

A Direct Solution Method for Stochastic Impulse Control Problems of One-dimensional Diffusions

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Abstract

We consider stochastic impulse control problems where the process is driven by one-dimensional diffusions. Impulse control problems are widely applied to financial engineering and decision-making problems including dividend payout problem, portfolio optimization with transaction costs, and inventory control. We shall show a new mathematical characterization of the value function in the continuation region as a linear function in certain transformed space. The merits of our approach are as follows: (1) one does not have to guess optimal strategies or verify the optimality via a verification lemma, (2) the method of finding the solution (based on the new characterization of the value function) is simple and direct and thereby (3) one can handle a broader class of reward and cost functions than the conventional methods that use quasi-variational inequalities.

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1 Introduction

This paper proposes a general solution method of stochastic impulse control problems for one dimensional diffusion processes. Stochastic impulse control problems have attracted a growing interest of many researchers for the last two decades. Under a typical setting, the controller faces some underlying process and reward/cost structure. There exist continuous and instantaneous components of reward/cost functions. By exercising impulse controls, the controller moves the underlying process from one point to another. At the same time, the controller receives rewards associated with the instantaneous shifts of the process. Then the controller's objective is to maximize the total discounted expected net income.

The mathematical framework to these types of problems is in Bensoussan and Lions (1984). Impulse control has been studied widely in inventory control (Harrison et al.(1983)), exchange rate problem (Jeanblanc-Picqué (1993), Mundaca and Øksendal (1998), Cadenillas and Zapatero (2000)), dividend payout problems (for example, Jeanblanc-Picqué and Shiryaev (1995)) and portfolio optimization with transaction costs (Korn (1998), Morton and Pliska (1995)). It is worth mentioning Korn (1999) surveys the applications in mathematical finance. Also see Chancelier et al.(2002) for a combination of optimal stopping and impulse control problems. In many economic and financial applications where the controlled process is described as an Itô diffusion, the solution to the problem demands a through study of a related Hamilton-Jacobi-Bellman equation and quasi-variational inequalities. The method based on quasi-variational inequalities is the following: One guesses the form of (a) continuation region and intervention region, (b) associated optimal policy, and (c) the value function. Then one specifies the value function by using appropriate boundary conditions and verifies optimality of the candidate policy. Both steps are often very difficult and the success depends heavily on the form of the controlled process, reward and cost functions.

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Alternatively, an impulse control problem can be viewed as a sequence of optimal stopping problems. The connection between impulse control and optimal stopping has been investigated by Davis (1992) and Øksendal and Sulem (2002) among others. In this setting, the value functions of a sequence of optimal stopping problems converge to the value function of the impulse control problem under suitable conditions. In this regard, we utilize the results of Dynkin (1965) (see e.g. Theorem 16.4) about the functional concavity characterization of α -excessive mappings and Dayanik and Karatzas (2003) that give a general characterization of optimal stopping times of one dimensional diffusions. We use these results to identify a new and useful characterization of the solution of the original impulse control problem. At the end, we get rid of the sequence of optimal stopping problems altogether and directly find the value function: the new characterization allows us to propose a new direct solution method for impulse control problems. Other works similar to our approach include Alvarez (2004), Alvarez and Lempa (2004) and Alvarez and Virtanen (2006).

Similar recursive formulations for one-dimensional diffusions appear in multiple stopping problems in the context of swing options, for example, Carmona and Dayanik (2003). See also Carmona and Touzi (2003) and Dahlgren and Korn (2005) for the valuation of swing options. It is worth pointing out the differences of this article from Carmona and Dayanik (2003), where given a reward function, the controller can exercise their options n times under the condition that the controller has to wait at least certain units of time between two exercising times. Using a recursive formulation, we are able to obtain the result that the value function has to be linear in the continuation region of the transformed space. This linear characterization enables us to present the three-step optimization procedure (described in section 3.3) that does not require us to use the recursive stopping scheme, while Carmona and Dayanik (2003) do not have this kind of geometric specification due to the nature of their problem; one needs to solve multiple stopping problems.

In the next section, we briefly go over the solution method for optimal stopping problems of one-dimensional diffusions. We describe the impulse control problem and its solution in Section 3. Examples are presented in Section 4. Finally, extensions and concluding remarks are in Section 5.

2 Summary of the Key Results of Optimal Stopping

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a standard Brownian motion $W = \{W_t; t \geq 0\}$ and consider the diffusion process X^0 with state space $\mathcal{I} \subseteq \mathbb{R}$ and dynamics

$$dX_t^0 = \mu(X_t^0)dt + \sigma(X_t^0)dW_t \quad (2.1)$$

for some Borel functions $\mu : \mathcal{I} \rightarrow \mathbb{R}$ and $\sigma : \mathcal{I} \rightarrow (0, \infty)$. We emphasize here that X^0 is an uncontrolled process. We assume that \mathcal{I} is an interval with endpoints $-\infty \leq c < d \leq +\infty$, and that X^0 is regular in (c, d) ; in other words, X^0 reaches y with positive probability starting at x for every x and y in (c, d) . We shall denote by $\mathbb{F} = \{\mathcal{F}_t\}$ the natural filtration generated by X^0 and assume that the usual condition is satisfied.

Let $\alpha \geq 0$ be a real constant and $h(\cdot)$ a Borel function such that $\mathbb{E}^x[e^{-\alpha\tau}h(X_\tau^0)]$ is well-defined for every \mathbb{F} -stopping time τ and $x \in \mathcal{I}$. We first consider

$$V(x) \triangleq \sup_{\tau \in \mathcal{S}} \mathbb{E}^x[e^{-\alpha\tau}h(X_\tau^0)], \quad x \in \mathcal{I} \quad (2.2)$$

the value function of the optimal stopping problem with the reward function $h(\cdot) : \mathcal{I} \rightarrow \mathbb{R}$ where the supremum is taken over the class \mathcal{S} of all \mathbb{F} -stopping times.

Let $z \in \mathcal{I}$ be a fixed point of the state space and τ_z be the first hitting time of $z \in \mathcal{I}$ by X^0 . Let us denote the infinitesimal generator of X^0 by \mathcal{A} and consider the ODE $(\mathcal{A} - \alpha)v(x) = 0$. This equation has two fundamental solutions, $\psi(\cdot)$ and $\varphi(\cdot)$. We set $\psi(\cdot)$ to be the increasing and $\varphi(\cdot)$ to be the decreasing

solution. They are linearly independent positive solutions and uniquely determined up to multiplication. It is well known that

$$\mathbb{E}^x[e^{-\alpha\tau_z}] = \begin{cases} \frac{\psi(x)}{\psi(z)}, & x \leq z, \\ \frac{\varphi(x)}{\varphi(z)}, & x \geq z. \end{cases}$$

For the complete characterization of $\psi(\cdot)$ and $\varphi(\cdot)$ corresponding to various types of boundary behavior, refer to Itô and McKean (1974). Let us now define

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)}, \quad x \in \mathcal{I}. \quad (2.3)$$

Then $F(\cdot)$ is continuous and strictly increasing. Next, following Dynkin (1965)(see pp. 238), we define concavity of a function with respect F as follows: A real valued function u is called F -concave on \mathcal{I} if, for every $x \in [l, r] \subseteq \mathcal{I}$,

$$u(x) \geq u(l) \frac{F(r) - F(x)}{F(r) - F(l)} + u(r) \frac{F(x) - F(l)}{F(r) - F(l)}.$$

We will use the notion of F -concavity to provide a new characterization of the value function of stochastic impulse control problems. Before doing that, the first step is to present the following results of optimal stopping problems, the proofs of which we refer to Dayanik and Karatzas (2003).

Proposition 2.1. The value function $V(\cdot)$ of (2.2) is the smallest nonnegative majorant of $h(\cdot)$ such that $V(\cdot)/\varphi(\cdot)$ is F -concave on \mathcal{I} .

Proposition 2.2. Let $W(\cdot)$ be the smallest nonnegative concave majorant of $H \triangleq (h/\varphi) \circ F^{-1}$ on $[F(c), F(d)]$, where $F^{-1}(\cdot)$ is the inverse of the strictly increasing function $F(\cdot)$ in (2.3). Then $V(x) = \varphi(x)W(F(x))$ for every $x \in \mathcal{I}$.

Proposition 2.3. Define

$$S \triangleq \{x \in [c, d] : V(x) = h(x)\}, \quad \text{and} \quad \tau^* \triangleq \inf\{t \geq 0 : X_t^0 \in S\}. \quad (2.4)$$

If $h(\cdot)$ is continuous on \mathcal{I} , then τ^* is an optimal stopping rule.

3 Impulse control problems and its solution

Suppose that at any time $t \in \mathbb{R}_+$ and any state $x \in \mathcal{I}$, we can intervene and give the system an impulse $\xi \in \mathbb{R}$. Once the system gets intervened, the point moves from x to $y \in \mathbb{R}$. An impulse control for the system is a double sequence,

$$\nu = (T_1, T_2, \dots, T_i, \dots; \xi_1, \xi_2, \dots, \xi_i, \dots)$$

where $0 \leq T_1 < T_2 < \dots$ are an increasing sequence of \mathbb{F} -stopping times and ξ_1, ξ_2, \dots are \mathcal{F}_{T_i} -measurable random variables representing impulses exercised at the corresponding intervention times T_i with $\xi_i \in Z$ for all i where $Z \in \mathbb{R}$ is a given set of admissible impulse amounts. The controlled process is, in general, described as follows:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad T_{i-1} \leq t < T_i \quad (3.1)$$

$$X_{T_i} = \Gamma(X_{T_i-}, \xi_i) \quad (3.2)$$

with some mapping $\Gamma : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$.

Let 0 be the absorbing state, without loss of generality, and $\tau_0 \triangleq \inf\{t : X_t = 0\}$ the ruin time. With the absorbing state at 0, it is natural to consider a set of problems where $Z \in \mathbb{R}_+$ (i.e., $\xi_i = x_i - y_i > 0$ for all i) and $X_{T_i} = X_{T_i-} - \xi_i$. (We shall comment on cases where interventions are allowed in both positive and negative directions in section 5.)

With each pair (T_i, ξ_i) , we associate the intervention reward

$$K(X_{T_i-}, X_{T_i}) \quad (3.3)$$

where $K(x, y) : \mathcal{I} \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function continuous in the first and second argument. Our result below does not depend on the specification of $K(\cdot)$. We assume that, for any point $x \in \mathcal{I}$,

$$K(x, x) < 0. \quad (3.4)$$

due to the existence of fixed costs. We consider the following performance measure with $\nu \in \mathcal{V}$, a collection of admissible strategies,

$$J^\nu(x) = \mathbb{E}^x \left[\int_0^{\tau_0} e^{-\alpha s} f(X_s) ds + e^{-\alpha \tau_0} P + \sum_{T_i < \tau_0} e^{-\alpha T_i} K(X_{T_i-}, X_{T_i}) \right] \quad (3.5)$$

where $P \in \mathbb{R}_-$ is a constant penalty¹ at the ruin time and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, satisfying :

$$\mathbb{E}^x \left[\int_0^\infty e^{-\alpha s} |f(X_s)| ds \right] < \infty. \quad (3.6)$$

We also assume the standard polynomial growth condition on $f(\cdot)$ and $K(x, y) = K(z)$ by setting $z := x - y$: That is, there exist constants C_i and m_i ($i = 1, 2$) such that, for all $x, z \in \mathbb{R}$,

$$|f(x)| \leq C_1(1 + |x|^{m_1}) \quad \text{and} \quad |K(z)| \leq C_2(1 + |z|^{m_2}). \quad (3.7)$$

Our goal is to find the optimal strategy $\nu^*(T_i, \xi_i)_{i \geq 0}$ and the corresponding value function,

$$v(x) \triangleq \sup_{\nu \in \mathcal{V}} J^\nu(x) = J^{\nu^*}(x). \quad (3.8)$$

Let us briefly go over our plan. In section 3.1 we introduce a recursive optimal stopping scheme that eventually solves the original impulse control problem as in Øksendal and Sulem (2002). In section 3.2, we consider a special case where the mapping $x \rightarrow \frac{K}{\varphi}(x) : \mathcal{I} \rightarrow \mathbb{R}$ is F -concave. We show, under this assumption, that the optimal intervention times T_i are characterized as exit times from an interval, say $(0, b^*)$, for every i . Then we characterize the value function for impulse control problems and present a solution method based on the characterization of the intervention times and value function. In section 3.4, we consider the general case where the F -concavity assumption above does not hold.

