Optimal Reinsurance Strategy under Fixed Cost and Delay

Masahiko Egami * Virginia R. Young †

Abstract

We consider an optimal reinsurance strategy in which the insurance company (1) monitors the dynamics of its surplus process, (2) optimally chooses a time to begin negotiating with a reinsurer to buy quota-share, or proportional, reinsurance, which introduces an implementation delay (denoted by $\Delta \geq 0$), (3) chooses the optimal proportion at the beginning of the negotiation period, and (4) pays a fixed transaction cost when the contract is signed (Δ units of time after negotiation begins). This setup leads to a combined problem of optimal stopping and stochastic control. We obtain a solution for the value function and the corresponding optimal strategy, while demonstrating the solution procedure in detail. It turns out that the optimal continuation region is a union of two intervals, a rather rare occurrence in optimal stopping. Numerical examples are given to illustrate our results and we discuss relevant economic insights from this model.

Key words: Reinsurance strategy, optimal stopping, implementation delay, transaction cost.
JEL Classification: G22, C61.
Mathematics Subject Classification (2000) : Primary: 62L15, Secondary: 60J60.

1 Introduction

The optimal quota-share, or proportional, reinsurance is one of the well-studied subjects in the literature. We mention Browne [3], Promislow and Young [14], Schmidli [16], and Taksar and Markussen [18]. These researchers study the minimization of the probability of ruin when claims follow a Brownian motion with drift, while Højgaard and Taksar [10] and Choulli et al. [5] analyze the maximization of dividend payout. Here we study an insurer who wants to maximize its total discounted value of surplus until the surplus process hits the ruin state. We consider an insurer facing a claim process modeled by a Brownian motion with drift and contemplating reinsurance subject to a fixed cost for buying reinsurance (in addition to a proportional load on the premium) and a time delay in completing the reinsurance

*Corresponding author. Tel:+81-75-753-3430; Fax:+81-75-753-3492. M. Egami is with Graduate School of Economics, Kyoto University, Yoshida-honmachi, Sakyo-ku, Kyoto, 606-8501, Japan; egami@econ.kyoto-u.ac.jp.

†V. R. Young is with Department of Mathematics, University of Michigan, 525 East University Avenue, Ann Arbor, MI 48109-1043, USA; vryoung@umich.edu. We thank Erhan Bayraktar for helpful conversations. We are also grateful to the Associate Editor and two anonymous referees for their comments that improved our manuscript.

transaction. It is expected that, even without any delay, the existence of a fixed transaction cost will force the insurer to postpone buying reinsurance until its surplus process hits a certain level. Therefore, the insurer's controls involve (1) the level of quota-share reinsurance and (2) the timing of when it will buy that reinsurance.

The insurer, after deciding on the level of reinsurance, spends a fixed length of time before the contract is signed. This delay time is necessitated by negotiating and other administrative work associated with implementing the reinsurance policy. Considering such a delay period makes the problem's model more realistic. Recently, delay has been explicitly addressed in the stochastic control literature, and we mention a few papers of interest: Peura and Keppo [13] consider the problem of a bank's recapitalization with a regulatory delay period. Bar-Ilan and Strange [1] study two-stage investment decision problems subject two sources of delay: one due to market analysis and the other due to construction of a production facility. Subramanian and Jarrow [17] consider a trader's problem where she is not a *price taker* and wants to liquidate her position and encounters execution delays in an illiquid market. Bayraktar and Egami [2] propose a direct solution method for delayed impulse control problems of one-dimensional diffusions and solve an optimal labor force problem with firing delay. We mention another paper that handles delay, while the set up is different from ours: Elsanosi et al. [8] study a harvesting problem where the dynamics of the controlled process depend on its own historical value as well as the present state.

It is expected that, even without fixed costs, the existence of delay makes the insurer's problem complicated because there is a positive probability that the process hits the ruin state during the delay period. We will explicitly write a reward function under fixed cost and implementation delay and solve the combined problem of optimal stopping and stochastic control. For this purpose, we rely on the work on Dynkin [7] (see, e.g., Theorem 16.4) and Dayanik and Karatzas [6] (Propositions 4.3 and 4.4). In this way, unlike arguments that rely on quasi-variational inequalities, we can avoid proving a verification lemma and the related guesswork in determining the optimal strategy.

We solve the problem in the following way: After defining the problem in Section 2.1, we calculate the necessary functionals involving delay time in Section A.1. We solve the problem completely in Section 2.2. In Section 3, we perform some numerical analysis and observe how the optimal solution changes as the length of the delay period changes. We discuss some economic implications concerning the fixed cost and delay. Furthermore, we extend to the case for which the insurer can purchase reinsurance infinitely many times and conclude with a summary of the results.

2 Optimal reinsurance strategy

2.1 Problem description

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a standard Brownian motion $W = \{W_t; t \geq 0\}$. We model the claim process C by a Brownian motion with drift:

$$dC_t = adt - \sigma dW_t, \tag{2.1}$$

where a and σ are positive constants. As is often assumed in the literature, this diffusion process approximates a compound Poisson model; see, for example, [9], [14], [16], and [18]. We assume that the premium is paid continuously at the constant rate $c = (1+\theta)a$ with $\theta > 0$. Therefore, before introducing reinsurance, the surplus process X^0 has state space $\mathcal{I} = \mathbb{R}$ (for Brownian motion with drift) with dynamics

$$dX_t^0 = cdt - dC_t = \theta adt + \sigma dW_t, \tag{2.2}$$

and with the initial value $X_0^0 = x \in \mathbb{R}_+$. We use "0" as a superscript to indicate that X^0 is the uncontrolled surplus process. The insurer reinsures a proportion of its claims to a reinsurer. Reinsurance is available for a proportional loading of $\eta > \theta$. In the literature, the reinsurance problem is often treated as a stochastic control problem for which the insurer determines the reinsurance level at time 0 with no delay in implementing the reinsurance and with no fixed transaction cost levied. (See Taksar and Markussen [18] and the references therein.) However, it is more realistic if we assume that the insurer pays a fixed transaction cost (whether the cost is due to dollars actually spent or employee hours used), and after the insurer decides on the reinsurance level, a certain length of time is required before the actual contract takes effect due to the time it takes to initialize the policy.

An *admissible reinsurance strategy* is a pair,

$$\pi = (\tau, \xi),$$

in which $0 \leq \tau$ is an $\mathcal{F}$-stopping time and ξ is a $\mathcal{F}_\tau$-measurable random variable representing the proportion reinsured at time $\tau + \Delta$. The proportion ξ is chosen at time τ, given the information available at that time, and at that time, is a specific number between 0 and 1. However, the reinsurance will not be implemented until time $\tau + \Delta$ due to the existence of a delay period. The state 0 is the absorbing state (ruin) without loss of generality and τ_0 is defined as the ruin time:

$$\tau_0 \triangleq \inf\{t \geq 0 : X_t \leq 0\}.$$

Assumption 2.1. We make the following assumptions in this paper:

(a) At the stopping time τ, the insurer chooses a proportion $\xi \in [0, 1]$ of its claims to reinsure and begins negotiating with the reinsurer. This negotiating takes a fixed amount of time $\Delta \geq 0$. After the time Δ elapses, if the surplus process has not hit the ruin level, the insurer pays a fixed transaction cost $K > 0$ and the proportional reinsurance takes effect at time $\tau + \Delta$. Hence, the surplus process X follows

$$\begin{cases} dX_t = \mu_0 dt + \sigma_0 dW_t, \quad 0 \le t < \tau + \Delta, \\ X_{\tau+\Delta} = X_{(\tau+\Delta)-} - K, \\ dX_t = \mu_1 dt + \sigma_1 dW_t, \quad \tau + \Delta \le t, \end{cases} \tag{2.3}$$

where $\mu_0 = \theta a$, $\sigma_0 = \sigma$,

$$\mu_1 = (\theta - \eta\xi)a, \quad \text{and} \quad \sigma_1 = \sigma(1 - \xi),$$

with $\xi \in [0, 1]$.

(b) When the insurer becomes insolvent, it has to pay a fixed cost $P \ge 0$.

(c) At time $\tau + \Delta$, if $X_{(\tau+\Delta)-} \le K$, the surplus process hits the ruin state at time $\tau + \Delta$, and the insurer becomes insolvent.

We consider the following performance measure associated with a reinsurance strategy $\pi \in \Pi$ (= the collection of admissible strategies),

$$J^\pi(x) \triangleq \mathbb{E}^x \left[\int_0^{\tau_0} e^{-\alpha s} f(X_s) ds - e^{-\alpha \tau_0} P \right], \tag{2.4}$$

in which X is the controlled surplus process and $\mathbb{E}^x[\cdot]$ is the expectation under the probability law when $X_0 = x$. Also, $f : \mathbb{R} \to \mathbb{R}$ denotes a continuous, nondecreasing (utility) function that satisfies

$$\mathbb{E}^x \left[\int_0^\infty e^{-\alpha s} |f(X_s^0)| ds \right] < \infty, \tag{2.5}$$

and $P \in \mathbb{R}_+$ is a constant that represents insolvency costs.

The objective is to find the optimal strategy $\pi^* \in \Pi$, if it exists, and the corresponding value function:

$$v(x) \triangleq \sup_{\pi \in \Pi} J^\pi(x) = J^{\pi^*}(x). \tag{2.6}$$

Next, we rewrite the problem (2.4) and (2.6) as a combination of an optimal stopping problem and a stochastic control problem. The possibility that the surplus process may hit the ruin state during the delay period complicates the expression. Note that in the following derivation, we only require that the continuous, nondecreasing function f satisfy (2.5).

First, define the function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) \triangleq \mathbb{E}^x \left[\int_0^\infty e^{-\alpha s} f(X_s^0) ds \right], \tag{2.7}$$

which corresponds to the expected total utility if the insurer does not implement any reinsurance. The following identity, which can be derived by using the strong Markov property of X^0 (see Karatzas and Shreve [11] for example), will prove useful in the computations below:

$$\mathbb{E}^x \left[\int_0^\tau e^{-\alpha s} f(X_s^0) ds \right] = g(x) - \mathbb{E}^x \left[e^{-\alpha \tau} g(X_\tau^0) \right], \tag{2.8}$$

for any stopping time τ, including τ_0, due to integrability condition in inequality (2.5).

