \infty$.

(iii) $\mathbb{E}_x^\gamma \left[e^{-(r+\lambda\gamma)\tau_{ab}} h(Y_{\tau_{ab}}^{X_0}) 1_{\{\tau_{ab} < \infty\}}\right] = \frac{\varphi_b(x)}{\varphi_b(a)} h(a) + \frac{\psi_a(x)}{\psi_a(b)} h(b)$ *for every* $0 < a \le x \le b < \infty$.

All three expectations are twice continuously differentiable on (a, b) and unique such solution of the ordinary differential equation $(\mathcal{A}_0 f)(y) - (r+\lambda\gamma)f(y) = 0$, $y \in (a, b)$ subject to boundary conditions (i) $f(a) = 1$, $f(b) = 0$, (ii) $f(a) = 0$, $f(b) = 1$, (iii) $f(a) = h(a)$, $f(b) = h(b)$, respectively.

Lemma 7. *For every bounded function $g : \mathbb{R}_+ \mapsto \mathbb{R}$ and $0 < a \le x \le b < \infty$, we have*

$$\begin{aligned}
&\mathbb{E}_x^\gamma \left[\int_0^{\tau_{ab}} e^{-(r+\lambda\gamma)t} g(Y_t^{X_0}) \mathrm{d}t + 1_{\{\tau_{ab} < \infty\}} e^{-(r+\lambda\gamma)\tau_{ab}} h(Y_{\tau_{ab}}^{X_0})\right] \\
&\quad = \int_a^b \frac{2\psi_a(x \wedge \xi)\varphi_b(x \vee \xi) g(\xi)}{p^2(\xi) W_{ab}(\xi)} \mathrm{d}\xi + \frac{\varphi_b(x)}{\varphi_b(a)} h(a) + \frac{\psi_a(x)}{\psi_a(b)} h(b) \\
&\quad = \varphi_b(x) \int_a^x \frac{2\psi_a(\xi) g(\xi)}{p^2(\xi) W_{ab}(\xi)} \mathrm{d}\xi + \psi_a(x) \int_x^b \frac{2\varphi_b(\xi) g(\xi)}{p^2(\xi) W_{ab}(\xi)} \mathrm{d}\xi + \frac{\varphi_b(x)}{\varphi_b(a)} h(a) + \frac{\psi_a(x)}{\psi_a(b)} h(b),
\end{aligned} \tag{3.20}$$

which is twice-continuously differentiable on (a, b) and uniquely solves the boundary value problem $(\mathcal{A}_0 f)(y) - (r+\lambda\gamma)f(y) + g(y) = 0$, $a < y < b$ with $f(a) = h(a)$ and $f(b) = h(b)$, where

$$p(\xi) = \sigma\xi \quad \textit{and} \quad q(\xi) = (r - \delta + \lambda\gamma y_0)\xi$$

are the diffusion and drift coefficients of diffusion Y in (3.10).

Corollary 8. *We have*

$$\begin{aligned}
(H_{ab} w)(x) &= \varphi_b(x) \int_a^x \frac{2\psi_a(\xi) w((1-y_0)\xi)}{p^2(\xi) W_{ab}(\xi)} \mathrm{d}\xi + \psi_a(x) \int_x^b \frac{2\varphi_b(\xi) w((1-y_0)\xi)}{p^2(\xi) W_{ab}(\xi)} \mathrm{d}\xi \\
&\quad + \frac{\varphi_b(x)}{\varphi_b(a)} h(a) + \frac{\psi_a(x)}{\psi_a(b)} h(b), \qquad 0 < a \le x \le b < \infty,
\end{aligned}$$

which is twice-continuously differentiable on (a, b) and uniquely solves the boundary value problem $(\mathcal{A}_0 f)(x) - (r+\lambda\gamma)f(x) + w((1-y_0)x) = 0$, $a < x < b$ with $f(a) = h(a)$ and $f(b) = h(b)$.

The proofs of Lemmas 6 and 7 can be checked by direct calculation and Itô's lemma; see also Karlin and Taylor [15, Chapter 15], and Corollary 8 immediately follows from Lemma 6 and 7. Finally, Lemma 9 follows from Corollary 8 by passing to limit as $a \downarrow 0$ and $b \uparrow \infty$ because 0 and ∞ are natural boundaries of diffusion Y^{X_0}.

Lemma 9. *For every $x>0$, we have*

$$(Hw)(x) \triangleq \mathbb{E}_x^{\gamma}\left[\int_0^\infty e^{-(r+\lambda\gamma)t} w((1-y_0)Y_t^{X_0})\mathrm{d}t\right] = \lim_{a\downarrow 0, b\uparrow\infty} (H_{ab}w)(x)$$
$$= \varphi(x)\int_0^x \frac{2\psi(\xi)w((1-y_0)\xi)}{p^2(\xi)W(\xi)}\mathrm{d}\xi + \psi(x)\int_x^\infty \frac{2\varphi(\xi)w((1-y_0)\xi)}{p^2(\xi)W(\xi)}\mathrm{d}\xi,$$

which is twice-continuously differentiable on $\mathbb{R}_+$ and satisfies the ordinary differential equation $(\mathcal{A}_0 f)(x) - (r+\lambda\gamma)f(x) + w((1-y_0)x) = 0$.

Using the potential theoretic direct methods of Dayanik and Karatzas [7] and Dayanik [6], we shall now solve the optimal stopping problem $(Gw)(\cdot)$ (3.14) with payoff function $(h-\lambda\gamma(Hw))(x) =$

$$(x-L)^+ - x + \frac{cL}{r} - \lambda\gamma\Big[\varphi(x)\int_0^x \frac{2\psi(\xi)w((1-y_0)\xi)}{p^2(\xi)W(\xi)}\mathrm{d}\xi + \psi(x)\int_x^\infty \frac{2\varphi(\xi)w((1-y_0)\xi)}{p^2(\xi)W(\xi)}\mathrm{d}\xi\Big]$$
$$= (x-L)^+ - x + \frac{cL}{r} - \frac{2\lambda\gamma}{\sigma^2(\alpha_1-\alpha_0)}\Big[x^{\alpha_0}\int_0^x \xi^{-1-\alpha_0}w((1-y_0)\xi)\mathrm{d}\xi + x^{\alpha_1}\int_x^\infty \xi^{-1-\alpha_1}w((1-y_0)\xi)\mathrm{d}\xi\Big],$$

where $\psi(x) = x^{\alpha_1}$, $\varphi(x) = x^{\alpha_0}$, $p^2(\xi) = \sigma^2\xi^2$, $W(\xi) = \psi'(\xi)\varphi(\xi) - \psi(\xi)\varphi'(\xi) = (\alpha_1-\alpha_0)\xi^{\alpha_0+\alpha_1-1}$.

We observe that $0 \le (Hw)(x) = \mathbb{E}_x^{\gamma}[\int_0^\infty e^{-(r+\lambda\gamma)t}w((1-y_0)Y_t^{X_0})\mathrm{d}t] \le \frac{cL}{r}\int_0^\infty e^{-(r+\lambda\gamma)t}\mathrm{d}t = \frac{cL}{r(r+\lambda\gamma)} < \infty$. Hence, $(h-\lambda\gamma(Hw))(\cdot)$ is bounded, and because $\psi(+\infty) = \varphi(0+) = +\infty$, we have

$$\limsup_{x\downarrow 0} \frac{(h-\lambda\gamma(Hw))^+(x)}{\varphi(x)} = 0 \qquad \text{and} \qquad \limsup_{x\uparrow\infty} \frac{(h-\lambda\gamma(Hw))^+(x)}{\psi(x)} = 0.$$

By Propositions 5.10 and 5.13 of Dayanik and Karatzas [7], value function $(Gw)(\cdot)$ is finite; the set

$$\Gamma[w] \triangleq \{x>0;\ (Gw)(x) = (h-\lambda\gamma(Hw))(x)\} = \{x>0;\ (Jw)(x) = h(x)\} \tag{3.21}$$

is the optimal stopping region, and

$$\tau[w] \triangleq \inf\{t\ge 0;\ Y_t^{X_0} \in \Gamma[w]\} \tag{3.22}$$

is an optimal stopping time for (3.14)—and for (3.12) because of (3.15). According to Proposition 5.12 of Dayanik and Karatzas [7], we have

$$(Gw)(x) = \varphi(x)(Mw)(F(x)),\ x\ge 0, \quad \text{and} \quad \Gamma[w] = F^{-1}(\{\zeta>0;\ (Mw)(\zeta) = (Lw)(\zeta)\}),$$

where $F(x) \triangleq \psi(x)/\varphi(x)$ and $(Mw)(\cdot)$ is the smallest nonnegative concave majorant on $\mathbb{R}_+$ of

$$(Lw)(\zeta) \triangleq \begin{cases} \dfrac{h-\lambda\gamma(Hw)}{\varphi}\circ F^{-1}(\zeta), & \zeta>0, \\ 0, & \zeta=0. \end{cases} \tag{3.23}$$

To describe explicitly the form of the smallest nonnegative concave majorant $(Mw)(\cdot)$ of $(Lw)(\cdot)$, we shall firstly identify a few useful properties of function $(Lw)(\cdot)$. Because $Y^{X_0} \equiv X_0 Y^1$ by (3.9) and $w(\cdot)$ is bounded, the bounded convergence theorem implies that

$$\lim_{x\uparrow\infty}(Hw)(x) = \mathbb{E}_1^{\gamma}\left[\int_0^\infty e^{-(r+\lambda\gamma)t}\lim_{x\uparrow\infty} w((1-y_0)xY_t^1)\mathrm{d}t\right] = \frac{w(+\infty)}{r+\lambda\gamma} \le \frac{cL}{r},$$

and $\lim_{x\uparrow\infty}(h-\lambda\gamma(Hw))(x) = \lim_{x\uparrow\infty}((x-L)^+ - x + \frac{cL}{r} - \lambda\gamma(Hw)(x)) \geq \frac{c-r}{r+\lambda\gamma}L > 0$. Therefore,

$$(Lw)(+\infty) = \lim_{x\uparrow\infty} \frac{(h-\lambda\gamma(Hw))(x)}{\varphi(x)} = +\infty. \tag{3.24}$$

Note also that

$$(Lw)'(\zeta) = \frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{h-\lambda\gamma(Hw)}{\varphi}\circ F^{-1}(\zeta)\right) = \left[\frac{1}{F'}\left(\frac{h-\lambda\gamma(Hw)}{\varphi}\right)'\right]\circ F^{-1}(\zeta).$$

Because $F(\cdot)$ is strictly increasing, we have $F'>0$. Because $w(\cdot)$ is nonincreasing, the mapping $x\mapsto \mathbb{E}^\gamma_x[\int_0^\infty e^{-(r+\lambda\gamma)t}w((1-y_0)Y_t^{X_0})\mathrm{d}t] = \mathbb{E}^\gamma_x[\int_0^\infty e^{-(r+\lambda\gamma)t}w((1-y_0)X_0Y_t^1)\mathrm{d}t]$ is decreasing. Then for $x>L$, because $h(\cdot)\equiv cL/r$ is constant, the mapping $x\mapsto(\frac{h-\lambda\gamma(Hw)}{\varphi})(x)$ is increasing.

