A DIRECT SOLUTION METHOD FOR PRICING OPTIONS INVOLVING MAXIMUM PROCESS

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ABSTRACT. One encounters options involving not only the stock price but also its running maximum. We provide, in a fairly general setting, explicit solutions for optimal stopping problems concerned with diffusion process and its running maximum. Our approach is to use the excursion theory for Markov processes and rewrite the original two-dimensional problem as an infinite number of one-dimensional ones. Our method is rather direct without presupposing optimal threshold or imposing the smooth-fit condition. We present a systematic solution method by illustrating it through classical and new examples.

Key words: American options with maximum process; Optimal stopping; excursion theory; diffusions.
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1. INTRODUCTION

We let $X = (X_t, t \geq 0)$ be one-dimensional diffusion and denote by Y the reflected process,

$$Y_t = S_t - X_t$$

where $S_t = \sup_{u\in[0,t]} X_u \vee s$ with $s = S_0$. Hence Y is the excursion of X from its running maximum S. We consider an optimal stopping problem that involves both X and S. That is,

$$\bar{V}(x,s) \quad = \quad \sup_{\tau} \mathbb{E}^{x,s}\left[\int_0^{\tau} e^{-qt} f(X_t,S_t)\mathrm{d}t + e^{-q\tau} g(X_\tau,S_\tau)\right] \tag{1.1}$$

where f and g are reward functions from $\mathbb{R}^2$ to $\mathbb{R}_+$. The rigorous mathematical definition of this problem is presented in Section 2. In this paper, we shall solve for optimal strategy and corresponding value function along with optimal stopping region in the (x,s)-plane.

For American option pricing that involve both S and X, we mention pioneering works of Shepp and Shiryaev [20] and Peskir [15]. In the former paper, the Russian option is solved and in the latter the author established the "maximum principle". There is also Ott [14] where the author solves problems including a capped version of the Shepp-Shiryaev [20]. We should mention Guo and Zervos [10], which makes another extension of [20] by treating the reward function $g(x,s) = (x^a s^b - K)^+$ with $a,b,K \geq 0$. This reward function includes perpetual call, lookback option, etc. as special cases. In many solved problems, Brownian motion or geometric Browninan motion is used as underlying process in an effort to obtain tractable solutions. A recent development in this area includes Alvarez and Matoäki [2] where a discretized approach is taken to find optimal solutions and a corresponding numerical

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algorithm is presented. In the context of risk management, an excursion from the running maximum is sometimes called *drawdown*. See Hadjiliadis and Zhang [22] and Zhang [21]. For example, the joint Laplace transform of the last visit time of the maximum of a process preceding the drawdown and the maximum of the process is calculated in the former, while a perturbation approach for obtaining the Laplace transform is used in the latter.

The idea of our solution method is the following: we look at excursions that occur from each level of S, during an excursion from level $S_t = s$, the value of S_t is fixed until X returns to s. By the dynamic programming principle the value there is $V(s,s)$. Using this, the problems reduce to an infinite number of one-dimensional optimal stopping problems. Note that this idea is mentioned in Ott [13] when the author argued the existence of optimal stopping time of hitting-time type. Then the author uses the smooth-fit principle for optimal stopping problems driven by spectrally negative Lévy processes. In this spirit, we attempt to rewrite the problem equation (1.1) in the form of sequences of excursions.

The difficulty is in finding $V(s,s)$. To this end, we employ the theory of excursion of Markov processes, in particular the excursion measure (also called characteristic measure for excursion) that is related to the height of excursions. (Refer to Bertoin [4] as a general reference.) The outcome is the representation of $V(s,s)$ in Proposition 3.1. To make it more explicit, we implement some limit-taking operation in Proposition 4.1. This part of the article (Sections 3 and 4) consists of the main contribution, describing a new solution method. For the excursion theory for spectrally negative Lévy processes (that have only downward jumps), we mention Avram et al. [3], Pistorius [18] [19] and Doney [8] where, among others, an exit problem of the reflected process Y is studied.

Having done that, we solve, at each level of S, one-dimensional optimal stopping problems by using the excessive characterization of the value function. This corresponds to the concavity of the value function after certain transformation, by which we can treat problems in a systemic way. We provide a kind of solution recipe in Section 5. We briefly review the aforementioned transformation in Section 2.2. See Dynkin[9], Alvarz[1] and Dayanik and Karatzas [7] for more details.

Our contributions in this paper may advance the literature in several respects: we do not assume any specific forms or properties in the reward functions (except for mild ones), and we provide explicit forms of the value function and illustrate the procedure of the solution method. In contrast to the literature, our approach is rather direct since we do not impose the smooth-fit principle in deciding optimal boundary. Accordingly, one does not have to prove so-called "verification lemma" (that is important in showing the presupposed candidate value function is in fact a solution) and hence may handle a broader set of problems.

The rest of the paper is organized as follows. In Section 2, we formulate a mathematical model with a review of some important facts of linear diffusions, and then find an optimal solution. The key step is to represent and compute the value $V(s,s)$, which is handled in Sections 3 and 4. Under the mild assumptions (Assumption 4.1), we present $V(s,s)$ in an explicit form. The next step is to find $V(x,s)$ in Section 5. Moreover, we shall demonstrate the methodology by using a new problem (Section 5.1) as well as some problems in the literature (Section 4.4.1 and 4.4.2). Let us stress that the new problem might not be easily handled by the conventional methods. The Appendix includes a proof of the technical lemma.

2. Mathematical Model

2.1. **Setup.** Let the diffusion process $X = \{X_t; t \geq 0\}$ represent the state variable defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where Ω is the set of all possible realizations of the stochastic economy, and $\mathbb{P}$ is a probability measure defined on $\mathscr{F}$. The state space of X is given by $(l, r) := \mathscr{I} \subseteq \mathbb{R}$, where l and r are *natural boundaries*. That is, X cannot start from and exit from l or r. We denote by $\mathbb{F} = \{\mathscr{F}_t\}_{t \geq 0}$ the filtration with respect to which X is adapted and with the usual conditions being satisfied. We assume that X satisfies the following stochastic differential equation:

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \quad X_0 = x,$$

where $B = \{B_t : t \geq 0\}$ is a standard Brownian motion and $\mu : \mathscr{I} \mapsto \mathbb{R}$ and $\sigma : \mathscr{I} \mapsto (0, \infty)$ satisfy the usual Lipschitz conditions ensuring the existence and uniqueness of a solution given an initial condition. The running maximum process $S = \{S_t; t \geq 0\}$ with $s = S_0$ is defined by $S_t = \sup_{u \in [0,t]} X_u \vee s$. In addition, we write Y for the reflected process defined by $Y_t = S_t - X_t$. We consider the following optimal stopping problem and the value function $\bar{V} : \mathbb{R}^2 \mapsto \mathbb{R}$ associated with initial values $X_0 = x$ and $S_0 = s$;

$$\bar{V}(x,s) = \sup_{\tau \in \mathscr{S}} \mathbb{E}^{x,s}\left[\int_0^\tau e^{-qt} f(X_t, S_t)\mathbb{1}_{\{\tau < +\infty\}}\mathrm{d}t + e^{-q\tau} g(X_\tau, S_\tau)\mathbb{1}_{\{\tau < +\infty\}}\right] \tag{2.1}$$

where $\mathbb{P}^{x,s}(\cdot) := \mathbb{P}(\cdot \,|\, X_0 = x, S_0 = s)$ and $\mathbb{E}^{x,s}$ is the expectation operator corresponding to $\mathbb{P}^{x,s}$, $q \geq 0$ is the constant discount rate and $\mathscr{S}$ is the set of all $\mathbb{F}$-adapted stopping times. The payoff is composed of two parts; the running income to be received continuously until stopped, and the terminal reward part. The running income function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ is a measurable function that satisfies

$$\mathbb{E}^{x,s}\left[\int_0^\infty e^{-qt}|f(X_t, S_t)|\mathrm{d}t\right] < \infty. \tag{2.2}$$

Regarding the integrability condition, it is worth mentioning that if f is continuous and satisfies the linear growth condition

$$|f(x,s)| \leq C_1(1 + |s|)$$

for some strictly positive constant $C_1 < \infty$, then it is guaranteed that

$$\mathbb{E}^{x,s}\left[\int_0^\infty e^{-qt}|f(X_t, S_t)|\mathrm{d}t\right] \leq C_2(1 + |s|)$$

for some C_2 when the discount rate is large enough (see Pham [16] page 191). The reward function $g : \mathbb{R}^2 \mapsto \mathbb{R}_+$ is assumed to be measurable and satisfied the linear growth condition. See also Pham [17] for these assumptions on f and g. Our main purpose is to calculate $\bar{V}$ and to find the stopping time τ^* which attains the supremum.

2.2. **Reduction to One-Dimensional Problem.** We will reduce the problem (2.1) to an infinite number of one-dimensional optimal stopping problem and discuss the optimality of the proposed strategy (2.12). Let us denote by $\bar{f} : \mathbb{R}^2 \mapsto \mathbb{R}$ the q-potential of f, that is, $\bar{f}(x,s) := \mathbb{E}^{x,s}[\int_0^\infty e^{-qt} f(X_t, S_t)\mathrm{d}t]$. From the strong Markov property of

(X,S), we have

$$\begin{aligned}\mathbb{E}^{x,s}\left[\int_0^\tau e^{-qt}f(X_t,S_t)\mathbb{1}_{\{\tau<+\infty\}}\mathrm{d}t\right] &= \mathbb{E}^{x,s}\left[\int_0^\infty e^{-qt}f(X_t,S_t)\mathrm{d}t - \int_\tau^\infty e^{-qt}f(X_t,S_t)\mathbb{1}_{\{\tau<+\infty\}}\mathrm{d}t\right]\\ &= \bar{f}(x,s) - \mathbb{E}^{x,s}\left[\mathbb{E}\left[\int_\tau^\infty e^{-qt}f(X_t,S_t)\mathbb{1}_{\{\tau<+\infty\}}\mathrm{d}t \;\middle|\; \mathscr{F}_\tau\right]\right]\\ &= \bar{f}(x,s) - \mathbb{E}^{x,s}\left[e^{-q\tau}\mathbb{E}^{X_\tau,S_\tau}\left[\int_0^\infty e^{-qt}f(X_t,S_t)\mathrm{d}t\right]\mathbb{1}_{\{\tau<+\infty\}}\right]\\ &= \bar{f}(x,s) - \mathbb{E}^{x,s}\left[e^{-q\tau}\bar{f}(X_\tau,S_\tau)\mathbb{1}_{\{\tau<+\infty\}}\right].\end{aligned}$$

Hence the value function $\bar{V}$ can be written as

$$\bar{V}(x,s) \quad = \quad \bar{f}(x,s) + V(x,s),$$

where

$$V(x,s) := \sup_{\tau\in\mathscr{S}} \mathbb{E}^{x,s}\left[e^{-q\tau}(g-\bar{f})(X_\tau,S_\tau)\mathbb{1}_{\{\tau<+\infty\}}\right]. \tag{2.3}$$

Since $\bar{f}(x,s)$ has nothing to do with the choice of τ, we concentrate on $V(x,s)$.