In Appendix A.1, we show that the original problem of finding the value function $v(x)$ in (2.6) reduces to solving,

$$v(x) - g(x) = \sup_{\xi \in [0,1]} \left(\sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[1_{\{\tau < \tau_0\}} e^{-\alpha \tau} h(X_\tau; \xi) \right] + \mathbb{E}^x \left[1_{\{\tau > \tau_0\}} e^{-\alpha \tau_0} \{-P - g(X_{\tau_0})\} \right] \right), \quad (2.9)$$

where $\mathcal{S}$ is the set of $\mathcal{F}$-stopping times. Recall that the proportion reinsured $\xi \in [0, 1]$ is chosen at time τ, as stated in Assumption 2.1(a). The optimization in (2.9) is a combined problem of an optimal stopping problem (inner optimization, with the ruin state at $x = 0$ and the payoff at the ruin $-P - g(0)$) and a stochastic control problem (outer optimization). Here the function h is defined by

$$h(z; \xi) \triangleq \mathbb{E}^z \left[1_{\{\Delta < \tau_0\}} e^{-\alpha \Delta} \left\{ J^\pi_\xi(X_\Delta) - g(X_{\Delta-}) \right\} + 1_{\{\Delta > \tau_0\}} e^{-\alpha \tau_0} \{-P - g(X_{\tau_0})\} \right], \quad (2.10)$$

which we can evaluate by using expressions (A.6), (A.7), and (A.8). The subscript ξ in J_ξ signifies that the surplus process now has new dynamics after the proportion of ξ is reinsured. When we want to emphasize the dependence of h on the delay period Δ, then we will write h_Δ or $h(\cdot; \xi, \Delta)$.

Remark 2.1. In the next section, we rely on work of Dynkin [7] and Dayanik and Karatzas [6] to determine the optimal strategy π^* in the case for which f is linear. Unlike arguments that rely on quasi-variational inequalities, we avoid proving a verification lemma and the related guesswork in determining the optimal strategy. The work of Dayanik and Karatzas [6](see especially, Proposition 4.3 and 4.4) in finding an optimal stopping time is applicable to one-dimensional diffusions more general that the Brownian motion we consider in (2.3). Complexity is added in our model due to the delay period Δ and due to the possibility that the surplus X hits zero during the delay period. Therefore, to obtain economically significant results, we require an explicit diffusion for the surplus process. Because in the insurance literature, Brownian motion with drift is quite common, we analyze our model under Brownian motion with drift, as in (2.3).

2.2 Solution when f is linear

In the previous subsection, we showed that if we solve (2.9) and (2.10), then we have effectively solved the original problem given by (2.4) and (2.6). Our setting for this result was quite general. To show a concrete result, in the work that follows, we suppose that f is linear: Namely,

$$f(x) = x.$$

We summarize our plan as follows: In Appendix A.2, we perform some preliminary computations. In Section 2.2.1, we solve the optimal-stopping problem conditional on the proportion reinsured and then optimize with respect to that proportion. We conclude Section 2.2 with a numerical example in Section 2.2.2. In each subsection, we consider both cases of $\Delta = 0$ and $\Delta > 0$. In fact, solving the special case of $\Delta = 0$ turns out to be helpful in analyzing the case of $\Delta > 0$.

2.2.1 Solution of the optimal stopping problem

To understand some characteristics of the optimization problem (2.9), it is useful to consider the special case for which $\Delta = 0$. If $\Delta = 0$, then from work in Appendix A.2, the expression in (2.9) becomes

$$
\begin{aligned}
v(x) - g(x) &= \sup_{\xi \in [0,1]} \Bigg(\sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[1_{\{\tau < \tau_0\}} e^{-\alpha \tau} \left(g_1(X_{\tau-} - K; \xi) - (P + g_1(0;\xi)) e^{\lambda(\xi)(X_{\tau-} - K)} - g(X_{\tau-}) \right) \right] \\
&\qquad + \mathbb{E}^x \left[1_{\{\tau > \tau_0\}} e^{-\alpha \tau_0} (-P - g(0)) \right] \Bigg) \\
&= \sup_{\xi \in [0,1]} \Bigg(\sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[1_{\{\tau < \tau_0\}} e^{-\alpha \tau} \left\{ \frac{1}{\alpha} \left(-K - \frac{\eta \xi a}{\alpha} \right) - \left(P + \frac{\mu_1}{\alpha^2} \right) e^{\lambda(\xi)(X_{\tau-} - K)} \right\} \right] \\
&\qquad + \mathbb{E}^x \left[1_{\{\tau > \tau_0\}} e^{-\alpha \tau_0} \left(-P - \frac{\theta a}{\alpha^2} \right) \right] \Bigg),
\end{aligned}
\tag{2.11}
$$

in which g_1 is given in (A.12). We denote the value function of the inner optimization for a given ξ by $U(x;\xi)$. We have to show the existence of a finite solution to the optimal stopping problem. For this purpose, we employ the characterization of the value function by Dynkin [7] and Dayanik and Karatzas [6] (which we explain below).

To this end, note that

$$
\begin{aligned}
h(x;\xi) = h(x;\xi,\Delta = 0) &= \frac{1}{\alpha} \left(-K - \frac{\eta \xi a}{\alpha} \right) - \left(P + \frac{\mu_1}{\alpha^2} \right) e^{\lambda(\xi)(x-K)} \\
&\triangleq A(\xi) - B(\xi) e^{\lambda(\xi)(x-K)}.
\end{aligned}
$$

Later in this section, we will write h_Δ when $\Delta > 0$ to distinguish it from h here when $\Delta = 0$.

Consider the infinitesimal generator $\mathcal{A}$ of X : $\mathcal{A}u(x) \triangleq (\sigma^2/2)u''(x) + \theta a u'(x)$ acting on a smooth function $u(\cdot)$. The (so-called fundamental) solutions of the ODE $(\mathcal{A} - \alpha)u(x) = 0$ are given by

$$
\psi(x) \triangleq e^{\gamma x} \quad \text{and} \quad \varphi(x) \triangleq e^{\rho x}, \tag{2.12}
$$

with

$$
\gamma = \frac{-\theta a + \sqrt{(\theta a)^2 + 2\sigma^2 \alpha}}{\sigma^2} > 0 \quad \text{and} \quad \rho = \frac{-\theta a - \sqrt{(\theta a)^2 + 2\sigma^2 \alpha}}{\sigma^2} < 0.
$$

Define the increasing function F by $F(x) \triangleq \psi(x)/\varphi(x)$. By the characterization of the value function in [6] (Propositions 4.3), if we find the smallest concave function $W(y;\xi)$ that passes through the point[1] $\left(F(0), -\frac{(P+\theta a/\alpha^2)}{\varphi(0)} \right) = \left(1, -\left(P + \theta a/\alpha^2 \right) \right)$ and majorizes $H(y;\xi)$ on $[F(0), \infty]$, where $y \triangleq F(x)$ and

$$
H(y;\xi) \triangleq h(F^{-1}(y);\xi)/\varphi(F^{-1}(y)), \tag{2.13}
$$

then the value function $U(x;\xi)$ is given by $\varphi(x)W(F(x);\xi)$. Moreover, from Proposition 4.4 in [6], the optimal stopping rule is given by $\tau^* = \inf\{t \geq 0 : X_t \in \Gamma\}$, in which $\Gamma \triangleq \{x \in \mathbb{R}_+ : U(x;\xi) = h(x;\xi)\}$.

[1] Recall that the ruin state 0 is transformed to $F(0)$.

The procedure is, therefore, (1) transforming h by (2.13) to H and identifying the smallest concave majorant of H, (2) finding the points where H and W meet to obtain the optimal boundaries in the transformed space, and (3) transforming back to recover the value function U in the original space. Hence, to show the existence of a finite value function U, we just need to show the existence of a finite concave majorant W by examining H. The major task now reduces to analyzing the function H in the transformed space. The analysis of the behavior of H is facilitated by the following observations: For $y = F(x)$, we have

$$H'(y;\xi) = \frac{1}{F'(x)}\left(\frac{h(x;\xi)}{\varphi(x)}\right)', \quad \text{and} \quad H''(y;\xi)[(\mathcal{A}-\alpha)h(x;\xi)] \geq 0, \tag{2.14}$$

with strict inequality if $H''(y;\xi) \neq 0$. The inequality in (2.14) is useful in identifying the concavity of H.

We next investigate the behavior of H.

(i) We first study $H(y;\xi)$ in a neighborhood of $y = F(0) = 1$. Assume that P is such that $P + \mu_1/\alpha^2 > 0$ (note that μ_1 might be negative) to ensure the following inequality holds: As $x \downarrow 0$,

$$\begin{aligned}
-\left(P + \frac{\theta a}{\alpha^2}\right) - h(0;\xi) &= \frac{1}{\alpha}\left(K + \frac{\eta\xi a}{\alpha}\right) + \left(P + \frac{\mu_1}{\alpha^2}\right)e^{-\lambda(\xi)K} - \left(P + \frac{\theta a}{\alpha^2}\right) \\
&> \frac{K}{\alpha} > 0
\end{aligned} \tag{2.15}$$

since $e^{-\lambda(\xi)K} > 1$. This means that the intercept at $F(0) = \varphi(0) = 1$, namely $-\left(P + \theta a/\alpha^2\right)$, is greater than $H(F(0);\xi)$; that is, $W(y;\xi) > H(y;\xi)$ in a neighborhood of $F(0) = 1$. See Remark 2.2(b) below if $W(y;\xi) > H(y;\xi)$ does not hold in a neighborhood of $y = F(0) = 1$.

