# Technical Appendix for
# Strategic Policy for Product R&D with Symmetric Costs

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**Lemma 1** *(1) $B(q^*; s) \neq q^*, \forall q^*$; (2) $B(q^*; s)$ is discontinuous at $q^* = \hat{q}^*(s)$; (3) $dB(q^*; s)/dq^* > 0, \forall q^* \neq \hat{q}^*(s)$; (4) $dB(q^*; s)/ds > 0$; and (5) $d\hat{q}^*/ds > 0$.*

*Proof.* It is shown that $\Pi^b(q^H(0; s), 0; s) = \Pi^b(1/8(1-s)k, 0; s) = 1/64(1-s)k > 0 = \Pi^b(q^L(0; s), 0; s) = \Pi^b(0, 0; s)$. On the other hand, since $\Pi^b(q^L(1/12(1-s)k; s), 1/12(1-s)k; s) \geq \Pi^b(q^L, 1/12(1-s)k; s), \forall q^L \leq 1/12(1-s)k$ and $\Pi^b(1/64(1-s)k, 1/12(1-s)k; s) > 0$, then $\Pi^b(q^L(1/12(1-s)k; s), 1/12(1-s)k; s) > 0$, which implies that $\Pi^b(q^L(q^*; s), q^*; s) > \Pi^b(q^H(q^*; s), q^*; s) = 0$ when $q^* = 1/12(1-s)k$. By the envelope theorem and the properties of the revenue function, $\Pi^b(q^H(q^*; s), q^*; s)$ ($\Pi^b(q^L(q^*; s), q^*; s)$) is decreasing (increasing) in $q^*$. By continuity of the profit function, there must exist $\hat{q}^*(s)$ which satisfies $\Pi^b(q^H(\hat{q}^*; s), \hat{q}^*(s); s) = \Pi^b(q^L(\hat{q}^*; s), \hat{q}^*(s); s)$. Then, (1) to (3) of the lemma follows directly from Lemma 3 in Aoki and Prusa (1997).

(4) Totally differentiate the home firm's first-order condition:

$$\partial \Pi^b(q, q^*; s)/\partial q \equiv \Pi^b_q(q, q^*; s) = R^b_q(q, q^*) - (1-s)C'(q) = 0 \tag{1}$$

to obtain $R^b_{qq}(q, q^*)dq + R^b_{qq^*}(q, q^*)dq^* - (1-s)C''(q)dq + C'(q)ds = 0$. Set $dq^* = 0$ and rearrange terms to yield $dq/ds = -C'(q)/\Pi^b_{qq}(q, q^*)$. Since $C' > 0$ and $\Pi^b_{qq}(q, q^*) < 0$ for $q = q^H$ and $q = q^L$, it follows that $dq^H/ds > 0$ and $dq^L/ds > 0$.

(5) By definition of $\hat{q}^*(s)$, $R^b(q^H(\hat{q}^*; s), \hat{q}^*) - (1-s)C(q^H(\hat{q}^*; s)) = R^b(q^L(\hat{q}^*; s), \hat{q}^*) - (1-s)C(q^L(\hat{q}^*; s))$. Totally differentiate it and rearrange terms to yield $d\hat{q}^*/ds = (C(q^L) - C(q^H))/(R^b_{q^*}(q^H, \hat{q}^*) - R^b_{q^*}(q^L, \hat{q}^*)) > 0$, because $C(q^L) < C(q^H)$, $R^b_{q^*}(q^H, \hat{q}^*) < 0$ and $R^b_{q^*}(q^L, \hat{q}^*) > 0$. □

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**Proposition 1** *When firms compete in prices in stage 3, the unilaterally optimal R&D policy for the home government is to implement the following subsidy schedule:*

$$s \begin{cases} = \hat{s} < 0, & \text{if } q > q^* \text{ and } q^* < \hat{q}^*, \\ = \underline{s} > 0, & \text{if } q < q^* \text{ and } q^* > \hat{q}^*, \\ < \hat{s}, & \text{if } q < q^* \text{ and } q^* < \hat{q}^*, \\ < \underline{s}, & \text{if } q > q^* \text{ and } q^* > \hat{q}^*, \end{cases}$$

*where* $\hat{s} \equiv \arg\max_s \{W(s) | q^* = B^*(q), q > q^*\}$ *and* $\underline{s}$ *eliminates the equilibrium in* $q < q^*$*.*

*Proof.* First, I show that if the home firm produces the higher quality product, the locally optimal R&D subsidy by the home government is negative. The first-order condition for the home government to maximize home welfare with respect to $s$ yields $dW/ds = \left(R^b_q(q, q^*) - C'(q)\right)(dq/ds) + R^b_{q^*}(q, q^*)(dq^*/ds) = 0$. Totally differentiate Eq. (1) and the foreign firm's first-order condition:

$$\Pi^{*b}_{q^*}(q, q^*; s^*) = R^{*b}_{q^*}(q, q^*) - (1 - s^*)C'(q^*) = 0 \tag{2}$$