For every $0<x<L$, we can calculate explicitly that $[\frac{1}{F'}(\frac{h-\lambda\gamma(Hw)}{\varphi})'](x) =$

$$\frac{x^{-\alpha_1}}{\alpha_1-\alpha_0}\left[(-\alpha_0)\frac{cL}{r} - (1-\alpha_0)x - \frac{\lambda\gamma(-\alpha_0)\alpha_1}{r+\lambda\gamma}x^{\alpha_1}\int_x^\infty \xi^{-1-\alpha_1}w((1-y_0)\xi)\mathrm{d}\xi\right],$$

and because $\lim_{x\downarrow 0} x^{\alpha_1}\int_x^\infty \xi^{-1-\alpha_1}w((1-y_0)\xi)\mathrm{d}\xi = \frac{w(0+)}{\alpha_1}$ and $\alpha_1>1$, we have

$$\lim_{x\downarrow 0}\left[\frac{1}{F'}\left(\frac{h-\lambda\gamma(Hw)}{\varphi}\right)'\right](x) = +\infty.$$

Let us also study the sign of the second derivative $(Lw)''(\cdot)$. For every $x\neq L$, Dayanik and Karatzas [7, page 192] show that

$$(Lw)''(F(x)) = \frac{2\varphi(x)}{p^2(x)W(x)F'(x)}(\mathcal{A}_0-(r+\lambda\gamma))(k-\lambda\gamma(Hw))(x) \tag{3.25}$$

and $\varphi(\cdot), p^2(\cdot), W(\cdot), F'(\cdot)$ are positive. Therefore,

$$\mathrm{sgn}[(Lw)''(F(x))] = \mathrm{sgn}[(\mathcal{A}_0-(r+\lambda\gamma))(h-\lambda\gamma(Hw))(x)].$$

Recall from Lemma 9 that $(\mathcal{A}_0-(r+\lambda\gamma))(Hw)(x) = -w((1-y_0)x)$ and because $h(x) = (-x+\frac{cL}{r})1_{\{x<L\}} + \frac{(c-r)L}{r}1_{\{x>L\}}$, we have $(\mathcal{A}_0-(r+\lambda\gamma))(h-\lambda\gamma(Hw))(x) =$

$$\left[\lambda\gamma(1-y_0)x - (r+\lambda\gamma)\frac{cL}{r} + \lambda\gamma w((1-y_0)x)\right]1_{\{x\leq L\}} \\ + \left[\lambda\gamma w((1-y_0)x) - (r+\lambda\gamma)\frac{(c-r)L}{r}\right]1_{\{x>L\}}.$$

Note that $\lim_{x\downarrow 0}(\mathcal{A}_0-(r+\lambda\gamma))(h-\lambda\gamma(Hw))(x) = -cL<0$ and $\lim_{x\uparrow\infty}(\mathcal{A}_0-(r+\lambda\gamma))(h-\lambda\gamma(Hw))(x) = -(c-r)L<0$. Note also that $x\mapsto(\mathcal{A}_0-(r+\lambda\gamma))(h-\lambda\gamma(Hw))(x)$ is convex and continuous on $x\in(0,L)$ and $x\in(L,\infty)$. Therefore, $(\mathcal{A}_0-(r+\lambda\gamma))(h-\lambda\gamma(Hw))(x)$ is strictly negative in some open neighborhoods of 0 and $+\infty$, and in the complement of their unions, whose closure contains L if it is not empty, it is nonnegative. Therefore, (3.25) implies that $(Lw)(\zeta)$ is strictly concave in some neighborhood of $\zeta=0$ and $\zeta=\infty$, and in the complement of their unions, whose closure contains $F(L)$ if it is not empty, this function is convex. Earlier we also showed that $\zeta\mapsto(Lw)(\zeta)$ is increasing at every $\zeta>F(L)$ and $(Lw)(+\infty) = (Lw)'(0+) = +\infty$. Moreover,

$$(Lw)'(F(L)-) - (Lw)'(F(L)+) = -\frac{L^{1-\alpha_1}}{\alpha_1-\alpha_0} < 0;$$

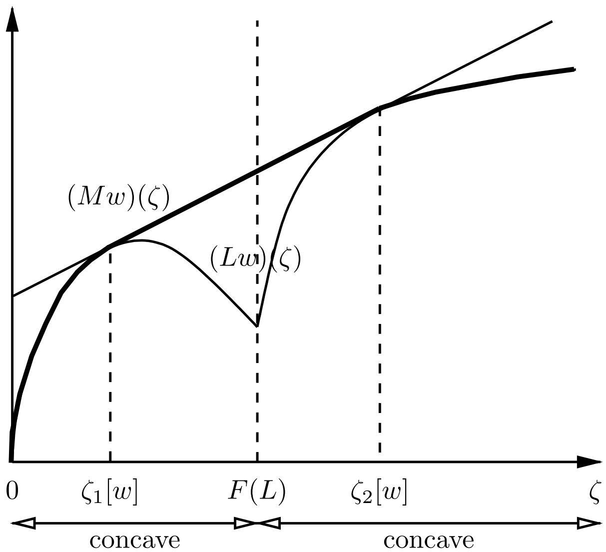

FIGURE 1. The sketches of two possible forms of $(Lw)(\cdot)$ and their smallest nonnegative concave majorants $(Mw)(\cdot)$.

namely, $(Lw)'(F(L)-) < (Lw)'(F(L)+)$. Two possible forms of $\zeta \mapsto (Lw)(\zeta)$ and their smallest nonnegative concave majorants $\zeta \mapsto (Mw)(\zeta)$ are depicted by two pictures of Figure 1.

The properties of the mapping $\zeta \mapsto (Lw)(\zeta)$ imply that there are unique numbers $0 < \zeta_1[w] < F(L) < \zeta_2[w] < \infty$ such that

$$(Lw)'(\zeta_1[w]) = \frac{(Lw)(\zeta_2[w]) - (Lw)(\zeta_1[w])}{\zeta_2[w] - \zeta_1[w]} = (Lw)'(\zeta_2[w]),$$

and the smallest nonnegative concave majorant $(Mw)(\cdot)$ of $(Lw)(\cdot)$ on $(0, \zeta_1[w]] \cup [\zeta_2[w], \infty)$ coincides with $(Lw)(\cdot)$, and on $(\zeta_1[w], \zeta_2[w])$ with the straight-line that majorizes $(Lw)(\cdot)$ everywhere on $\mathbb{R}_+$ and is tangent to $(Lw)(\cdot)$ exactly at $\zeta = \zeta_1[w]$ and $\zeta_2[w]$; see Figure 1. More precisely,

$$(Mw)(\zeta) = \begin{cases} (Lw)(\zeta), & \zeta \in (0, \zeta_1[w]] \cup [\zeta_2[w], \infty), \\ \dfrac{\zeta_2[w] - \zeta}{\zeta_2[w] - \zeta_1[w]}(Lw)(\zeta_1[w]) + \dfrac{\zeta - \zeta_1[w]}{\zeta_2[w] - \zeta_1[w]}(Lw)(\zeta_2[w]), & \zeta \in (\zeta_1[w], \zeta_2[w]). \end{cases}$$

Let us define $x_1[w] \triangleq F^{-1}(\zeta_1[w])$ and $x_2[w] \triangleq F^{-1}(\zeta_2[w])$. Then by Proposition 5.12 of Dayanik and Karatzas [7], the value function of the optimal stopping problem in (3.14) equals

$$\begin{aligned} (3.26) \quad (Gw)(x) &= \varphi(x)(Mw)(F(x)) \\ &= \begin{cases} (h - \lambda\gamma(Hw))(x), & x \in (0, x_1[w]] \cup [x_2[w], \infty), \\ \dfrac{(x_2[w])^{\alpha_1-\alpha_0} - x^{\alpha_1-\alpha_0}}{(x_2[w])^{\alpha_1-\alpha_0} - (x_1[w])^{\alpha_1-\alpha_0}}(h - \lambda\gamma(Hw))(x_1[w]) & \\ + \dfrac{x^{\alpha_1-\alpha_0} - (x_1[w])^{\alpha_1-\alpha_0}}{(x_2[w])^{\alpha_1-\alpha_0} - (x_1[w])^{\alpha_1-\alpha_0}}(h - \lambda\gamma(Hw))(x_2[w]), & x \in (x_1[w], x_2[w]). \end{cases} \end{aligned}$$

The optimal stopping region in (3.21) becomes $\Gamma[w] = \{x > 0;\ (Gw)(x) = (h - \lambda\gamma)(Hw)(x)\} = (0, x_1[w]] \cup [x_2[w], \infty)$, and the optimal stopping time in (3.22) becomes

$$\tau[w] = \inf\{t \geq 0;\ Y_t^{X_0} \in (0, x_1[w]] \cup [x_2[w], \infty)\}.$$

Proposition 10. *The value function $x \mapsto (Gw)(\cdot)$ of (3.14) is continuously differentiable on $\mathbb{R}_+$ and twice-continuously differentiable on $\mathbb{R}_+ \setminus \{x_1[w], x_2[w]\}$. Moreover, $(Gw)(\cdot)$ satisfies*

$$\begin{aligned}
&(i) && (\mathcal{A}_0 - (r + \lambda\gamma))(Gw)(x) = 0, && x \in (x_1[w], x_2[w]),\\
&(ii) && (Gw)(x) > h(x) - \lambda\gamma(Hw)(x), && x \in (x_1[w], x_2[w]),\\
&(iii) && (\mathcal{A}_0 - (r + \lambda\gamma))(Gw)(x) < 0, && x \in (0, x_1[w]) \cup (x_2[w], \infty),\\
&(iv) && (Gw)(x) = h(x) - \lambda\gamma(Hw)(x), && x \in (0, x_1[w]] \cup [x_2[w], \infty).
\end{aligned}$$

The differentiability of $(Gw)(\cdot)$ is clear from (3.26). The variational inequalities can be verified directly. For (iii) note that, if $x \in (0, x_1[w]) \cup (x_2[w], \infty)$, then $\mathrm{sgn}\{(\mathcal{A}_0 - (r + \lambda\gamma))(Gw)(x)\} = \mathrm{sgn}\{(\mathcal{A}_0 - (r + \lambda\gamma))(h - \lambda\gamma(Hw))(x)\} = \mathrm{sgn}\{(Lw)''(F(x))\} < 0$.

Because $(Hw)(\cdot)$ is twice-continuously differentiable and $(\mathcal{A}_0 - (r+\lambda\gamma)(Hw))(x) = -w((1-y_0)x)$ for every $x > 0$ by Proposition 9, Proposition 10 and (3.15) lead directly to the next proposition.

Proposition 11. *The value function $x \mapsto (Jw)(\cdot)$ of (3.12) is continuously differentiable on $\mathbb{R}_+$ and twice-continuously differentiable on $\mathbb{R}_+ \setminus \{x_1[w], x_2[w]\}$. Moreover, $(Jw)(\cdot)$ satisfies*

$$\begin{aligned}
&(i) && (\mathcal{A}_0 - (r + \lambda\gamma))(Jw)(x) + \lambda\gamma w((1 - y_0)x) = 0, && x \in (x_1[w], x_2[w]),\\
&(ii) && (Jw)(x) > h(x), && x \in (x_1[w], x_2[w]),\\
&(iii) && (\mathcal{A}_0 - (r + \lambda\gamma))(Jw)(x) + \lambda\gamma w((1 - y_0)x) < 0, && x \in (0, x_1[w]) \cup (x_2[w], \infty),\\
&(iv) && (Jw)(x) = h(x), && x \in (0, x_1[w]] \cup [x_2[w], \infty).
\end{aligned}$$

By Lemma 3, every $v_n(\cdot)$, $n \geq 0$ and $v_\infty(\cdot)$ are nonincreasing, convex, and bounded between $h(\cdot)$ and cL/r. Moreover, by using induction on n, we can easily show that $v_n(0+) = cL/r$ and $v_n(+\infty) = (c - r)L/r$ for every $n \in \{0, 1, \ldots, \infty\}$. Therefore, Proposition 11, applied to $w = v_\infty$, and Proposition 4 directly lead to the next theorem.

