Simplified Alonso-Mills-Muth Model with a Monopoly Vendor∗

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Abstract

One important but unrealistic assumption in the Alonso-Mills-Muth (AMM) model is that the composite good is ubiquitous and thus there is zero shopping cost for residents in an open city. In our simplified AMM model, we assume that the composite good is only sold by a monopoly vendor inside the city and hence a positive shopping cost is inevitable for residents. This paper shows that the vendor will locate at the city boundary in both “one-group” and “two-group” models. In contrast to the symmetric land rent pattern in the AMM model, our model offers an asymmetric land rent pattern in equilibrium. Moreover, it is shown that the central business district (CBD) is the social optimal location for the vendor from the standpoint of minimizing the average transportation cost.

Keywords: Alonso-Mills-Muth Model, location, bid rents, urban configuration

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Abstract

One important but unrealistic assumption in the Alonso-Mills-Muth (AMM) model is that the composite good is ubiquitous and thus there is zero shopping cost for residents in an open city. In our simplified AMM model, we assume that the composite good is only sold by a monopoly vendor inside the city and hence a positive shopping cost is inevitable for residents. This paper shows that the vendor will locate at the city boundary in both “one-group” and “two-group” models. In contrast to the symmetric land rent pattern in the AMM model, our model offers an asymmetric land rent pattern in equilibrium. Moreover, it is shown that the central business district (CBD) is the social optimal location for the vendor from the standpoint of minimizing the average transportation cost.

1 Introduction

The urban residential pattern was originally developed by Alonso (1964), Mills (1967), and Muth (1969). Those models describe the equilibrium residential pattern (land rent, housing height, housing size) in a monocentric city. In their models, people commute to the central business district (CBD) with a transportation cost depending on the commuting distance. Moreover, each resident maximizes his/her utility by allocating his/her income to the consumption between a composite good and land (housing). Their models successfully explain some urban landscapes, such as buildings being taller near the CBD, while housing sizes are larger in the suburbs (Brueckner (1986)).

An important assumption in the Alonso-Mills-Muth (AMM) model is that the composite good is available everywhere (ubiquitous) with an identical price, while residents commute to the CBD in order to earn income and thus a transportation cost is inevitable. However, many people in reality go shopping (for a composite good) in a shopping center once a week. This behavior entails an important transportation cost which is ignored in the AMM model. The absence of store location and shopping cost in the AMM model may result in a misunderstanding of the real urban structure.\(^1\)

The purpose of this paper is to introduce a monopoly vendor into a simplified AMM model

\(^1\)For example, the AMM model cannot explain why large shopping centers often locate at the city fringe.
(fixed land consumption for each resident)\(^2\) in order to analyze the vendor location as well as the (bid) land rent pattern, and to finally compare the land rent pattern with and without a vendor.

Shopping cost is intuitively no longer zero in this paper, and the bid rent function is expected to be lower than the AMM model, while the equilibrium city size is also expected to be smaller. Moreover, poor people traditionally are assumed to live in the inner city (the rich live in the suburbs) in the two-group AMM model. If the monopoly chooses to locate in the suburbs, then the poor are much worse off when the vendor is introduced, and at the extreme, even cannot survive in the city. This viewpoint has never been raised in the literature.

The rest of the paper is organized as follows. Section 2 introduces the models (one-group model and two-group model) and Section 3 discusses the economic sense of the current model. The last section provides some concluding remarks.

2 The Model

Suppose there is a small open linear city on a plain, where all land in this city is owned by an absentee landowner. Each resident in this city has a job in the CBD with identical income \(y\) and lives inside the city with one unit of land. A resident living \(x\) distance away from the CBD pays a total commuting cost \(tx\), where \(t\) is the unit commuting cost. Each resident consumes one unit of a composite good (with price \(p\)) produced by a monopoly vendor located at a point \((z)\) which is inside the city. Every resident goes shopping to the vendor location \((z)\) and pays for a unit shopping cost \(k\) \((k < t)\)\(^3\) times the distance between his/her home and \(z\). When \(k = 0\), the current model is reduced to the original AMM model.\(^4\)

\(^2\)Basically, the original the AMM model allows residents to use the composite good to substitute for land consumption, and vice versa. Thus, people living near the central city consume less land and raise their bid rents and consequently the bid rent curve is convex (non-linear). A non-linear bid rent system may increase the complexity of mathematics and is also hard to achieve analytical solutions. In order to focus the economic sense of the model, fixed land consumption (as with linear bid rent curves) is an appropriate assumption.