and apply Cramer's rule to obtain $dq/ds = -\Pi^{*b}_{q^*q^*} C'/|D|$ and $dq^*/ds = \Pi^*_{q^*q} C'/|D|$, where $|D| = \Pi^b_{qq}\Pi^{*b}_{q^*q^*} - \Pi^b_{qq^*}\Pi^{*b}_{q^*q}$. Since $R^b_{qq}R^{*b}_{q^*q^*} - R^b_{qq^*}R^{*b}_{q^*q} = 0$, then $|D|$ can be rewritten as $|D| = -R^{*b}_{q^*q^*}(1 - s)C''(q) - \Pi^b_{qq}C''(q^*)$. Since $R^{*b}_{q^*q^*} < 0$ and $\Pi^b_{qq} < 0$ for both $q > q^*$ and $q < q^*$ and $C''(\cdot) > 0$, then $|D| > 0$. Hence, the locally optimal $s$, $\hat{s}$, is given by

$$\hat{s} = -R^b_{q^*}\Pi^{*b}_{q^*q} / \left(C'(q)\Pi^{*b}_{q^*q^*}\right) < 0, \tag{3}$$

because $R^b_{q^*} < 0$ and $\Pi^{*b}_{q^*q} > 0$ for $q > q^*$, and $C' > 0$ and $\Pi^{*b}_{q^*q^*} < 0$. To show $d^2W/ds^2|_{s=\hat{s}} < 0$, since $dW/ds = (d\Pi^b/dq)(dq/ds)$, it follows that $d^2W/ds^2 = (d^2\Pi^b/dq^2)(dq/ds)^2 + (d\Pi^b/dq)(d^2q/ds^2)$. Since $d\Pi^b/dq\big|_{s=\hat{s}} = 0$, it remains to show that $d^2\Pi^b/dq^2 < 0$. It is shown that $d^2\Pi^b/dq^2 = C''\{(R^b_{qq} - C'')C'' - R^{*b}_{q^*q^*}(R^{*b}_{q^*q^*} - 2C'')\}/(\Pi^{*b}_{q^*q^*})^2 < 0$, because $C'' > 0$, $R^b_{qq} < 0$, and $R^{*b}_{q^*q^*} < 0$.

Second, I show that there exists a range of $s$ that eliminates the equilibrium where $q < q^*$. I know that $1/8k < q^{*H}(q) < 7/48k$ for $q \leq 1/18k$ and that $1/18k \leq \hat{q} \leq 1/12k$ (Aoki, 1995). I also know $dB(q^*)/dq^* > 0$. Thus, if $B(1/8k) > 1/18k$, there is no equilibrium where $q < q^*$. Since $B(q^*)$ satisfies the first-order condition: $(q^*)^2(4q^* - 7q) - 2(1 - s)kq(4q^* - q)^3 = 0$, it is easy to show that $B(1/8k) > 1/18k$ if and only if $s > 0.771$, which implies that $s > 0.771$ can eliminate the equilibrium where $q < q^*$.

Third, in order to make sure that the equilibrium in $q > q^*$ is retained and the equilibrium in $q < q^*$ is eliminated, $s < \hat{s}$ for $q < q^* < \hat{q}^*$ and $s < \underline{s}$ for $q > q^* > \hat{q}^*$ are required.

By implementing a sufficiently high R&D subsidy for $q < q^*$ and $q^* > \hat{q}^*$ and an R&D tax given by Eq. (3) with other two elements, the remaining unique NE is at $(q_N^H(\hat{s}), q_N^{*L}(\hat{s}))$.

I now show that the first-order condition for the home firm in the simultaneous choice game with the optimal R&D tax is exactly the same as that for the home firm as a Stackelberg leader in the sequential choice game without R&D tax. Note that $q > q^*$ holds at the global maximum. The home firm's problem as a Stackelberg leader is given by $\max_q \quad \Pi^b(q, q^*; 0)$ subject to $q^* \in B^*(q)$. The first-order condition is given by

$$\Pi_q^b(q, q^*; 0) + \Pi_{q^*}^b(q, q^*; 0)(dq^{*L}/dq) = 0. \tag{4}$$

On the other hand, the home firm's problem in the simultaneous choice game with the optimal R&D tax is given by $\max_q \quad \Pi^b(q, q^*; \hat{s})$. Since $\Pi^b(q, q^*; \hat{s}) = \Pi^b(q, q^*; 0) + \hat{s}C(q)$, then the first-order condition is given by

$$\Pi_q^b(q, q^*; 0) + \hat{s}C'(q) = 0. \tag{5}$$

Substitute the formula of the optimal R&D subsidy, Eq. (3), into Eq. (5) to yield

$$\Pi_q^b(q, q^*; 0) - R_{q^*}^b \Pi_{q^*q}^* / \Pi_{q^*q^*}^{*b} = 0. \tag{6}$$

When $q > q^*$, totally differentiate the foreign firm's first-order condition and rearrange terms to obtain $dq^{*L}/dq = -R_{q^*q}^{*b}/\Pi_{q^*q^*}^{*b}$. Since $\Pi_{q^*}^b(q, q^*; 0) = R_{q^*}^b(q, q^*)$ and $\Pi_{q^*q}^b(q, q^*; 0) = R_{q^*q}^b(q, q^*)$, Eq. (6) is exactly the same as Eq. (4), which implies that $q_N^H(\hat{s})$ is equal to $q_S^H$.