Let us first define the first passage times of X:

$$T_a := \inf\{t\geq 0 : X_t > a\} \quad \text{and} \quad T_a^- := \inf\{t\geq 0 : X_t < a\}. \tag{2.4}$$

Under the assumptions on f and g in the last subsection, by the dynamic programming principle, we can write $V(x,s)$ as

$$V(x,s) = \sup_{\tau\in\mathscr{S}} \mathbb{E}^{x,s}\left[\mathbb{1}_{\{\tau<\theta\}}e^{-q\tau}(g-\bar{f})(X_\tau,S_\tau) + \mathbb{1}_{\{\theta\leq\tau<+\infty\}}e^{-q\theta}V(X_\theta,S_\theta)\right], \tag{2.5}$$

for any stopping time $\theta\in\mathscr{S}$. See, for example, Pham [17] page 97. Now we set $\theta = T_s$ in (2.5). For each level $S=s$ from which an excursion $Y=S-X$ occurs, the value S does not change during the excursion. Hence, during the first excursion interval from $S_0=s$, we have $S_t=s$ for any $t\leq T_s$, and (2.5) can be written as the following one-dimensional problem for the state process X;

$$V(x,s) \quad = \quad \sup_{\tau\in\mathscr{S}} \mathbb{E}^{x,s}\left[\mathbb{1}_{\{\tau<T_s\}}e^{-q\tau}(g-\bar{f})(X_\tau,s) + \mathbb{1}_{\{T_s\leq\tau<+\infty\}}e^{-qT_s}V(s,s)\right]. \tag{2.6}$$

Now we can look at *only* the process X and find $\tau^*\in\mathscr{S}$. In relation to (2.6), we consider the following one-dimensional optimal stopping problem as for X and its value function $\widehat{V}:\mathbb{R}^2\mapsto\mathbb{R}$;

$$\widehat{V}(x,s) = \sup_{\tau\in\mathscr{S}} \mathbb{E}^{x,s}\left[\mathbb{1}_{\{\tau<T_s\}}e^{-q\tau}(g-\bar{f})(X_\tau,s) + \mathbb{1}_{\{T_s\leq\tau<+\infty\}}e^{-qT_s}K\right], \tag{2.7}$$

where $K\geq 0$ is a constant. Note that $V=\widehat{V}$ holds when $K=V(s,s)$, and we shall present how to characterize and compute $V(s,s)$ in Sections 3 and 4.

Before presenting the solution method, we recall the fundamental facts about one-dimensional diffusions; let the differential operator $\mathscr{A}$ be the infinitesimal generator of the process X defined by

$$\mathscr{A}v(\cdot) = \frac{1}{2}\sigma^2(\cdot)\frac{\mathrm{d}^2v}{\mathrm{d}x^2}(\cdot) + \mu(\cdot)\frac{\mathrm{d}v}{\mathrm{d}x}(\cdot)$$

and consider the ODE $\mathscr{A}v - qv = 0$. This equation has two fundamental solutions: $\psi(\cdot)$ and $\varphi(\cdot)$. We set $\psi(\cdot)$ to be the increasing and $\varphi(\cdot)$ to be the decreasing solution. They are linearly independent positive solutions and uniquely determined up to multiplication. It is well known that

$$\mathbb{E}^x[e^{-\alpha\tau_z}] = \begin{cases} \frac{\psi(x)}{\psi(z)}, & x \le z, \\ \frac{\varphi(x)}{\varphi(z)}, & x \ge z. \end{cases} \tag{2.8}$$

For the complete characterization of $\psi(\cdot)$ and $\varphi(\cdot)$, refer to Itô and McKean [11]. Let us now define

$$F(x) := \frac{\psi(x)}{\varphi(x)}, \qquad x \in \mathscr{I}. \tag{2.9}$$

Then $F(\cdot)$ is continuous and strictly increasing. Next, following Dynkin (pp. 238, [9]), we define concavity of a function with respect F as follows: A real-valued function u is called *F-concave* on $\mathscr{I}$ if, for every $x \in [l,r] \subseteq \mathscr{I}$,

$$u(x) \ge u(l)\frac{F(r)-F(x)}{F(r)-F(l)} + u(r)\frac{F(x)-F(l)}{F(r)-F(l)}.$$

Now consider the optimal stopping problem:

$$V(x) = \sup_{\tau\in\mathscr{S}} \mathbb{E}^x[e^{-q\tau}h(X_\tau)]$$

where h: $[c,d] \mapsto \mathbb{R}_+$. Let $W(\cdot)$ be the smallest nonnegative concave majorant of

$$H := \frac{h}{\varphi} \circ F^{-1} \qquad \text{on } [F(c), F(d)] \tag{2.10}$$

where F^{-1} is the inverse of F. Then we have $V(x) = \varphi(x)W(F(x))$ and the optimal stopping region Γ is

$$\Gamma := \{x \in [c,d] : V(x) = h(x)\} \quad \text{and} \quad \tau^* := \inf\{t \ge 0 : X_t \in \Gamma\}.$$

Note that for the rest of this article, the term "transformation" should be understood as (2.10).

When both boundaries l and r are natural, $V(x) < +\infty$ for all $x \in (l,r)$ if and only if

$$\xi_l := \limsup_{x\downarrow l} \frac{h^+(x)}{\varphi(x)} \quad \text{and} \quad \xi_r := \limsup_{x\uparrow r} \frac{h^+(x)}{\psi(x)} \tag{2.11}$$

are both finite.

2.3. **Optimal Strategy.**

We reproduce (2.7) here:

$$\widehat{V}(x,s) = \sup_{\tau\in\mathscr{S}} \mathbb{E}^{x,s}\left[\mathbb{1}_{\{\tau<T_s\}}e^{-q\tau}(g-\bar{f})(X_\tau,s) + \mathbb{1}_{\{T_s\le\tau<+\infty\}}e^{-qT_s}K\right], x \in (-\infty, s].$$

Note that the right absorbing boundary is s where one receives reward K. Then we can use the general theory of one-dimensional optimal stopping problem:

Proposition 2.1. *The optimal stopping region $\Gamma(s)$ and optimal strategy $\tau^*(s)$ for each s fixed in (2.7) are*

$$\Gamma(s) := \{x < s : V(x) = (g-\bar{f})(x,s)\} \quad \textit{and} \quad \tau^*(s) := \inf\{t \ge 0 : X_t \in \Gamma(s)\}.$$

Proof. Apply Proposition 4.4 of Dayanik and Karatzas [7]. $\square$

In the (x,s)-plane, for a Borel measurable set $D \in \mathscr{B}(\mathbb{R}^2)$ and $m \in \mathbb{R}$, define a set $D(m) \in \mathscr{B}(\mathbb{R})$ such that

$$D(m) \times \{m\} = (\mathbb{R} \times \{m\}) \cap D$$

holds where $\mathscr{B}(\mathbb{R}^d)$ denotes the Borel σ-algebra in $\mathbb{R}^d$. Hence $D(m)$ is the section of D by the horizontal line $s = m$ in the (x,s)-plane and we hence write $D = \bigcup_s (D(s) \times \{s\})$ now. Define a set of strategies

$$\tau(D) := \inf\{t \geq 0 : S_t - X_t \in D(S_t)\}. \tag{2.12}$$

In other words, $\tau(D)$ is the first time the excursion $S - X$ from level, say $S = s$, enters the region $D(s)$. We collect this type of strategies to form a set

$$\mathscr{S}' := \{\tau(D)\}_{D \in \mathbb{R}^2} \subset \mathscr{S}.$$

In particular, suppose that $D(m) = (c, +\infty)$ for any $m \in \mathbb{R}$, we write

$$\tau_c := \inf\{t \geq 0 : S_t - X_t > c\}.$$

Then, from Proposition 2.1, by setting $D(s) = \Gamma(s)$ for each s, we can obtain an optimal stopping strategy $\tau^* \in \mathscr{S}'$.

As for the calculation of $\widehat{V}$, the following propositions are available also from [7]:

Proposition 2.2. *Fix $s \in \mathbb{R}$. The value function $\widehat{V}(\cdot)$ of (2.7) is the smallest nonnegative majorant of $(g - \bar{f})(\cdot)$ and the point (s, K) such that $\widehat{V}(\cdot)/\varphi(\cdot)$ is F-concave.*

Proposition 2.3. *Fix $s \in \mathbb{R}$. Let $W(\cdot)$ be the smallest nonnegative concave majorant of $H := ((g - \bar{f})/\varphi) \circ F^{-1}$ and the point $(F(s), K/\varphi(s,s)$ on $[F(-\infty), F(s)]$. Then $\widehat{V}(x) = \varphi(x) W(F(x))$, for every $x < s$.*

Once we have $K = V(s,s)$, we can compute the global solution $V(x,s)$ from these propositions. However, *the real difficulty lies in how to obtain $V(s,s)$.*

3. Representation of $V(s,s)$

Now we look to an explicit solution of $\bar{V}$ in $\mathscr{S}'$. The first step is to find $K = V(s,s)$ in (2.7). That is, we consider the case $S_0 = X_0$ and want to compute the right-hand side of (2.3) with $s = x$. Set stopping times $T_m = \inf\{t \geq 0 : X_t > m\}$ as in (2.4). and define a function $l_D : \mathbb{R}_+ \mapsto \mathbb{R}_+$ by

$$l_D(m) := \inf D(m). \tag{3.1}$$

for which $\tau(D) \in \mathscr{S}'$. Since we have shown that optimal strategy belongs to threshold strategies in the previous section, we now focus on the set of $\tau(D)$ in (2.12). Hence we can write from (2.3)

$$V(s,s) = \sup_{\tau(D) \in \mathscr{S}'} \mathbb{E}^{s,s}[\mathbb{1}_{\{\tau(D) < +\infty\}} e^{-q\tau} (g - \bar{f})(X_{\tau(D)}, S_{\tau(D)})] \tag{3.2}$$

and try to simplify the expectation on the right-hand side. This is done in the next proposition. Note that at this stage we do not specify the values of l_D, which we shall do in Section 4.