Theorem 12. *The function $x \mapsto v_\infty(x) = (Jv_\infty)(x)$ is continuously differentiable on $\mathbb{R}_+$ and twice-continuously differentiable on $\mathbb{R}_+ \setminus \{x_1[v_\infty], x_2[v_\infty]\}$ and satisfies the variational inequalities*

$$\begin{aligned}
&(i) && (\mathcal{A}_0 - (r + \lambda\gamma))v_\infty(x) + \lambda\gamma v_\infty((1 - y_0)x) = 0, && x \in (x_1[v_\infty], x_2[v_\infty]),\\
&(ii) && v_\infty(x) > h(x), && x \in (x_1[v_\infty], x_2[v_\infty]),\\
&(iii) && (\mathcal{A}_0 - (r + \lambda\gamma))v_\infty(x) + \lambda\gamma v_\infty((1 - y_0)x) < 0, && x \in (0, x_1[v_\infty]) \cup (x_2[v_\infty], \infty),\\
&(iv) && v_\infty(x) = h(x), && x \in (0, x_1[v_\infty]] \cup [x_2[v_\infty], \infty),
\end{aligned}$$

which can be expressed in terms of the generator $\mathcal{A}^\gamma$ in (3.5) of the jump-diffusion process X as

$$\begin{array}{lll} (i)' & (\mathcal{A}^\gamma - r)v_\infty(x) = 0, & x \in (x_1[v_\infty], x_2[v_\infty]), \\ (ii)' & v_\infty(x) > h(x), & x \in (x_1[v_\infty], x_2[v_\infty]), \\ (iii)' & (\mathcal{A}^\gamma - r)v_\infty(x) < 0, & x \in (0, x_1[v_\infty]) \cup (x_2[v_\infty], \infty), \\ (iv)' & v_\infty(x) = h(x), & x \in (0, x_1[v_\infty]] \cup [x_2[v_\infty], \infty). \end{array}$$

The next theorem identifies the value function and an optimal stopping time for the optimal stopping problem in (3.7). For every $w : \mathbb{R}_+ \mapsto \mathbb{R}$ satisfying Assumption 5 let us denote by $\widetilde{\tau}[w]$ the stopping time of jump-diffusion process X defined by

$$\widetilde{\tau}[w] \triangleq \inf\{t \geq 0;\ X_t \in (0, x_1[w]] \cup [x_2[w], \infty)\}.$$

Theorem 13. *For every $x \in \mathbb{R}_+$, we have $V(x) = v_\infty(x) = \mathbb{E}_x^\gamma\left[e^{-r\widetilde{\tau}[v_\infty]}h(X_{\widetilde{\tau}[v_\infty]})\right]$, and $\widetilde{\tau}[v_\infty]$ is an optimal stopping time for (3.7).*

Proof. Let $\widetilde{\tau}_{ab} = \inf\{t \geq 0;\ X_t \in (0, a] \cup [b, \infty)\}$ for every $0 < a < b < \infty$. By Itô's rule, we have

$$\begin{aligned} e^{-r(t\wedge\tau\wedge\widetilde{\tau}_{ab})}v_\infty(X_{t\wedge\tau\wedge\widetilde{\tau}_{ab}}) &= v_\infty(X_0) + \int_0^{t\wedge\tau\wedge\widetilde{\tau}_{ab}} e^{-rs}(\mathcal{A}^\gamma - r)v_\infty(X_s)\mathrm{d}s \\ &+ \int_0^{t\wedge\tau\wedge\widetilde{\tau}_{ab}} e^{-rs}v_\infty(X_s)\sigma X_s \mathrm{d}B_s^\gamma + \int_0^{t\wedge\tau\wedge\widetilde{\tau}_{ab}} e^{-rs}[v_\infty((1-y_\infty)X_{s-}) - v_\infty(X_{s-})](\mathrm{d}N_s - \lambda\gamma\mathrm{d}s) \end{aligned}$$

for every $t \geq 0$, $\tau \in \mathcal{S}$, and $0 < a < b < \infty$. Because $v_\infty(\cdot)$ and $v'_\infty(\cdot)$ are continuous and bounded on every compact subinterval of $(0, \infty)$, both stochastic integrals are square-integrable martingales, and taking expectations of both sides gives

$$\mathbb{E}_x^\gamma[e^{-r(t\wedge\tau\wedge\widetilde{\tau}_{ab})}v_\infty(X_{t\wedge\tau\wedge\widetilde{\tau}_{ab}})] = v_\infty(x) + \mathbb{E}_x^\gamma\Big[\int_0^{t\wedge\tau\wedge\widetilde{\tau}_{ab}} e^{-rs}(\mathcal{A}^\gamma - r)v_\infty(X_s)\mathrm{d}s\Big]. \tag{3.27}$$

Because $(\mathcal{A}^\gamma - r)v_\infty(\cdot) \leq 0$ and $v_\infty(\cdot) \geq h(\cdot)$ by the variational inequalities of Theorem 12, we have $\mathbb{E}^\gamma[e^{-r(t\wedge\tau\wedge\widetilde{\tau}_{ab})}v_\infty(X_{t\wedge\tau\wedge\widetilde{\tau}_{ab}})] \leq v_\infty(x)$ for every $t \geq 0$, $\tau \in \mathcal{S}$, and $0 < a < b < \infty$. Because $\lim_{a\downarrow 0, b\uparrow\infty}\widetilde{\tau}_{ab} = \infty$ a.s. and $h(\cdot)$ is continuous and bounded, we can take limits of both sides as $t \uparrow \infty$, $a \downarrow 0$, $b \uparrow \infty$ and use the bounded convergence theorem to get $\mathbb{E}^\gamma[e^{-r\tau}v_\infty(X_\tau)] \leq v_\infty(x)$ for every $\tau \in \mathcal{S}$. Taking supremum over all $\tau \in \mathcal{S}$ gives $V(x) = \sup_{\tau\in\mathcal{S}}\mathbb{E}^\gamma[e^{-r\tau}v_\infty(X_\tau)] \leq v_\infty(x)$.

In order to show the reverse inequality, we replace in (3.27) τ and $\widetilde{\tau}_{ab}$ with $\widetilde{\tau}[v_\infty]$. Because $(\mathcal{A}^\gamma - r)v_\infty(x) = 0$ for every $x \in (x_1[v_\infty], x_2[v_\infty])$ by Theorem 12 $(i)'$, we have

$$\mathbb{E}_x^\gamma[e^{-r(t\wedge\widetilde{\tau}[v_\infty])}v_\infty(X_{t\wedge\widetilde{\tau}[v_\infty]})] = v_\infty(x) + \mathbb{E}_x^\gamma\Big[\int_0^{t\wedge\widetilde{\tau}[v_\infty]} e^{-rs}(\mathcal{A}^\gamma - r)v_\infty(X_s)\mathrm{d}s\Big] = v_\infty(x)$$

for every $t \geq 0$. Since v_∞ is bounded and continuous, taking limits as $t \uparrow \infty$ and the bounded convergence theorem gives $v_\infty(x) = \mathbb{E}_x^\gamma[e^{-r\widetilde{\tau}[v_\infty]}v_\infty(X_{\widetilde{\tau}[v_\infty]})] = \mathbb{E}_x^\gamma[e^{-r\widetilde{\tau}[v_\infty]}h(X_{\widetilde{\tau}[v_\infty]})] \leq V(x)$ by Theorem 12 $(iv)'$, which completes the proof. □

Proposition 14. *The optimal stopping regions* $\Gamma[v_n] = \{x > 0;\ (Jv_n)(x) \le h(x)\} = (0, x_1[v_n]] \cup [x_2[v_n], \infty)$, $n \in \{0, 1, \ldots, \infty\}$ *are decreasing; namely,* $\Gamma[v_0] \supseteq \Gamma[v_1] \supseteq \ldots \supseteq \Gamma[v_\infty]$, *and* $0 < x_1[v_\infty] \le \ldots \le x_1[v_1] \le x_1[v_0] \le L \le x_2[v_0] \le x_2[v_1] \le \ldots \le x_2[v_\infty] < \infty$. *Moreover,* $x_1[v_\infty] = \lim_{n\to\infty} x_1[v_n]$ *and* $x_2[v_\infty] = \lim_{n\to\infty} x_2[v_n]$.

The proof follows from the monotonicity of operator J and that $v_n(x) \uparrow v_\infty(x)$ as $n \to \infty$ uniformly in $x > 0$. The next proposition and its corollary identify the optimal expected reward and nearly optimal stopping strategies for the asset manager in the first problem.

Proposition 15. *For all* $n \ge 0$, *we have* $v_\infty(x) \le \mathbb{E}^\gamma_x[e^{-r\widetilde{\tau}[v_n]} h(X_{\widetilde{\tau}[v_n]})] + \frac{cL}{r}(\frac{\lambda}{r+\lambda\gamma})^{n+1}$. *Hence, for every* $\varepsilon > 0$ *and* $n \ge 0$ *such that* $\frac{cL}{r}(\frac{\lambda}{r+\lambda\gamma})^{n+1} \le \varepsilon$, *the stopping time* $\widetilde{\tau}[v_n]$ *is* ε*-optimal for (3.7).*

Proof. Recall that $\tilde{\tau}[v_n] = \inf\{t \ge 0;\ X_t \in \Gamma[v_n]\} = \inf\{t \ge 0;\ X_t \in (0, x_1[v_n]] \cup [x_2[v_n], \infty)$. If we replace τ and $\widetilde{\tau}_{ab}$ in (3.27) with $\widetilde{\tau}[v_n]$, then for every $t \ge 0$ we obtain

$$\mathbb{E}^\gamma_x[e^{-r(t\wedge\widetilde{\tau}[v_n])} v_\infty(X_{t\wedge\widetilde{\tau}[v_n]})] = v_\infty(x) + \mathbb{E}^\gamma_x\Big[\int_0^{t\wedge\widetilde{\tau}[v_n]} e^{-rs}(\mathcal{A}^\gamma - r)v_\infty(X_s)\mathrm{d}s\Big] = v_\infty(x),$$

because, for every $0 < t < \widetilde{\tau}[v_n]$ we have $X_t \in (x_1[v_n], x_2[v_n]) \subseteq (x_1[v_\infty], x_2[v_\infty])$, at every element x of which $(\mathcal{A}^\gamma - r)v_\infty(x)$ equals 0 according to 12 $(i)'$. Because $v_\infty(\cdot)$ is continuous and bounded, taking limits as $t \uparrow \infty$ and the bounded convergence theorem give $v_\infty(x) = \mathbb{E}^\gamma_x[e^{-r\widetilde{\tau}[v_n]} v_\infty(X_{\widetilde{\tau}[v_n]})]$. By Proposition 4,

$$\begin{aligned} v_\infty(x) \le \mathbb{E}^\gamma_x\Big[e^{-r\widetilde{\tau}[v_n]}\Big(v_{n+1}(X_{\widetilde{\tau}[v_n]}) + \frac{cL}{r}\Big(\frac{\lambda\gamma}{r+\lambda\gamma}\Big)^{n+1}\Big)\Big] \le \mathbb{E}^\gamma_x\Big[e^{-r\widetilde{\tau}[v_n]}\Big((Jv_n)(X_{\widetilde{\tau}[v_n]})\Big)\Big] \\ + \frac{cL}{r}\Big(\frac{\lambda\gamma}{r+\lambda\gamma}\Big)^{n+1} = \mathbb{E}^\gamma_x\Big[e^{-r\widetilde{\tau}[v_n]}\Big(h(X_{\widetilde{\tau}[v_n]})\Big)\Big] + \frac{cL}{r}\Big(\frac{\lambda\gamma}{r+\lambda\gamma}\Big)^{n+1}, \end{aligned}$$

because $(Jv_n)(\cdot) = h(\cdot)$ on $\Gamma[v_n] \ni X_{\widetilde{\tau}[v_n]}$ on $\{\widetilde{\tau}[v_n] < \infty\}$. ☐

Corollary 16. *The maximum expected reward of the asset manager is given by* $U(x) = x - \frac{cL}{r} + V(x) = x - \frac{cL}{r} + v_\infty(x)$ *for every* $x \ge 0$. *The stopping rule* $\widetilde{\tau}[v_\infty]$ *is optimal, and* $\widetilde{\tau}[v_n]$ *is* ε*-optimal for every* $\varepsilon > 0$ *and* $n \ge 0$ *such that* $\frac{cL}{r}(\frac{\lambda\gamma}{r+\lambda\gamma})^{n+1} < \varepsilon$, *in the sense that for every* $x > 0$

$$\begin{aligned} U(x) &= \mathbb{E}^\gamma_x\Big[e^{-r\widetilde{\tau}[v_\infty]}(X_{\widetilde{\tau}[v_\infty]} - L)^+ + \int_0^{\widetilde{\tau}[v_\infty]} e^{-rt}(\delta X_t - cL)\mathrm{d}t\Big], \\ U(x) - \varepsilon &\le \mathbb{E}^\gamma_x\Big[e^{-r\widetilde{\tau}[v_n]}(X_{\widetilde{\tau}[v_n]} - L)^+ + \int_0^{\widetilde{\tau}[v_n]} e^{-rt}(\delta X_t - cL)\mathrm{d}t\Big]. \end{aligned}$$

