OPTIMAL STOPPING WHEN THE ABSORBING BOUNDARY IS FOLLOWING AFTER

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ABSTRACT. We consider a new type of optimal stopping problems where the absorbing boundary moves as the state process X attains new maxima S. More specifically, we set the absorbing boundary as $S-b$ where b is a certain constant. This problem is naturally connected with excursions from zero of the reflected process $S-X$. We examine this constrained optimization with the state variable X as a spectrally negative Lévy process. The problem is in nature a two-dimensional one. The threshold strategy given by the path of X is not in fact optimal. It turns out, however, that we can reduce the original problem to an infinite number of one-dimensional optimal stopping problems, and we find explicit solutions.

This work is motivated by the bank's profit maximization with the constraint that it maintain a certain level of leverage ratio. When the bank's asset value severely deteriorates, the bank's required capital requirement shall be violated. This situation corresponds to $X < S-b$ in our setting. This model may well describe a real-life situation where even a big bank can fail because the absorbing boundary is keeping up with the size of the bank.

Key words: Optimal stopping; excursion theory; spectrally negative Lévy processes; scale functions.
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1. INTRODUCTION

The literature about optimal stopping problems and their applications is immense. In an infinite horizon problem, with one-dimensional continuous diffusions as the state variable, a full characterization of the value function and of optimal stopping rule is known and the methodology for solution has been established. See, for example, Dynkin [12], Alvarez [2], Dayanik and Karatzas [10]. For *spectrally negative Lévy processes*, or Lévy processes with only negative jumps, a number of authors have succeeded in extending the classical results by using the scale functions. We just name a few here : [5, 6] for stochastic games, [4, 17, 19] for the optimal dividend problem, [1, 3] for American and Russian options, and [14, 18] for credit risk. However, the solution techniques presented in each paper are more or less problem-specific and no characterization of the value function is yet known. If the problem involves *two* state variables, then even for continuous diffusions, very few things are known in the literature.

We study a new type of optimal stopping problems. We let $X = (X_t, t \geq 0)$ be a spectrally negative Lévy process and denote by Y the reflected process

$$Y_t = S_t - X_t$$

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where $S_t = \sup_{u\in[0,t]} X_u \vee s$. We then consider an optimal stopping problem for both X and S in which the absorbing boundary is defined by $(S_t - b, t \geq 0)$ with b as a positive constant. This means that while X grows and keeps attaining new maxima, the absorbing boundary is accompanying with S. Hence an excursion from S, if greater than b, would bring X to ruin. This situation is seen in the real world; for example, several large financial institutions failed in the last crisis in 2007-2008. One of the reasons is that, while becoming big banks, they maintain high leverage and accordingly, the banks are not so far way from the bankruptcy threshold. Instead, the bankruptcy threshold keeps up with the size of the banks. That is, despite the size of the bank, the risk of bankruptcy is not so much mitigated. This paper is motivated by this phenomenon. See Section 4 for details. While we take the example of banking, one can come up with other applications of this type, as long as the absorbing boundary is determined in relation to the state process' running maxima. For instance, a gambler may have a policy that he stops betting when his wealth X goes below a certain level b from that day's running maxima S.

An excursion theory for spectrally negative Lévy processes has been developed recently. See Bertoin [7] as a general reference. More specifically, an exit problem of the reflected process Y was studied by Avram et al. [3], Pistrorius [21] [22] and Doney [11].

In the above cited papers on optimal stopping problems, the optimal strategy is usually obtained by so-called "threshold strategy". That is, the player should stop and receive rewards on the first occasion when the state process enters a stopping region. In Lévy and other jump models, the authors first find the optimal threshold level and then prove its optimality by verifying the 'quasi-variational inequalities'. See Øksendal and Sulem [20]. Since the problem at hand involves two dimensions; one is X and the other is S, finding and proving the overall optimal strategy may be challenging (as mentioned, no characterization in two-dimensional problems has been found). But in our particular situation, by using the independence of excursions that occur at each level of S, we reduce the problem to an infinite number of one-dimensional optimal stopping problems. We shall then find an explicit form of the value function, thanks to the results by Pistroius [22]. It turns out that the optimal stopping region can be shown in a diagram created by various values of S and $S - X$.

The rest of the paper is organized as follows. In Section 2, we formulate a mathematical model with a review of some important facts of spectrally negative Lévy processes, and then find an optimal threshold level in Section 3. We shall take the example of a bank's optimization in Section 4 and provide an explicit calculation, and the optimality of the threshold strategy is discussed in Section 5 that follows.

2. Mathematical Model

Let the spectrally negative Levy process $X = \{X_t; t \geq 0\}$ represent the state variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the set of all possible realization of the stochastic economy, and $\mathbb{P}$ is a probability measure defined on $\mathcal{F}$. We denote by $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ the filtration with respect to which X is adapted and with the usual conditions being satisfied. The Laplace exponent ψ of X is given by

$$\psi(\lambda) = \mu\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} (e^{\lambda x} - 1 - \lambda x \mathbb{1}_{(x>-1)})\Pi(\mathrm{d}x),$$

where $\mu \geq 0$, $\sigma \geq 0$, and Π is a measure concentrated on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$. It is well-known that ψ is zero at the origin, convex on $\mathbb{R}_+$ and has a right-continuous inverse:

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}, \quad q \geq 0.$$

The running maximum process $S = \{S_t; t \geq 0\}$ is defined by $S_t = \sup_{u \in [0,t]} X_u \vee s$. In addition, we write Y for the reflected process defined by $Y_t = S_t - X_t$, and let ζ be the stopping time defined by

$$\zeta := \inf\{t \geq 0 : Y_t \geq b\},$$

the time of ruin. The payoff is composed of three parts; the running income to be received continuously until stopped or absorbed, the terminal reward part to be received when the process is stopped, and the penalty part incurred when the process is absorbed.