(ii) Next, note that $\lim_{x\to\infty} h(x;\xi) = -\frac{1}{\alpha}\left(K + \frac{\eta\xi a}{\alpha}\right) < 0$, and $\lim_{x\to\infty} h'(x;\xi) = 0$ for all $\xi \in [0,1]$. It follows from the first expression in (2.14) and direct calculation, that H' changes sign at most once (from $+$ to $-$) and $\lim_{y\to\infty} H'(y;\xi) = -\infty < 0$. If H' does not change sign, then $H'(y;\xi) < 0$ for all $y \geq 1$. Additionally, we compute

$$(\mathcal{A}-\alpha)h(x;\xi) = -\left(P + \frac{\mu_1}{\alpha^2}\right)e^{\lambda(\xi)(x-K)}\left(\frac{\sigma^2}{2}\lambda^2(\xi) + \theta a\lambda(\xi) - \alpha\right) + \left(K + \frac{\eta\xi a}{\alpha}\right).$$

By combining this expression with the second one in (2.14), we conclude that H'' changes sign at most once and $\lim_{y\to\infty} H''(y;\xi) > 0$.

Hence, for a concave majorant of H to exist, H' has to change sign. Assuming H' changes sign (if it does, it does just once from $+$ to $-$), the next task is to find the smallest concave function that majorizes H. From the facts we gathered, it follows that the smallest concave majorant W is described as follows:

(1) For $y \in [F(0), F(b))$, W is the linear function (call it $W_1(y;\xi)$) that intersects $\left(F(0), -\frac{P+\theta a/\alpha^2}{\varphi(0)}\right) = \left(1, -(P + \theta a/\alpha^2)\right)$ and is tangent to $H(y;\xi)$ at $y = F(b)$.

(2) For $y \in [F(b), F(d)]$, $W(y;\xi) = H(y;\xi)$.

(3) For $y \in (F(d), \infty)$, W is the horizontal line, with value, say, $\delta(\xi)$ equal to the global maximum of $H(y;\xi)$.

See the graph (b) of Figure 1 for an example. When this majorant exists, the optimal stopping rule (see [6]) (Proposition 4.4) says that these two points b and d are the threshold values: In other words, the insurer should *not* buy reinsurance when the surplus belongs to either $(0, b)$ or (d, ∞), and the insurer *should* buy reinsurance contracts when the surplus belongs to $[b, d]$. When the surplus is small, the insurance company waits until the surplus becomes large enough due to the existence of the fixed cost. (See Remark 2.2(a) below in the case for which $K = 0$.) On the contrary, if the surplus is large, the insurer can bear 100% of the risk since its surplus is far enough from the ruin state. Note that b (and hence d) is necessarily positive when (2.15) holds.

(iii) We now derive a necessary and sufficient condition for the existence of a linear majorant $W_1(y;\xi)$ with positive slope, say $\beta(\xi) > 0$. By direct calculation, the sole critical point of $H(F(x);\xi)$ is given by

$$\bar{x} \triangleq \frac{1}{\lambda(\xi)} \ln\left(\frac{A(\xi)\rho}{B(\xi)(\rho-\lambda(\xi))}\right) + K, \tag{2.16}$$

which exists and is greater than K when $0 < \frac{A(\xi)\rho}{B(\xi)(\rho-\lambda(\xi))} < 1$. Then, a linear majorant with positive slope exists if

$$H(F(\bar{x});\xi) = \frac{h(\bar{x};\xi)}{\varphi(\bar{x})} > -\left(P + \frac{\theta a}{\alpha^2}\right) = W(F(0);\xi),$$

which is equivalent to

$$\frac{1}{\alpha}\left(K + \frac{\eta\xi a}{\alpha}\right)\frac{\lambda(\xi)}{\rho-\lambda(\xi)} > -\left(P + \frac{\theta a}{\alpha^2}\right) e^{\rho\bar{x}}. \tag{2.17}$$

Suppose that (2.17) holds. We see that H is concave in a neighborhood of $\{y > 0 : H'(y;\xi) = 0\}$. Recall that we always have $\lim_{y\to\infty} H''(y;\xi) > 0$. Together with the fact that H' changes sign at most once, the specifications of $W_1(y;\xi)$ and $\delta(\xi)$ in (ii) above are also justified. Indeed, $H(y;\xi)$ is increasing and concave in y first and attains the global maximum at $y = F(\bar{x})$ and then becomes convex eventually.

When (2.17) holds, the smallest concave majorant of H on $[F(0), F(b))$ is necessarily a linear function with a positive slope that is tangent to H at $y = F(b)$ and equal to H itself on $[F(b), F(\bar{x})]$ until $y = F(\bar{x})$, after which it must be a horizontal line. Note that

$$d = \bar{x} \quad \text{and} \quad \delta(\xi) = H(F(\bar{x});\xi).$$

Remark 2.2. We make some comments in relation to the above analysis:

(a) Note that if $K = 0$, then we have $\left(-P - \frac{\theta a}{\alpha^2}\right) - h(0;\xi) = 0$ from (2.15). That is, the linear majorant is tangent to $H(y;\xi)$ at the point $\left(F(0), -\frac{(P+\theta a/\alpha^2)}{\varphi(0)}\right)$. Hence in the case of no transaction costs or no delay, the continuation region is of the form (d, ∞).

(b) If the inequality (2.15) does not hold but (2.17) still holds, it follows that the smallest concave majorant W of H is described as follows: On $[F(0), F(d)]$, it is $H(y;\xi)$ itself and on $(F(d), \infty)$, it is the horizontal line $W_2(y;\xi) = \delta(\xi)$. Hence, in this case, if the initial surplus is less than d, it is optimal to buy reinsurance immediately. But, in general, it is considered that the penalty at ruin should be sufficiently high for risk management purposes. Therefore, for the subsequent argument, we assume that (2.15) holds.

(c) Moreover, due to the behavior of H as determined in items (i) and (ii), (2.17) is also a necessary condition for the existence of an optimal stopping time. Indeed, if (2.17) does not hold, the smallest concave majorant is the horizontal line starting at $\left(F(0), -\frac{P+\theta a/\alpha^2}{\varphi(0)}\right)$. Then, the set $\{y : W(y) = H(y;\xi)\}$ is empty. It follows that there is no optimal stopping time; see Proposition 4.4 in [6].

We summarize our work up to this point in the following lemma for the inner optimization of (2.11):

Lemma 2.1. *The following optimal stopping problem, given* $\xi \in [0,1]$,

$$U(x;\xi) \triangleq \sup_{\tau\in\mathcal{S}} \mathbb{E}^x \left[1_{\{\tau<\tau_0\}} e^{-\alpha\tau} \left\{\frac{1}{\alpha}\left(-K - \frac{\eta\xi a}{\alpha}\right) - \left(P + \frac{(\theta-\eta\xi)a}{\alpha^2}\right) e^{\lambda(\xi)(X_{\tau-}-K)}\right\}\right] + \mathbb{E}^x\left[1_{\{\tau>\tau_0\}} e^{-\alpha\tau_0}\left(-P - \frac{\theta a}{\alpha^2}\right)\right] \tag{2.18}$$

has a solution with $\tau^*(\xi) \triangleq \inf\{t \geq 0 : X_t \notin (0, b^*(\xi)) \cup (d^*(\xi), \infty)\}$, *for some constants* b^* *and* d^* *dependent on* ξ, *if and only if the parameters* $(\alpha, \theta, \eta, a, K, P)$ *and* $\xi \in [0,1]$ *satisfy* (2.17) *with* $\bar{x}$ *given by* (2.16). *In particular, if* $K = 0$, *we have* $b^*(\xi) = 0$ *for any* $\xi \in [0,1]$ *that satisfies condition* (2.17).

From this lemma, we can specify the value function.

Corollary 2.1. *Let* $\psi(x)$ *and* $\varphi(x)$ *be defined in* (2.12), *and assume that* (2.17) *holds.*

(i) *The value function* $U(x;\xi)$ *on* $(0, b^*(\xi))$ *is increasing in* x *and is of the form*

$$U(x;\xi) = \beta(\xi)(\psi(x) - \varphi(x)) - \left(P + \theta a/\alpha^2\right)\varphi(x), \tag{2.19}$$

with $\beta(\xi) > 0$.

(ii) *The value function* $U(x;\xi)$ *on* $(d^*(\xi), \infty)$ *is of the form* $U(x;\xi) = \delta(\xi)\varphi(x)$, *for some* $\delta(\xi) \in \mathbb{R}$.

Proof. (i) By following the argument for Lemma 2.1 and the preceding discussion, the smallest concave majorant is a linear function with slope $\beta(\xi) > 0$ and is of the form:

$$W_1(F(x);\xi) = \beta(\xi)(F(x) - F(0)) - \frac{\left(P + \theta a/\alpha^2\right)}{\varphi(0)}.$$

By transforming back to the original space via $U(x;\xi) = \varphi(x)W(F(x);\xi)$ and by noting that $F(0) = \varphi(0) = 1$, we get the desired result. The function is increasing in x due to the positivity of β.

(ii) Similarly, the horizontal line can be described by $W_2(y;\xi) = \delta(\xi) = H(F(\bar{x});\xi)$. In the original space, by multiplying by $\varphi(x)$, we get the result. □

In the case of $\Delta > 0$, (2.9) becomes

$$v^{\Delta}(x) - g(x) = \sup_{\xi \in [0,1]} \left(\sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[1_{\{\tau < \tau_0\}} e^{-\alpha \tau} h_{\Delta}(X_\tau; \xi) \right] + \mathbb{E}^x \left[1_{\{\tau > \tau_0\}} e^{-\alpha \tau_0} \left(-P - \frac{\theta a}{\alpha^2} \right) \right] \right), \tag{2.20}$$

for which the analysis is similar to that of (2.18). In this case, we denote the value function by v^{Δ} to distinguish from the case of $\Delta = 0$. Here h_{Δ} is given by

$$h_{\Delta}(x;\xi) \triangleq h(x;\xi,\Delta > 0) = I_1(x;\xi) - I_2(x) + I_3(x) + I_4(x) \tag{2.21}$$

where the specific forms of $I_1(\cdot)$, $I_2(\cdot)$, $I_3(\cdot)$ and $I_4(\cdot)$ are given in (A.14), (A.15), (A.16), and (A.17). Due to the delay, note that given $\xi \in [0,1]$,

$$h_{\Delta}(x;\xi) < h(x;\xi) \quad \text{for} \quad x \in \mathbb{R}_+. \tag{2.22}$$

If we define $D(x;\xi) \triangleq h(x;\xi) - h_{\Delta}(x;\xi)$, then $D(x;\xi) \downarrow 0$ monotonically as $x \uparrow \infty$. This result can be checked directly, but a simpler argument is that the probability that the surplus process hits zero during the delay period monotonically decreases to zero as x increases. In other words, $h_{\Delta}(x;\xi)$ behaves like $h(x;\xi)$ as $x \to \infty$. In fact, we can directly verify $\lim_{x\to\infty} h_{\Delta}(x;\xi) = -\alpha \left(K + \frac{\eta \xi a}{\alpha} + \eta \xi a \Delta \right) < 0$, $\lim_{x\to\infty} h'_{\Delta}(x;\xi) = 0$ and $\lim_{y\to\infty} H''_{\Delta}(y;\xi) > 0$ for all $\xi \in [0,1]$ in which H_{Δ} is defined as in (2.13) with h replaced by h_{Δ}. It follows from the first equality in (2.14) that $H'_{\Delta}(y;\xi)$ changes sign at most once (from $+$ to $-$), $\lim_{y\to\infty} H'_{\Delta}(y;\xi) = -\infty < 0$, and H_{Δ} becomes convex eventually.