3.1 A sequence of optimal stopping problems

In this subsection, we consider a recursive optimal stopping with a view to characterizing intervention times for the impulse control problems. Here we assume that no absorbing boundary exists. As we will see in the

¹Equation (3.5) can be rewritten as $J^\nu(x) = P + \mathbb{E}^x \left[\int_0^{\tau_0} e^{-\alpha s} (f(X_s) - \alpha P) ds + \sum_{T_i < \tau_0} e^{-\alpha T_i} K(X_{T_i-}, X_{T_i}) \right]$. Hence we could define $\tilde{f}(x) \triangleq f(x) - \alpha P$ to get rid of P . We, however, maintain the original formulation to make the penalty term explicit.

next subsection, the existence of an absorbing state is easily incorporated. Hence by using the same $v(x)$, we consider the problem,

$$v(x) = \sup_{\nu \in \mathcal{V}} \mathbb{E}^x \left[\int_0^\infty e^{-\alpha s} f(X_s) ds + \sum_i e^{-\alpha T_i} K(X_{T_i-}, X_{T_i}) \right]. \quad (3.9)$$

Let us also define the set S_n and the objective function v_n as follows:

$$S_n \triangleq \{\nu \in S; \nu = (T_1, T_2, \dots, T_{n+1}; \xi_1, \xi_2, \dots, \xi_n); T_{n+1} = +\infty\},$$

and

$$v_n(x) \triangleq \sup_{\nu \in S_n} \mathbb{E}^x \left[\int_0^\infty e^{-\alpha s} f(X_s) ds + \sum_{T_i} e^{-\alpha T_i} K(X_{T_i-}, X_{T_i}) \right]. \quad (3.10)$$

In other words, we are allowed to make at most n interventions. For this recursive approach, see, for example, Davis (1992) and Øksendal and Sulem (2002). We use the following notation for convenience:

$$g(x) \triangleq \mathbb{E}^x \left[\int_0^\infty e^{-\alpha s} f(X_s^0) ds \right]. \quad (3.11)$$

Let \mathcal{H} denote the space of all Borel functions. Define the two operators $\mathcal{M} : \mathcal{H} \rightarrow \mathcal{H}$ and $\mathcal{L} : \mathcal{H} \rightarrow \mathcal{H}$ as follows:

$$\mathcal{M}u(x) \triangleq \sup_{y \in \mathbb{R}} [K(x, y) - (g(x) - g(y)) + u(y)], \quad (3.12)$$

and

$$\mathcal{L}u(x) \triangleq \sup_{\tau \in \mathcal{S}} \mathbb{E}^x [e^{-\alpha \tau} \mathcal{M}u(X_{\tau-})], \quad (3.13)$$

for $u \in \mathcal{H}$. From the definition of the two operators, $a_1(x) \leq a_2(x)$ for $x \in \mathbb{R}$, $a_1(\cdot), a_2(\cdot) \in \mathcal{H}$ implies $\mathcal{M}a_1(x) \leq \mathcal{M}a_2(x)$ and $\mathcal{L}a_1(x) \leq \mathcal{L}a_2(x)$ for all $x \in \mathbb{R}$.

On the other hand, consider the following recursive formula:

$$\begin{cases} w_0(x) &= g(x) \\ w_n(x) &= \sup_{\tau \in \mathcal{S}, \xi \in \mathbb{R}_+} \mathbb{E}^x \left[\int_0^\tau e^{-\alpha s} f(X_s) ds + e^{-\alpha \tau} (K(X_{\tau-}, X_\tau) + w_{n-1}(X_\tau)) \right] \end{cases} \quad (3.14)$$

for $n = 1, 2, 3, \dots$. It should be noted that, for each n , this is an optimal stopping problem over τ . The second line of (3.14) is equivalent to

$$w_{n+1}(x) - g(x) = \sup_{\tau \in \mathcal{S}, \xi \in \mathbb{R}_+} \mathbb{E}^x [e^{-\alpha \tau} (K(X_{\tau-}, X_\tau) - g(X_{\tau-}) + w_n(X_\tau))] \quad (3.15)$$

by applying the strong Markov property (with (3.6)) at time τ to the integral term. Note that on $\{\omega : 0 \leq t \leq \tau(\omega) - \}$, we have $X(\omega) = X^0(\omega)$ almost surely. In fact, this derivation is explained in detail in subsection 3.2. By defining

$$\phi \triangleq w - g,$$

and adding and subtracting $g(X_\tau)$ on the right hand side of (3.15), it becomes

$$\phi_{n+1}(x) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^x [e^{-\alpha \tau} \mathcal{M}\phi_n(X_{\tau-})].$$

Then it can be further simplified, by using the operator \mathcal{L} defined in (3.13),

$$\phi_{n+1}(x) = \mathcal{L}\phi_n(x). \quad (3.16)$$

Let us start this recursive scheme with $w_0(x) = g(x)$ (i.e., no interventions are allowed, equivalently $\phi_0(x) = 0$) and define recursively $\phi_n(x) \triangleq w_n(x) - g(x) = \mathcal{L}(w_{n-1}(x) - g(x)) = \mathcal{L}\phi_{n-1}$. Clearly,

$$\begin{aligned}\phi_1(x) &= \mathcal{L}\phi_0(x) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^x [e^{-\alpha\tau} (\mathcal{M}(w_0(X_{\tau-}) - g(X_{\tau-})))] \\ &= \sup_{\tau \in \mathcal{S}, \xi \in \mathbb{R}_+} \mathbb{E}^x [e^{-\alpha\tau} \{K(X_{\tau-}, X_\tau) - g(X_{\tau-}) + g(X_\tau)\}].\end{aligned}$$

On the other hand, by looking at the first (and the sole) intervention time,

$$\begin{aligned}v_1(x) - g(x) &= \sup_{\nu \in \mathcal{S}_1} \mathbb{E}^x \left[\int_0^\infty e^{-\alpha s} f(X_s) ds + e^{-\alpha\nu} K(X_{\nu-}, X_\nu) \right] - \mathbb{E}^x \left[\int_0^\infty e^{-\alpha s} f(X_s^0) ds \right] \\ &= \sup_{\tau \in \mathcal{S}, \xi \in \mathbb{R}_+} \mathbb{E}^x \left[\int_0^\tau e^{-\alpha s} f(X_s) ds + e^{-\alpha\tau} \left(\mathbb{E}^{X_\tau} \left[\int_0^\infty e^{-\alpha s} f(X_s) ds \right] + K(X_{\tau-}, X_\tau) \right) \right. \\ &\quad \left. - \int_0^\tau e^{-\alpha s} f(X_s^0) ds - e^{-\alpha\tau} g(X_{\tau-}) \right] \\ &= \sup_{\tau \in \mathcal{S}, \xi \in \mathbb{R}_+} \mathbb{E}^x [e^{-\alpha\tau} \{K(X_{\tau-}, X_\tau) - g(X_{\tau-}) + g(X_\tau)\}].\end{aligned}$$

The last equation is due to the fact that only one intervention is allowed. Hence we have $w_1(x) = v_1(x)$. By the definition of the recursive scheme, w_n is an increasing sequence (i.e., $w_1(x) \leq w_2(x) \leq \dots$ for all $x \in \mathbb{R}$). In fact, we shall prove that $w_n = v_n$ for all n in Lemma 3.2. Before that, we need the following lemma to relate this recursive scheme with the method described in Section 2.

Lemma 3.1. *The mapping $x \rightarrow \frac{\mathcal{L}\phi(x)}{\varphi(x)} : \mathcal{I} \rightarrow \mathbb{R}$ is F -concave.*

Proof. We shall fix some $x \in (l, r) \subseteq \mathcal{I}$. Since $\mathcal{M}\phi(\cdot)$ is bounded there, for a given $\varepsilon > 0$, there are admissible ε -optimal intervention pairs $(\sigma_\varepsilon^l, \xi_\varepsilon^l)$ and $(\sigma_\varepsilon^r, \xi_\varepsilon^r)$ such that

$$\mathbb{E}^l [e^{-\alpha\sigma_\varepsilon^l} \mathcal{M}\phi(X_{\sigma_\varepsilon^l})] > \mathcal{L}\phi(l) - \varepsilon, \quad \text{and} \quad \mathbb{E}^r [e^{-\alpha\sigma_\varepsilon^r} \mathcal{M}\phi(X_{\sigma_\varepsilon^r})] > \mathcal{L}\phi(r) - \varepsilon.$$

Define another stopping time $\sigma_\varepsilon^{lr} \in \mathcal{S}$ with

$$\sigma_\varepsilon^{lr} \triangleq \begin{cases} \tau^l + \sigma_\varepsilon^l \circ \theta_{\tau^l}, & \text{if } \tau^l < \tau^r, \\ \tau^r + \sigma_\varepsilon^r \circ \theta_{\tau^r}, & \text{if } \tau^l > \tau^r. \end{cases}$$

Putting all together, with the strong Markov property of X , which is X^0 a.s. until the stopping time τ ,

$$\begin{aligned}\mathcal{L}\phi(x) &\geq \mathbb{E}^x [e^{-\alpha\sigma_\varepsilon^{lr}} \mathcal{M}\phi(X_{\sigma_\varepsilon^{lr}})] \\ &> (\mathcal{L}\phi(l) - \varepsilon) \mathbb{E}^x [e^{-\alpha\tau^l} \mathbf{1}_{\{\tau^l < \tau^r\}}] + (\mathcal{L}\phi(r) - \varepsilon) \mathbb{E}^x [e^{-\alpha\tau^r} \mathbf{1}_{\{\tau^l > \tau^r\}}] \\ &\geq \frac{\mathcal{L}\phi(l)}{\varphi(l)} \varphi(x) \frac{F(r) - F(x)}{F(r) - F(l)} + \frac{\mathcal{L}\phi(r)}{\varphi(r)} \varphi(x) \frac{F(x) - F(l)}{F(r) - F(l)} - \varepsilon.\end{aligned}$$

Since ε is arbitrary, we have an F -concavity. □

This lemma guarantees that we can use Proposition 2.1 to 2.3 to identify the value function and an optimal stopping rule for each of the recursive optimal stopping problems (3.14).

3.2 Characterization of the Intervention Times and the Value Function: F -Concave Reward Case

Based on the results in the previous subsection, we first consider a special case where the mapping $x \rightarrow \frac{\bar{K}}{\varphi}(x) : \mathcal{I} \rightarrow \mathbb{R}$ is F -concave. Let us define, for notational convenience,

$$\bar{K}(x, y) \triangleq K(x, y) - (g(x) - g(y)). \quad (3.17)$$

Further, we prove the following properties of the recursive optimization scheme.

Lemma 3.2. *If we define w_n by (3.14) (with $w_0 = g$) and v_n by (3.10), then*

$$w_n(x) = v_n(x) \quad \text{for each } n \quad \text{and} \quad v(x) = \lim_{n \rightarrow \infty} w_n(x).$$

Moreover, $w(x) \triangleq \lim_{n \rightarrow +\infty} w_n(x)$ is the smallest solution majorizing g of the functional equation $w - g = \mathcal{L}(w - g)$.

Proof. The proof is given in Appendix. □

Hence if we solve the optimal stopping problem

$$\phi_{n+1}(x) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^x [e^{-\alpha\tau} \mathcal{M}\phi_n(X_{\tau-})] \quad (3.18)$$

recursively for each n , then we obtain $\phi(x) \triangleq \lim_{n \rightarrow \infty} \phi_n(x) = \lim_{n \rightarrow \infty} v_n(x) - g(x) = v(x) - g(x)$. Summarizing the above argument, we have the following proposition:

Proposition 3.1. *The value function $v(x)$ for (3.9) is given by the smallest solution majorizing g of the functional equation $v - g = \mathcal{L}(v - g)$, and $\frac{v-g}{\varphi} (= \frac{\phi}{\varphi})$ is always F -concave.*

Proof. The first statement comes from Lemma 3.2. By the recursive method that we described above, we are solving a series of optimal stopping problems for each ϕ_n . Hence Lemma 3.1 and Proposition 2.1 give the second statement. □

Note that the functional equation $v - g = \mathcal{L}(v - g)$ is in essence the same as (2.8) in Alvarez (2004) where the optimal boundary is found by ordinary optimization techniques in a case where the size of the control is exogenously given. Similar ideas are also in Alvarez and Virtanen (2006) and Alvarez and Lempa (2004) that extend Alvarez (2004) by finding optimal size of control as well under some practically reasonable assumptions on the reward and cost function. These papers identify the function (along with optimal control) that satisfies this relationship, using the idea of α -excessive mappings and proving optimality with the verification of a weaker version of quasi-variational inequalities under the assumptions on the reward function. Now we consider different paths to reach the solution and develop a new method.

The argument in the previous subsection is modified to incorporate the existence of the ruin state. Instead of (3.10) and (3.14), we define, respectively,

$$\begin{aligned} v_n(x) &\triangleq \sup_{\nu \in \tilde{\mathcal{S}}_n} \mathbb{E}^x \left[\int_0^{\tau_0} e^{-\alpha s} f(X_s) ds + e^{-\alpha\tau_0} P + \sum_{T_i < \tau_0} e^{-\alpha T_i} K(X_{T_i-}, X_{T_i}) \right] \\ w_{n+1}(x) &\triangleq \sup_{\tau \in \mathcal{S}, \xi} \mathbb{E}^x \left[\int_0^{\tau_0 \wedge \tau} e^{-\alpha s} f(X_s) ds + e^{-\alpha\tau_0} P 1_{\{\tau_0 < \tau\}} \right. \\ &\quad \left. + e^{-\alpha\tau} \{K(X_{\tau-}, X_{\tau}) + w_n(X_{\tau})\} 1_{\{\tau < \tau_0\}} \right] \end{aligned}$$

with

$$w_0(x) = \mathbb{E}^x \left[\int_0^\infty e^{-\alpha s} f(X_s^0) 1_{\{s < \tau_0\}} ds + e^{-\alpha \tau_0} P \right] \triangleq g_0(x).$$

Then by defining the operator $\mathcal{L} : \mathcal{H} \rightarrow \mathcal{H}$ instead of (3.13),

$$\mathcal{L}u(x) \triangleq \sup_{\tau \in \mathcal{S}} \mathbb{E}^x [e^{-\alpha \tau} \mathcal{M}u(X_{\tau-}) 1_{\{\tau < \tau_0\}} + e^{-\alpha \tau_0} (P - g(0)) 1_{\{\tau_0 < \tau\}}], \quad (3.19)$$

we have the same recursion formula as in (3.16). We can obtain the same results as in Lemma 3.1 and Lemma 3.2. Proposition 3.1 also holds with one change that the value function is given by the smallest solution majorizing g_0 of the functional equation $v - g = \mathcal{L}(v - g)$ where $\mathcal{L} : \mathcal{H} \rightarrow \mathcal{H}$ is given by (3.19). Now we consider the characterization of the intervention times.