We consider the following optimal stopping problem and the value function $V : \mathbb{R}^2 \mapsto \mathbb{R}$ associated with initial values $X_0 = x$ and $S_0 = s$;

$$V(x,s) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^{x,s}\left[\int_0^{\tau \wedge \zeta} e^{-qt} f(X_t, S_t)\mathrm{d}t + 1\!\!1_{\{\tau < \zeta\}} e^{-q\tau} g(X_\tau, S_\tau) - 1\!\!1_{\{\tau > \zeta\}} e^{-q\zeta} k(X_\zeta, S_\zeta)\right]$$

where $q \geq 0$ is the constant discount rate and $\mathcal{S}$ is a set of stopping times. The running income function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ is a measurable function that satisfies

$$\mathbb{E}^x\left[\int_0^\infty e^{-qt}|f(X_t, S_t)|\mathrm{d}t\right] < \infty.$$

The reward function $g : \mathbb{R}^2 \mapsto \mathbb{R}$ and the penalty function $k : \mathbb{R}^2 \mapsto \mathbb{R}$ are assumed to be measurable.Our main purpose is to calculate V and to find the stopping time τ^* which attains the supremum.

For each function $l : \mathbb{R} \mapsto \mathbb{R}_+$, we define a stopping time $\tau(l)$ by

$$\tau(l) := \inf\{t \geq 0 : S_t - X_t > l(S_t)\}. \tag{2.1}$$

This is the first time the excursion $S - X$ from level, say $S = s$, becomes greater than some value $l(s)$. When l is constant, for example, $\bar{l} \equiv c$ on $\mathbb{R}$, we write

$$\tau_c := \inf\{t \geq 0 : S_t - X_t > c\}.$$

In particular, if $\bar{l} \equiv b$, then $\tau_b = \zeta$. Next we let $\mathcal{S}'$ be the set of stopping times defined by

$$\mathcal{S}' := \{\tau(l) : l(m) \leq b \text{ for all } m \in \mathbb{R}\}.$$

Note that if $\tau \in \mathcal{S}'$, then $\tau \leq \zeta$. Usually, $\mathcal{S}$ in the definition of the value function V should be the set of all possible $\mathbb{F}$-stopping times, but since the time of ζ is observable and the game is over once it happens, we can restrict $\mathcal{S}$ to $\mathcal{S}'$. Moreover, we shall show that it suffices to consider stopping times of the form (2.1), i.e., $\tau^* \in \mathcal{S}'$ later in Section 5.

2.1. **Scale functions.** We review some mathematically important facts before solving the problem. Associated with every spectrally negative Lévy process, there exists a (q-)scale function

$$W^{(q)} : \mathbb{R} \mapsto \mathbb{R}; \quad q \geq 0,$$

that is continuous and strictly increasing on $[0, \infty)$ and is uniquely determined by

$$\int_0^\infty e^{-\beta x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\beta) - q}, \qquad \beta > \Phi(q).$$

Fix $a > x > 0$ and define

$$T_a := \inf\{t \geq 0 : X_t \geq a\} \quad \text{and} \quad T_0^- := \inf\{t \geq 0 : X_t < 0\}. \tag{2.2}$$

then we have

(2.3)
$$\mathbb{E}^x\left[e^{-qT_a}1_{\{T_a<T_0^-,\,T_a<\infty\}}\right]=\frac{W^{(q)}(x)}{W^{(q)}(a)}\quad\text{and}\quad\mathbb{E}^x\left[e^{-qT_0^-}1_{\{T_a>T_0^-,\,T_0^-<\infty\}}\right]=Z^{(q)}(x)-Z^{(q)}(a)\frac{W^{(q)}(x)}{W^{(q)}(a)},$$

where

$$Z^{(q)}(x):=1+q\int_0^x W^{(q)}(y)\mathrm{d}y,\quad x\in\mathbb{R}.$$

Here we have

$$W^{(q)}(x)=0\quad\text{on}\quad(-\infty,0)\quad\text{and}\quad Z^{(q)}(x)=1\quad\text{on}\quad(-\infty,0]. \tag{2.4}$$

We also have

$$\mathbb{E}^x\left[e^{-qT_0^-}\right]=Z^{(q)}(x)-\frac{q}{\Phi(q)}W^{(q)}(x),\quad x>0. \tag{2.5}$$

In particular, $W^{(q)}$ is continuously differentiable on $(0,\infty)$ if Π does not have atoms and $W^{(q)}$ is twice-differentiable on $(0,\infty)$ if $\sigma>0$; see, e.g., [9]. Throughout this paper, we assume the former.

Assumption 2.1. *We assume that Π does not have atoms.*

Fix $q>0$. The scale function increases exponentially;

$$W^{(q)}(x)\sim\frac{e^{\Phi(q)x}}{\psi'(\Phi(q))}\quad\text{as}\ x\uparrow\infty. \tag{2.6}$$

There exists a (scaled) version of the scale function $W_{\Phi(q)}=\{W_{\Phi(q)}(x);x\in\mathbb{R}\}$ that satisfies

$$W_{\Phi(q)}(x)=e^{-\Phi(q)x}W^{(q)}(x),\quad x\in\mathbb{R} \tag{2.7}$$

and

$$\int_0^\infty e^{-\beta x}W_{\Phi(q)}(x)\mathrm{d}x=\frac{1}{\psi(\beta+\Phi(q))-q},\quad\beta>0.$$

Moreover $W_{\Phi(q)}(x)$ is increasing, and as is clear from (2.6),

$$W_{\Phi(q)}(x)\uparrow\frac{1}{\psi'(\Phi(q))}\quad\text{as}\ x\uparrow\infty. \tag{2.8}$$

Regarding its behavior in the neighborhood of zero, it is known that

(2.9)
$$W^{(q)}(0)=\left\{\begin{array}{ll}0, & \text{unbounded variation}\\ \frac{1}{\mu}, & \text{bounded variation}\end{array}\right\}\quad\text{and}\quad W^{(q)\prime}(0+)=\left\{\begin{array}{ll}\frac{2}{\sigma^2}, & \sigma>0\\ \infty, & \sigma=0\text{ and }\Pi(0,\infty)=\infty\\ \frac{q+\Pi(0,\infty)}{\mu^2}, & \text{compound Poisson}\end{array}\right\};$$

see Lemmas 4.3-4.4 of [18]. For a comprehensive account of the scale function, see [7, 8, 16, 18]. See [13, 23] for numerical methods for computing the scale function.