Now, we can proceed with the same argument as in the case of $\Delta = 0$. The value function along with a continuation region of the form $(0, b^{\Delta}) \cup (d^{\Delta}, \infty)$ exists if and only if $h_{\Delta}(\bar{x};\xi) > -(P + \frac{\theta a}{\alpha^2}) e^{\rho \bar{x}}$ holds, in which $\bar{x}$ is the root of $h'_{\Delta}(x;\xi)\varphi(x) - h_{\Delta}(x;\xi)\varphi'(x) = 0$. If this condition is met, then $H_{\Delta}(y;\xi)$ is concave in a neighborhood of $\{y > 0 : H'_{\Delta}(y;\xi) = 0\}$. We have thereby solved the first-stage of the optimization in (2.9) for any $\xi \in [0,1]$.

Due to the described characterization of the value function in the transformed space, the second-stage optimization in (2.9) can be solved by finding ξ that maximizes the parameterized (by ξ) value function, namely by maximizing the slope, $\beta(\xi)$ of the linear majorant and the horizontal line $\delta(\xi)$. This stage can be easily implemented numerically. Note that since the dependence of $U(x;\xi)$ on $\xi \in [0,1]$ is rather complicated, there is no guarantee that ξ that maximizes $\beta(\xi)$ necessarily simultaneously maximizes $\delta(\xi)$. To choose an optimal ξ, the insurance company could compute the two value functions corresponding to the two ξ's. Then, depending on the initial surplus level $X_0 = x$, the insurer could choose the ξ that provides the higher value for that given x. In this paper, hereafter, for the sake of simplicity of the argument, we suppose that the insurance company wishes to maximize the slope $\beta(\xi)$ since it is more concerned with reinsurance policies to avoid ruin.

In summary, for the case of $\Delta > 0$, we have the following proposition. Note that we denote $b^{\Delta}(\xi)$ as the optimal threshold level for a given ξ to distinguish it from b^{Δ}, the overall optimal level among all the possible ξ's, and similarly for $d^{\Delta}(\xi)$ and d^{Δ}.

Proposition 2.1. *1. The following optimal stopping problem for a given $\xi \in [0,1]$,*

$$U^{\Delta}(x;\xi) \triangleq \sup_{\tau \in \mathcal{S}} \mathbb{E}^{x}\left[1_{\{\tau<\tau_0\}} e^{-\alpha\tau} h_{\Delta}(X_\tau;\xi) + 1_{\{\tau>\tau_0\}} e^{-\alpha\tau_0}\left(-P - \frac{\theta a}{\alpha^2}\right)\right]$$

has a solution $\tau^{\Delta}(\xi) \triangleq \inf\{t \geq 0 : X_t \notin (0, b^{\Delta}(\xi)) \cup (d^{\Delta}(\xi), \infty)\}$, for some constants $b^{\Delta}(\xi)$ and $d^{\Delta}(\xi)$, if and only if the parameters $(\alpha, \theta, \eta, a, K, P)$ and $\xi \in [0,1]$ satisfy $h_{\Delta}(\bar{x};\xi) > -(P + \frac{\theta a}{\alpha^2})e^{\rho\bar{x}}$, with $\bar{x}$ being the unique solution of $h'_{\Delta}(x;\xi)\varphi(x) = h_{\Delta}(x;\xi)\varphi'(x)$.

2. The value function $v^{\Delta}(x) \triangleq \sup_{\pi\in\Pi} \mathbb{E}^{x}[\int_0^{\tau_0} e^{-\alpha s} X_s ds - e^{-\alpha\tau_0}P]$ is of the form:

$$v^{\Delta}(x) = \begin{cases} -P, & x = 0, \\ \beta(\xi^{\Delta})(\psi(x) - \varphi(x)) - \left(P + \frac{\theta a}{\alpha^2}\right)\varphi(x) + \left(\frac{x}{\alpha} + \frac{\theta a}{\alpha^2}\right), & 0 < x < b^{\Delta}, \\ h_{\Delta}(x;\xi^{\Delta}) + \left(\frac{x}{\alpha} + \frac{\theta a}{\alpha^2}\right), & b^{\Delta} \leq x \leq d^{\Delta}, \\ \delta(\xi^{\Delta})\varphi(x) + \left(\frac{x}{\alpha} + \frac{\theta a}{\alpha^2}\right), & d^{\Delta} < x, \end{cases} \tag{2.23}$$

where ξ^{Δ} maximizes $\beta(\xi)$ over all the possible values of $\xi \in [0,1]$ that satisfy $h_{\Delta}(\bar{x};\xi) > -(P + \frac{\theta a}{\alpha^2})e^{\rho\bar{x}}$. The optimal time to buy reinsurance is given by $\tau^{\Delta} = \inf\{t \geq 0 : X_t \notin (0, b^{\Delta}) \cup (d^{\Delta}, \infty)\}$.

Remark 2.3. Owing to inequality (2.22) and the fact $h(0;\xi) = -\left(P + \frac{\theta a}{\alpha^2}\right)$ when $K = 0$, we have

$$\lim_{x\to 0} h_{\Delta}(x;\xi) < -\left(P + \frac{\theta a}{\alpha^2}\right),$$

with $K = 0$ (no transaction cost). This implies that in the presence of a delay period, there exists for any $\xi \in [0,1]$, a continuation region of the form: $(0, b^{\Delta}) \cup (d^{\Delta}, \infty)$, for some constants $d^{\Delta} \geq b^{\Delta} > 0$. The positivity of b^{Δ} follows by the same argument in the paragraph between equations (2.15) and (2.16). We point out the contrast to the last statement in Lemma 2.1 for the no-delay case.

2.2.2 A numerical example

Figure 1 shows a numerical experiment with parameters $(a, \sigma, \theta, \eta, P, \alpha, K) = (0.2, 0.3, 0.1, 0.25, 20, 0.1, 0.03)$ and $\Delta = 0$. Note that condition (2.17) is satisfied with $\xi > 0.25$. The first graph shows the slopes $\beta(\xi)$ for various ξ and indicates that the slope is maximized by $\xi^* = 0.815$ and that the corresponding slope is $\beta^* \triangleq \beta(\xi^*) = 2.977$. (Note that the horizontal line is maximized by $\xi = 0.810$.) The second graph shows the concave majorant $W(y;\xi^*)$ with $b^* = 0.201$ and $d^* = 0.448$.

Figure 2 shows the corresponding results when $\Delta > 0$. With a delay of $\Delta = 0.5$, the solution changes to $(\xi^{\Delta}, \beta^{\Delta}, b^{\Delta}, d^{\Delta}) = (0.770, 0.823, 0.461, 0.565)$, in which $\beta^{\Delta} \triangleq \beta(\xi^{\Delta})$. Note that $b^* < b^{\Delta}$, $d^* < d^{\Delta}$, and $\beta^* > \beta^{\Delta}$. The reinsurance proportion drops from $\xi = 0.815$ to $\xi^{\Delta} = 0.770$. The smaller slope with delay is expected. The interval $[b^{\Delta}, d^{\Delta}]$ shifts to the right with delay due to the positive probability of ruin during the delay period. That is, the insurer becomes more cautious in the presence of delay.

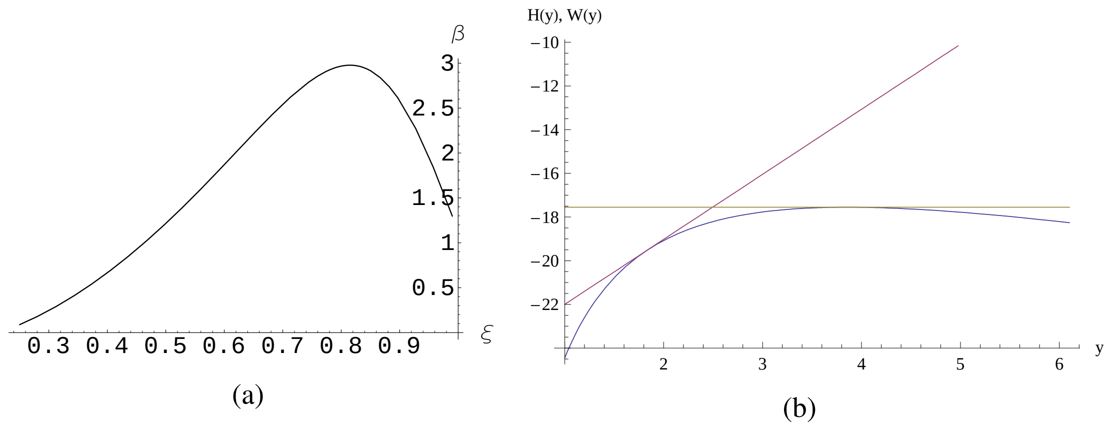

Figure 1: Numerical example with parameters $(a, \sigma, \theta, \eta, P, \alpha, K) = (0.2, 0.3, 0.1, 0.25, 20, 0.1, 0.03)$ and $\Delta = 0$: (a) $\beta(\xi)$ for various $\xi \in [0, 1]$. (b) The linear function with positive slope $W_1(y; \xi^*)$ (red line), $H(y; \xi^*)$ itself (blue line), and the horizontal line $W_2(y; \xi) = \delta(\xi^*)$ (yellow line) with the optimal ξ^*.