Proposition 3.1. *When $S_0 = X_0$, the function $V(s,s)$, finite or infinite, for $\tau \in \mathscr{S}$ can be represented by*

$$\begin{aligned} V(s,s) \quad = \quad & \sup_{l_D} \int_s^\infty \frac{\varphi(s)}{\varphi(m - l_D(m))} \exp\left(- \int_s^m \frac{F'(u)\mathrm{d}u}{F(u) - F(u - l_D(u))} \right) \\ & \times \frac{F'(m)(g - \bar{f})(m - l_D(m), m)}{F(m) - F(m - l_D(m))} \mathrm{d}m. \end{aligned} \tag{3.3}$$

Proof. The notation in (3.1) is to identify an exact point of stopping. In other words, if $S_0 = X_0$, given a threshold strategy $\tau = \tau(D)$ where $D(m)$ is in the form of $[a,c] \subset [0,+\infty)$, the value $l_{D(m)}$ is equal to a. Accordingly, $S_{\tau_{l_{D(m)}}} = S_{\tau_a}$ on the set $\{S_{\tau_{l_{D(m)}}} \in \mathrm{d}m\}$. Due to the continuity of sample paths of X and $X_0 = S_0$, we are looking at continuous paths of excursion from $S_0 = s$. It follows that $X_{\tau(D)} = m - l_{D(m)}$ on $\{S_{\tau(D)} = m\}$ and therefore, for the purpose of computing $V(s,s)$, it suffices to look to these points $l_{D(m)}$.

From the strong Markov property of (X,S), when $\tau(D) \in \mathscr{S}'$ and $S_0 = X_0 = s$, it becomes

$$
\begin{aligned}
(3.4)\qquad & \mathbb{E}^{s,s}\left[\mathbb{1}_{\{\tau(D)<+\infty\}} e^{-q\tau(D)} (g-\bar{f})(X_{\tau(D)}, S_{\tau(D)})\right] \\
= & \int_s^\infty \mathbb{E}^{s,s}\left[\mathbb{1}_{\{\tau(D)<+\infty, S_{\tau(D)}\in \mathrm{d}m\}} e^{-q\tau(D)} (g-\bar{f})(X_{\tau(D)}, S_{\tau(D)})\right] \\
= & \int_s^\infty \mathbb{E}^{s,s}\left[\mathbb{1}_{\{T_m \le \tau(D)\}} e^{-qT_m} \mathbb{E}^{m,m}\left[e^{-q\tau_{l_{D(m)}}} (g-\bar{f})(X_{\tau_{l_{D(m)}}}, S_{\tau_{l_{D(m)}}})\right.\right. \\
& \left.\left. \times \mathbb{1}_{\{S_{\tau_{l_{D(m)}}} \in \mathrm{d}m\}}\right]\right] \\
= & \int_s^\infty \mathbb{E}^{s,s}\left[\mathbb{1}_{\{S_{\tau(D)} \ge m\}} e^{-qT_m}\right] (g-\bar{f})(m - l_D(m), m) \\
& \times \mathbb{E}^{m,m}\left[e^{-q\tau_{l_{D(m)}}} \mathbb{1}_{\{S_{\tau_{l_{D(m)}}} \in \mathrm{d}m\}}\right].
\end{aligned}
$$

Now we calculate these expectations by changing probability measure. We introduce the probability measure $\mathbb{P}^{\varphi,q}_{x,s}$ defined by

$$
(3.5)\qquad \mathbb{P}^{\varphi,q}_{x,s}(A) := \frac{1}{\varphi(x)} \mathbb{E}^{x,s}\left[e^{-qt} \varphi(X_t) \mathbb{1}_A\right], \text{for every } A \in \mathscr{F}.
$$

Then under the new measure $\mathbb{P}^{\varphi,q}_{x,s}$, the scale function of X is equal to F, which means that

$$
\mathbb{P}^{\varphi,q}_{x,s}(\tau_a < \tau_b) = \frac{F(b) - F(x)}{F(b) - F(a)}, \text{ for } a < x < b,
$$

and in other words that the process $F(X)$ is in natural scale. Note that processes in natural scale include a standard Brownian motion. See Borodin and Salminen [5] (page 33) and Dayanik and Karatzas [7] (Chapter 8) for detailed explanations. Since $F(X)$ is a diffusion, we can define the process $\eta := \{\eta_t; t \ge 0\}$ of the height of the excursion as

$$
\eta_u := \sup\{(S-X)_{T_{u-}+w} : 0 \le w \le T_u - T_{u-}\}, \text{ if } T_u > T_{u-},
$$

and $\eta_u = 0$ otherwise, where $T_{u-} := \inf\{t \ge 0 : X_t \ge u\} = \lim_{m\to u-} T_m$. Then η is a Poisson point process, and we denote its excursion measure under $\mathbb{P}^{\varphi,q}_{x,s}$ by $\nu : \mathscr{F} \mapsto \mathbb{R}_+$ of $F(X)$. It is well known that

$$
\nu[u,\infty) = \frac{1}{u}, \text{ for } u \in \mathbb{R}_+ \backslash \{0\}.
$$

See, for example, Çinlar [6] (pp. 416), where we used the fact that $f(X)$ is in natural scale under $\mathbb{P}^{\varphi,q}_{x,s}$. By using these notations, we have[1]

$$\begin{aligned}\mathbb{P}^{\varphi,q}_{s,s}(S_{\tau(D)} > m) &= \exp\left(-\int_{F(s)}^{F(m)} \nu[y - F(F^{-1}(y) - l_D(F^{-1}(y))), \infty)\mathrm{d}y\right) \\ &= \exp\left(-\int_s^m \frac{F'(u)\mathrm{d}u}{F(u) - F(u - l_D(u))}\right). \end{aligned} \tag{3.6}$$

On the other hand, from the definition of the measure $\mathbb{P}^{\varphi,q}_{x,s}$, we have

$$\begin{aligned}\mathbb{P}^{\varphi,q}_{s,s}(S_{\tau(D)} > m) &= \frac{1}{\varphi(s)}\mathbb{E}^{s,s}\left[e^{-qT_m}\varphi(X_{T_m})\mathbb{1}_{\{S_{\tau(D)}>m\}}\right] \\ &= \frac{\varphi(m)}{\varphi(s)}\mathbb{E}^{s,s}\left[e^{-qT_m}\mathbb{1}_{\{S_{\tau(D)}>m\}}\right].\end{aligned}$$

Combining these two things together,

$$\mathbb{E}^{s,s}\left[e^{-qT_m}\mathbb{1}_{\{S_{\tau(D)}>m\}}\right] = \frac{\varphi(s)}{\varphi(m)}\exp\left(-\int_s^m \frac{F'(u)\mathrm{d}u}{F(u) - F(u - l_D(u))}\right). \tag{3.7}$$

Similarly, by changing the measure and noting that $X_{\tau_{l_D(m)}} = m - l_D(m)$, we have

$$\begin{aligned}&\mathbb{E}^{m,m}\left[e^{-q\tau_{l_D(m)}}\mathbb{1}_{\{S_{\tau_{l_D(m)}}\in \mathrm{d}m\}}\right] \\ &= \frac{\varphi(m)}{\varphi(m - l_D(m))}\cdot\frac{1}{\varphi(m)}\mathbb{E}^{m,m}\left[e^{-q\tau_{l_D(m)}}\varphi(X_{\tau_{l_D(m)}})\mathbb{1}_{\{S_{\tau_{l_D(m)}}\in \mathrm{d}m\}}\right] \\ &= \frac{\varphi(m)}{\varphi(m - l_D(m))}\mathbb{P}^{\varphi,q}_{m,m}(F(S_{\tau_{l_D(m)}}) \in \mathrm{d}F(m))\end{aligned} \tag{3.8}$$

To compute the last probability, differentiate (3.6) with respect to m, multiply by (-1), and let $s \to m$ to obtain

$$\mathbb{P}^{\varphi,q}_{m,m}(F(S_{\tau_{l_D(m)}}) \in \mathrm{d}F(m)) = \frac{F'(m)\mathrm{d}m}{F(m) - F(m - l_D(m))}. \tag{3.9}$$

Plugging (3.7), (3.8), and (3.9) in (3.4), we have (3.3) in view of (3.2). □

Note that the probability (3.9) coincides with the result derived from Theorem 2 in Pistorius [19].

The representation of $V(s,s)$ in (3.3) applies to general cases. Given s, the integrand of (3.3) represents the expected reward that one receives when he stops the during the excursion of X from the level $m \geq s$. In the next section, we shall use (3.3) to derive explicit formula of $V(s,s)$ for various locations of $s \in \mathbb{R}$. Let us stress that this representation is new in the literature and a key to direct solution method (without the smooth-fit principle).

[1] Note that when the diffusion X is a standard Brownian motion B, then $F(x) = x$ and the right-hand side reduces to $\exp\left(-\int_s^m \frac{\mathrm{d}u}{l_D(u)}\right)$.

4. Computing $V(s,s)$ and $l^*(s)$

In solving an optimal stopping problem involving S and X, one of the aims is to draw a diagram like Figure 3. For distinct values in the (x,s)-diagram, we need to determine whether a point in $\mathbb{R}^2$ is in the continuation region (C) or stopping region (Γ). The task in this subsection[2] is to compute the value $V(s,s)$ at a point (s,s) on the diagonal and to determine whether it belongs to C or Γ.

As stressed before, once we fix $S=s$, the problem reduces to one-dimensional problems in X. Accordingly, as an intermediate step, we set $S=s$ and attempt to find the smallest nonnegative concave majorant $w_s(\cdot)$ of

$$H_s(y) := \frac{(g-\bar{f})(F^{-1}(y),s)}{\varphi(F^{-1}(y))}, \quad y \in F(\mathscr{I}) \tag{4.1}$$

in the neighborhood of s. Recall that F is defined in (2.9).