\(^3\)Readers may think that people go to their office 5 times a week, while they go shopping only once a week. Thus, if the traffic situation is fixed, then \(t\) is 5 times more than \(k\).

\(^4\)The zero shopping cost in the AMM model can be interpreted by two viewpoints: First, there are ubiquitous competitive vendors (price takers), and thus shopping cost is zero. Vendors play no roles in the model. Second, there is only one monopoly vendor somewhere in the city and the price of the composite good is determined by a simple cost-marked-up pricing (thus the vendor plays a passive role in the model and every resident faces the same price, say 1 dollar per unit of the composite good) with zero unit shopping cost \((k = 0)\). The current model adopts the second viewpoint and allows the monopoly to choose a mill pricing based on profit maximization.
Since residents are free to be mobile between both the inter-city and intra-city, everyone’s utility is thus identical in equilibrium. The monopoly vendor also rents out one unit of the land at the store location. The land is used by a resident (or the vendor) with the highest bid rent. First, the equilibrium analysis and the optimal location analysis for the one-group model (all residents having identical income) as well as land size will be discussed in Section 2.1. Second, the basic one-group model will be extended to the two-group model (Section 2.2) where one group (the rich) has a higher income than the other group (the poor).

2.1 The city with one-group renters

Suppose all residents are identical except for their locations (i.e., all renters are in one group). The agriculture land rent is assumed to be zero. In the AMM model (unit shopping cost $k$ is zero, or stores selling the composite good are ubiquitous), the city boundary ($\hat{x}_R$ and $\hat{x}_L$, where “R” (“L”) represents “right (left) side of the city,” in Figure 1) should be at the point where the bid rent is equal to zero, while the bid rent at the CBD (denoted by 0) should be $y - p$, because people do not need to pay any transportation cost at CBD. Once the shopping cost and the location of the vendor are introduced, the bid rent curve will be different for different vendor locations.

Suppose the vendor locates at $z_1$ as shown in Figure 1 (thin (bold) lines represent the land rent of $k = 0$ ($k > 0$)). The bid rent at CBD should then be $y - p - kz_1$, while the city right boundary should be at $\tilde{x}_R < \hat{x}_R$. This is because people should pay the shopping cost before the land rent is paid. On the left part of the city, the city’s left boundary should be $\tilde{x}_L$ when $k = 0$ and $\tilde{x}_L^*$ when $k > 0$. For a certain point $x_2 < 0$, the bid rents should be $y - p + tx_2$ (when $k = 0$) or $y - p - kz_1 + (t + k)x_2$ (when $k > 0$). Therefore, the city size ($\tilde{x}_R - \tilde{x}_L$) with $z = z_1$ is smaller than the AMM model ($\hat{x}_R - \hat{x}_L^*$). Readers may note that the land rent pattern is asymmetric in the current model ($k > 0$) if the vendor does not locate at CBD ($z > 0$), while it is symmetric in the original AMM model ($k = 0$).

2.1.1 Equilibrium analysis

Suppose each resident consumes one unit of land directly as well as one unit of the composite good. Since this city is open, the overall utility is exogenous, which is assumed to be $u = \ldots$ Thus, more realistically, the monopoly vendor plays an active role in the current model. Precisely speaking, the AMM model is based on the first viewpoints. For the sake of comparison and without loss of generality, this paper assumes that $k = 0$ represents the AMM model.
Each resident earns the same income $y$ from working a job in the CBD. Therefore, every resident must commute to the CBD and pay a transportation cost (unit commuting cost $t$ times the distance to the CBD).

Suppose people must go shopping (buy one unit of the composite good) from a monopoly vendor who is located at $z \in [0, x_R]$, where $x_R$ is the city’s right boundary and the left boundary is $x_L < 0$. Suppose each resident consumes “one unit of land”, and then $x_R - x_L$ is also the total number of residents ($N$) in the city. The total shopping cost is $k \cdot |z - x|$ for people living at $x$, where $k$ is the unit shopping cost. The vendor is free to choose location $z$ and free to set price $p$ so as to maximize its profits.