As is reflected by $\beta^{\Delta} < \beta^*$, the value function without delay $v(x)$ is greater than the value function with delay, denoted by $vD(x)$ in the graph. As x becomes larger, the probability of ruin becomes negligible. For this reason, the two value functions become indistinguishable (see graphs (e) and (f)).

3 Discussions and concluding remarks

Before concluding this paper, we perform a sensitivity analysis in the length of the delay period and briefly comment on possible extensions of this problem.

3.1 Sensitivity analysis

We change the length of delay period Δ while keeping the other parameters as in the original example. The table below shows the optimal threshold values and proportions reinsured for different values of the delay period. The phenomenon of the rightward shift of $[b^{\Delta}, d^{\Delta}]$ is consistently observed here, too. Namely, the longer the delay period, the interval gets further from the ruin state.

As expected, the longer the delay period, the smaller the slope. Geometrically this occurs because the global maximum of H_{Δ} (due to the increased chance of hitting the ruin state) decreases, while the vertical intercept $\left(F(0), -\frac{P+\theta a/\alpha^2}{\varphi(0)}\right)$ is fixed.

More interestingly, the last column $d^{\Delta} - b^{\Delta}$ shows that the longer delay period results in an interval of smaller length. Since the slope flattens as the delay period increases, at a certain delay level, say Δ^*, we shall have $\beta^{\Delta^*} = 0$ with $b^{\Delta^*} = d^{\Delta^*}$. At this level, the action (reinsurance) region is the singleton $\{b^{\Delta^*}(= d^{\Delta^*})\}$ and the continuation region is $(0, b^{\Delta^*}) \cup (b^{\Delta^*}, \infty)$.

Now, if we further increase the delay period beyond Δ^*, then the slope β^{Δ} becomes negative. In other

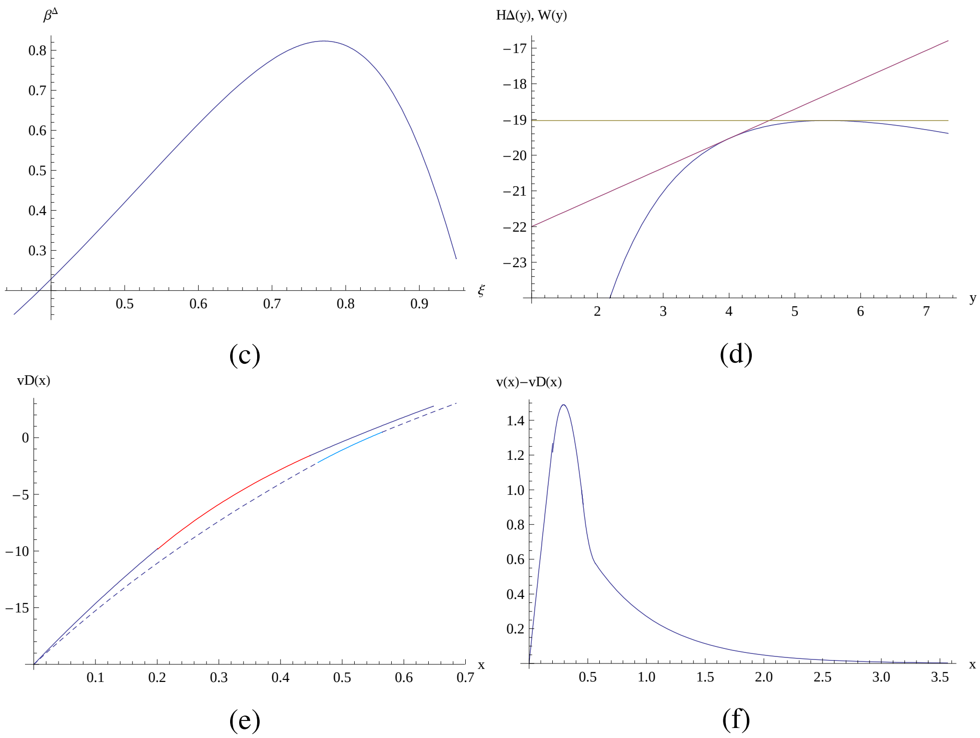

Figure 2: Numerical example with parameters $(a, \sigma, \theta, \eta, P, \alpha, K) = (0.2, 0.3, 0.1, 0.25, 20, 0.1, 0.03)$ and $\Delta = 0.5$: (c) $\beta^{\Delta}(\xi)$ for various $\xi \in [0, 1]$. (d) The linear function with positive slope $W_1(y; \xi)$ (red line), $H(y; \xi)$ itself (blue line), and the horizontal line $W_2(y; \xi) = \delta(\xi)$ (yellow line) with the optimal ξ^{Δ}. (e) The value functions $v(x)$ (above) and $vD(x)$ (below), without and with delay, respectively. (f) Plot of the difference, $v(x) - vD(x)$ whose value converges to zero, as expected, when x gets larger.

words, the condition (2.17) is now violated. But, we know that $H_{\Delta}(y; \xi)$ becomes eventually convex for all $\xi \in [0, 1]$. It follows that the smallest concave majorant is just the horizontal line starting at $\left(F(0), -\frac{P+\theta a/\alpha^2}{\varphi(0)}\right)$, and no optimal stopping time exists. See Remark 2.2(c). No purchase of reinsurance makes sense in this case because the delay period is too long to risk the surplus hitting the ruin during that period.

3.2 Multiple-step analysis

As an extension to the work in this paper, we briefly consider a multiple-step problem: Namely, an admissible strategy is a double sequence,

$$\pi = (\tau_1, \tau_2, \ldots, \tau_i, \ldots; \xi_1, \xi_2, \ldots, \xi_i, \ldots),$$

in which $0 \le \tau_1 < \tau_2 < \ldots$ is an increasing sequence of $\mathcal{F}$-stopping times such that $\tau_{i+1} - \tau_i \ge \Delta$, and $\xi_1, \xi_2, \ldots$ are $\mathcal{F}_{\tau_i}$-measurable random variables representing the proportion reinsured at time $\tau_i + \Delta$. The proportion $\xi_i \in [0, 1]$ is determined at time τ_i and implemented at time $\tau_i + \Delta$ due to the existence of a delay period. Then, Assumption 2.1 (a) and (c) become

Table 1: Reinsurance proportion ξ^{Δ} and thresholds b^{Δ} and d^{Δ} for various delay times Δ.

Δ	ξ^{Δ}	β^{Δ}	b^{Δ}	d^{Δ}	$d^{\Delta}-b^{\Delta}$
0	0.815	2.977	0.201	0.448	0.247
0.1	0.807	2.350	0.257	0.466	0.209
0.15	0.802	2.043	0.292	0.475	0.183
0.25	0.793	1.552	0.352	0.497	0.145
0.375	0.781	1.124	0.411	0.531	0.120
0.5	0.770	0.823	0.461	0.565	0.104
0.6	0.761	0.642	0.496	0.589	0.093
0.75	0.747	0.436	0.545	0.622	0.077

(a)′ At the stopping time τ_i, the insurer begins negotiating with the reinsurer. This negotiating takes a fixed amount of time $\Delta \geq 0$. After the time Δ elapses, if the surplus process has not hit the ruin level, the insurer pays a fixed transaction cost $K > 0$ and reinsures a proportion $\xi_i \in [0, 1]$ of its claims at time $\tau_i + \Delta$. Hence the surplus process X follows

$$\begin{cases} dX_t = \mu_{i-1}dt + \sigma_{i-1}dW_t, & \tau_{i-1} + \Delta \leq t < \tau_i + \Delta, \\ X_{\tau_i+\Delta} = X_{(\tau_i+\Delta)-} - K, \end{cases} \tag{3.1}$$

for $i = 1, 2, \ldots$, where $\mu_0 = \theta a$, $\sigma_0 = \sigma$,

$$\mu_i = (\theta - \eta\xi_i)a, \quad \text{and} \quad \sigma_i = \sigma(1 - \xi_i),$$

with $\xi_i \in [0, 1]$ for $i = 1, 2, \ldots$, and $\xi_0 = 0$.

(c)′ At time $\tau_i + \Delta$, if $X_{(\tau_i+\Delta)-} \leq K$, the surplus process hits the ruin state at time $\tau_i + \Delta$, and the insurer becomes insolvent.

Let ξ_i^{Δ} denote the optimal proportion reinsured at step i of the multiple-step reinsurance problem for $i = 1, 2, \ldots$. At step i, we assume that the insurer buys reinsurance only if its surplus lies in an interval $[b_i, d_i]$. Then, we solve the optimal stopping problem (which we shall define below) recursively by using the two sets of drift and volatility parameters: More precisely, in (A.14), (A.15), (A.16), and (A.17), we replace μ_0, μ_1, and σ by $\mu(\xi) := (\theta - \eta\xi)a$, $\mu(y) := (\theta - \eta y)a$, and $\sigma(\xi) = \sigma(1 - \xi)$, respectively. In other words, the old fraction is denoted by ξ and the new fraction is denoted by y in each iteration.

In the next lemma, under this assumption for the form of the reinsurance strategy, we show that the sequence $\left(\xi_i^{\Delta}\right)$ converges to a limit ξ^{Δ}.