3. Solution

To calculate the value function V, let us introduce the probability measure $\widetilde{\mathbb{P}}^{x,s}$ such that the Radon-Nikodym derivative between $\widetilde{\mathbb{P}}^{x,s}$ and $\mathbb{P}^{x,s}$ is defined by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}^{x,s}}{\mathrm{d}\mathbb{P}^{x,s}}\bigg|_{\mathcal{F}_t} = e^{-qt+\Phi(q)(X_t-x)}.$$

Under $\widetilde{\mathbb{P}}^{x,s}$, X has the Laplace exponent $\widetilde{\psi}$ defined by

$$\begin{aligned}\widetilde{\psi}(\lambda) &= \psi(\lambda+\Phi(q))-\psi(\Phi(q)) \\ &= \left(\sigma^2\Phi(q)+\mu+\int_{(-\infty,0)} x(e^{\Phi(q)x}-1)1\!\!1_{\{x>-1\}}\Pi(\mathrm{d}x)\right)\lambda \\ &\quad +\frac{1}{2}\sigma^2\lambda^2+\int_{(-\infty,0)}(e^{\lambda x}-1-\lambda x 1\!\!1_{\{x>-1\}})e^{\Phi(q)x}\Pi(\mathrm{d}x).\end{aligned}$$

Note that since $\widetilde{\psi}'(0+)=\psi'(\Phi(q)+)>0$, X drifts to ∞ for $q\geq 0$.

Let $W:\mathbb{R}_+\mapsto\mathbb{R}_+$ be the scale function of X under $\widetilde{\mathbb{P}}^{x,s}$, that is, W has the Laplace transform

$$\int_0^\infty e^{-\lambda x}W(x)dx = \frac{1}{\widetilde{\psi}(\lambda)}.$$

In addition, we define the process $\eta=\{\eta_t;t\geq 0\}$ of the height of the excursion as

$$\eta_u := \sup\{(S-X)_{T_u(-)+w} : 0\leq w\leq T_u - T_u(-)\},$$

then η is a Poisson point process, and we denote its characteristic measure on $\widetilde{\mathbb{P}}^{x,s}$ by $\tilde{\nu}$. It is known that there is a relation between $W_{\Phi(q)}$ and $\tilde{\nu}$:

$$W_{\Phi(q)}(x) = c\exp\left(-\int_x^\infty \tilde{\nu}[u,\infty)\mathrm{d}u\right), \tag{3.1}$$

where c is some positive constant.

Denote by $\bar{f}:\mathbb{R}^2\mapsto\mathbb{R}$ the q-potential of f

$$\bar{f}(x,s)=\mathbb{E}^{x,s}\left[\int_0^\infty e^{-qt}f(X_t,S_t)\mathrm{d}t\right].$$

From the strong Markov property of (X,S), we have

$$\begin{aligned}\mathbb{E}^{x,s}\left[\int_0^{\tau\wedge\zeta} e^{-qt}f(X_t,S_t)\mathrm{d}t\right] &= \mathbb{E}^{x,s}\left[\int_0^\infty e^{-qt}f(X_t,S_t)\mathrm{d}t-\int_{\tau\wedge\zeta}^\infty e^{-qt}f(X_t,S_t)\mathrm{d}t\right] \\ &= \bar{f}(x,s)-\mathbb{E}^{x,s}\left[\mathbb{E}\left[\int_{\tau\wedge\zeta}^\infty e^{-qt}f(X_t,S_t)\mathrm{d}t\,\bigg|\,\mathcal{F}_{\tau\wedge\zeta}\right]\right] \\ &= \bar{f}(x,s)-\mathbb{E}^{x,s}\left[e^{-q(\tau\wedge\zeta)}\mathbb{E}^{X_{\tau\wedge\zeta},S_{\tau\wedge\zeta}}\left[\int_0^\infty e^{-qt}f(X_t,S_t)\mathrm{d}t\right]\right] \\ &= \bar{f}(x,s)-\mathbb{E}^{x,s}\left[e^{-q(\tau\wedge\zeta)}\bar{f}(X_{\tau\wedge\zeta},S_{\tau\wedge\zeta})\right] \\ &= \bar{f}(x,s)-\mathbb{E}^{x,s}\left[1\!\!1_{\{\tau\leq\zeta\}}e^{-q\tau}\bar{f}(X_\tau,S_\tau)+1\!\!1_{\{\tau>\zeta\}}e^{-q\zeta}\bar{f}(X_\zeta,S_\zeta)\right].\end{aligned}$$

If $\tau \in \mathcal{S}'$, we can write

$$\mathbb{E}^{x,s}\left[\int_0^{\tau\wedge\zeta} e^{-qt} f(X_t,S_t)\mathrm{d}t\right] = \bar{f}(x,s) - \mathbb{E}^{x,s}\left[\mathbb{1}_{\{\tau<\zeta\}}e^{-q\tau}\bar{f}(X_\tau,S_\tau) + \mathbb{1}_{\{\tau=\zeta\}}e^{-q\tau}\bar{f}(X_\tau,S_\tau)\right].$$

Hence the value function V can be written by

$$V(x,s) = \bar{f}(x,s) + \sup_{\tau\in\mathcal{S}'}\mathbb{E}^{x,s}\left[\mathbb{1}_{\{\tau<\zeta\}}e^{-q\tau}(g-\bar{f})(X_\tau,S_\tau) - \mathbb{1}_{\{\tau=\zeta\}}e^{-q\tau}(k+\bar{f})(X_\tau,S_\tau)\right]. \tag{3.2}$$