Suppose the agriculture land rent is zero. Thus, land rent at the city boundary $(x_R, x_L)$ should be zero. That is,

$$y - p - kx_R - k(x_R - z) = 0, \quad y - p - k(x_R - z) = 0,$$

(1)

Since the land rent pattern is symmetric to the CBD in the AMM model, for the sake of simplicity, we assume that the vendor only locates at the right part of the city. To assume the vendor will locate at the left side of the city is similar to this setting.

It is important to note that each resident as well as the monopoly consumes “one unit of land” which is only “a point” in the city. As the reader can see in Figure 1, the land rents are linear functions along the linear city. This is the basic assumption of the AMM model. If the resident location is measured by real distance, then the land rents should be piecewise (discrete) and the number of lots (odd or even) will affect the land rent pattern. Moreover, the commuting and shopping distance must be redefined; say, from the center point of one lot to the center point of another lots is the distance between these two lots. Finally, the CBD should also be redefined; say, when the lot number is even, then there are two center lots, but where is the CBD? Even though a positive measure of lots is much more realistic than the point measure of lots, such a modeling is far apart from the original AMM model. Thus, it can be excluded in the current paper.
In other words, the city must have \( N(=\overline{x}^R - \overline{x}^L) \) people, such that
\[
\overline{x}^R = \frac{y - p + kz}{t + k}, \quad \overline{x}^L = \frac{- (y - p - kz)}{t + k}.
\] (2)

The profit function of the vendor is as follows:
\[
\pi = p(\overline{x}^R - \overline{x}^L) - (y - p - tz),
\] (3)

where \( p \) is the price of the composite good (its production cost is assumed to be zero) and \( y - p - tz \) is the bid rent at \( z \). The vendor can choose \( p \) and \( z \) to maximize its profit, subject to every resident inside the city being served.\(^7\) Thus, the objective function of the vendor is
\[
\max \pi = p(\overline{x}^R - \overline{x}^L) - (y - p - tz)
\]  
\[ 0 \leq z \leq \overline{x}^R. \] (4)

A Lagrangian function can be set up as follows.
\[
L = \pi + \mu(\overline{x}^R - z), \quad \mu \geq 0,
\] (5)

where the constraint is that the vendor cannot locate outside the city. The Kuhn-Tucker conditions require that the following conditions must hold.
\[
\frac{\partial L}{\partial z} \cdot z = 0,
\] (6)
\[
\frac{\partial L}{\partial p} \cdot p = 0,
\] (7)
\[
\frac{\partial L}{\partial \mu} \cdot \mu = 0.
\] (8)

Solving (6)-(8) simultaneously yields two feasible answers:
\[
\{ z = 0, \; p = \frac{1}{2}(y + t + k) \}
\] (9)

and
\[
\{ z = \frac{y}{2t}, \; p = \frac{y}{2} \}.
\] (10)

Plugging (9) and (10) into \( \pi \), respectively, yields
\[
\pi = \frac{1}{8} \frac{(2y - t - k)^2}{t + k}
\] (11)

\(^7\)It is assumed that each resident must consume one and only one composite good and thus the vendor is not allowed to partially serve the residents or conduct a price discrimination.
and
\[ \pi = \frac{y^2}{2(t + k)}. \] (12)

Equation (12) is obviously greater than (11). In other words, \( z^* = y/2t \) and \( p^* = y/2 \) are the unique equilibrium solutions. After some calculations, the city boundaries are then
\[ \overline{x}_R^* = \frac{y}{2t} = z, \quad \text{and} \quad \overline{x}_L^* = \frac{-y(t - k)}{2(t + k)}. \]

That is, the vendor locates at the city’s right boundary and pays a zero land rent. Note that the city is asymmetric (symmetric) in equilibrium, because \( |\overline{x}_R^*| > (=) |\overline{x}_L^*| \) if \( k > 0 \) (\( k = 0 \), in the AMM model). The total number of residents and the profits are respectively:
\[ N^* = \overline{x}_R^* - \overline{x}_L^* = \frac{y}{t + k}. \] (13)
\[ \pi^* = \frac{y^2}{2(t + k)}. \] (14)

Note that \( \overline{x}_R^* \) is independent of \( k \), which means the city’s right boundary is identical to the AMM model, while \( N \) and \( \pi \) are smaller when \( k > 0 \) is considered. This is because \( \overline{x}_L^* \) is a function of \( k \). Comparing to the AMM model, only the absentee landowner receives fewer land rents, because the bid rent curve decreases as \( y - p - tx - k(\overline{x} - x) \) (instead of \( y - p - tx \) in the AMM model).

