AN OPTIMAL STOPPING PROBLEM FOR MULTIPLE ASSETS IN A MEAN-REVERTING MODEL

MASAHIKO EGAMI AND TADAO ORYU

Abstract. We solve optimal stopping problems whose payoff functions are the maximum of two state variables driven by the Ornstein-Uhlenbeck processes. We consider a class of problems where we obtain analytical solutions. By making use of the analytical results we study some properties of exercise regions including convexity, symmetry, and continuity. It turns out that the exercise regions shall become remarkably different in shape as a result of slight changes in the reward function, an interesting phenomenon attributable to the multiple dimensionality of the problem.

Key words: Optimal stopping, Mean-reverting diffusion, Explicit solutions for multiple dimension problem, Scale functions.
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1. Introduction

The main contribution of this paper is (1) to provide an analytical solution to a two-dimensional optimal stopping problem where the state variables are modeled by Ornstein-Uhlenbeck (OU, hereafter) processes and (2) to study some properties of the exercise regions in the problem. For practical purposes, we shall find exercise regions for American-type derivatives associated with the higher value of two assets. One can think of derivatives, in financial engineering, associated with interest rates, or in the theory of investment under uncertainty, certain state variables whose dynamics are appropriately modeled by mean-reverting processes. For the latter case, see for example Cadenillas et al. [3] and Bayraktar and Egami [1]. To our knowledge, this is a first attempt to handle optimal stopping problems for multiple assets driven by mean-reverting diffusion processes. In the current literature, there are no analytical solutions found for multi-dimension optimal stopping problems, or for finite-time horizon problems. Therefore, it is important to identify the shape of exercise regions for better understanding of optimal stopping problems in two dimensions and we believe that this direction of research is quite meaningful. Indeed, we shall observe that in Figure 9 and 10, although the parameters in the reward functions are only slightly altered, the resulting exercise regions are dramatically different. In this situation, if we were to try to solve problems only numerically, we should encounter serious difficulties in putting appropriate boundary conditions.

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The benefits of obtaining analytical solutions include the following: we believe that the results here can serve as a benchmark against which one can compare more general cases (that may require numerical solutions), shedding lights to the theoretical aspect of the multi-dimensional optimal stopping. Moreover, we can make a rigorous analysis on the shape of exercise boundaries by taking advantage of explicit formulae. For analytical solutions, we introduce “the difference of the two processes” as a new process and represent the payoff solely by this new process, so that we reduce the dimension of the problem to one, an effective technique, e.g. Davis and Norman [4]. Indeed, we have a family of optimal stopping problems in one-dimension parameterized by a function of the initial values of the two OU processes. After solving the family of problems, we then come back to the original problem and identify the exact shape of the exercise boundary that turns out to consist of several segments. The continuity of these segments are also rigorously proved.

We review some literature in this vein. There are numerous financial derivatives that relate to two or more assets; for example, the one pays difference of prices of two assets, or the one brings the asset whose price is higher than the other. There are no general methods to solve the pricing and hedging problems for American-type derivatives related to multiple assets. Broadie and Detemple [2] characterize the option exercise regions and provide valuation formulae for several kinds of payoff functions. Under the Black-Scholes setup, for example, they consider American call options on the maximum of two stocks with payoff function $f(x_1, x_2) = (\max(x_1, x_2) - K)^+$ and showed, among other things, that it is not optimal to exercise when the prices of the underling assets are equal and that each of the two subregions of early exercise regions is convex (page 247 in [2]). Following this, Villeneuve [11] extended [2] by analyzing additional payoff functions and studied the notion of critical surface for which one can extend some results in the one-dimensional case. More recently, Detemple et al. [7] show, for call options on the minimum of two dividend-paying assets, that the optimal exercise boundary consists of three components, two continuous curves and one component along the diagonal with empty interior (page 955, Figure 1). Guillaume [9] further extends in the direction that more assets are included in the payoff with knock-in and knock-out provisions and also that the pricing of these contracts is made in a multivariate jump-diffusion framework allowing for a stochastic two-factor term structure of interest rates. Also see Detemple [6] and the references therein.

This paper is constructed as follows. In section 2, we show our model setup. We solve the optimal stopping problems with payoff function $(x_1 - K_1) \vee (x_2 - K_2)$. While our reward function may seem to be specific, the class of the reward functions (to which the method in this paper can be applied) is in fact rich. First, note that this class of functions contains the function in Broadie and Detemple [2] by setting $K_1 = K_2 = K$. Secondly, we can solve the problems with payoff functions in the form of any linear combinations of the maximum and minimum of x_1 and x_2, or in the form of $f(x_1 - x_2)$ for any real function f. Moreover, these method can be used to solve the problems with some other processes, such as Brownian motions.

In section 3, we solve a simple case where $K_1 = K_2 = 0$ to analyze the characteristics of the problem, presenting numerical examples and discussing the properties of the exercise regions. The method presented in section 3 can be utilized also in more general cases in section 4. We shall show that if the difference of K_1 and K_2 is greater than some specific value, then the exercise region shall change its shape dramatically. One can not any longer predict such a peculiar shape beforehand and hence it should be very difficult to obtain the result merely with numerical computations. This is another contribution of this article.

In section 5, we conclude and give some ideas of possible extensions of this paper. Section 6 is an appendix which explains the outline of the solution method (for optimal stopping) based on Dynkin [8] and Dayanik and Karatzas [5].

2. The Model Setup

Let the stochastic processes $X^i = \{X^i_t; t \geq 0\}$ $(i = 1, 2)$ represent two state variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the set of all possible realization of the stochastic economy, and $\mathbb{P}$ is a risk-neutral measure defined on $\mathcal{F}$. We denote by $\mathbb{F} = \{\mathcal{F}_t\}$ the filtration with respect to which X^1 and X^2 are adapted and with the usual conditions being satisfied. X^1 and X^2 satisfy the stochastic differential equations, with $X^1_0 = x_1$ and $X^2_0 = x_2$,

$$
\begin{aligned}
\mathrm{d}X^i_t &= -\alpha X^i_t \mathrm{d}t + \sigma_i \mathrm{d}B^i_t \quad (i = 1, 2), \\
\mathrm{d}[B^1, B^2]_t &= \rho \mathrm{d}t,
\end{aligned}
\tag{2.1}
$$

where α and σ_i are positive constants, and B^1 and B^2 are standard Brownian Motions on $(\Omega, \mathcal{F}, \mathbb{P})$. We set the common parameter α. Note that the reduction of the dimension seems impossible if they are distinct, in which case we have to resort to numerical solutions. But our main purpose is to obtain and analyze the *exact* description (with rigor) of the exercise region. The OU processes is frequently used for representing the dynamics of the assets that have the tendency of mean-reversion. Hence one can understand that our problem here is associated with interest rates or prices of cyclical products. Note that we set the mean-reverting level zero here. Since a simple translation can handle the non-zero mean level easily, we solve our problem based on the dynamics (2.1). It is well known that X^1 and X^2 are represented of the closed form,

$$
X^i_t = e^{-\alpha t} x_i + \sigma_i \int_0^t e^{-\alpha(t-s)} \mathrm{d}B^i_s, \quad i = 1, 2.
$$

The payoff function is $\Phi(x_1, x_2) = (x_1 - K_1) \vee (x_2 - K_2)$, and the value function $V : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is defined by

$$
V(x_1, x_2) = \sup_{\tau \in \mathcal{S}} \mathbb{E}_{x_1, x_2}[e^{-r\tau} \Phi(X^1_\tau, X^2_\tau)],
\tag{2.2}
$$

where r is constant discount rate $(r > 0)$, and $\mathcal{S}$ is the set of all possible $\mathbb{F}$-stopping times. Note that the process $X^i - K_i$ can be seen as an OU process reverting to $-K_i$, hence it doesn't affect generality that we restricted the mean of processes to 0.