Let us denote by $\Sigma_s \subseteq \mathbb{R}$ (resp. $\mathscr{C}_s$) the stopping region (resp. continuation region) with respect to the reward $H_s(y)$ in (4.1), corresponding to this s. Let us emphasize that that this Σ_s (resp. $\mathscr{C}_s$) should be distinguished from the stopping region $\Gamma \subseteq \mathbb{R}^2$ (resp. C) of the problem (2.1), the final object to figure out. Note that this part (i.e., to find the smallest concave majorant w_s of H_s and to identify $(\Sigma_s, \mathscr{C}_s)$ pair) can be easily done by Propositions 2.1 to 2.3 with the transformation (2.10). Indeed, once we make the transformation, we just check if $F(s)$ belongs in

$$\Sigma_s = \{y : H_s(y) = w_s(y)\} \quad \text{or} \quad \mathscr{C}_s = \{y : w_s(y) > H_s(y)\}, \quad y \in F(\mathscr{I}).$$

See Dayanik and Karatzas [7] and note that it contains a number of examples of obtaining optimal policy by this geometric method.

Fix $s \in \mathbb{R}$ and denote by $x^*(s)$ the threshold point, if exists, which separates $\mathscr{C}_s$ and Σ_s with respect to the reward H_s associated with this s. The difficulty here is that due to the dependence of the reward on s, however, there are certain situations where we need to be careful. To discuss, we shall hereafter (for the rest of this section) assume the following:

Assumption 4.1. *Then we assume*

(i) $(g-\bar{f})(x,s)$ *is increasing in s, and*
(ii) (2.11) *holds with* h^+ *replaced by* H_s^+.

The first assumption is merely to restrict our problems to practical ones because we are solving maximization problems. For the second, since our main concern is to find a finite value function, we shall consider the case where (2.11) holds.

Although it is easy to find Σ_s and $\mathscr{C}_s$, and hence $x^*(s)$, as exemplified in Chapter 6 of [7], there are so many patterns depending on the shape of H_s function in the transformed space. Hence it may not be practical to go through all of them. Rather, to present our solution method clearly in a general setting, we shall show building-block cases. More specifically, for $s \in \mathscr{I} = (l,r)$, we work on the case where the continuation region $\mathscr{C}_s$ corresponding to $H_s(y)$ in (4.1) is in the form of $A_s := (x^*(s), r)$ or $B_s := (l, x^*(s))$:

$$\text{Case (1): } s \in \Sigma_s, \qquad \text{Case (2): } s \in A_s = (x^*(s), r), \quad \text{and} \quad \text{Case (3): } s \in B_s = (l, x^*(s))$$

[2] Once this is done, then the next task is to examine the points (x,s) by moving leftwards to $x=0$ from the diagonal $s=x$. We take this in Section 5 and solve an example in 5.1.

based on whether s belongs to Σ_s or $\mathscr{C}_s$. For each case, we shall provide a direct way of solution. In fact, Case (1) is the most challenging of those, and hence significant part of this section is devoted to the explanation for this case.

While we consider here the cases in which there are at most one $x^*(s)$, these are building-block cases in the sense that more complex structure (e.g. multiple $x^*(s)$) can be handled by combinations of them: Suppose that we have two $x^*(s)'s$ for a particular s, say $x_1^*(s)$ and $x_2^*(s)$ in the ascending order. Then another case $s \in (x_1^*(s), x_2^*(s))$ should arise. Since $(x_1^*(s), x_2^*(s)) = (x_1^*(s), r) \cap (l, x_2^*(s))$, this case is seen as the combination of Case (2) and Case (3), so that one can compare two values derived from each case and take the greater one as $V(s,s)$ for this s. If we were to have another $x_3^*(s)$, we again simply split the real line into segments and do the same tasks for s in each segment.

4.1. **Case (1): $s \in \Sigma_s$.** If s belongs to Σ_s, we have $w_s(y) = H_s(y)$; however, instead of stopping immediately, there is a possibility that a greater value can be attained if one stops X during the excursion from some upper level $s' > s$. Recall that Proposition 3.1 has incorporated this. See also Remark 4.2 below. Now we wish to obtain more explicit formulae for $V(s,s)$ from the general representation (3.3). For this purpose, let us denote

$$P(u; l_D) := \frac{F'(u)}{F(u) - F(u - l_D(u))}, \quad \text{and} \quad G(u; l_D) := (g - \bar{f})(u - l_D(u), u),$$

to avoid the long expression and rewrite (3.3) in the following way: for any $\varepsilon > 0$,

$$\begin{aligned} V(s,s) &= \sup_{l_D} \Bigg[\int_s^{s+\varepsilon} \frac{\varphi(s)}{\varphi(m - l_D(m))} \exp\left(-\int_s^m P(u; l_D) \mathrm{d}u \right) P(m; l_D) G(m; l_D) \mathrm{d}m \\ &\quad + \frac{\varphi(s)}{\varphi(s+\varepsilon)} \exp\left(-\int_s^{s+\varepsilon} P(u; l_D) \mathrm{d}u \right) \\ &\qquad \times \int_{s+\varepsilon}^{\infty} \frac{\varphi(s+\varepsilon)}{\varphi(m - l_D(m))} \exp\left(-\int_{s+\varepsilon}^m P(u; l_D) \mathrm{d}u \right) P(m; l_D) G(m; l_D) \mathrm{d}m \Bigg] \\ &= \sup_{l_D} \Bigg[\int_s^{s+\varepsilon} \frac{\varphi(s)}{\varphi(m - l_D(m))} \exp\left(-\int_s^m P(u; l_D) \mathrm{d}u \right) P(m; l_D) G(m; l_D) \mathrm{d}m \\ &\quad + \frac{\varphi(s)}{\varphi(s+\varepsilon)} \exp\left(-\int_s^{s+\varepsilon} P(u; l_D) \mathrm{d}u \right) V(s+\varepsilon, s+\varepsilon) \Bigg] \end{aligned}$$

where in the last equation we used the form of $V(\cdot,\cdot)$ in (3.3) and evaluate at $(s+\varepsilon, s+\varepsilon)$ by recalling that the right-hand side of (3.3) looks at all the levels of $m \geq s$ in finding optimal $l_D^*(m)$. This expression naturally motivates us to set $V_\varepsilon : \mathbb{R} \mapsto \mathbb{R}$ as

$$V_\varepsilon(s) := \sup_{l_D(s)} \left[\frac{\varphi(s)}{\varphi(s+\varepsilon)} \exp(-\varepsilon P(s; l_D)) V(s+\varepsilon, s+\varepsilon) + \frac{\varphi(s)}{\varphi(s - l_D(s))} \cdot \varepsilon P(s; l_D) G(s; l_D) \right] \tag{4.2}$$

and we have $\lim_{\varepsilon \downarrow 0} V_\varepsilon(s) = V(s,s)$. Now the problem has reduced in such a way that we only need to determine one maximizer $l_D^*(s)$ at level s. Divide both sides by $\varphi(s)$ and set a maximizer $l_D^*(s) \in [0, \infty)$ of the right hand side of (4.2) to obtain the equality for any s and $\varepsilon > 0$:

$$\frac{V_\varepsilon(s)}{\varphi(s)} = \left[\frac{V(s+\varepsilon, s+\varepsilon)}{\varphi(s+\varepsilon)} e^{-\varepsilon P(s; l_D^*)} + \frac{G(s; l_D^*)}{\varphi(s - l_D^*(s))} \varepsilon P(s; l_D^*) \right]. \tag{4.3}$$

To see what has been done, let us recall the transformation of a Borel function z defined on $-\infty \le c \le x \le d \le \infty$ through

$$Z(y) := \frac{z}{\varphi} \circ F^{-1}(y) \tag{4.4}$$

on $[F(c), F(d)]$ where F^{-1} is the inverse of the strictly increasing $F(\cdot)$ in (2.9). If we evaluate Z at $y = F(x)$, we obtain $Z(F(x)) = \frac{z(x)}{\varphi(x)}$, which is the form that appears in (4.3). Note that

$$Z'(y) = q'(x) \quad \text{where} \;\; q'(x) = \frac{1}{F'(x)} \left(\frac{z}{\varphi}\right)'(x) \tag{4.5}$$

Hence, in view of (3.7), the intuitive meaning of (4.3) is that if stopping occurs before the maximum is renewed from s to $s+\varepsilon$, we receive $G(s, l_D)$ but otherwise, we continue with value $V(s+\varepsilon, s+\varepsilon)$. An optimal choice of $l_D(s)$ brings us the value $V_\varepsilon(s)$ and, by letting $\varepsilon \downarrow 0$, $V(s,s)$.

Proposition 4.1. *Fix $s \in \mathscr{I}$. If (1) the reward function $(g-\bar{f})(x,s)$ is increasing in the second argument and (2) if $\log\varphi(\cdot)$ is strictly convex on $\mathscr{I}$, we have*

$$V(s,s) = \frac{\varphi(s)}{\varphi(s - l_D^*(s))} \cdot Q(s; l_D^*) \cdot (g-\bar{f})(s - l_D^*(s), s), \tag{4.6}$$

where

$$Q(u; l_D) := \frac{F'(u)\varphi'(u)}{\varphi''(u)[F(u) - F(u - l_D^*(u))] + F'(u)\varphi'(u)}$$

and $l_D^*(s)$ is the maximizer of the map

$$z \mapsto \frac{\varphi(s)}{\varphi(s-z)} \cdot \frac{F'(s)\varphi'(s)}{\varphi''(s)[F(s) - F(s-z)] + F'(s)\varphi'(s)} \cdot (g-\bar{f})(s-z, s) \quad \text{on} \;\; [0,\infty). \tag{4.7}$$

If (2)' $\log\varphi(\cdot)$ is linear on $\mathscr{I}$, then $Q(\cdot; l_D)$ is to be replaced by

$$\tilde{Q}(u; l_D) := \frac{F'(u)\varphi(u)}{(\varphi'(u) - \varphi(u))[F(u) - F(u - l_D^*(u))] + F'(u)\varphi(u)}.$$

Remark 4.1. (i) If we evaluate at $l_D(s) = 0$, then $Q(u;0)$ (and $\tilde{Q}(u;0)$) becomes 1 and the right-hand side of (4.6) is $(g-\bar{f})(s,s)$ as expected.

