Convertible Bond Financing as an Optimal Capital Structure *

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1 Introduction

In this paper, two problems are examined. The first is the reason why firms issue convertible debts and the second is the difference between equity with debt financing and convertible debt financing. In previous studies it has been shown that convertible bond financing can settle various difficulties related to debt financing or equity financing. Nevertheless the difference of roles between convertible debt financing and mixed financing is still not clear. What is exactly the difference between convertible debt financing and mixed financing? This point is investigated here and the following results are shown. First, if the entrepreneur-manager’s decision depends both on his private benefit and on financing methods, mixed financing can be preferable to pure-equity or pure-debt financing. Second, convertible debt financing can be more efficient at mitigating conflicts between equity-holders and debt-holders than mixed financing.

In convertible debt financing, investors who also own the bonds can decide to convert the bond to equity at will. In this sense equity-holders coincide with debt-holders. Therefore, what benefits equity-holders may also benefits bondholders, which promotes the settlement of conflicts between equity-holders and bondholders. From this point of view, we show the advantage of convertible bond over mixed financing.

There has been several theoretical explanations for reasons to issue convertible bonds. Green (1984) argues that convertible bond financing can mitigate “asset substitution” in debt financing. Our model bases on asset substitution and non-verifiability of investment returns.

The asset substitution problem is well-known for distorting the efficient investment decision. Once considerable amount of debts are issued, equity-holders have an incentive to substitute risky for less risky project at the expense of debt-holders’ benefit, even if the firm’s market value were to decline. Therefore, debt financing is not desirable when the asset substitution problem is serious.

It is possible that non-verifiable returns, which one can take as the private benefit of the entrepreneur-manager, distorts the efficient investment decision. In general, the level of an entrepreneur-manager’s private benefit varies with projects, and thus the entrepreneur-managers are prone to choose projects with higher private benefits \textit{ceteris paribus}. When the private benefit problem is serious, debt financing with renegotiation may be preferable to equity financing. The reason for this will be shown later.

Taking distortions, private benefit and the asset substitution problem into account, issuing convertible debt (bonds) can be the most efficient financing method when future problems are uncertain.

In order to explain the advantage of convertible debt financing, the following procedure is adopted. First, equity and debt financings are compared. It is shown that equity financing is preferable to debt financing if the asset substitu-
tion problem is more serious than distortion from the entrepreneur-managers’
private benefit, and vice versa. Second, we demonstrate that mixed financing
can be more efficient than pure-equity or pure-debt financings. Finally, we prove
that convertible debt financing can be more efficient than mixed financing.

The asset substitution problem can be resolved by issuing equities, but the
inefficient effect of private benefit still remains. Note that the larger the equity
share of outside investors, the smaller is the entrepreneur-manager’s incentive
to choose an efficient project since, then, his share of the return is smaller.
In equity financing, the incentive for the entrepreneur-manager to choose an
efficient project is reduced when his share is small. In this case, the entrepreneur-
manager makes a decision based on his private benefit.

Suppose that the level of private benefit of the entrepreneur-manager is large
but that it is uncertain whether the private benefit distorts the project execution
or not. In other words, the occurrence of the private benefit problem depends
on the state of the world. Because the entrepreneur-manager’s incentive to
choose an efficient project increase as outside investors’ equity shares decrease,
there exists an upper bound for outside investor’s share in order to prevent an
inefficient project to be chosen.

This upper bound is ineffective if it so happens that the private benefit
does not distort the project execution. However, it is in fact impossible to vary
shares according to the state of the world once they are issued. Therefore, equity
financing can limit the possibility of raising enough funds from the market.

On the other hand, under debt financing, the face value in the original con-
tract can be renegotiated after the state is revealed. Debt financing has an
advantage that via renegotiation, patterns of payoff distribution over states ex-
tends when the influence of the entrepreneur-manager’s private benefit is more
serious than the asset substitution problem. Under this circumstance debt fi-
nancing prevails over equity financing.

Moreover, mixed financing through equity and debt can be more desirable
than pure-equity or pure-debt financing, because the merits of both methods can
be exploited. By using equity and debt renegotiation, the return distribution
over states can be expanded further. Thus, mixed financing can resolve future
asset substitution and private benefit-oriented distortion problems.

Convertible debt financing play the same role as mixed financing, with the
additional advantage that it has a potential to resolve conflicts among investors.
The most apparent difference between convertible debt financing and combined
equity and debt financing is as follows. In convertible debt financing, investors
who own bonds can decide to convert their bonds into equities at will. In this
sense equity-holders coincide with debt-holders. On the other hand equity-
holders are usually not debt-holders under mixed financing. Therefore, there is
no conflict between equity-holders and debt-holders when convertible bonds are
The conflict between equity-holders and debt-holders will arise when renegotiation of the debt contract is needed. Sometimes renegotiation of the debt contract is efficient. The occurrence of renegotiation depends on whether it is desirable for both debt-holders and the entrepreneur-manager. Thus, if either one of these parts does not gain from it, renegotiation will not occur, even if it is efficient.

This inefficiency occurs because the surplus arising from renegotiation between debt-holders and the entrepreneur-manager is distributed not only to both of them but also to shareholders, i.e., renegotiation surplus spills over to shareholders. Therefore, whether or not shareholders coincide with debt-holders can affect the occurrence of renegotiation. The larger the surplus produced by debt-holders’ concession and spilled over into equity-holders, the less incentive for debt-holders to renegotiate the original debt contract. However, if debt-holders also own equities, they may permit larger concessions because the spill-over problem is less of a matter. We can regard the simultaneous possession of equity and debt as a contributor to bringing closer the interests of equity-holders and debt-holders.

In practice, nevertheless, it is difficult to force debt-holders (or equity-holders) to possess equity (or debt). If debt-holders do not have equity, then efficient renegotiation may not always be attained. This difficulty can be resolved by convertible debt financing. In this sense, convertible bond financing is superior to mixed equity and debt financing.

To the best of the author’s knowledge there are no previous works that focus on simultaneous possession of equity and bond. In related literature, however, there are works that advocate that banks should possess equities of the companies to which they loan. James (1995) and Berlin and Saunders (1996) emphasize that equity possessing by a creditor bank strengthens the bank’s role, i.e., it stimulates creditor banks to rescue promising companies which are temporarily in financial crisis. Rescuing a good company benefit not only the bank but also it’s equity-holders. Only the creditor banks, however, pay the costs of rescue operations, such as those of the investigation to determine whether the firm deserves to be rescued or not. This can cause a conflict between creditor banks and equity-holders, which might be mitigated by the banks’ possession of equity.

Focusing on the fact that firms and banks tend to establish a long term relationship, Smith (2000b) shows that banks’ partial holding of equities as well as loans can be optimal. This is because then creditor banks have means for gathering borrowers’ information and know about the firms’ future cash flows which are superior to those of outside investors. Considering this fact, it is likely that outside investors without precise information will not provide funds
to a firm that has been denied additional loans by creditor banks, because then investors will conclude that the firm’s future returns are too low. Therefore, in the long-run, only creditor banks will provide additional funds to the firms. Such a relationship raises banks’ power, and enables them to acquire excess profit, which must be paid by the firms. This relationship cost for firms can be decreased by banks’ simultaneous possession of loans and equities, which can be socially efficient.