2.1. **Reduction of the Dimension.**

To reduce the dimension of the problem, we define the new process $X^e = X^1 - X^2$, then X^e satisfies the stochastic differential equation

$$
\mathrm{d}X^e_t = -\alpha X^e_t \mathrm{d}t + \sigma_e \mathrm{d}B^e_t,
$$

where

$$\sigma_e = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}, \quad \text{and} \quad B_t^e = \frac{1}{\sigma_e}(\sigma_1 B_t^1 - \sigma_2 B_t^2).$$

B^e is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ because it is a continuous martingale that starts at 0 and

$$\begin{aligned} \mathrm{d}[B^e, B^e]_t &= \frac{1}{\sigma_e^2}(\sigma_1^2 \mathrm{d}[B^1, B^1]_t - 2\sigma_1\sigma_2 \mathrm{d}[B^1, B^2]_t + \sigma_2^2 \mathrm{d}[B^2, B^2]_t) \\ &= \frac{1}{\sigma_e^2}(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)\mathrm{d}t \\ &= \mathrm{d}t. \end{aligned}$$

Since $\alpha, r > 0$ and $\tau \geq 0$ a.s.,

$$\begin{aligned} \mathbb{E}_{x_1,x_2}\left[\left(e^{-r\tau}\int_0^\tau e^{-\alpha(\tau-s)}\mathrm{d}B_t^i\right)^2\right] &= \mathbb{E}_{x_1,x_2}\left[e^{-2r\tau}\int_0^\tau e^{-2\alpha(\tau-s)}\mathrm{d}s\right] \\ &= \mathbb{E}_{x_1,x_2}\left[e^{-2(\alpha+r)\tau}\int_0^\tau e^{2\alpha s}\mathrm{d}s\right] \\ &= \frac{1}{2\alpha}\mathbb{E}_{x_1,x_2}[e^{-r\tau}(1-e^{-2\alpha\tau})] \leq \frac{1}{2\alpha} < \infty, \end{aligned}$$

so for any $\tau \in \mathcal{S}$, $e^{-r\tau}\int_0^\tau e^{-\alpha(\tau-s)}\mathrm{d}B_t^i$ is integrable and then

$$\mathbb{E}_{x_1,x_2}\left[e^{-r\tau}\int_0^\tau e^{-\alpha(\tau-s)}\mathrm{d}B_t^i\right] = 0.$$

Therefore the following equations are satisfied;

$$\begin{aligned} \mathbb{E}_{x_1,x_2}[e^{-r\tau}(X_\tau^2)] &= \mathbb{E}_{x_1,x_2}\left[e^{-(\alpha+r)\tau}x_2 + \sigma_2 e^{-r\tau}\int_0^\tau e^{-\alpha(\tau-s)}\mathrm{d}B_s^2\right] \\ &= x_2\mathbb{E}_{x_1,x_2}[e^{-(\alpha+r)\tau}], \\ \mathbb{E}_{x_1,x_2}[e^{-r\tau}(X_\tau^e)] &= \mathbb{E}_{x_1,x_2}\left[e^{-(\alpha+r)\tau}(x_1-x_2) + \sigma_e e^{-r\tau}\int_0^\tau e^{-\alpha(\tau-s)}\mathrm{d}B_s^e\right] \\ &= (x_1-x_2)\mathbb{E}_{x_1,x_2}[e^{-(\alpha+r)\tau}]. \end{aligned}$$

By these equations, when $x_1 \neq x_2$, we can reduce the dimension, that is, we can represent the value function

$$\begin{aligned} V(x_1,x_2) &= \sup_{\tau\in\mathcal{S}}\mathbb{E}_{x_1,x_2}[e^{-r\tau}((X_\tau^1 - K_1)\vee(X_\tau^2 - K_2))] \\ &= \sup_{\tau\in\mathcal{S}}\mathbb{E}_{x_1,x_2}[e^{-r\tau}(X_\tau^e - K_1 + K_2)^+ + e^{-r\tau}X_\tau^2 - e^{-r\tau}K_2] \\ &= \sup_{\tau\in\mathcal{S}}\mathbb{E}_{x_1,x_2}\left[e^{-r\tau}\left((X_\tau^e - K_1 + K_2)^+ + \frac{x_2}{x_1-x_2}X_\tau^e - K_2\right)\right] \\ &= \sup_{\tau\in\mathcal{S}}\mathbb{E}_{x_1,x_2}[e^{-r\tau}h(X_\tau^e)], \end{aligned}$$

where

$$h(x) = \begin{cases} \frac{x_1}{x_1-x_2}x - K_1 & (x \geq K_1 - K_2) \\ \frac{x_2}{x_1-x_2}x - K_2 & (x < K_1 - K_2) \end{cases}.$$

The case $x_1 = x_2$ shall be discussed later (in Remark 3.1).

The original problem (2.2) is now represented by X^e alone. For brevity, let

$$p := \frac{x_1}{x_1 - x_2} \quad \text{and} \quad p - 1 := \frac{x_2}{x_1 - x_2} \tag{2.3}$$

through the remainder of this article. We have thus reduced the original problem to a family of one-dimensional optimal stopping problems parameterized by p. Accordingly, we need to examine the value function by considering the cases $p \geq 1$, $0 < p < 1$, and $p \leq 0$, separately and we shall undertake this below.

3. The simple case

First, we solve the problem for the special case of $K_1 = K_2 = 0$, because the solution for this case contains the essence of the method we provide in this paper and plays an important role in more general case to be discussed later.

3.1. Solution.

The outline of a general solution method for optimal stopping problem of one-dimensional diffusions is described in Appendix. In this section, we will use the method to solve our problem. The differential operator $\mathcal{A}$ for X^e is defined by

$$\mathcal{A}u(\cdot) = \frac{\sigma_e^2}{2}\frac{d^2u}{dx^2}(\cdot) - \alpha x \frac{du}{dx}(\cdot).$$

The increasing and decreasing solution ψ and φ for the ordinary differential equation $(\mathcal{A} - r)u(x) = 0$ (with $u \in C^2$) are known as

$$\psi(x) = e^{\alpha x^2/2}\mathcal{D}_{-r/\alpha}\left(\frac{-x\sqrt{2\alpha}}{\sigma_e}\right) \quad \text{and} \quad \varphi(x) = e^{\alpha x^2/2}\mathcal{D}_{-r/\alpha}\left(\frac{x\sqrt{2\alpha}}{\sigma_e}\right),$$

where $\mathcal{D}_\nu$ is the parabolic cylinder function denoted by

$$\mathcal{D}_\nu(z) = 2^{-\nu/2}e^{-z^2/4}\mathcal{H}_\nu(z/\sqrt{2}),\ z \in \mathbb{R},$$

and $\mathcal{H}_\nu$ is the Hermite function of degree ν denoted by the integral representation

$$\mathcal{H}_\nu(z) = \frac{1}{\Gamma(-\nu)}\int_0^\infty e^{-t^2-2tz}t^{-\nu-1}dt,\ \text{Re}\nu < 0.$$

For these special functions, see Lebedev [10]. Let us define the transformation

$$F(x) := \psi(x)/\varphi(x) \quad \text{and} \quad H(y) := (h/\varphi) \circ F^{-1}(y).$$

By the characterization of the value function as the nonnegative smallest majorant of h/φ (see Proposition 6.2 in the appendix), we investigate the shape of the H function in the transformed space.

Since $\lim_{x\to-\infty}(h^+/\varphi)(x) = 0$, then $H(0) = 0$, and it is clear from $F(0) = 1$ and $h(0) = 0$ that $H(1) = (h/\varphi)(0) = 0$ (see Proposition 6.3). It is well-known that $H''(y)$ and $[(\mathcal{A}-r)h](F^{-1}(y))$ have the same sign. To calculate $[(\mathcal{A}-r)h](x)$ is easy;

$$[(\mathcal{A}-r)h](x) = \begin{cases} -p(\alpha+r)x & (x \geq 0), \\ -(p-1)(\alpha+r)x & (x < 0). \end{cases}$$

Because $F(x)$ is a monotone increasing function and $F(0) = 1$, the sign of $H(y)$ and $H''(y)$ are determined as is shown in Table 1. In addition, it can be checked that these equations are satisfied;

$$\lim_{y\to\infty} H(y) = \begin{cases} \infty & (p > 0) \\ -\infty & (p \leq 0) \end{cases}, \quad \lim_{y\to\infty} H'(y) = 0.$$

Now we have identified the complete description of $H(y)$, the next step is to find the smallest concave majorant of H. Recall however that the function H depends on the parameter p.

Case (a) $p \geq 1$. In this case, the function $H(y)$ is convex and negative on $[0, 1)$, concave and positive on $(1, +\infty)$. It can be checked from $\lim_{y\to\infty} H'(y)$ that there exists unique $z_a > 1$, which is the unique solution of $yH'(y) = H(y), y > 0$. In our problem, the value of z_a is independent of p. This can be known by setting $k = 0$ in the following proposition.

Proposition 3.1. *When h is in the form of $h(x) = p(x-k)$, the solution z_a of $yH'(y) = H(y)$ is independent of p.*

cases		the sign of $H(y)$	the sign of $H''(y)$
(a) $p \geq 1$	$y \geq 1$	$+$	$-$
	$0 < y < 1$	$-$	$+$
(b) $0 < p < 1$	$y \geq 1$	$+$	$-$
	$0 < y < 1$	$+$	$-$
(c) $p \leq 0$	$y \geq 1$	$-$	$+$
	$0 < y < 1$	$+$	$-$

TABLE 1.