4. The solution of the asset manager's second problem

In the *asset manager's second problem*, the investors' assets have limited protection. In the presence of the limited *protection at level* $\ell > 0$, the contract terminates at time $\widetilde{\tau}_{\ell,\infty} \triangleq \inf\{t \ge 0 : X_t \notin (\ell, \infty)\}$ automatically. The asset manager wants to maximize her expected total discounted earnings as in (2.2), but now the supremum has to be taken over all stopping times $\tau \in \mathcal{S}$ which are less than or equal to $\widetilde{\tau}_{\ell,\infty}$ almost surely. Namely, we would like to solve the problem

$$U_\ell(x) \triangleq \sup_{\tau\in\mathcal{S}} \mathbb{E}_x^\gamma\Big[e^{-r(\widetilde{\tau}_{\ell,\infty}\wedge\tau)}(X_{\widetilde{\tau}_{\ell,\infty}\wedge\tau}-L)^+ + \int_0^{\widetilde{\tau}_{\ell,\infty}\wedge\tau} e^{-rt}(\delta X_t - cL)\mathrm{d}t\Big], \quad x\in\mathbb{R}_+. \tag{4.1}$$

If $\ell < x_1[v_\infty]$, then $U_\ell(x) = U(x) = \mathbb{E}_x^\gamma[e^{-r(\widetilde{\tau}[v_\infty])}(X_{\widetilde{\tau}_{\ell,\infty}[v_\infty]}-L)^+ + \int_0^{\widetilde{\tau}[v_\infty]} e^{-rt}(\delta X_t - cL)\mathrm{d}t]$ for every $x>0$. On the one hand, because for every $\tau\in\mathcal{S}$, $\widetilde{\tau}[v_\infty]\wedge\tau$ also belongs to $\mathcal{S}$, we have $U_\ell(x)\le U(x)$. On the other hand, because $\ell \le x_1[v_\infty]$, we have $\widetilde{\tau}[v_\infty] = \widetilde{\tau}_{\ell,\infty}\wedge\widetilde{\tau}[v_\infty]$ a.s. and

$$\begin{aligned} U_\ell(x) &\ge \mathbb{E}_x^\gamma\Big[e^{-r(\widetilde{\tau}_{\ell,\infty}\wedge\widetilde{\tau}[v_\infty])}(X_{\widetilde{\tau}_{\ell,\infty}\wedge\widetilde{\tau}[v_\infty]}-L)^+ + \int_0^{\widetilde{\tau}_{\ell,\infty}\wedge\widetilde{\tau}[v_\infty]} e^{-rt}(\delta X_t - cL)\mathrm{d}t\Big] \\ &= \mathbb{E}_x^\gamma\Big[e^{-r\widetilde{\tau}[v_\infty]}(X_{\widetilde{\tau}[v_\infty]}-L)^+ + \int_0^{\widetilde{\tau}[v_\infty]} e^{-rt}(\delta X_t - cL)\mathrm{d}t\Big] = U(x) \quad \text{for every } x. \end{aligned}$$

Therefore, $U_\ell(x) = U(x)$ for every $x>0$ if $\ell\le x_1[v_\infty]$.

Assumption 17. *In the remainder, we shall assume that the protection level ℓ satisfies the inequalities $x_1[v_\infty] < \ell \le L$.*

The strong Markov property of X can be used to similarly show that

$$U_\ell(x) = x - \frac{cL}{r} + V_\ell(x), \quad x\ge 0, \tag{4.2}$$

where

$$V_\ell(x) \triangleq \sup_{\tau\in\mathcal{S}} \mathbb{E}_x^\gamma\left[e^{-r(\widetilde{\tau}_{\ell,\infty}\wedge\tau)}h(X_{\widetilde{\tau}_{\ell,\infty}\wedge\tau})\right], \quad x>0 \tag{4.3}$$

is the discounted optimal stopping problem for the stopped jump-diffusion process $X_{\widetilde{\tau}_{\ell,\infty}\wedge t}$, $t\ge 0$ with the same terminal payoff function $h(\cdot)$ as in (3.8).

Let us define stopping time

$$\tau_{\ell,\infty} \triangleq \inf\{t\ge 0;\ Y_t^{X_0}\notin(\ell,\infty)\}$$

of diffusion process Y^{X_0} and the operator

$$\begin{aligned} (J_\ell w)(x) &\triangleq \sup_{\tau\in\mathcal{S}_B} \mathbb{E}_x^\gamma\left[e^{-r\tau}h(X_{\tau_{\ell,\infty}\wedge\tau})1_{\{\tau_{\ell,\infty}\wedge\tau<T_1\}} + e^{-rT_1}w(X_{T_1})1_{\{\tau_{\ell,\infty}\wedge\tau\ge T_1\}}\right] \\ &= \sup_{\tau\in\mathcal{S}_B} \mathbb{E}_x^\gamma\Big[e^{-(r+\lambda\gamma)(\tau_{\ell,\infty}\wedge\tau)}h(Y^{X_0}_{\tau_{\ell,\infty}\wedge\tau}) + \int_0^{\tau_{\ell,\infty}\wedge\tau}\lambda\gamma e^{-(r+\lambda\gamma)t}w((1-y_0)Y_t^{X_0})\mathrm{d}t\Big], \quad x\ge 0. \end{aligned} \tag{4.4}$$

We expect that $V_\ell(\cdot) = (JV_\ell)(\cdot)$; namely, that $V_\ell(\cdot)$ is one of the fixed points of operator J_ℓ. We can find one of the fixed points of J_ℓ by taking limit of successive approximations defined by

$$\begin{aligned} v_{\ell,0}(x) &\triangleq h(x) \equiv (x-L)^+ - x + \frac{cL}{r}, && x>0, \\ v_{\ell,n}(x) &\triangleq (J_\ell v_{\ell,n-1})(x), && x>0,\ n\ge 1. \end{aligned}$$

The results of previous section can be adapted to the new problem, and we state only the differences.

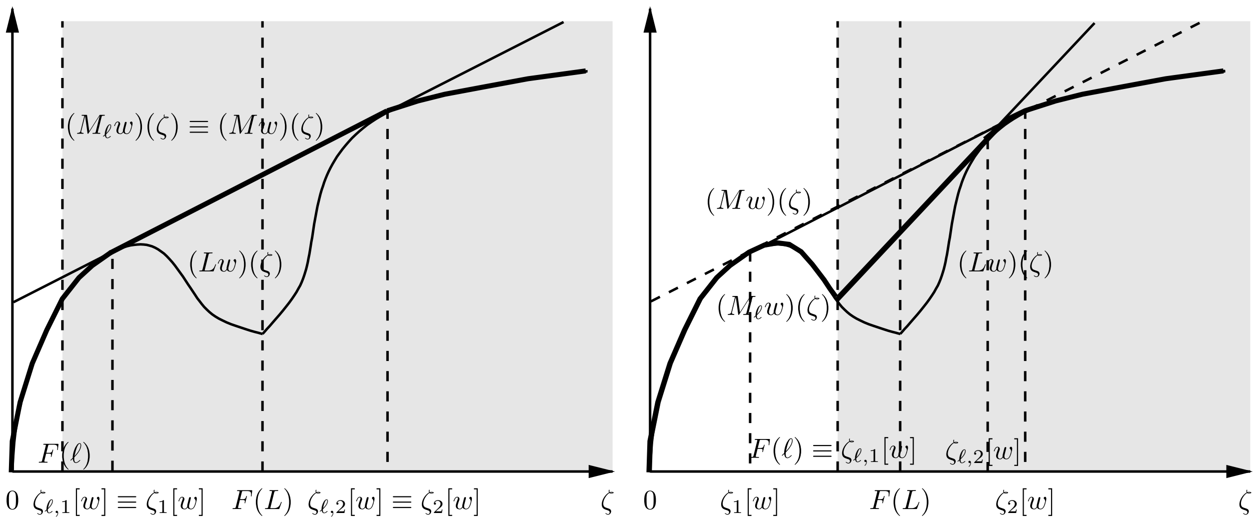

FIGURE 2. The sketches of $(Lw)(\cdot)$ and $(M_\ell w)(\cdot)$. On the left, because $F(\ell) \leq \zeta_1[w]$, $\zeta_{\ell,1}[w] \equiv \zeta_1[w]$ and $\zeta_{\ell,2}[w] \equiv \zeta_2[w]$ and $(M_\ell w)(\cdot) \equiv (Mw)(\cdot)$. On the right, because $\zeta_1[w] < F(\ell) \leq F(L)$, $\zeta_1[w] < F(\ell) = \zeta_{\ell,1}[w] < \zeta_{\ell,2}[w] < \zeta_2[w]$, and $(M_\ell w)(x) < (Mw)(x)$ for every $x \in (\zeta_1[w], \zeta_2[w])$.

Lemma 18. *Let $w_1, w_2 : \mathbb{R}_+ \mapsto \mathbb{R}$ be bounded. If $w_1(\cdot) \leq w_2(\cdot)$, then $(J_\ell w_1)(\cdot) \leq (J_\ell w_2)(\cdot)$. If $w(\cdot)$ is nonincreasing and convex such that $h(\cdot) \leq w(\cdot) \leq cL/r$, then $(J_\ell w)(\cdot)$ has the same properties.*

Proposition 19. *For every bounded $w_1, w_2 : \mathbb{R}_+ \mapsto \mathbb{R}$, we have $\|J_\ell w_1 - J_\ell w_2\| \leq \frac{\lambda\gamma}{r+\lambda\gamma}\|w_1 - w_2\|$; namely, J_ℓ acts as a contraction mapping on the collection of bounded functions.*

Lemma 20. *The sequence $(v_{\ell,n})_{n\geq 0}$ of successive approximations is increasing. Therefore, the pointwise limit $v_{\ell,\infty}(x) = \lim_{n\to\infty} v_{\ell,n}(x)$, $x > 0$ exists. Every $v_{\ell,n}(\cdot)$, $n \geq 0$ and $v_{\ell,\infty}(\cdot)$ are nonincreasing, convex, and bounded between $h(\cdot)$ and cL/r.*

Proposition 21. *The limit $v_{\ell,\infty}(\cdot) = \lim_{n\to\infty} v_{\ell,n}(\cdot) = \sup_{n\geq 0} v_{\ell,n}$ is the unique bounded fixed point of J_ℓ. Moreover, $0 \leq v_{\ell,\infty}(x) - v_{\ell,n}(x) \leq \frac{cL}{r}(\frac{\lambda\gamma}{r+\lambda\gamma})^n$ for every $x > 0$ and $n \geq 0$.*

Let $w : \mathbb{R}_+ \mapsto \mathbb{R}$ be a function as in Assumption 5. Then

$$(J_\ell w)(x) = \lambda\gamma(Hw)(x) + (G_\ell w)(x), \quad x > 0, \tag{4.5}$$

where $(G_\ell w)(\cdot)$ is the value function of the discounted optimal stopping problem

$$(G_\ell w)(x) \triangleq \sup_{\tau\in\mathcal{S}_B} \mathbb{E}^\gamma_x \left[e^{-(r+\lambda\gamma)\tau_{\ell,\infty}\wedge\tau} \{h - \lambda\gamma(Hw)\} (Y^{X_0}_{\tau_{\ell,\infty}\wedge\tau}) \right], \quad x > 0, \tag{4.6}$$

for the stopped diffusion process $Y^{X_0}_{\tau_{\ell,\infty}}$, $t \geq 0$ at stopping time $\tau_{\ell,\infty}$.

We obviously have $(G_\ell w)(x) = h(x)$ for every $x \in (0, \ell]$. If the initial state X_0 of $Y^{X_0}_{\tau_{\ell,\infty}}$, $t \geq 0$ is in (ℓ, ∞), then ℓ becomes an absorbing left-boundary for the stopped process $Y^{X_0}_{\tau_{\ell,\infty}}$, $t \geq 0$.