This paper is organized as follows. Section 2 presents the model. In Section 3, the optimal capital structure is analyzed when the entrepreneur-managers’ private benefit occurs in risker project execution. It is shown that either equity financing or debt financing with renegotiation is chosen, depending on the level of the entrepreneur-manager’s private benefit. Section 4 shows that convertible bond financing is better than mixed equity and debt financing, which is better than equity or debt financing. In Section 5, the optimal capital structure is analyzed when the entrepreneur-managers’ private benefit occurs in safer project execution. In Section 6, the optimal capital structure is analyzed when the entrepreneur-managers’ private benefit is independent on the project risk. And finally, in Section 7, conclusions are drawn.

2 The Model

The model have three dates $t = \{0, 1, 2\}$ and two types of risk neutral agents: an entrepreneur-manager, who has a profitable investment opportunity but no funds to run it, and investors with plenty of capital but no investment opportunity except investment for risk-free asset. For simplicity, it is assumed that the risk-free rate is zero. Suppose there are two states of the world, and state 1 occurs with probability $\pi$ and state 2 with probability $1 - \pi$. The probability distribution is commonly known to the entrepreneur-manager and the investors.

At date 0, the entrepreneur-manager raises the amount of funds $I$ in order to set up the firm. The raised fund is expended to acquire the asset. Four financing methods, equity, debt, combined equity and debt, and convertible debt financing, are considered in this paper.

At date 1, the entrepreneur-manager observes the state of the world and executes the project. States are assumed observable at date 1 but non-verifiable, so that contracts cannot be written on it. The entrepreneur-manager choose between two projects each with a different risk: risky project and safe project. The risk of the safe project is normalized to zero.

The return consists of verifiable and non-verifiable returns. Denote the ver-
ifable return of a risky project by $R_r$ and non-verifiable return by $b_r$ and that of safe project by $R_s$ and $b_s$ respectively. $b_r$ and $b_s$ can be regarded as the entrepreneur-manager’s private benefit.

The efficient projects are different depending on the state. Denote the probability that the return of a risky project occur in state 1 by $p$, and in state 2 by $q$, where $p > q$. We assume that the expected return of the risky project is higher (lower) than that of the safe project in state 1 (state 2),

$$pR_r + b_r > R_s + b_s > qR_r + b_r.$$  \hfill (1)

Thus, executing a risky project is efficient in state 1 while a safe project is efficient in state 2.

We also assume that $R_r$ is larger than $R_s$

$$R_r > R_s,$$  \hfill (2)

and $R_r - b_r$ is larger than $R_s - b_s$.

$$R_r - b_r > R_s - b_s.$$  \hfill (3)

At time 2, the return is realized and the firm is liquidated.

The entrepreneur-manager is inclined to choose the project with larger private benefit. In general, the level of entrepreneur-manager’s private benefit will differ with the project. We analyze the optimal capital structure when $b_r > b_s$ in Section 3, that of when $b_r < b_s$ in Section 5, and that of when $b_r = b_s$ in Section 6.

## 3 Private benefit with a risky project

In this section we discuss the optimal capital structure when the entrepreneur-manager receives private benefit with a risky project and does not receive any

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2For example, producing a commodity whose demand is relatively stable is desirable during depression, while during a boom, producing a commodity whose demand is dependant on fashion or popularity might be efficient.

3This condition is included for Proposition 5 so as to exclude an unreasonable solution.
private benefit with safe project, that is, \( b_r > 0 \) and \( b_s = 0 \). We set \( b = b_r \) for notational simplicity.

If the entrepreneur-manager has enough capital to execute the investment project and does not need to raise money from investors, the best solution will be attained. However, if the entrepreneur-manager has no capital, in order to raise capital from the market he must promise that some portion of the project’s return will be distributed to investors, which can distort his decision. The more money the entrepreneur-manager needs to raise from the market (i.e., the larger \( I \)), the more it is likely to distort the entrepreneur-manager’s decision. Given such a contract, it is possible that the entrepreneur manager will not choose the efficient project. The more serious the distortion, the more unlikely an efficient project will be chosen.

In order to analyze the optimal capital structure, we derive the range of parameters under which an efficient project is chosen by the entrepreneur-manager for each financial method. Specifically we derive the maximum \( I \) under which efficient projects are chosen in each state. Then we compare the range of parameters (the maximum \( I \)) for each financing method. We can recognize the financing method with the larger parameter as being the more efficient one.

From the assumptions, the efficient choice of project is the risky one in state 1, and the safe one in state 2. Because of the existence of private benefit, on the other hand, the entrepreneur-manager tends to choose the risky project in state 2. We use the term “efficient project choice” to describe the entrepreneur-manager chooses a risky project in state 1 and a safe project in state 2, and the term “inefficient project choice” to describe he chooses a risky project in both states.
3.1 Equity financing

In this section equity financing is discussed. The sub-game perfect equilibrium is considered. Let us consider which project the entrepreneur-manager will chose at date 1, given that equities are issued at date 0. Let $\alpha$ be the investors' share, and $1 - \alpha$ be that of the entrepreneur-manager.

There exists a threshold $\alpha$ over which the entrepreneur-manager chooses the inefficient project, while below it he chooses the efficient project. The condition that the entrepreneur-manager chooses the safe project in state 2 for a given $\alpha$ is

$$\alpha \leq 1 - \frac{b}{\rho},$$  
(4)

where $\rho = R_s - qR_e$. Thus we see that the higher $b$ is, the higher $\alpha$, or the lower $\rho$ is, the more the asset substitution that occurs.

Suppose that (4) is satisfied. $\alpha$ must satisfy the following equation in order to increase $I$ when investors anticipate that the entrepreneur-manager will choose the efficient project.

$$\alpha = \frac{I}{\pi p R_e + (1 - \pi) R_s},$$  
(5)

If the conditions (4) and (5) are satisfied, this investment is feasible and the efficient projects are chosen. We can rewrite these conditions as the following inequality.

$$I \leq (\pi p R_e + (1 - \pi) R_s) \left(1 - \frac{b}{\rho}\right).$$  
(6)

Next, let us consider the maximum $I$ under which the inefficient projects are chosen. The condition that the entrepreneur-manager chooses inefficient projects is opposite to that of (4), that is $\alpha > 1 - b/\rho$. Note that the limit of $\alpha$ is 1, and the inefficient projects are chosen under the following condition.

$$1 \geq \alpha > 1 - \frac{b}{\rho}.$$  
(7)

In addition, $\alpha$ must satisfy the following equation in order to raise $I$ when investors anticipate that the entrepreneur-manager will choose inefficient projects

$$\alpha = \frac{I}{\pi p + (1 - \pi)q R_e}.$$  
(8)

If the conditions (7) and (8) are satisfied, this investment is feasible and the inefficient projects are chosen. We can rewrite these conditions as the following inequality.

$$(\pi p + (1 - \pi)q R_e) \geq I > \frac{(\rho - b)(\pi p + (1 - \pi)q R_e)}{\rho}.$$  
(9)
\[ (\pi p R_e + (1 - \pi) R_s) \left( 1 - \frac{b}{p} \right) \]

\[ (\pi p + (1 - \pi) q) R_e \]

\[ \frac{(\rho - b)(\pi p + (1 - \pi) q) R_e}{\rho} \]

**Figure 3: Equity financing – Case 1**

(5) and (8) are shown on axes of \( I \) and \( \alpha \), in Figure 3. The following equation is satisfied under Case 1, and not satisfied under Case 2.

\[ (\pi p R_e + (1 - \pi) R_s) \left( 1 - \frac{b}{p} \right) \geq (\pi p + (1 - \pi) q) R_e \]

If \( I \) satisfies both (6) and (9), the entrepreneur-manager prefers the contract on (5) to that on (8) \(^4\).