For $(q_N^H(\hat{s}), q_N^{*L}(\hat{s}))$ to be globally stable, it is required that the home firm, taking $q_N^{*L}(\hat{s})$ and $\hat{s}$ as given, would prefer to choose $q_N^H(\hat{s}) = q^H(q_N^{*L}(\hat{s}); \hat{s})$ rather than to choose $q^L(q_N^{*L}(\hat{s}); s)$, where $s$ is high enough to eliminate the equilibrium in which $q < q^*$. It is also required that the foreign firm, taking $q_N^H(\hat{s})$ as given, would prefer to choose $q_N^{*L}(\hat{s}) = q^{*L}(q_N^H(\hat{s}))$ rather than to choose $q^{*H}(q_N^H(\hat{s}))$.

When $s = 1$, $q^L(q^*; 1) = (4/7)q^*$ and $\Pi^b(q^L(q^*; 1), q^*; 1) = q^*/48$. The numerical result shows that $q_N^H(\hat{s}) = 0.122232/k$ and $q_N^{*L}(\hat{s}) = 0.023894/k$. Using the formula of $\hat{s}$ given by Eq. (3), the value of $\hat{s}$ can be numerically calculated, which is $-0.034156$. Then, $\Pi^b(q^H(q_N^{*L}(\hat{s});\hat{s}), q_N^{*L}(\hat{s});\hat{s}) = 0.011725/k$. On the other hand, $\Pi^b(q^L(q_N^{*L}(\hat{s});1), q_N^{*L}(\hat{s});1) = q_N^{*L}(\hat{s})/48 = 0.000498/k$. It follows that $\Pi^b(q^H(q_N^{*L}(\hat{s});\hat{s}), q_N^{*L}(\hat{s});\hat{s}) > \Pi^b(q^L(q_N^{*L}(\hat{s});1), q_N^{*L}(\hat{s});1)$. Using the envelope theorem, $\partial\Pi^b(q^L(q^*;s), q^*;s)/\partial s = C(q^L(q^*;s)) > 0$, which implies that $\Pi^b(q^L(q^*;s), q^*;s)$ is increasing in $s$. Thus, for any $s < 1$ which is high enough to eliminate the equilibrium where $q < q^*$, $\Pi^b(q^H(q_N^{*L}(\hat{s});\hat{s}), q_N^{*L}(\hat{s});\hat{s}) > \Pi^b(q^L(q_N^{*L}(\hat{s});s), q_N^{*L}(\hat{s});s)$ holds, which implies that $(q_N^H(\hat{s}), q_N^{*L}(\hat{s}))$ is globally stable for the home firm. For the foreign firm, since $q_N^H(\hat{s}) > \hat{q}$,[1] by the definition of $B^*(q)$ it follows that $B^*(q_N^H(\hat{s})) = q^{*L}(q_N^H(\hat{s}))$. Thus, $(q_N^H(\hat{s}), q_N^{*L}(\hat{s}))$ is globally stable for the foreign firm. $\square$

**Lemma 2** *In stage 1, the following combination of subsidy schedules is one class of NEs:*

$$s \begin{cases} = \hat{s}_N < 0, & \text{if } q > q^* \text{ and } q^* < \hat{q}^*, \\ = \underline{s}_N > 0, & \text{if } q < q^* \text{ and } q^* > \hat{q}^*, \\ < \hat{s}_N, & \text{if } q < q^* \text{ and } q^* < \hat{q}^*, \\ < \underline{s}_N, & \text{if } q > q^* \text{ and } q^* > \hat{q}^*, \end{cases} \tag{7}$$

$$s^* \begin{cases} = \hat{s}^*_N > 0, & \text{if } q > q^* \text{ and } q > \hat{q}, \\ = \underline{s}^*_N, & \text{if } q < q^* \text{ and } q < \hat{q}, \\ < \hat{s}^*_N, & \text{if } q < q^* \text{ and } q > \hat{q}, \\ < \underline{s}^*_N, & \text{if } q > q^* \text{ and } q < \hat{q}, \end{cases} \tag{8}$$

*where* $\hat{s}_N \equiv \arg\max_s\{W(s,s^*) \mid q^* = B^*(q;\hat{s}^*_N), q > q^*\}$, $\hat{s}^*_N \equiv \arg\max_{s^*}\{W^*(s,s^*) \mid q = B(q^*;\hat{s}_N), q > q^*\}$, *and* $\underline{s}_N$ *and* $\underline{s}^*_N$ *jointly eliminate the equilibrium where* $q < q^*$. *There is another class of NEs where* $s$ *and* $s^*$ *(and* $q$ *and* $q^*$*) are switched in the previous case.*