Let $(M_\ell w)(\cdot)$ be the smallest concave majorant on $[F(\ell),\infty)$ of $(Lw)(\cdot)$ defined by (3.23) and equal on $(0,F(\ell))$ identically to $(Lw)(\cdot)$. Then by Proposition 5.5 of Dayanik and Karatzas [7]

$$(G_\ell w)(x)=\varphi(x)(M_\ell w)(F(x)),\ x>0 \quad\text{and}\quad \Gamma_\ell[w]=F^{-1}(\{\zeta>0;\ (M_\ell w)(\zeta)=(Lw)(\zeta)\})$$

are value function and optimal stopping region for (4.6). The analysis of the shape of $(Lw)(\cdot)$ prior to Figure 1 implies that there are unique numbers $0<\zeta_{\ell,1}[w]<F(L)<\zeta_{\ell,2}[w]<\infty$ such that

$$\begin{cases} \left\{\begin{array}{c} (Lw)'(\zeta_{\ell,1}[w])=\dfrac{(Lw)(\zeta_{\ell,2}[w])-(Lw)(\zeta_{\ell,1}[w])}{\zeta_{\ell,2}[w]-\zeta_{\ell,1}[w]}=(Lw)'(\zeta_{\ell,2}[w]) \\ \text{namely,}\quad \zeta_{\ell,1}[w]\equiv\zeta_1[w]\quad\text{and}\quad \zeta_{\ell,2}[w]\equiv\zeta_2[w] \end{array}\right\} & \text{if } F(\ell)\le\zeta_1[w], \\ \zeta_{\ell,1}[w]=\ell\quad\text{and}\quad \dfrac{(Lw)(\zeta_{\ell,2}[w])-(Lw)(\zeta_{\ell,1}[w])}{\zeta_{\ell,2}[w]-\zeta_{\ell,1}[w]}=(Lw)'(\zeta_{\ell,2}[w]) & \text{if } F(\ell)>\zeta_1[w], \end{cases}$$

and

$$(M_\ell w)(\zeta)=\begin{cases} (Lw)(\zeta), & \zeta\in(0,\zeta_{\ell,1}[w]]\cup[\zeta_{\ell,2}[w],\infty), \\ \dfrac{\zeta_{\ell,2}[w]-\zeta}{\zeta_{\ell,2}[w]-\zeta_{\ell,1}[w]}(Lw)(\zeta_{\ell,1}[w]) \\ \quad+\dfrac{\zeta-\zeta_{\ell,1}[w]}{\zeta_{\ell,2}[w]-\zeta_{\ell,1}[w]}(Lw)(\zeta_{\ell,2}[w]), & \zeta\in(\zeta_{\ell,1}[w],\zeta_{\ell,2}[w]). \end{cases}$$

Let us define $x_{\ell,1}[w]=F^{-1}(\zeta_{\ell,1}[w])$ and $x_{\ell,2}[w]=F^{-1}(\zeta_{\ell,2}[w])$. Then the value function equals

$$\begin{aligned} &(G_\ell w)(x)=\varphi(x)(M_\ell w)(F(x)) \\ &=\begin{cases} (h-\lambda\gamma(Hw))(x), & x\in(0,x_{\ell,1}[w]]\cup[x_{\ell,2}[w],\infty), \\ \dfrac{(x_{\ell,2}[w])^{\alpha_1-\alpha_0}-x^{\alpha_1-\alpha_0}}{(x_{\ell,2}[w])^{\alpha_1-\alpha_0}-(x_{\ell,1}[w])^{\alpha_1-\alpha_0}}(h-\lambda\gamma(Hw))(x_{\ell,1}[w]) \\ \quad+\dfrac{x^{\alpha_1-\alpha_0}-(x_{\ell,1}[w])^{\alpha_1-\alpha_0}}{(x_{\ell,2}[w])^{\alpha_1-\alpha_0}-(x_{\ell,1}[w])^{\alpha_1-\alpha_0}}(h-\lambda\gamma(Hw))(x_{\ell,2}[w]), & x\in(x_{\ell,1}[w],x_{\ell,2}[w]) \end{cases} \end{aligned} \tag{4.7}$$

and the optimal stopping time becomes

$$\Gamma_\ell[w]=\{x>0;\ (G_\ell w)(x)=(h-\lambda\gamma(Hw))(x)\}=(0,x_{\ell,1}[w]]\cup[x_{\ell,2}[w],\infty), \tag{4.8}$$

and an optimal stopping time is given by

$$\tau_\ell[w]\triangleq\inf\{x>0;\ Y_t^{X_0}\in\Gamma_\ell[w]\}=\inf\{x>0;\ Y_t^{X_0}\in(0,x_{\ell,1}[w]]\cup[x_{\ell,2}[w],\infty)\} \tag{4.9}$$

for the problem in (4.6). A direct verification together with the chain of equalities $\mathrm{sgn}\{(\mathcal{A}_0-(r+\lambda\gamma))(G_\ell w)(x)\}=\mathrm{sgn}\{(\mathcal{A}_0-(r+\lambda\gamma))(h-\lambda\gamma(Hw))(x)\}=\mathrm{sgn}\{(Lw)''(F(x))\}<0$ for every $x\in(\ell,x_{\ell,1}[w])\cup(x_{\ell,2}[w],\infty)$ from Dayanik and Karatzas [7, page 192] proves the next proposition.

Proposition 22. *The value function $x\mapsto(G_\ell w)(x)$ is continuously differentiable on $[\ell,\infty)$ and twice-continuously differentiable on $[\ell,\infty)\setminus\{x_{\ell,1}[w],x_{\ell,2}[w]\}$. The function $(G_\ell w)(x)$, $x\in[\ell,\infty)$*

solves the variational inequalities

$$\begin{aligned}
&(i) && (\mathcal{A}_0-(r+\lambda\gamma))(G_\ell w)(x)=0, && x\in(x_{\ell,1}[w],x_{\ell,2}[w]),\\
&(ii) && (G_\ell w)(x)>h(x)-\lambda\gamma(Hw)(x), && x\in(x_{\ell,1}[w],x_{\ell,2}[w]),\\
&(iii) && (\mathcal{A}_0-(r+\lambda\gamma))(G_\ell w)(x)<0, && x\in(\ell,x_{\ell,1}[w])\cup(x_{\ell,2}[w],\infty),\\
&(iv) && (G_\ell w)(x)=h(x)-\lambda\gamma(Hw)(x), && x\in[\ell,x_{\ell,1}[w]]\cup[x_{\ell,2}[w],\infty).
\end{aligned}$$

Because $(J_\ell w)(x)=\lambda\gamma(Hw)(x)+(G_\ell w)(x)$ for every $x>0$, $(Hw)(\cdot)$ is twice-continuously differentiable, and $(\mathcal{A}_0-(r+\lambda\gamma))(Hw)(x)=-w((1-y_0)x)$ for every $x>0$, the next proposition immediately follows from Proposition 22.

Proposition 23. *The value function $x\mapsto(J_\ell w)(x)$ in (4.4) is continuously differentiable on $[\ell,\infty)$, twice-continuously differentiable on $[\ell,\infty)\setminus\{x_{\ell,1}[w],x_{\ell,2}[w]\}$, and satisfies*

$$\begin{aligned}
&(i) && (\mathcal{A}_0-(r+\lambda\gamma))(J_\ell w)(x)+\lambda\gamma w((1-y_0)x)=0, && x\in(x_{\ell,1}[w],x_{\ell,2}[w]),\\
&(ii) && (J_\ell w)(x)>h(x), && x\in(x_{\ell,1}[w],x_{\ell,2}[w]),\\
&(iii) && (\mathcal{A}_0-(r+\lambda\gamma))(J_\ell w)(x)+\lambda\gamma w((1-y_0)x)<0, && x\in(\ell,x_{\ell,1}[w])\cup(x_{\ell,2}[w],\infty),\\
&(iv) && (J_\ell w)(x)=h(x), && x\in[\ell,x_{\ell,1}[w]]\cup[x_{\ell,2}[w],\infty).
\end{aligned}$$

As in the asset manager's first problem, the successive approximations $v_{\ell,n}(\cdot)$, $n\geq 0$ and their limit $v_{\ell,\infty}(\cdot)$ satisfy Assumption 5. Therefore, Propositions 21 and 23 lead to the next theorem.

Theorem 24. *The function $x\mapsto v_{\ell,\infty}(x)=(Jv_\infty)(x)$ is continuously differentiable on $[\ell,\infty)$, twice-continuously differentiable on $[\ell,\infty)\setminus\{x_{\ell,1},x_{\ell,2}\}$ and satisfies the variational inequalities*

$$\begin{aligned}
&(i) && (\mathcal{A}_0-(r+\lambda\gamma))v_{\ell,\infty}(x)+\lambda\gamma v_{\ell,\infty}((1-y_0)x)=0, && x\in(x_{\ell,1}[v_{\ell,\infty}],x_{\ell,2}[v_{\ell,\infty}]),\\
&(ii) && v_{\ell,\infty}(x)>h(x), && x\in(x_{\ell,1}[v_{\ell,\infty}],x_{\ell,2}[v_{\ell,\infty}]),\\
&(iii) && (\mathcal{A}_0-(r+\lambda\gamma))v_{\ell,\infty}(x)+\lambda\gamma v_{\ell,\infty}((1-y_0)x)<0, && x\in(\ell,x_{\ell,1}[v_{\ell,\infty}])\cup(x_{\ell,2}[v_{\ell,\infty}],\infty),\\
&(iv) && v_{\ell,\infty}(x)=h(x), && x\in[\ell,x_{\ell,1}[v_{\ell,\infty}]]\cup[x_{\ell,2}[v_{\ell,\infty}],\infty),
\end{aligned}$$

which can be expressed in terms of the generator $\mathcal{A}^\gamma$ in (3.5) of the jump-diffusion process X as

$$\begin{aligned}
&(i)' && (\mathcal{A}^\gamma-r)v_{\ell,\infty}(x)=0, && x\in(x_{\ell,1}[v_{\ell,\infty}],x_{\ell,2}[v_{\ell,\infty}]),\\
&(ii)' && v_{\ell,\infty}(x)>h(x), && x\in(x_{\ell,1}[v_{\ell,\infty}],x_{\ell,2}[v_{\ell,\infty}]),\\
&(iii)' && (\mathcal{A}^\gamma-r)v_{\ell,\infty}(x)<0, && x\in(\ell,x_{\ell,1}[v_{\ell,\infty}])\cup(x_{\ell,2}[v_{\ell,\infty}],\infty),\\
&(iv)' && v_{\ell,\infty}(x)=h(x), && x\in[\ell,x_{\ell,1}[v_{\ell,\infty}]]\cup[x_{\ell,2}[v_{\ell,\infty}],\infty).
\end{aligned}$$

Note again that the second part follows from the first part and from the equality $(\mathcal{A}_0-(r+\lambda\gamma))v_{\ell,\infty}(x)+\lambda\gamma v_{\ell,\infty}((1-y_0)x)=(\mathcal{A}^\gamma-r)v_{\ell,\infty}(x)$ for every $x\in(\ell,\infty)\setminus\{x_{\ell,1}[v_{\ell,\infty}],x_{\ell,2}[v_{\ell,\infty}]\}$.

By the next theorem, optimal stopping time for asset manager's second problem is of the form

$$\widetilde{\tau}_\ell[w]\triangleq\inf\{t\geq 0;\ X_t\in(0,x_{\ell,1}[w]]\cup[x_{\ell,2}[w],\infty)\}.$$

Theorem 25. *For every $x \in \mathbb{R}_+$, we have $V_\ell(x) = v_{\ell,\infty}(x) = \mathbb{E}_x^\gamma\big[e^{-r\widetilde{\tau}_\ell[v_{\ell,\infty}]}h(X_{\widetilde{\tau}_\ell[v_{\ell,\infty}]})\big]$, and $\widetilde{\tau}_\ell[v_{\ell,\infty}]$ is an optimal stopping time for (4.3).*

Since $v_{\ell,\infty}(x) = h(x) = V(x)$ for every $x \in (0,\ell]$, Theorem 25 has to be proved on (ℓ,∞), which can be done as in the proof of Theorem 13 but with localizing stopping rules $\widetilde{\tau}_{\ell b}$ for $b > \ell$.

The proof of the next proposition is similar to that of Proposition 14.