In Case 1, either the efficient projects are chosen or the investment is not feasible. In Case 2, three possibilities exist, the efficient projects are chosen, the inefficient projects are chosen, or the investment is not feasible. Therefore we make the following proposition.

**Proposition 1** Under equity financing,

1. If (10) is satisfied, the efficient projects are chosen when \( 0 \leq I \leq (\pi p R_e + (1 - \pi) R_s) \left( 1 - \frac{b}{p} \right) \) (\( \pi p R_e + (1 - \pi) R_s \)).

\[^4\]Under equity financing, the payoff of the entrepreneur-manager is as follows. From (5),

\[ \left( 1 - \frac{I}{\pi p R_e + (1 - \pi) R_s} \right) (\pi p R_e + (1 - \pi) R_s) = (\pi p R_e + (1 - \pi) R_s) - I. \]

From (8),

\[ \left( 1 - \frac{I}{\pi p + (1 - \pi) q R_e} \right) (\pi p + (1 - \pi) q) R_e = (\pi p + (1 - \pi) q) R_e - I. \]

It is obvious that \( (\pi p R_e + (1 - \pi) R_s) - I > (\pi p + (1 - \pi) q) R_e - I. \)

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(1 - \pi)R_s(1 - b/\rho), the investment is not feasible when \( I > (\pi p R_r + (1 - \pi) R_s)(1 - b/\rho) \).

2. If (10) is satisfied, the efficient projects are chosen when 0 \leq I \leq (\pi p R_r + (1 - \pi) R_s)(1 - b/\rho), the inefficient projects are chosen when (\pi p R_r + (1 - \pi) R_s)(1 - b/\rho) < I \leq (\pi p + (1 - \pi) q)R_r, and the investment is not feasible when \( I > (\pi p + (1 - \pi) q)R_r \).

### 3.2 Debt financing without renegotiation

In this section, we analyze debt financing. We denote the face-value of a debt bond as \( D \).

For a given \( D \), the condition that the entrepreneur-manager chooses a safe project in state 2 is

\[
D \leq \frac{\rho - b}{1 - q}.
\]

(11)

\( D \) must satisfy the following equation in order to raise \( I \) if investors anticipate that the entrepreneur-manager will choose the efficient projects

\[
D = \frac{I}{\pi p + 1 - \pi}.
\]

(12)

If (11) and (12) are satisfied, the investment is feasible and the efficient projects
are chosen. We can rewrite these conditions as follows:

\[ I \leq \frac{(\rho - b)(\pi p + 1 - \pi)}{1 - q}. \quad (13) \]

Next, we consider the maximum \( I \) when the inefficient projects are chosen. The condition that the entrepreneur-manager chooses an inefficient project is the opposite of (11), that is, \( D > \frac{\rho - b}{1 - q} \). Note that the maximum \( D \) is \( R_r \). The entrepreneur-manager chooses inefficient projects under the following condition:

\[ R_r \geq D > \frac{\rho - b}{1 - q}. \quad (14) \]

On the other hand, \( D \) must satisfy the following condition in order to raise \( I \) if the investors anticipate that the entrepreneur-manager will choose the inefficient projects.

\[ D = \frac{I}{\pi p + (1 - \pi)q} \quad (15) \]

If (14) and (15) are satisfied, the investment is feasible and the inefficient projects are chosen. We can rewrite these conditions as follows:

\[ (\pi p + (1 - \pi)q)R_r \geq I > \frac{(\rho - b)(\pi p + (1 - \pi)q)}{1 - q}. \quad (16) \]

Equations (12) and (15) are illustrated in Figure 5.

If \( I \) satisfies both (13) and (16), the entrepreneur-manager prefers the contract on (12) to that on (15). Comparing Figure 3 and Figure 4, we derive the following proposition.

**Proposition 2** The efficient range of parameters of equity financing are larger than that of debt financing.

**Proof.** It is obvious from Figure 3 and 4, as well as the following inequality \(^5\).

\[ (\pi p R_r + (1 - \pi)R_s)\left(1 - \frac{b}{\rho}\right) > \frac{(\rho - b)(\pi p + 1 - \pi)}{1 - q}. \]

This proposition shows that equity financing is efficient because of the seriousness of the asset substitution problem involved in debt financing. Although equity financing is a more efficient method than debt financing, equity financing does not settle the problems completely. It is possible that the inefficient projects are chosen under equity financing if \( \alpha \) is too large, due to the effect of \( b \).

\[ (\pi p R_r + (1 - \pi)R_s)\left(1 - \frac{b}{\rho}\right) - \frac{(\rho - b)(\pi p + 1 - \pi)}{1 - q} = \frac{(\rho - b)(R_r - R_s)(\pi p + q - \pi q)}{\rho(1 - q)} > 0. \]

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3.3 Debt financing with renegotiation

We consider the case that $I > \frac{(\rho - b)(\pi p + (1 - \pi) q)}{1 - q}$, and analyze the effect of renegotiation of the debt contract.

It is well-known that the renegotiation of a debt contract has a function to avoid the inefficient project choice. The asset substitution problem is caused by the large face value of the debt contract. This problem arises in state 2. Executing the risky project in state 2, however, is not Pareto-efficient. Thus it is possible that for both investors and entrepreneur-managers decreasing the face value is still profitable. In this case, renegotiation of a debt contract between investors and the entrepreneur-manager will happen. Especially if there are not too many investors, renegotiation can be easy. The equilibrium will differ from that of the previous section when considering renegotiation. We will show that firstly renegotiable debt financing is a more efficient method than that of nonnegotiable, and secondly that the efficiency of equity financing or renegotiable debt financing, depends on the level of the entrepreneur-manager’s private benefit.

Before analyzing the renegotiation, we explain the procedures of renegotiation of a debt contract. Let us assume that investors have all the negotiating power, that is, investors make a take-it- or-leave-it offer, then the entrepreneur-manager decides to accept or reject it. If the entrepreneur-manager accepts the offer, the new contract $D_2$ is established, otherwise, original contract $D_1$ is still
valid.

First we investigate $D_2$ settled in the renegotiation subgame. Note that if $I > \frac{(\rho - b)(\pi p + \pi)}{1 - q}$ and $D_1$ is given at date 1, the risky project will be chosen in state 2 unless renegotiation occurs. Under the original contract, the expected payoff for investors is $qD_1$ and that of the entrepreneur-manager is $q(R_r - D_1) + b$ in state 2.