Proof. From the definition of $H(y)$ and the chain rule of derivative,

$$
\begin{aligned}
H(y) &= \left(\frac{h}{\varphi}\right) \circ F^{-1}(y) \qquad (3.1)\\
&= p\left(\frac{I-k}{\varphi}\right) \circ F^{-1}(y)\\
yH'(y) &= \frac{y}{F'(F^{-1}(y))}\left(\frac{h}{\varphi}\right)' \circ F^{-1}(y) \qquad (3.2)\\
&= y \cdot \left(\frac{h'\varphi - h\varphi'}{F'\varphi^2}\right) \circ F^{-1}(y)\\
&= py \cdot \left(\frac{\varphi - (I-k)\varphi'}{F'\varphi^2}\right) \circ F^{-1}(y).
\end{aligned}
$$

where $I(x) = x$. Therefore $H(y) = yH'(y)$ if and only if

$$
\left(\frac{I-k}{\varphi}\right) \circ F^{-1}(y) = y \cdot \left(\frac{\varphi - (I-k)\varphi'}{F'\varphi^2}\right) \circ F^{-1}(y). \qquad (3.3)
$$

This equation is independent of p. □

The smallest nonnegative concave majorant $W_a(y)$ of $H(y)$ for the case of $p \geq 1$ is thus

$$
W_a(y) = \begin{cases} \dfrac{yH(z_a)}{z_a} & (0 < y \leq z_a), \\ H(y) & (y > z_a). \end{cases}
$$

If we define $x_a = F^{-1}(z_a)$, then $x_a > 0$ and is also independent of p by Proposition 3.1. The function $V_a(x) = \varphi(x)W_a(F(x))$ (see Proposition 6.3) is

$$
V_a(x) = \begin{cases} \dfrac{px_a}{\psi(x_a)}\psi(x) & (x < x_a), \\ px & (x \geq x_a). \end{cases}
$$

Case (b) 0<*p*<1. In this case, the function $H(y)$ is concave and positive on $[0, 1)$ and on $(1, \infty)$. There exists unique pair $(z_{b,1}, z_{b,2})$ with $z_{b,1} < 1 < z_{b,2}$ which is the solution of simultaneous equations:

$$
\begin{cases} H'(z_{b,1}) = H'(z_{b,2}), \\ H(z_{b,2}) - H(z_{b,1}) = H'(z_{b,1})(z_{b,2} - z_{b,1}). \end{cases}
$$

In contrast to z_a, the value of $(z_{b,1}, z_{b,2})$ varies with p. The smallest nonnegative concave majorant $W_b(y)$ of $H(y)$ for the case $0 < p < 1$ is

$$
W_b(y) = \begin{cases} H(y) & (0 \leq y < z_{b,1}, z_{b,2} < y), \\ \dfrac{H(z_{b,2}) - H(z_{b,1})}{z_{b,2} - z_{b,1}}(y - z_{b,1}) + H(z_{b,1}) & (z_{b,1} \leq y \leq z_{b,2}). \end{cases}
$$

If we define $z_{b,1} = F(x_{b,1})$ and $z_{b,2} = F(x_{b,2})$, then the function $V_b(x) = \varphi(x)W_b(F(x))$ is

$$V_b(x) = \begin{cases} px, & (x > x_{b,2}), \\ (p-1)x & (x < x_{b,1}), \\ \dfrac{px_{b,2}\varphi(x_{b,1}) - (p-1)x_{b,1}\varphi(x_{b,1})}{\varphi(x_{b,1})\psi(x_{b,2}) - \varphi(x_{b,2})\psi(x_{b,1})}\varphi(x) - \dfrac{px_{b,2}\psi(x_{b,1}) - (p-1)x_{b,1}\psi(x_{b,2})}{\varphi(x_{b,1})\psi(x_{b,2}) - \varphi(x_{b,2})\psi(x_{b,1})}\psi(x) & (x_{b,1} \le x \le x_{b,2}). \end{cases}$$

Case(c) $p \le 0$. In this case, the function $H(y)$ is concave and positive on $[0, 1)$, and convex and negative on $(1, \infty)$. There exists unique $z_c < 1$ which is the solution of $H'(y) = 0$. The value of z_c is also independent of p as z_a is.

Proposition 3.2. *When h is in the form of $h(x) = p(x - k)$, the solution z_c of $H'(y) = 0$ is independent of p.*

Proof. It is clear from (2.4) that $H'(y) = 0$ if and only if

$$\left(\frac{\varphi - (I-k)\varphi'}{F'\varphi^2}\right) \circ F^{-1}(y) = 0.$$

This is independent of p. □

The smallest nonnegative concave majorant $W_c(y)$ of $H(y)$ for the case $p \le 0$ is

$$W_c(y) = \begin{cases} H(y) & (0 \le y < z_c), \\ H(z_c) & (y \ge z_c). \end{cases}$$

If we define $z_c = F(x_c)$, then $x_c > 0$ and is independent of p by Proposition 3.2. The function $V_c(x) = \varphi(x)W_c(F(x))$ is

$$V_c(x) = \begin{cases} (p-1)x & (x \le x_c), \\ \dfrac{(p-1)x_c}{\varphi(x_c)}\varphi(x) & (x > x_c). \end{cases}$$

The functions $V_a(x)$, $V_b(x)$, and $V_c(x)$ are the value functions denoted by $x = x_1 - x_2$ when p is fixed. Since $px_2 = (p-1)x_1$ is satisfied from the definition of p, p is constant on a line in (x_1, x_2)-plane through the origin. Therefore, $V_a(x)$, $V_b(x)$, and $V_c(x)$ can be regarded as the cross sections of $V(x_1, x_2)$ cut by the plane $px_2 = (p-1)x_1$. Indeed, it is difficult to draw the graphs of $V(x_1, x_2)$ and $\mathcal{E}$, but the method mentioned above is enough to know the value of $V(x_1, x_2)$ and make decisions as to "exercise" or "continue".

3.2. **Numerical Example.** In this subsection, we solve a numerical example by the method described in previous subsection for the case $r = 0.05$, $\alpha = 1$, $\sigma_1 = 0.15$, $\sigma_2 = 0.125$, $\rho = 0.75$, and then $\sigma_e = 0.1$. Note that the solution in the previous section does not depend on the parameters. *Hence the analysis here is applicable to a general situation of our problem.* The values of $x_a = 0.134435$ and $x_c = -0.134435$ are independent of p as is shown in Proposition 2.1 and Proposition 2.2. Table 2 shows the values of $x_{b,1}$ and $x_{b,2}$ for five different p. The graphs of $W_a(y)$, $W_b(y)$, and $W_c(y)$ are shown with $H(y)$ in Figure 1,

Figure 2, and Figure 3, respectively. We also draw the graphs of $V_a(x)$, $V_b(x)$, and $V_c(x)$ with $h(x)$ in each figure.

Now, we shall go back to the original two-dimensional space to identify the exercise region in terms of the initial values x_1 and x_2. The exercise region $\mathcal{E}$ is shown in Figure 4(i) as the upper and lower regions of the curves. Hence the continuation region is the region between the two curves. The two curves in the first quadrant (in Figure 4(i)) are straight lines, reflecting the fact that z_a and z_c are independent of p. In other words, the upper straight line that passes $(0, -x_c)$ corresponds to the case $p \le 0$ and the lower straight line that passes $(0, -x_a)$ corresponds to the case $p \ge 1$. The other parts of the curves (rather than the straight lines) in Figure 4(i) correspond to the case $0 < p < 1$ and are drawn in the following steps (see also Figure 4(ii)):

step (i): Fix some $p \in (0, 1)$.
step (ii): In view of (2.3), draw the line $l_p : px_2 = (p-1)x_1$ on the (x_1, x_2)-plane.
step (iii): Calculate $x_{b,1}$ and $x_{b,2}$, and draw the lines $l_1 : x_1 - x_2 = x_{b,1}$ and $l_2 : x_1 - x_2 = x_{b,2}$.
step (iv): Plot the intersection point of " l_p and l_1", and that of " l_p and l_2". (in fact, these points are $(px_{b,1}, (p-1)x_{b,1})$ and $(px_{b,2}, (p-1)x_{b,2})$ to be computed from easy calculations.)
step (v): Return to step (i) with another $p \in (0, 1)$.

p	0.3	0.4	0.5	0.6	0.7
$x_{b,1}$	-0.134808	-0.139114	-0.150892	-0.174689	-0.225930
$x_{b,2}$	0.225930	0.174689	0.150892	0.139114	0.1348098

TABLE 2.