Proposition 26. *The optimal stopping regions $\Gamma_\ell[v_{\ell,n}] = \{x > 0;\ (Jv_{\ell,n})(x) \le h(x)\} = (0, x_{\ell,1}[v_{\ell,n}]] \cup [x_{\ell,2}[v_{\ell,n}],\infty)$, $n \in \{0,1,\ldots,\infty\}$ are decreasing; namely, $\Gamma_\ell[v_{\ell,0}] \supseteq \Gamma_\ell[v_{\ell,1}] \supseteq \ldots \supseteq \Gamma_\ell[v_{\ell,\infty}]$, and $0 < x_{\ell,1}[v_{\ell,\infty}] \le \ldots \le x_{\ell,1}[v_{\ell,1}] \le x_{\ell,1}[v_{\ell,0}] \le L \le x_{\ell,2}[v_{\ell,0}] \le x_{\ell,2}[v_{\ell,1}] \le \ldots \le x_{\ell,2}[v_{\ell,\infty}] < \infty$. Moreover, $x_{\ell,1}[v_{\ell,\infty}] = \lim_{n\to\infty} x_{\ell,1}[v_{\ell,n}]$ and $x_{\ell,2}[v_{\ell,\infty}] = \lim_{n\to\infty} x_{\ell,2}[v_{\ell,n}]$.*

Proposition 27. *For every $n \ge 0$, we have $v_{\ell,\infty}(x) \le \mathbb{E}_x^\gamma[e^{-r\widetilde{\tau}_\ell[v_{\ell,n}]}h(X_{\widetilde{\tau}_\ell[v_{\ell,n}]})] + \frac{cL}{r}(\frac{\lambda}{r+\lambda\gamma})^{n+1}$. Hence, for every $\varepsilon > 0$ and $n \ge 0$ such that $\frac{cL}{r}(\frac{\lambda}{r+\lambda\gamma})^{n+1} \le \varepsilon$, $\widetilde{\tau}_\ell[v_{\ell,n}]$ is ε-optimal for (4.3).*

The proof is similar to that of Proposition 15 if we replace localizing stopping times $\widetilde{\tau}_{ab}$ with $\widetilde{\tau}_{\ell b}$. Finally, Corollary 28 identifies the maximum expected reward and nearly optimal stopping strategies of the asset manager for the second problem.

Corollary 28. *The maximum expected reward of the asset manager is given by $U_\ell(x) = x - \frac{cL}{r} + V_\ell(x) = x - \frac{cL}{r} + v_{\ell,\infty}(x)$ for every $x \ge 0$. The stopping rule $\widetilde{\tau}_\ell[v_{\ell,\infty}]$ is optimal, and $\widetilde{\tau}_\ell[v_{\ell,n}]$ is ε-optimal for every $\varepsilon > 0$ and $n \ge 0$ such that $\frac{cL}{r}(\frac{\lambda\gamma}{r+\lambda\gamma})^{n+1} < \varepsilon$, in the sense that for every $x > 0$*

$$U_\ell(x) = \mathbb{E}_x^\gamma\Big[e^{-r\widetilde{\tau}_\ell[v_{\ell,\infty}]}(X_{\widetilde{\tau}_\ell[v_{\ell,\infty}]} - L)^+ + \int_0^{\widetilde{\tau}_\ell[v_{\ell,\infty}]} e^{-rt}(\delta X_t - cL)\mathrm{d}t\Big],$$
$$U_\ell(x) - \varepsilon \le \mathbb{E}_x^\gamma\Big[e^{-r\widetilde{\tau}_\ell[v_{\ell,n}]}(X_{\widetilde{\tau}_\ell[v_{\ell,n}]} - L)^+ + \int_0^{\widetilde{\tau}_\ell[v_{\ell,n}]} e^{-rt}(\delta X_t - cL)\mathrm{d}t\Big].$$

We expect that the value of the limited protection at level ℓ to increase as ℓ decreases. We also expect that the asset manager quits early as the protection limit ℓ increases to L. This expectations are validated later, and they are backed up by the findings of the next lemma.

Lemma 29. *Let $w : \mathbb{R}_+ \mapsto \mathbb{R}$ be as in Assumption 5. Suppose that $0 < \ell < u < L$. Then*

(i) $\quad (M_\ell w)(\cdot) \ge (M_u w)(\cdot)$ *on* $\mathbb{R}_+$,

(ii) $\quad 0 < \zeta_{\ell,1}[w] < \zeta_{u,1}[w] < F(L) < \zeta_{u,2}[w] < \zeta_{\ell,2}[w] < \infty$,

(iii) $\quad (J_\ell w)(\cdot) \ge (J_u w)(\cdot)$ *on* $\mathbb{R}_+$,

(iv) $\quad 0 < x_{\ell,1}[w] < x_{u,1}[w] < L < x_{u,2}[w] < x_{\ell,2}[w] < \infty$.

Recall that $(M_\ell w)(\cdot)$ and $(M_u w)(\cdot)$ coincide, respectively, on $(0, F(\ell)]$ and $(0, F(u)]$ with $(Lw)(\cdot)$ and on $(F(\ell),\infty)$ and $(F(u),\infty)$ with the smallest nonnegative concave majorants of $(Lw)(\cdot)$, respectively, over $(F(\ell),\infty)$ and $(F(u),\infty)$. Therefore, *(i)* and *(ii)* of Lemma 29 immediately follow; see Figure 2. Finally, *(iii)* and *(iv)* follow from *(i)* and *(ii)* by the relation (4.5): $(J_\ell w)(x) =$

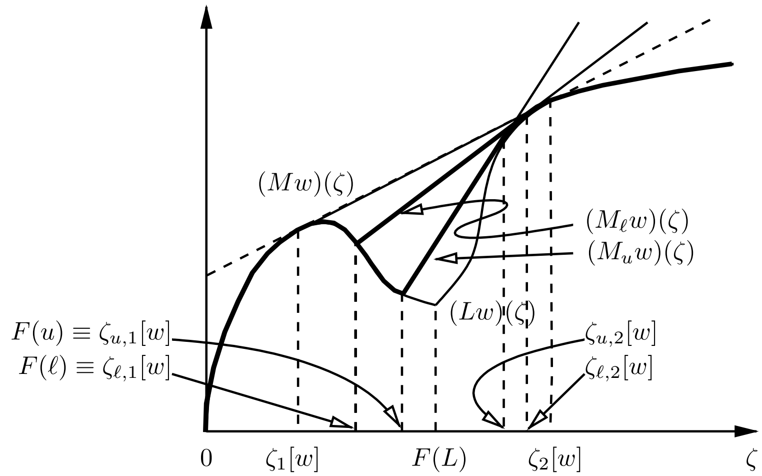

FIGURE 3. Comparison of $(M_\ell w)(\cdot)$ and $(M_u w)(\cdot)$ for $\zeta_1[w] < F(\ell) < F(u) < F(L)$. Observe that $\zeta_1[w] < \zeta_{\ell,1}[w] < \zeta_{u,1}[w] < F(L) < \zeta_{u,2}[w] < \zeta_{\ell,2}[w]$ and $(M_\ell w)(\cdot) \geq (M_u w)(\cdot)$.

$\lambda\gamma(Hw)(x) + (G_\ell w)(x) = \lambda\gamma(Hw)(x) + \varphi(x)(M_\ell w)(F(x))$ for every x; $x_{\ell,1}[w] = F^{-1}(\zeta_{\ell,1}[w])$, $x_{\ell,2}[w] = F^{-1}(\zeta_{\ell,2}[w])$, and that $F(\cdot)$ is strictly increasing.

Proposition 30 shows that demanding higher portfolio insurance or limiting more severely the downward risks or losses also limits the upward potential and reduces the total value of the portfolio.

Proposition 30. *For every $0 < \ell < u < L$, we have*

$$(i) \qquad v_{\ell,n}(x) \geq v_{u,n}(x), \quad x \in \mathbb{R}_+,\ n \in \{0, 1, \ldots, \infty\},$$

$$(ii) \qquad U_\ell(x) \geq U_u(x), \quad x \in \mathbb{R}_+,$$

$$(iii) \qquad 0 < x_{\ell,1}[v_{\ell,n}] \leq x_{u,1}[v_{u,n}] < L < x_{u,2}[v_{u,n}] \leq x_{\ell,2}[v_{\ell,n}] < \infty.$$

Proof. Note first that $v_{\ell,0}(x) = h(x) = v_{u,0}(x)$ for every $x \in \mathbb{R}_+$. Suppose that for some $n \geq 0$ we have $v_{\ell,n}(\cdot) \geq v_{u,n}(\cdot)$ on $\mathbb{R}_+$. Then by Lemmas 18 and 29 (iii), $v_{\ell,n+1}(\cdot) = (J_\ell v_{\ell,n})(\cdot) \geq (J_\ell v_{u,n})(\cdot) \geq (J_u v_{u,n})(\cdot) = v_{u,n+1}(\cdot)$. Therefore, for every $n \geq 0$, we have $v_{\ell,n}(\cdot) \geq v_{u,n}(\cdot)$ and $v_{\ell,\infty}(\cdot) = \lim_{n\to\infty} v_{\ell,n}(\cdot) \geq \lim_{n\to\infty} v_{u,n}(\cdot) = v_{u,\infty}(\cdot)$, which proves (i). By (4.2), $U_\ell(x) = x - \frac{cL}{r} + v_{\ell,\infty}(x) \geq x - \frac{cL}{r} + v_{u,\infty}(x) = U_u(x)$ for every $x > 0$, and (ii) follows. Finally, (4.8) and (i) imply

$$\begin{aligned}(0, x_{\ell,1}[v_{\ell,\infty}]] \cup [x_{\ell,1}[v_{\ell,\infty}], \infty) &= \Gamma_\ell[v_{\ell,\infty}] = \{x > 0;\ (J_\ell v_{\ell,\infty})(x) \leq h(x)\} = \{x > 0;\ v_{\ell,\infty}(x) \leq h(x)\} \\ &\subseteq \{x > 0;\ v_{u,\infty}(x) \leq h(x)\} = \{x > 0;\ (J_u v_{u,\infty})(x) \leq h(x)\} = (0, x_{u,1}[v_{u,\infty}]] \cup [x_{u,1}[v_{u,\infty}], \infty).\end{aligned}$$

Hence, $0 < x_{\ell,1}[v_{\ell,\infty}] \leq x_{u,1}[v_{u,\infty}] < L < x_{u,2}[v_{u,\infty}] \leq x_{\ell,2}[v_{\ell,\infty}] < \infty$. Similarly,

$$\begin{aligned}(0, x_{\ell,1}[v_{\ell,n}]] \cup [x_{\ell,1}[v_{\ell,n}], \infty) &= \Gamma_\ell[v_{\ell,n}] = \{x > 0;\ (J_\ell v_{\ell,n})(x) \leq h(x)\} = \{x > 0;\ v_{\ell,n+1}(x) \leq h(x)\} \\ &\subseteq \{x > 0;\ v_{u,n+1}(x) \leq h(x)\} = \{x > 0;\ (J_u v_{u,n})(x) \leq h(x)\} = (0, x_{u,1}[v_{u,n}]] \cup [x_{u,1}[v_{u,n}], \infty),\end{aligned}$$

which implies $0 < x_{\ell,1}[v_{\ell,n}] \leq x_{u,1}[v_{u,n}] < L < x_{u,2}[v_{u,n}] \leq x_{\ell,2}[v_{\ell,n}] < \infty$ for every finite $n \geq 0$. $\square$

5. Numerical algorithms

The following algorithm describes how one can calculate maximum expected reward and optimal stopping strategy of the asset manager in the first problem.