Lemma 3.1. *The mapping $T : [0, 1] \to [0, 1]$ has a fixed point ξ: $T(\xi) = \xi$, in which*

$$T(\xi) \triangleq \sup_{y\in[0,1]} \left(\sup_{\tau\in\mathcal{S}} \mathbb{E}^x \left[1_{\{\tau<\tau_0\}} e^{-\alpha\tau} h_{\Delta}(X_\tau; \xi, y) + 1_{\{\tau>\tau_0\}} e^{-\alpha\tau_0} (-P - g(0; \xi)) \right] \right) \tag{3.2}$$

with

$$h_{\Delta}(x;\xi,y) = e^{-\alpha\Delta}\{I_1(x;\xi,y) - I_2(x;\xi,y) + I_3(x;\xi,y) + I_4(x;\xi,y)\}.$$

$I_1, I_2, I_3,$ *and* I_4 *are defined in* $(A.14), (A.15), (A.16),$ *and* $(A.17)$ *with* ξ_0 *and* ξ_1 *replaced by* ξ *and* y*, respectively.* X *satisfies the stochastic differential equation*

$$\begin{cases} dX_t = (\theta - \eta\xi)dt + \sigma(1-\xi)dW_t, & 0 \le t < \tau + \Delta, \\ X_{\tau+\Delta} = X_{(\tau+\Delta)-} - K, \end{cases}$$

with initial value $X_0 = x$.

Proof. See Appendix A.3. □

Note that probabilistically, $T(\xi) = \xi$ is attained when the condition (2.17) is violated. In fact, the parameter set used in the example in Section 2.2.2 shows that, with and without delay, the first ξ_1^{Δ} is optimal in this multiple-step problem and, hence, $\tau_2 = \infty$. The following is an open problem: Under what conditions does the optimal $\tau_2 = \infty$?

See Carmona and Touzi [4] for an example of a mutiple-optimal stopping problem in the setting of so-called swing options. They prove the existence of a solution of their problem, for which they are allowed a finite number of exercise times. Note that in our problem, one can implement reinsurance arbitrarily many times. Another difference between their work and ours is in the timing of the rewards.

3.3 Concluding Remarks

In this paper, we explicitly incorporated fixed costs and time delay into the optimal reinsurance problem. We identified that the optimal stopping problem has a two-sided continuation region. We summarize our findings:

(a) Without any fixed cost or delay, it is optimal to buy reinsurance when the surplus lies in an interval of the form $(0, d]$.

(b) In the presence of a fixed cost but no delay, it is optimal to buy reinsurance when the surplus lies in an interval of the form $[b, d]$.

(c) In the presence of a fixed cost and a small enough delay, it is optimal to buy reinsurance when the surplus lies in an interval of the form $[b^{\Delta}, d^{\Delta}]$. The continuation region is $(0, b^{\Delta}) \cup (d^{\Delta}, \infty)$. When the delay period is large enough, say Δ^*, the reinsurance region is only a singleton set (the maximizer of H_{Δ}). When $\Delta > \Delta^*$, it is optimal not to purchase reinsurance.

Recall that we assumed if $X_{(\tau_i+\Delta)-} \le K$, then the surplus process hits the ruin state at time $\tau_i + \Delta$ and the insurer becomes insolvent (Assumption 2.1(c)). Another possibility is that, if this happens, the insurer does not have to fulfill its obligation of buying a reinsurance but could restart its business with surplus $X_{\tau_i+\Delta}$. The problem becomes more difficult to solve since one loses tractability. But, based on our results, we expect that the reinsurance threshold will decrease in this case compared with the one in our current model because the ruin probability decreases. Other extensions include (1) allowing the insurance company to invest its surplus in a risky asset and (2) maximizing some utility function other than the surplus itself (i.e., replace $f(x) = x$ with a different utility function.)

A Proofs and Derivations

A.1 Derivation of Expression (2.9)

We simplify J^π by splitting the terms in (2.4). We can write the first term as

$$\begin{aligned}
&\mathbb{E}^x\left[\int_0^{\tau_0} e^{-\alpha s} f(X_s)ds\right] \\
&\qquad = \mathbb{E}^x\left[1_{\{\tau+\Delta<\tau_0\}}\left\{\int_0^{\tau+\Delta} e^{-\alpha s} f(X_s^0)ds + e^{-\alpha(\tau+\Delta)}\mathbb{E}_\xi^{X_{\tau+\Delta}}\int_0^{\tau_0} e^{-\alpha s} f(X_s)ds\right\}\right] \\
&\qquad\qquad + \mathbb{E}^x\left[1_{\{\tau+\Delta>\tau_0\}}\int_0^{\tau_0} e^{-\alpha s} f(X_s)ds\right] \\
&\qquad = \mathbb{E}^x\left[1_{\{\tau+\Delta<\tau_0\}}e^{-\alpha(\tau+\Delta)}\left\{\mathbb{E}_\xi^{X_{\tau+\Delta}}\int_0^{\tau_0} e^{-\alpha s} f(X_s)ds - g(X^0_{\tau+\Delta})\right\}\right] \\
&\qquad\qquad - \mathbb{E}^x\left[1_{\{\tau+\Delta>\tau_0\}}e^{-\alpha\tau_0}g(X_{\tau_0})\right] + g(x) \\
&\qquad = \mathbb{E}^x\left[1_{\{\tau+\Delta<\tau_0\}}e^{-\alpha(\tau+\Delta)}\left\{\mathbb{E}_\xi^{X_{\tau+\Delta}}\int_0^{\tau_0} e^{-\alpha s} f(X_s)ds - g(X_{(\tau+\Delta)-})\right\}\right] \\
&\qquad\qquad - \mathbb{E}^x\left[1_{\{\tau+\Delta>\tau_0\}}e^{-\alpha\tau_0}g(X_{\tau_0})\right] + g(x),
\end{aligned}$$

where we use $\mathbb{E}_\xi^y[\cdot]$ to stress that the insurer has reinsured a proportion ξ and the process starts with state y, so that the surplus process has dynamics with drift $(\theta-\eta\xi)a$ and volatility $\sigma(1-\xi)$. Note that X_s and τ mean different things in different parts of this expression; their meaning is clarified by the conditions on the corresponding expectations.

The second term in (2.4), namely the penalty term, can be developed as

$$\begin{aligned}
\mathbb{E}^x[e^{-\alpha\tau_0}P] &= \mathbb{E}^x\left[1_{\{\tau+\Delta<\tau_0\}}e^{-\alpha(\tau+\Delta)}\mathbb{E}^x[e^{-\alpha(\tau_0-(\tau+\Delta))}P|\mathcal{F}_{\tau+\Delta}]\right] + \mathbb{E}^x\left[1_{\{\tau+\Delta>\tau_0\}}e^{-\alpha\tau_0}P\right] \\
&= \mathbb{E}^x\left[1_{\{\tau+\Delta<\tau_0\}}e^{-\alpha(\tau+\Delta)}\mathbb{E}^x[e^{-\alpha(\tau_0\circ s(\tau+\Delta))}P|\mathcal{F}_{\tau+\Delta}]\right] + \mathbb{E}^x\left[1_{\{\tau+\Delta>\tau_0\}}e^{-\alpha\tau_0}P\right] \\
&= \mathbb{E}^x\left[1_{\{\tau+\Delta<\tau_0\}}e^{-\alpha(\tau+\Delta)}\mathbb{E}_\xi^{X_{\tau+\Delta}}[e^{-\alpha\tau_0}P]\right] + \mathbb{E}^x\left[1_{\{\tau+\Delta>\tau_0\}}e^{-\alpha\tau_0}P\right],
\end{aligned}$$

where $s(\cdot)$ is the shift operator (see Karatzas and Shreve [11]). By combining the two terms together, we have

$$\begin{aligned}
J^\pi(x) = \mathbb{E}^x&\Big[1_{\{\tau+\Delta<\tau_0\}}e^{-\alpha(\tau+\Delta)}\left(J_\xi^\pi(X_{\tau+\Delta}) - g(X_{(\tau+\Delta)-})\right)\Big] \\
&+ \mathbb{E}^x\left[1_{\{\tau+\Delta>\tau_0\}}e^{-\alpha\tau_0}\{-P-g(X_{\tau_0})\}\right] + g(x),
\end{aligned} \tag{A.1}$$

where J^{π}_{ξ} signifies that the surplus process now has new dynamics after the proportion of ξ is reinsured:

$$J^{\pi}_{\xi}(x) \triangleq \mathbb{E}^{x}_{\xi}\left[\int_{0}^{\tau_0} e^{-\alpha s} f(X_s)ds - e^{-\alpha\tau_0}P\right] \tag{A.2}$$

¿From Assumption 2.1(a), the post-transaction value of the surplus is $X_{\tau+\Delta} = X_{(\tau+\Delta)-} - K$.

By taking into account the positive probability that the surplus process hits the ruin state during the delay period Δ, we rewrite the expression in (A.1) as follows:

$$\begin{aligned}
&J^{\pi}(x) - g(x)\\
&= \mathbb{E}^{x}\left[1_{\{\tau+\Delta<\tau_0\}} e^{-\alpha(\tau+\Delta)}\left\{J^{\pi}_{\xi}(X_{\tau+\Delta}) - g(X_{(\tau+\Delta)-})\right\}\right] + \mathbb{E}^{x}\left[1_{\{\tau+\Delta>\tau_0\}} e^{-\alpha\tau_0}\{-P - g(X_{\tau_0})\}\right]\\
&= \mathbb{E}^{x}\left[\mathbb{E}^{x}\left[1_{\{\tau+\Delta<\tau_0\}} e^{-\alpha(\tau+\Delta)}\left\{J^{\pi}_{\xi}(X_{\tau+\Delta}) - g(X_{(\tau+\Delta)-})\right\} \mid \mathcal{F}_{\tau}\right]\right]\\
&\quad + \mathbb{E}^{x}\left[1_{\{\tau>\tau_0\}} e^{-\alpha\tau_0}\{-P - g(X_{\tau_0})\}\right]\\
&\quad + \mathbb{E}^{x}\left[\mathbb{E}^{x}\left[1_{\{\tau<\tau_0\}} 1_{\{\tau+\Delta\circ s(\tau)>\tau_0\}} e^{-\alpha\tau_0}\{-P - g(X_{\tau_0})\} \mid \mathcal{F}_{\tau}\right]\right]\\
&= \mathbb{E}^{x}\left[1_{\{\tau<\tau_0\}} e^{-\alpha\tau}\mathbb{E}^{X_\tau}\left[1_{\{\Delta<\tau_0\}} e^{-\alpha\Delta}\left\{J^{\pi}_{\xi}(X_{\Delta}) - g(X_{\Delta-})\right\} + 1_{\{\Delta>\tau_0\}} e^{-\alpha\tau_0}\{-P - g(X_{\tau_0})\}\right]\right]\\
&\quad + \mathbb{E}^{x}\left[1_{\{\tau>\tau_0\}} e^{-\alpha\tau_0}\{-P - g(X_{\tau_0})\}\right].
\end{aligned} \tag{A.3}$$