3.1. **When** $X_0 = S_0$**.** As a first step, we consider the case $X_0 = S_0$. Set stopping times T_m as $T_m = \inf\{t \geq 0 : X_t \geq m\}$ (Recall (2.2)). From the strong Markov property of (X,S), when $\tau(l) \in \mathcal{S}'$ and $S_0 = X_0 = s$, we have,

$$\begin{aligned}
&\mathbb{E}^{s,s}\left[\mathbb{1}_{\{\tau(l)<\zeta\}}e^{-q\tau(l)}(g-\bar{f})(X_{\tau(l)},S_{\tau(l)})\right] \\
=& \int_s^\infty \mathbb{E}^{s,s}\left[\mathbb{1}_{\{\tau(l)<\zeta,S_{\tau(l)}\in\mathrm{d}m\}}e^{-q\tau(l)}(g-\bar{f})(X_{\tau(l)},S_{\tau(l)})\right] \\
=& \int_s^\infty \mathbb{E}^{s,s}\left[\mathbb{1}_{\{T_m\leq\tau(l)\}}e^{-qT_m}\mathbb{E}^{m,m}\left[e^{-q\tau_{l(m)}}(g-\bar{f})(X_{\tau_{l(m)}},S_{\tau_{l(m)}})\mathbb{1}_{\{S_{\tau_{l(m)}}-X_{\tau_{l(m)}}\leq b,S_{\tau_{l(m)}}\in\mathrm{d}m\}}\right]\right] \\
=& \int_s^\infty \mathbb{E}^{s,s}\left[\mathbb{1}_{\{S_{\tau(l)}\geq m\}}e^{-qT_m}\right]\Bigg(\mathbb{E}^{m,m}\left[e^{-q\tau_{l(m)}}\mathbb{1}_{\{Y_{\tau_{l(m)}-}=l(m),S_{\tau_{l(m)}}\in\mathrm{d}m\}}\right](g-\bar{f})(m-l(m),m) \\
+& \iint_A \mathbb{E}^{m,m}\left[e^{-q\tau_{l(m)}}\mathbb{1}_{\{X_{\tau_{l(m)}}-X_{\tau_{l(m)}-}\in\mathrm{d}h,S_{\tau_{l(m)}}\in\mathrm{d}m,Y_{\tau_{l(m)}-}\in\mathrm{d}y\}}\right](g-\bar{f})(m-y+h,m)\Bigg),
\end{aligned} \tag{3.3}$$

where

$$A = \{(y,h)\in\mathbb{R}^2; y-h\in[l(m),b], h<0, y\in[0,l(m)]\}.$$

Now we examine each term in the last line of (3.3). Since X is a spectrally negative process and S is its running maximum process, by (3.1) we have, for $m \geq s$,

$$\begin{aligned}
\mathbb{E}^{s,s}\left[\mathbb{1}_{\{S_{\tau(l)}\geq m\}}e^{-qT_m}\right] &= \widetilde{\mathbb{E}}^{s,s}\left[e^{-(m-s)\Phi(q)}\mathbb{1}_{\{S_{\tau(l)}\geq m\}}\right] \\
&= e^{-(m-s)\Phi(q)}\widetilde{\mathbb{P}}^{s,s}(S_{\tau(l)}\geq m) \\
&= \exp\left(-\int_s^m\left(\frac{W'_{\Phi(q)}(l(u))}{W_{\Phi(q)}(l(u))}+\Phi(q)\right)\mathrm{d}u\right) \\
&= \exp\left(-\int_s^m\frac{W^{(q)\prime}(l(u))}{W^{(q)}(l(u))}\mathrm{d}u\right).
\end{aligned}$$

From Theorems 1 and 2 in Pistorius [22], we have

$$\begin{aligned}
&\mathbb{E}^{m,m}\left[e^{-q\tau_{l(m)}}\mathbb{1}_{\{X_{\tau_{l(m)}}-X_{\tau_{l(m)}-}\in\mathrm{d}h,S_{\tau_{l(m)}}\in\mathrm{d}m,Y_{\tau_{l(m)}-}\in\mathrm{d}y\}}\right] \\
&= \mathbb{1}_{\{y-h>l(m)\}}\Lambda(\mathrm{d}h)\left(W^{(q)\prime}(y)-\frac{W^{(q)\prime}(l(m))}{W^{(q)}(l(m))}W^{(q)}\right)\mathrm{d}y\mathrm{d}m,
\end{aligned}$$

and

$$\mathbb{E}^{m,m}\left[e^{-q\tau_{l(m)}}\mathbb{1}_{\{Y_{\tau_{l(m)}-}=l(m),S_{\tau_{l(m)}}\in\mathrm{d}m\}}\right] = \frac{\sigma^2}{2}\left(\frac{W^{(q)\prime}(l(m))^2}{W^{(q)}(l(m))}-W^{(q)\prime\prime}(l(m))\right).$$

Putting together, if $\tau(l) \in \mathcal{S}'$, (3.3) becomes

$$
\begin{aligned}
& \mathbb{E}^{s,s}\left[\mathbb{1}_{\{\tau(l)<\zeta, S_{\tau(l)}\in \mathrm{d}m\}} e^{-q\tau(l)}(g-\bar{f})(X_{\tau(l)}, S_{\tau(l)})\right] \\
= \; & \exp\left(-\int_s^m \frac{W^{(q)\prime}(l(u))}{W^{(q)}(l(u))}\mathrm{d}u\right)\left(\frac{\sigma^2}{2}\left(\frac{W^{(q)\prime}(l(m))^2}{W^{(q)}(l(m))} - W^{(q)\prime\prime}(l(m))\right)(g-\bar{f})(m-l(m),m)\right. \\
& \left. + \int_0^{l(m)}\mathrm{d}y\int_{y-b}^{y-l(m)}\Lambda(\mathrm{d}h)(g-\bar{f})(m-y+h,m)\left(W^{(q)\prime}(y) - \frac{W^{(q)\prime}(l(m))}{W^{(q)}(l(m))}W^{(q)}(y)\right)\right)\mathrm{d}m.
\end{aligned}
$$

