Beta Smile and Coskewness: Theoretical and Empirical Results in Options Markets

Masahiko Egami, Yanming Shu, Larry W. Taylor, and Wenlong Weng*

Abstract

This paper investigates the characteristics of the higher moment risks of option returns, namely beta and coskewness. Under mild assumptions, the investors' decreasing absolute risk aversion can result in a U-shaped pattern (*a beta smile*) for put option betas, though call option betas are always non-decreasing in the strike price. The coskewness for call options can be an inverted U-shape when the underlying returns are negatively skewed. We discuss some implications of these results in the context of the investor's risk preference for skewness. We also provide empirical evidence that supports our theoretical results.

Key Words: Options return, beta, coskewness, smiles, stochastic discount factor
JEL Classification: G12, G13

1 Introduction

The importance of call and put options cannot be exaggerated since these options are usually included as part of an investment portfolio. Hence the knowledge of options beta (i.e., risk-return analysis) is very helpful for investment purposes. To this end, Cox and Rubinstein [5] show that call options beta (resp. put options beta) is an increasing (resp. decreasing) function of strike price in the binomial tree model and the Black-Scholes model under the assumption that the single-factor capital asset pricing model (CAPM) holds true (see pp.189-196). While the CAPM of Sharpe [17]-Lintner [14] marks an important milestone in the development of modern asset pricing theory, it suffers from poor empirical support.[1] The binomial tree model and the Black-Scholes model as the limiting case of the former assume the completeness of the market for which there exists a unique martingale measure (and hence stochastic discount factor (SDF)). Equivalently, all the derivative securities are replicable.

In this paper, we shall show, as one of our main contributions, that under mild assumptions but without assuming the validity of the CAPM or the completeness of the market (see Assumption 2.1), call options beta is monotonically increasing in the strike price (hence confirming the result of Cox and Rubinstein

*M. Egami is at the Graduate School of Economics, Kyoto University, Kyoto, Japan, Email: egami@econ.kyoto-u.ac.jp. Y. Shu is at Department of Finance, St. John's University, Taiwan, Email: yms2@mail.sju.edu.tw. L. W. Taylor is at the College of Business and Economics, Lehigh University, Bethlehem, PA 18015, USA, Email: lwt0@lehigh.edu, and W. Weng is at the College of Business and Economics, Lehigh University, Bethlehem, PA 18015, USA, wew4@lehigh.edu.

[1]Evidence of failed empirical records may be found in Campbell, Lo, and MacKinlay [3].

[5] in a more general sense) and put options beta can reveal a U-shaped pattern (a beta smile) in the strike price. Our results are supported by empirical evidence.

These findings can also shed light on the issue of the investor's preference for skewness and on the related problem of proper specification of the SDF. In other words, our theoretical analysis shows that the characteristics of higher-moment risks depend on information about investors' risk preference embedded in the SDF. The empirical failure of CAPM can be traced to its restrictive assumptions, among them, the normality assumption of the return distribution. Empirical evidence indicates that returns of financial assets are not normally distributed, and in fact, the return distribution tends to be fat-tailed and skewed. Skewness in the return distribution will have an impact on the portfolio selection of investors with a preference for it.[2] Arditti [1] shows that under decreasing absolute risk aversion, investors will exhibit a preference for positively skewed portfolios.[3] The issue of investor's preference for skewness is related to the specification of the SDF, which we denote by m. Let us define the beta and coskewness risk factors as, respectively, $\beta_i := \text{cov}(r_i, r_M)/\sigma_M^2$ and $\gamma_i := \text{cov}(r_i, r_M^2)/\sigma_M^3$. From the fundamental pricing equation, it follows that,

$$\mathbb{E}[(1 + r_i)m] = 1 \tag{1.1}$$

where $\mathbb{E}$ denotes the expectation operator, r_i is the net return on any asset and m is strictly positive. Likewise, from (1.1),

$$-\text{Cov}(1 + r_i, m) + \mathbb{E}(1 + r_i)\mathbb{E}(m) = 1$$

which by assuming the existence of a risk-free asset with rate of return r_f,can be written as,

$$-\text{Cov}(r_i, m) = \frac{1}{1 + r_i}(\mathbb{E}(r_i) - r_f). \tag{1.2}$$

It is clear that a linear SDF $m = a + br_M$ results in the monotonicity (in the strike price) for call and put betas. Indeed, with a linear SDF, we can rewrite (1.2) as

$$\mathbb{E}(r_i) - r_f = \frac{\text{Cov}(r_i, r_M)}{\text{Var}(r_M)}(\mathbb{E}(r_M) - r_f)$$

Given that expected option returns are monotonically increasing in the strike price (Coval and Shumway[4]), this equation suggests that the option beta must increase with the strike price. On the other hand, Harvey and Siddique [9], among others, assume that the SDF is quadratic in the market return

$$m = a + br_M + cr_M^2 \tag{1.3}$$

with $a > 0, b < 0$ and $c > 0$. They show that the quadratic form can be linked to an important property of non-increasing absolute risk aversion which can be explicitly modeled as skewness in a two-period model. Consider that substituting (1.3) into (1.2) yields

$$\mathbb{E}(r_i) - r_f = -b(1 + r_f)\text{Cov}(r_i, r_M) - c(1 + r_f)\text{Cov}(r_i, r_M^2). \tag{1.4}$$

[2] It is intuitively accepted that investors have a preference for skewness. Compare the following two fair games, A and B. Game A costs \$1 to play and has one-in-one-million chance of paying out a million dollars. Game B pays \$1 up front but has a one-in-one-million chance of costing the player a million dollars. Although both games have identical first and second moments, the third moments are of opposite sign. The return from game A is positively skewed and the return from game B is negatively skewed. Presumably, most people would prefer game A to B. That is, most people would wager a few dollars for a small chance of wining a large fortune.

[3] Additionally, Scott and Horvath [15] point out that under the assumptions of 1) an increasing utility function, 2) decreasing absolute risk aversion and 3) strict consistency for moment preference (i.e. all moments are always associated with the same preference direction for any wealth level), investors will exhibit preference for odd moments and aversion for even moments.

Equation (1.4) can be considered as the three-moment CAPM model and is an unconditional version of Harvey and Siddique[9]. The equation is assumed to hold true for *all* risky assets, including options written on the market portfolio. As the strike price increases, the left hand side of (1.4) increases and so does the right hand side. However, the expected option return depends not only on the beta but also on systematic skewness, that is, any coskewness captured by the term $\mathrm{Cov}(r_i, r_M^2)$. Hence, if the SDF is quadratic in the market return, it is not clear whether the option beta is necessarily increasing in the strike price – even though the expected option return is definitely increasing in the strike price. We are motivated to understand these interrelated issues (option beta as a function of the strike price, preference for skewness, and specifications of SDF's).

We can summarize our results as follows:

1. Put-option betas can reveal a U-shaped pattern (a beta smile) in the strike price. If there is a beta smile for put-option betas, decreasing absolute risk aversion is a necessary condition for producing the beta smile, and systematic skewness must be priced in the market. (Proposition 2.1).[4] Our theoretical result is consistent with the general belief that the SDF includes a preference for skewness (Harvey and Siddique [9], Dittmar [6]) and cannot be linear in the market return.[5]

2. For call options, we prove that beta is monotonically increasing in the strike price irrespective of the specification of the SDF (Proposition 3.1). Hence we extend the classical results in Cox and Rubinstein [5] to incomplete markets where the single-factor CAPM does not necessarily hold. With respect to the skewness issue, the options beta, in turn, is not informative. We thus look directly to the higher moment, that is, systematic skewness. We find that coskewness is an inverted U-shape pattern in the strike price when the underlying is negatively skewed (Proposition 3.3).

3. Our empirical results show that call option betas are increasing in the strike and put option betas exhibit a beta smile. The call options beta is uninformative in the sense that they always increase with the strike price regardless of whether skewness is priced. But the put options beta smile indicates that systematic skewness must be priced in the put option market. The empirical results support the theoretical analysis about the call and put betas. Our empirical results on the coskewness of the call option returns show an increasing pattern in the strike price. Although it is not clear whether the skewness preference is priced in the call options market, this monotonic coskewness is consistent with our sample in that the underlying is positive skewed.

The rest of this paper is organized as follows. In section 2, we examine put-option betas to show the possibility of a beta smile. If so, skewness is priced. In section 3, we examine call-option betas and coskewness. We derive some general theoretical results that describe the relationship between the beta risk and the strike price, and between the systematic skewness and the strike price. In section 4, we provide some empirical evidence that supports our theoretical results. Section 5 then concludes with a brief discussion of our results. Technical proofs are relegated to the Appendix.[6]

[4] Accordingly, investor's risk preference cannot be modeled with utility functions that result in a linear SDF (such as the logarithmic utility).

[5] In a recent article, Coval and Shumway [4] show that expected option returns are too low to be consistent with CAPM predictions and suggest additional risk factor should be priced. Our findings suggest that coskewness risk is a valid candidate.

[6] We note that Kraus and Litzenberger [12] develop a three-moment capital asset pricing model that identifies two systematic

2 Put options

We work under the following assumptions in the entire paper:

Assumption 2.1. *1. $d\mathbb{E}[m|S]/\,\mathrm{d}S < 0$,*

2. $d^2\mathbb{E}[m|S]/\,\mathrm{d}S^2 \geq 0$.