The social welfare can be measured by
\[ SW = CS + \pi, \] (15)
where \( CS \) is the consumer surplus for all residents. Every resident earns identical income \( y \) and enjoys identical utility level \( (\bar{u}) \) which arises directly from consuming one unit of land as well as one unit of the composite good \( (u(c = 1, l = 1) = \bar{u}) \). In other words, any location of the vendor and any pattern of land rent do not change a consumer’s utility, but there is just a re-distribution of the “cake” \( ((\overline{x}_R^* - \overline{x}_L^*) y) \) among total land rent and the vendor’s profit. Maximizing the social welfare is identical to minimizing total transportation cost (TTC). Since the total population is different under different \( z \)’s, this model chooses average transportation cost \( (ATC = TTC/(\overline{x}_R^* - \overline{x}_L^*)) \) to evaluate the social welfare index. The objective of the government is to minimize \( ATC \), because all transportation costs are sunk costs.

\[ ATC = \left[ t \left( \int_{0}^{\overline{x}_R^*} m \, dm + \int_{0}^{-\overline{x}_L^*} m \, dm \right) \right. \]
\[ + k \left( \int_{0}^{\overline{x}_R^*} (z - m) \, dm + \int_{\overline{x}_R^*}^{\overline{x}_L^*} (m - z) \, dm + \int_{\overline{x}_L^*}^{z} (z + m) \, dm \right) \left/ (\overline{x}_R^* - \overline{x}_L^*) \right], \] (16)
where \( m \) denotes the location of a consumer and \( z \) is the choice variable for the government.\(^8\)

Plugging the equilibrium \( z \) and \( p \) in (10) into \( ATC \) yields the equilibrium \( ATC^* \):

\[
ATC^* = \frac{y(t + k)}{4t}.
\]  
(17)

The equilibrium result is shown as Figure 2.

2.1.2 Location regulation

Suppose the government assigns a business point \( z_g \), where the subscript “g” means regulations, for the vendor in the first stage, while the vendor decides price \( p_g \) at the second stage. A backward induction will then be used to solve this problem.

In the second stage, the firm decides the price of the composite good depending on the given \( z_g \), and then (1) - (3) can be applied directly. Solving \( \partial \pi_g / \partial p_g = 0 \) for \( p_g \) yields

\[
p_g^* = \frac{1}{4}(2y + t + k).
\]  
(18)

Note that \( p_g^* \) is independent to \( z_g \). In the first stage, the government sets up a \( z \) to minimize \( ATC \) (i.e., maximize \( SW \)).

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\(^8\)Here is a mathematical technicality problem: The actual total transportation cost must subtract \( tz \) from above TTC, because the vendor does not commute to CBD everyday. However, the consumers are in a continuum which cannot be measured individually. Thus, if one assumes \( ATC = (TTC - tz)/(x^R - x^L - 1) \), then it is totally wrong. Similarly, \( ATC = (TTC - tz)/(x^R - x^L) \) is also wrong. This is because the consistence problem arises when “continuous consumers” with “a discrete vender” (“one vendor” is obviously discrete) are embodied in one model. To avoid this technicality, with loss of generality, we assume the vendor has a daily trip to the CBD. Thus, everyone in the city should pay a commuting cost and ATC can be expressed as 17.
We set up a Lagrangian as
\[ L = ATC + \lambda(z_g - \pi^R), \quad \lambda \geq 0. \]
Solving \( \partial L / \partial z_g = 0 \) yields a unique solution of \( z_g \) such that \( z_g^* = 0 \) and consequently the optimal ATC is
\[ ATC_g^* = \frac{1}{8}(2y - t - k). \]
Comparing \( ATC^* \) and \( ATC_g \) yields
\[ ATC_g^* - ATC^* = \frac{-2yk + tk + t^2}{8t} < 0. \]
(19)
Since \( ATC_g^* < ATC^* \), it means that there exists a market failure and that location regulation is justified. These results can be summarized as the following proposition.