Proposition 3.2. *If the mapping $x \rightarrow \frac{\bar{K}}{\varphi}(x) : \mathbb{R}_+ \rightarrow \mathbb{R}$ is F -concave and 0 is an absorbing state, then the optimal intervention times T_i^* are given, for some $b^* \in \mathbb{R}_+$, by*

$$T_i^* = \inf\{t > T_{i-1}^*; X_t \notin (0, b^*), \quad i = 1, 2, \dots\}.$$

Proof. Our proof is constructive, describing the procedure of recursive optimization steps. For any $n \geq 1$, in view of Lemma 3.1, $\phi_n(x)/\varphi(x)$ is the smallest F -concave majorant of $\mathcal{L}\phi_{n-1}(x)/\varphi(x)$. We claim that this majorant (that passes $(F(0), \frac{P-g(0)}{\varphi(0)})$ in the transformed space) always exists. Indeed, since we consider the case of $\xi_i > 0$, i.e. $x > y$ for $K(x, y)$ and

$$\mathcal{M}\phi_0(x) = \sup_{y \in \mathbb{R}_+} [K(x, y) - (g(x) - g(y)) + \phi_0(y)] = \sup_{y \in \mathbb{R}_+} [K(x, y) - (g(x) - g(y)) + (g_0(y) - g(y))],$$

we should check whether the concave majorant exists, namely,

$$\lim_{x \downarrow 0} (K(x, y) - g(x) + g_0(y)) < P - g(0) \quad (3.20)$$

holds when $y \downarrow 0$. Note that $\lim_{y \downarrow 0} g_0(y) = P$ and $g(x) \rightarrow g(0)$ as $x \rightarrow 0$ due to the continuity of f . Hence (3.20) holds in the neighborhood of $y = 0$ because of (3.4). In the subsequent iterations, we consider

$$\mathcal{M}\phi_1(x) = \sup_{y \in \mathbb{R}_+} [K(x, y) - (g(x) - g(y)) + \phi_1(y)].$$

We should check if the expression inside the supremum operator becomes less than $P - g(0)$ as $x \downarrow 0$ and $y \downarrow 0$. Since $\lim_{y \downarrow 0} \phi_1(y) = \phi_1(0) = P - g(0)$ by the concavity (hence continuity) of ϕ_1 and since $\lim_{x \downarrow 0} g(x) = \lim_{y \downarrow 0} g(y)$, we have in the neighborhood of $y = 0$,

$$\lim_{x \downarrow 0} K(x, 0) + P - g(0) < P - g(0)$$

holds. Hence the concave majorant always exist also in the subsequent iterations.

Now the F -concavity of ϕ_n is obviously maintained for all n . The limit function of the increasing sequence of functions, $\phi(x) = \lim_{n \rightarrow \infty} \phi_n(x)$ exists and is also F -concave. Accordingly, $\bar{K}(x, y)/\varphi(x) + \phi(y)/\varphi(x)$ is F -concave for all y . Hence $\phi(x)/\varphi(x)$ and $(\bar{K}(x, y) + \phi(y))/\varphi(x)$ meet once and only once in the transformed space. Recall that the value function satisfies $\phi = \mathcal{L}\phi$. This implies that the continuous region is in the form of $(0, b^*)$ for some $b^* \in \mathbb{R}_+$, which completes the proof. \square

By using the above characterization of intervention times, we next want to characterize the value function and reduce the impulse control problem (3.8) to some optimal stopping problem. Moreover, we shall present a method that does not have to go through the iteration scheme. Let us first simplify J^ν :

$$J^\nu(x) = \mathbb{E}^x \left[\int_0^{\tau_0} e^{-\alpha s} f(X_s) ds + e^{-\alpha \tau_0} P + \sum_{T_i < \tau_0} e^{-\alpha T_i} K(X_{T_i-}, X_{T_i}) \right]. \quad (3.21)$$

This is just a reproduction of (3.5). Let us split the right hand side of (3.21) into pieces and use the strong Markov property of the uncontrolled process X^0 at the first intervention time T_1 (together with the shift operator $\theta(\cdot)$) to each of them. Note that on $\{\omega : 0 \leq t \leq T_1(\omega)-\}$, we have $X(\omega) = X^0(\omega)$ almost surely. The first term becomes

$$\begin{aligned} \mathbb{E}^x \left[\int_0^{\tau_0} e^{-\alpha s} f(X_s) ds \right] &= \mathbb{E}^x \left[\mathbf{1}_{\{T_1 < \tau_0\}} \left\{ \int_0^{T_1} e^{-\alpha s} f(X_s) ds + e^{-\alpha T_1} \mathbb{E}^{X_{T_1}} \int_0^\infty e^{-\alpha s} f(X_s) \mathbf{1}_{\{s < \tau_0\}} ds \right\} \right] \\ &\quad + \mathbb{E}^x \left[\mathbf{1}_{\{T_1 > \tau_0\}} \int_0^{\tau_0} f(X_s) ds \right] \end{aligned}$$

since $\int_{T_1-}^{T_1} e^{-\alpha s} f(X_s) ds = 0$. The second and third terms become

$$\begin{aligned} \mathbb{E}^x [e^{-\alpha \tau_0} P] &= \mathbb{E}^x \left[\mathbf{1}_{\{T_1 < \tau_0\}} e^{-\alpha T_1} \mathbb{E}^x [e^{-\alpha(\tau_0 - T_1)} P | \mathcal{F}_{T_1}] \right] + \mathbb{E}^x [\mathbf{1}_{\{T_1 > \tau_0\}} e^{-\alpha \tau_0} P] \\ &= \mathbb{E}^x \left[\mathbf{1}_{\{T_1 < \tau_0\}} e^{-\alpha T_1} \mathbb{E}^x [e^{-\alpha(\tau_0 \circ \theta(T_1))} P | \mathcal{F}_{T_1}] \right] + \mathbb{E}^x [\mathbf{1}_{\{T_1 > \tau_0\}} e^{-\alpha \tau_0} P] \\ &= \mathbb{E}^x \left[\mathbf{1}_{\{T_1 < \tau_0\}} e^{-\alpha T_1} \mathbb{E}^{X_{T_1}} (e^{-\alpha \tau_0} P) \right] + \mathbb{E}^x [\mathbf{1}_{\{T_1 > \tau_0\}} e^{-\alpha \tau_0} P] \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}^x \left[\sum_{T_i < \tau_0} e^{-\alpha T_i} K(X_{T_i-}, X_{T_i}) \right] \\ &= \mathbb{E}^x \left[\mathbf{1}_{\{T_1 < \tau_0\}} \left\{ e^{-\alpha T_1} K(X_{T_1-}, X_{T_1}) + e^{-\alpha T_1} \sum_{i=2} e^{-\alpha(T_i - T_1)} K(X_{T_i-}, X_{T_i}) \mathbf{1}_{\{T_i < \tau_0\}} \right\} \right] \\ &= \mathbb{E}^x \left[\mathbf{1}_{\{T_1 < \tau_0\}} \left\{ e^{-\alpha T_1} K(X_{T_1-}, X_{T_1}) + e^{-\alpha T_1} \mathbb{E}^x \left[\sum_{T_i < \tau_0} e^{-\alpha(T_i \circ \theta(T_1))} K(X_{S_i-}, X_{S_i}) \mid \mathcal{F}_{T_1} \right] \right\} \right] \\ &= \mathbb{E}^x \left[\mathbf{1}_{\{T_1 < \tau_0\}} e^{-\alpha T_1} \left\{ K(X_{T_1-}, X_{T_1}) + \mathbb{E}^{X_{T_1}} \sum_{i=1} e^{-\alpha T_i} K(X_{T_i-}, X_{T_i}) \mathbf{1}_{\{T_i < \tau_0\}} \right\} \right], \end{aligned}$$

where $S_i \triangleq T_1 + T_i \circ \theta(T_1)$ and the index i runs from 1 for the sum in the second equality. Combining the three terms and rearranging, we have

$$\begin{aligned} J^\nu(x) &= \mathbb{E}^x \left[\mathbf{1}_{\{T_1 < \tau_0\}} \left\{ \int_0^{T_1} e^{-\alpha s} f(X_s) ds + e^{-\alpha T_1} K(X_{T_1-}, X_{T_1}) + e^{-\alpha T_1} J^\nu(X_{T_1}) \right\} \right] \\ &\quad + \mathbb{E}^x \left[\mathbf{1}_{\{T_1 > \tau_0\}} \left\{ \int_0^{\tau_0} e^{-\alpha s} f(X_s) ds + e^{-\alpha \tau_0} P \right\} \right]. \quad (3.22) \end{aligned}$$

For any \mathbb{F} stopping time τ , the strong Markov property of X^0 , with our assumption (3.6), gives us

$$\mathbb{E}^x \left[\int_0^\tau e^{-\alpha s} f(X_s^0) ds \right] = g(x) - \mathbb{E}^x [e^{-\alpha \tau} g(X_\tau^0)]$$

where $g(\cdot)$ is defined as in (3.11). We apply this result to (3.22) by reading $\tau = T_1$ and $\tau = \tau_0$ to derive

$$J^\nu(x) = \mathbb{E}^x [1_{\{T_1 < \tau_0\}} e^{-\alpha T_1} \{K(X_{T_1-}, X_{T_1}) - g(X_{T_1}^0) + J^\nu(X_{T_1})\}] \\ + \mathbb{E}^x [1_{\{T_1 > \tau_0\}} e^{-\alpha \tau_0} \{P - g(X_{\tau_0})\}] + g(x). \quad (3.23)$$

Noting that $g(X_{T_1}^0) = g(X_{T_1-})$, adding and subtracting $g(X_{T_1})$ and further defining

$$u(x) \triangleq J^\nu(x) - g(x),$$

(3.23) finally becomes

$$u(x) = \mathbb{E}^x [1_{\{T_1 < \tau_0\}} e^{-\alpha T_1} \{K(X_{T_1-}, X_{T_1}) + u(X_{T_1}) - g(X_{T_1-}) + g(X_{T_1})\}] \\ + \mathbb{E}^x [1_{\{T_1 > \tau_0\}} e^{-\alpha \tau_0} \{P - g(X_{\tau_0})\}], \quad (3.24)$$

and we consider the maximization of this $u(\cdot)$ function and add back $g(x)$ since $\sup u(x) = \sup \{J^\nu(x) - g(x)\} = \sup J^\nu(x) - g(x)$. Note that this simplification leading to (3.24) does not depend on the F -concavity assumption.

Since we have confirmed that optimal intervention times are exit times of the process from an interval, let us use a simpler notation: $X_{T_i-} = b$ and $X_{T_i} = a$ for all $i = 0, 1, 2, \dots$. We can denote $T_i- = \tau_b \triangleq \inf\{t > T_{i-1}; X_t \geq b\}$. By observing (3.24),

$$u(b) = u(X_{T_1-}) = K(X_{T_1-}, X_{T_1}) + g(X_{T_1}) - g(X_{T_1-}) + u(X_{T_1}) \\ = K(b, a) + g(a) - g(b) + u(a) = \bar{K}(b, a) + u(a) \quad (3.25) \\ u(0) = u(X_{\tau_0}) = P - g(X_{\tau_0}) = P - g(0)$$

we have

$$u(x) = \begin{cases} u_0(x), & x \in [0, b) \\ \bar{K}(x, a) + u_0(a), & x \in [b, \infty). \end{cases} \quad (3.26)$$

where

$$u_0(x) \triangleq \mathbb{E}^x [1_{\{\tau_b < \tau_0\}} e^{-\alpha \tau_b} u(b)] + \mathbb{E}^x [1_{\{\tau_b > \tau_0\}} e^{-\alpha \tau_0} u(0)].$$

The second equation of (3.26) is obtained from (3.24) by noticing that, on $x \in [b, \infty)$, $\mathbb{P}^x(T_1 < \tau_0) = 1$. Indeed, in this case, we immediately jump to a , so that $X_{T_1-} = x$ and $X_{T_1} = a$. Since $a \in (0, b)$, $u(a) = u_0(a)$. Now let us note that we have the following representations in (3.24)

$$\mathbb{E}^x [e^{-\alpha \tau_r} 1_{\{\tau_r < \tau_l\}}] = \frac{\psi(l)\varphi(x) - \psi(x)\varphi(l)}{\psi(l)\varphi(r) - \psi(r)\varphi(l)}, \quad x \in [l, r]$$

where $\tau_l \triangleq \inf\{t > 0; X_t = l\}$ and $\tau_r \triangleq \inf\{t > 0; X_t = r\}$ and $\varphi(\cdot)$ and $\psi(\cdot)$ defined in the previous section. Finally, with $F(\cdot)$ being defined as in (2.3), we have a characterization of $u(x)$ in the continuation region,

$$\frac{u(x)}{\varphi(x)} = \frac{u(b)(F(x) - F(0))}{\varphi(b)(F(b) - F(0))} + \frac{u(0)(F(b) - F(x))}{\varphi(0)(F(b) - F(0))}, \quad x \in [0, b]. \quad (3.27)$$

Define the transformation

$$W \triangleq \frac{u}{\varphi} \circ F^{-1} \quad (3.28)$$

(3.27) becomes, for any $a > 0$ and $b > 0$,

$$W(F(x)) = W(F(b)) \frac{F(x) - F(0)}{F(b) - F(0)} + W(F(0)) \frac{F(b) - F(x)}{F(b) - F(0)}, \quad x \in [0, b], \quad (3.29)$$

which represents a *linear function* that passes a fixed point, $(F(0), W(F(0)))$.