We denote by S the price of the underlying market portfolio (on which the option is written) at the maturity of the option. The first assumption is based on the fact that risk averse investors have decreasing marginal utility on wealth, and the second requires that investors exhibit non-increasing absolute risk aversion. The non-increasing absolute risk aversion implies that $\mathbb{E}[m|S]$ must be either a linear or convex function in S. For $\mathbb{E}[m|S]$ to be convex in S, decreasing absolute risk aversion must hold true.

Suppose that the distribution of asset price S at the maturity has density $f(\cdot)$. We respectively denote by X and S_0 the strike price and today's price of the underlying. The put option beta is defined as $\beta_p := \mathrm{Cov}(r_p, r_M)/\sigma_M^2$. Given the current value of the market portfolio S_0 and the SDF, m, we have

$$\mathrm{Cov}(r_p, r_s) = \frac{\int^{S=X}(X-S)(S-\bar{S})f(S)\,\mathrm{d}S}{\int^{S=X} S_0(X-S)\mathbb{E}[m|S]f(S)\,\mathrm{d}S} \tag{2.1}$$

The derivative of put option beta with respect to the strike price can be obtained as

$$\frac{\int^{S=X} S(S-\bar{S})f(S)\,\mathrm{d}S \int^{S=X} S_0\mathbb{E}[m|S]f(S)\,\mathrm{d}S - \int^{S=X}(S-\bar{S})f(S)\,\mathrm{d}S \int^{S=X} SS_0\mathbb{E}[m|S]f(S)\,\mathrm{d}S}{\left(\int^{S=X} S_0(X-S)\mathbb{E}[m|S]f(S)\,\mathrm{d}S\right)^2 \sigma_M^2}$$

The denominator is positive, but a brief check of the numerator reveals that it can be either positive or negative. That is, the derivative of the put option beta does not provide a conclusive result. The implication of this uncertainty is that the shape of put option betas depends on the exact shape of $\mathbb{E}[m|S]$, or in other words, the investors' representative utility function.

We state one of our main results here which helps us to better understand *how* put option betas reflect investors' risk preference:

Proposition 2.1. *For a put option written on the market portfolio as the underlying, beta can either be monotonically increasing in the strike price or exhibit a U-shaped pattern, i.e., a beta smile. However, decreasing absolute risk aversion is a necessary condition for producing the beta smile.*

3 Call Options

Here we show that beta smiles do not occur in the returns of call options. Suppose that the distribution of price S at the maturity has density $f(\cdot)$. We respectively denote by X and S_0 the strike price and today's price of the underlying.

risk factors, namely, market beta and coskewness, to account for the respective second- and third-moment risks. Also, empirical results from these models suggest that they are much better at explaining cross-sectional variation in expected returns than the single-factor CAPM. See, for example, Krause and Litzenberger [12], Friend and Westerfield [8], Sears and Wei [16], Lim [13], and Harvey and Siddique [9].

3.1 Characteristic of Options Beta

Similar to the previous section, the beta of the call option is then defined as $\beta_c := \text{Cov}(r_c, r_M)/\sigma_M^2$. The following Proposition states the relationship of changes in the call option beta with respect to changes in the strike price X:

Proposition 3.1. *For a call option with the market portfolio as the underlying, beta is monotonically increasing in the strike price; that is*

$$\partial\beta_c/\partial X > 0 \tag{3.1}$$

where X is the strike price.

Three immediate consequences follow from Proposition 3.1. First, the call option beta is always positive and greater than one. This can be seen by setting $X = 0$, since when $X = 0$, the option is just the underlying market portfolio and its beta is one. Second, when the underlying is a security, not the market portfolio, Proposition 3.1 is still true as long as the security itself has a positive beta. Furthermore, call option returns always have greater betas, in absolute value, than their respective underlying assets.[7] Finally, the variance of any call option returns is greater than that of the underlying returns regardless of its distribution. The last statement is true because the proof for Proposition 3.1 shows that $\text{Cov}(r_c, r_s)$ is increasing in the strike price regardless of whether the underlying is the market portfolio or an individual stock. Therefore, $\text{Cov}(r_c, r_s) > \text{Var}(r_s)$ for any positive strike price[8] and we have $\text{Cov}(r_c, r_s) \leq \sqrt{\text{Var}(r_c)}\sqrt{\text{Var}(r_s)}$. That is, combining the two inequalities, we obtain $\text{Var}(r_c) > \text{Var}(r_s)$.

The monotonicity of call option betas in Proposition 3.1 concurs with the CAPM model. Since we assume neither a specific stochastic process for the underlying nor a specific utility function for the investors' risk preference, Proposition 3.1 implies that the monotonicity property of call option betas can be generalized and is universally true under the assumptions stated therein.

3.2 Coskewness of Call Options Returns

In this section we investigate the characteristics of the third moment of call option returns, namely coskewness, to better understand if the skewness preference is priced. If investors prefer positive skewness, coskewness can be important for option investors. We show that the coskewness of call options can show an inverted U-shaped pattern as a function of the strike price. To show how coskewness of option returns depends on the skewness of the underlying security and the strike price, we assume, again for simplicity, that the market portfolio is the underlying asset.

Proposition 3.2. *If the SDF factor is negatively correlated with the market, the coskewness of a call option with its underlying security, $\mathbb{E}[(r_c - \bar{r}_c)(r_s - \bar{r}_s)^2]/\mathbb{E}[(r_s - \bar{r}_s)^3]$, is increasing in the strike price X when $\mathbb{E}[(r_s - \bar{r}_s)^3] > 0$ where r_c and r_s are option and underlying security returns, respectively.*

[7] In particular, no specific distribution is assumed for the underlying assets.

[8] The result is well known when the underlying follows a geometric Brownian motion.

On the other hand, when $\mathbb{E}[(r_s - \bar{r}_s)^3] < 0$, the coskewness is increasing in the strike price at first, and then decreasing in the strike price. To be more specific, we define $I_1(X) \triangleq \int_{S=X} \mathbb{E}(m|S)f(S)dS$ and $I_2(X) \triangleq \int_{S=X} (S - \bar{S})^2 f(S)dS$, and we obtain

Proposition 3.3. *If the SDF is negatively correlated with the market, the coskewness of a call option with its underlying security,* $\mathbb{E}[(r_c - \bar{r}_c)(r_s - \bar{r}_s)^2]/\mathbb{E}[(r_s - \bar{r}_s)^3]$ *is first increasing and eventually decreasing in the strike price if* $\mathbb{E}[(r_s - \bar{r}_s)^3] < 0$*. The coskewness has an inverted U-shape if*

$$I_1(X)(V - (X - \bar{S})^2) = (V(1 - F(X)) - I_2(X))\mathbb{E}[m|X] \tag{3.2}$$

has only one root.

These results show that when the underlying market return is positively skewed, systematic skewness for the call option will be positive and increase with the strike price. On the other hand, when the market is negatively skewed, the systematic skewness will be positive and first increase, but then decrease, to be negative. As we know, the market premium for asset coskewness is negative when market return is positively skewed, and is positive when market returns are negatively skewed. Therefore, when the market is positively skewed, an increase in the strike price will increase the coskewness and hence decrease the expected option return. On the other hand, when the market is negatively skewed, an increase in the strike price will, initially increase the coskewness and hence increase the expected option return. However, further increase in the strike price will eventually decrease the coskewness and hence decrease the expected option return.

4 Empirical evidence

This section applies previous theoretical framework to the real data. The theoretical analysis does not assert whether the investors' risk references are embedded in the markets. It only tells that if there are such risk preferences, how they should be reflected by the options' betas and coskewness. It thus provides a way for us to identify and examine them.

We use European S&P500 index options (SPX) data to examine the beta and coskewness for call and put options. The data is taken from the Chicago Board Options Exchange, and the sample period is from June 1990 to December 2002. The option and underlying S&P500 index returns are computed for the holding periods from one to six weeks. The ending dates are always the expiration dates. For one-week return, the holding period is four days, starting on the third Tuesday and ending on the third Friday of the month in which the options expire. Due to the fact that options held to expiration often have net returns of -1, the conventional log return commonly used in equity studies renders infinite negative values. Thus, in this study net returns are used.

For each holding period, we sort the options based on their moneyness, the ratio of the level of the S&P500 index to the strike price. Twenty groups are selected for the call and put options, respectively. For call options, groups 1 - 9 are in-the-money (ITM) options, group 10 is at-the-money (ATM) option, and groups 11 - 20 are out-of-the-money (OTM) options. On the other hand, for put options, groups 1 - 9 are out-of-the-money (OTM) options, group 10 is at-the-money (ATM) option and groups 11 - 20 are in-the-money (ITM) options. At-the-money options are classified as those whose moneyness is between 0.9985 and 1.0015. A 0.3 percentage point incremental interval is used for each group. The tight moneyness interval allows the grouping method to maintain consistent risk characteristics for the options within a group and permits only one option within each group at a set point in time. We never reclassify an option once it is placed within a group.