Next we consider what level of face-value investors will offer. If investors choose renegotiation, it must be $D_2 \geq qD_1$. In addition, $D_2$ will never be over $D_1$, because the entrepreneur-manager will refuse. Thus, the outcome $D_2$ from renegotiation satisfies the following equations.

$$D_1 > D_2 \geq qD_1$$

$$\frac{\rho - b}{1 - q} \geq D_2$$

Among these contracts, the most favorable $D_2$ for investors will be chosen, that is, the maximum $D_2$. If $D_1 > \frac{\rho - b}{1 - q}$, the renegotiation does not happen. Thus, the following equation must be satisfied.

$$D_1 \leq \frac{\rho - b}{q(1 - q)}.$$  \hspace{1cm} (17)

Therefore, the outcome of renegotiation is

$$D_2 = \frac{\rho - b}{1 - q}.$$  \hspace{1cm} (18)

In order to raise $I$ from the market, $D_1$ and $D_2$ must satisfy

$$\pi pD_1 + (1 - \pi)D_2 = I.$$  \hspace{1cm} (19)

Note that $D_2$ can never be over $R_s$, therefore we can rewrite (17) and (19) as follow.

$$I \leq \frac{\rho - b}{1 - q} \left( \frac{\pi p}{q} + 1 - \pi \right)$$  \hspace{1cm} (20)

Considering the renegotiation of a debt contract, Figure 4 can be modified to Figure 5. In Case 1, the following condition is fulfilled, while in Case 2 it is not.

$$R_r \leq \frac{\rho - b}{q(1 - q)}.$$  \hspace{1cm} (21)

**Proposition 3** Considering the renegotiation, the efficient range of parameters of debt financing is grater than that of without renegotiation. Under debt financing with renegotiation,
1. If \( R_r \leq \frac{\rho - b}{q(1 - q)} \), the efficient investments are carried out when \( 0 \leq I \leq \pi p R_r + (1 - \pi) \frac{\rho - b}{1 - q} \), while the investment is not possible when \( I > \pi p R_r + (1 - \pi) \frac{\rho - b}{1 - q} \).

2. If \( R_r > \frac{\rho - b}{q(1 - q)} \), the efficient investments are carried out when \( 0 \leq I \leq \frac{\rho - b}{1 - q} \left( \pi p + 1 - \pi \right) \), the inefficient investments are carried out when \( \frac{\rho - b}{1 - q} \left( \pi p + 1 - \pi \right) < I \leq (\pi p + (1 - \pi)q) R_r \), or the investment is not possible when \( I > (\pi p + (1 - \pi)q) R_r \).

Proof. It is obvious from Figure 4 and 5.

Let us compare debt financing with renegotiation and equity financing. For the sake of simplicity, we assume that (10) and (21) are satisfied from here \(^{6}\).

Proposition 4 (Comparison of debt financing with renegotiation and equity financing) Suppose that \[ B \equiv \frac{\rho q(1 - \pi)(R_r - R_s)}{\pi p(1 - q) R_r + q(1 - \pi)(R_r - R_s)}. \] (22)

If \( b < B \), the efficient range of parameters of equity financing are larger than that of debt financing with renegotiation, and vice versa. If \( b = B \), the efficient

\(^{6}\)Even if these assumptions are not satisfied, we can make the similar conclusions.
Proposition 4 shows that the preferability of equity financing or debt financing depends on the effect of the private benefit of the entrepreneur-manager. From (4), we can see that a larger $b$, involves the lower limit of $\alpha$ under which the efficient investments are chosen. It means that if the entrepreneur-manager’s private benefit is not negligible, the share of the outside investors can not be large, otherwise the inefficient project will be chosen in state 2. A smaller $\alpha$ means that less return belongs to the equity-holders. That is, a larger private benefit leads to the larger cash flow which can not be distributed to investors in equity financing, and results in the upper bound of the investment to shrink the efficient projects to be chosen.

On the other hand as for debt financing, noting that $\pi R_r + (1 - \pi) \frac{\rho - b}{1 - q}$, $b$ affects the the cash flow distributed to debt-holders in state 2, but not that of in state 1. In debt financing, return allocated to investors shrinks due to private
benefit only in state 2, while all return $R_e$ can be allocated to investors in state 1.

If the private benefit is large, the advantage of equity financing becomes smaller, while if the private benefit is small, the asset substitution problem becomes more serious relatively, and the merit of debt financing will decrease. That is, there is a threshold value of private benefit $B$, over which value the debt financing with renegotiation has a larger range of parameters under which efficient investments are chosen in the equilibrium than that of equity financing, otherwise equity financing has more parameters than that of debt financing. If $b = B$, the range of parameter under which the efficient investments are chosen in the equilibrium is the same between equity financing and debt financing with renegotiation.

In general, debt financing and equity financing have no difference as for efficiency as long as the asset substitution problem is settled by renegotiation. In this model, however, these two financing methods are different even though asset substitution can be settled by renegotiation, because we focus on the range of parameter under which the efficient investment are chosen. To see this, suppose that the private benefit of the entrepreneur-manager is nearly zero. In debt financing, the face value decided after renegotiation will be set to a level which can avoid asset substitution. On the other hand, in equity financing, the share of investors $\alpha$ can be made nearly 1. In other words, a feature of debt financing still remains even though we consider the renegotiation. The feature is that the face value which can avoid the asset substitution problem is small. The low renegotiated face value causes the maximum amount of capital the entrepreneur-manager can raise to decrease, which leads the investment itself to be impractical. Thus, if the asset substitution problem is serious, the equity financing will be chosen, even thought renegotiation of the debt contract can avoid the asset substitution problem. It is well known that equity financing can avoid the asset substitution problem, however, it is remarkable that equity financing can have more advantage than debt financing with renegotiation.

4 Convertible debt financing and the optimal capital structure

In this section, we show that convertible debt financing has more merit than pure-debt, pure-equity financing, or mixed financing of equity and debt. In the previous section, we saw that the range of parameters under which the efficient investments are chosen varies with financing methods. In this section, first, we show that the mixed financing of equity and debt has a larger range of parameters for efficient investment. Second, we show that convertible debt financing has a larger range of parameters than that of mixed financing. It is obvious
from proposition 1 that the mixture of equity and debt without renegotiation have no more merit than pure-equity financing. Thus the mixed financing we analyze here is the combination of equity and debt with renegotiation.

Moreover it is significant here whether the same investors who own bonds also have equities or not. This is because renegotiation of the debt contract profits not only debt-holders but also equity-holders. The outcome of renegotiation depends on how much equities the debt-holders have. Thus, we define \( \theta \) as the share of issued equity which belongs to debt-holders.

Let us describe the results which are introduced in this section first. Mixed financing of equity and debt has merit under a certain range of parameter if \( \theta \) is larger than a threshold value, otherwise it is of no merit. Specifically, if debt holders can abandon their obligation, mixed financing can be a more efficient financing method. In addition, the more equities the debt-holders have, the larger the range of parameters of efficient investment. Partial abandonment of a debt contract is not in equilibrium. This is because, in this model, reducing the face value of a debt contract when it is needed has a monotonic relation with the upper bound of the feasible investment \( I \), so if reducing the face value is preferable, then complete reducing i.e, abandonment is the most efficient.

Moreover, convertible debt financing has a larger efficient parameter set than mixed financing. In convertible debt financing, the same investors who posses bonds take equities when they decide to convert the convertible bonds, and the bonds disappear. Remember that in mixed financing, simultaneous possession of equity and debt is preferable because it prompts debt-holders to abandon the debt contract in state 2. We can regard convertible debt financing as having the same function with simultaneous possession. A higher value of equity induces conversion, which means debt abandonment can be more possible if it is efficient. In general, however, it is difficult to oblige investors to possess both equities and bonds. Thus, convertible debt financing can have more advantage than mixed financing of equity and debt. In other words, convertible debt financing has a function that coincides equity-holders and bond-holders, and in this sense, convertible debt financing settle the conflict between equity-holders and bond-holders and can be the most efficient financing method. This function has not been discussed so far, but deserve attention.