To discuss how to find the optimal pair (a^*, b^*) , we write $u(x)$ as $u_{a,b}(x)$ to emphasize the dependence on a, b , then on $x \in [0, b]$,

$$\sup_{a \in \mathbb{R}_+, b \in \mathbb{R}_+} u_{a,b}(x) = \sup_{a \in \mathbb{R}_+} \sup_{b \in \mathbb{R}_+} \{ \mathbb{E}^x [1_{\{\tau_b < \tau_0\}} e^{-\alpha \tau_b} (\bar{K}(b, a) + u_{a,b}(a))] + \mathbb{E}^x [1_{\{\tau_b > \tau_0\}} e^{-\alpha \tau_0} u_{a,b}(0)] \}. \quad (3.30)$$

This can be considered as a two-stage optimization problem. First, let a be fixed. For each a , the inner maximization of (3.30) becomes

$$V_a(x) \triangleq \sup_{\tau_b \in \mathcal{S}} \{ \mathbb{E}^x [1_{\{\tau_b < \tau_0\}} e^{-\alpha \tau_b} (\bar{K}(b, a) + V_a(a))] + \mathbb{E}^x [1_{\{\tau_b > \tau_0\}} e^{-\alpha \tau_0} (P - g(0))] \} \quad (3.31)$$

and, among a 's, choose an optimal a in the sense, $\tilde{v}(x) \triangleq \sup_a V_a(x)$ for any x . It should be pointed out that $V_a(x)$ may take negative values if $P - g(0)$ does. Now, we discuss a solution method of the first stage optimization (3.31). For this purpose, we need a technical lemma:

Lemma 3.3. *If we define*

$$G(x, \gamma) \triangleq \sup_{\tau \in \mathcal{S}} \mathbb{E}^x [e^{-\alpha \tau} (h(X_\tau^0) + \gamma)], \quad x \in \mathbb{R}, \gamma \in \mathbb{R}$$

for some Borel function $h : \mathbb{R} \rightarrow \mathbb{R}$ and with condition (3.6), then, for $\gamma_1 > \gamma_2 \geq 0$,

$$G(x, \gamma_1) - G(x, \gamma_2) \leq \gamma_1 - \gamma_2, \quad (3.32)$$

for any x .

Proof. The left hand side of (3.32) is well-defined due to (3.6). It is clear that $G(x, \gamma)$ is convex in γ for any x . Then $D_\gamma^+ G(x, \gamma_0) \triangleq \lim_{\gamma_1 \downarrow \gamma_0} \frac{G(x, \gamma_1) - G(x, \gamma_0)}{\gamma_1 - \gamma_0}$ exists at every $\gamma_0 \in \mathbb{R}$, and

$$\frac{G(x, \gamma_1) - G(x, \gamma_2)}{\gamma_1 - \gamma_2} \leq D_\gamma^+ G(x, \gamma_1). \quad (3.33)$$

Consider the bound of $G(x, \gamma)$ for x fixed:

$$G(x, \gamma) \leq \sup_{\tau \in \mathcal{S}} \mathbb{E}^x [e^{-\alpha \tau} |h(X_\tau^0)|] + |\gamma| \sup_{\tau \in \mathcal{S}} \mathbb{E}^x [e^{-\alpha \tau}].$$

The first term on the right hand side is constant in γ and the second term is linear in γ and the $\mathbb{E}^x [e^{-\alpha \tau}] \leq 1$ for any $\tau \in \mathcal{S}$. Due to the convexity of $G(x, \gamma)$ in γ , for the above inequality to hold, $D_\gamma^+ G(x, \gamma) \leq 1$ for all $\gamma \in \mathbb{R}$. On account of (3.33), we have (3.32). \square

Coming back to (3.31), we need some care because the value function $V_a(x)$ contains its value at a , i.e., $V_a(a)$ in the definitive equation. Let us consider a family of optimal stopping problem parameterized by $\gamma \in \mathbb{R}$.

$$\begin{aligned} V_a^\gamma(x) &\triangleq \sup_{\tau \in \mathcal{S}} \{ \mathbb{E}^x [1_{\{\tau < \tau_0\}} e^{-\alpha \tau} (\bar{K}(X_\tau, a) + \gamma)] + \mathbb{E}^x [1_{\{\tau > \tau_0\}} e^{-\alpha \tau_0} (P - g(0))] \} \\ &= \sup_{\tau \in \mathcal{S}} \mathbb{E}^x [e^{-\alpha \tau} r^\gamma(X_\tau, a)] \end{aligned} \quad (3.34)$$

where

$$r^\gamma(x, a) = \begin{cases} P - g(0), & x = 0, \\ \bar{K}(x, a) + \gamma, & x > 0. \end{cases}$$

Obviously, this parameterized problem can be solved by using Proposition 2.1 to 2.3. Now we link this parameterized optimal stopping problem to (3.31).

Lemma 3.4. *For $a > 0$ given, if there exists a solution to (3.34), then there always exists unique γ such that $\gamma = V_a^\gamma(a)$ holds, provided that (3.4) holds.*

Proof. Without loss of generality, we need only to consider the case where

$$\sup_{x \in \mathbb{R}_+} \bar{K}(x, a) > 0 \quad (3.35)$$

for some $a > 0$. Indeed, suppose that there is no such a and let us consider a sequence of optimal stopping scheme. In each iteration, the value function for the optimal stopping problem takes negative values, so that $\phi_n(\cdot) < 0$ for all n . Then in the next iteration, $\bar{K}(x, y)$ function will be shifted downwards, leading to $\phi_{n+1}(\cdot) < 0$. Hence the “no interventions” strategy is trivially optimal.

In (3.34), since γ is some constant parameter, we benefit from Proposition 2.1 and claim that $V_a^\gamma(x)$ is characterized as the smallest F -concave majorant of $r^\gamma(\cdot, a)$ that passes $\left(F(0), \frac{P-g(0)}{\varphi(0)}\right)$. In terms of the notation of Proposition 2.3, if we define $W_a^\gamma(\cdot)$ such that

$$V_a^\gamma(x) = \varphi(x)W_a^\gamma(F(x)),$$

then $W_a^\gamma(\cdot)$ passes through the fixed point $A = (F(0), W_a^\gamma(F(0)))$ and is the smallest concave majorant of $H^\gamma(\cdot, a) \triangleq \frac{r^\gamma(F^{-1}(\cdot), a)}{\varphi(F^{-1}(\cdot))} = \frac{\bar{K}(F^{-1}(\cdot), a)}{\varphi(F^{-1}(\cdot))} + \frac{\gamma}{\varphi(F^{-1}(\cdot))}$.

Now fix a . Our approach here is by starting with $\gamma = 0$, we move γ upwards and evaluate $V_a^\gamma(a)$ and try to find γ such that $\gamma = V_a^\gamma(a)$. Due to (3.35), we have $W_a^0(F(a)) > 0$. By the monotonicity of F , it is equivalent to saying that $V_a^0(a) > 0 = \gamma$. As γ increases, $W_a^\gamma(F(a))$ increases monotonically: see the right hand side of (3.34).

Lemma 3.3 implies that for $\gamma_1 > \gamma_2 \geq 0$,

$$V_a^{\gamma_1}(x) - V_a^{\gamma_2}(x) \leq \gamma_1 - \gamma_2 \quad (3.36)$$

for any $x \in \mathbb{R}_+$. Note that $W_a^\gamma(F(a)) \geq H^\gamma(F(a), a)$. However, since V_a^γ has less than the linear growth in γ as demonstrated by (3.36), there is a certain γ' large enough such that $W_a^\gamma(F(a)) = H^\gamma(F(a), a)$ for $\gamma \geq \gamma'$. This implies

$$\begin{aligned} \varphi(a)W_a^{\gamma'}(F(a)) &= \varphi(a)H^{\gamma'}(F(a), a) \\ \Leftrightarrow V_a^{\gamma'}(a) &= \bar{K}(a, a) + \gamma' < \gamma' \end{aligned}$$

where the inequality is due to the assumption (3.4). For this γ' , we have $V_a^{\gamma'}(a) < \gamma'$.

The monotonicity and continuity of $W^\gamma(F(a))$ (due to the convexity of $V_a^\gamma(\cdot)$) with respect to γ , together with (3.36), implies that, for any a , there exists one and only one γ such that $V_a^\gamma(a) = \gamma$. \square

3.3 Methodology to find $v(x)$ and (a^*, b^*)

Using (3.27), namely the characterization of $u_{a,b}$, we describe an optimization procedure based on Propositions 2.2 and 2.3.

1. Fix $a > 0$. Consider the function

$$R(\cdot, a) \triangleq \frac{\bar{K}(F^{-1}(\cdot), a)}{\varphi(F^{-1}(\cdot))} \quad (3.37)$$

Define $W_a(\cdot)$ such that $V_a(x) = \varphi(x)W_a(F(x))$ and by the linear characterization (3.29), it is a straight line with a slope, say $\beta(a)$ and passes through $(F(0), W_a(F(0))) = \left(F(0), \frac{P-g(0)}{\varphi(0)}\right)$. We can write the linear majorant, in general,

$$W_a(y) = \beta(a)y + \delta. \quad (3.38)$$

2. First stage optimization: For each slope $\beta(a)$, we can calculate the value of $W_a(F(a))$, but we have to find the $W_a(\cdot)$ function such that, at some point $F(b(a))$, we have

$$W_a(F(b)) = R(F(b), a) + W_a(F(a)) \frac{\varphi(a)}{\varphi(b)}. \quad (3.39)$$

where we write $b(a) \equiv b$ for notational simplicity. This requirement is equivalent to finding γ in (3.34) in Lemma 3.4 such that

$$\frac{\gamma}{\varphi(a)} = W_a^\gamma(F(a)).$$

Let us denote the right hand side of (3.39) by

$$\Phi(y, a) \triangleq R(y, a) + W_a(F(a)) \frac{\varphi(a)}{\varphi(F^{-1}(y))}, \quad y \in (F(c), F(d)). \quad (3.40)$$

By Proposition 3.2, $(0, b(a))$ is the continuation region. If R is a differentiable function with respect to the first argument, we can find the optimal point $b(a)$ analytically. In fact, it is to find a point $b(a)$ such that the linear majorant and the shifted function $\Phi(y, a)$ have a tangency point. This is equivalent to calculating the smallest linear majorant of $\Phi(y, a)$ (due to Proposition 3.1 and the linear characterization (3.29)). Explicitly, we solve

$$\left(\frac{\bar{K}(b, a)}{\varphi(b)}\right)' - \frac{\varphi'(b)\varphi(a)}{\varphi(b)^2}\delta = \beta(a) \left(F'(b) + \frac{\varphi'(b)\varphi(a)}{\varphi(b)^2}F(a)\right) \quad (3.41)$$

for $b(a)$ where $\beta(a)$ is

$$\beta(a) = \frac{\varphi(b)R(F(b), a) - \delta(\varphi(b) - \varphi(a))}{F(b)\varphi(b) - F(a)\varphi(a)}. \quad (3.42)$$

For the absorbing boundary case, these equations can be easily modified. Let us denote $D \triangleq W_a(F(0)) = (P - g(0))/\varphi(0)$. Then (3.41) and (3.42) become, by substituting

$$\delta = D - \beta(a)F(0) \quad (3.43)$$

in (3.41),

$$\left(\frac{\bar{K}(b, a)}{\varphi(b)}\right)' - \frac{\varphi'(b)\varphi(a)}{\varphi(b)^2}D = \beta(a) \left(F'(b) + \frac{\varphi'(b)\varphi(a)}{\varphi(b)^2}(F(a) - F(0))\right) \quad (3.44)$$

and

$$\beta(a) = \frac{\varphi(b)R(F(b), a) - D(\varphi(b) - \varphi(a))}{(F(b) - F(0))\varphi(b) - (F(a) - F(0))\varphi(a)}, \quad (3.45)$$

respectively. Note that $\varphi(b)R(F(b), a) = \bar{K}(b, a)$ from (3.37).

3. Second stage optimization: To summarize up to this point, we set a and find $b(a)$ and in turn $\beta(a)$. Now, let a vary and choose, among $\beta(a)$, find the largest slope $\beta^* \triangleq \max_{a \in \mathbb{R}_+} \beta(a)$, if exists, and also the corresponding a^* and $b(a^*)$. Due to the characterization of the value function with (3.30), these a^* and $b^* \triangleq b(a^*)$ must be the solution to (3.8).