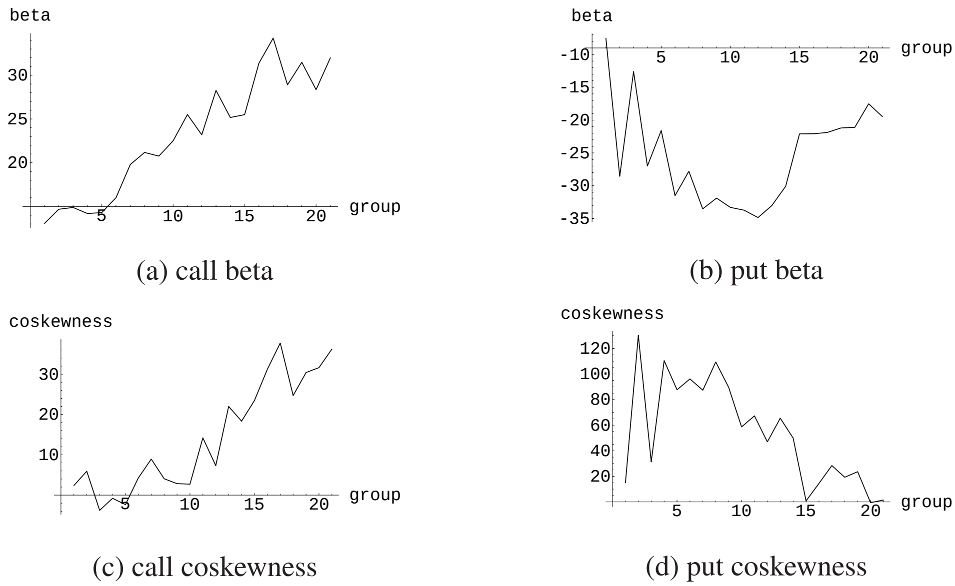

Figure 1: Empirical evidence from S&P 500 index options with maturities 1-3 weeks.

Figure 1 shows the empirical results computed from S&P 500 index options with the time to maturity being one, two and three weeks. Computation of beta and coskewness follows the definitions given in the previous sections. The empirical evidence from S&P 500 index options are consistent with previous proofs on the betas and coskewness of call and put options. The beta of call option in Figure 1 (a) suggests a monotonic increase with strike price, but the beta of put options exhibits an U-shaped pattern. This is clearly shown in Figure 1 (b). This suggests that the S&P500 index option market reflects our Case 3 in previous section. Figure 1 (c) shows the coskewness of the call options. Given that the skewness of the sample returns of the underlying S&P 500 index is positive, the coskewness of the call options indicates a monotonically increase in the strike price. The monotonicity here is consistent with Proposition 2.5. The pattern of the coskewness of the put options is less clear, though it might suggest a monotonic decreasing pattern, as indicated in Figure 1 (d).

To further check the above results, we also examine the betas and coskewness by using four-, five- and six-week returns data for call and put options. The calculated betas and coskewness, shown in Figure 2, largely confirm our previous results. For call options, beta and coskewness show a similar pattern as in Figure 1. For put options, beta clearly demonstrates a U-shaped pattern. But put-option coskewness suggests a weak inverse U-shaped pattern.

Although it is counterintuitive to the results from the CAPM, a beta smile for put options is a direct result from a broader set of investors' preferences. The SDF may be expressed as $U'(S_{t+1})/U'(S_t)$, where S is aggregate wealth represented by the value of market portfolio in our context. A first order Taylor approximation gives $m = 1 + \frac{U''(S_t)}{U'(S_t)} r_M$, linear in the market return. Alternatively, a second

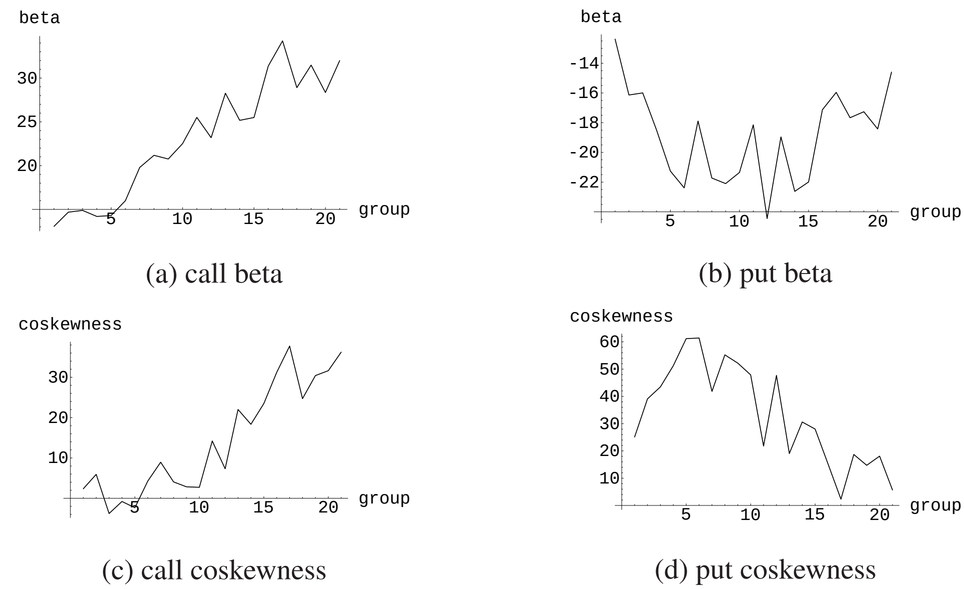

Figure 2: Empirical evidence from S&P 500 index options with maturities 4-6 weeks.

order approximation gives $m = 1 + \frac{U''(S_t)}{U'(S_t)} r_M + \frac{U'''(S_t)}{U'(S_t)} r_M^2$. The third derivative of the utility captures the curvature and represents the decreasing absolute risk aversion for investors. As shown in Case 3 in section 3, a large curvature is necessary to generate the beta smile. It is also clear that the SDF in the later specification incorporates a preference for skewness in the return generating equation. So, the pattern of put option beta reflects the strength of investors' preference for skewness. If the preference for skewness is weak, the last term on the right hand side of the second order approximation becomes irrelevant and the SDF specification reverts back to the CAPM; if so, the beta smile pattern would not be apparent.

5 Conclusion

It should not be surprising to find empirical evidence that investors' skewness preference is embedded in option pricing since it is generally believed that investors have a preference for skewness. Our investigation here, however, has shown the asymmetric effects of skewness preference on the structures of call and put options.

It is well known that the market is incomplete. Indeed, if the market were to be complete, there would be a unique equivalent martingale measure, and that measure would capture the investor's risk preference across all the markets (stocks, options, and bonds). In this paper, we developed a theoretical framework that can be used to investigate how markets reflect investors' risk preferences without assuming the completeness for the market. To our knowledge, our paper is the first one to consider the case of incomplete markets under the mild assumptions, as for example, Cox and Rubinstein (1985) consider only the case of complete markets.

Our empirical results indicate that the investors with different risk preference participate in different markets. For call options, the design of the contract truncates the effect of negative skewness and call option investors are mainly concerned with variance of the underlying market portfolio. In contrast, put option investors are mainly concerned with negative skewness of the underlying portfolio. Put options protect their holders from a significant loss and can compensate loss averse investors when the underlying market drops. This asymmetric pattern implies that the strength of skewness preference is stronger for put option investors than for call option investors. Our empirical result indicates a higher premium demanded in the put options market for the risk protection benefits. Furthermore, the beta smile pattern may be consistent with Kahneman and Tversky [11] that investors tend to be more sensitive to reductions in their levels of well-being than to increments.

The put-option beta smile found in the S&P500 index option markets is thus consistent with the general belief that the SDF includes a preference for skewness and cannot be linear in the market return.[9] Therefore, the investors' risk preference cannot be modeled by those utility functions that would result in a linear discount factor such as quadratic or logarithmic utility functions (Harvey and Siddique [9]). In fact, arbitrage opportunities may indeed occur in quadratic utility economies. See, for instance, Vanden [18], Dybvig and Ingersoll [7], and Jarrow and Madan [10].

6 Appendix

6.1 Proof of Proposition 2.1

Our proof consists of a series of lemmas:

Lemma 6.1. *Let us fix $S_1 > 0$. For any positive convex function $F(S)$ such that $dF(S)/dS \le 0$ for all S, if there exists a point $S^* < S_1$, such that at $S = S^*$, $dF(S)/dS = F(S^*)/(S^* - S_1)$, then $F(S)/(S_1 - S)$ is decreasing in S for $S < S^*$ and increasing in S for $S^* < S < S_1$.*

Proof. By taking the derivative of $F(S)/(S_1 - S)$, for $S^* < S < S_1$, it suffices to show $F(S)/(S - S_1) < F'(S)$. Indeed, since F is convex in S, we have

$$F(S) - F(S^*) > F'(S^*)(S - S^*).$$

By substituting $F'(S^*) = F(S^*)/(S^* - S_1)$, we obtain, after simple algebra,

$$\frac{F(S)}{S - S_1} < \frac{F(S^*)}{S^* - S_1} = F'(S^*) < F'(S)$$

where the last inequality is due to the convexity of F. The case of $S < S^*$ is similar. □

Lemma 6.2. *If there exists $S^* < \bar{S}$ such that at $S = S^*$, $d\mathbb{E}[m|S]/ds = -\mathbb{E}[m|S]/(\bar{S} - S)$, then $d\mathrm{Cov}(r_p, r_M)/dX < 0$ for $X < S^*$.*

Proof. Take the derivative of $\mathrm{Cov}(r_p, r_M)$ with respect to X to obtain

$$\frac{\int^{S=X} S(S-\bar{S})f(S)\,\mathrm{d}S \int^{S=X} S_0\mathbb{E}[m|S]f(S)\,\mathrm{d}S - \int^{S=X}(S-\bar{S})f(S)\,\mathrm{d}S \int^{S=X} SS_0\mathbb{E}[m|S]f(S)\,\mathrm{d}S}{\left(\int^{S=X} S_0(X-S)\mathbb{E}[m|S]f(S)\,\mathrm{d}S\right)^2}. \tag{6.1}$$

[9] The SDF might also be path-dependent. See Bondarenko [2] for a discussion.