**Proposition 1.** The monopoly vendor in the simplified AMM model locates at one of the city’s boundaries and pays zero land rent. However, the social optimal location which minimizes the average transportation cost is at CBD.

The equilibrium of total residents and the vendor’s profit are then obtained, respectively, as follows:
\[ N_g^* = \frac{2y - t - k}{2(t + k)} < \frac{y}{t + k} = N^*, \] (20)
\[ \pi_g^* = \frac{(2y - t - k)^2}{8(t + k)} < \frac{y^2}{2(t + k)} = \pi^*. \] (21)
These results can be summarized as the following proposition.

**Proposition 2.** The size of the total population under optimal location regulation (\( z_g^* = 0 \)) is smaller than the population size under market equilibrium. The monopoly vendor earns less profit under optimal location regulation than without location regulation.

### 2.2 The city with two-group renters

The AMM model also focuses on the city’s bid rent with two groups of renters: one is poor people (group 1) and the other group is rich people (group 2). The difference between the rich and the poor is that the former group has higher income as well as a larger size of land consumption than the latter. Similarly, the two-group AMM model ignores the shopping cost for residents. Suppose each resident of group 1 consumes \( \alpha < 1 \) unit of land, while each one of
Figure 3: The bid rent pattern in the two-group model ($\hat{x} \leq z \leq \pi^R$)

group 2 consumes 1 unit of land. The thin lines in Figure 4 represent the bid rents of groups 1 and 2 ($\hat{r}^R_1, \hat{r}^R_2, \hat{r}^L_1, \hat{r}^L_2$, and the subscripts represent the group number) without shopping cost ($k = 0$). Suppose the monopoly vendor locates at $z$, where $z$ is in the poor area ($z \in [0, \hat{x}]$). The general bid rent curves are then the bold lines as shown in Figure 4. If $z$ locates in the rich area, then the bid rent curves are as the bold lines in Figure 3.

2.2.1 Market equilibrium analysis

According to the location of the vendor, there are two cases: The first one is the vendor locating in the rich area ($\hat{x} \leq z \leq \pi^R_2$), while the second is in the poor area ($0 \leq z \leq \hat{x}$). Suppose now the vendor locates in the rich area $\hat{x} \leq z \leq \pi^R_2$. The right border of the city ($\pi_2^R$) then satisfies

$$y_2 - p - t\pi_2^R - k(\pi_2^R - z) = 0.$$ \hspace{1cm} (22)

Thus,

$$\pi_2^R = \frac{y_2 - p + kz}{t - k} > 0,$$ \hspace{1cm} (23)

and the left border of the city is

$$\pi_2^L = \frac{-(y_2 - p - kz)}{t + k} < 0.$$ \hspace{1cm} (24)
Figure 4: The land rent pattern in the two-group model \((0 \leq z \leq \hat{x})\)

The profit of the vendor is then

\[
\pi = p(\bar{x}_2^R - \bar{x}_2^L) - (y_2 - p - tz).
\]  \hfill (25)

We now set up a Lagrangian as

\[
L = \pi + \mu(\bar{x}_2^R - z).
\]  \hfill (26)

Solving \(\frac{\partial L}{\partial z} \cdot z = 0, \frac{\partial L}{\partial \mu} \cdot \mu = 0, \) and \(\frac{\partial L}{\partial p} \cdot p = 0\) yield two feasible solutions. Choosing the highest profit resulting from these two feasible solutions yields the equilibrium price and equilibrium location as follows.

\[
p^* = \frac{y_2}{2},
\]  \hfill (27)

\[
z^* = \frac{y_2}{2t} = \bar{x}_2^R.
\]  \hfill (28)

Note that the equilibrium location of the vendor is also at the city’s right border. The term \(ATC\) in equilibrium is then

\[
ATC^* = \frac{y_2(t + k)}{4t},
\]  \hfill (29)

and equilibrium profit is

\[
\pi^* = \frac{y_2^2}{2(t + k)}.
\]  \hfill (30)
and total population is
\[ N^* = \frac{y_2}{t + k}. \]  
(31)

As \( k > 0 \), then \( ATC^* \) is higher than that in the \( AMM \) model (letting \( k = 0 \) in (29)), while \( \pi^* \) and \( N^* \) are smaller than in the \( AMM \) model.