If $\bar{K}(x, y)$ is a differentiable function with respect to both the first and second arguments, then we can find a^* analytically. If, for any $a \in \mathbb{R}_+$, $\Phi(y, a)$ is strictly concave in y at $F(b)$, a^* must satisfy

$$\left(\frac{\partial \bar{K}(b, a)}{\partial a} + \delta \varphi'(a) \right) (\psi(b) - \psi(a)) = - (\bar{K}(b, a) - \delta(\varphi(b) - \varphi(a))) (F'(a)\varphi(a) + F(a)\varphi'(a)). \quad (3.46)$$

Therefore, in this case, our non-linear optimization procedure is just to solve (3.41) and (3.46) with (3.42), simultaneously. We postpone the derivation of (3.46) to Appendix.

In particular, for our case of absorbing boundary, the corresponding condition is

$$\begin{aligned} & \left(\frac{\partial \bar{K}(b, a)}{\partial a} + D\varphi'(a) \right) \left((F(b) - F(a))\varphi(b) - (F(a) - F(0))\varphi(a) \right) \\ & = - (\bar{K}(b, a) - D(\varphi(b) - \varphi(a))) \left(F'(a)\varphi(a) + (F(a) - F(0))\varphi'(a) \right) \end{aligned}$$

to be solved simultaneously with (3.44) and (3.45). For the case of natural boundary, δ is obtained in Section 3.5.

Remark 3.1. With respect to the third point of the proposed method above, we should check if there exists a concave majorant as $a \downarrow 0$. Namely, we consider whether

$$\lim_{x \downarrow 0} (K(x, a) - (g(x) - g(a)) + u(a)) < P - g(0)$$

holds in the neighborhood of $a = 0$. Since $\lim_{x \downarrow 0} g(x) = \lim_{a \downarrow 0} g(a)$ and $\lim_{a \rightarrow 0} u(a) = u(0) = P - g(0)$ by the continuity of u , the last inequality holds due to (3.4). \diamond

3.4 Characterization of the Intervention Times and the Value Function: General Case

Let us move on to a general case where the mapping $x \rightarrow \frac{\bar{K}}{\varphi}(x) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is not necessarily F -concave. First, we extend Proposition 3.2 to characterize optimal intervention times.

Proposition 3.3. *The value function $v(x)$ for (3.8) is given by the smallest solution majorizing g of $v - g = \mathcal{L}(v - g)$ and optimal intervention times T_i^* are given by exit times from an interval if and only if, for all $y \in \mathbb{R}_+$,*

$$x \rightarrow \bar{K}(x, y) \text{ is continuous and } q \triangleq \limsup_{x \rightarrow \infty} D^- \left(\frac{\bar{K}}{\varphi} \circ F^{-1} \right) (x) \text{ is finite.} \quad (3.47)$$

where $D^- f(x_0) \triangleq \limsup_{x \uparrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$.

Proof. For any given $a \in \mathbb{R}_+$, if we can find the smallest linear majorant of $\frac{\bar{K}(F^{-1}(\cdot), a) + \gamma}{\varphi(F^{-1}(\cdot))}$ for an arbitrary $\gamma \in \mathbb{R}_+$, we can find $\gamma = \varphi(a)W_a(F(a))$ by Lemma 3.4. Due to the constancy of γ , it suffices to show that condition (3.47) is necessary and sufficient for the existence of concave majorant of $\frac{\bar{K}}{\varphi} \circ F^{-1}$ on $F(\mathcal{I})$. The sufficiency is immediate. For the necessity, we assume that $q = +\infty$. We can take a sequence of points $\{x^k\} \subset \mathbb{R}$ such that $x^k \rightarrow \infty$ and $D^- \left(\frac{\bar{K}}{\varphi} \circ F^{-1} \right) (x^k) \rightarrow \infty$ as $k \rightarrow \infty$. If necessary, by

taking a subsequence, we can make this sequence $\{x^k\}$ monotone. Consider the smallest concave majorant of $\frac{\bar{K}}{\varphi} \circ F^{-1}$ on $[F(0), F(x^k)]$. Call it $v^k(x)$. It is clear that $v^k(x)$ is monotone increasing in k for all $x \in [F(0), F(x^k)]$. As $k \rightarrow \infty$, $x^k \rightarrow \infty$ and $v(x) \geq v^k(x)$. We thus have $v(x) = \lim_{k \rightarrow \infty} v^k(x) = \infty$ for all $x \in \mathbb{R}_+$. There is no optimal intervention policy. \square

Suppose that the F -concavity of the reward function is violated, so that the intervention point may be multiple. For the rest of this subsection, as is practically the case (see our examples in Section 4), we study the case where $\Phi(y, a)$ with $y = F(x)$ is increasing to infinity and becomes eventually concave and then derive linear characterization as in Section 3.2.

Let us consider a strategy that we have two intervention points, b_1 and b_2 being arbitrarily chosen such that $0 < b_1 < b_2$. We want to characterize function $J^\nu(x)$ as in (3.5) again. Recall that there are no controls in a way that the process is pulled up to avoid ruin. In other words, $\mathbb{P}^x[\tau_0 < \infty] = 1$. Assume, for the moment, that we always apply control at these boundaries b_1 and b_2 and then, once applied, the process moves to $a_1 < b_1$ and $a_2 < b_2$, respectively.

If we start with a point $x \in [0, b_1]$, the problem is equivalent to the case we considered already, since the process cannot go beyond the level b_1 . Hence following (3.26), we have for $x \in [0, b_1]$

$$J_1^\nu(x) \triangleq \mathbb{E}^x \left[\int_0^{\tau_0} e^{-\alpha s} f(X_s) ds + e^{-\alpha \tau_0} P + \sum_{T_i < \tau_0} e^{-\alpha T_i} K(X_{T_i-}, X_{T_i}) \right]$$

and

$$u_1(x) = \mathbb{E}^x [1_{\{\tau_{b_1} < \tau_0\}} e^{-\alpha \tau_{b_1}} u_1(b_1)] + \mathbb{E}^x [1_{\{\tau_{b_1} > \tau_0\}} e^{-\alpha \tau_0} u_1(0)], \quad x \in [0, b_1]$$

by defining $u_1(x) \triangleq J_1^\nu(x) - g(x)$. If we start with a point $x \in [b_1, b_2]$, there are two strategies available:

- (A) Let X_t move along. (It either hits b_1 or b_2 first.)
- (B) Apply the control immediately ($t = 0$) by moving the process from x to a_1 (the post-control point that corresponds to b_1) and let the process start at a_1 . (Recall that we do not let X enter into (b_1, ∞) after moving to a_1 .)

Consider strategy (A) first. Let us define

$$J_2^\nu(x) \triangleq \mathbb{E}^x \left[\int_0^{\tau_{b_1}} e^{-\alpha s} f(X_s) ds + \sum_{T_i < \tau_{b_1}} e^{-\alpha T_i} K(X_{T_i-}, X_{T_i}) \right] \quad x \in [b_1, b_2].$$

Using the strong Markov property at the first intervention, we can reduce J_2^ν to a simpler form. For any (a_1, b_1) and (a_2, b_2) , we have

$$\begin{aligned} J_2^\nu(x) &= \mathbb{E}^x [1_{\{\tau_{b_1} < \tau_{b_2}\}} e^{-\alpha \tau_{b_1}} K(X_{\tau_{b_1}-}, X_{\tau_{b_1}}) - g(X_{\tau_{b_1}}) + J_1^\nu(X_{\tau_{b_1}}) + g(X_{\tau_{b_1}}) - g(X_{\tau_{b_1}-}) \\ &\quad + \mathbb{E}^x [1_{\{\tau_{b_1} > \tau_{b_2}\}} e^{-\alpha \tau_{b_2}} K(X_{\tau_{b_2}-}, X_{\tau_{b_2}}) - g(X_{\tau_{b_2}}) + J_2^\nu(X_{\tau_{b_2}}) + g(X_{\tau_{b_2}}) - g(X_{\tau_{b_2}-})] + g(x). \end{aligned}$$

We shall use $u_1(x) = J_1^\nu(x) - g(x)$ in the first term. Now let us define $u_2(x) \triangleq J_2^\nu(x) - g(x)$. Then the last equation becomes

$$\begin{aligned} u_2(x) &= \mathbb{E}^x [1_{\{\tau_{b_1} < \tau_{b_2}\}} e^{-\alpha \tau_{b_1}} K(X_{\tau_{b_1}-}, X_{\tau_{b_1}}) + u_1(X_{\tau_{b_1}}) + g(X_{\tau_{b_1}}) - g(X_{\tau_{b_1}-}) \\ &\quad + \mathbb{E}^x [1_{\{\tau_{b_1} > \tau_{b_2}\}} e^{-\alpha \tau_{b_2}} K(X_{\tau_{b_2}-}, X_{\tau_{b_2}}) + u_2(X_{\tau_{b_2}}) + g(X_{\tau_{b_2}}) - g(X_{\tau_{b_2}-})] \\ &= \mathbb{E}^x [1_{\{\tau_{b_1} < \tau_{b_2}\}} e^{-\alpha \tau_{b_1}} (\bar{K}(b_1, a_1) + u_1(a_1))] + \mathbb{E}^x [1_{\{\tau_{b_1} > \tau_{b_2}\}} e^{-\alpha \tau_{b_2}} (\bar{K}(b_2, a_2) + u_2(a_2))] \quad (3.48) \end{aligned}$$

on $x \in [b_1, b_2]$. By identifying $\bar{K}(b_2, a_2) + u_2(a_2) = u_2(b_2)$ and $u_2(b_1) = \bar{K}(b_1, a_1) + u_1(a_1) = u_1(b_1)$ (the latter shows $u_1(x)$ and $u_2(x)$ are connected at $x = b_1$),

$$u_2(x) = \frac{\varphi(x)}{\varphi(b_1)} \frac{F(b_2) - F(x)}{F(b_2) - F(b_1)} u_2(b_1) + \frac{\varphi(x)}{\varphi(b_2)} \frac{F(x) - F(b_1)}{F(b_2) - F(b_1)} u_2(b_2), \quad x \in [b_1, b_2]. \quad (3.49)$$

To summarize this result, if we define $W_i(\cdot) \triangleq \frac{u_i}{\varphi} \circ F^{-1}(\cdot)$ for $i = 1, 2$ on $F(\mathcal{I})$, this is again a linear function for each i . Hence by defining

$$W_A(F(x)) \triangleq \begin{cases} W_1(F(x)) = W_1(F(0)) \frac{F(b_1) - F(x)}{F(b_1) - F(0)} + W_1(F(b_1)) \frac{F(x) - F(0)}{F(b_1) - F(0)}, & x \in [0, b_1] \\ W_2(F(x)) = W_2(F(b_1)) \frac{F(b_2) - F(x)}{F(b_2) - F(b_1)} + W_2(F(b_2)) \frac{F(x) - F(b_1)}{F(b_2) - F(b_1)}, & x \in [b_1, b_2], \end{cases}$$

we have a piecewise linear function on $F(\mathcal{I})$.

Next consider strategy (B), whose value function is

$$W_B(F(x)) \triangleq \begin{cases} W_1(F(x)), & 0 \leq x \leq b_1 \\ \bar{W}_1(F(x)) \triangleq \frac{\varphi(a_1)}{\varphi(x)} W_1(F(a_1)) + R(F(x), a_1), & b_1 < x. \end{cases} \quad (3.50)$$

Lemma 3.5. (A) is better than (B) only if

$$\beta_1 \triangleq \frac{W(F(b_1)) - W(F(0))}{F(b_1) - F(0)} < \frac{W(F(b_2)) - W(F(b_1))}{F(b_2) - F(b_1)} \triangleq \beta_2.$$

Proof. Since the value function of strategy (B) is (3.50), choosing (A) over (B) is equivalent to

$$\bar{W}_1(F(x)) < W_2(F(x)) \quad \text{on } x > b_1.$$

If $W_1(F(x))$ majorizes $\bar{W}_1(F(x))$ on $x \in [0, \infty)$, then this problem reduces to F -concavity case discussed in the previous subsection. Hence we consider the case where there exists some $x \in [b_1, \infty)$ such that

$$W_1(F(x)) < \bar{W}_1(F(x)).$$

Now suppose that we have $\beta_1 \geq \beta_2$. Then it is clear that we cannot have $W_2(F(x)) > \bar{W}_1(F(x))$ on $x \in [b_1, \infty)$. \square

There are two cases to consider:

- (1) If $W_2(F(x))$ majorizes $\bar{W}_1(F(x))$ on $x \in [b_1, \infty)$, then we adopt the point b_2 as an intervention point. In this case, $\beta_2 > \beta_1$ holds. However, this implies that if we connect $A \triangleq (F(0), W_1(F(0)))$ and $C \triangleq (F(b_2), W_2(F(b_2)))$, then this line segment AC is above the line segment connecting, piece by piece, points $A, B \triangleq (F(b_1), W_1(F(b_1)))$ and C . We can show that there exists a point $b' \geq b_2$ such that its corresponding linear majorant $W'(F(x))$ satisfies $W'(F(x)) > W_1(F(x))$ on $x \in [0, b_1]$ and $W'(F(x)) > W_2(F(x))$ on $[b_1, b_2]$. The proof of the existence of a post-intervention point a' corresponding to this point b' follows in a similar manner to Lemma 3.4.
- (2) If $W_2(F(x))$ does not majorize $\bar{W}_1(F(x))$, we can find another point \bar{b} , instead of b_1 , such that the linear (not piecewise linear) function $W(F(x))$ corresponds to \bar{b} majorizes $R(F(x), \bar{a}) + W(F(\bar{a})) \frac{\varphi(\bar{a})}{\varphi(x)}$ on $x \in \mathbb{R}_+$ by Proposition 3.3.