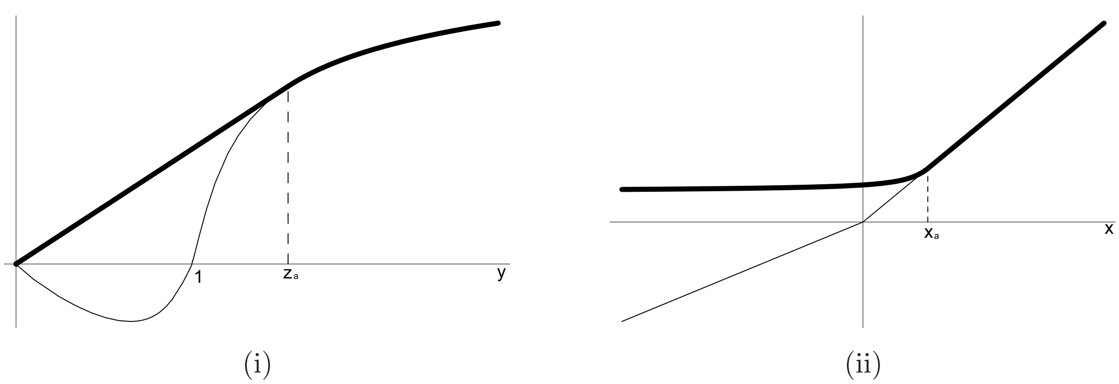
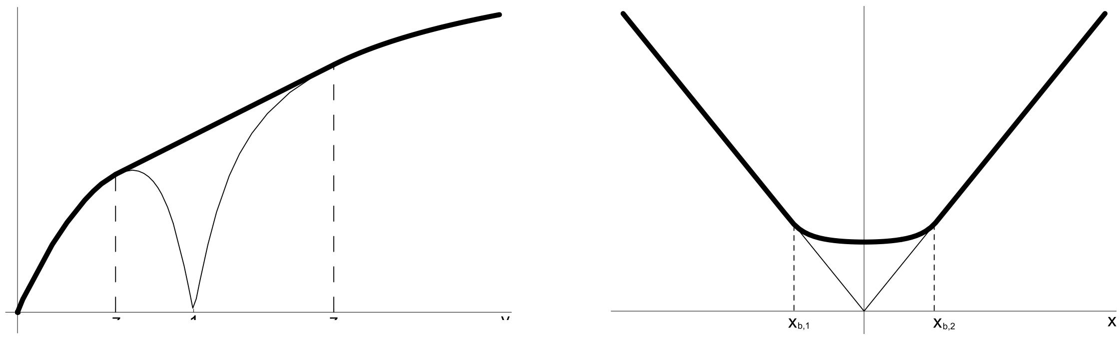
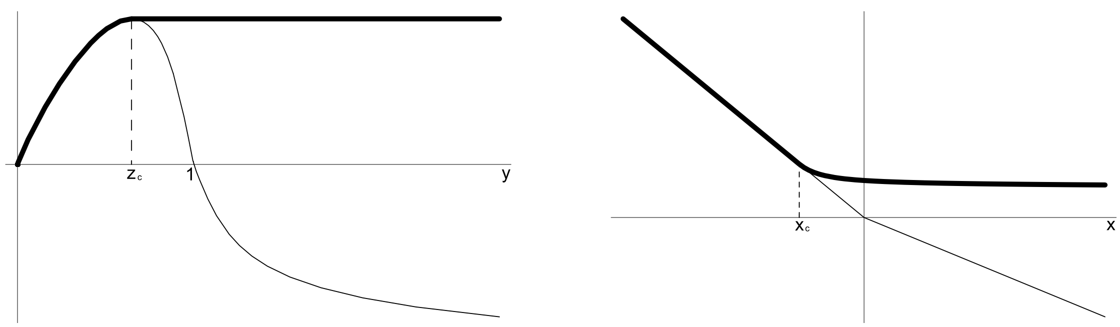


FIGURE 1. $\boldsymbol{p \geq 1}$. These are the graphs of the case $p = 2$. The solid line in (i) is $H_a(y)$ and the thick line is $W_a(y)$. The solid line in (ii) is $h(x)$ and the thick line is $V_a(x)$. $z_a = 1.54432$, and $x_a = F^{-1}(z_a) = 0.134435$.



(i) (ii)

FIGURE 2. **0<p<1.** These are the graphs of the case $p = 0.5$ The solid line in (i) is $H_b(y)$ and the thick line is $W_b(y)$. The solid line in (ii) is $h(x)$ and the thick line is $V_b(x)$. $z_{b,1} = 0.556868$, $z_{b,2} = 1.79576$, $x_{b,1} = F^{-1}(z_{b,1}) = -0.150892$, and $x_{b,2} = F^{-1}(z_{b,2}) = 0.150892$.



(i) (ii)

FIGURE 3. **p≤0.** The solid line in (i) is $H_c(y)$ and the thick line is $W_c(y)$. The solid line in (ii) is $h(x)$ and the thick line is $V_c(x)$. $z_c = 0.647536$, and $x_c = F^{-1}(z_c) = -0.134435$.

3.3. **Properties of Exercise Region.** By viewing Figure 4, the exercise region, denoted by $\mathcal{E}$, is divided into two parts. In the $x_1 - x_2$ plane, $x_2 = E_1(x_1)$ and $x_1 = E_2(x_2)$ are the two curves which are the boundaries between the continuation and exercise regions. That is, $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, where $\mathcal{E}_1 := \{(x_1, x_2) \in \mathbb{R}^2; x_2 \geq E_1(x_1)\}$ and $\mathcal{E}_2 := \{(x_1, x_2) \in \mathbb{R}^2; x_1 \geq E_2(x_2)\}$. It seems that $\mathcal{E}_1$, $\mathcal{E}_2$, $E_1(x_1)$, and $E_2(x_2)$ have following properties.

- **(i) Convexity :** If $(x_1, x_2) \in \mathcal{E}_1$ and $(x_1', x_2') \in \mathcal{E}_1$, then $(\lambda x_1 + (1-\lambda)x_1', \lambda x_2 + (1-\lambda)x_2') \in \mathcal{E}_1$ for all $\lambda \in [0, 1]$ (and the same thing about $\mathcal{E}_2$ is true).
- **(ii) Symmetry :** $(x_1, x_2) \in \mathcal{E}_1$ if and only if $(x_2, x_1) \in \mathcal{E}_2$.
- **(iii) Continuity :** $E_1(x_1)$ and $E_2(x_2)$ are continuous at $x_1 = 0$ and $x_2 = 0$, respectively.

Note about (ii) symmetry that this should be true even when $\sigma_1 \neq \sigma_2$ (actually, Figure 4 is the case $\sigma_1 = 0.15$, $\sigma_2 = 0.125$).

Proposition 3.3. *$\mathcal{E}_1$ and $\mathcal{E}_2$ are convex-sets.*

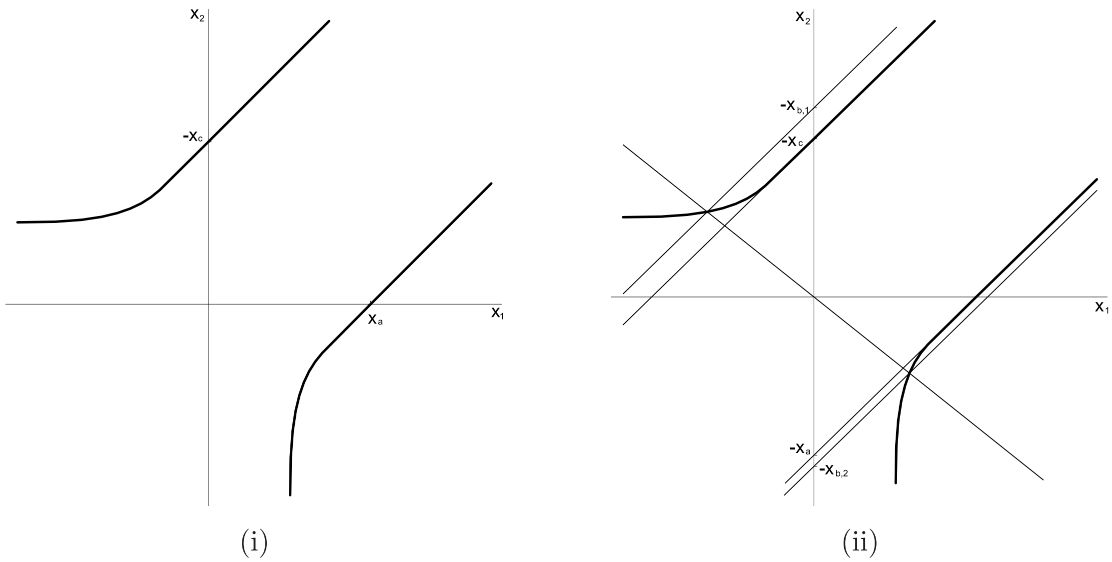


FIGURE 4. (i)The left line is $x_2 = E_1(x_1)$, and the right line is $x_1 = E_2(x_2)$. The exercise region $\mathcal{E}$ is $\{(x_1,x_2) \in \mathbb{R}^2 | x_2 \geq E_1(x_1)\} \cup \{(x_1,x_2) \in \mathbb{R}^2 | x_1 \geq E_2(x_2)\}$. (ii)The straight line with negative slope is $l_p : px_2 = (p-1)x_1$ for $p = 0.55$. The line intercepting x_2-axes at $-x_{b,1} = 0.160738$ is $l_1 : x_1 - x_2 = x_{b,1}$. The line intercepting x_2-axes at $-x_{b,2} = -0.143904$ is $l_2 : x_1 - x_2 = x_{b,2}$. Make sure l_p, l_i are crossed on the thick line.