In the same way as above, if $\tau(l) \in \mathcal{S}'$, we obtain for the second term of the expectation in (3.2)

$$
\begin{aligned}
& \mathbb{E}^{x,s}\left[\mathbb{1}_{\{\tau(l)=\zeta, S_{\tau(l)}\in \mathrm{d}m\}} e^{-q\tau(l)}(\bar{f}+k)(X_{\tau(l)}, S_{\tau(l)})\right] \\
= \; & \int_s^\infty \mathbb{E}^{s,s}\left[\mathbb{1}_{\{T_m\le\tau(l)\}}e^{-qT_m}\mathbb{E}^{m,m}\left[e^{-q\tau_{l(m)}}(\bar{f}+k)(X_{\tau_{l(m)}}, S_{\tau_{l(m)}})\mathbb{1}_{\{S_{\tau_{l(m)}}-X_{\tau_{l(m)}}>b, S_{\tau_{l(m)}}\in \mathrm{d}m\}}\right]\right] \\
= \; & \exp\left(-\int_s^m \frac{W^{(q)\prime}(l(u))}{W^{(q)}(l(u))}\mathrm{d}u\right)\times \\
& \iint_B \mathbb{E}^{m,m}\left[e^{-q\tau_{l(m)}}\mathbb{1}_{\{X_{\tau_{l(m)}}-X_{\tau_{l(m)}-}\in \mathrm{d}h, S_{\tau_{l(m)}}\in\mathrm{d}m, Y_{\tau_{l(m)}-}\in\mathrm{d}y\}}\right](\bar{f}+k)(m-y+h,m) \\
= \; & \exp\left(-\int_s^m \frac{W^{(q)\prime}(l(u))}{W^{(q)}(l(u))}\mathrm{d}u\right)\times \\
& \left(\int_0^{l(m)}\mathrm{d}y\int_{-\infty}^{y-b}\Lambda(\mathrm{d}h)(\bar{f}+k)(m-y+h,m)\left(W^{(q)\prime}(y) - \frac{W^{(q)\prime}(l(m))}{W^{(q)}(l(m))}W^{(q)}(y)\right)\right)\mathrm{d}m
\end{aligned}
$$

where

$$B = \{(y,h)\in\mathbb{R}^2; y-h>b, h<0, y\in[0,l(m)].\}$$

For notational simplicity, let the function $F_m(z):\mathbb{R}_+\mapsto\mathbb{R}$ defined by

$$
\begin{aligned}
F_m(z) := \; & \frac{\sigma^2}{2}\left(\frac{W^{(q)\prime}(z)^2}{W^{(q)}(z)} - W^{(q)\prime\prime}(z)\right)(g-\bar{f})(m-z,m) \\
& + \int_0^z\mathrm{d}y\int_{y-b}^{y-z}\Lambda(\mathrm{d}h)(g-\bar{f})(m-y+h,m)\left(W^{(q)\prime}(y) - \frac{W^{(q)\prime}(z)}{W^{(q)}(z)}W^{(q)}(y)\right) \\
& - \int_0^z\mathrm{d}y\int_{-\infty}^{y-b}\Lambda(\mathrm{d}h)(\bar{f}+k)(m-y+h,m)\left(W^{(q)\prime}(y) - \frac{W^{(q)\prime}(z)}{W^{(q)}(z)}W^{(q)}(y)\right).
\end{aligned} \tag{3.4}
$$

Hence we have, up to this point, proved the following:

Proposition 3.1. *When $X_0 = S_0$, the value function $V(s,s)$ can be represented by*

$$V(s,s) = \sup_l \int_s^\infty \exp\left(-\int_s^m \frac{W^{(q)}(l(u))}{W^{(q)\prime}(l(u))}\mathrm{d}u\right)F_m(l(m))\mathrm{d}m$$

where $F_m(\cdot)$ is defined in (3.4)

Recall that $l(s)$ denotes the height of the excursion $Y = S - X$ when $S = s$. We wish to find, given s, the optimal height $l^*(s)$ to stop the process, and to calculate $V(s,s)$ explicitly.

Proposition 3.2. *Under Assumption 2.1 and $q > 0$, suppose further that $F_m : \mathbb{R}_+ \mapsto \mathbb{R}$ is continuous. Then we have*

$$V(s,s) = \frac{F_s(l^*(s))W^{(q)}(l^*(s))}{W^{(q)\prime}(l^*(s))}, \tag{3.5}$$

and $l^*(s)$ is the maximizer of the map $z \mapsto \frac{F_s(z)W^{(q)}(z)}{W^{(q)\prime}(z)}$.

Note that a sufficient condition for the continuity of F_m is the continuity of f, g and k. This is a consequence of the continuity of $x \mapsto \mathbb{E}^{x,s}[f(X_t, S_t)]$ for all $t \geq 0$ and $s \in \mathbb{R}_+$, $W^{(q)} \in C^1$ (by recalling Assumption 2.1) and

$$|\bar{f}(x,s) - \bar{f}(y,s)| \leq q^{-1}|f(x,s) - f(y,s)|$$

for all $x, y \in \mathbb{R}$.

Proof. When F_m is continuous, this integral can be approximated by the limit of Riemann sum. Set $V_n : \mathbb{R}^2 \mapsto \mathbb{R}$ as

$$V_n(s,s) = \sup_l \frac{1}{n}\left[F_s(l(s)) + \sum_{i=1}^{\infty} \exp\left(-\frac{1}{n}\sum_{j=0}^{n-1}\frac{W^{(q)\prime}\left(l\left(s+\frac{j}{n}\right)\right)}{W^{(q)}\left(l\left(s+\frac{j}{n}\right)\right)}\right)F_{s+\frac{i}{n}}\left(l\left(s+\frac{i}{n}\right)\right)\right].$$

Then we have $\lim_{n\to\infty} V_n(s,s) = V(s,s)$. From the Riemann sum above, we can write

$$V_n(s,s) = \sup_z \left[\frac{F_s(z)}{n} + \exp\left(-\frac{W^{(q)\prime}(z)}{nW^{(q)}(z)}\right)V_n\left(s+\frac{1}{n}, s+\frac{1}{n}\right)\right].$$

Since $\lim_{n\to\infty} V_n\left(s+\frac{1}{n}, s+\frac{1}{n}\right) = V(s,s)$, the optimal threshold $l^*(s)$ should satisfy

$$\lim_{n\to\infty} V_n(s,s) = \lim_{n\to\infty}\left[\frac{F_s(l^*(s))}{n} + \exp\left(-\frac{W^{(q)\prime}(l^*(s))}{nW^{(q)}(l^*(s))}\right)V_n\left(s+\frac{1}{n}, s+\frac{1}{n}\right)\right].$$