In either case, the value function in the transformed space should be a linear function that attains the largest slope among all the possible linear majorant. This argument holds true for any b_1 and b_2 with $b_1 < b_2$. We can continue this argument inductively to the case of n intervention points, (b_1, \dots, b_n) . We here summarize our argument up to this point as a main proposition:

Proposition 3.4. *Suppose that (3.47) holds and the optimal continuation region is connected. The value function corresponding to (3.5) of the impulse problem described in (3.3)~ (3.8) is written as*

$$v(x) = \begin{cases} v_0(x) \triangleq \varphi(x)W^*(F(x)) + g(x), & 0 \leq x \leq b^* \\ v_0(a^*) + K(x, a^*), & b^* \leq x. \end{cases} \quad (3.51)$$

where $W^*(\cdot)$ is the line segment connects $(F(0), W^*(F(0)))$ and $(F(b^*), W^*(F(b^*)))$ and satisfy the following:

1. $W^*(F(\cdot))$ is the smallest linear majorant of $W^*(F(a^*))\frac{\varphi(a^*)}{\varphi(\cdot)} + R(F(\cdot), a^*)$ and meets with $W^*(F(a^*)) + R(F(\cdot), a^*)\frac{\varphi(a^*)}{\varphi(\cdot)}$ at point $F(b^*)$ and passes $(F(0), \frac{P-g(0)}{\varphi(0)})$. If R is differentiable, (a^*, b^*) satisfy (3.41).
2. The slope of $W^*(\cdot)$, denoted as β^* , is the largest slope among $\beta(a)$'s of all the possible linear majorants $W_a(\cdot)$.

Moreover, if the mapping $x \rightarrow \frac{\bar{K}}{\varphi}(x) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is F -concave, then the optimal continuation region $(0, b^*)$ is uniquely determined.

Note that, at $x = 0$,

$$v(0) = \varphi(0)W^*(F(0)) + g(0) = \varphi(0)\frac{P - g(0)}{\varphi(0)} + g(0) = P$$

as expected.

Remark 3.2. If the F -concavity of \bar{K}/φ is violated, there are at least two possible cases (and combination of them) where we have multiple continuation regions with linear value function in the continuation region of the transformed space.

1. For some a_i^* with $i = 1, 2, \dots$, we have the common β^* . This is the case which we shall show in the next example. In this case, the continuation region is $C = \{(0, b_1^*), (b_1^*, b_2^*), (b_2^*, b_3^*) \dots\}$ where b_i^* corresponds to a_i^* for each i , and the intervention region is $\Gamma = \{\{b_1^*\}, \{b_2^*\}, \{b_3^*\} \dots\}$. Each time the process hits one of the points $\{b_i^*\}$, the control pulls the process back to the corresponding a_i^* .
2. Another case is that, for the unique optimal a^* , there exists non-unique b_1^* and b_2^* . In this case, the continuation region is $C = \{(0, b_1^*), (b_1^*, b_2^*)\}$, and the stopping region is $\Gamma = \{\{b_1^*\}, [b_2^*, \infty)\}$. If the process hits b_1^* or b_2^* , then the control pulls the process back to a^* in either situation. It makes sense to continue in the region (b_1^*, b_2^*) because there is positive probability that one can extract $\bar{K}(b_2^*, a^*) (> \bar{K}(b_1^*, a^*))$ within a finite time.

3.5 No absorbing boundary case

Next, we extend our argument to a problem without the absorbing boundary. Hence the process can move along in the state space in an infinite amount of time. The problem becomes

$$J^\nu(x) = \mathbb{E}^x \left[\int_0^\infty e^{-\alpha s} f(X_s) ds + \sum_{i=1} e^{-\alpha T_i} K(X_{T_i-}, X_{T_i}) \right] \quad (3.52)$$

We can characterize intervention times as exit times from certain boundary and simplify the performance measure (3.52)

$$\begin{aligned} J^\nu(x) &= \mathbb{E}^x \left[\int_0^\infty e^{-\alpha s} f(X_s) ds + \sum_{i=1} e^{-\alpha T_i} K(X_{T_i-}, X_{T_i}) \right] \\ &= \mathbb{E}^x [e^{-\alpha T_1} \{K(X_{T_1-}, X_{T_1}) - g(X_{T_1-}) + J^\nu(X_{T_1})\}] + g(x). \end{aligned}$$

The second equation is easily obtained in the same way as in the previous section by noting $\mathbb{P}^x(T_1 < \infty) = 1$. The last term does not depend on controls, so we define $u(x) \triangleq J^\nu(x) - g(x)$:

$$u(x) = \mathbb{E}^x [e^{-\alpha T_1} \{K(X_{T_1-}, X_{T_1}) - g(X_{T_1-}) + g(X_{T_1}) + u(X_{T_1})\}].$$

Again, we consider the F -concave case with the notation $T_i- = \tau_b$ for all i and we have

$$u(x) = \mathbb{E}^x [e^{-\alpha \tau_b} (K(b, a) - g(b) + g(a) + u(a))] = \mathbb{E}^x [e^{-\alpha \tau_b} (\bar{K}(b, a) + u(a))].$$

By defining $W = (u/\varphi) \circ F^{-1}$, we have

$$W(F(x)) = W(F(c)) \frac{F(b) - F(x)}{F(b) - F(c)} + W(F(b)) \frac{F(x) - F(c)}{F(b) - F(c)}, \quad x \in (c, b].$$

We should note that $F(c) \triangleq F(c+) = \psi(c+)/\varphi(c+) = 0$ and

$$W(F(c)) = l_c \triangleq \limsup_{x \downarrow c} \frac{\bar{K}(x, a)^+}{\varphi(x)}$$

for any $a \in (c, d]$. For more detailed mathematical meaning of this value l_c , we refer the reader to Dayanik and Karatzas(2003). We can effectively consider $(F(c), l_c)$ as the absorbing boundary.

4 Examples

In this section, we work out some examples from financial engineering problems. As described in Section 3.3, the main task to find solutions now reduces to analyzing the reward function in the transformed space and find the smallest linear majorant. Let us recall that for given $a \in \mathbb{R}$, the shifted reward function in the transformed space is (3.40), that is,

$$\Phi(y, a) = R(y, a) + W_a(F(a)) \frac{\varphi(a)}{\varphi(F^{-1}(y))}, \quad y \in (F(c), F(d)) \quad (4.1)$$

in which $R(\cdot, a)$ is given by (3.37) and $W_a(\cdot)$ is given by (3.38). For the purpose of analyzing $\Phi(\cdot, a)$, we recall some useful observations: If $h(\cdot)$ is twice-differentiable at $x \in \mathcal{I}$ and $y \triangleq F(x)$, then $H'(y) = m(x)$ and $H''(y) = m'(x)/F'(x)$ with

$$m(x) = \frac{1}{F'(x)} \left(\frac{h}{\varphi} \right)'(x), \quad \text{and} \quad H''(y)(\mathcal{A} - \alpha)h(x) \geq 0, \quad y = F(x) \quad (4.2)$$

with strict inequality if $H''(y) \neq 0$. These identities are of practical use in identifying the concavities of $H(\cdot)$ when it is hard to calculate its derivatives explicitly. Before the transformation defined by (3.28), (4.1) is of the form $\bar{K}(x, a) + u(a)$. Hence, for a fixed a , we read $h(x) = \bar{K}(x, a)$ for the first term of (4.1) and $h(x) = \text{constant}$ for the second term of (4.1) to apply (4.2).

Remark 4.1. It is worth examining $h(x) = k$ where $k \in \mathbb{R}_+$ is a constant. The first equation of (4.2) is $m(x) = -\frac{k\varphi'(x)}{F'(x)\varphi(x)^2} > 0$ since $\varphi'(x) < 0$ and $F'(x) > 0$ for all $x \in \mathcal{I}$. As to the second inequality of (4.2), $(\mathcal{A} - \alpha)k = -\alpha k < 0$. These facts imply that the second term of (4.1) is always increasing and concave in the transformed space for an a that makes $W_a(F(a)) > 0$.

Example 4.1. Øksendal (1999) considers the following problem:

$$J'_o(x) = \mathbb{E}^x \left[\int_0^\infty e^{-\alpha s} X_s^2 ds + \sum_i^\infty e^{-\alpha T_i} (c + \lambda \xi_i) \right] \quad (4.3)$$

where $X_t^0 = B_t$ is a standard Brownian motion and $c > 0$ and $\lambda \geq 0$ are constants. The Brownian motion represents the exchange rate of some currency and each impulse represents an interventions taken by the central bank in order to keep the exchange rate in a given target zone. Here we are only allowed to give the system impulses ζ with values in $(0, +\infty)$. By reducing a level from b to a (i.e., $b > a$) through interventions, one can save continuously incurred cost (which is high if the process is at a high level). The problem is to minimize the expected total discounted cost $v_o(x) = \inf_\nu J'_o(x)$. We want to solve its sup version and change the sign afterwards (i.e. $v_o(x) = -v(x)$):

$$v(x) = \sup_\nu \mathbb{E}^x \left[\int_0^\infty e^{-\alpha s} (-X_s^2) ds - \sum_i^\infty e^{-\alpha T_i} (c + \lambda \xi_i) \right].$$

Data: The continuous cost rate $f(x) = -x^2$ and the intervention cost is $K(x, y) = -c - \lambda(x - y)$ in our terminology. By solving the equation $(\mathcal{A} - \alpha)v(x) = \frac{1}{2}v''(x) - \alpha v(x) = 0$, we find $\psi(x) = e^{x\sqrt{2\alpha}}$ and $\varphi(x) = e^{-x\sqrt{2\alpha}}$. Hence $F(x) = e^{2x\sqrt{2\alpha}}$ and $F^{-1}(x) = \frac{\log x}{2\sqrt{2\alpha}}$. $g(x)$ can be calculated by Fubini's theorem:

$$g(x) = \mathbb{E}^x \left[\int_0^\infty -e^{-\alpha s} (x + B_s)^2 ds \right] = - \left(\frac{x^2}{\alpha} + \frac{1}{\alpha^2} \right).$$

Note that when $b > a$, $g(a) - g(b) > 0$ is the source of cost savings. Hence $\bar{K}(x, a) = -c - \lambda(x - a) + \frac{x^2 - a^2}{\alpha}$.

Analysis of the reward function: Let us fix $a > 0$ and consider $h(x) \triangleq \bar{K}(x, a) = -c - \lambda(x - a) + \frac{x^2 - a^2}{\alpha}$ and $H(y) \triangleq (h/\varphi)(F^{-1}(y))$, $y > 0$. By the first equation in (4.2), the sign of $\left(\frac{h}{\varphi}\right)'(x)$ will lead us to conclude that $H(F(x))$ is increasing to infinity from a certain point, say $x = p$ on (p, ∞) , so is $H(F(x))$. Also, by direct calculation, $H'(+\infty) = 0$, from which we can assert that the value function is finite by Proposition 3.3.

If we set $p(x) \triangleq -x^2 + a^2 + \lambda\alpha(x - a) + \alpha c + 1/\alpha$, then $(\mathcal{A} - \alpha)h(x) = p(x)$ for every $x > 0$. This quadratic function $p(x)$ possibly has one or two positive roots. Let k be the largest one. Since $\lim_{x \rightarrow \infty} p(x) = -\infty$, by the second inequality in (4.2), $H(\cdot)$ is concave on $(F(k), +\infty)$. Hence $H(y, a)$ is increasing (to infinity) and concave on $y \in (F(k), \infty)$. Since it is obvious that there exists a such that $W(F(a)) > 0$, due to Remark 4.1, the second term of (4.1) is increasing and concave on \mathbb{R}_+ .

Since the cost function in the transformed space is increasing and concave from a certain point on, there is a linear majorant that touches the cost function once and only once. We can conclude that for any $a > 0$ and the parameter set, we have a connected continuation region in the form of $(0, b^*)$.