Denote the numerator as a function of the strike price, i.e., $H(X)$ and note that $H(0) = 0$. We show that $H(X)$ is decreasing in X when $X < S^*$. Taking the derivative of $H(X)$, we obtain

$$\begin{aligned}
H'(X) &= X(X-\bar{S})f(X)\int^{S=X} S_0\mathbb{E}[m|S]f(S)\,\mathrm{d}S + \int^{S=X} S(S-\bar{S})f(S)\,\mathrm{d}S(S_0\mathbb{E}[m|X]f(X)) \\
&\quad - (X-\bar{S})f(X)\int^{S=X} SS_0\mathbb{E}[m|S]f(S)\,\mathrm{d}S - \int^{S=X}(S-\bar{S})f(S)\,\mathrm{d}S(S_0X\mathbb{E}[m|X]f(X)) \\
&= S_0f(X)\int^{S=X}(X-S)\Big((\bar{S}-S)\mathbb{E}[m|X] - (\bar{S}-X)\mathbb{E}[m|S]\Big)f(S)\,\mathrm{d}S.
\end{aligned}$$

Lemma 6.1 suggests that $\frac{\mathbb{E}[m|S]}{\bar{S}-S}$ is decreasing in S for $S < S^*$. Hence for any X such that $X < S^*$, we have

$$\frac{\mathbb{E}[m|X]}{\bar{S}-X} < \frac{\mathbb{E}[m|S]}{\bar{S}-S}$$

for all $S < X$. So $H'(X) < 0$ which shows that $H(X)$ is decreasing in X when $X < S^*$. Since $H(0) = 0$, we have $H(X) < 0$ for $X < S^*$. □

Lemma 6.3. *Assuming that $H(X)$ as in Lemma 6.2, then $H(\infty) > 0$ and if $H(X) > 0$ for some $X = X_0$, then $H(X) > 0$ for all $X > X_0$.*

Proof. First, we show that $H(\infty) > 0$. This can be checked directly:

$$\begin{aligned}
H(\infty) &= \int_0^\infty S(S-\bar{S})f(S)\,\mathrm{d}S\int_0^\infty S_0\mathbb{E}[m|S]f(S)\,\mathrm{d}S - \int_0^\infty (S-\bar{S})f(S)\,\mathrm{d}S\int_0^\infty SS_0\mathbb{E}[m|S]f(S)\,\mathrm{d}S \\
&= S_0\mathrm{Var}(S)\mathbb{E}[m] > 0.
\end{aligned}$$

Next we show that $H(X)$ remains positive once it is positive. From Lemma 6.2, it is clear that $X_0 > S^*$. Now, consider case: $X_0 < X < \bar{S}$. Dividing interval $(0, X)$ into $(0, X_0]$ and (X_0, X), we can write $H(X)$ as

$$\begin{aligned}
H(X) &= \int_0^{X_0} S(S-\bar{S})f(S)\,\mathrm{d}S\int_0^{X_0} S_0\mathbb{E}[m|S]f(S)\,\mathrm{d}S - \int_0^{X_0}(S-\bar{S})f(S)\,\mathrm{d}S\int_0^{X_0} SS_0\mathbb{E}[m|S]f(S)\,\mathrm{d}S \\
&\quad + \int_{X_0}^{X} S(S-\bar{S})f(S)\,\mathrm{d}S\int_{X_0}^{X} S_0\mathbb{E}[m|S]f(S)\,\mathrm{d}S - \int_{X_0}^{X}(S-\bar{S})f(S)\,\mathrm{d}S\int_{X_0}^{X} SS_0\mathbb{E}[m|S]f(S)\,\mathrm{d}S.
\end{aligned}$$

The sum of the first two terms of the above equation is $H(X_0)$, assumed to be positive. So, if the sum of the last two terms is positive, we have $H(X) > 0$ for $X > X_0$. Denote by $G(X)$ the sum of the last two terms. It is clear that $G(X_0) = 0$. If $G(X)$ is increasing in X for $X \in [X_0, \bar{S})$, then $G(X) > 0$ for all $X \in [X_0, \bar{S})$. Taking the derivative with respect to X, we obtain

$$G'(X) = S_0f(X)\int_{X_0}^{X}(X-S)\Big((\bar{S}-S)\mathbb{E}[m|X] - (\bar{S}-X)\mathbb{E}[m|S]\Big)f(S)\,\mathrm{d}S.$$

Lemma 6.1 suggests that $\frac{\mathbb{E}[m|X]}{\bar{S}-X} \geq \frac{\mathbb{E}[m|S]}{\bar{S}-S}$ for all $S \in [X_0, X)$. It follows that $G'(X) > 0$, which in turn shows $H(X) > 0$ for all $X \in [X_0, \bar{S})$.

Finally, we show $H(X) > 0$ for all $X \geq \bar{S}$. For $X = \bar{S}$, we have

$$H'(X) = S_0 f(X) \int^{S=X} (X-S)(\bar{S}-S)\mathbb{E}[m|X] f(S)\, \mathrm{d}S > 0.$$

For $X > \bar{S}$, we write $H'(X)$ as

$$\begin{aligned} H'(X) &= S_0 f(X) \int^{S=X} (X-S)\Big((\bar{S}-S)\mathbb{E}[m|X] - (\bar{S}-X)\mathbb{E}[m|S]\Big) f(S)\, \mathrm{d}S \\ &= S_0 f(X) \int_0^{\bar{S}} (X-S)\Big((\bar{S}-S)\mathbb{E}[m|X] + (X-\bar{S})\mathbb{E}[m|S]\Big) f(S)\, \mathrm{d}S \\ &+ S_0 f(X) \int_{\bar{S}}^{X} (X-S)\Big((X-\bar{S})\mathbb{E}[m|S] - (S-\bar{S})\mathbb{E}[m|X]\Big) f(S)\, \mathrm{d}S. \end{aligned}$$

The first term of the last equation is positive. In the second term, $\mathbb{E}[m|S] > \mathbb{E}[m|X] > 0$ and $X - \bar{S} > S - \bar{S} > 0$. Therefore, the second term is also positive. Combining all the results, since $H(X)$ is increasing in X for $X > X_0$, $H(X_0) > 0$ implies that $H(X) > 0$ for all $X > X_0$. For the case, $X_0 \geq \bar{S}$, the conclusion is also true since $H(X)$ is increasing in X for all $X \geq \bar{S}$ and $H(X_0) > 0$. □

Proof of Proposition 2.1: The above three lemmas tell us that there are three possible cases, two of which would have a monotonic increasing beta and the third has a U-shaped beta in X. We present the three cases below.

Case 1: $X < \bar{S}$ and $\mathbb{E}[m|S]$ is linear in S: This case is demonstrated in Figure 1, where $\mathbb{E}[m|S]$ is a linear function in S. $\mathbb{E}[m|X]/(\bar{S}-X)$ and $\mathbb{E}[m|S]/(\bar{S}-S)$ are the absolute values of two slopes. Figure 1 shows that $\mathbb{E}[m|X]/(\bar{S}-X)$ is always greater than $\mathbb{E}[m|S]/(\bar{S}-S)$. Thus, $H'(X)$ is always positive and put option beta is monotonically increasing in the strike price. Since the CAPM assumes that the stochastic discount SDF is linear in the market return, Case 1 is analogous to the CAPM economy.

Figure A about here.

Case 2: $X < \bar{S}$ and $\mathbb{E}[m|S]$ is convex in S, but $\mathbb{E}[m|X]/(\bar{S}-X) > \mathbb{E}[m|S=0]/\bar{S}$ for all X: This case implies that $\frac{\mathbb{E}[m|S]}{\bar{S}-S}$ is increasing in S. We must have $\frac{\mathbb{E}[m|X]}{\bar{S}-X} \geq \frac{\mathbb{E}[m|S]}{\bar{S}-S}$ for all $S < X$. Hence $H'(X) > 0$ for all X and therefore, $H(X) > 0$ for all X with $H(0) = 0$. This is shown in Figure 2. Note that $\mathbb{E}[m|S]$ has the steepest slope at point $S = 0$. Figure 2 shows that for any $S < X$, $\mathbb{E}[m|X]/(\bar{S}-X)$ is always greater than $\mathbb{E}[m|S]/(\bar{S}-S)$. As was true for Case 1, $\mathbb{E}[m|S]$ decreases slowly as S increases and we observe a monotonically increasing function for the put option beta in X.

Figure B about here.