Hence we have

$$\begin{aligned} V(s,s) &= \lim_{n\to\infty}\frac{V_n(s,s) - \exp\left(-\frac{W^{(q)\prime}(l^*(s))}{nW^{(q)}(l^*(s))}\right)V_n\left(s+\frac{1}{n}, s+\frac{1}{n}\right)}{\left(1-\exp\left(-\frac{W^{(q)\prime}(l^*(s))}{nW^{(q)}(l^*(s))}\right)\right)} \\ &= \lim_{n\to\infty}\frac{F_s(l^*(s))}{n\left(1-\exp\left(-\frac{W^{(q)\prime}(l^*(s))}{nW^{(q)}(l^*(s))}\right)\right)} = \frac{F_s(l^*(s))W^{(q)}(l^*(s))}{W^{(q)\prime}(l^*(s))}, \end{aligned}$$

and $l^*(s)$ is the value which gives supremum to $\frac{F_s(z)W^{(q)}(z)}{W^{(q)\prime}(z)}$. $\square$

3.2. **When $S_0 > X_0$.** Finally, let us consider the case of $X_0 < S_0$. In this case, V can be represented in terms of $V(s,s)$ as follows:

$$V(x,s) = \bar{f}(x,s) + \sup_{\tau \in \mathcal{S}'} \mathbb{E}^{x,s}\left[\mathbb{1}_{\{T_s<\tau\}} e^{-qT_s}(V-\bar{f})(s,s) + \mathbb{1}_{\{\tau<\tau_s\wedge\zeta\}} e^{-q\tau}(g-\bar{f})(X_\tau,s) \right. \\ \left. - \mathbb{1}_{\{\zeta=\tau<\tau_s\}} e^{-q\tau}(k+\bar{f})(X_\tau,s)\right] \tag{3.6}$$

Set $\tau = \tau(l)$. Then, from (2.3), the first term in (3.6) can be written by

$$\mathbb{E}^{x,s}\left[\mathbb{1}_{\{T_s<\tau\}} e^{-qT_s}(V-\bar{f})(s,s)\right] = \frac{W^{(q)}(l(s)+x-s)}{W^{(q)}(l(s))}(V-\bar{f})(s,s).$$

For the second term, we use Theorem 1 and 2 in Pistorius [22] again to obtain

$$\begin{aligned}
& \mathbb{E}^{x,s}\left[\mathbb{1}_{\{\tau<T_s\wedge\zeta\}} e^{-q\tau}(g-\bar{f})(X_\tau,s)\right] \\
= \; & \mathbb{E}^{x,s}\left[e^{-q\tau_{l(s)}} \mathbb{1}_{\{Y_{\tau_{l(s)}-}=l(s), S_{\tau_{l(s)}}=s\}}\right](g-\bar{f})(s-l(s),s) \\
+ \; & \iint_A \mathbb{E}^{x,s}\left[e^{-q\tau_{l(s)}} \mathbb{1}_{\{X_{\tau_{l(s)}}-X_{\tau_{l(s)}-}\in dh, S_{\tau_{l(s)}}=s, Y_{T_{\tau_{l(s)}}-}\in dy\}}\right](g-\bar{f})(s-y+h,s) \\
= \; & \frac{\sigma^2}{2}\left(\frac{W^{(q)\prime}(l(s))^2}{W^{(q)}(l(s))} - W^{(q)\prime\prime}(l(s))\right)(g-\bar{f})(s-l(s),s) \\
+ \; & \int_0^{l(s)} dy \int_{y-b}^{y-l(s)} \Lambda(dh)\left(\frac{W^{(q)}(l(s)+x-s)}{W^{(q)}(l(s))}W^{(q)}(y) - W^{(q)}(y+x-s)\right)(g-\bar{f})(s-y+h,s).
\end{aligned}$$

On the third term, we have, in the same way as above,

$$\begin{aligned}
& \mathbb{E}^{x,s}\left[\mathbb{1}_{\{\zeta=\tau<T_s\}} e^{-q\tau}(k+\bar{f})(X_\tau,s)\right] \\
= \; & \iint_B \mathbb{E}^{x,s}\left[e^{-q\tau_{l(s)}} \mathbb{1}_{\{X_{\tau_{l(s)}}-X_{\tau_{l(s)}-}\in dh, S_{\tau_{l(s)}}=s, Y_{\tau_{l(s)}-}\in dy\}}\right](k+\bar{f})(s-y+h,s) \\
= \; & \int_0^{l(s)} dy \int_{-\infty}^{y-b} \Lambda(dh)\left(\frac{W^{(q)}(l(s)+x-s)}{W^{(q)}(l(s))}W^{(q)}(y) - W^{(q)}(y+x-s)\right)(k+\bar{f})(s-y+h,s).
\end{aligned}$$

Combing all of these terms, we can write

$$\begin{aligned}
V(x,s) \; = \; & \bar{f}(x,s) + \frac{W^{(q)}(l^*(s)+x-s)}{W^{(q)}(l^*(s))}(V-\bar{f})(s,s) \\
+ \; & \frac{\sigma^2}{2}\left(\frac{W^{(q)\prime}(l^*(s))^2}{W^{(q)}(l^*(s))} - W^{(q)\prime\prime}(l^*(s))\right)(g-\bar{f})(s-l^*(s),s) \\
+ \; & \int_0^{l^*(s)} dy \int_{y-b}^{y-l^*(s)} \Lambda(dh)\left(\frac{W^{(q)}(l^*(s)+x-s)}{W^{(q)}(l^*(s))}W^{(q)}(y) - W^{(q)}(y+x-s)\right)(g-\bar{f})(s-y+h,s) \\
- \; & \int_0^{l(s)} dy \int_{-\infty}^{y-b} \Lambda(dh)\left(\frac{W^{(q)}(l^*(s)+x-s)}{W^{(q)}(l^*(s))}W^{(q)}(y) - W^{(q)}(y+x-s)\right)(k+\bar{f})(s-y+h,s).
\end{aligned}$$

4. BANK'S OPTIMIZATION UNDER CAPITAL REQUIREMENTS

In this section, we solve an example. Imagine that a bank's total asset value is represented by e^X. We set that the *leverage ratio*, defined as (Debt)/(Total Asset), cannot exceed e^{-b}. For example, if the bank has the initial asset of $e^x = 100$ with $e^{-b} = 0.8$, it has total asset of 100 financed by debt 80 and equity 20. We can think of this ratio as the maximum leverage ratio that is allowed by the banking regulations. We assume that the bank increases its asset base as long as $X = S$ where S is the running maximum of X and that the bank's leverage ratio is maintained at 0.8. Hence if the asset value appreciates to 120, then this would provide the bank with more lending opportunity since the equity value is now 40. With this new equity level, the bank increases its leverage up to 0.8, that is, total asset increasing to 200 financed by debt 160 and equity 40. Note that $e^S = e^X = 200$ and the debt level is $e^{-b}e^S = e^{S-b} = 160$. Now if the bank's asset deteriorates due to defaults in the lending portfolio, we would have $S - X > 0$. In other words, there appears an excursion from the level of $e^S = 200$. Since the asset level has been pegged at $e^S = 200$, the bank's equity would be wiped out when $e^{S-b} = e^X$. That is, when $e^X = 160$ and the process is absorbed.