If the vendor locates in the poor area \((0 \leq z \leq \hat{x})\), then one can follow the process as above and set up a Lagrangian:
\[ L = p(x^R - x^L) - \frac{y_1 - p - tz}{\alpha} + \mu(\hat{x} - z), \quad \mu \geq 0. \]  
(32)

Note that the size of the vendor is always one unit of land, no matter if it is in the “poor area” or the “rich area.” Solving for the Kuhn-Tucker conditions \( \frac{\partial L}{\partial z} \cdot z = 0 \), \( \frac{\partial L}{\partial p} \cdot p = 0 \), and \( \frac{\partial L}{\partial \mu} \cdot \mu = 0 \) yields two feasible solutions:
\[ z = \frac{2y_1 - \alpha y_2 + y_2}{2t(1 - \alpha)}, \quad p = \frac{y_2}{2} \quad \text{and} \quad z = 0, \quad p = \frac{2y_2 \alpha + t + k}{4\alpha}. \]  
(33)

Plugging (33) into the profit function and comparing with the equilibrium profit as \( z \) in the rich area prompts:
\[ \pi(z = \frac{2y_1 - \alpha y_2 + y_2}{2t(1 - \alpha)}, p = \frac{y_2}{2}) - \pi(z = x^R, p = \frac{y_2}{2}) = \frac{y_1 - y_2}{1 - \alpha} < 0 \]  
(34)

and
\[ \pi = (z = 0, p = \frac{2y_2 \alpha + t + k}{4\alpha}) - \pi(z = x^R, p = \frac{y_2}{2}) = \frac{t + k - 8\alpha y_1 + 4\alpha y_2}{8\alpha^2}. \]  
(35)

Only the second solution \((z = 0)\) above is plausible. However, if \( z = 0 \) is the potential equilibrium, then it must satisfy some conditions. First, the income of the poor \((y_1)\) must be greater than the price of the composite good \((p)\). Solving \( p - y_1 < 0 \) yields
\[ y_1 > \frac{2\alpha y_2 + t + k}{4\alpha} \stackrel{\text{def}}{=} \bar{y}_1, \]  
(36)

and secondly, (35) must be positive such that
\[ \frac{t + k - 8\alpha y_1 + 4\alpha y_2}{8\alpha^2} \geq 0. \]  
(37)

Rearranging (37) yields
\[ y_1 \leq \frac{4\alpha y_2 + t + k}{8\alpha} \stackrel{\text{def}}{=} \underline{y}_1. \]  
(38)

Hence,
\[ y_1 - \underline{y}_1 = \frac{24\alpha^2 y_1 + (1 - \alpha)(t + k)}{4\alpha(1 + 3\alpha)} > 0. \]  
(39)
Since the upperbound income is lower than the lowerbound income ($\bar{y}_1 < y_1$), then (36) and (38) cannot be satisfied simultaneously. Therefore, $z = 0$ cannot be a solution, and thus the only feasible equilibrium solution should be at $z^* = x^*_{R^2}$ (see also a numerical example in Figure 5).

**Proposition 3.** In the two-group model the equilibrium location of the vendor is at one of the borders of the city.

After the equilibrium location of the vendor is calculated ($z^* = x^*_{R^2}$), another interesting points arises such that shopping cost may force the poor out of the city. In order to live inside the city, the bid rents of the poor in the CBD must be higher than the bid rents of the rich at CBD. In the AMM model ($k = 0$),

$$\frac{y_1 - p}{\alpha} > (y_2 - p),$$

which means that

$$y_1 > \alpha y_2 + (1 - \alpha)p \overset{\text{def}}{=} y_0 = \frac{y_2(1 + \alpha)}{2}.$$

When shopping cost is considered ($k > 0$), the existence of the poor must be

$$\frac{y_1 - p - kz}{\alpha} > (y_2 - p - kz),$$

which means that

$$y_1 > \alpha y_2 + (1 - \alpha)p + (1 - \alpha)kz = \frac{y_2(t + k + \alpha t - \alpha k)}{2t}.$$

As with the result in Proposition 2, $z^* = x^*_{R^2} > 0$, therefore when $y_0^0 < y_1 < y_0^k$, there are some poor people living in the city in the AMM model, while no poor people live in the city in the current two-group model (reduced to the one-group model). This striking result can be summarized as the following proposition (see also a numerical example in Figure 6).