Case 3: $X < \bar{S}$ and $\mathbb{E}[m|S]$ is convex in S, but there exists some X such that $\mathbb{E}[m|X]/(\bar{S}-X) < \mathbb{E}^x[m|S=0]/\bar{S}$ holds: For this X, it follows that $\frac{\mathbb{E}[m|X]}{\bar{S}-X} < \frac{\mathbb{E}[m|S]}{\bar{S}-S}$ for all $S < X$. This implies that $H'(X) < 0$ at that point. But Lemma 6.3 shows that $H(X)$ becomes positive eventually. Hence there must exist a point X_0 such that $H(X) > 0$ for $X \in [X_0, \bar{S})$. It follows that there exists a critical point X_c, which is less than $\bar{S}$, such that put option beta is decreasing in the strike price for

$X < X_c$ and increasing for $X > X_c$. In this case, put option beta exhibits a U-shaped pattern, i.e., a beta smile. Under such condition, assume a second order Taylor approximation is legitimate, we obtain $m = 1 + \frac{U''(S_t)}{U'(S_t)} r_M + \frac{U'''(S_t)}{U'(S_t)} r_M^2$. Thus, it is evident that under this circumstance the coefficient $\frac{U'''(S_t)}{U'(S_t)}$ cannot be zero. Thus our analysis indicates that decreasing absolute risk aversion is only a necessary condition for generating the beta smile.

Note that there exists some $X_1 < X_c$ such that $\mathbb{E}[m|X]/(\bar{S} - X_1)$ equals the absolute value of the derivative of $\mathbb{E}[m|S]$ at $S = X_1$. Otherwise, we would have $\frac{\mathbb{E}[m|X]}{\bar{S}-X} < \frac{\mathbb{E}[m|S]}{\bar{S}-S}$ for all $X > 0$. This implies that for $H'(X) < 0$ for all X, and as such, contradicts $H(\infty) > 0$.

Figure C about here.

6.2 Proof of Proposition 3.1

Proof. The monotonicity of call option betas can easily be shown by taking the derivative with respect to X. Given the current value of the market portfolio S_0 and the SDF m, we have

$$\mathrm{Cov}(r_c, r_s) = \frac{\int_{S=X} (S - X)(S - \bar{S}) f(S)\, \mathrm{d}S}{\int_{S=X} S_0 (s - X) \mathbb{E}[m|S] f(S)\, \mathrm{d}S} \tag{6.2}$$

Therefore, the derivative of β_c with respect to X becomes, after some simplifications,

$$\partial\beta_c/\partial X$$
$$= \frac{\int_{S=X} S(S - \bar{S}) f(S)\, \mathrm{d}S \int_{S=X} S_0 \mathbb{E}[m|S] f(S)\, \mathrm{d}S - \int_{S=X} (S - \bar{S}) f(S)\, \mathrm{d}S \int_{S=X} S_0 S \mathbb{E}[m|S] f(S)\, \mathrm{d}S}{\left(\int_{S=X} S_0 (S - X)\mathbb{E}[m|S] f(S)\, \mathrm{d}S\right)^2 \sigma_M^2} \tag{6.3}$$

where $\bar{S} := \mathbb{E}(S)$. Since the denominator is positive, we only need to show that the numerator is positive, which can be written as

$$\begin{aligned} &\int_{S=X} S^2 f(S)\, \mathrm{d}S \int_{S=X} S_0 \mathbb{E}[m|S] f(S)\, \mathrm{d}S - \int_{S=X} S f(S)\, \mathrm{d}S \int_{S=X} S S_0 \mathbb{E}[m|S] f(S)\, \mathrm{d}S \\ &+ \int_{S=X} \bar{S} f(S)\, \mathrm{d}S \int_{S=X} S S_0 \mathbb{E}[m|S] f(S)\, \mathrm{d}S - \int_{S=X} S \bar{S} f(S)\, \mathrm{d}S \int_{S=X} S_0 \mathbb{E}[m|S] f(S)\, \mathrm{d}S. \end{aligned} \tag{6.4}$$

Defining $F(S)$ as the cumulative density function that corresponds to $f(\cdot)$, the last two terms in (6.4) can be combined as

$$S_0 \bar{S} (1 - F(X))^2 \mathrm{Cov}(\mathbb{E}[m|S], S|_{S>X}) \tag{6.5}$$

and the first two terms in (6.4) can be written as

$$\begin{aligned} &S_0 (1 - F(X))^2 \Big(\mathbb{E}(S^2|_{S>X}) \mathbb{E}(m|_{S>X}) - \mathbb{E}(S|_{S>X}) \mathbb{E}[\mathbb{E}(m|S) S|_{S>X}] \Big) \\ &= S_0 (1 - F(X))^2 \Big(\mathbb{E}(m|_{S>X}) \mathrm{Var}(S|_{S>X}) - \mathbb{E}(S|_{S>X}) \mathrm{Cov}(\mathbb{E}[m|S], S|_{S>X}) \Big). \end{aligned} \tag{6.6}$$

Combining (6.5) and (6.6), we obtain

$$S_0 (1 - F(X))^2 \Big(\mathbb{E}(m|_{S>X}) \mathrm{Var}(S|_{S>X}) - \mathrm{Cov}(\mathbb{E}[m|S], S|_{S>X}) (\mathbb{E}(S|_{S>X}) - \bar{S}) \Big). \tag{6.7}$$

Since $(\mathbb{E}(S|_{S>X}) - \bar{S}) \geq 0$ and m is negatively correlated with the market, the above expression is positive. $\square$

6.3 Proof of Proposition 3.2

Before we derive our result about call options coskewness, we first present the following four general mathematical propositions:

Lemma 6.4. *If a nonnegative random variable S is positively skewed, i.e.,* $\mathbb{E}[(S-\mathbb{E}(S))^3] > 0$*, then* *Cov*$[S,(S-\mathbb{E}(S))^2|_{S>X}] > 0$ *for any* $X \geq 0$.

Proof. First, let us consider the case 1: $X \geq \mathbb{E}(S)$. We know that

$$\begin{aligned}
&\text{Cov}[S,(S-\mathbb{E}(S))^2|S>X] \\
&= \text{Cov}[S,(S-\mathbb{E}(S|S>X)+\mathbb{E}(S|S>X)-\mathbb{E}(S))^2|S>X] \\
&= \text{Cov}[S-\mathbb{E}(S),(S-\mathbb{E}(S|S>X))^2|S>X] + 2(\mathbb{E}(S|S>X)-\mathbb{E}(S))\text{Cov}[S,S|S>X].
\end{aligned}$$

Since $\text{Cov}[S,S|S>X] = \mathbb{E}[(S-\mathbb{E}(S|S>X))^2|S>X]$, we obtain

$$\begin{aligned}
&\text{Cov}[S,(S-\mathbb{E}(S))^2|S>X] \\
&= \mathbb{E}[(S-\mathbb{E}(S))(S-\mathbb{E}(S|S>X))^2|S>X] + [\mathbb{E}(S|S>X)-\mathbb{E}(S)]\mathbb{E}[(S-\mathbb{E}(S|S>X))^2|S>X].
\end{aligned} \tag{6.8}$$

Equation (6.8) indicates that as long as $X \geq \mathbb{E}(S)$, $\text{Cov}[S,(S-\mathbb{E}(S))^2|S>X] > 0$ (regardless of whether $\mathbb{E}[(S-\mathbb{E}(S))^3] > 0$ or not).

To consider the case 2: $X < \mathbb{E}(S)$, let $\psi : \mathbb{R}_+ \mapsto \mathbb{R}$ be defined as

$$\psi(X) \triangleq \text{Cov}[S,(S-\mathbb{E}(S))^2|S>X],$$

then we have

$$\psi(X) = \mathbb{E}[(S-\mathbb{E}(S))^3|S>X] - [\mathbb{E}(S|S>X)-\mathbb{E}(S)]\mathbb{E}[(S-\mathbb{E}(S))^2|S>X]. \tag{6.9}$$

Since

$$\begin{aligned}
\frac{\partial \mathbb{E}[(S-\mathbb{E}(S))^3|S>X]}{\partial X} &= \frac{f(X)}{1-F(X)}\{-(X-\mathbb{E}(S))^3 + \mathbb{E}[(S-\mathbb{E}(S))^3|S>X]\} \\
\frac{\partial \mathbb{E}[(S-\mathbb{E}(S))^2|S>X]}{\partial X} &= \frac{f(X)}{1-F(X)}\{-(X-\mathbb{E}(S))^2 + \mathbb{E}[(S-\mathbb{E}(S))^2|S>X]\},
\end{aligned} \tag{6.10}$$

and

$$\frac{\partial \mathbb{E}[S|S>X]}{\partial X} = \frac{f(X)}{1-F(X)}\{-X+\mathbb{E}(S|S>X)\},$$

we have

$$\frac{\partial\psi(X)}{\partial X} = \frac{f(X)}{1-F(X)}\{-(X-\mathbb{E}(S))^3 + \mathbb{E}[(S-\mathbb{E}(S))^3|S>X]\}$$

$$- \frac{f(X)}{1-F(X)}[\mathbb{E}(S|S>X) - \mathbb{E}(S)]\{-(X-\mathbb{E}(S))^2 + \mathbb{E}[(S-\mathbb{E}(S))^2|S>X]\}$$

$$- \frac{f(X)}{1-F(X)}\{-X + \mathbb{E}(S|S>X)\}\mathbb{E}[(S-\mathbb{E}(S))^2|S>X]$$

$$= \frac{f(X)}{1-F(X)}\{\psi(X) - (X-\mathbb{E}(S))^3 + [\mathbb{E}(S|S>X) - \mathbb{E}(S)](X-\mathbb{E}(S))^2$$

$$- [-X + \mathbb{E}(S|S>X)]E[(S-\mathbb{E}(S))^2|S>X]\}$$

$$= \frac{f(X)}{1-F(X)}\{\psi(X) + [\mathbb{E}(S|S>X) - X](X-\mathbb{E}(S))^2$$

$$- [-X + \mathbb{E}(S|S>X)]\mathbb{E}[(S-\mathbb{E}(S))^2|S>X]\}$$

$$= \frac{f(X)}{1-F(X)}\{\psi(X) - [\mathbb{E}(S|S>X) - X][\mathbb{E}((S-\mathbb{E}(S))^2|S>X) - (X-\mathbb{E}(S))^2]\}. \qquad (6.11)$$