Initialization. Set $n=1$, $v_0(x)=h(x)$, $x\geq 0$. Calculate $\varphi(x)=x^{\alpha_0}$, $\psi(x)=x_1^{\alpha}$, $F(x)=\psi(x)/\varphi(x)=x^{\alpha_1}-x^{\alpha_0}$, where

$$\alpha_{0,1}=\frac{-(r-\delta+\lambda\gamma y_0-\frac{\sigma^2}{2})\mp\sqrt{(r-\delta+\lambda\gamma y_0-\frac{\sigma^2}{2})^2+2\sigma^2(r+\lambda\gamma)}}{\sigma^2},\quad \alpha_0<\alpha_1. \tag{5.1}$$

Step 1. Calculate

$$(Lv_n)(\zeta)=\begin{cases}\dfrac{h-\lambda\gamma(Hv_n)}{\varphi}\circ F^{-1}(\zeta), & \zeta>0,\\ 0, & \zeta=0.\end{cases}$$

Step 2. Calculate the critical boundaries $\zeta_1[v_n]<F(L)<\zeta_2[v_n]$, which are unique solutions of

$$(Lv_n)'(\zeta_1[v_n])=\frac{(Lv_n)(\zeta_2[v_n])-(Lv_n)(\zeta_2[v_n])}{\zeta_1[v_n]-\zeta_1[v_n]}=(Lv_n)'(\zeta_2[v_n]),$$

and the smallest nonnegative concave majorant $(Mv_n)(\cdot)$ of $(Lv_n)(\cdot)$ on $\mathbb{R}_+$ by

$$(Mv_n)(\zeta)=\begin{cases}(Lv_n)(\zeta), & \zeta\in(0,\zeta_1[v_n]]\cup[\zeta_2[v_n],\infty),\\ \dfrac{\zeta_2[v_n]-\zeta}{\zeta_2[v_n]-\zeta_1[v_n]}(Lv_n)(\zeta_1[v_n]) & \\ \quad+\dfrac{\zeta-\zeta_1[v_n]}{\zeta_2[v_n]-\zeta_1[v_n]}(Lv_n)(\zeta_2[v_n]), & \zeta\in(\zeta_1[v_n],\zeta_2[v_n]).\end{cases}$$

Step 3. Calculate $x_1[v_n]=F^{-1}(\zeta_1[v_n])$, $x_2[v_n]=F^{-1}(\zeta_2[v_n])$, and

$$(Gv_n)(\zeta)=\begin{cases}(h-\lambda\gamma(Hv_n))(x), & x\in(0,x_1[v_n]]\cup[x_2[v_n],\infty),\\ \dfrac{(x_2[v_n])^{\alpha_1-\alpha_0}-x^{\alpha_1-\alpha_0}}{(x_2[v_n])^{\alpha_1-\alpha_0}-(x_1[v_n])^{\alpha_1-\alpha_0}}(h-\lambda\gamma(Hv_n))(x_1[v_n]) & \\ \quad+\dfrac{x^{\alpha_1-\alpha_0}-(x_1[v_n])^{\alpha_1-\alpha_0}}{(x_2[v_n])^{\alpha_1-\alpha_0}-(x_1[v_n])^{\alpha_1-\alpha_0}}(h-\lambda\gamma(Hv_n))(x_2[v_n]), & x\in(x_1[v_n],x_2[v_n]).\end{cases}$$

Step 4. Calculate $v_{n+1}(x)=\lambda\gamma(Hv_n)(x)+(Gv_n)(x)$ for every $x>0$.

Step 5. If some stopping criterion has not yet been satisfied (for example, the uniform bound $\frac{cL}{r}(\frac{\lambda\gamma}{r+\lambda\gamma})^{n+1}$ on $\|v_\infty-v_n\|$ has not yet been reduced below some desired error level), then set n to $n+1$ and got to Step 1, otherwise stop.

Outcome. After the algorithm terminates with v_{n+1}, $x_1[v_n]$, and $x_2[v_n]$,

(i) we have $x-\frac{cL}{r}+v_n(x)\leq U(x)\leq x-\frac{cL}{r}+v_n(x)+\frac{cL}{r}(\frac{\lambda\gamma}{r+\lambda\gamma})^n$ for every $x>0$,

(ii) the stopping time $\widetilde{\tau}[v_n]=\inf\{t\geq 0;\ X_t\notin(x_1[v_n],x_2[v_n])\}$ is ε-optimal for every $\varepsilon>\frac{cL}{r}(\frac{\lambda\gamma}{r+\lambda\gamma})^n$ for the portfolio manager's first problem; namely, for every $x>0$

$$U(x)-\frac{cL}{r}\left(\frac{\lambda\gamma}{r+\lambda\gamma}\right)^n\leq\mathbb{E}_x^{\gamma}\Big[e^{-r\widetilde{\tau}[v_n]}(X_{\widetilde{\tau}[v_n]}-L)^+ +\int_0^{\widetilde{\tau}[v_n]}e^{-rt}(\delta X_t-cL)\mathrm{d}t\Big]\leq U(x).$$

The following algorithm calculates the maximum expected reward and optimal stopping strategy of the asset manager in the second problem with portfolio protection level set at some $0 < \ell < L$.

Initialization. Set $n = 1$, $v_{\ell,0} = h(x)$, $x > 0$. Calculate $\varphi(x) = x^{\alpha_0}$, $\psi(x) = x_1^{\alpha}$, $F(x) = \psi(x)/\varphi(x) = x^{\alpha_1} - x^{\alpha_0}$, where α_0 and α_1 are as in (5.1).

Step 1. Calculate

$$(Lv_{\ell,n})(\zeta) = \begin{cases} \dfrac{h - \lambda\gamma(Hv_{\ell,n})}{\varphi} \circ F^{-1}(\zeta), & \zeta > 0, \\ 0, & \zeta = 0. \end{cases}$$

Step 2. Calculate $(M_\ell v_{\ell,\infty})(\cdot)$, which equals $(Lv_{\ell,n})(\cdot)$ on $(0, F(\ell)]$ and coincides on $(F(\ell), \infty)$ with the smallest nonnegative concave majorant of the restriction of $(Lv_{\ell,n})(\cdot)$ to $[F(\ell), \infty)$. Let $0 < \zeta_{\ell,1}[v_{\ell,n}] < F(L) < \zeta_{\ell,2}[v_{\ell,n}]$ be the endpoints of interval $\{\zeta; (M_\ell v_{\ell,\infty})(\zeta) > (Lv_{\ell,n})(\zeta)$. Then

$$(M_\ell v_{\ell,n})(\zeta) = \begin{cases} (Lv_{\ell,n})(\zeta), & \zeta \in (0, \zeta_{\ell,1}[v_{\ell,n}]] \cup [\zeta_{\ell,2}[v_{\ell,n}], \infty), \\ \dfrac{\zeta_{\ell,2}[v_{\ell,n}] - \zeta}{\zeta_{\ell,2}[v_{\ell,n}] - \zeta_{\ell,1}[v_{\ell,n}]}(Lv_{\ell,n})(\zeta_{\ell,1}[v_{\ell,n}]) \\ \quad + \dfrac{\zeta - \zeta_{\ell,1}[v_{\ell,n}]}{\zeta_{\ell,2}[v_{\ell,n}] - \zeta_{\ell,1}[v_{\ell,n}]}(Lv_{\ell,n})(\zeta_{\ell,2}[v_{\ell,n}]), & \zeta \in (\zeta_{\ell,1}[v_{\ell,n}], \zeta_{\ell,2}[v_{\ell,n}]). \end{cases}$$

Step 3. Calculate $x_{\ell,1}[v_{\ell,n}] = F^{-1}(\zeta_{\ell,1}[v_{\ell,n}])$, $x_{\ell,2}[v_{\ell,n}] = F^{-1}(\zeta_{\ell,2}[v_{\ell,n}])$, and $(Gv_{\ell,n})(\zeta) =$

$$\begin{cases} (h - \lambda\gamma(Hv_{\ell,n}))(x), & x \in (0, x_{\ell,1}[v_{\ell,n}]] \cup [x_{\ell,2}[v_{\ell,n}], \infty), \\ \dfrac{(x_{\ell,2}[v_{\ell,n}])^{\alpha_1-\alpha_0} - x^{\alpha_1-\alpha_0}}{(x_{\ell,2}[v_{\ell,n}])^{\alpha_1-\alpha_0} - (x_{\ell,1}[v_{\ell,n}])^{\alpha_1-\alpha_0}}(h - \lambda\gamma(Hv_{\ell,n}))(x_{\ell,1}[v_{\ell,n}]) \\ \quad + \dfrac{x^{\alpha_1-\alpha_0} - (x_{\ell,1}[v_{\ell,n}])^{\alpha_1-\alpha_0}}{(x_{\ell,2}[v_{\ell,n}])^{\alpha_1-\alpha_0} - (x_{\ell,1}[v_{\ell,n}])^{\alpha_1-\alpha_0}}(h - \lambda\gamma(Hv_{\ell,n}))(x_{\ell,2}[v_{\ell,n}]), & x \in (x_{\ell,1}[v_{\ell,n}], x_{\ell,2}[v_{\ell,n}]). \end{cases}$$

Step 4. Calculate $v_{\ell,n+1}(x) = \lambda\gamma(Hv_{\ell,n})(x) + (Gv_{\ell,n})(x)$ for every $x > 0$.

Step 5. If some stopping criterion has not yet been satisfied (for example, the uniform bound $\frac{cL}{r}(\frac{\lambda\gamma}{r+\lambda\gamma})^{n+1})$ on $\|v_{\ell,\infty} - v_{\ell,n}\|$ has not yet been reduced below some desired error level), then set n to $n+1$ and got to Step 1, otherwise stop.

Outcome. After the algorithm terminates with $v_{\ell,n+1}$, $x_{\ell,1}[v_{\ell,n}]$, and $x_{\ell,2}[v_{\ell,n}]$,

(i) we have $x - \frac{cL}{r} + v_{\ell,n}(x) \le U_\ell(x) \le x - \frac{cL}{r} + v_{\ell,n}(x) + \frac{cL}{r}(\frac{\lambda\gamma}{r+\lambda\gamma})^n$ for every $x > 0$,

(ii) the stopping time $\widetilde{\tau}_\ell[v_{\ell,n}] = \inf\{t \ge 0;\ X_t \notin (x_{\ell,1}[v_{\ell,n}], x_{\ell,2}[v_{\ell,n}])\}$ is ε-optimal for every $\varepsilon > \frac{cL}{r}(\frac{\lambda\gamma}{r+\lambda\gamma})^n$ for the portfolio manager's second problem; namely, for every $x > 0$

$$U_\ell(x) - \frac{cL}{r}\left(\frac{\lambda\gamma}{r+\lambda\gamma}\right)^n \le \mathbb{E}_x^\gamma\Big[e^{-r\widetilde{\tau}_\ell[v_{\ell,n}]}(X_{\widetilde{\tau}_\ell[v_{\ell,n}]} - L)^+ + \int_0^{\widetilde{\tau}_\ell[v_{\ell,n}]} e^{-rt}(\delta X_t - cL)\mathrm{d}t\Big] \le U_\ell(x).$$

6. Numerical illustration

For illustration, we take $L = 1$, $\sigma = 0.275$, $r = 0.03$, $c = 0.05$, $\delta = 0.08$, $\lambda\gamma = 0.01$, $y_0 = 0.03$. Observe that $0 < r < c < \delta$. We obtain $\alpha_0 = -0.3910$ and $\alpha_1 = 2.7054$. We implemented the numerical algorithms of Section 5 in R in order to use readily available routines to calculate the smallest nonnegative concave majorants of functions. We have used `gcmlcm` function from the R

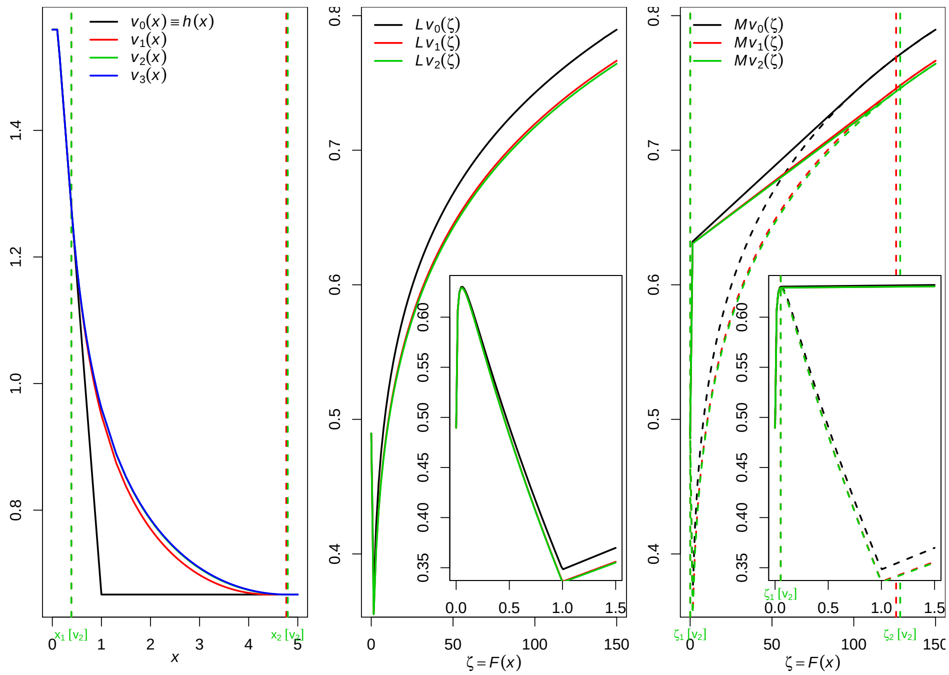

FIGURE 4. Numerical illustration of the solution of the auxiliary optimal stopping problem (3.7) in the first problem.

package `fdrtool` developed by Korbinian Strimmer for that purpose. The approximation functions `approxfun` and `splinefun` were also useful to compactly represent the functions we evaluated on appropriate grids placed on state space and its F-transformation. By trial-and-error, we find out that optimal continuation region lies strictly inside $[0, 10L]$. Because $F(L)$ turns out to be significantly smaller than the upper bound $10L$, for the accuracy of the results it proved useful to put a grid on the interval $[0, F(L)]$ one hundred times finer than the grid put on $[F(L), F(10L)]$.