(ii) The strict convexity of $\log\varphi(\cdot)$ implies

$$\frac{\varphi(s)}{\varphi(s+\varepsilon)} \frac{\varphi'(s+\varepsilon)}{\varphi'(s)} < 1 \quad \forall s \in \mathscr{I} \quad \text{and} \quad \forall \varepsilon > 0. \tag{4.8}$$

On the other hand, if $\log\varphi(\cdot)$ is linear, then the inequality in (4.8) is replaced by the equality:

$$\frac{\varphi(s)}{\varphi(s+\varepsilon)} \frac{\varphi'(s+\varepsilon)}{\varphi'(s)} = 1 \quad \forall s \in \mathscr{I} \quad \text{and} \quad \forall \varepsilon > 0. \tag{4.9}$$

We shall use this property in the proof of Lemma 4.1. Note that it is easily proved that geometric Brownian motion satisfies (4.8) and Brownian motion (with or without drift) does the other one. The $\varphi(\cdot)$ functions of Orstein-Uhlenbeck process $\mathrm{d}X_t = k(m - X_t)\mathrm{d}t + \sigma \mathrm{d}B_t$ (with $k > 0, \sigma > 0$ and $m \in \mathbb{R}$) and its exponential version $\mathrm{d}X_t = \mu X_t(\alpha - X_t)\mathrm{d}t + \sigma X_t \mathrm{d}B_t$ (where μ, α, σ are positive constant) involve special functions; the parabolic cylinder function and the confluent hypergeometric function of the second kind, respectively (see Lebedev [12]). While it is hard to prove the log convexity of these special functions, it is numerically confirmed that both processes satisfy (4.8). Moreover, the equality condition (4.9) implies that $(\log\varphi(s+\varepsilon))' = (\log\varphi(s))'$ so that $\varphi(s)$ is an exponential function. Hence it is confirmed that this case includes Brownian motion.

Proof. (of Proposition 4.1) Let us first take the strictly convex case of $\log\varphi(\cdot)$. For taking limits of $\varepsilon\downarrow 0$ in (4.2), we need the following lemma whose proof is postponed to Appendix A:

Lemma 4.1. *Under the assumption of Proposition 4.1 with convex* $\log\varphi(\cdot)$, *for* $\varepsilon > 0$ *sufficiently close to zero, we have*

$$\frac{V_\varepsilon(s)}{\varphi(s)} = \alpha_s(\varepsilon)\cdot\frac{V(s+\varepsilon,s+\varepsilon)}{\varphi(s+\varepsilon)} \qquad \textit{where} \ \ \alpha_s(\varepsilon) := \frac{\varphi'(s+\varepsilon)}{\varphi'(s)}. \tag{4.10}$$

Note that $\alpha_s(\varepsilon)\in(0,1)$ for all $s\in\mathscr{I}$ and $\varepsilon>0$ and that $\alpha_s(\varepsilon)\uparrow 1$ for all $s\in\mathscr{I}$.

Suppose that the lemma is proved, let us continue the proof of Proposition 4.1. By using (4.10) in Lemma 4.1, we can write, for ε small,

$$V_\varepsilon(s) - \frac{\varphi(s)}{\varphi(s+\varepsilon)}\exp(-\varepsilon P(s;l_D^*))V(s+\varepsilon,s+\varepsilon) = \left(1-\frac{\varphi'(s)}{\varphi'(s+\varepsilon)}\exp(-\varepsilon P(s;l_D^*))\right)V_\varepsilon(s). \tag{4.11}$$

Moreover, since $\lim_{\varepsilon\downarrow 0}V(s+\varepsilon,s+\varepsilon)=V(s,s)$, the optimal threshold $l_D^*(s)$ should satisfy

$$V(s,s)=\lim_{\varepsilon\downarrow 0}V_\varepsilon(s)=\lim_{\varepsilon\downarrow 0}\left[\frac{\varphi(s)}{\varphi(s+\varepsilon)}\exp(-\varepsilon P(s;l_D^*))V(s+\varepsilon,s+\varepsilon)+\frac{\varphi(s)}{\varphi(s-l_D^*(s))}\cdot\varepsilon P(s;l_D^*)G(s;l_D^*)\right],$$

from which equation, in view of (4.11), we obtain

$$\begin{aligned}
V(s,s) &= \lim_{\varepsilon\downarrow 0}\frac{V_\varepsilon(s)-\frac{\varphi(s)}{\varphi(s+\varepsilon)}\exp(-\varepsilon P(s;l_D^*))V(s+\varepsilon,s+\varepsilon)}{1-\frac{\varphi'(s)}{\varphi'(s+\varepsilon)}\exp(-\varepsilon P(s;l_D^*))}\\
&= \lim_{\varepsilon\downarrow 0}\frac{\frac{\varphi(s)}{\varphi(s-l_D^*(s))}\cdot\varepsilon P(s;l_D^*)G(s;l_D^*)}{1-\frac{\varphi'(s)}{\varphi'(s+\varepsilon)}\exp(-\varepsilon P(s;l_D^*))}\\
&= \frac{\varphi(s)}{\varphi(s-l_D^*(s))}\,\frac{F'(s)\varphi'(s)}{\varphi''(s)[F(s)-F(s-l_D^*(s))]+F'(s)\varphi'(s)}(g-\bar f)(s-l_D^*(s),s),
\end{aligned}$$

where the last equality is obtained by L'Hôpital's rule. From the last equality, $l_D^*(s)$ must give the supremum to $\frac{\varphi(s)}{\varphi(s-z)}Q(s;z)(g-\bar f)(s-z,s)$ since $l_D^*(s)$ was set to be a maximizer of $V_\varepsilon(s)$ in (4.2) and $V(s,s)=\lim_{\varepsilon\downarrow 0}V_\varepsilon(s)=\lim_{\varepsilon\downarrow 0}V(s+\varepsilon,s+\varepsilon)$. Hence the claim of the proposition is proved.

When $\log\varphi(\cdot)$ is linear, in lieu of (4.10), we claim that

$$\frac{V_\varepsilon(s)}{\varphi(s)} = \frac{1}{1+\varepsilon}\cdot\frac{V(s+\varepsilon)}{\varphi(s+\varepsilon)} \tag{4.12}$$

for $\varepsilon>0$ sufficiently small. The intuition here is the following: since $\alpha_s(\varepsilon)$ in Lemma 4.1 can be written as $\left(1+\varepsilon\cdot\frac{\varphi''(s)}{\varphi'(s)}\right)\uparrow 1$ (as $\varepsilon\downarrow 0$) and in case of Brownian motion, $\frac{\varphi''(s)}{\varphi'(s)}=$ (const), the factor should be independent of s. It can be easily seen that the proof of Lemma 4.1 holds in this case, too. Accordingly, instead of (4.2), we have

$$V_\varepsilon(s)-\frac{\varphi(s)}{\varphi(s+\varepsilon)}\exp(-\varepsilon P(s;l_D^*))V(s+\varepsilon,s+\varepsilon)=\left(1-\frac{\varphi(s)}{\varphi(s+\varepsilon)}(1+\varepsilon)\exp(-\varepsilon P(s;l_D^*))\right)V_\varepsilon(s).$$

For the rest, we just proceed as in the proof of Proposition 4.1 to obtain $\tilde{Q}(\cdot; l_D)$. □

In summary, in Case (1), we should resort to Proposition 4.1 to compute $V(s,s)$. If $l_D^*(s) \geq 0$ for this s, we shall have $(g-\bar{f})(s,s) \leq V(s,s)$.

Remark 4.2.

(i) Let us slightly abuse the notation by writing $\frac{F'(s)\varphi'(s)}{\varphi''(s)[F(s)-F(s-z)]+F'(s)\varphi'(s)} = Q(s;z)$ to avoid the long expression. Note that $\frac{\varphi(s)}{\varphi(s-z)}Q(s;z)(g-\bar{f})(s-z,s)$ is the value corresponding to the strategy D with $l_D(s) = z$ and $l_D(m) = l_D^*(m)$ for every $m > s$; that is, this amount is obtained when we stop if X goes below $s-z$ in the excursion at level $S = s$ and behave optimally at all the higher levels $S > s$.

(ii) Note that there may be several maximizers for the maps in Proposition 4.1. In that case, every maximizer is indifferent in the sense that every local maximizer leads to the identical optimal value, so we can choose any of those as $l_D^*(s)$.

4.2. **Case (2):** $s \in A_s = [x^*(s), r)$**.** Recall that A_s is defined in Assumption 4.1. In this case, similar to Case (1), a positive $l_D^*(s)$ may lead to improvement of the value of $V(s,s)$, so that we use Proposition 4.1. While the next example does not have s explicitly in the reward function, it should be beneficial to see how to treat problems in a general setting.

4.3. **Case (3):** $s \in B_s = (l, x^*(s)]$**.** B_s is defined in Assumption 4.1. For this case, the typical situation is that $x^*(s)$ is monotonically decreasing in s. See Figure 3. The curve separating the region Γ and C_2 corresponds to the function $x^*(s)$. Then define the point $\hat{s}$ such that

$$s = x^*(s)$$

holds. One receives $(g-\bar{f})(x^*(\hat{s}),\hat{s}) = (g-\bar{f})(\hat{s},\hat{s})$ when stops there. In contrast to the previous Cases (1) and (2), s is located to the left of $x^*(s)$, we are not supposed to stop during the excursions from the level $u \in [s,\hat{s})$. Mathematically, it means that we let $u - l_D^*(u) \to l$ in (3.3) of Proposition 3.1. The left boundary l is assumed to be natural and hence $F(u - l_D^*(u)) \to 0$ and $S_{\tau_{l_D}(m)} \in \mathrm{d}m = \delta_{\hat{s}}\mathrm{d}m$, the Dirac measure sitting at $\hat{s}$. Now, from (3.9), (3.3) simplifies to

$$V(s,s) = \int_s^\infty \frac{\varphi(s)}{\varphi(m)} \exp\left(-\int_s^m \frac{F'(u)}{F(u)}\mathrm{d}u\right)(g-\bar{f})(m,m)\delta_{\hat{s}}\mathrm{d}m = \frac{\psi(s)}{\psi(\hat{s})}(g-\bar{f})(\hat{s},\hat{s}), \tag{4.13}$$

which is, in view of (2.8), simply the expected discounted value of $(g-\bar{f})(\hat{s},\hat{s})$. Note that for $s \leq \hat{s}$, the reward $(g-\bar{f})(x^*(\hat{s}),\hat{s})$ does not depend on s and thereby with respect to this reward, $x^*(s) = x^*(\hat{s})$ for $s \leq \hat{s}$.
An example that involves this case is presented in Section 5.1.

Up to this point, we have shown how to find $V(s,s)$ in (2.6). Before presenting the complete solution to (2.6) and hence to (2.1), we illustrate our method by solving classical problems.

4.4. **Examples.** Before moving, it should be beneficial to briefly review some special cases in finding $V(s,s)$. In this section, the diffusion X is geometric Brownian motion $\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}B_t$ and $(\mathscr{A} - q)v(x) = 0$ provides $\varphi(x) = x^{\gamma_0}$ and $\psi(x) = x^{\gamma_1}$ with $\gamma_0 < 0$ and $\gamma_1 > 1$. The parameters are $(\mu, \sigma, q, K, k) = (0.05, 0.25, 0.15, 5, 0.5)$. The values of the options here are computed under the physical measure $\mathbb{P}$.