*Proof.* First, I show that when the home government implements (7), the best response (BR) of the foreign government is (8). Let $\bar{s}$ be the lowest level of subsidy that makes $W^*(q,q^*)$ at any point on $B(q^*;s)$ for $q < q^*$ lower than $W^*(\hat{q}, q^{*L}(\hat{q}))$. Then, if $s = \underline{s}_N > \bar{s}$ for $q < q^*$ and $q^* > \hat{q}^*$, the foreign government does not have any incentive to retain an equilibrium in $q < q^*$. Thus, it is optimal to implement $s^* = \underline{s}^*_N$ that eliminates the equilibrium in $q < q^*$ jointly with $s = \underline{s}_N$. For $q > q^*$, set $s^*$ to maximize $W^*$ to yield that $dW^*/ds^* = (R^{*b}_{q^*}(q,q^*) - C'(q^*))(dq^*/ds^*) + R^{*b}_q(q,q^*)(dq/ds^*) = 0$. Totally differentiate Eqs. (1) and (2) and apply Cramer's rule to obtain $dq^*/ds^* = -\Pi^b_{qq}C'(q^*)/|D|$ and $dq/ds^* = \Pi^b_{qq^*}C'(q^*)/|D|$. The locally optimal $s^*$ is given by $\hat{s}^* = -R^{*b}_q\Pi^b_{qq^*}/(\Pi^b_{qq}C'(q^*)) >$

[1] The numerical result shows that $q_N^H(\hat{s}) = 0.122232/k$. Since $\hat{q} \leq 1/12k = 0.083333/k$, it is easy to show that $q_N^H(\hat{s}) > \hat{q}$.

0 because $R^{*b}_q > 0$ and $\Pi^b_{qq^*} > 0$ for $q > q^*$ and $\Pi^b_{qq} < 0$ and $C'(q^*) > 0$. To show $d^2W^*/ds^{*2}|_{s^*=\hat{s}^*} < 0$, since $dW^*/ds^* = (d\Pi^{*b}/dq^*)(dq^*/ds^*)$, it follows that $d^2W^*/ds^{*2} = (d^2\Pi^{*b}/dq^{*2})(dq^*/ds^*)^2 + (d\Pi^{*b}/dq^*)(d^2q^*/ds^{*2})$. Since $d\Pi^{*b}/dq^*\Big|_{s^*=\hat{s}^*} = 0$, it remains to show that $d^2\Pi^{*b}/dq^{*2} < 0$. It is shown that $d^2\Pi^{*b}/dq^{*2} = C''\{(R^{*b}_{q^*q^*} - C'')(1-s)^2C'' - R^b_{qq}\Pi^b_{qq} + (1-s)C''R^b_{qq}\}/(\Pi^b_{qq})^2 < 0$, because $C'' > 0$, $R^{*b}_{q^*q^*} < 0$, $R^b_{qq} < 0$, and $\Pi^b_{qq} < 0$. The elements for $\hat{q} < q < q^*$ and $\hat{q} > q > q^*$ in (8) are required to retain the equilibrium in $q > q^*$ and to eliminate the equilibrium in $q < q^*$.

Second, I show that when the foreign government implements (8), the BR of the home government is to implement (7). For $q > q^*$ from Eq. (3) the locally optimal $s$ is given by $\hat{s} = -R^b_{q^*}\Pi^*_{q^*q}/(C'(q)\Pi^{*b}_{q^*q^*})$, which is still negative for $s^* > 0$. For $q < q^*$, the equilibrium can be eliminated by $s = \underline{s}_N > \bar{s}$ when $s^* = \underline{s}^*_N$. The elements for $q < q^* < \hat{q}^*$ and $q > q^* > \hat{q}^*$ in (7) are required to retain the equilibrium in $q > q^*$ and to eliminate the equilibrium in $q < q^*$. Since the home country is better off by producing a high quality product, this policy schedule is the best response, which implies that the combination of (7) and (8) is one class of NEs.

The symmetry implies that the combination of the subsidy schedules in which $s$ and $s^*$ (and $q$ and $q^*$) are switched is another class of NEs. It can be easily checked that there are no other NEs. □

**Proposition 2** *When firms compete in prices in stage 3, there are two SPNE outcomes, which are identical except for the identity of the countries. In these SPNEs, the two governments implement the policy schedules that are specified in Lemma 2.*

*Proof.* The existence of SPNEs follows from Lemma 2. In one set of SPNEs, the home firm produces a high quality product and the foreign firm produces a low quality product. Another set of SPNEs are obtained by switching the identity of firms. □

**Lemma 3** *(1) $\tilde{B}(q^*; s) \neq q^*, \forall q^*$; (2) $\tilde{B}(q^*; s)$ is discontinuous at $q^* = \tilde{q}^*(s)$; (3) $d\tilde{B}(q^*; s)/dq^* > (<)\ 0$ for $q^* \leq (\geq)\ \tilde{q}^*(s)$; and (4) $d\tilde{B}(q^*; s)/ds > 0$.*

*Proof.* When $q^* = 5/54(1-s)k$, $\Pi^c(\tilde{q}^H(q^*; s), q^*; s) \geq \Pi^c(q^H, q^*; s)$, $\forall q^H \geq q^*$ and $\Pi^c(\tilde{q}^L(q^*; s), q^*; s) = \Pi^c(q^*, q^*; s) = 5/2916(1-s)k < \Pi^c(7/54(1-s)k, q^*; s) = 4697/1542564(1-s)k$. It follows that $\Pi^c(\tilde{q}^H(q^*; s), q^*; s) > \Pi^c(\tilde{q}^L(q^*; s), q^*; s)$ when $q^* = 5/54(1-s)k$. As Aoki (1995 pp. 18-19) has shown, on the other hand,

$\Pi^c(\tilde{q}^H(q^*;s), q^*;s) < \Pi^c(\tilde{q}^L(q^*;s), q^*;s)$ when $q^* = 1/9(1-s)k$. By the envelope theorem and the properties of the revenue function, $\Pi^c(\tilde{q}^H(q^*;s), q^*;s)$ and $\Pi^c(\tilde{q}^L(q^*;s), q^*;s)$ are both decreasing in $q^*$. By continuity of the profit function, there must exist $\tilde{q}^*(s)$ such that $\Pi^c(\tilde{q}^H(\tilde{q}^*;s), \tilde{q}^*(s);s) = \Pi^c(\tilde{q}^L(\tilde{q}^*;s), \tilde{q}^*(s);s)$.