This model well describes a real situation where even a large bank can fail easily as we have experienced several times, the recent and magnified shock being the last financial crisis in 2007-2008. After becoming a big bank, it may still have an incentive to increase assets, seeking for profits. The danger of becoming insolvent is still $X = S - b$ if the bank continues to use leverage ratio of e^{-b}. The absorbing boundary is coming after.

Moreover, note that this model can incorporate the regulatory requirements that the bank, when experiencing asset deterioration, need to sell the assets in order to reduce the leverage.[1] For example, assume that when the bank loses one dollar of asset, the bank loses its equity by α and reduces its debt by $1 - \alpha$, where $\alpha \in (0, 1]$. Then, at the time the equity is wiped out, we have

$$e^X \le e^S \left(1 - \frac{1 - e^{-b}}{\alpha}\right),$$

that is, the process is absorbed when the excursion $S - X$ reaches to $\log\left(\frac{1-e^{-b}}{\alpha} - 1\right)$.

We consider this problem with

$$X_t = x + \mu t + \sigma B_t + \sum_{i=1}^{N(t)} \xi_i,$$

where B is a standard Brownian motion, N is a Poisson process with intensity a, and ξ_i $(i = 1, 2, \dots)$ are independent identically distributed random variables whose distributions are exponential with parameter ρ under $\mathbb{P}$. The reward functions are set by $f(x, s) = e^{x/2}$, $g(x, s) = e^x$, and $k(x, s) = 0$. In this case, the Laplace exponent ψ of X is given by

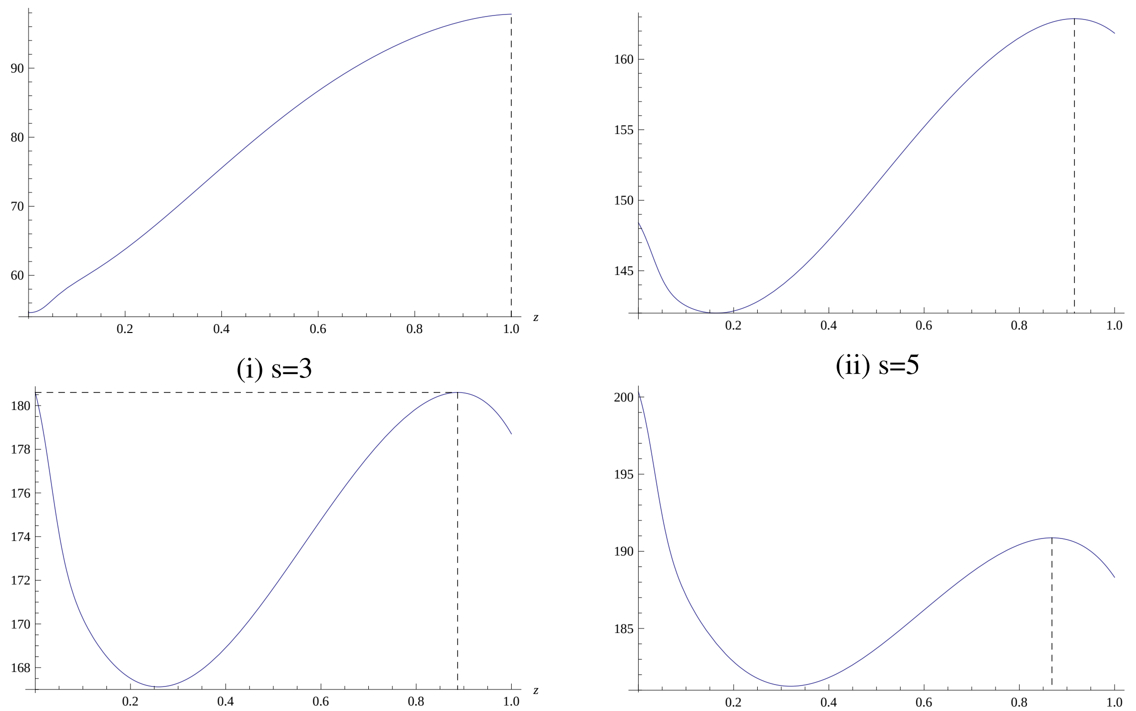
$$\psi(\lambda) = \mu\lambda + \frac{\sigma^2\lambda^2}{2} - \frac{a\lambda}{\rho + \lambda}.$$

$\psi(\lambda) = q$ has three solutions $\Phi(q)$, α, and β (in decreasing order) and q-scale function $W^{(q)}$ of X is represented with these values;

$$W^{(q)}(x) = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} + \frac{e^{\alpha x}}{\psi'(\alpha)} + \frac{e^{\beta x}}{\psi'(\beta)}.$$

[1]We are thankful to Nan Chen for pointing out this requirement.

We computed the value and an optimal strategy for this problem with $\mu = 0.25$, $\sigma = 0.1$, $a = 2$, $\rho = 10$, $q = 0.1$, and $b = 1$. Panels (i), (ii), (iii), and (iv) in Figure 1 are the graphs of $\frac{F_s(z)W^{(q)}(z)}{W^{(q)\prime}(z)}$ with $s = 3, 5, 5.1963$ and 5.3, respectively. As shown in Proposition 3.2, the maximum value in each graph corresponds to $V(s,s)$ for alternative values of s, and $z = l(s)$ are the maximizers.