4.4.1. *Lookback Option.* The reward function is $(g-\bar{f}) = s - kx$ where $k \in [0,1]$. Set $s = 5$. By setting $y = F(x)$ in (4.1), $H_s(F(x)) = \frac{s-kx}{\varphi(x)}$. The graph of $\frac{s-kx}{\varphi(x)}$ against the horizontal axis $F(x)$ is in Figure 1-(a). It can be seen that $s \in \Sigma_s$ and that Case (1) applies for the entire region $x \in \mathbb{R}_+$. The optimal threshold $l_D^*(s)$ can be found by Proposition 4.1: the optimal level x^* is given by $x^* = \beta s$ where $\beta = 0.701636$, independent of s, so that $l_D^*(s) = (1-\beta)\cdot s$.

Once $l_D^*(s)$ is obtained, we can compute $V(s,s)$ from (4.6). While we shall discuss the general method of computing $V(x,s)$ for $x \leq s$ in Section 5, it is appropriate to touch upon this issue here. For this fixed $s = 5$, we examine the smallest concave majorant of $H_s(y)$. But the majorant must pass the point

$$\left(F(s), \frac{V(s,s)}{\varphi(s)}\right) \quad \text{and} \quad \left(F(s-l_D^*(s)), H_s(s-l_D^*(s))\right).$$

The red line L_s is drawn connecting these points with a positive slope. In fact, the smooth-fit principle holds at $F(s-l_D^*(s))$ as is discussed in [20]. Accordingly, $(s-l_D^*(s), s) \subset \mathbb{R}$ is in the continuation region. Let us stress again that we do *not* assume the smooth-fit condition.

4.4.2. *Perpetual Put.* The reward function is $(g-\bar{f})(x,s) = g(x) = (K-x)^+$ which does not depend on s. The graph of $g(x)/\varphi(x)$ is in Figure 1-(b) which is drawn against the horizontal axis of $y = F(x)$ when $s = x = 5$. For this s, Case (2) applies. The function $H_s(F(x))$ attains *unique maximum* at $F(x^*)$ where $x^* = 3.57604$, so that $l_D^*(s) = s - x^* = 1.42396$. Since g is independent of s, so is x^*.

By using this fact, we can use Proposition 3.1 and compute $V(s,s)$ easily. In fact, an observation of (3.3) reveals that if $F(u - l_D(u))$ is constant for $u > s$, then we have

$$\exp\left(-\int_s^\infty \frac{F'(u)}{F(u)-F(u-l_D(u))}\mathrm{d}u\right) = 0,$$

which in turn makes

$$\int_s^\infty \exp\left(-\int_s^m \frac{F'(u)\mathrm{d}u}{F(u)-F(u-l_D(u))}\right)\frac{F'(m)}{F(m)-F(m-l_D(m))}\mathrm{d}m = 1.$$

Then (3.3) reduces to

$$V(s,s) = \sup_{l_D(s)} \frac{\varphi(s)}{\varphi(s-l_D(s))}(g-\bar{f})(s-l_D(s),s), \tag{4.14}$$

and $l_D^*(s)$ is the maximizer of the map $z \mapsto \frac{\varphi(s)}{\varphi(s-z)}(g-\bar{f})(s-z,s)$.

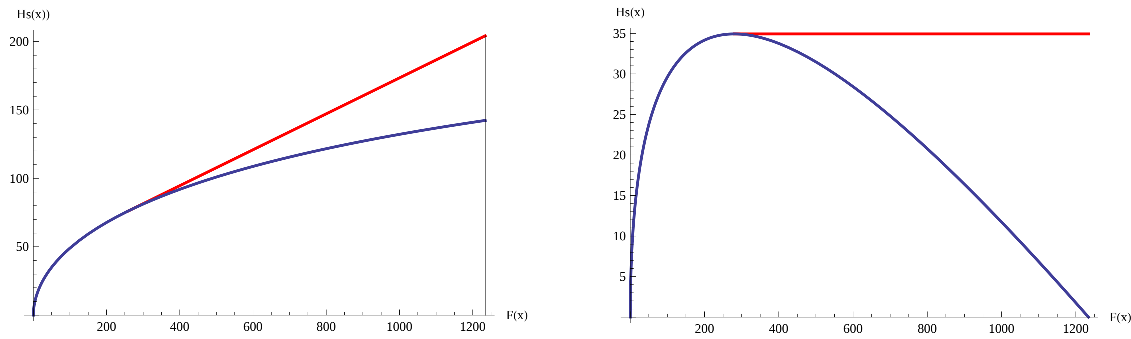
At this point the tangent line has slope zero. See the red horizontal line connecting two points $(F(x^*), H_s(x))$ and $(F(s), \frac{V(s,s)}{\varphi(s)})$. At $F(x^*)$, we have the smooth-fit principle hold and $(x^*, s) \subset \mathbb{R}$ is in the continuation region.

5. Global Solution $V(x,s)$

Since we calculated $V(s,s)$, we can represent $V(x,s)$ by (2.6), which we recall here

$$V(x,s) \quad = \quad \sup_{\tau\in\mathscr{S}} \mathbb{E}^{x,s}\left[\mathbb{1}_{\{\tau<T_s\}}e^{-q\tau}(g-\bar{f})(X_\tau,s) + \mathbb{1}_{\{T_s<\tau<+\infty\}}e^{-qT_s}V(s,s)\right].$$

As we noted in Section 2, this can be seen as just an one-dimensional optimal stopping problem for the process X. In terms of the (x,s)-diagram like Figure 3, we fix $s = \bar{s}$, say and use the information of $V(\bar{s},\bar{s})$, compute $V(x,\bar{s})$



(a) Lookback Option $(k = 1/2)$

(b) Perpetual Put $(K = 5)$

FIGURE 1. **The graphs of $g(x,s)/\varphi(x)$ against the horizontal axis $F(x)$:** We fix $s = 5$ in both problems. The vertical lines show the position of $\varphi(s)$. In the perpetual put case, the optimal exercise threshold is well-known: $x^* = \frac{\gamma_0 K}{\gamma_0 - 1} = 3.57604$, which does not depend on s.

and tell, by moving down from the diagonal point $(\bar{s},\bar{s})$, whether a point $(\bar{s},x)$ belongs to C or Γ. After discussing generality here, we shall study an example in Section 5.1 by showing how to implement the method.

Now suppose that we have found $V(s,s)$ for each $s \in \mathbb{R}_+$. The next step is to solve (2.6). Consider an excursion from the level $S = s$. Recall that $V(s,s)$ represents the value that one would obtain when X would return to that level s. If there is no absorbing boundary, we can let the height of excursions arbitrarily large. Since we are assuming (2.11), by Proposition 5.12 in [7], the value function in the transformed space must pass the points:

$$\left(0, \xi_l\right) \quad \text{and} \quad \left(F(s), \frac{V(s,s)}{\varphi(s)}\right).$$

We shall summarize how one can solve (2.6). While Step 1 and 2 were already discussed in Section 4, we nonetheless repeat here to make our recipe complete.

Step 1: For each s, solve an auxiliary problem. That is, to find the smallest concave majorant $w_s(y)$ of

$$H_s(y) := \frac{(g - \bar{f})(F^{-1}(y), s)}{\varphi(F^{-1}(y))}$$

in (4.1) and to identify the region $\{y : H_s(y) = w_s(y)\}$ as Σ_s.

Step 2: Once we identify Σ_s and $\mathscr{C}_s$ for each s, the next step is to tell which Case (1), (2), or (3) applies. For example, if Case (1) does, refer to Proposition 4.1 where we presented how to find $V(s,s)$ and $l_D^*(s)$. This provides the local solution in the neighborhood of s.

Step 3: Let us stress that up to Step 2, we have found *local* solution around s. By Propsitions 2.1 to 2.3, for all Cases (1), (2) and (3), the global solution, denoted by W_s, must satisfy the following conditions:

(i) $W_s(y) \geq H_s(y)$ on $[0, F(s)]$,
(ii) $W_s(F(s)) = \frac{V(s,s)}{\varphi(s)}$,
(iii) $W_s(0) = \xi_l$,
(iv) W_s is concave on $[0, F(s)]$, and
(v) for any functions $\overline{W}_s$ which satisfies four conditions above, $W_s \leq \overline{W}_s$ on $[0, F(s)]$.

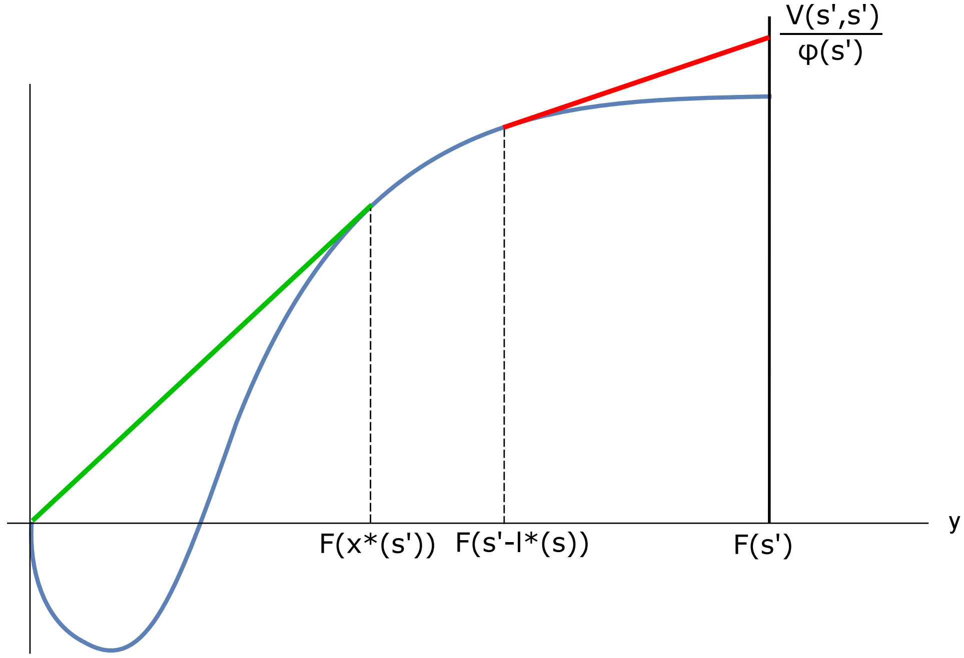


FIGURE 2. A typical example of W_s and H_s. Fix $s = s'$. Find $V(s,s)$ and then specify the optimal strategy on $(l,s]$ based on $H_{s'}$.