Let us concentrate on the inner expectation of the first term in (A.3). Recall that if $X_{(\tau+\Delta)-} \le K$, the surplus process hits the ruin state by Assumption 2.1(c). We further divide the inner expectation as follows:

$$\begin{aligned}
&\mathbb{E}^{X_\tau}\left[1_{\{\Delta<\tau_0\}} e^{-\alpha\Delta}\left\{J^{\pi}_{\xi}(X_{\Delta}) - g(X_{\Delta-})\right\} + 1_{\{\Delta>\tau_0\}} e^{-\alpha\tau_0}\{-P - g(X_{\tau_0})\}\right]\\
&= \mathbb{E}^{X_\tau}\Big[1_{\{\inf_{0\le u<\Delta} X_u>0\}} 1_{\{X_{\Delta-}>K\}} e^{-\alpha\Delta}\left\{J^{\pi}_{\xi}(X_{\Delta}) - g(X_{\Delta-})\right\}\\
&\qquad\qquad + 1_{\{\inf_{0\le u<\Delta} X_u>0\}} 1_{\{X_{\Delta-}\le K\}} e^{-\alpha\Delta}\{-P - g(0)\} + 1_{\{\inf_{0\le u<\Delta} X_u\le 0\}} e^{-\alpha\tau_0}\{-P - g(0)\}\Big]\\
&\triangleq I_1(X_\tau;\xi) - I_2(X_\tau) + I_3(X_\tau) + I_4(X_\tau),
\end{aligned} \tag{A.4}$$

where

$$\begin{aligned}
I_1(x;\xi) &= \mathbb{E}^{x}[1_{\{\inf_{0\le u<\Delta} X_u>0\}} 1_{\{X_{\Delta-}>K\}} e^{-\alpha\Delta} J^{\pi}_{\xi}(X_{\Delta})],\\
I_2(x) &= \mathbb{E}^{x}[1_{\{\inf_{0\le u<\Delta} X_u>0\}} 1_{\{X_{\Delta-}>K\}} e^{-\alpha\Delta} g(X_{\Delta-})],\\
I_3(x) &= \mathbb{E}^{x}[1_{\{\inf_{0\le u<\Delta} X_u>0\}} 1_{\{X_{\Delta-}\le K\}} e^{-\alpha\Delta}\{-P - g(0)\}],\\
I_4(x) &= \mathbb{E}^{x}[1_{\{\inf_{0\le u<\Delta} X_u\le 0\}} e^{-\alpha\tau_0}\{-P - g(0)\}].
\end{aligned}$$

To evaluate these expectations, we use the following well known result for a Brownian motion with drift ν and volatility σ (see, for example, Musiela and Rutkowski [12]):

$$\mathbb{P}^{x}\left(X^0_{\Delta} \ge z, \min_{0\le u\le\Delta} X^0_u \ge y\right) = N\left(\frac{x - z + \nu\Delta}{\sigma\sqrt{\Delta}}\right) - e^{2\nu(y-x)/\sigma^2} N\left(\frac{2y - x - z + \nu\Delta}{\sigma\sqrt{\Delta}}\right) \tag{A.5}$$

for $y \le x$ and $y \le z$, and $N(\cdot)$ is the cumulative distribution function of the standard normal random variable. We can calculate the joint density function $p(y,z)$ of $(\min_{0\le u\le t} X_u, X_t)$ from (A.5).

For I_1 and I_2 in (A.4), we need to calculate

$$\begin{aligned}
&\mathbb{E}^x\left[1_{\{\inf_{0\le u<\Delta} X_u>0\}}1_{\{X_{\Delta-}>K\}}h(X_{\Delta-})\right] = \int_K^\infty \int_0^x h(z)p(y,z)dydz \\
&= \frac{1}{\sigma\sqrt{\Delta}}\int_K^\infty h(z)\left[\phi\left(\frac{x-z+\nu\Delta}{\sigma\sqrt{\Delta}}\right) - e^{-2\nu x/\sigma^2}\phi\left(\frac{-x-z+\nu\Delta}{\sigma\sqrt{\Delta}}\right)\right]dz \\
&= \int_{-\infty}^{\frac{x+\nu\Delta-K}{\sigma\sqrt{\Delta}}} h(x+\nu\Delta-w\sigma\sqrt{\Delta})\phi(w)dw + e^{-2\nu x/\sigma^2}\int_{-\infty}^{\frac{-x+\nu\Delta-K}{\sigma\sqrt{\Delta}}} h(-x+\nu\Delta-w\sigma\sqrt{\Delta})\phi(w)dw,
\end{aligned} \tag{A.6}$$

in which $\phi(\cdot)$ is the probability density function of the standard normal random variable and h is any continuous function $h:\mathbb{R}_+\to\mathbb{R}$. For $I_3(\cdot)$ and $I_4(\cdot)$ in (A.4), we use

$$\mathbb{P}^x(\tau_0>t) = \mathbb{P}^x\left(\min_{0\le u\le t} X_u^0 \ge 0\right) = N\left(\frac{x+\nu t}{\sigma\sqrt{t}}\right) - e^{-2\nu x/\sigma^2}N\left(\frac{-x+\nu t}{\sigma\sqrt{t}}\right).$$

It follows that the expectation of the indicator function in $I_3(\cdot)$ is

$$\begin{aligned}
&\mathbb{P}^x\left(\inf_{0\le u<\Delta} X_u>0, X_{\Delta-}\le K\right) = \mathbb{P}^x(\inf_{0\le u<\Delta} X_u>0) - \mathbb{P}^x(\inf_{0\le u<\Delta} X_u>0, X_{\Delta-}>K) \\
&= N\left(\frac{x+\nu\Delta}{\sigma\sqrt{\Delta}}\right) - e^{-2\nu x/\sigma^2}N\left(\frac{-x+\nu\Delta}{\sigma\sqrt{\Delta}}\right) - N\left(\frac{x-K+\nu\Delta}{\sigma\sqrt{\Delta}}\right) + e^{-2\nu x/\sigma^2}N\left(\frac{-x-K+\nu\Delta}{\sigma\sqrt{\Delta}}\right),
\end{aligned} \tag{A.7}$$

and the Laplace transform of τ_0 in $I_4(\cdot)$ can be written as

$$\mathbb{E}^x[1_{\{\inf_{0\le u<\Delta} X_u\le 0\}}e^{-\alpha\tau_0}] = \int_0^\Delta e^{-\alpha t}\mathbb{P}^x(\tau_0\in dt). \tag{A.8}$$

With these preparations, we can evaluate the inner expectation of (A.3) explicitly.

In summary, the original problem of finding the value function $v(x)$ in (2.6) reduces to solving,

$$v(x)-g(x) = \sup_{\pi\in\Pi}\mathbb{E}^x\left[1_{\{\tau<\tau_0\}}e^{-\alpha\tau}h(X_\tau;\xi)\right] + \mathbb{E}^x\left[1_{\{\tau>\tau_0\}}e^{-\alpha\tau_0}\{-P-g(X_{\tau_0})\}\right] \tag{A.9}$$

where

$$\begin{aligned}
h(z;\xi) &\triangleq \mathbb{E}^z\left[1_{\{\Delta<\tau_0\}}e^{-\alpha\Delta}\left\{J_\xi^\pi(X_\Delta) - g(X_{\Delta-})\right\} + 1_{\{\Delta>\tau_0\}}e^{-\alpha\tau_0}\{-P-g(X_{\tau_0})\}\right] \\
&= I_1(z;\xi) - I_2(z) + I_3(z) + I_4(z),
\end{aligned} \tag{A.10}$$

which we can evaluate by using expressions (A.6), (A.7), and (A.8).

A.2 Preliminary computations for Section 2.2

With the specification of $f(x)=x$, we compute (2.9). For each $\xi\in[0,1]$, we first consider

$$J_\xi^\pi(x) = \mathbb{E}_\xi^x\left[\int_0^{\tau_0} e^{-\alpha s}f(X_s)ds - e^{-\alpha\tau_0}P\right] = g_1(x;\xi) - (P+g_1(0;\xi))\mathbb{E}_\xi^x[e^{-\alpha\tau_0}].$$

in which $g_1 : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ is defined by

$$g_1(x;\xi) \triangleq \mathbb{E}_\xi^x \left[\int_0^\infty e^{-\alpha s} f(X_s) ds \right].$$

Similar to $g(x)$, the function $g_1(x;\xi)$ denotes the expected total utility if the insurer starts with the reinsurance level ξ and does not change its reinsurance thereafter. The last expectation can be written

$$\mathbb{E}_\xi^x[e^{-\alpha \tau_0}] = B\varphi_\xi(x),$$

in which $\varphi_\xi(x)$ is the decreasing solution of

$$(\mathcal{A}_\xi - \alpha)v(x) \triangleq \frac{1}{2}\sigma^2(1-\xi)^2 v''(x) + (\theta - \eta\xi)a v'(x) - \alpha v(x) = 0,$$

and $B = 1$ by the boundary condition at $x = 0$. The solution of the above ODE is given by $\varphi_\xi(x) = e^{\lambda(\xi)x}$, with

$$\lambda(\xi) \triangleq \frac{-(\theta-\eta\xi)a - \sqrt{(\theta-\eta\xi)^2 a^2 + 2\sigma^2(1-\xi)^2\alpha}}{\sigma^2(1-\xi)^2} < 0.$$

Therefore, by combining these results, we have

$$J_\xi^\pi(x) = g_1(x;\xi) - (P + g_1(0;\xi))e^{\lambda(\xi)x}. \tag{A.11}$$

Since we assume that $f(x) = x$, by Fubini's theorem,

$$g_1(x;\xi) = \frac{x}{\alpha} + \frac{(\theta-\eta\xi)a}{\alpha^2}. \tag{A.12}$$

We can find $g(x)$ in (2.7) by setting $\xi = 0$, so that $g(x) = \frac{x}{\alpha} + \frac{\theta a}{\alpha^2}$.