Proof. We will show the convexity of $\mathcal{E}_1$ here. The case of $\mathcal{E}_2$ can be shown in the same way. The proof here is based on Broadie and Detemple [2]. First we show the convexity of the value function $V(x_1,x_2)$. Let $x = (x_1,x_2) \in \mathcal{E}_1$, $x' = (x'_1,x'_2) \in \mathcal{E}_1$, $\lambda \in [0,1]$ and $x(\lambda) = (x_1(\lambda), x_2(\lambda)) := (\lambda x_1 + (1-\lambda)x'_1, \lambda x_2 + (1-\lambda)x'_2)$. In addition, we define the stochastic process $N_t = (N^1_t, N^2_t)$, where $N^i_t := \sigma_i \int_0^t e^{-\alpha(t-s)} \mathrm{d}B^i_s (i=1,2)$, then $X_t = (X^1_t, X^2_t) = X_0 e^{-rt} + N_t$. Because the payoff function $\Phi(x)$ is convex and N_t is independent of X_0,

$$
\begin{aligned}
V(x(\lambda)) &= \sup_{\tau \in \mathcal{S}} \mathbb{E}_{x(\lambda)}[e^{-r\tau}\Phi(\lambda(xe^{-\alpha\tau} + N_\tau) + (1-\lambda)(x'e^{-\alpha\tau} + N_\tau))] \\
&\leq \sup_{\tau \in \mathcal{S}} \mathbb{E}_{x(\lambda)}[e^{-r\tau}(\lambda\Phi(xe^{-\alpha\tau} + N_\tau) + (1-\lambda)\Phi(x'e^{-\alpha\tau} + N_\tau))] \\
&\leq \sup_{\tau \in \mathcal{S}} \mathbb{E}_{x(\lambda)}[e^{-r\tau}\lambda\Phi(xe^{-\alpha\tau} + N_\tau)] + \sup_{\tau \in \mathcal{S}} \mathbb{E}_{x(\lambda)}[e^{-r\tau}(1-\lambda)\Phi(x'e^{-\alpha\tau} + N_\tau)] \\
&= \lambda \sup_{\tau \in \mathcal{S}} \mathbb{E}_{x}[e^{-r\tau}\Phi(xe^{-\alpha\tau} + N_\tau)] + (1-\lambda) \sup_{\tau \in \mathcal{S}} \mathbb{E}_{x'}[e^{-r\tau}\Phi(x'e^{-\alpha\tau} + N_\tau)] \\
&= \lambda V(x) + (1-\lambda)V(x').
\end{aligned}
$$

Because $x \in \mathcal{E}_1$ and $x' \in \mathcal{E}_1$,

$$
\begin{aligned}
V(x(\lambda)) &\leq \lambda V(x) + (1-\lambda)V(x') \\
&= \lambda x_2 + (1-\lambda)x'_2 = x_2(\lambda).
\end{aligned}
$$

On the other hand, by the definition of $V(x)$,

$$V(x(\lambda)) \geq x_1(\lambda) \vee x_2(\lambda) = x_2(\lambda).$$

Hence it follows that $V(x(\lambda)) = x_2(\lambda)$, which implies that $x(\lambda) \in \mathcal{E}_1$. □

Proposition 3.4. *$(x_1, x_2) \in \mathcal{E}_1$ if and only if $(x_2, x_1) \in \mathcal{E}_2$.*

Proof. The lines l_p's for $p = p^*$ and for $p = 1 - p^*$ are symmetric about $x_1 = x_2$. Then it suffices to show the following two things:

(i) $x_a = -x_c$,

(ii)$\{x \in \mathbb{R} | V_b(x) > h(x)\} = (x_{b,1}, x_{b,2})$ for $p = p^* \in (0,1)$ if and only if $\{x \in \mathbb{R} | V_b(x) > h(x)\} = (-x_{b,2}, -x_{b,1})$ for $p = 1 - p^*$.

The stochastic process $-X^e$ satisfies the stochastic differential equation

$$\mathrm{d}(-X^e_t) = -\alpha(-X^e_t)\mathrm{d}t + \sigma_e \mathrm{d}(-B^e_t).$$

Because $-B^e$ has the same distribution as B^e, $-X^e$ is an OU process with the same parameters as X^e except the sign of the initial value. In addition, the equation

$$\begin{aligned} h(x) &= \begin{cases} px & (x \geq 0), \\ (p-1)x & (x < 0), \end{cases} \\ &= \begin{cases} ((1-p)-1)(-x) & (x \geq 0), \\ (1-p)(-x) & (x < 0) \end{cases} \end{aligned}$$

shows that the graph of $h(x)$ with $p = p^*$ and that with $p = 1 - p^*$ are symmetric about $x = 0$. From these observations, it is clear that (i) and (ii) hold. □

Proposition 3.5. *$E_1(x_1)$ and $E_2(x_2)$ are continuous at $x_1 = 0$ and $x_2 = 0$, respectively.*

Proof. We show only the continuity of $E_1(x_1)$ here since that of $E_2(x_2)$ can be shown in the same way. The right hand limit $\lim_{x_1 \downarrow 0} E_1(x_1) = -x_c$ is clear from Proposition 3.2 and then what remains to be shown is the left hand limit $\lim_{x_1 \uparrow 0} E_1(x_1) = -x_c$. From the procedures to draw Figure 4 and continuity of the function F, $\lim_{x_1 \uparrow 0} E_1(x_1) = -x_c$ is equivalent to

$$\lim_{p \downarrow 0} z_{b,1} = z_c, \tag{3.4}$$

which we shall show in the following. For brevity, let $\zeta := \lim_{p \downarrow 0} z_{b,1}$ through the reminder of this proof.

(i) If $\zeta > z_c$, then $\lim_{p \downarrow 0} H'(z_{b,1}) = H'(\zeta) < 0$, since $H'(z_c) = 0$, $H''(y) < 0$ for all $y \in (0,1)$, and $H'(y)$ is continuous on $(0,1)$. Because $H'(y) > 0$ for $y \in (1, \infty)$ and $H(1) = 0 < H(\zeta)$, there is unique $z_{b,2} \in (1, \infty)$ such that $H'(\zeta)(z_{b,2} - \zeta) + H(\zeta) = H(z_{b,2})$. For these $z_{b,2}$ and ζ(the limit of $z_{b,1}$), $W_b(y)$ defined in section 3 is not concave, a contradiction to the fact $W_b(y)$ is the smallest concave majorant of $H(y)$.

(ii) If $\zeta < z_c$, then there exists some $\zeta' \in (\zeta, z_c)$, and $H'(\zeta) > H'(\zeta') > 0$ because $H'(z_c) = 0$ and $H''(y) < 0$ on $(0,1)$. We define the functions $W_\zeta(y)$ and $W_{\zeta'}(y)$ by

$$W_\zeta(y) := \begin{cases} H'(\zeta)(y-\zeta) & (\zeta \le y), \\ H(y) & (0 < y < \zeta), \end{cases}$$

and

$$W_{\zeta'}(y) := \begin{cases} H'(\zeta')(y-\zeta') & (\zeta' \le y), \\ H(y) & (0 < y < \zeta'). \end{cases}$$

It is obvious from this definition that W_ζ and $W_{\zeta'}$ are concave on $\mathbb{R}$. Additionally, the inequality $W_\zeta(y) \ge W_{\zeta'}(y) \ge H(y)$ is satisfied since $\lim_{p\downarrow 0} H(y) = 0$ for $y \in (1,\infty)$. Then $W_\zeta(y)$ and $W_{\zeta'}(y)$ both are concave majorants of $H(y)$ and $W_\zeta(y) \ge W_{\zeta'}(y)$. This contradicts to the fact that $W_\zeta(y)$ is the smallest concave majorant of $H(y)$. Therefore, (3.4) must be the case and this completes the proof. □

Finally, let us mention the case $x_1 = x_2$, which is eliminated in the argument in section 3.

Remark 3.1. *When $x_1 = x_2$, to exercise this option is not optimal, i.e. $\{(x_1,x_2) | x_1 = x_2\} \subset \mathcal{E}^c$.*

Proof. Proposition 3.3 implies that $E_1(x_1)$ and $E_2(x_2)$ are convex functions. Therefore, the inequalities

$$E_1(x_1) \ge x_1 + x_a \text{ and } E_2(x_2) \ge x_2 - x_c$$

hold on $\mathbb{R}$, and then $\{(x_1,x_2) | x_2 \in (x_1 + x_c, x_1) \cup (x_1, x_1 + x_a)\} \subset \mathcal{E}^c$.

Fix $\bar{x_1} \in \mathbb{R}$. It follows from $\{(x_1,x_2) | x_2 \in (x_1 + x_c, x_1) \cup (x_1, x_1 + x_a)\} \subset \mathcal{E}^c$ that

$$V(\bar{x_1}, x_2) > h(\bar{x_1}, x_2) \text{ on } x_2 \in (\bar{x_1} + x_c, \bar{x_1}) \cup (\bar{x_1}, \bar{x_1} + x_a).$$

In addition, since $h(x_1,x_2)$ is continuous on $\mathbb{R}^2$, $V(x_1,x_2)$ is also continuous, and hence so is $V(\bar{x_1}, x_2)$ in x_2 on $\mathbb{R}$. Due to continuity and increasingness of $V(\bar{x_1}, x_2)$ in x_2, $V(\bar{x_1}, x_2) > h(\bar{x_1}, x_2)$ leads to $V(\bar{x_1}, \bar{x_1}) > h(\bar{x_1}, \bar{x_1})$ by taking limit $x_2 \downarrow \bar{x_1}$. This shows $(\bar{x_1}, \bar{x_1}) \notin \mathcal{E}$. □

4. The general case

In this section, we'll solve the problem for more general case. We examine the case $K_1 \ge K_2 \ge 0$. Note that, however, because of symmetry of the model, the solution explained here also applies to the case of $K_2 \ge K_1 \ge 0$. The signs of $H(y)$ and $H''(y)$ can be checked in the same way as section 3.1, and the results are shown in Table 3. Moreover, these equations still hold:

$$\lim_{y\to\infty} H(y) = \begin{cases} \infty & (p > 0) \\ -\infty & (p \le 0) \end{cases}, \quad \lim_{y\to\infty} H'(y) = 0. \tag{4.1}$$

We found out that when $K_1 \ge K_2 \ge 0$, the properties of the problems are different between the cases of $0 \le K_1 - K_2 \le q$ and $K_1 - K_2 \ge q$, where q is the same as x_a calculated in the section 3.1. Recall that $F(q)$ is the solution to $yH'(y) = H(y)$, independent of p, that appeared in Proposition 3.1.

cases		the sign of $H(y)$	the sign of $H''(y)$
(a) $p \geq 1$	$y \geq \min(F(\frac{K_1}{p}), F(\frac{K_2}{p-1}))$	$+$	$-$
	$0 < y < \min(F(\frac{K_1}{p}), F(\frac{K_2}{p-1}))$	$-$	$+$
(b) $0 < p < 1$	$y \geq F(\frac{K_1}{p})$	$+$	$-$
	$F(\frac{K_2}{p-1}) \leq y < F(\frac{K_1}{p})$	$-$	$+$
	$0 < y < F(\frac{K_2}{p-1})$	$+$	$-$
(c) $p \leq 0$	$y \geq F(\frac{K_1}{p})$	$-$	$+$
	$0 < y < F(\frac{K_1}{p})$	$+$	$-$

TABLE 3.