Now once we have done with one s, we then move on to another $\tilde{s}$, say, and find $W_{\tilde{s}}$ in the new interval $[0, F(\tilde{s})]$. As seen from this discussion, $W_s(y) = w_s(y)$ does not hold in general.

Figure 2 illustrates a typical example of the graphs of W_s and H_s in transformed space. Fix $s = s'$. Note that in the neighborhood of $F(s')$, the reward function $H_s(y)$ is concave and we have $s \in \Sigma_s$ (Case (1)). We find $V(s', s')$ and $l_D^*(s')$ at the same time by Proposition 4.1.

Now take the point $F(s')$ on the horizontal axis and find $W_{s'}(y)$ that satisfies the above conditions. For this purpose, three vertical lines are drawn at $y = F(x^*(s')), F(s' - l_D^*(s'))$, and $F(s')$ from the left to right. Starting with the point $\left(F(s'), \frac{V(s',s')}{\varphi(s')}\right)$, the concave majorant near that point is the line that is tangent to $H_{s'}(y)$. The tangency point is $(F(s' - l_D^*(s'), H_{s'}(F(s' - l_D^*(s')))$. In the region $[F(x^*(s'), F(s' - l_D^*(s')]$, the value function is the reward function itself. On the other hand, the smallest concave majorant of $H_{s'}(y)$ on $(0, F(x^*(s'))$ is the line, from the origin, tangent to $H_{s'}(y)$ at $F(x^*(s'))$.

For this s', optimal strategy reads as follows: If it happens that $x \in (0, s')$ belongs to $(s' - l_D^*(s'), s')$, one should see if X reaches $s' - l_D^*(s')$ or s', whichever comes first. If the former point is the case, one should stop and receive the reward, otherwise one should continue with $s > s'$. If $x \in (0, s')$ belongs to $(x^*(s'), s' - l_D^*(s'))$, one should immediately stop X and receive $g(x, s')$. Finally, in $x \in (0, x^*(s'))$, one should wait until X reaches $x^*(s')$.

5.1. **Illustration: A New Problem.** To illustrate how to implement the solution method for a problem that involves both S and X, we postulate the reward function as

$$g(x,s) = s^a + kx^b - K, \quad a,b,k,K > 0 \tag{5.1}$$

and $f(x,s) \equiv 0$. For concreteness, we set $a = 1/2, b = 1, k = 1/2$, and $K = 5$. We assume that the underlining process X is geometric Brownian motion:

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}B_t, \quad t \in \mathbb{R},$$

where μ and σ are constants and B is a standard Brownian motion under $\mathbb{P}$. In this case,

$$\psi(x) = x^{\gamma_1} \quad \text{and} \quad \varphi(x) = x^{\gamma_0}$$

where

$$\gamma_0 = \frac{1}{2}\left(-\left(\frac{2\mu}{\sigma^2} - 1\right) - \sqrt{\left(\frac{2\mu}{\sigma^2} - 1\right)^2 + \frac{8q}{\sigma^2}}\right) < 0,$$

and

$$\gamma_1 = \frac{1}{2}\left(-\left(\frac{2\mu}{\sigma^2} - 1\right) + \sqrt{\left(\frac{2\mu}{\sigma^2} - 1\right)^2 + \frac{8q}{\sigma^2}}\right) > 1.$$

For this reward $g - \bar{f}$, the value of ξ_l in (2.11) is zero for any $s \in \mathscr{I}$.

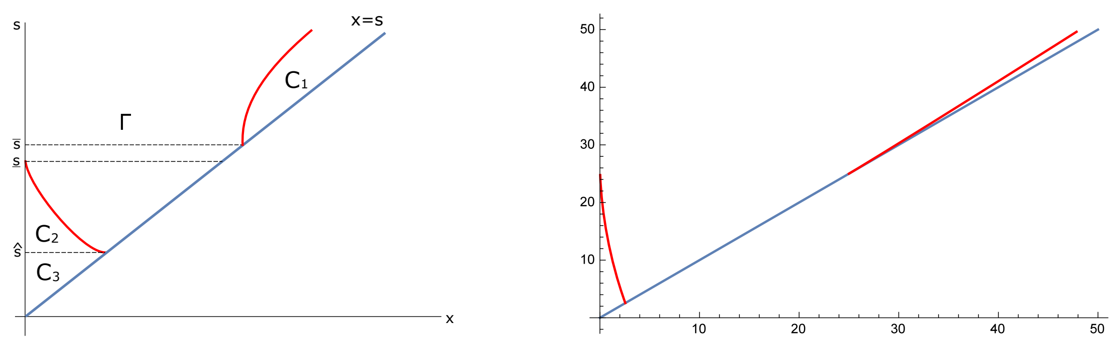


FIGURE 3. The solution (x,s)-diagram to the reward function (5.1) in a schematic presentation (left) and in the real scale (right).

It should be helpful to preview the entire state space at the beginning. See Figure 3 for optimal strategy: continuation and stopping region in the two-dimensional diagram. The left panel is for presenting in a schematic drawing. For different values of (x,s), we see continuation regions $\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3$ and stopping region Γ. The right panel is the real solution for this problem with parameters $(\mu, \sigma, q) = (0.05, 0.25, 0.15)$. According to the values of s, we have four regions: (i) $s > \bar{s}$, (iii) $\underline{s} \le s \le \bar{s}$, (ii) $\hat{s} < s < \underline{s}$, and (iv) $s \le \hat{s}$. We shall explain how to find the value function and optimal strategy for each region. Note that for the ease of exposition, we handle $\hat{s} < s < \underline{s}$ before we do $\underline{s} \le s \le \bar{s}$.

(i) Let us start with $s > \bar{s}$. First we need to find $V(s,s)$. Plug this s in (5.1) and examine the reward function in the transformed space:

$$H_s(y) = \frac{(g-\bar{f})(F^{-1}(y),s)}{\varphi(F^{-1}(y))} = \frac{\sqrt{s} + ky^{\frac{1}{\gamma_1-\gamma_0}} - K}{y^{\frac{\gamma_0}{\gamma_1-\gamma_0}}}, \quad y \in (0,\infty).$$

Figure 4 -(i) shows the function $H_s(y)$ on $[0,F(s)]$. For this s, $H_s(y)$ is concave in the neighborhood of $F(s)$ and Case (1) in Section 4 applies. The map

$$A(l) : l \mapsto \frac{\varphi(s)}{\varphi(s-l)} \cdot \frac{F'(s)\varphi'(s)}{\varphi''(s)[F(s)-F(s-l)] + F'(s)\varphi'(s)} \cdot g(s-l,s)$$

in (4.6) is in Figure 4-(b) and $l_D^*(s) = 0.5371$ when $s = 35$.

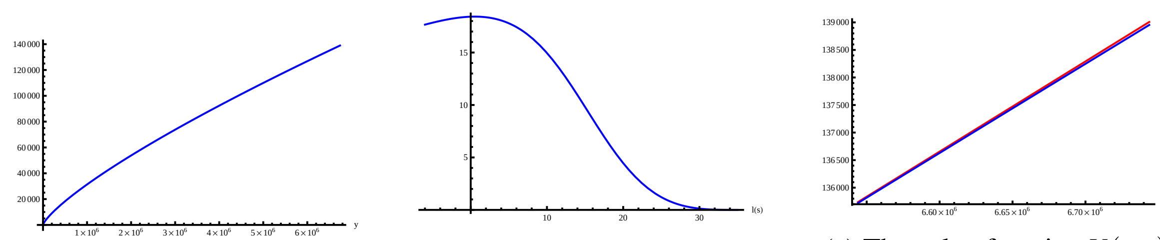
Now we can find $V(x,s)$ for $x \in [0,s]$. Following Section 5, we shall find the smallest concave majorant of $H_s(y)$ that passes the origin and $\left(F(s), \frac{V(s,s)}{\varphi(s)}\right)$. For this particular s, a diagram similar to Figure 1-(a) can be drawn: see Figure 4-(c). The value function on the continuation region C_1 (red line in the graph) is

$$V(x,s) = W(F(x))\varphi(x) = \Big(\beta_1(F(x) - F(s-l_D^*(s))) + H_s(F(s-l_D^*(s)))\Big) \cdot \varphi(x)$$

where $\beta_1 = \frac{\mathrm{d}H_s(y)}{\mathrm{d}y}\Big|_{y=F(s-l_D^*(s))}$.

In summary, the value function is

$$V(x,s) = \begin{cases} g(x,s), & x \in (0, s-l_D^*(s)), \\ \Big(\beta_1(F(x) - F(s-l_D^*(s))) + H_s(F(s-l_D^*(s)))\Big) \cdot \varphi(x), & x \in [s-l_D^*(s), s]. \end{cases}$$



(a) Graph of $H_s(y)$ when $s = 35$

(b) The map $A(l)$ to find $l_D^*(s) = 0.5371$.

(c) The value function $V(x,s)$ (red) and $H_s(y)$ (blue)

FIGURE 4. The region $s \in (\bar{s}, +\infty)$. Note in panel (c) for a better picture, the lower-left corner of the graph is not the origin.

(ii) Let us move on to the region $\hat{s} < s < \underline{s}$ (before we examine $\underline{s} \le s \le \bar{s}$). See Figure 5-(a) for the graph of $H_s(y)$ in the transformed space. In the neighborhood of $F(s)$, the reward $H_s(y)$ is concave, i.e., $s \in \Sigma_s$ and hence Case (1) applies. But We have $l_D^*(s) = 0$ for s in this region, so that $V(s,s) = g(s,s)$. On the other hand, for the reward function $g(x,s)$ with this fixed $s \in (\hat{s}, \underline{s}]$, there is a point $x^*(s)$ such that $(0, x^*(s))$ is the continuation region. Let us see this situation in the transformed space. See Figure 5-(a) again. At the point $F(x^*(s)) > F(s)$, the smallest concave majorant $W_s(y)$ of $H_s(y)$ is the line, from the origin, tangent to $H_s(y)$. Hence the value function is

$$V(x,s) = \begin{cases} (\beta_2 F(x)) \cdot \varphi(x) = \beta_2 \psi(x), & x \in (0, x^*(s)], \\ g(x,s), & x \in (x^*(s), s], \end{cases}$$

where $\beta_2 = \frac{\mathrm{d}H_s(y)}{\mathrm{d}y}\Big|_{y=F(x^*(s))}$. Hence $\beta_2\psi(x)$ is the value function in region C_2 in Figure 3.