(i) s=3 (ii) s=5

(iii) s=5.1963 (iv) s=5.3

FIGURE 1. the graphs of $\frac{F_s(z)W^{(q)}(z)}{W^{(q)\prime}(z)}$.

In the case (i) $s = 3$, there is the boundary solution $l(3) = 1$. This means that, in the excursion which occurs at level $S = 3$, it is optimal to stop when X goes below $3 - l(3) = 2$. Since we set $b = 1$ here, if X creeps over the level 2, we can obtain the terminal reward. Instead, if X jumps over the level 2, we should pay the penalty (but we set this as 0 here) and cannot gain the terminal reward. In the case (ii) $s = 5$, there is the internal solution $l(5) = 0.915551 < 1 = b$. Therefore, in the excursion which occurs at level $S = 5$, it is optimal to stop immediately that X goes below $5 - l(5) = 4.08445$. Since $l(5) < b$, if X creeps over the level $5 - l(5)$ or jumps onto in the area between $5 - l(5)$ and $5 - b = 4$, we can obtain the terminal reward. But if X jumps across the level 4, we cannot obtain the terminal reward.

The case (iii) $s = 5.1963$ is a special point in some sense. Unlike the case $s < 5.1963$, there are two solutions $l(5.1963) = 0$ and 0.886898. If we choose the former strategy $l(5.1963) = 0$, this means, when X reaches level 5.1963 for the first time, we stop it immediately and gain the terminal reward. If we choose the latter one $l(5.1963) = 0.886898$, that means we should behave like the case (ii). In the case (iv), there is the boundary solution $l(5.3) = 0$. That is, when X reaches level 5.3 for the first time, we should exit immediately and gain the terminal reward.

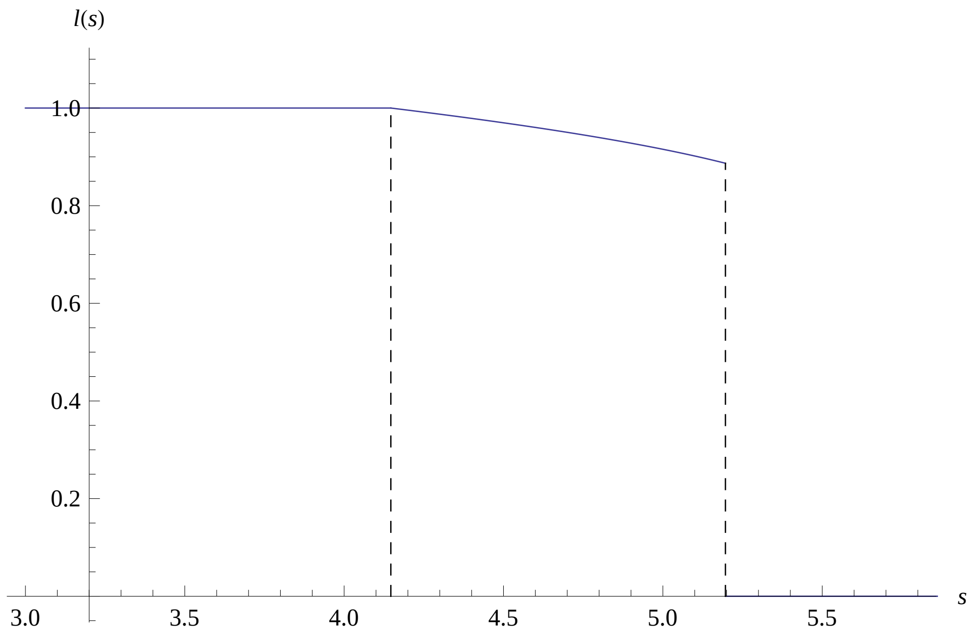


FIGURE 2. The graph of an optimal strategy l.

These arguments are summarized in Figure 2 that illustrates an optimal strategy l over the whole region of $s \in \mathbb{R}_+$. Two dashed lines are drawn at $s = 4.1464$ and $s = 5.1963$, which indicate the turning points of the strategies. For $s < 4.1464$, l constantly takes the value of 1. In this region, it is optimal to stop when the height of the excursion is 1. When s lies between 4.1464 and 5.1963, l has the form of concave curve started at 1. In this region, one should stop once the height of excursion is greater than or equal to $l(s) < 1$. Finally, for $s > 5.1963$, l constantly takes on 0. In this region, our strategy reduces to the classical threshold strategy by observing the path of process X. That is, stop at the first passage time of level 5.1963 by the process X.

5. NOTE ON OPTIMALITY

In this section, we will comment on the optimality of the strategy derived above. It suffices to consider (3.3), which we reproduce here, while we replacing $\tau(l)$ by a generic stopping time τ.

$$\mathbb{E}^{s,s}\left[1\!\!1_{\{\tau<\zeta\}} e^{-q\tau}(g-\bar{f})(X_\tau, S_\tau)\right] = \int_s^\infty \mathbb{E}^{s,s}\left[1\!\!1_{\{\tau<\zeta, S_\tau \in \mathrm{d}s\}} e^{-q\tau}(g-\bar{f})(X_\tau, S_\tau)\right].$$

For each level $S = s$ from which an excursion occurs, the value S does not change during the excursion. During the excursion interval, the problem can be thought as one-dimensional problem for the state process X. Now we look at *only* the process X over the time horizon $[0, \zeta \wedge T_m)$ and find $\tau^* \in \mathcal{S}'$. We have seen in Section 4 that the solution of this optimal stopping varies depending on $S = s$. There are three regions $l(s) = b, 0 < l(s) < b$, and $l(s) = 0$. *Hence we have the optimality of our strategy presented in Section 3 to the extent that the threshold strategy is optimal for the one-dimensional spectrally negative Lévy process.* For the case of optimality being given by the threshold strategy, see Egami and Yamazaki [15], which contains the formula for finding optimal threshold levels as well as certain sufficient conditions for optimality. As mentioned in Section 1, a full characterization of optimal stopping rules has not yet been found in the literature.

For problems with one-dimensional continuous diffusion, however, a full characterization of the value function and of optimal stopping rule is known and the methodology for solution has been established; an optimal stopping rule is given by the threshold strategy. So at least, if X has no jump (that is, X is Brownian motion with drift), the solution we derived should be an optimal strategy for our problem.

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