### 4.1 Mixed financing of equity and debt

Before analyzing convertible debt financing, we investigate if mixed financing of equity and debt has some benefit to the entrepreneur-manager. As in the previous section, we check the range of parameters, especially the maximum amount of \( I \) under which efficient investments are chosen. Denote the initial face value of debt by \( D_1 \), the renegotiated face value of debt by \( D_2 \), the investors’ share of equity by \( \alpha \), and the entrepreneur-manager’s share of equity by \( 1 - \alpha \).
From the definition of $\theta$, the equity share which belongs to bond-holders becomes $\theta\alpha$.

If the following equation is satisfied because of the high value of $D_1$, renegotiation take place in state 2.

$$(1 - \alpha)(R_s - D_1) < (1 - \alpha)q(R_r - D_1) + b$$  \hspace{1cm} (23)

If the entrepreneur-manager chooses efficient investments in state 2, $D_2$ must satisfy the following relation.

$$(1 - \alpha)(R_s - D_2) \geq (1 - \alpha)q(R_r - D_2) + b$$  \hspace{1cm} (24)

Equation (23) means that the risky project is chosen without renegotiation in state 2, and (24) means that the safe project is chosen under the new contract $D_2$ in state 2. For a given $\theta$, the contract which bondholders offer to the entrepreneur-manager in state 2 must satisfy the following equation.

$$D_2 + \theta\alpha(R_s - D_2) \geq q(D_1 + \theta\alpha(R_r - D_1))$$  \hspace{1cm} (25)

This means that renegotiation is rational for bondholders in state 2. If the entrepreneur-manager accepts, the new face value must satisfy the following relation.

$$(1 - \alpha)(R_s - D_2) \geq (1 - \alpha)q(R_r - D_1) + b$$  \hspace{1cm} (26)

Note that $D_2 < D_1$, if (24) is satisfied, then (26) is satisfied. Thus, if there exists a $D_2$ which satisfies (23), (24) and (25), renegotiation occurs in the equilibrium.

In order to raise fund $I$ from the market, $\alpha$, $D_1$, and $D_2$ must satisfy the following relation.

$$I = \pi p(D_1 + \alpha (R_r - D_1)) + (1-\pi) (D_2 + \alpha (R_s - D_2))$$  \hspace{1cm} (27)

Now we derive the maximum $I$ under which the efficient investments are chosen by solving the following equation.

$$\max_{D_1, D_2, \alpha} \pi p(D_1 + \alpha (R_r - D_1)) + (1-\pi) (D_2 + \alpha (R_s - D_2))$$  \hspace{1cm} (28)

s.t. $$(23), (24), (25)$$

$$D_1 \leq R_r$$

$$D_2 \leq R_s$$

$$\alpha \leq 1$$

$$D_1, D_2, \alpha \geq 0$$

From the solution, we get Proposition (5) \footnote{See Appendix 1.}. 

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Proposition 5 The mixed financing of equity and debt is a more efficient financing method than pure-equity or pure-debt financing if one or more of the following four conditions are satisfied. (i) \( \delta \geq \gamma \), \( \gamma \leq 1 \), and \( \theta \in [\gamma, 1] \). (ii) \( \delta < 1 < \gamma \) and \( \theta \in [\delta, 1] \). (iii) \( \delta < \gamma \leq 1 \) and \( \theta \in (\delta, \gamma) \). (iv) \( \delta < \gamma \leq 1 \) and \( \theta \in [\gamma, 1] \). \( \gamma \) and \( \delta \) are defined in (53) and (54) in appendix 1 respectively.

In the equilibrium of case (i) and (iv), the contract \( C_1 \) is

\[
C_1 = \left\{ D_1 = R_r, \ D_2 = 0, \ \alpha = 1 - \frac{b}{\rho} \right\}
\]
while that of case (ii) and (iii), which we call \( C_2 \) is

\[
C_2 = \left\{ D_1 = \frac{\theta \rho (\rho - b)}{q (\rho + \theta (\rho - b))}, \ D_2 = 0, \ \alpha = 1 - \frac{b}{\rho} \right\}.
\]

\( I \) in \( C_1 \) is expressed as

\[
\pi p R_r + (1 - \pi) \left( 1 - \frac{b}{\rho} \right) R_s,
\]
and \( I \) in \( C_2 \) is expressed as

\[
\left( 1 - \frac{b}{\rho} \right) \left( \pi p R_r + (1 - \pi) R_s \right) + \frac{\pi p \theta (\rho - b)}{q (\rho + \theta (\rho - b))}.
\]

\( I \) in \( C_1 \) is larger than that of \( C_2 \), and \( I \) in both \( C_1 \) and \( C_2 \) are larger than \( (\pi p R_r + (1 - \pi) R_s)(1 - \frac{b}{\rho}) \), which is the maximum value of \( I \) in pure equity financing under which efficient investments are chosen, and are also larger than \( \pi p R_r + (1 - \pi) \frac{b}{\rho} \), which is the maximum value of \( I \) in pure debt financing.\(^8\)

Proposition 6 As for \( C_2 \), the larger \( \theta \), the larger amount of \( I \) under which efficient investments are chosen.

Proof

\[
\left( \pi p R_r + (1 - \pi)(1 - \frac{b}{\rho}) R_s \right) - \left( \pi p R_r + (1 - \pi) \frac{\rho - b}{1 - q} \right) = \frac{(1 - \pi) q (R_r - R_s)(R_s - q R_r - b)}{(1 - q)(R_s - q R_r)} > 0.
\]
If \( \theta = \delta \),

\[
\left( \left( 1 - \frac{b}{\rho} \right) \left( \pi p R_r + (1 - \pi) R_s \right) + \frac{\pi p \delta (\rho - b)}{q (\rho + \theta (\rho - b))} \right) - \left( \pi p R_r + (1 - \pi) \frac{\rho - b}{1 - q} \right) = \frac{(1 - \pi) q (R_r - R_s)(\rho - b)}{(1 - q) \rho} > 0.
\]
Denote the amount of investment in $C_2$ by $U_2$. Proposition 6 is obvious from the following relation.

$$\frac{\partial U_2}{\partial \theta} = \frac{\pi pb \rho (\rho - b)}{q(\rho + b\theta - \theta \rho)^2} > 0$$

These two propositions mean that mixed financing can be more efficient than pure-equity or pure-debt financing, which have some characters that whole bonds are abandoned in state 2 and efficiency increases by investors’ holding equity and debt simultaneously. Proposition 5 shows that there are two types of contract $C_1$ and $C_2$ in mixed financing, but these contracts are different only in the face value of the original contract. It deserves attention that complete abandonment of a debt contract in state 2 is common to them both. Bondholders’ concession in state 2 prompts the entrepreneur-manager to choose the efficient project and leads to a higher value of the firm. In this model, the value of the firm relates to the extent of the concession made by bond-holders monotonically, thus partial abandonment is not in equilibrium. We can see this from (46) and (49) in appendix 1. Bond-holders concessions increase the firm’s value, while surplus of the renegotiation spills over to equity-holders. If bondholders own equity at the same time, their loss of concession can be covered by an advancement of the equity value. Therefore, there exists a critical value of $\theta$ which enables bond-holders to abandon their contract completely in state 2, and leads to an increase in the firm’s value.