(iii) For $\underline{s} \le s \le \bar{s}$, we have $l_D^*(s) = 0$ so that the point (s,s) is in the stopping region. Moreover, there exist no points $x^*(s)$ where the line from the origin becomes tangent to $H_s(y)$. Accordingly, the value function is

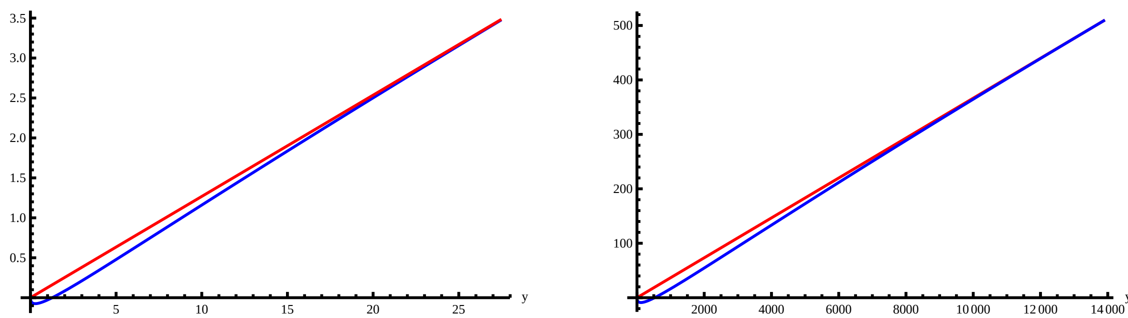
$$V(x,s) = g(x,s) \quad x \in (0,s].$$

In our parameters, $\underline{s} = 25$ and $\bar{s}$ is very close; $\bar{s} > 25$.

(iv) Now we shall examine $s \le \hat{s}$. As explained in Case (3) in Section 4, this $\hat{s} = 8.6420$ satisfies $\hat{s} = x^*(\hat{s})$. See (4.13). Following the argument there, $V(s,s) = \frac{\psi(s)}{\psi(\hat{s})} g(x^*(\hat{s}), \hat{s})$. Accordingly, the value function is

$$V(x,s) = \frac{\psi(x)}{\psi(\hat{s})} g(x^*(\hat{s}), \hat{s}), \quad x \in (0,s]. \tag{5.2}$$

This corresponds to region C_3 in Figure 3.



(a) Graph of $H_s(y)$ and the tangent line $W_s(y)$ when $s = 20$. $x^*(s) = 2.1242$.

(b) Graph of $H_{\hat{s}}(y)$ and the tangent line $W_{\hat{s}}(y)$ with $\hat{s} = x^*(\hat{s}) = 8.6420$

FIGURE 5. The region $s \in (\hat{s}, \underline{s})$ (left) and $s \in (0, \bar{s}]$ (right).

APPENDIX A. PROOF OF LEMMA 4.1

Lemma 4.1 is the following claim: *Under the assumption of Proposition 4.1 with convex* $\log\varphi(\cdot)$*, we have*

$$\frac{V_\varepsilon(s)}{\varphi(s)} = \alpha_s(\varepsilon) \cdot \frac{V(s+\varepsilon, s+\varepsilon)}{\varphi(s+\varepsilon)} \qquad \textit{where} \;\; \alpha_s(\varepsilon) := \frac{\varphi'(s+\varepsilon)}{\varphi'(s)}. \tag{A.1}$$

Proof. (of the lemma) Recall (3.1) for the definition of $l_D(s)$. In view of (3.6), the probabilistic meaning of (4.2) is that $V_\varepsilon(s)$ is attained when one chooses the excursion level $l_D(s)$ optimally in the following optimal stopping:

$$V_\varepsilon(s) = \sup_{l_D(s)} \mathbb{E}^{s,s}[e^{-qT_{s+\varepsilon}} \mathbb{1}_{\{T_{s+\varepsilon} \le \tau_{s-l_D(s)}\}} V(s+\varepsilon, s+\varepsilon) + e^{-q\tau_{s-l_D(s)}} \mathbb{1}_{\{T_{s+\varepsilon} > \tau_{s-l_D(s)}\}} (g - \bar{f})(s - l_D(s), s)], \tag{A.2}$$

that is, if the excursion from s does not reach the level of $l_D(s)$ before X reaches $s+\varepsilon$, one shall receive $V(s+\varepsilon,s+\varepsilon)$ and otherwise, one shall receive the reward. By using the transformation (4.4), one needs to consider the function $\frac{(g-\bar{f})(x,s)}{\varphi(x)}$ and the point $\left(F(s+\varepsilon),\frac{V(s+\varepsilon,s+\varepsilon)}{\varphi(s+\varepsilon)}\right)$ in the $(F(x),z(x)/\varphi(x))$-plane. Then the value function of (A.2) in this plane is the smallest concave majorant of $\frac{(g-\bar{f})(x,s)}{\varphi(x)}$ which passes through the point $\left(F(s+\varepsilon),\frac{V(s+\varepsilon,s+\varepsilon)}{\varphi(s+\varepsilon)}\right)$. It follows that $\frac{V_\varepsilon(s)}{\varphi(s)}\le\frac{V(s+\varepsilon,s+\varepsilon)}{\varphi(s+\varepsilon)}$. As $\varepsilon\downarrow 0$, it is clear that $\frac{V(s+\varepsilon,s+\varepsilon)}{\varphi(s+\varepsilon)}\downarrow\frac{V(s,s)}{\varphi(s)}$ and $\frac{V_\varepsilon(s)}{\varphi(s)}\downarrow\frac{V(s,s)}{\varphi(s)}$. Suppose, for a contradiction, that we have

$$\alpha_s(\varepsilon)\frac{V(s+\varepsilon,s+\varepsilon)}{\varphi(s+\varepsilon)}<\frac{V_\varepsilon(s)}{\varphi(s)}<\frac{V(s+\varepsilon,s+\varepsilon)}{\varphi(s+\varepsilon)},\tag{A.3}$$

for all $\varepsilon>0$. This implies that the first term in (A.3) goes to $\frac{V(s,s)}{\varphi(s)}$ from below and the third term goes to the same limit from above. While the second inequality always hold, the first inequality leads to a contradiction to the fact that the function $\varepsilon\mapsto(1-\alpha_s(\varepsilon))\frac{V(s+\varepsilon,s+\varepsilon)}{\varphi(s+\varepsilon)}$ is continuous for all s.

Indeed, due to the monotonicity of $\alpha_s(\varepsilon)\frac{V(s+\varepsilon,s+\varepsilon)}{\varphi(s+\varepsilon)}$ in ε, we would have $\frac{V_\varepsilon(s)}{\varphi(s)}>\frac{V(s,s)}{\varphi(s)}>\alpha_s(\varepsilon)\frac{V(s+\varepsilon,s+\varepsilon)}{\varphi(s+\varepsilon)}$ for all $\varepsilon>0$. Hence one cannot make the distance between $\frac{V(s+\varepsilon,s+\varepsilon)}{\varphi(s+\varepsilon)}$ and $\alpha_s(\varepsilon)\frac{V(s+\varepsilon,s+\varepsilon)}{\varphi(s+\varepsilon)}$ arbitrarily small without violating (A.3). This shows that there exists an $\varepsilon'=\varepsilon'(s)$ such that $\varepsilon<\varepsilon'$ implies that $\frac{V_\varepsilon(s)}{\varphi(s)}\le\alpha_s(\varepsilon)\frac{V(s+\varepsilon,s+\varepsilon)}{\varphi(s+\varepsilon)}$.

On the other hand, in (A.2), one could choose a stopping time $\tau_{l_D(s)}$ that visits the left boundary l, then by reading (3.6) with $l_D(u)=u$ and $m=s+\varepsilon$, (A.2) becomes

$$\begin{aligned}V_\varepsilon(s)&\ge\frac{\varphi(s)}{\varphi(s+\varepsilon)}\exp\left(-\int_s^{s+\varepsilon}\frac{F'(u)\mathrm{d}u}{F(u)-F(0+)}\right)V(s+\varepsilon,s+\varepsilon)\\&>\frac{\varphi(s)}{\varphi(s+\varepsilon)}\exp\left(-\int_s^{s+\varepsilon}\frac{F'(u)\mathrm{d}u}{F(u)}\right)V(s+\varepsilon,s+\varepsilon)\\&>\frac{\varphi(s)}{\varphi(s+\varepsilon)}\frac{F(s)}{F(s+\varepsilon)}V(s+\varepsilon,s+\varepsilon)=\frac{\psi(s)}{\psi(s+\varepsilon)}V(s+\varepsilon,s+\varepsilon)=\mathbb{E}^{s,s}(e^{-qT_{s+\varepsilon}})V(s+\varepsilon,s+\varepsilon)\end{aligned}$$

for *any* $\varepsilon>0$. Since $s\in\mathscr{I}$ is a regular point, the last expectation can be arbitrarily close to unity, monotonically in ε (see page 89 [11]). Now suppose that there were no ε's such that $V_\varepsilon(s)\ge V(s+\varepsilon,s+\varepsilon)$. It follows that for any ε, we would have

$$V_\varepsilon(s)>\mathbb{E}^{s,s}(e^{-qT_{s+\varepsilon}})V(s+\varepsilon,s+\varepsilon)>\mathbb{E}^{s,s}(e^{-qT_{s+\varepsilon}})V_\varepsilon(s).$$

Then by letting $\varepsilon\downarrow 0$, it would be $V(s)>V(s)$ for all $s\in\mathscr{I}$, which is absurd. Since the convergence of $\mathbb{E}^{s,s}(e^{-qT_{s+\varepsilon}})\uparrow 1$ is monotone in ε, there exists an $\varepsilon''=\varepsilon''(s)>0$ such that $\varepsilon<\varepsilon''$ implies that $V_\varepsilon(s)\ge V(s+\varepsilon,s+\varepsilon)$. By using the second assumption in the statement of Proposition, in particular (4.8), for any s, we have $\frac{V_\varepsilon(s)}{\varphi(s)}\ge\alpha_s(\varepsilon)\frac{V(s+\varepsilon,s+\varepsilon)}{\varphi(s+\varepsilon)}$ for $\varepsilon<\varepsilon''$. This completes the proof of Lemma 4.1. □

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