The first and the second parts of the lemma are easily shown from the fact that $\tilde{B}(q^*;s) = \tilde{q}^H(q^*;s)$ for $q^* \leq \tilde{q}^*(s)$ and that $\tilde{B}(q^*;s) = \tilde{q}^L(q^*;s)$ for $q^* \geq \tilde{q}^*(s)$.

Now, I show the third part of the lemma. Given $s$, it follows from Lemma 7 in Aoki (1995) that $\tilde{q}^H(q^*;s) > q^*$ for $q^* < 7/54(1-s)k$ and $\tilde{q}^H(q^*;s) = q^*$ for $q^* \geq 7/54(1-s)k$, and that $d\tilde{q}^H(q^*;s)/dq^* > 0, \forall q^*$. On the other hand, from the first-order condition for $\tilde{q}^L(\tilde{q}^*;s)$: $\partial\Pi^c(\tilde{q}^L(q^*;s), q^*;s)/\partial\tilde{q}^L = 0$, it can be shown that $\tilde{q}^L(q^*;s) = q^*$ when $q^* = 5/54(1-s)k$. Then, it follows that $\tilde{q}^L(q^*;s) < q^*$ for $q^* > 5/54(1-s)k$. For $q^* \leq 5/54(1-s)k$, there is no $q^L < q^*$ which satisfies the first-order condition, and hence the solution is given by corner solutions. That is, $\tilde{q}^L(q^*;s) = q^*$ for $q^* \leq 5/54(1-s)k$. It follows from Lemma 8 in Aoki (1995) that $d\tilde{q}^L(q^*;s)/dq^* < 0$ for $q^* \geq 5/54(1-s)k$.

As I have shown, $5/54(1-s)k \leq \tilde{q}^*(s) \leq 1/9(1-s)k$. Since $\tilde{B}(q^*;s) = \tilde{q}^H(q^*;s)$ for $q^* \leq \tilde{q}^*(s)$ and $\tilde{B}(q^*;s) = \tilde{q}^L(q^*;s)$ for $q^* \geq \tilde{q}^*(s)$, then $d\tilde{B}(q^*;s)/dq^* = d\tilde{q}^H(q^*;s)/dq^* > 0$ for $q^* \leq \tilde{q}^*(s)$ and $d\tilde{B}(q^*;s)/dq^* = d\tilde{q}^L(q^*;s)/dq^* < 0$ for $q^* \geq \tilde{q}^*(s) \geq 5/54(1-s)k$.

To prove the fourth part of the lemma, totally differentiate the home firm's first-order condition:

$$\Pi^c_q(q, q^*;s) = R^c_q(q, q^*) - (1-s)C'(q) = 0 \tag{9}$$

to obtain $R^c_{qq}(q,q^*)dq + R^c_{qq^*}(q,q^*)dq^* - (1-s)C''(q)dq + C'(q)ds = 0$. Set $dq^* = 0$ and rearrange terms to yield $dq/ds = -C'(q)/\Pi^c_{qq}(q,q^*)$. Since $C'(q) > 0$ and $\Pi^c_{qq}(q,q^*) < 0$ for $q > q^*$, $d\tilde{q}^H/ds > 0$. For $\tilde{B}(q^*) = \tilde{q}^L(q^*)$, while $R^c_{qq}(\tilde{q}^L, q^*) > 0$, it must hold that $\Pi^c_{qq}(\tilde{q}^L, q^*) < 0$ at $\tilde{B}(q^*) = \tilde{q}^L(q^*)$ (otherwise, $\tilde{B}(q^*) = q^*$). Thus, $d\tilde{q}^L/ds > 0$. □

**Proposition 3** *When firms compete in quantities in stage 3, the unilaterally optimal R&D policy for the home government is to implement the following subsidy schedule:*

$$s \begin{cases} = \tilde{s} > 0, & \text{if } q > q^* \text{ and } q^* < \tilde{q}^*, \\ = \underline{s}' > 0, & \text{if } q < q^* \text{ and } q^* > \tilde{q}^*, \\ < \tilde{s}, & \text{if } q < q^* \text{ and } q^* < \tilde{q}^*, \\ < \underline{s}', & \text{if } q > q^* \text{ and } q^* > \tilde{q}^*, \end{cases}$$

*where* $\tilde{s} \equiv \arg\max_s\{W^c(s)|q^* = \tilde{B}^*(q), q > q^*\}$ *and* $\underline{s}'$ *eliminates the equilibrium in* $q < q^*$