Let us define

$$\psi_2(X) \triangleq \mathbb{E}[(S-\mathbb{E}(S))^2|S>X] \quad \text{and} \quad g(X) \triangleq (X-\mathbb{E}(S))^2$$

and note that, from (6.10), for $X \in [0,\infty)$, $\psi_2(X) > g(X)$ implies that $\psi_2(X)$ is increasing and $\psi_2(X) < g(X)$ implies that $\psi_2(X)$ is decreasing. On the other hand, we know that $g(X)$ is decreasing in X on $X \in [0,\mathbb{E}(S))$. It follows that $\psi_2(X)$ and $g(X)$ can intersect at most once at point, say $X = X'$ with $X' \in [0,\mathbb{E}(S))$ and that, if they intersect,

$$\psi_2(X) < g(X) \quad \text{for} \quad X \in [0,X') \quad \text{and} \quad \psi_2(X) > g(X) \quad \text{for} \quad X \in (X',\infty) \qquad (6.12)$$

since if they intersect again, $\psi_2(X)$ would become negative eventually. Recall that $g(X)$ is increasing on $X \in (\mathbb{E}(S),\infty)$.

Now let us further define $R(X) \triangleq (\psi_2(X) - g(X))(\mathbb{E}[S|S>X] - X)$. From (6.11), we note that the function $\psi(X)$ is increasing when $\psi(X) - R(X) > 0$ and decreasing when $\psi(X) - R(X) < 0$.

Let us consider the case where $\psi_2(X)$ and $g(X)$ intersect once. Since $\mathbb{E}[S|S>X] - X > 0$, $R(X) < 0$ on $X \in [0,X')$ and $R(X) > 0$ on $X \in (X',\mathbb{E}(S))$. Due to our assumption $\mathbb{E}(S-\mathbb{E}(S))^3 > 0$, we have $\psi(0) > 0 > R(0)$ from (6.9) and therefore $\psi(X) > R(X)$ in the neighborhood of 0 and $\psi(X)$ is increasing. It follows that $\psi(X)$ cannot become negative on $X \in [0,\mathbb{E}(S))$: Indeed, if $\psi(X)$ and $R(X)$ intersect at some point $X'' \in [0,\mathbb{E}(S))$, then we must have $X'' \in (X',\infty)$ and $\psi(X)$ starts decreasing from X''. But if it becomes negative at some point, $\psi(X)$ remains negative thereafter because $R(X) > 0$ on $X \in (X',\infty)$ by (6.12). It would contradict the fact $\psi(X) > 0$ for $X \in [\mathbb{E}(S),\infty)$ from Case 1. This proves $\psi(X) > 0$ on $X \in [0,\mathbb{E}(S))$.

Finally, let us consider the case where $\psi_2(X)$ and $g(X)$ do not intersect. In this case, we have $\psi_2(X) > g(X)$ on $X \in [0,\mathbb{E}(S))$: Indeed, if $\psi_2(X) < g(X)$, then $\psi_2(X)$ is decreasing in that region $[0,\mathbb{E}(S))$ and we must have $\psi_2(\mathbb{E}(S)) = 0 = g(\mathbb{E}(S))$ since $\psi_2(X) \geq 0$ on $X \in [0,\infty)$. However, $\psi_2(\mathbb{E}(S)) = 0$ cannot happen due to the form of $\psi_2(X)$. Hence we have $R(X) > 0$ on $X \in [0,\mathbb{E}(S))$. By the same argument as in the previous case, $\psi(X)$ cannot become negative because once it does, then it remains negative, a contradiction again. □

Lemma 6.5. *Let S be a nonnegative random variable, and $V \triangleq \mathbb{E}[(S-\mathbb{E}(S))^2]$. If $V - \mathbb{E}(S)^2 > 0$ then $\mathbb{E}[(S-\mathbb{E}(S))^2|S > X] - V$ is an increasing function of X for $X \geq 0$. If $V - \mathbb{E}(S)^2 < 0$, $\mathbb{E}[(S-\mathbb{E}(S))^2|S > X] - V$ is a U shape function of X, and attains its minimum at $\bar{X} \in (0, \mathbb{E}(S))$.*

Proof. In the proof of Lemmas 6.4, we showed that $\psi_2(X)$ and $g(X)$ can intersect at most once at point, say $X = X'$ with $X' \in [0, \mathbb{E}(S))$ and that, if they intersect, (6.12) will happen. On the other hand, if $\psi_2(X)$ and $g(X)$ do not intersect, we have $\psi_2(X) > g(X)$ and $\psi_2(X)$ is increasing, for all $X \in [0, \infty)$.

Since $V = \lim_{X\to 0} \psi_2(X)$ and $\mathbb{E}(S)^2 = \lim_{X\to 0} g(X)$, in the case $\psi_2(X)$ and $g(X)$ intersect, we have $V < \mathbb{E}(S)^2$ and in the case where $\psi_2(X) > g(X)$, we have $V > \mathbb{E}(S)^2$. The results follow from the analysis of $\psi_2(X)$ and $g(X)$ in the preceding paragraph since $\mathbb{E}[(S-\mathbb{E}(S))^2|S > X] - V$ is obtained by vertically shifting $\psi_2(X) = \mathbb{E}[(S-\mathbb{E}(S))^2|S > X]$. □

Lemma 6.6. *If a nonnegative random variable S is positively skewed, i.e., $\mathbb{E}[(S-\mathbb{E}(S))^3] > 0$, then $\mathbb{E}[(S-X)(S-\mathbb{E}(S))^2|_{S>X}] - \mathbb{E}[(S-X)|_{S>X}]\mathbb{E}[(S-\mathbb{E}(S))^2] > 0$ for $X \geq 0$.*

Proof. Let us define $\phi : \mathbb{R}_+ \mapsto \mathbb{R}$ as

$$\phi(X) \triangleq E[(S-X)(S-E(S))^2|S > X] - E[(S-X)|S > X]E[(S-E(S))^2],$$

then we have

$$\begin{aligned}\phi(X) = {} & \mathrm{Cov}[S, (S-\mathbb{E}(S))^2|S > X] \\ & + [\mathbb{E}(S|S > X) - X]\{\mathbb{E}[(S-\mathbb{E}(S))^2|S > X] - \mathbb{E}[(S-\mathbb{E}(S))^2]\}.\end{aligned} \tag{6.13}$$

Therefore,

$$\begin{aligned}\frac{\partial \phi(X)}{\partial X} & = \frac{f(X)}{1-F(X)}\{\mathrm{Cov}[S, (S-\mathbb{E}(S))^2|S > X]\} \\ & - \frac{f(X)}{1-F(X)}\{-X + [\mathbb{E}(S|S > X)][\mathbb{E}((S-\mathbb{E}(S))^2|S > X) - (X-\mathbb{E}(S))^2]\} \\ & + \frac{f(X)}{1-F(X)}\{-X + \mathbb{E}(S|S > X)\}\{\mathbb{E}[(S-\mathbb{E}(S))^2|S > X] - \mathbb{E}[(S-\mathbb{E}(S))^2]\} \\ & - \{\mathbb{E}[(S-E(S))^2|S > X] - \mathbb{E}[(S-\mathbb{E}(S))^2]\} \\ & + \frac{f(X)}{1-F(X)}[-X + \mathbb{E}(S|S > X)]\{-(X-\mathbb{E}(S))^2 + \mathbb{E}[(S-\mathbb{E}(S))^2|S > X]\} \\ & = \frac{f(X)}{1-F(X)}\left(\phi(X) - E[(S-\mathbb{E}(S))^2|S > X] + \mathbb{E}[(S-\mathbb{E}(S))^2]\right)\end{aligned} \tag{6.14}$$