### 4.2 Convertible debt financing

In the previous section, we show that mixed financing of equity and debt can be more efficient than pure-equity or pure-debt financing. In order to attain this outcome, investors’ simultaneous holding of equity and debt is required, which is unrealistic in reality. In general, we can not force investors to own both equity and debt, and the complete abandonment of a debt contract hardly ever occurs. This fact may be interpreted as mixed financing having no merit.

Convertible debt financing, however, settles this problem and is an alternative financing method for mixed financing of equity and debt. In convertible debt financing, we do not need an unrealistic assumption, and it can attain the same or better efficiency as mixed financing. In this sense, we can regard convertible debt financing as a substantial financing method which obtains the preconditions needed in mixed financing. In this section, we show that convertible bond financing has a property that coincides equity-holders with debt-holders, and can settle the conflict between equity-holders and debt-holders, leading to an increase in the firm’s value.
Let us denote the face value of convertible-debt by $D$, and the equity share of investors when they convert a convertible bond into equities by $\alpha$. Whether the entrepreneur-manager chooses the safe project in state 2 or not depends on whether investors convert their convertible bonds into equities in state 2 or not. The condition under which investors are willing to convert their convertible bonds into equities is \((1 - \alpha)qR_r + b \leq (1 - \alpha)R_s\), while they will not choose conversion if \(q(R_r - D) + b \leq R_s - D\). The situation is the same as normal debt financing when there is no possibility that conversion occurs in the equilibrium. Let us focus on the case in which conversion occurs in the equilibrium. Thus we ignore the condition \(q(R_r - D) + b \leq R_s - D\).

Suppose that \((1 - \alpha)qR_r + b \leq (1 - \alpha)R_s\), \(qD \leq \alpha R_s\), and \(q(R_r - D) + b > R_s - D\). The first condition means that the safe investment is chosen in state 2, the second means that conversion occurs in state 2, and the third means that the risky project is chosen if conversion does not occur in state 2. In order to raise the fund $I$ from market, $D$ and $\alpha$ must satisfy the following relation.

\[
I = \pi pD + (1 - \pi)\alpha R_s \tag{29}
\]

In convertible debt financing, the maximum amount of investment $I$ under which efficient projects are chosen is the solution of the following equation.

\[
\max_{D, \alpha} \pi pD + (1 - \pi)\alpha R_s \tag{30}
\]

s.t. \[
\begin{align*}
D &> \frac{\rho - b}{1 - q} \\
\rho(1 - \alpha) - b &\geq 0 \\
\alpha R_s - qD &\geq 0 \\
D &\leq R_r \\
\alpha &\leq 1 \\
D, \alpha &\geq 0
\end{align*}
\]

The first condition means that the risky project is chosen in state 2 if the convertible debt is not converted into equity, the second means that the safe project is chosen in state 2 if the convertible debt is converted to equity. The third condition means that conversion is rational for investors. The forth condition sets the limit of the face value of convertible debt and the fifth one sets the limit of $\alpha$.

The solution is \(^9\)

\[
D = R_r, \quad \alpha = 1 - \frac{b}{\rho}.
\]

\(^9\)See appendix 2.
The amount of investment is

\[ \pi pR_r + (1 - \pi) \left( 1 - \frac{b}{\rho} \right) R_s. \]  \( (31) \)

(31) is the same investment value \( I \) as \( C_1 \) in previous section. Therefore we can see that the maximum value of \( I \) under which efficient projects are chosen can never be lower than that of mixed financing. That is, the range of parameters under which efficient projects are chosen with convertible debt financing is larger than that of mixed financing of equity and debt. From this, we can make the following proposition.

**Proposition 7** Convertible debt financing method is as efficient or more efficient than that of mixed financing of equity and debt.

The convertible debt financing is equal to that of mixed financing which can attain the contract \( C_1 \) in the equilibrium. In order to execute \( C_1 \) using mixed financing, parameters must satisfy certain conditions, which are not required in convertible debt financing. The range of parameters for convertible debt financing under which the efficient project are chosen is larger than that of mixed financing. This accounts for the issuing of convertible debt.

In mixed financing, for the sake of efficiency, whole debt abandonment in state 2 is important, while in convertible bond financing, conversion of convertible bonds has the same effect as this abandonment. This is because the debt disappears when convertible debt are converted into equity. Moreover, the higher the equity value, the easier it is to convert the convertible bond, which mean that investors are willing to chose conversion so as to raise the value of equity. This relation means that convertible debt financing can settle the conflict between equity-holders and debt-holders easily. Actually whole abandonment of debt is rare because debt-holders suffer a lot of damage from it. Nevertheless, it can be significant that debt-holders make a concession in order to run the firm more efficiently. This conflict can be settled by investors’ simultaneous possession of equity and debt, or by issuing convertible bonds. Noting that simultaneous possession is unrealistic, issuing convertible debt is a rational option for the entrepreneur-manager.

From the discussion above, it is shown that convertible debt financing has a role to coincide equity-holders and debt-holders, and in this sense, convertible debt financing has an advantage for settling the conflict between equity-holders and debt-holders.
5 Private benefit with a safe project

If $b_s > 0$ and $b_r = 0$, i.e., the private benefit of the entrepreneur-manager occurs with safe project, there exists a certain range of parameters under which convertible bond financing is the most efficient financing method among pure-debt, pure-equity, a mixture of them, and convertibles. We can analyze this case in the same manner as the previous section.

Frankly speaking, in this case, the problem is less serious than that of the previous section. The reason is that the entrepreneur-manager has an incentive to execute a safe project, so the problem occurs in state 1. The large amount of investment $I$ will cause the entrepreneur-manager to choose a safe project, which is inefficient in state 1. In order to settle this problem, the convertible debt financing has a similar benefit to the previous section. In this model, however, noting that $p$ is large, the entrepreneur-manager has more incentive to choose a risky project in state 1, compared to the case of $b_s = 0$ and $b_r > 0$. Thus, the problem is not so serious here. We omit analyzing this case as we can draw the same conclusion by tracking the same procedure presented in the previous sections.

6 Equal private benefit with risky and safe projects

In this section we analyze the case of $b_r = b_s$, and show that equity financing is the most efficient method. We consider the sub-game perfect equilibrium. Let us consider which project the entrepreneur-manager will chose at date 1, given that equities are issued at date 0. Let $\alpha$ be the investors’ share, and $1-\alpha$ be that of the entrepreneur-manager.

The entrepreneur-manager chooses the project which maximize his expected payoff given $\alpha$ and the state. In state 1, for any $\alpha$, $(1-\alpha)pR_r + b_r > (1-\alpha)R_s + b_s$. The left term means that the entrepreneur-manager’s expected payoff when he chooses a risky project while the right term denotes that of a safe project, meaning that he always chooses a risky project in state 1 for any $\alpha$. Analogously, in state 2, $(1-\alpha)qR_r + b_r < (1-\alpha)R_s + b_s$, for any $\alpha$, a safe project is chosen. Therefore the efficient project are executed under both states for any $\alpha$.