*Proof.* First, I show that if the home firm produces the higher quality product, the locally optimal R&D subsidy by the home government is positive. The first-order condition for the home government to maximize home welfare with respect to $s$ yields $dW^c/ds = (R^c_q(q,q^*) - C'(q))(dq/ds) + R^c_{q^*}(q,q^*)(dq^*/ds) = 0$. Totally differentiate Eq. (9) and the foreign firm's first-order condition:

$$\Pi^{*c}_{q^*}(q,q^*;s^*) = R^{*c}_{q^*}(q,q^*) - (1-s^*)C'(q^*) = 0 \tag{10}$$

and apply Cramer's rule to obtain $dq/ds = -\Pi^{*c}_{q^*q^*}C'/|D|$ and $dq^*/ds = \Pi^{*c}_{q^*q}C'/|D|$, where $|D| = \Pi^c_{qq}\Pi^{*c}_{q^*q^*} - \Pi^c_{qq^*}\Pi^{*c}_{q^*q}$. Since $\Pi^c_{qq} < 0$ and $\Pi^{*c}_{q^*q^*} < 0$ for both $q > q^*$ and $q < q^*$, and since $\Pi^c_{qq^*} > 0$ and $\Pi^{*c}_{q^*q} < 0$ for $q > q^*$ and $\Pi^c_{qq^*} < 0$ and $\Pi^{*c}_{q^*q} > 0$ for $q < q^*$, then $|D| > 0$ in both cases. The locally optimal $s$ is given by

$$\tilde{s} = -R^c_{q^*}\Pi^{*c}_{q^*q}/C'(q)\Pi^{*c}_{q^*q^*} > 0, \tag{11}$$

because $R^c_{q^*} < 0$ and $\Pi^{*c}_{q^*q} < 0$ for $q > q^*$, and $C' > 0$ and $\Pi^{*c}_{q^*q^*} < 0$. To show $d^2W^c/ds^2|_{s=\tilde{s}} < 0$, since $dW^c/ds = (d\Pi^c/dq)(dq/ds)$, it follows that $d^2W^c/ds^2 = (d^2\Pi^c/dq^2)(dq/ds)^2 + (d\Pi^c/dq)(d^2q/ds^2)$. Since $d\Pi^c/dq|_{s=\tilde{s}} = 0$, it remains to show that $d^2\Pi^c/dq^2 < 0$. It is shown that $d^2\Pi^c/dq^2 = R^c_{qq} + 2R^c_{qq^*}(dq^*/dq) + R^c_{q^*q^*}(dq^*/dq)^2 - C'' < 0$, because $C'' > 0$, $R^c_{qq} < 0$, $R^c_{qq^*} > 0$, $R^c_{q^*q^*} < 0$, and $dq^*/dq = -\Pi^{*c}_{q^*q}/\Pi^{*c}_{q^*q^*} < 0$.

Second, I show that there exists a range of $s$ which eliminates the equilibrium in which $q < q^*$. Under the specific functional forms I employ in this paper, it is shown that $1/8k < \tilde{q}^{*H}(q) < 7/54k$ for $q \leq 5/54k$ and that $5/54k \leq \tilde{q} \leq 1/9k$. I know that $d\tilde{B}(q^*)/dq^* < 0$ for $q^* \geq 1/9k$. Thus, if $\tilde{B}(7/54k) > 5/54k$, there is no equilibrium in which $q < q^*$. Since $\tilde{B}(q^*)$ satisfies the first-order condition: $(q^*)^2(4q^* + q) - 2(1-s)kq(4q^* - q)^3 = 0$, it is easy to show that $\tilde{B}(7/54k) > 5/54k$ if and only if $s > 0.282$, which implies that an R&D subsidy $s > 0.282$ can eliminate the equilibrium in which $q < q^*$.

The numerical result for the optimal subsidy rate given by Eq. (11) shows that $\tilde{s} = 0.077610$ in my case, which does not satisfy the condition given above. The subsidy $\tilde{s}$ thus cannot eliminate the equilibrium in which $q < q^*$. Thus, it is optimal for the home government to set a sufficiently higher R&D subsidy for $q < q^*$ and $q^* > \tilde{q}^*$ in order to eliminate an equilibrium in which $q < q^*$. In order to make sure that the equilibrium in $q > q^*$ is retained and the equilibrium in $q < q^*$ is eliminated, $s < \tilde{s}$ for $q < q^* < \tilde{q}^*$ and $s < \underline{s}'$ for $q > q^* > \tilde{q}^*$ are required. The unique NE is then given by $(\tilde{q}^H_N(\tilde{s}), \tilde{q}^{*L}_N(\tilde{s}))$.

Use the approach that I used in the proof of Proposition 1 to show that the first-order condition for the home firm in the simultaneous choice game with the optimal R&D subsidy is exactly the same as that for the home firm as a Stackelberg leader in the sequential choice game without R&D subsidy. That is, $\tilde{q}_N^H(\tilde{s})$ is equal to $\tilde{q}_S^H$.