Let $h(X) \triangleq \mathrm{Cov}[S, (S-\mathbb{E}(S))^2|S > X]$, $m(X) \triangleq \mathbb{E}(S|S > X) - X$, $k(X) \triangleq \mathbb{E}[(S-\mathbb{E}(S))^2|S > X] - \mathbb{E}[(S-\mathbb{E}(S))^2]$, and $u(X) \triangleq f(X)/(1-F(X))$, then (6.13) and (6.14) can be represented as

$$\phi(X) = h(X) + m(X)k(X) \tag{6.15}$$

and

$$\frac{\partial \phi(X)}{\partial X} = u(X)\phi(X) - k(X), \tag{6.16}$$

respectively. Substituting $k(X)$ from (6.15) into (6.16) and rearranging, we obtain the following first order differential equation

$$\frac{\partial \phi(X)}{\partial X} + \left(\frac{1}{m(X)} - u(X)\right)\phi(X) = \frac{h(X)}{m(X)}$$

m Given the boundary condition $\phi(0)$, $\phi(X)$ can be found as

$$\phi(X) = e^{-\int \frac{1}{m(X)} - u(X) dX}\left[\phi(0) + \int \frac{h(X)}{m(X)} e^{\int \frac{1}{m(X)} - u(X) dX} dX\right]$$

Since $\phi(X) > 0$ at $X = 0$, and $h(X) > 0$ and $m(X) > 0$ for all $X > 0$, $\phi(X)$ is positive for all $X > 0$. □

Proof of Proposition 3.2: We denote $\bar{S} \triangleq \mathbb{E}(S)$ (i.e., $\bar{S}/S_0$ is the expected market return). Let S be the price of the underlying at maturity with density function $f(\cdot)$ and C be the price of a call option on S with a strike price of X. Then

$$\begin{aligned}\mathbb{E}[(r_c - \bar{r}_c)(r_s - \bar{r}_s)^2] &= \frac{\mathbb{E}[C(S-\bar{S})^2] - \bar{C}\mathbb{E}[(S-\bar{S})^2]}{C_0 S_0^2} \\ &= \frac{\int_{S=X}(S-X)(S-\bar{S})^2 f(S)\,\mathrm{d}S - \mathbb{E}(S-\bar{S})^2\int_{s=X}(S-X)f(S)\,\mathrm{d}S}{\int_{S=X} S_0^2(S-X)\mathbb{E}(m|S)f(S)\,\mathrm{d}S}\end{aligned} \tag{6.17}$$

where m is the strictly positive SDF with $d\mathbb{E}[m|S]/dS < 0$.

Denote $V = \mathbb{E}(S-\bar{S})^2$ and take the derivative of equation (6.17) with respect to strike price. The denominator is

$$D^2(X) \triangleq \left[\int_{S=X} S_0^2(S-X)\mathbb{E}(m|S)f(S)\,\mathrm{d}S\right]^2$$

which is positive, and the numerator is

$$\begin{aligned}&\left[-\int_{S=X}(S-\bar{S})^2 f(S)\,\mathrm{d}S + \int_{S=X} V f(S)\,\mathrm{d}S\right]\int_{S=X}(S-X)S_0^2\mathbb{E}(m|S)f(S)\,\mathrm{d}S \\ &+ \int_{S=X} S_0^2\mathbb{E}(m|S)f(S)\,\mathrm{d}S\left[\int_{S=X}(S-X)(S-\bar{S})^2 f(S)\,\mathrm{d}S - \int_{S=X} V(S-X)f(S)\,\mathrm{d}S\right].\end{aligned} \tag{6.18}$$

Defining $F(s)$ as the cumulative density function that corresponds to $f(s)$ and rearranging (6.18) gives

$$\begin{aligned}&S_0^2(1-F(X))^2\Big\{V\int_{S=X} S\mathbb{E}(m|S)\frac{f(S)}{1-F(X)}\,\mathrm{d}S - V\int_{S=X} S\frac{f(S)}{1-F(X)}\,\mathrm{d}S\int_{S=X}\mathbb{E}(m|S)\frac{f(S)}{1-F(X)}\,\mathrm{d}S \\ &\quad + \int_{S=X}\mathbb{E}(m|S)\frac{f(S)}{1-F(X)}\,\mathrm{d}S\int_{S=X} S(S-\bar{S})^2\frac{f(S)}{1-F(X)}\,\mathrm{d}S \\ &\quad - \int_{S=X} S\mathbb{E}(m|S)\frac{f(S)}{1-F(X)}\,\mathrm{d}S\int_{S=X}(S-\bar{S})^2\frac{f(S)}{1-F(X)}\,\mathrm{d}S\Big\} \\ &= S_0^2(1-F(X))^2\Big\{V\cdot\mathrm{Cov}\left[S,\mathbb{E}(m|S)|_{S>X}\right] + \mathbb{E}\{\mathbb{E}(m|S)|_{S>X}\}\mathbb{E}[S(S-\bar{S})^2|_{S>X}] \\ &\quad - \mathbb{E}\{S\mathbb{E}(m|S)|_{S>X}\}\mathbb{E}[(S-\bar{S})^2|_{S>X}]\Big\}.\end{aligned} \tag{6.19}$$

The sign of the numerator will be determined by the terms in the braces of (6.19), which can be simplified to

$$\begin{aligned}
&-\mathrm{Cov}\left[S,\mathbb{E}(m|S)|_{S>X}\right](\mathbb{E}[(S-\bar{S})^2|_{S>X}]-V)\\
&+\mathbb{E}\{\mathbb{E}(m|S)|_{S>X}\}\mathbb{E}[S(S-\bar{S})^2|_{S>X}]-\mathbb{E}[S|_{S>X}]\mathbb{E}[(S-\bar{S})^2|_{S>X}]\\
&=-\mathrm{Cov}\left[S,\mathbb{E}(m|S)|_{S>X}\right](\mathbb{E}[(S-\bar{S})^2|_{S>X}]-V)+\mathbb{E}\{\mathbb{E}(m|S)|_{S>X}\}\mathrm{Cov}\left[S,(S-\bar{S})^2|_{S>X}\right].
\end{aligned} \tag{6.20}$$

We know that $\mathbb{E}\{\mathbb{E}(m|S)|_{S>X}\}>0$ and $-\mathrm{Cov}\left[S,\mathbb{E}(m|S)|_{S>X}\right]>0$. So, to determine the sign of the above expression, we need to know

1. the sign of $\mathbb{E}[(S-\bar{S})^2|_{S>X}]-V$, where $V=E[(S-\bar{S})^2]$, and
2. the sign of $\mathrm{Cov}\left[S,(S-\bar{S})^2|_{S>X}\right]$.

Lemma 6.4 shows that when the underlying security is positive skewed, the sign of the second term, $\mathrm{Cov}\left[S,(S-\bar{S})^2|_{S>X}\right]$, is positive, but the sign of the first term can be positive or negative. Hence we cannot determine the sign of (6.20) when the two terms have different signs. However, the numerator in (6.17) is

$$\begin{aligned}
&\mathbb{E}[C(S-\bar{S})^2]-\bar{C}\mathbb{E}[(S-\bar{S})^2]\\
&=(1-F(X))\{\mathbb{E}[(S-X)(S-\bar{S})^2|_{S>X}]-\mathbb{E}[S-X|_{S>X}]\mathbb{E}[(S-\bar{S})^2]\}\\
&=(1-F(X))\{\mathrm{Cov}\left[S,(S-\bar{S})^2|_{S>X}\right]+\mathbb{E}[S-X|_{S>X}]\{\mathbb{E}[(S-\bar{S})^2|_{S>X}]-V\}\}.
\end{aligned} \tag{6.21}$$

The sign of equation (6.21) is the same as

$$\begin{aligned}
&\mathbb{E}\{\mathbb{E}(m|S)|_{S>X}\}\Big\{\mathrm{Cov}\left[S,(S-\bar{S})^2|_{S>X}\right]+\mathbb{E}[(S-X)|_{S>X}](\mathbb{E}[(S-\bar{S})^2|_{S>X}]-V)\Big\}\\
&=-\mathrm{Cov}\left[S,\mathbb{E}(m|S)|_{S>X}\right](\mathbb{E}[(S-\bar{S})^2|_{S>X}]-V)+\mathbb{E}\{\mathbb{E}(m|S)|_{S>X}\}\mathrm{Cov}\left[S,(S-\bar{S})^2|_{S>X}\right]\\
&+\mathbb{E}\{\mathbb{E}(m|S)(S-X)|_{S>X}\}\{\mathbb{E}[(S-\bar{S})^2|_{S>X}]-V\}.
\end{aligned} \tag{6.22}$$

Lemmas 6.4 and 6.6 show that when $E[(r_s-\bar{r}_s)^3]$ is positive,

$$\mathrm{Cov}\left[S,(S-\bar{S})^2|_{S>X}\right]>0$$

and

$$\mathbb{E}[(S-X)(S-\bar{S})^2|_{S>X}]-\mathbb{E}[(S-X)|_{S>X}]V>0,$$

respectively. The second equation implies that

$$\begin{aligned}
&\mathrm{Cov}\left[S-X,(S-\bar{S})^2|_{S>X}\right]+\mathbb{E}[(S-X)|_{S>X}]\mathbb{E}[(S-\bar{S})^2|_{S>X}]-\mathbb{E}[(S-X)|_{S>X}]V\\
&=\mathrm{Cov}\left[S,(S-\bar{S})^2|_{S>X}\right]+\mathbb{E}[S-X|_{S>X}](\mathbb{E}[(S-\bar{S})^2|_{S>X}]-V)>0
\end{aligned}$$

which shows the right hand side of (6.21) is positive. Hence (6.22) is positive. Now let us come back to (6.20). If $\mathbb{E}[(S-\bar{S})^2|_{S>X}]-V>0$, then (6.20) is positive. If negative, by comparing (6.22) (which is positive) with (6.20), we conclude that equation (6.20) is positive. In other words, when the underlying security is positively skewed, the coskewness is positive (in fact, greater than one), and is increasing in the strike price.