Next let us consider which $\alpha$ is chosen at date 0. In order to raise the amount of $I$, $\alpha$ must satisfy the following equation.

$$\alpha = \frac{I}{\pi p R_r + (1-\pi) R_s}$$

Note that the limit of $\alpha$ is 1, as long as

$$I \leq \pi p R_r + (1-\pi) R_s,$$

then efficient projects are chosen and it is feasible.
Let us discuss debt financing. Denote the face value of the debt by \( D \). Note that for any \( D, \) the entrepreneur-manager’s expected payoff in state 1 is 
\[
p(R_r - D) + b_r > R_s - D + b_s,
\]
so he chooses a risky project. On the other hand, in state 2, if 
\[
D \leq \frac{R_s - qR_r}{1 - q},
\]
then \( q(R_r - D) + b_r \leq R_s - D + b_s \), so the entrepreneur-manager chooses a safe project, while if \( D > \frac{R_s - qR_r}{1 - q} \), then \( q(R_r - D) + b_r > R_s - D + b_s \), and he chooses a risky project. In other words, as for debt financing, when the face value is larger than a certain critical value, the entrepreneur-manager chooses an inefficient project.

If investors expect that efficient projects are executed, \( D \) must satisfy the following relation in order to raise the fund \( I \).

\[
D = \frac{I}{\pi p + 1 - \pi}
\]

Note that the limit of \( D \) is \( \frac{R_s - qR_r}{1 - q} \), the investment is feasible and the efficient projects are chosen in the equilibrium if

\[
I \leq \frac{(\pi p + 1 - \pi)(R_s - qR_r)}{1 - q}.
\]  

(33)

The right hand term of equation (32) is bigger than that of (33) \(^{10}\).

Therefore, when \( b_r = b_s \), efficient projects are chosen under equity financing while inefficient can be chosen if \( I \) is larger than the left-hand term of equation (33) under debt financing.

7 Conclusion

In this paper, we have presented two results. When the entrepreneur-manager’s project choice depends on both financing method and his private benefit a problem arises stochastically, (i) mixed financing is more efficient than pure-equity or pure-debt financing, (ii) convertible bond financing is more efficient than mixed financing. A mixture of equity financing and debt financing prompts the entrepreneur-manager to choose an efficient project and convertible financing can mitigate the conflict between equity-holders and debt-holders in the renegotiation of a debt contract, since equity-holders coincide with debt-holders in convertible debt financing.

\[
\pi p R_r + (1 - \pi) R_s - \frac{(\pi p + 1 - \pi)(R_s - qR_r)}{1 - q} = \frac{(\pi \pi + q - \pi q)(R_r - R_s)}{1 - q} > 0
\]  

\(^{10}\)
Appendix

Appendix 1 Proof of Proposition 5:
The Lagrangian of the problem (28) can be written as

\[ L = \pi p (D_1 + \alpha (R_r - D_1)) + (1 - \pi) (D_2 + \alpha (R_s - D_2)) + \lambda_0 ((1 - \alpha) (-\rho + (1 - q) D_1) + b) + \lambda_e ((1 - \alpha)(\rho - (1 - q) D_2) - b) + \lambda_i ((D_2 - q D_1)(1 - \theta \alpha) + \theta \alpha \rho) + \lambda_3 (R_r - D_1) + \lambda_2 (R_s - D_2) + \lambda_3 (1 - \alpha) \]

where \( \lambda_0, \lambda_e, \lambda_i, \lambda_1, \lambda_2 \) and \( \lambda_3 \) are the Lagrange multipliers on each constraint.

Then, the first-order conditions are given by

\[ \pi p (1 - \alpha) - \lambda_i (1 - \theta \alpha) - \lambda_1 \leq 0 \] (34)
\[ (1 - \pi)(1 - \alpha) - \lambda_e (1 - \alpha)(1 - q) + \lambda_i (1 - \theta \alpha) - \lambda_2 \leq 0 \] (35)
\[ \pi p (R_r - D_1) + (1 - \pi)(R_s - D_2) - \lambda_e (\rho - (1 - q) D_2) + \lambda_i (-\theta (D_2 - q D_1) + \theta \rho) - \lambda_3 \leq 0 \] (36)
\[ (1 - \alpha)(-\rho + (1 - q) D_1) + b \geq 0 \] (37)
\[ (1 - \alpha)(\rho - (1 - q) D_2) - b \geq 0 \] (38)
\[ (D_2 - q D_1)(1 - \theta \alpha) + \theta \alpha \rho \geq 0 \] (39)
\[ R_r - D_1 \geq 0 \] (40)
\[ R_s - D_2 \geq 0 \] (41)
\[ 1 - \alpha \geq 0 \] (42)

where \( \lambda_0 = 0 \) because \((1 - \alpha)(-\rho + (1 - q) D_1) + b > 0 \) from (23).

Suppose that \( \lambda_2 > 0 \) and \( \lambda_3 > 0 \), then it is in contradiction with condition (38), thus, \( \lambda_2 = 0, R_r > D_2, \lambda_3 = 0, 1 > \alpha \). Note that \( \lambda_0 = 0 \), from (35) \( \lambda_e = 0 \) is not possible. Thus, \( \lambda_e > 0 \) and the inequality in (38) must be equal.

\[ (1 - \alpha)(\rho - (1 - q) D_2) = b \] (43)

From the equation (34), \( \lambda_i = \lambda_1 = 0 \) is not possible.

Therefore there are three cases remaining.

**Case A**: \( \lambda_1 > 0, \lambda_i = 0, \lambda_e > 0, \lambda_0 = 0, \lambda_2 = 0 \) and \( \lambda_3 = 0 \).

**Case B**: \( \lambda_1 > 0, \lambda_i > 0, \lambda_e > 0, \lambda_0 = 0, \lambda_2 = 0 \) and \( \lambda_3 = 0 \).

**Case C**: \( \lambda_1 = 0, \lambda_i > 0, \lambda_e > 0, \lambda_0 = 0, \lambda_2 = 0 \) and \( \lambda_3 = 0 \).

Strictly speaking, these classifications are explained as follows. In case A, the parameters are set as equation (39) becomes a strict inequality when \( D_2 = 0 \).
and $D_1 = R_r$. This setting of parameters is equivalent to the following equation.

$$\theta > \frac{R_r(R_r - R_s - b)}{R_s(R_r - R_s)}$$

(44)

On the other hand, in case B, parameters are set as equation (44) becomes the equality when $D_2 = 0$ and $D_1 = R_r$. In case C, parameters are set as equation (44) is not satisfied when $D_2 = 0$. Case A and case B are basically the same. In case B, if (38) is equal, then (39) happens to also be equal.