For $(\tilde{q}_N^H(\tilde{s}), \tilde{q}_N^{*L}(\tilde{s}))$ to be globally stable, it is required that the home firm, taking $\tilde{q}_N^{*L}(\tilde{s})$ and $\tilde{s}$ as given, would prefer to choose $\tilde{q}_N^H(\tilde{s}) = \tilde{q}^H(\tilde{q}_N^{*L}(\tilde{s}); \tilde{s})$ rather than to choose $\tilde{q}^L(\tilde{q}_N^{*L}(\tilde{s}); s)$, where $s$ is high enough to eliminate the equilibrium in which $q < q^*$. It is also required that the foreign firm, taking $\tilde{q}_N^H(\tilde{s})$ as given, would prefer to choose $\tilde{q}_N^{*L}(\tilde{s}) = \tilde{q}^{*L}(\tilde{q}_N^H(\tilde{s}))$ rather than to choose $\tilde{q}^{*H}(\tilde{q}_N^H(\tilde{s}))$.

When $s = 1$, $\tilde{q}^L(q^*; 1) = q^*$ and $\Pi^c(\tilde{q}^L(q^*; 1), q^*; 1) = q^*/9$. The numerical result shows that $\tilde{q}_N^H(\tilde{s}) = 0.136350/k$ and $\tilde{q}_N^{*L}(\tilde{s}) = 0.043198/k$. Using the formula of $\tilde{s}$ given by Eq. (11), the value of $\tilde{s}$ can be numerically calculated, which is 0.077610. Then, $\Pi^c(\tilde{q}^H(\tilde{q}_N^{*L}(\tilde{s}); \tilde{s}), \tilde{q}_N^{*L}(\tilde{s}); \tilde{s}) = 0.011327/k$. On the other hand, $\Pi^c(\tilde{q}^L(\tilde{q}_N^{*L}(\tilde{s}); 1), \tilde{q}_N^{*L}(\tilde{s}); 1) = \tilde{q}_N^{*L}(\tilde{s})/9 = 0.004800/k$. It follows that $\Pi^c(\tilde{q}^H(\tilde{q}_N^{*L}(\tilde{s}); \tilde{s}), \tilde{q}_N^{*L}(\tilde{s}); \tilde{s}) > \Pi^c(\tilde{q}^L(\tilde{q}_N^{*L}(\tilde{s}); 1), \tilde{q}_N^{*L}(\tilde{s}); 1)$. Using the envelope theorem, $\partial \Pi^c(\tilde{q}^L(q^*; s), q^*; s)/\partial s = C(\tilde{q}^L(q^*; s)) > 0$, which implies that $\Pi^c(\tilde{q}^L(q^*; s), q^*; s)$ is increasing in $s$. Thus, for any $s < 1$ which is high enough to eliminate the equilibrium in which $q < q^*$, $\Pi^c(\tilde{q}^H(\tilde{q}_N^{*L}(\tilde{s}); \tilde{s}), \tilde{q}_N^{*L}(\tilde{s}); \tilde{s}) > \Pi^c(\tilde{q}^L(\tilde{q}_N^{*L}(\tilde{s}); s), \tilde{q}_N^{*L}(\tilde{s}); s)$ holds, which implies that $(\tilde{q}_N^H(\tilde{s}), \tilde{q}_N^{*L}(\tilde{s}))$ is globally stable for the home firm. For the foreign firm, since $\tilde{q}_N^H(\tilde{s}) > \tilde{q}$,[2] by the definition of $\tilde{B}^*(q)$ it follows that $\tilde{B}^*(\tilde{q}_N^H(\tilde{s})) = \tilde{q}^{*L}(\tilde{q}_N^H(\tilde{s}))$. Thus, $(\tilde{q}_N^H(\tilde{s}), \tilde{q}_N^{*L}(\tilde{s}))$ is globally stable for the foreign firm. □

**Lemma 4** *In stage 1, the following combination of subsidy schedules is one class of NEs:*

$$s \begin{cases} = \tilde{s}_N > 0, & \textit{if } q > q^* \textit{ and } q^* < \tilde{q}^*, \\ = \underline{s}'_N > 0, & \textit{if } q < q^* \textit{ and } q^* > \tilde{q}^*, \\ < \tilde{s}_N, & \textit{if } q < q^* \textit{ and } q^* < \tilde{q}^*, \\ < \underline{s}'_N, & \textit{if } q > q^* \textit{ and } q^* > \tilde{q}^*, \end{cases} \tag{12}$$

$$s^* \begin{cases} = \tilde{s}^*_N < 0, & \textit{if } q > q^* \textit{ and } q > \tilde{q}, \\ = \underline{s}^{*\prime}_N, & \textit{if } q < q^* \textit{ and } q < \tilde{q}, \\ < \tilde{s}^*_N, & \textit{if } q < q^* \textit{ and } q > \tilde{q}, \\ < \underline{s}^{*\prime}_N, & \textit{if } q > q^* \textit{ and } q < \tilde{q}, \end{cases} \tag{13}$$

[2]The numerical result shows that $\tilde{q}_N^H(\tilde{s}) = 0.136350/k$. Since $\tilde{q} \leq 1/9k = 0.111111/k$, it is easy to show that $\tilde{q}_N^H(\tilde{s}) > \tilde{q}$.