Now, we can explicitly compute (2.10). Let

$$\mu_0 \triangleq (\theta - \eta\xi_0)a = \theta a \quad \text{and} \quad \mu_1 \triangleq (\theta-\eta\xi)a, \tag{A.13}$$

by setting $\xi_0 = 0$. Then, from (A.6), we have

$$\begin{aligned}
&e^{\alpha\Delta} I_1(x;\xi) \\
&= \frac{1}{\sigma\sqrt{\Delta}} \int_K^\infty J_\xi(z-K) \left(\phi\left(\frac{x - z + \mu_0\Delta}{\sigma\sqrt{\Delta}}\right) + e^{-2\mu_0 x/\sigma^2} \phi\left(\frac{-x-z+\mu_0\Delta}{\sigma\sqrt{\Delta}}\right)\right) dz \\
&= \frac{1}{\alpha}\Bigg\{ \left(x + \mu_0\Delta - K + \frac{\mu_1}{\alpha}\right) N(d_1) + \sigma\sqrt{\Delta}\phi(d_1) \\
&\quad + e^{-2\mu_0 x/\sigma^2}\left(\left(-x + \mu_0\Delta - K + \frac{\mu_1}{\alpha}\right) N(d_2) + \sigma\sqrt{\Delta}\phi(d_2)\right)\Bigg\} \\
&\quad - \left(P + \frac{\mu_1}{\alpha^2}\right) e^{\lambda^2(\xi)\Delta\sigma^2/2}\left(N(d_3)e^{\lambda(\xi)(x+\mu_0\Delta-K)} + e^{-2\mu_0 x/\sigma^2} N(d_4) e^{\lambda(\xi)(-x+\mu_0\Delta-K)}\right),
\end{aligned} \tag{A.14}$$

in which

$$d_1, d_2 \triangleq \frac{\pm x + \mu_0\Delta - K}{\sigma\sqrt{\Delta}}, \qquad d_3, d_4 \triangleq \frac{\pm x + \mu_0\Delta + \lambda(\xi)\sigma^2\Delta - K}{\sigma\sqrt{\Delta}},$$

and

$$e^{\alpha\Delta}I_2(x) = \frac{1}{\alpha\sigma\sqrt{\Delta}}\int_K^\infty \left(z+\frac{\mu_0}{\alpha}\right)\left(\phi\left(\frac{x-z+\mu_0\Delta}{\sigma\sqrt{\Delta}}\right)+e^{-2\mu_0 x/\sigma^2}\phi\left(\frac{-x-z+\mu_0\Delta}{\sigma\sqrt{\Delta}}\right)\right)dz$$

$$= \frac{1}{\alpha}\left\{\left(x+\mu_0\Delta+\frac{\mu_0}{\alpha}\right)N\left(d_1\right)+\sigma\sqrt{\Delta}\phi\left(d_1\right)+e^{-2\mu_0 x/\sigma^2}\left(\left(-x+\mu_0\Delta+\frac{\mu_0}{\alpha}\right)N\left(d_2\right)+\sigma\sqrt{\Delta}\phi\left(d_2\right)\right)\right\}. \tag{A.15}$$

Also, from (A.7) and (A.8),

$$e^{\alpha\Delta}(I_3(x)+I_4(x)) = \left(-P-\frac{\mu_0}{\alpha^2}\right)\left\{N\left(\frac{x+\mu_0\Delta}{\sigma\sqrt{\Delta}}\right)-e^{-2\mu_0 x/\sigma^2}N\left(\frac{-x+\mu_0\Delta}{\sigma\sqrt{\Delta}}\right)-N\left(d_1\right)\right.$$
$$\left.+e^{-2\mu_0 x/\sigma^2}N\left(d_2\right)+\int_0^\Delta e^{-\alpha t}\mathbb{P}^x(\tau_0\in dt)\right\}, \tag{A.16}$$

where the last integral is given by

$$\int_0^\Delta e^{-\alpha t}\mathbb{P}^x(\tau_0\in dt) = \int_0^\Delta e^{-\alpha t}\left\{\frac{x-\mu_0 t}{2\sigma t^{\frac{3}{2}}}\phi\left(\frac{x+\mu_0 t}{\sigma\sqrt{t}}\right)+e^{-2\mu_0 x/\sigma^2}\frac{x+\mu_0 t}{2\sigma t^{\frac{3}{2}}}\phi\left(\frac{-x+\mu_0 t}{\sigma\sqrt{t}}\right)\right\}dt. \tag{A.17}$$

We next substitute I_1, I_2, I_3, and I_4 into h in (2.9) and solve the corresponding two-stage optimization in that expression.

A.3 Proof of Lemma 3.1

Proof. First, we sketch a proof that the mapping T is continuous. Note that $\frac{\partial}{\partial x}h_\Delta(x;\xi,y)$ is continuous both in ξ and y. As we discussed in Section 2.2.1, the inner expectation in (3.2) can be solved by finding the smallest linear majorant of $h_\Delta(x;\xi,y)/\varphi(x;\xi)$ in the transformed space.

Let us fix $y=\bar{y}$ that satisfies the condition for the existence of positive β. Recall that since for any $\xi\in[0,1]$, $h_\Delta(x;\xi,y)/\varphi(x;\xi)$ has the sole local maximum, the slope of linear majorant does not jump as we vary ξ. This fact, together with the continuity of $\frac{\partial}{\partial x}h_\Delta(x;\xi,y)$ in ξ, implies that the slope $\beta(\xi,\bar{y})$ of the linear majorant is also continuous in ξ. Similarly, when we fix $\xi=\bar{\xi}$, $\beta(\bar{\xi},y)$ is continuous in y. It follows that $\beta(\xi,y)$ is continuous both in ξ and y.

Now, suppose that we let ξ change to ξ' and correspondingly $T(\xi)=y$ moves to $T(\xi')=y'$, and let $c>0$ be given. Suppose that for all $\delta>0$ with $|\xi-\xi'|<\delta$, we have $|y-y'|>c$; that is, suppose that T is not continuous at ξ. This contradicts the continuity of $\beta(\xi,y)$ in both arguments. Indeed, for any $\epsilon>0$, there exist $\delta_1>0$ and $\delta_2>0$ such that if $|\xi-\xi'|<\delta_1$ and $|y-y'|<\delta_2$, then

$$|\beta(\xi,y)-\beta(\xi',y')| \le |\beta(\xi,y)-\beta(\xi',y)|+|\beta(\xi',y)-\beta(\xi',y')|<\epsilon. \tag{A.18}$$

Now suppose that, for this ϵ, no matter how small we make $|\xi-\xi'|$, we cannot make $|y-y'|$ smaller than

c. In this case, (A.18) is violated because $|\beta(\xi', y) - \beta(\xi', y')|$ cannot be small enough, a contradiction.[2] Since T is a mapping from a closed bounded convex set in $\mathbb{R}$ into itself, the continuity of T guarantees the existence of a fixed point due to Brower's fixed point theorem; see Rudin [15], Theorem 5.28. $\square$

References

[1] A. Bar-Ilan and W. C. Strange. A model of sequential invetment. *Journal of Economic Dynamics and Control*, 22:437–463, 1998.

[2] E. Bayraktar and M. Egami. The effects of implementation delay on decision-making under uncertainty. *Stochastic Processes and Their Applications*, 117 (3):333–358, 2007.

[3] S. Browne. Optimal investment policies for a firm with a random risk process: Exponential utility and minimization of the probability of ruin. *Mathematics of Operations Research*, 20:937–958, 1995.

[4] R. Carmona and N. Touzi. Optimal multiple stopping and valuation of swing options. *Mathematical Finance*, 8 (2):239–268, 2008.

[5] T. Choulli, M. Taksar, and X. Y. Zhou. Excess-of-loss reinsurance for a company with debt liability and constraints on risk reduction. *Quantitative Finance*, 1:573–596, 2001.

[6] S. Dayanik and I. Karatzas. On the optimal stopping problem for one-dimensional diffusions. *Stochastic Processes and their Applications*, 107 (2):173–212, 2003.

[7] E. Dynkin. *Markov Processe Volume II*. Springer-Verlag, Berlin, 1965.

[8] I. Elsanosi, B. Øksendal, and A. Sulem. Some solvable stochastic control problems with delay. *Stochastics and Stochastics Reports*, 71(1-2):69–89, 2000.

[9] J. Grandell. *Aspects of Risk Theory*. Springer Verlag, Berlin, 1990.

[10] B. Højgaard and M. Taksar. Controlling risk exposure and dividends payout schemes: Insurance company examples. *Mathematical Finance*, 9(2):153–182, 1999.

[11] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus, 2nd Edition*. Springer-Verlag, New York, 1991.

[12] M. Musiela and M. Rutkowski. *Martingale Methods in Financial Modelling*. Springer, New York, 1997.

[13] S. Peura and J. Keppo. Optimal bank capital with costly recapitalization. *Journal of Business*, 79 (4):2163–2202, 2006.

[14] S. D. Promislow and V. R. Young. Minimizing the probability of ruin when claims follow Brownian motion with drift. *North American Actuarial Journal*, 9(3):109–128, 2005.

[15] W. Rudin. *Functional Analysis*. McGraw-Hill, 2nd Edition, New York, 1991.

[16] H. Schmidli. Diffusion approximations for a risk process with the possibility of borrowing and interest. *Communications in Statistics. Stochastic Models*, 10:365–388, 2001.

[2] Refer to Figure 2(c) for a graphical interpretation. If we change ξ, the graph of $\beta(\xi)$ will shift, but the point $\xi^{\triangle}$ that gives the largest slope (denoted by y in the current paragraph) cannot move far away. If it did, then the two graphs corresponding to ξ and ξ' would be vertically far apart, contradicting the continuity of the slope in ξ.

[17] A. Subramanian and R. A. Jarrow. The liquidity discount. *Mathematical Finance*, 11:447–474, 2001.

[18] M. I. Taksar and C. Markussen. Optimal dynamic reinsurance policies for large insurance portfolios. *Finance Stochast.*, 7:97–121, 2003.