Let us investigate each case. First, we consider the solution of case A and B. From $\lambda_1 > 0$, it must be that $D_1 = R_r$. Denote the relation between $\alpha$ and $D_2$ given from (43) by $\alpha(D_2)$. Substituting $\alpha(D_2)$ and $D_1 = R_r$ for $\alpha$ and $D$ in the objective function (28), we get

$$\Pi = \pi p R_r + (1 - \pi)(D_2 + \alpha(D_2)(R_s - D_2)).$$

(45)

Partial differentiation of $\Pi$ with respect to $D_2$ is strictly negative.

$$\frac{\partial \Pi}{\partial D_2} = \frac{-b q (1 - \pi) (R_r - R_s)}{(-D_2(1 - q) - q R_r + R_s)^2} < 0$$

(46)

Thus, $D_2 = 0$ is chosen and $\alpha$ satisfies the following equation.

$$\alpha = 1 - \frac{b}{\rho}$$

$D_2 = 0$ and $\alpha$ in this equation satisfies (39) if $\theta \geq \frac{R_r(R_r - R_s - b)}{R_s(R_r - R_s)}$. Therefore we can get the solution in case A and B.

$$D_1 = R_r, \quad D_2 = 0, \quad \alpha = 1 - \frac{b}{\rho}$$

(47)

Next, let us investigate case C, in which parameters are set as $\theta < \frac{R_r(R_r - R_s - b)}{R_s(R_r - R_s)}$. Note that from $\lambda_1 = 0$ then $D_1 < R_r$, and from $\lambda_1 > 0$ then (39) is equal. Using (43) and (39), the investors expected payoff function $\Pi$ can be rewritten as a function of $D_2$.

$$\Pi(D_2) = \pi p R_r + (1 - \pi)R_s - \frac{(1 - \pi + p \pi) b (R_s - D_2)}{\rho - D_2(1 - q)} + \frac{p \pi}{q} \frac{bp}{(\rho - D_2(1 - q))(1 - \theta) + b \theta}$$

(48)

Partial differentiation of this equation with respect to $D_2$ is strictly negative.

$$\frac{\partial \Pi(D_2)}{\partial D_2} = \frac{-b (p \pi + q - \pi q) (R_r - R_s)}{((\rho - (1 - q) D_2)(1 - \theta) + b \theta)^2} < 0$$

(49)

Thus $D_2 = 0$ is chosen and $\alpha = 1 - \frac{b}{\rho}$ must be satisfied. Substituting these into (39) and solving it with respect to $D_1$, we get

$$D_1 = \frac{\theta \rho (\rho - b)}{q (b \theta + \rho (1 - \theta))}.$$
Note that $D_1 < R_r$, parameters must be set as follows for the sake of an existing of solution.

$$\frac{q R_r (R_s - q R_r)}{R_s (R_s - q R_r - b)} < \theta < \frac{R_r (R_r - R_s - b)}{R_s (R_r - R_s)} \tag{51}$$

The existence of $\theta$ which satisfies (51) depends on other parameters; $R_r$, $R_s$, $b$ and $q$. If it exists, the solution satisfies the followin equation.

$$D_1 = \frac{\theta\rho(\rho - b)}{q(b\theta + \rho(1 - \theta))}, \quad D_2 = 0, \quad \alpha = 1 - \frac{b}{\rho} \tag{52}$$

On the other hand, no solution exists if the parameters are set so that either one of the following equations are satisfied.

- $\theta \leq \frac{q R_r (R_s - q R_r)}{R_s (R_s - q R_r - b)}$
- $\frac{q R_r (R_s - q R_r)}{R_s (R_s - q R_r - b)} > \frac{R_r (R_r - R_s - b)}{R_s (R_r - R_s)}$, and if there dose not exist $\theta$ which satisfies (51), $\theta < \frac{R_r (R_r - R_s - b)}{(R_r - R_s) R_r}$.

From this and $\theta \in [0, 1]$, we get the following relation. For the sake of simplicity, we define $\gamma$ and $\delta$ as follows, they are always positive from assumptions.

$$\gamma = \frac{R_r (R_r - R_s - b)}{R_s (R_r - R_s)} \quad \delta = \frac{q R_r (R_s - q R_r)}{R_s (R_s - q R_r - b)} \tag{53, 54}$$

1. If $\delta \geq \gamma > 1$, then mixed financing has no merit.

2. If $\delta \geq \gamma$ and $\gamma \leq 1$, then if $\theta \in [0, \gamma)$, mixed financing has no merit, if $\theta \in [\gamma, 1]$, mixed financing expressed by (47) is more efficient than pure-equity or pure debt financing.

3. If $\delta < 1 < \gamma$, then if $\theta \in [0, \delta)$, mixed financing has no merit, if $\theta \in (\delta, 1]$, mixed financing expressed by (52) is more efficient than pure-equity or pure debt financing.

4. If $\delta < \gamma \leq 1$, then if $\theta \in [0, \delta)$, mixed financing has no merit, if $\theta \in (\delta, \gamma)$, mixed financing expressed by (52) is more efficient than pure-equity or pure debt financing, if $\theta \in [\gamma, 1]$, mixed financing expressed by (47) is more efficient than pure-equity or pure debt financing.

**Appendix 2** Proof of Proposition 7:
The Lagrangian of the problem (30) can be written as

\[ L = \pi pD + (1 - \pi)\alpha R_s + \lambda_0 \left( D - \frac{\rho - b}{1 - q} \right) + \lambda_e (\rho(1 - \alpha) - b) + \lambda_i (\alpha R_s - qD) + \lambda_1 (R_r - D) + \lambda_2 (1 - \alpha), \]

where \(\lambda_0, \lambda_e, \lambda_i, \lambda_1\) and \(\lambda_2\) are the Lagrange multipliers on each constraint. The first-order conditions are given by

1. \(\pi p - \lambda_i q - \lambda_1 \leq 0 \) \quad (55)
2. \((1 - \pi)R_s - \lambda_e \rho + \lambda_i R_s - \lambda_2 \leq 0 \) \quad (56)
3. \(D - \frac{\rho - b}{1 - q} > 0 \) \quad (57)
4. \(\rho(1 - \alpha) - b \geq 0 \) \quad (58)
5. \(\alpha R_s - qD \geq 0 \) \quad (59)
6. \(R_r - D \geq 0 \) \quad (60)
7. \(1 - \alpha \geq 0, \) \quad (61)

where \(\lambda_0 = 0\).

\(\lambda_2 > 0\) is not possible since it implies \(\alpha = 1\), which contradicts with (58). Thus, \(\lambda_2 = 0\) and \(\alpha < 1\). \(\lambda_e = \lambda_i = 0\) is not possible, since from \(\lambda_2 = 0\), it contradicts with (56). \(\lambda_e > 0\) and \(\lambda_i > 0\) is not possible, since from (58) we get \((1 - \alpha)R_s - q(R_r - D) - b = 0\), and form (59) we get \(\alpha R_s - qD = 0\), meaning \(R_s = qR_r + b\), but this contradicts with the assumption. \(\lambda_e = 0\) and \(\lambda_i > 0\) is not possible, since noting that \(\lambda_2 = 0\) contradicts with (56). Thus, \(\lambda_e > 0\) and \(\lambda_i = 0\). \(\lambda_1 = 0\) is not possible, since, noting that \(\lambda_e > 0\) and \(\lambda_i = 0\), it contradicts with (55). Thus, \(\lambda_1 > 0\) and \(D = R_r\). Noting that \(D = R_r\), from the assumptions we get \(R_r > \frac{\rho - b}{1 - q}\), and (57) is satisfied. Since \(D > 0\), noting the equation (59), \(\alpha > 0\).

Therefore it is possible that \(\lambda_e > 0, \lambda_i = 0, \lambda_1 > 0, \) and \(\lambda_2 = 0\). In this case, we derive

\[ D = R_r, \quad \alpha = 1 - \frac{b}{\rho}. \]

References