4.1. $0 \leq K_1 - K_2 \leq q$. First, we consider the case of $0 \leq K_1 - K_2 \leq q$ whose solution is mostly the same as that of the case of $K_1 = K_2 = 0$. As in the previous section, we need to split into three regions in terms of p in order to deal with the optimal stopping problem parameterized by p:

Case (a) $p \geq 1$. The difference from section 3.1 is that, in this case, the solution z_a of $yH'(y) = H(y)$ is not independent of p. This is because the value k in Proposition 3.1 is now K_1/p, which depends on p. Define $x_a := F^{-1}(z_a)$, then by recalling that the functions W_a and V_a are the smallest concave majorant of H for the case $p \geq 1$ and its corresponding function in the original space, respectively, we have

$$W_a(y) = \begin{cases} \dfrac{yH(z_a)}{z_a} & (0 < y \leq z_a), \\ H(y) & (y > z_a), \end{cases}$$

and

$$V_a(x) = \begin{cases} \dfrac{px_a - K_1}{\psi(x_a)}\psi(x) & (x < x_a), \\ px - K_1 & (x \geq x_a). \end{cases}$$

Let us write $x_a(p)$ to indicate the dependence on p. We shall show later (in the proof of Lemma 4.1) that x_a monotonically decreases to converge to q as $p \to \infty$. That is,

$$x_a(p) \searrow q, \quad (p \to \infty). \tag{4.2}$$

Case (b) $0 < p < 1$. In this case, there are no differences from section 3.1. Let $(z_{b,1}, z_{b,2})$ with $z_{b,1} < z_{b,2}$ be the solution of simultaneous equations

$$\begin{cases} H'(z_{b,1}) = H'(z_{b,2}), \\ H(z_{b,2}) - H(z_{b,1}) = H'(z_{b,1})(z_{b,2} - z_{b,1}). \end{cases}$$

Then define $(x_{b,1}, x_{b,2}) := (F^{-1}(z_{b,1}), F^{-1}(z_{b,2}))$ so that the functions W_b and V_b are written as

$$W_b(y) = \begin{cases} H(y) & (0 \leq y < z_{b,1}, z_{b,2} < y), \\ \dfrac{H(z_{b,2}) - H(z_{b,1})}{z_{b,2} - z_{b,1}}(y - z_{b,1}) + H(z_{b,1}) & (z_{b,1} \leq y \leq z_{b,2}), \end{cases}$$

and

$$V_b(x) = \begin{cases} px - K_1 & (x > x_{b,2}), \\ (p-1)x - K_2 & (x < x_{b,1}), \\ \frac{(px_{b,2}-K_1)\varphi(x_{b,1}) - ((p-1)x_{b,1}-K_2)\varphi(x_{b,1})}{\varphi(x_{b,1})\psi(x_{b,2}) - \varphi(x_{b,2})\psi(x_{b,1})}\varphi(x) & \\ -\frac{(px_{b,2}-K_1)\psi(x_{b,1}) - ((p-1)x_{b,1}-K_2)\psi(x_{b,2})}{\varphi(x_{b,1})\psi(x_{b,2}) - \varphi(x_{b,2})\psi(x_{b,1})}\psi(x) & (x_{b,1} \le x \le x_{b,2}). \end{cases}$$

Case(c) $p \le 0$. Note that, like case (a), the solution z_c of $H'(y) = 0$ is no longer independent of p. The functions W_c and V_c are

$$W_c(y) = \begin{cases} H(y) & (0 \le y < z_c), \\ H(z_c) & (y \ge z_c), \end{cases}$$

and

$$V_c(x) = \begin{cases} (p-1)x - K_2 & (x \le x_c), \\ \\ \dfrac{((p-1)x_c - K_2)}{\varphi(x_c)}\varphi(x) & (x > x_c). \end{cases}$$

4.2. $K_1 - K_2 > q$**.** In this case, this problem becomes a bit more complex. For the argument on this case, we define functions $h_1(x) := px - K_1$ and $h_2(x) := (p-1)x - K_2$, so that h can be seen as $\max(h_1, h_2)$. Define $H_i, i = 1, 2$ as $H_i(y) := (h_i/\varphi) \circ F^{-1}(y)$, then H is equal to $\max(H_1, H_2)$ similarly. In addition, let us define W_i as the smallest concave nonnegative majorant of H_i for $i = 1, 2$. As usual, W_a denotes the smallest concave majorant of H for the case $p \ge 1$.

Lemma 4.1. *There exists some $\bar{p} > 1$ such that, when $1 < p \le \bar{p}$, H and its smallest nonnegative concave majorant W_a come in contact at only one point, and when $p > \bar{p}$, they do at three points.*

Proof. Let y_i^* be the solution of $yH_i'(y) = H_i(y)$, then

$$W_i(y) = \begin{cases} \dfrac{yH(y_i^*)}{y_i^*} & (0 < y \le y_i^*), \\ H_i(y) & (y > y_i^*). \end{cases}$$

If $\{y : W_1(y) < W_2(y)\} = \emptyset$, then $W_a = W_1$ and the number of tangent point of W_a and H is just one at $y = y_1^*$. See Figure 5.

On the other hand, if $\{y : W_1(y) < W_2(y)\} \neq \emptyset$, we need to first show that y_1^* (which depends on p) is monotonously increasing in $k := K_1/p$. Similarly, it can be shown in the same way that y_2^* is monotonously increasing in $K_2/(p-1)$. From (3.3), for any y^* that solves $yH'(y) = H(y)$,

$$\begin{aligned} k &= F^{-1}(y^*) - \frac{y^*\varphi(F^{-1}(y^*))}{\varphi(F^{-1}(y^*))F'(F^{-1}(y^*)) + y^*\varphi'(F^{-1}(y^*))} \\ &= x^* - \frac{F(x^*)\varphi(x^*)}{\varphi(x^*)F'(x^*) + F(x^*)\varphi'(x^*)} \\ &= x^* - \frac{\psi(x^*)}{\psi'(x^*)}, \end{aligned}$$

where $x^* := F^{-1}(y^*)$, and then

$$\frac{\mathrm{d}k}{\mathrm{d}x^*} = \frac{\psi(x^*)\psi''(x^*)}{(\psi'(x^*))^2}.$$

We claim that the derivative is positive. Indeed, since ψ is the increasing solution of $(\mathcal{A} - r)u = 0$,

$$\psi''(x^*) = \frac{2}{\sigma_e^2}(\alpha x^* \psi'(x^*) + r\psi(x^*)).$$

It is clear from Table 3 that $y^* > F(k) > F(0)$ and $x^* > 0$. Therefore $\psi''(x^*) > 0$, and then $\frac{\mathrm{d}k}{\mathrm{d}x^*} > 0$, which establishes the claim.

Because F is monotonously increasing,

$$\frac{\mathrm{d}k}{\mathrm{d}y^*} > 0, \text{hence } \frac{\mathrm{d}y^*}{\mathrm{d}k} > 0.$$

.