Solution: For a fixed a , let us define $W_a(\cdot)$ such that $V_a(x) = \varphi(x)W_a(F(x))$ and $r(x, a) = -c$ if $x < a$ and $r(x, a) = h(x) = -c - \lambda(x - a) + \frac{x^2}{\alpha} - \frac{a^2}{\alpha}$ if $x \geq a$. Then we have for any $a > 0$,

$$l_{-\infty} = \limsup_{x \downarrow -\infty} \frac{r(x, a)^+}{\varphi(x)} = 0. \quad (4.4)$$

Recall that the left boundary $-\infty$ is natural for a Brownian motion. Hence $W_a(y)$ that passes the origin of the transformed space is the straight-line majorant of $R(\cdot, a) + W_a(F(a))/\varphi(F^{-1}(\cdot))$ where $R(\cdot, a)$ is defined in (3.37):

$$R(y, a) = \begin{cases} -c\sqrt{y}, & 0 \leq y \leq F(a), \\ H(y, a) = \sqrt{y} \left(-c - \frac{\lambda}{2\sqrt{2\alpha}} \log y + \lambda a + \frac{(\log y)^2}{8\alpha^2} - \frac{a^2}{\alpha} \right), & y > F(a). \end{cases}$$

We can represent W_a as $W(y) = \beta y$. Since $R(x, a)$ is differentiable with respect to x on $x \geq a$, we can use (3.41) with $\delta = 0$ in (3.38) to find $b(a)$ and corresponding $\beta(a)$. Then varying a , one can find the optimal (a^*, b^*, β^*) . Going back to the original space, on $x \in (-\infty, b^*]$

$$\tilde{v}(x) \triangleq \sup u(x) = \varphi(x)W^*(F(x)) = \varphi(x)(\beta^*)F(x) = \beta^* e^{x\sqrt{2\alpha}}.$$

To get $v(x) = \sup_{\nu} J^{\nu}(x)$, we add back $g(x)$,

$$v(x) = \tilde{v}(x) + g(x) = \beta^* e^{x\sqrt{2\alpha}} - \left(\frac{x^2}{\alpha} + \frac{1}{\alpha^2} \right).$$

Finally, flip the sign and obtain the optimal cost function

$$v_o(x) = \begin{cases} \hat{v}_o(x) \triangleq \left(\frac{x^2}{\alpha} + \frac{1}{\alpha^2} \right) - \beta^* e^{x\sqrt{2\alpha}}, & 0 \leq x \leq b^*, \\ \hat{v}_o(a^*) + c + \lambda(x - a^*). & b^* \leq x. \end{cases}$$

which coincides with the solution given by Øksendal (1999). Figure 1 displays the solution with parameters $(c, \lambda, \alpha) = (150, 50, 0.2)$.

Example 4.2. This example is a dividend payout problem where the underlying process follows an Ornstein-Uhlenbeck process. This problem was originally studied by Cadenillas et al. (2007). Suppose that X^0 has the dynamics

$$dX_t^0 = \delta(m - X_t)dt + \sigma dW_t, \quad t \geq 0,$$

where $\delta > 0$, $\sigma > 0$ and $m \in \mathbb{R}$. Only positive impulse is allowed in this problem. We consider the impulse control problem,

$$v(x) \triangleq \sup_{\nu \in S} \mathbb{E}^x \left[\sum_{T_i < \tau_0}^{\infty} e^{-\alpha T_i} (-K + k\xi_i^{\gamma}) \right]$$

with some positive constant K , k and the risk-aversion parameter $\gamma \in (0, 1]$.

Data: $x = 0$ is an absorbing boundary. Since $\xi \in \mathbb{R}_+$, we have

$$\bar{K}(x, y) = k(x - y)^{\gamma} - K, \quad x > y > 0.$$

Since $f(x) = 0$ for all $x \in \mathbb{R}$, we have $g(x) = 0$. The functions $\psi(\cdot)$ and $\varphi(\cdot)$ are positive, increasing and decreasing solutions of the differential equation $(\mathcal{A} - \alpha)v(x) = (1/2)\sigma^2 v''(x) + \delta(m - x)v'(x) - \alpha v(x) = 0$. We denote, by $\tilde{\psi}(\cdot)$ and $\tilde{\varphi}(\cdot)$, the functions of the fundamental solutions for the auxiliary process $Z_t \triangleq (X_t - m)/\sigma$, $t \geq 0$, which satisfies $dZ_t = -\delta Z_t dt + dW_t$. For every $x \in \mathbb{R}$,

$$\tilde{\psi}(x) = e^{\delta x^2/2} \mathcal{D}_{-\alpha/\delta}(-x\sqrt{2\delta}) \quad \text{and} \quad \tilde{\varphi}(x) = e^{\delta x^2/2} \mathcal{D}_{-\alpha/\delta}(x\sqrt{2\delta}),$$

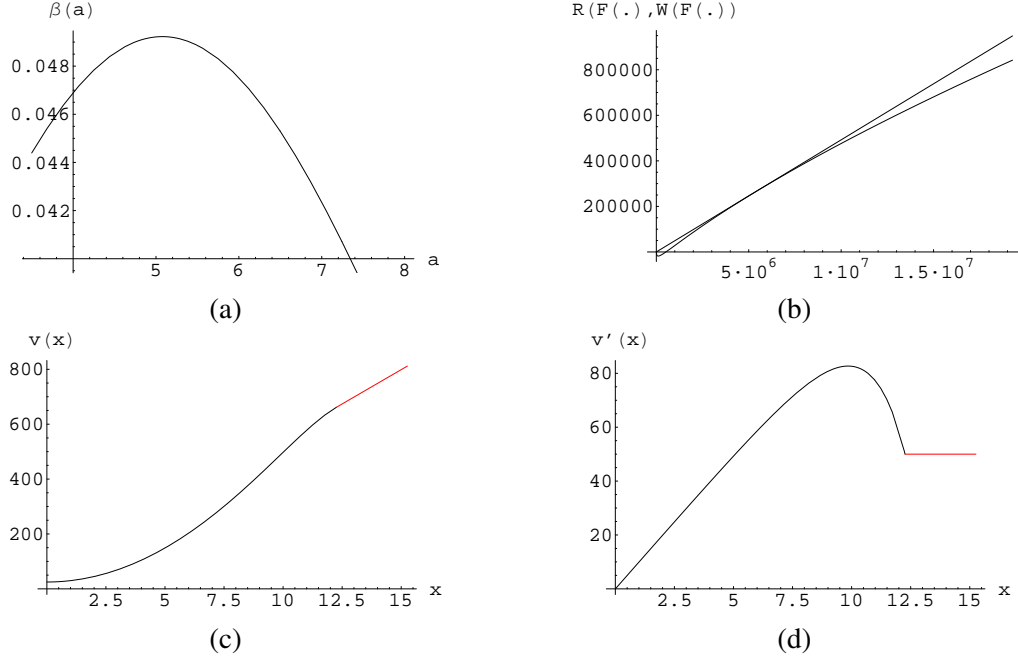


Figure 1: (a) The plot of $\beta(a)$ against a , the former being maximized at $a^* = 5.077$ with $\beta^* = 0.0492$. (b) The functions $R(F(\cdot), a^*)$ shifted by the amount $W_{a^*}(F(a^*)) \frac{\varphi(a)}{\varphi(x)}$ (lower curve) and the majorant $W_{a^*}(F(\cdot))$ (upper curve) corresponding to a^* , giving us $b^* = 12.261$. (c) The cost function $v_o(x)$. (d) The derivative of $v_o(x)$, showing that the smooth-fit principle holds at b^* .

and $\psi(x) = \tilde{\psi}((x-m)/\sigma)$ and $\varphi(x) = \tilde{\varphi}((x-m)/\sigma)$, where $\mathcal{D}_\nu(\cdot)$ is the parabolic cylinder function; (see Borodin and Salminen (2002, Appendices 1.24 and 2.9) and Carmona and Dayanik (2003, Section 6.3)). By using the relation

$$\mathcal{D}_\nu(z) = 2^{-\nu/2} e^{-z^2/4} \mathcal{H}_\nu(z/\sqrt{2}), \quad z \in \mathbb{R} \quad (4.5)$$

in terms of the Hermite function \mathcal{H}_ν of degree ν and its integral representation

$$\mathcal{H}_\nu(z) = \frac{1}{\Gamma(-\nu)} \int_0^\infty e^{-t^2-2tz} t^{-\nu-1} dt, \quad \text{Re}(\nu) < 0, \quad (4.6)$$

(see for example, Lebedev(1972, pp. 284, 290)).

Analysis of the reward function: Let us consider the function

$$h(x) \triangleq k(x-a)^\gamma - K, \quad x > 0, \gamma \in (0, 1].$$

Since the function $h(\cdot)$ is positive when x is large enough and it is increasing to infinity on the whole real line, so is the function $H(y) = (h/\varphi) \circ F^{-1}(y)$, $y \in (0, \infty)$.

Since it is obvious that there exists a such that $W(F(a)) > 0$, due to Remark 4.1, the second term of (4.1) is increasing and concave on $(F(0), \infty)$. Let us concentrate on the first term and define the function

$$p(x) \triangleq \frac{1}{2} \sigma^2 k \gamma (\gamma - 1) x^{\gamma-2} + m \delta k \gamma x^{\gamma-1} - k(\delta \gamma + \alpha) x^\gamma + \alpha K$$

which satisfies $(\mathcal{A} - \alpha)h(x) = p(x)$. By using (4.2), $H''(y)$ and $p(F^{-1}(y))$ have the same sign at every y where h is twice-differentiable. Hence we study the (positive) roots of $p(x) = 0$. We have to divide two

cases: (1) $\gamma = 1$ and (2) $\gamma < 1$. In either case, it can be shown that $H'(+\infty) < \infty$ by using (4.5) and (4.6) and the identity $\mathcal{H}'_\nu(z) = 2\nu\mathcal{H}_{\nu-1}(z)$, $z \in \mathbb{R}$. Therefore, the finiteness of the value function is proved.

- (1) $\gamma = 1$: $h(\cdot)$ reduces to a linear function and the $p(x) = 0$ always has a one positive root, say $p > 0$. $H(\cdot)$ function is convex on $[0, F(p))$ and concave on $(F(p), +\infty)$. Hence we have a connected continuation region $(0, b^*)$.
- (2) $\gamma < 1$: We observe that $\lim_{x \downarrow 0} p(x) = -\infty$, $\lim_{x \uparrow +\infty} p(x) = -\infty$, $\lim_{x \downarrow 0} p'(x) = +\infty$, and $\lim_{x \uparrow +\infty} p'(x) = 0-$. A direct analysis of $p'(x)$ shows that there is only one stationary point in $(0, \infty)$ and the number of the roots of $p(x) = 0$ is either 0, 1 or 2. Hence in the first two cases, $H(\cdot)$ is concave on $[0, \infty)$ and the continuation region is connected. In the last case where there are two roots, say $0 < p_1 < p_2$. The $H(\cdot)$ function is then concave on $[0, F(p_1)) \cup (F(p_2), +\infty)$ and is convex on $(F(p_1), F(p_2))$. Since $H(\cdot)$ increases and concave on $y \in (F(p_2), \infty)$, we can conclude that the continuation region is connected in this case as well.

Solution: Let us move on to finding an optimal continuation region. Unlike the previous example, it is not easy (at least analytically) to find $F^{-1}(y)$ explicitly. But it is not necessary. We can solve (3.44) for $b(a)$ with $D = 0$:

$$\left(\frac{\bar{K}(b, a)}{\varphi(b)} \right)' = \frac{\bar{K}(b, a)}{\varphi(b)(F(b) - F(0)) - \varphi(a)(F(a) - F(0))} \left(F'(b) + \frac{\varphi'(b)\varphi(a)}{\varphi(b)^2}(F(a) - F(0)) \right).$$

As in the previous examples, $W_a(\cdot)$ is a straight line passing $(F(0), 0)$ in the form of $W_a(y) = \beta(y - F(0))$. The value function $v(x)$ in $x \in (0, b^*)$ is

$$\begin{aligned} \hat{v}(x) &= \varphi(x)W(F(x)) = \beta(F(x) - F(0))\varphi(x) \\ &= \beta^*(\psi(x) - F(0)\varphi(x)) = \beta^* e^{\frac{\delta}{2} \frac{(x-m)^2}{\sigma^2}} \left\{ \mathcal{D}_{-\alpha/\delta} \left(- \left(\frac{x-m}{\sigma} \right) \sqrt{2\delta} \right) - F(0) \mathcal{D}_{-\alpha/\delta} \left(\left(\frac{x-m}{\sigma} \right) \sqrt{2\delta} \right) \right\}. \end{aligned}$$

Therefore, the solution to the problem is

$$v(x) = \begin{cases} \hat{v}(x), & 0 \leq x \leq b^*, \\ \hat{v}(a^*) + k(x - a^*)^\gamma - K, & b^* \leq x. \end{cases}$$

This solves the problem. See Figure 2-(b) for the value function in case of parameters $\delta = 0.1$, $m = 0.9$, $\sigma = 0.35$, $\alpha = 0.105$ for the diffusions. As for the reward/cost function parameters, $k = 0.7$, $K = 0.1$ and $\gamma = 0.75$. The solution is $(a^*, b^*, \beta) = (0.2192, 0.6220, 0.5749)$.

Example 4.3. We show a simple example where we have multiple continuation regions, the first case of Remark 3.2. Let the uncontrolled process is a standard Brownian motion B_t and let $\alpha = 0$, $f = 0$ and

$$K(x, y) = -c(\sin x - \sin y) - \delta$$

with $c \in \mathbb{R}_+$ and $\delta \in \mathbb{R}_+$ being some constant parameters. We want to solve

$$v(x) = \sup_{\nu \in S} \mathbb{E}^x \left[\sum_{T_i < \tau_0} (\xi_i - \delta) \right].$$

In this case $F(x) = x$ and let us define

$$R(x, a) = r(x, a) = \begin{cases} 0, & x = 0, \\ -c(\sin x - \sin a) - \delta, & x > 0. \end{cases}$$

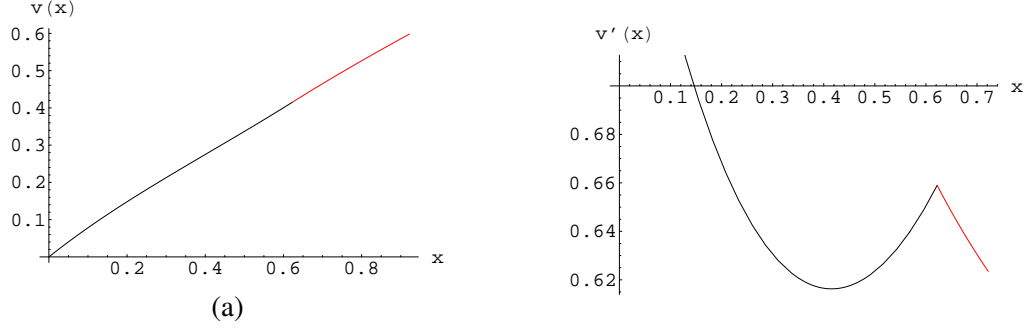


Figure 2: (a) The value function for Cadenillas et al.(2007) problem and (b) its derivative.