6.4 Proof of Proposition 3.3

Proof. When underlying security return is negatively skewed, it is easy to check that (6.21) is negative at $X = 0$, and positive in the limit as $X \to \infty$. Therefore, initially, the coskewness is positive (since the underlying is negatively skewed), and in the limit, is negative.

To see how the coskewness would change as the strike price increases, we denote (6.17) by

$$C(X) = (1 - F(X))\phi(X)/D(X)$$

where $\phi(X)$ is defined as in (6.13) and $D(X) = \int_{S=X} S_0^2(S - X)E(m|S)f(S)dS$. We have

$$\begin{aligned} C'(X) &= \frac{1 - F(X)}{D(X)} \left\{ V - \mathbb{E}[(S - \bar{S})^2|S > X] - \phi(X)\frac{D'(X)}{D(X)} \right\} \\ &= \frac{1 - F(X)}{D(X)} \{S_0^2 C(X)\mathbb{E}\{\mathbb{E}(m|S)|_{S>X}\} - (\mathbb{E}[(S - \bar{S})^2|S > X] - V)\} \end{aligned} \tag{6.23}$$

where V is the variance defined above. At $X = 0$, it is clear that $C(X) < 0$ and $C'(X) < 0$. We expect that initially, as X increases, $C(X)$ decreases. Hence the coskewness is positive and increases.

Next, let us define

$$M(X) \triangleq \frac{S_0^2 \int_{S=X} \mathbb{E}(m|S)f(S)dS}{1 - F(X)} \quad \text{and} \quad L(X) \triangleq \frac{\int_{S=X}(S - \bar{S})^2 f(S)dS}{1 - F(X)} - V.$$

The sign of $C'(X)$ depends on that of $C(X) - L(X)/M(X)$. Note that

$$C'(X) \gtrless 0 \quad \text{when} \quad C(X) - L(X)/M(X) \gtrless 0. \tag{6.24}$$

We have

$$M'(X) = u(X)(M(X) - S_0^2\mathbb{E}[m|X]) \quad \text{and} \quad L'(X) = u(X)(L(X) + V - \mathbb{E}[(X - \bar{S})^2])$$

where $u(X) = f(X)/1{-}F(X)$ as defined above. Note that $M'(X) < 0$ for $X \geq 0$ since $\mathrm{d}\mathbb{E}[m|S]/\,\mathrm{d}S < 0$. From the proof of Lemmas 6.4 and 6.5, we know $L(X)$ is either U-shaped or strictly increasing. (Recall that the analysis of $\psi_2(X) - g(X)$ does not rely on the sign of $\mathbb{E}(S - \bar{S})^3$). However, if $L(X)$ is increasing, then $L(X)/M(X)$ (and $L(0)/M(0) = 0$, assuming $\mathbb{E}[m]$ does not vanish) is positive and increasing for all $X \geq 0$ since $M(X) > 0$ and $M'(X) < 0$ for all $X \geq 0$. Since $C(0) < 0$ and therefore, $C'(X) < 0$ for all $X \geq 0$, this case would contradict the fact that $C(X) > 0$ for a sufficiently large X. Therefore, $L(X)$ must be U-shaped and the positive real line is dividend into three regions: (1) $L' < 0$ and $L < 0$, (2) $L' > 0$ and $L < 0$, and (3)$L' > 0$ and $L > 0$. We can prove by using the signs of M and M' that in region (1), $(L/M) < 0$ and $(L/M)' < 0$ and in region (3), $(L/M) > 0$ and $(L/M)' > 0$. It is clear that in region (3), $C(X) > L(X)/M(X)$ and hence $C(X)$ is increasing monotonically since otherwise $C(X)$ would become decreasing from some point, and thus contradict that $\lim_{X\to\infty} C(X) > 0$. Moreover, it follows that $C(X)$ must intersect $L(X)/M(X)$ at least once. By using (6.24) together with the facts $C(0) < L(0)/M(0)$ and $C(X) > L(X)/M(X)$ on $X \in (X', \infty)$ for some X' sufficiently large, we can conclude that $C'(X) = 0$ has only one root if $(L/M)' = 0$ has only one root.

In region (2), $L(X) < 0$ and $L'(X) > 0$ imply that

$$(X - \bar{S})^2 < \frac{I_2(X)}{1 - F(X)} < V \tag{6.25}$$

where $I_1(X) = \int_{S=X} \mathbb{E}(m|S) f(S) dS$ and $I_2(X) = \int_{S=X} (S - \bar{S})^2 f(S) dS$. Then the condition that $(L/M)'$ vanishes at one point is equivalent to saying that

$$I_1(X)(V - (X - \bar{S})^2) = (V(1 - F(X)) - I_2(X))\mathbb{E}[m|X]$$

has only one root. Note that both sides of the equation is positive due to (6.25). This equation depends on the quantity of $\mathbb{E}[m|X]$ and $\bar{S}$. $\square$

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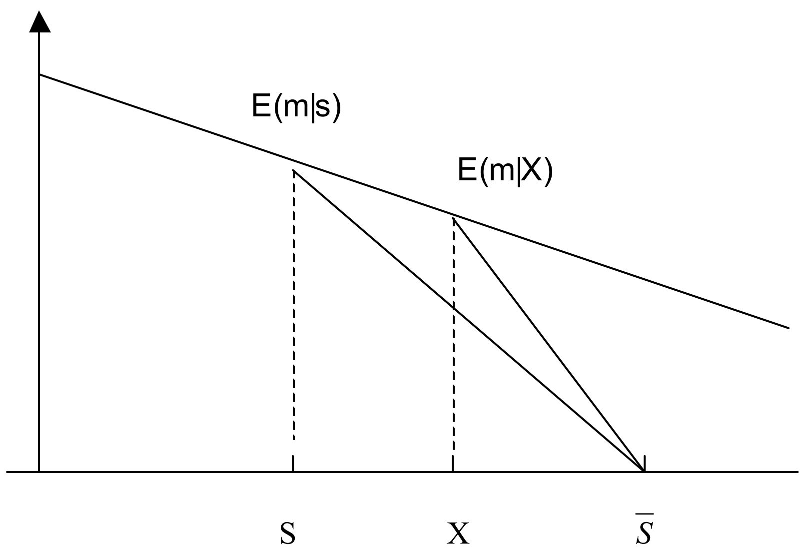

Figure A

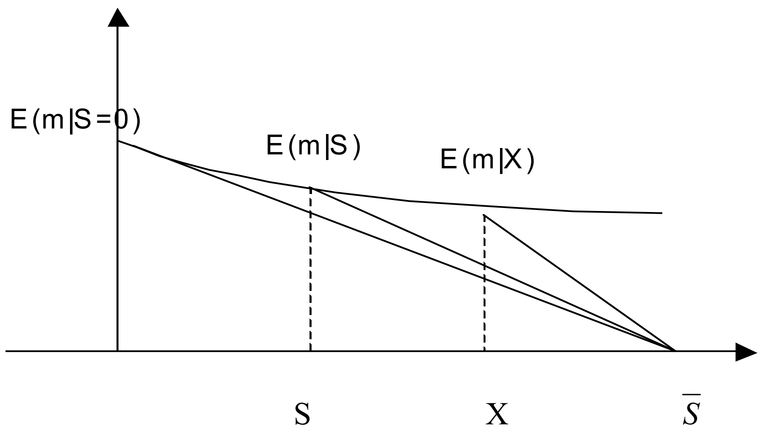

Figure B

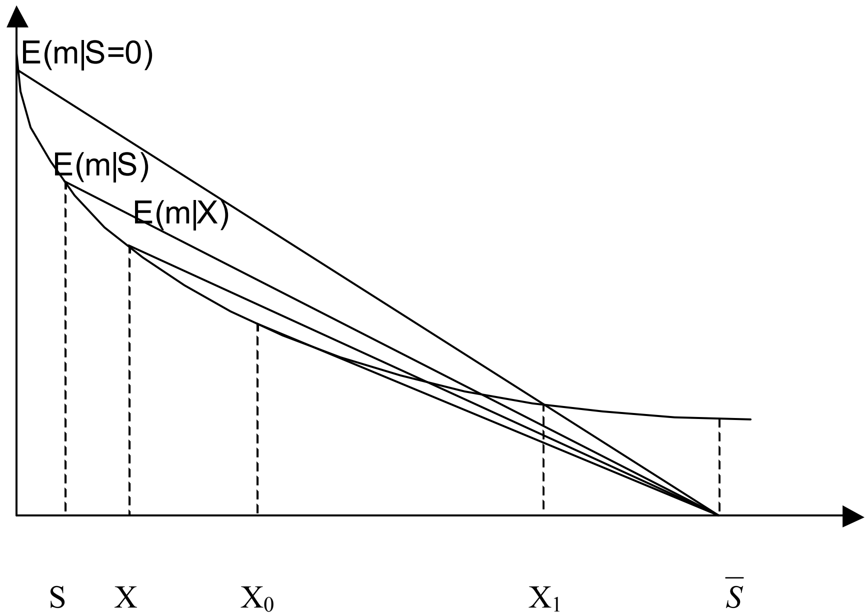

Figure C