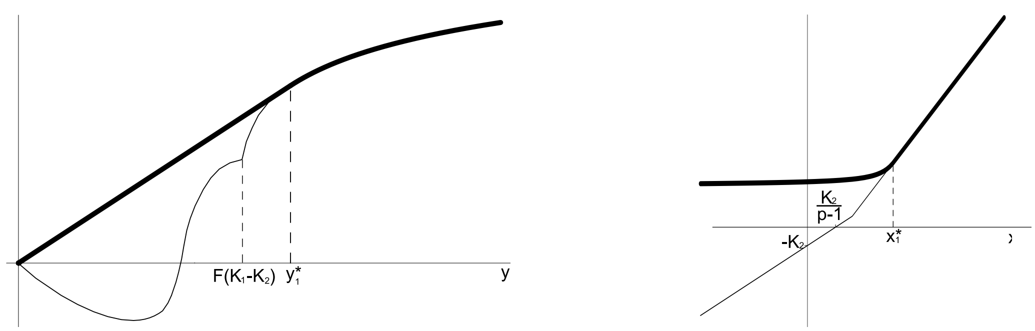
From the argument in section 3.1, if $k = 0$, then $y^* = F(q)$. Therefore, $k \downarrow 0$ and $y^* \downarrow F(q) < F(K_1 - K_2)$, as $p \to \infty$. The last inequality is due to $K_1 - K_2 > q$. From the definition of H_i, $H_1(y) < H_2(y)$ on $(0, F(K_1 - K_2))$, so $\{y : W_1(y) < W_2(y)\} = (0, F(K_1 - K_2))$ if p is large enough, that is, greater than or equal to, say $\bar{p}$. Hence the smallest concave majorant W_a of $H = \max(H_1, H_2)$ is

$$W_a(y) = \begin{cases} H(y) & (y_2^* < y < z_{a,1}, z_{a,2} < y) \\ \dfrac{yH(y_2^*)}{y_2^*} & (0 < y \le y_2^*) \\ \dfrac{H(z_{a,2}) - H(z_{a,1})}{z_{a,2} - z_{a,1}}(y - z_{a,1}) + H(z_{a,1}) & (z_{a,1} \le y \le z_{a,2}), \end{cases}$$

where $(z_{a,1}, z_{a,2})$ is the solution of

$$\begin{cases} H'(z_{a,1}) = H'(z_{a,2}) \\ H(z_{a,2}) - H(z_{a,1}) = H'(z_{a,1})(z_{a,2} - z_{a,1}), \end{cases}$$

and the number of tangent points of W_a and H is three, namely $y = y_2^*, z_{a,1}$, and $z_{a,2}$. See Figure 6. □



(i) (ii)

FIGURE 5. **$1 \le p \le \bar{p}$.** The solid line in (i) is $H_a(y)$ and the thick line is $W_a(y)$. The solid line in (ii) is $h(x)$ and the thick line is $V_a(x)$, and $x_1^* := F^{-1}(y_1^*)$.

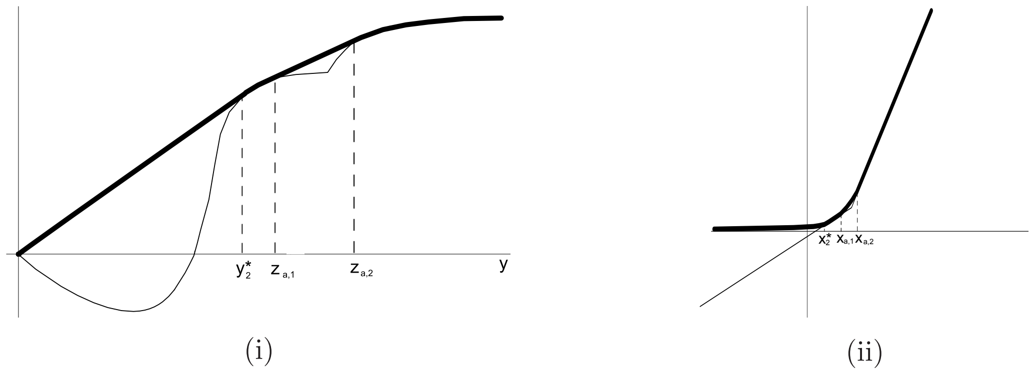


FIGURE 6. $\boldsymbol{p \geq \bar{p}}$. The solid line in (i) is $H_a(y)$ and the thick line is $W_a(y)$. The solid line in (ii) is $h(x)$, the thick line is $V_a(x)$, $x_2^* := F^{-1}(y_2^*)$, and $x_{a,i} := F^{-1}(z_{a,i})$.

4.3. **Numerical Example.** We solved two numerical examples with (I)$K_1 = 0.4, K_2 = 0.3$ and (II)$K_1 = 0.5, K_2 = 0.3$. The other parameters are not changed from those in section 3.2. Recall that $q = 0.134435$ in our example in section 3.2. Hence $K_1 - K_2 = 0.1 < q$ in case (I) and, on the other hand, $K_1 - K_2 = 0.2 > q$ in case (II). In the same way as in section 3.2, the graphs of $W_a(y)$, $W_b(y)$, and $W_c(y)$ are shown with $H(y)$ in Figure 5, 6, 7, and 8 for various p values. We omit, however, these graphs for case (I) because the graphs are essentially the same as in case (II). Indeed, in case (I), $p \geq 1, 0 < p < 1$, and $p \leq 0$ correspond to Figure 5, Figure 7, and Figure 8, respectively. Note that, in case (I), we do not need to consider the possibilities that $p \geq 1$ may be split into two sub-cases, and hence a graph like Figure 6 shall not occur. See section 4.1 for details.

Now, for case (II), Figure 5 and 6 correspond to $p \geq 1$ and show two patterns of W_a and H_a as discussed in section 4.2, depending on whether $1 \leq p \leq \bar{p}$ or $p \geq \bar{p} \geq 1$. Note that Figures 5, 6, 7, and 8 are sketches of the transformed functions while the shape of the curves remains true to the real functions. The purpose is merely to make the characteristics of the graphs easier to see. In each Figure, we also plot the functions in the original space.

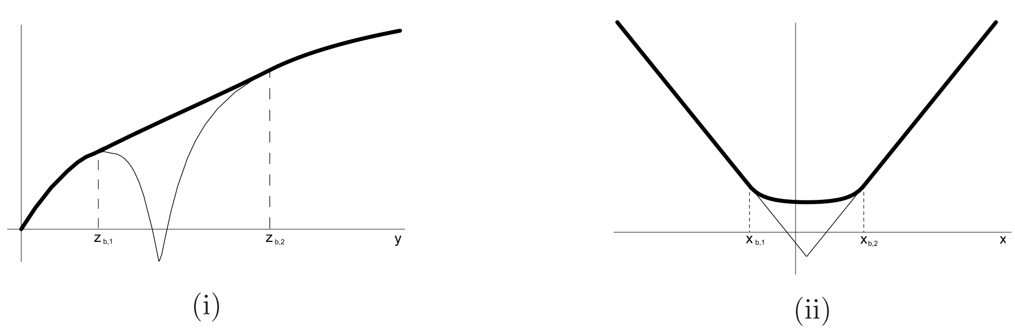


FIGURE 7. $\boldsymbol{0 < p < 1}$. The solid line in (i) is $H_b(y)$ and the thick line is $W_b(y)$. The solid line in (ii) is $h(x)$, and the thick line is $V_b(x)$.

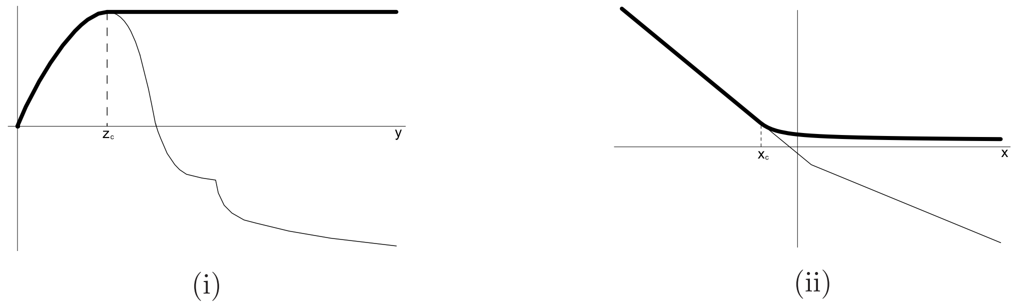


FIGURE 8. $\boldsymbol{p \leq 0}$. The solid line in (i) is $H_c(y)$ and the thick line is $W_c(y)$. The solid line in (ii) is $h(x)$, and the thick line is $V_c(x)$.

The exercise regions $\mathcal{E}$ are shown in Figure 9 (for case (I)) and Figure 10 (for case (II)). As discussed in 4.1, Figure 9 is similar to Figure 4(i) while the symmetry in Figure 4(i) is lost due to $K_1 \neq K_2$. On the other hand, Figure 10 shows a considerably strange shape of exercise region. In case (I), $\mathcal{E}$ consists of two convex sets, $\mathcal{E}_1$ and $\mathcal{E}_2$. Note that each curve has the asymptotic lines, $x_1 - x_2 = q$ and $x_1 - x_2 = -q$. On the other hand, $\mathcal{E}$ is decomposed into three convex sets, $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ in case (II). These regions are divided by three lines $x_1 - x_2 = q$, $x_1 - x_2 = -q$, and $x_1 - x_2 = K_1 - K_2$, each of which are the asymptotic lines. The fact that $\mathcal{E}$ does not contain the line $x_1 - x_2 = K_1 - K_2$ can be considered as the counterpart of Remark 3.1 where we had $K_1 = K_2 = 0$. This shape of $\mathcal{E}$ may not be *a priori* predicted, so this result may contribute to numerical solutions as well.