By solving (3.41) with some parameter $(c, \delta) = (10, 0.35)$, we find that $a_k^* = 2.75 + 4k\pi$ and $b_k^* = 3.52 + 4k\pi$ with $k = 0, 1, 2, \dots$. For all these pairs, β^* has a common value of 9.30. Hence all these pairs are optimal. This implies that if the initial state $x \in (b_k^*, b_{k+1}^*)$, then we let the process move until it reaches b_k^* or b_{k+1}^* . If it reaches b_k^* first, then an intervention is made to a_k^* . Now we are in the interval (b_{k-1}^*, b_k^*) . We continue until the process is absorbed at $x = 0$.

5 Conclusions

Before we conclude this article, we shall mention an immediate extension to two boundary impulse control problems:

$$J^\nu(x) = \mathbb{E}^x \left[\int_0^\infty e^{-\alpha s} f(X_s) ds + \sum_{i=1} e^{-\alpha T_i} C_1(X_{T_i-}, X_{T_i}) + \sum_{j=1} e^{-\alpha S_j} C_2(X_{S_j-}, X_{S_j}) \right] \quad (5.1)$$

and

$$v(x) = \sup_{\nu} J^\nu(x) = J^{\nu^*}(x) \quad (5.2)$$

for all $x \in \mathbb{R}$, where

$$\nu = (T_1, T_2, \dots; \zeta_1, \zeta_2, \dots; S_1, S_2, \dots; \eta_1, \eta_2, \dots)$$

with $\zeta_i > 0$ corresponds to interventions at the upper boundary at intervention time T_i and $\eta_j < 0$ at the lower boundary at intervention time S_j .

Examples of this type include the storage model analyzed by Harrison et al. (1983) and foreign exchange rate model studied by Jeanblanc-Picqué (1993). The former problem, for example, is that a controller continuously monitors the inventory so that the inventory level will not fall below the zero level. He is allowed to make interventions by increasing and decreasing the inventory by paying costs associated with interventions. In this case, the process remains within some band(s). In other words, the optimal intervention times are characterized as exit times from an interval in the form of (p^*, b^*) for $0 \leq p^* < b^*$. See Korn (1999) for a survey.

Under suitable assumptions, we can develop an argument similar to the previous chapters. Among others, the intervention times can be characterized as exit times from an interval (p^*, b^*) . We can also simplify the performance measure,

$$J^\nu(x) = \mathbb{E}^x [1_{\{T_1 < S_1\}} e^{-\alpha T_1} \{C_1(X_{T_1-}, X_{T_1}) - g(X_{T_1-}) + J^\nu(X_{T_1})\}] \\ + \mathbb{E}^x [1_{\{T_1 > S_1\}} e^{-\alpha S_1} \{C_2(X_{S_1-}, X_{S_1}) - g(X_{S_1-}) + J^\nu(X_{S_1})\}] + g(x)$$

where $g(x) = \mathbb{E}^x \int_0^\infty e^{-\alpha s} f(X_s^0) ds$ as usual. Again, the last term does not depend on controls, we define $u(x)$ as $u(x) = J^\nu(x) - g(x)$,

$$u(x) = \mathbb{E}^x [1_{\{\tau_b < \tau_p\}} e^{-\alpha \tau_b} u(b)] + \mathbb{E}^x [1_{\{\tau_b > \tau_p\}} e^{-\alpha \tau_p} u(p)], \quad x \in [p, b] \quad (5.3)$$

where $T_1 = \tau_b$ and $S_1 = \tau_p$ and it follows that

$$\frac{u(x)}{\varphi(x)} = \frac{u(b)(F(x) - F(p))}{\varphi(b)(F(b) - F(p))} + \frac{u(p)(F(b) - F(x))}{\varphi(p)(F(b) - F(p))}, \quad x \in [p, b]. \quad (5.4)$$

Hence if we define $W \triangleq \frac{u}{\varphi} \circ F^{-1}$, we have linear characterization again in the transformed space;

$$W(F(x)) = W(F(b)) \frac{F(x) - F(p)}{F(b) - F(p)} + W(F(p)) \frac{F(b) - F(x)}{F(b) - F(p)}, \quad x \in [p, b], \quad (5.5)$$

and the solution to the problem is described as

$$u(x) = \begin{cases} \bar{C}_2(x, q) + u_0(q), & x \leq p, \\ u_0(x) \triangleq \mathbb{E}^x [1_{\{\tau_b < \tau_p\}} e^{-\alpha \tau_b} u(b)] + \mathbb{E}^x [1_{\{\tau_b > \tau_p\}} e^{-\alpha \tau_p} u(p)], & p \leq x \leq b, \\ \bar{C}_1(x, a) + u_0(a), & b \leq x, \end{cases}$$

where $\bar{C}_i(x, y) = C_i(x, y) - g(x) + g(y)$ for $i = 1$ and 2 .

We have studied impulse control problems. The intervention times are characterized as exit times of the process from a finite union of disjoint intervals on the real line. A sufficient condition is given for the connectedness of the continuation region. The value function is shown to be linear in the continuation region of the transformed space and a direct calculation method is described. This method can handle impulse control problems with non-smooth reward and cost functions. The finiteness of the value function is shown to be equivalent to the existence of a concave majorant of the suitable transformed reward function. The latter is easier to check by using elementary geometric arguments.

The new characterization of the value function and optimal strategies can be extended to other optimization problems, such as optimal switching and combined problems of optimal stopping and impulse control. If an optimal strategy exists in the class of exit times, then the problem can be reduced to a sequence of optimal stopping problems and an effective characterization of the value function is possible.

6 Appendix

6.1 Proof of Lemma 3.2

To make the proof more intuitive, we will work with (3.14) rather than with (3.15) where the integration part is converted to g functions. For this purpose, it is convenient to define the following two operators $\mathcal{M}_o : \mathcal{H} \rightarrow \mathcal{H}$ and $\mathcal{L}_o : \mathcal{H} \rightarrow \mathcal{H}$:

$$\mathcal{M}_o u(x) = \sup_{y \in \mathbb{R}} [K(x, y) + u(y)] \quad (6.1)$$

and

$$\mathcal{L}_o u(x) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[\int_0^\tau e^{-\alpha s} f(X_s) ds + e^{-\alpha \tau} \mathcal{M}_o u(X_{\tau-}) \right]. \quad (6.2)$$

Hence we can proceed with the arguments developed in Davis (1992). In terms of the two operators just defined, (3.14) becomes

$$w_{n+1}(x) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[\int_0^\tau e^{-\alpha s} f(X_s) ds + e^{-\alpha \tau} \mathcal{M}_o w_n(X_{\tau-}) \right] \quad (6.3)$$

$$= \mathcal{L}_o w_n(x). \quad (6.4)$$

(1) $w_n = v_n$ for all n : Note that F -concavity of the function \bar{K}/φ guarantees the value $u(y)$ in (3.12) is a finite number as is shown in Proposition 3.2. Hence f and $\mathcal{M}w_n$ satisfies the condition (polynomial growth) of Theorem 3.2 in Øksendal and Sulem (2002), which shows $w_n = v_n$.

(2) $v(x) = \lim_{n \rightarrow \infty} w_n(x)$: Since w_n is monotone increasing, the limit $w(x) = \lim_{n \rightarrow \infty} w_n(x)$ exists. Since $\mathcal{S}_n \subset \mathcal{S}$, $w_n(x) \leq v(x)$. Hence $w(x) \leq v(x)$. To show the reverse inequality, we define S^* be a set of interventions such that

$$S^* = \{\nu \in \mathcal{S} : J^\nu(x) < \infty \text{ for all } x \in \mathbb{R}\}.$$

Let us assume that $\underline{v(x) < +\infty}$ and consider strategy $\nu^* \in S^*$ such that

$$J^{\nu^*}(x) \geq v(x) - \epsilon \quad (6.5)$$

for some $\epsilon > 0$ and another strategy ν_n that coincides with ν^* up to and including time T_n and then takes no further interventions.

$$J^{\nu^*}(x) - J^{\nu_n}(x) = \mathbb{E}^x \left[\int_{T_n}^\infty e^{-\alpha s} (f(X_s) - f(X_s^0)) ds + \sum_{i \geq n+1} e^{-\alpha T_i} K(X_{T_i-}, X_{T_i}) \right],$$

which implies

$$|J^{\nu^*}(x) - J^{\nu_n}(x)| \leq \mathbb{E}^x \left[\int_{T_n}^\infty e^{-\alpha s} (|f(X_s)| + |f(X_s^0)|) ds + \sum_{i \geq n+1} e^{-\alpha T_i} |K(X_{T_i-}, X_{T_i})| \right].$$

As $n \rightarrow +\infty$, $T_n \rightarrow +\infty$ and the first term of the right hand side can be arbitrary small due to (3.6) and so is the second term by the finiteness of $v(x)$ with (3.7). Hence it is shown $|J^{\nu^*}(x) - J^{\nu_n}(x)| < \epsilon$ for n large enough, which implies with (6.5) that

$$\liminf_{n \rightarrow +\infty} J^{\nu_n}(x) \geq J^{\nu^*}(x) - \epsilon \geq v(x) - 2\epsilon$$

so that $v(x) \leq \lim_{n \rightarrow +\infty} v_n(x)$ since ϵ is arbitrary. Now we have established $v(x) = \lim_{n \rightarrow \infty} w_n(x)$ when $v(x) < +\infty$. Next, consider the case of $\underline{v(x) = +\infty}$. Then by the recursive method described in Section 3.1, we see that $v_1(x) = w_1(x) = \infty$. By the first statement of this lemma, we can conclude $v_n(x) = w_n(x) = \infty$ for all $n \in \mathbb{N}$, obtaining $v(x) = \lim_{n \rightarrow \infty} w_n(x)$. This completes the proof of the second statement.

(3) $w = \mathcal{L}_o w$: Since $w_n \uparrow w$, we have the following chain of equalities:

$$\begin{aligned} \mathcal{M}_o w(x) &= \sup_{y \in \mathbb{R}} [K(x, y) + w(y)] = \sup_{y \in \mathbb{R}} \sup_{n \in \mathbb{N}} [K(x, y) + w_n(y)] \\ &= \sup_{n \in \mathbb{N}} \sup_{y \in \mathbb{R}} [K(x, y) + w_n(y)] = \sup_{n \in \mathbb{N}} \mathcal{M}_o w_n(x). \end{aligned}$$

In view of this, if we take the limit on the both sides of (6.3) as $n \rightarrow \infty$, by the monotone convergence theorem,

$$w(x) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[\int_0^\tau e^{-\alpha s} f(X_s) ds + e^{-\alpha \tau} \mathcal{M}_o w(X_{\tau-}) \right].$$

This shows that $w = \mathcal{L}_o w$. Suppose $w'(x)$ satisfies $w' = \mathcal{L}_o w'$ and majorizes $g(x) = v_0(x)$. Then $w' = \mathcal{L}_o w' \geq \mathcal{L}_o v_0 = w_1$. If we assume that $w' \geq v_n$, then

$$w' = \mathcal{L}_o w' \geq \mathcal{L}_o v_n = v_{n+1} = w_{n+1}.$$

This shows that, by the induction argument, we have $w' \geq w_n$ for all n , leading to $w' \geq \lim_{n \rightarrow \infty} w_n = w$. Thus it shows that w is the smallest solution majorizing g of the functional equation, $w - g = \mathcal{L}(w - g)$. This completes the third statement of the lemma.

6.2 Derivation of (3.46)

The first order condition of the optimality with respect to a is

$$d\beta(a)/da = d\beta/db \times db/da = 0. \quad (6.6)$$

For a fixed a , from (3.41) by viewing β as a function of state $x = F^{-1}(y)$, that is, $\beta = \beta(F^{-1}(y))$, it is clear that $d\beta(F^{-1}(y))/dy = \frac{1}{F'(x)} d\beta/dx$. Since $F'(x) > 0$ for all $x \in \mathcal{I}$, we have $d\beta(F^{-1}(y))/dy = 0$ if and only if $d\beta/dx = 0$. But $d\beta(F^{-1}(y))/dy = 0$ at $x = F^{-1}(y) = b$ implies the following: At $x = b(a)$, where the shifted function $\Phi(y, a)$ becomes tangent to the linear function $W_a(F(x)) = \beta(a)F(x) + \delta$, the second derivative of the shifted function vanishes. But it is clearly impossible if $\Phi(y, a)$ is strictly concave at $F(b)$. Hence in view of (6.6), we can claim that $d\beta(a)/da = 0$ only if $db/da = 0$, provided that $\Phi(y, a)$ is strictly concave at $y = F(b)$. Now we differentiate (3.42) with respect to a with noting $db/da = 0$, we obtain (3.46).

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