*where* $\tilde{s}_N \equiv \arg\max_s\{W^c(s,s^*)|q^* = \tilde{B}^*(q;\tilde{s}_N^*), q > q^*\}$*,* $\tilde{s}_N^* \equiv \arg\max_{s^*}\{W^{*c}(s,s^*)|q = \tilde{B}(q^*;\tilde{s}_N), q > q^*\}$*, and* $\underline{s}_N'$ *and* $\underline{s}_N^{*\prime}$ *jointly eliminate the equilibrium where* $q < q^*$*. There is another class of NEs where* $s$ *and* $s^*$ *(and* $q$ *and* $q^*$*) are switched in the previous case.*

*Proof.* First, I show that when the home government implements (12), the BR of the foreign government is (13). Let $\bar{s}'$ be the lowest level of subsidy that makes $W^{*c}(q,q^*)$ at any point on $\tilde{B}(q^*;s)$ for $q < q^*$ lower than $W^{*c}(\tilde{q},\tilde{q}^{*L}(\tilde{q}))$. Then, if $s = \underline{s}_N' > \bar{s}'$ for $q < q^*$ and $q^* > \tilde{q}^*$, the foreign government does not have any incentive to retain an equilibrium in $q < q^*$. Thus, it is optimal to implement $s^* = \underline{s}_N^{*\prime}$ that eliminates the equilibrium in $q < q^*$ jointly with $s = \underline{s}_N'$. For $q > q^*$, set $s^*$ to maximize $W^{*c}$ to yield that $dW^{*c}/ds^* = (R_{q^*}^{*c}(q,q^*) - C'(q^*))(dq^*/ds^*) + R_q^{*c}(q,q^*)(dq/ds^*) = 0$. Totally differentiate Eqs. (9) and (10) and apply Cramer's rule to obtain $dq^*/ds^* = -\Pi_{qq}^c C'(q^*)/|D|$ and $dq/ds^* = \Pi_{qq^*}^c C'(q^*)/|D|$. The locally optimal $s^*$ is given by $\tilde{s}^* = -R_q^{*c}\Pi_{qq^*}^c/(\Pi_{qq}^c C'(q^*)) < 0$, because $R_q^{*c} < 0$ and $\Pi_{qq^*}^c > 0$ for $q > q^*$ and $\Pi_{qq}^c < 0$ and $C'(q^*) > 0$. To show $d^2W^{*c}/ds^{*2}|_{s^*=\tilde{s}^*} < 0$, since $dW^{*c}/ds^* = (d\Pi^{*c}/dq^*)(dq^*/ds^*)$, it follows that $d^2W^{*c}/ds^{*2} = (d^2\Pi^{*c}/dq^{*2})(dq^*/ds^*)^2 + (d\Pi^{*c}/dq^*)(d^2q^*/ds^{*2})$. Since $d\Pi^{*c}/dq^*|_{s^*=\tilde{s}^*} = 0$, it remains to show that $d^2\Pi^{*c}/dq^{*2} < 0$. It is shown that $d^2\Pi^{*c}/dq^{*2} = C''\{(R_{q^*q^*}^{*c} - C'')(1-s)^2C'' - R_{qq}^c\Pi_{qq}^c + (1-s)C''R_{qq}^c\}/(\Pi_{qq}^c)^2$. Since $C'' > 0$, $R_{q^*q^*}^{*c} > 0$, $R_{qq}^c < 0$, and $\Pi_{qq}^c < 0$, I assume $(R_{q^*q^*}^{*c} - C'')\Big|_{s^*=\tilde{s}^*} < 0$ to ensure $d^2\Pi^{*c}/dq^{*2} < 0$. The elements for $\tilde{q} < q < q^*$ and $\tilde{q} > q > q^*$ in (13) are required to retain the equilibrium in $q > q^*$ and to eliminate the equilibrium in $q < q^*$.

Second, I show that when the foreign government implements (13), the BR of the home government is to implement (12). For $q > q^*$ from Eq. (11) the locally optimal $s$ is given by $\tilde{s} = -R_{q^*}^c\Pi_{q^*q}^{*c}/C'(q)\Pi_{q^*q^*}^{*c}$, which is still positive for $s^* < 0$. For $q < q^*$, the equilibrium can be eliminated by $s = \underline{s}_N' > \bar{s}'$ when $s^* = \underline{s}_N^{*\prime}$. The elements for $q < q^* < \tilde{q}^*$ and $q > q^* > \tilde{q}^*$ in (12) are required to retain the equilibrium in $q > q^*$ and to eliminate the equilibrium in $q < q^*$. Since the home country is better off by producing a high quality product, this policy schedule is the best response, which implies that the combination of (12) and (13) is one class of NEs.

The symmetry implies that the combination of the subsidy schedules in which $s$ and $s^*$ (and $q$ and $q^*$) are switched is another class of NEs. It can be easily checked that there are no other NEs. □

**Proposition 4** *When firms compete in quantities in stage 3, there are two SPNE outcomes, which are identical except for the identity of the countries. In these SPNEs, the two governments implement the policy schedules that are specified in Lemma 4.*

*Proof.* The existence of SPNEs follows from Lemma 4. In one set of SPNEs, the home firm produces a high quality product and the foreign firm produces a low quality product. Another set of SPNEs are obtained by switching the identity of firms. □

## References


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[2] Aoki, R. and T.J. Prusa (1997) 'Sequential versus simultaneous choice with endogenous quality,' *International Journal of Industrial Organization* 15, 103-21.