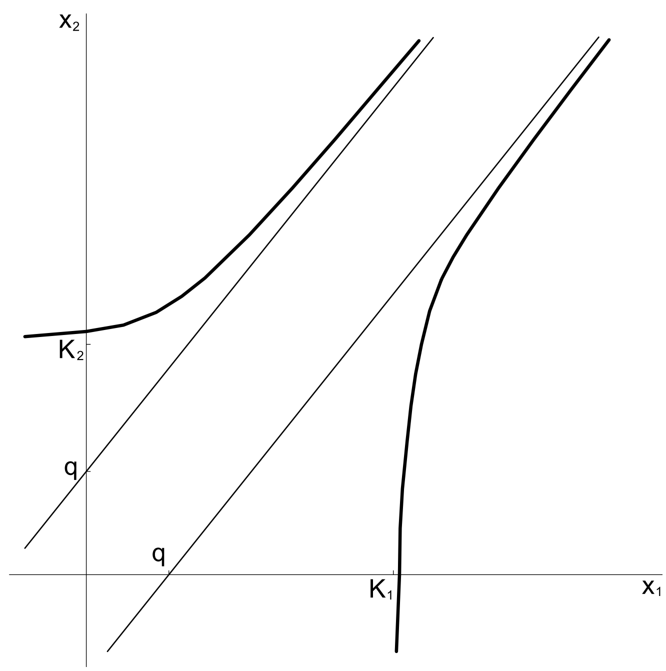


FIGURE 9. $\boldsymbol{K_1 - K_2 < q}$. The exercise region is the union of the region $\mathcal{E}_1$ above the left curve and the region $\mathcal{E}_2$ below the right curve.

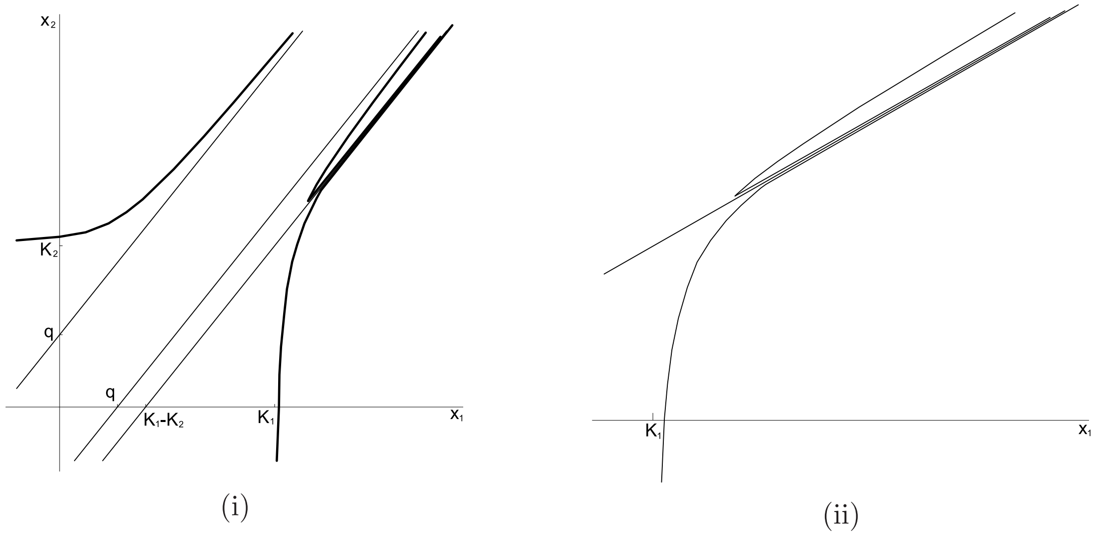


Figure 10. $\boldsymbol{K_1 - K_2 \geq q}$. (i) The exercise region $\mathcal{E}$ is the union of the region $\mathcal{E}_1$ above the leftmost curve, the region $\mathcal{E}_2$ below the rightmost curve, and the narrow region $\mathcal{E}_3$ between the two lines $x_1 - x_2 = K_1 - K_2$ and $x_1 - x_2 = q$. (ii) The graph in which the regions $\mathcal{E}_2$ and $\mathcal{E}_3$ are magnified (the line $x_1 - x_2 = q$ is omitted). Make sure that $\mathcal{E}_2$ and $\mathcal{E}_3$ doesn't contain the line $x_1 - x_2 = K_1 - K_2$.

5. Conclusion Remarks

In this paper, we solved the optimal stopping problem on the maximum of the two OU processes, by reducing the dimension. Our setting may seem specific, but indeed the class of the problems to which the method in this paper can be applied is rich. For example, we can solve the problems with payoff functions in the form of any linear combinations of maximum and minimum of x_1 and x_2, or in the form of $f(x_1 - x_2)$ for any real function f. Moreover, these method can be used to solve the problems with some other processes, such as Brownian motions. We believe these results can be the benchmarks against the problems to which the method we offerred cannot be applied, such as the problems with $\alpha_1 \neq \alpha_2$.

There are also some future tasks about this problem. We check the effect of changing σ_e to exercise region $\mathcal{E}$, and find, interestingly, that the following observation seems to be true from the results of computer calculations.

Observation 5.1. *Let $\mathcal{E}$ be the optimal exercise region with $\sigma_e = \sigma^*$, and let $\mathcal{E}'$ be that with $\sigma_e = m\sigma^*$ $(m \in \mathbb{R}_+)$. Then $(x_1, x_2) \in \mathcal{E}$ if and only if $(mx_1, mx_2) \in \mathcal{E}'$.*

In other words, if σ_e changed from σ^* to $m\sigma^*$, then the shape of exercise region doesn't change, and the scale is increased m times. Note that, in this model, the changes of σ_i and ρ affect the problem only through the change of σ_e, hence this phenomenon explains the effect of changing σ_i and ρ as well. We shall leave a rigorous proof of this phenomenon an open problem.

6. Appendix

This appendix explains the outline of the method used in section 3.1. The proofs and detailed explanation of the following propositions are in Dayanik and Karatzas [5]. See also Dynkin [8]. Let the stochastic process $X = \{X_t; t \geq 0\}$ with state space $(a, b) \subset \mathbb{R}$ satisfy the stochastic differential equation,

$$\mathrm{d}X_t = \alpha(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t.$$

Let the continuous function $h(x)$ be the reward function, then the value function $V(\cdot)$ is given by

$$V(x) = \sup_{\tau \in \mathcal{S}} \mathbb{E}_x[e^{-\beta\tau} h(X_\tau)]. \tag{6.1}$$

We shall define the differential operator $\mathcal{A}$ by

$$\mathcal{A}u(\cdot) = \frac{1}{2}\sigma^2(\cdot)\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(\cdot) + \alpha(\cdot)\frac{\mathrm{d}u}{\mathrm{d}x}(\cdot), \quad u \in \mathcal{C}^2 \tag{6.2}$$

The increasing and decreasing solutions $\psi(\cdot)$ and $\varphi(\cdot)$ of ordinary differential equation $(\mathcal{A} - \beta)u(\cdot) = 0$ are denoted by

$$\psi(x) = \begin{cases} \mathbb{E}_x[e^{-\beta\tau_c} 1_{\tau_c < \infty}] & (x \leq c) \\ 1/\mathbb{E}_c[e^{-\beta\tau_x} 1_{\tau_x < \infty}] & (x > c) \end{cases},$$

$$\varphi(x) = \begin{cases} 1/\mathbb{E}_c[e^{-\beta\tau_x} 1_{\tau_x < \infty}] & (x \leq c) \\ \mathbb{E}_x[e^{-\beta\tau_c} 1_{\tau_c < \infty}] & (x > c) \end{cases},$$

for arbitrary fixed $c \in (a, b)$, where $\tau_z = \inf\{t \geq 0; X_t = z\}$ for $z \in (a, b)$. A function u is called F-concave if, for every $a \leq l < r \leq b$ and $x \in [l, r]$, we have

$$u(x) \geq u(l)\frac{F(r) - F(x)}{F(r) - F(l)} + u(r)\frac{F(x) - F(l)}{F(r) - F(l)}.$$

Proposition 6.1. *$U(x)/\varphi(x)$ is F-concave if and only if $U(x)$ is β-excessive, i.e,*

$$U(x) \geq \mathbb{E}[e^{-\beta\tau} U(X_\tau)],\ \forall \tau \in \mathcal{S}, \forall x \in (a, b).$$

Proposition 6.2. *$V(x)$ is the smallest nonnegative majorant of $h(x)$ such that $V(x)/\varphi(x)$ is F-concave.*

We shall define the value l_a and l_b by

$$l_a := \limsup_{x \downarrow a}(h^+/\varphi)(x), \quad l_b := \limsup_{x \uparrow b}(h^+/\psi)(x).$$

Proposition 6.3. *Let $W(y) : [0, \infty) \to \mathbb{R}$ be the smallest nonnegative concave majorant of*

$$H(y) := \begin{cases} (h/\varphi) \circ F^{-1}(y) & (y > 0) \\ l_a & (y = 0) \end{cases},$$

then the value function V is denoted by $V(x) = \varphi(x)W(F(x))$.

Proposition 6.4. *The value function V(x) is continuous. If $l_a = l_b = 0$, then the optimal stopping rule of (5,1) is*

$$\tau^* = \inf\{t \geq 0; X_t \in \mathcal{E}\},$$

where the exercise region $\mathcal{E}$ is given by

$$\mathcal{E} = \{x \in [c,d] : V(x) = h(x)\}.$$

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