In the implementation of Step 5 of the numerical algorithms of Section 5, we decided to stop the iterations as soon as the maximum absolute difference between the last two approximations on the grid placed on $[0, 10L]$ is less than 0.01. The first algorithm stops after three iterations with the maximum absolute difference $\|v_3 - v_2\| \approx 0.0011$ and returns $v_3(\cdot)$, $(0, x_1[v_2]] \cup [x_2[v_2], \infty) = (0, 0.3874] \cup [4.7968, \infty)$, and $\widetilde{\tau}[v_3] = \inf\{t \geq 0;\ X_t \notin (0, 0.3874] \cup [4.7968, \infty)\}$ as the approximate value function, approximate stopping region, and nearly optimal stopping rule for (3.7). The bound in (i) on page 5 also guarantees that $\|V(\cdot) - v_3(\cdot)\| \leq \frac{cL}{r}(\frac{\lambda\gamma}{r+\lambda\gamma})^3 = 0.026$. The leftmost picture in Figure 4 suggests that the algorithm actually converges faster than what this upper bound suggests. The middle and rightmost pictures illustrate how the solution of each auxiliary problem is found by constructing the smallest nonnegative concave majorants M of the transformations with operator L. The insets give closer look over the small interval $[0, F(L)]$ at the same pictures which are otherwise harder to identify. All of the pictures in Figure 4 are consistent with the general form sketched in Figure 1.

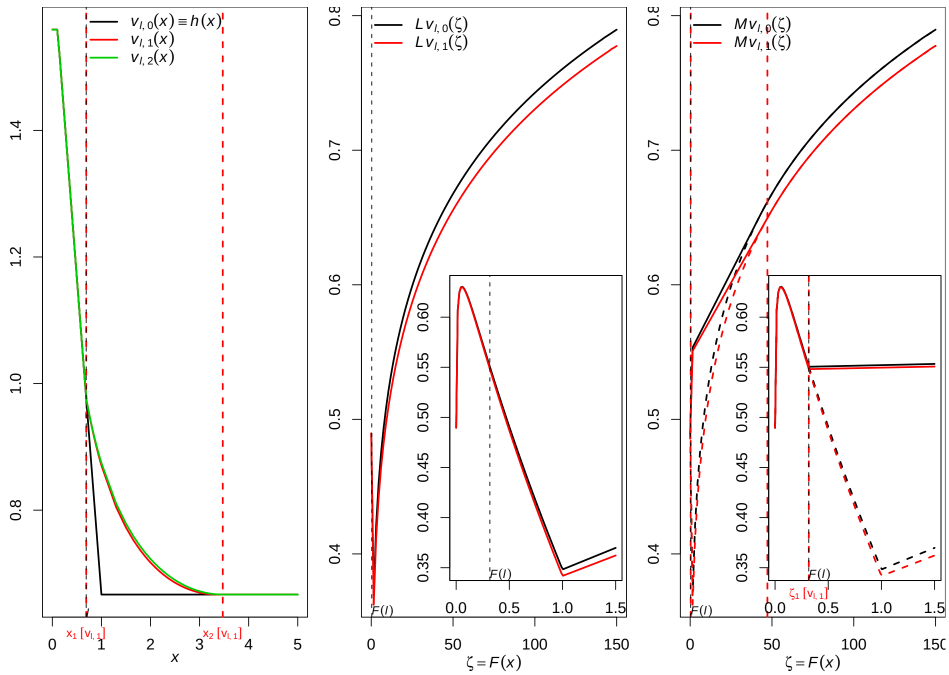

FIGURE 5. Numerical illustration of the solution of the auxiliary optimal stopping problem (4.3) in the second problem when the protection level equals $\ell = 0.69$.

Figure 5 similarly illustrates the solution of the second problem of the asset manager when the investors hold a limited protection of their assets with lower bound $\ell = 0.69$ on the market value of the asset manager's portfolio. Because $x_1[v_\infty] \approx x_1[v_2] = 0.3874 < \ell < 4.7968 = x_2[v_2] \approx x_2[v_\infty]$, the unconstrained solution of Problem 1 (corresponding to $\ell = 0$) is not any more optimal. Therefore, we run the second algorithm of Section 5, which converges in two iterations because $\|v_{\ell,2} - v_{\ell,1}\| \approx 0.0063 < 1/100$. Hence, $v_{\ell,2}(\cdot)$, $(0, x_{\ell,1}[v_{\ell,1}]] \cup [x_{\ell,2}[v_{\ell,1}], \infty) = (0, 0.69] \cup [3.4724, \infty)$, and $\widetilde{\tau}_\ell[v_{\ell,1}] = \inf\{t \geq 0;\ X_t \notin (0, 0.69] \cup [3.4724, \infty)\}$ are approximate value function, approximate stopping region, and nearly optimal stopping rule for (4.3).

Observe that the stopping region of Problem 2 contains the stopping region of Problem 1: $(0, x_{\ell,1}[v_{\ell,1}]] \cup [x_{\ell,2}[v_{\ell,1}], \infty) = (0, 0.69] \cup [3.4724, \infty) \supset (0, x_1[v_2]] \cup [x_2[v_2], \infty) = (0, 0.3874] \cup [4.7968, \infty)$. Thus, asset manager stops early in the presence of portfolio protection at level $\ell = 0.69$. Because $U(x) \approx x - \frac{cL}{r} + v_2(x)$ and $U_\ell(x) \approx x - \frac{cL}{r} + v_{\ell,1}(x)$ are approximately the value functions of Problems 1 and 2, the value of the limited protection at level ℓ when stock price is x equals $U(x) - U(\ell)(x) \approx v_3(x) - v_{\ell,2}(x)$, which is plotted on the left in Figure 6. Therefore, the no-difference price of this protection at the initiation of the contract equals $U(L) - U(\ell)(L) \approx v_3(L) - v_{\ell,2}(L) = 0.087$. The plot on the right in Figure 6 shows the no-difference prices of the protection at levels ℓ changing between 0 and $L = 1$. The protection has no value at the protection levels less than or equal to $x_1[v_\infty] \approx x_1[v_2]$, because the optimal policy, even in the

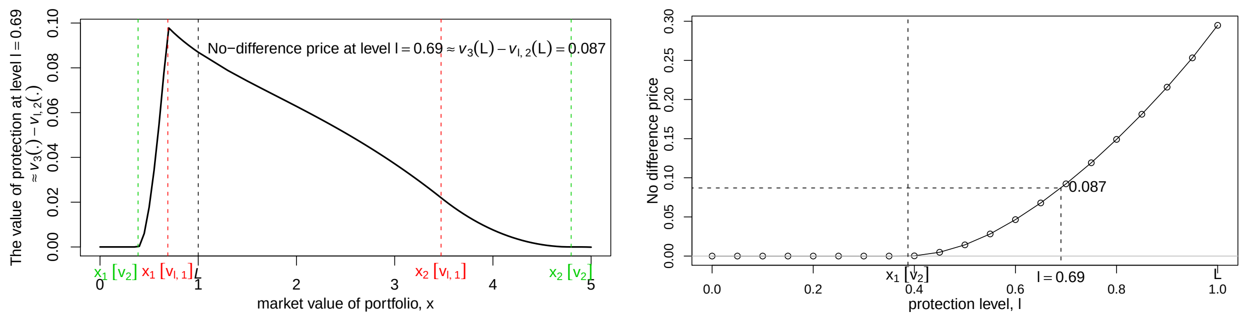

FIGURE 6. On the left, the value of the limited protection at level $\ell = 0.69$ as the market value of portfolio changes, and on the right, no-difference prices of the protections for different protection limits.

absence of protection clause, instructs the asset manager to quit as soon as the market value of the portfolio goes below $x_1[v_\infty] \approx x_1[v_2]$.

REFERENCES

[1] S. Asmussen, F. Avram, and M. R. Pistorius. Russian and American put options under exponential phase-type Lévy models. *Stochastic Process. Appl.*, 109(1):79–111, 2004.

[2] S. I. Boyarchenko and S. Z. Levendorskiĭ. Perpetual American options under Lévy processes. *SIAM J. Control Optim.*, 40(6):1663–1696 (electronic), 2002.

[3] T. Chan. Pricing contingent claims on stocks driven by Lévy processes. *Ann. Appl. Probab.*, 9(2):504–528, 1999.

[4] D. Colwell and R. Elliott. Discontinuous asset prices and non-attainable contingent claims. *Mathematical Finance*, 3(3):295–308, 2006.

[5] R. Cont and P. Tankov. *Financial modelling with jump processes.* Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[6] S. Dayanik. Optimal stopping of linear diffusions with random discounting. *Math. Oper. Res.*, 33(3):645–661, 2008.

[7] S. Dayanik and I. Karatzas. On the optimal stopping problem for one-dimensional diffusions. *Stochastic Process. Appl.*, 107(2):173–212, 2003.

[8] S. Dayanik, H. V. Poor, and S. O. Sezer. Multisource Bayesian sequential change detection. *Ann. Appl. Probab.*, 18(2):552–590, 2008.

[9] S. Dayanik and S. O. Sezer. Multisource bayesian sequential hypothesis testing. Preprint, 2009.

[10] D. Duffie. *Dynamic asset pricing theory.* Princeton University Press, Princeton, NJ, 1996.

[11] D. Duffie and N. Garleanu. Risk and valuation of collateralized debt obligations. *Financial Analysts Journal*, 57(1):41–59, 2001.

[12] D. Duffie and K. Singleton. Modeling term structures of defaultable bonds. *Review of Financial Studies*, 12(4):687, 1999.

[13] M. Egami and K. Esteghamat. An approximation method for analysis and valuation of credit correlation derivatives. *Journal of Banking & Finance*, 30(2):341–364, 2006.

[14] J. Hull and A. White. Valuation of a CDO and an n-th to default CDS without Monte Carlo simulation. *The Journal of Derivatives*, 12(2):8–23, 2004.

[15] S. Karlin and H. M. Taylor. *A second course in stochastic processes*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1981.

[16] S. Kou and H. Wang. Option pricing under a double exponential jump diffusion model. *Management Science*, pages 1178–1192, 2004.

[17] H. Leland. Corporate debt value, bond covenants, and optimal capital structure. *Journal of finance*, pages 1213–1252, 1994.

[18] D. Lucas, L. Goodman, and F. Fabozzi. *Collateralized debt obligations: structures and analysis*. John Wiley & Sons Inc, 2006.

[19] E. Mordecki. Optimal stopping for a diffusion with jumps. *Finance Stoch.*, 3(2):227–236, 1999.

[20] E. Mordecki. Optimal stopping and perpetual options for Lévy processes. *Finance Stoch.*, 6(4):473–493, 2002.

[21] H. Pham. Optimal stopping, free boundary, and American option in a jump-diffusion model. *Appl. Math. Optim.*, 35(2):145–164, 1997.

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