**Proposition 4.** When $y_0^0 < y_1 < y_0^k$, there are some poor residents living in the inner city in the AMM model ($k = 0$), while the poor are forced out of this city when ($k > 0$).

Proposition 4 is straightforward, but to the authors’ best knowledge, it has never been mentioned in the literature. The implication behind Proposition 4 is that several decades ago, the poor bid up higher rents in the inner city by way of reducing land consumption and they went shopping to many neighboring small stores. In recent decades, after more and more large shopping malls were installed in suburban areas with easy parking, many local small stores
closed and this situation has forced the poor to pay a higher shopping cost to survive in the city. If their house sizes cannot be reduced further (as with a fixed $\alpha$ in the current paper), high shopping costs will force them out of the city.\(^9\)

### 2.2.2 Optimal location regulation

Suppose the government assigns a location ($z_g$) of a shopping center to minimize ATC in the first stage, and then the monopoly vendor sets up a price to maximize its profit in the second stage. To solve the equilibrium, a backward induction is employed. In the second stage the monopolist decides its price given an exogenous variable $z_g$. There are two possible $z_g$. That is, $0 \leq z_g \leq \hat{x}$ (in the poor area) and $\hat{x} \leq z_g \leq \bar{x}_2$ (in the rich area).

First, suppose now $0 \leq z_g \leq \hat{x}$ (poor area), and then the profit of the vendor is

$$\pi = p(\bar{x}_2 - \bar{x}_2^L) - \frac{y_1 - p - tz}{\alpha}. \quad (44)$$

We set up Lagrangian as

$$L = \pi + \lambda(y_2 - p - kz + t\bar{x}_L^L + k\bar{x}_L^L), \quad (45)$$

where the constraint after $\lambda$ means that the remotest resident ($\bar{x}_L$) is served by the vendor. Solving for the optimal price yields

$$p = \frac{2\alpha y_2 + t + k}{4\alpha}. \quad (46)$$

The government minimizes ATC by setting up a Lagrangian as

$$L = ATC + \lambda(z - \hat{x}), \quad \lambda \geq 0. \quad (47)$$

Solving $\frac{\partial L}{\partial z} : z = 0$ and $\frac{\partial L}{\partial \lambda} : \lambda = 0$ yields a unique solution as

$$z_g^* = 0, \quad (48)$$

and ATC is then

$$ATC_{z_g=0} = \frac{2\alpha y_2 - t - k}{8\alpha}. \quad (49)$$

Second, if $\hat{x} \leq z_g \leq \bar{x}_2$ (rich area), then the profit of the firm is

$$\pi = p(\bar{x}_2 - \bar{x}_2^L) - (y_2 - p - tz). \quad (50)$$

\(^9\)Roughly speaking, with the assumption of the monopoly vendor, this model endows a vitality to the traditional AMM model to explain the recent trends in urban development.
Now set up a Lagrangian as
\[ L = \pi + \lambda(y_2 - p - kz + \ell x_2^L + kx_2^F). \] (51)

Solving \( \frac{\partial L}{\partial p} \cdot p = 0 \) yields
\[ p = \frac{1}{2}y_2 + \frac{1}{4}t + \frac{1}{4}k. \] (52)

Minimizing \( ATC \) results in
\[ L = ATC + \lambda(\hat{x} - z) + \mu(z - x^R_2), \quad \lambda \geq 0. \] (53)

Solving \( \frac{\partial L}{\partial z} \cdot z = 0 \) and \( \frac{\partial L}{\partial \lambda} \cdot \lambda = 0 \) yields two solutions as
\[ z_{g1} = \frac{2y_2 - t - k}{4t}, \quad z_{g2} = \frac{4y_1 - 2y_2 - t - k - 2\alpha y_2 + t\alpha + k\alpha}{4(1 - \alpha)} = \hat{x}. \] (54)

Without loss of generality, we assume that \( 2\alpha y_2 - t - k > 0 \) and recall \( y_1 > \alpha y_2 \) in (43), and thus
\[ ATC_{z_{g2}} - ATC_{z_{g1}} = -\frac{k(y_1 - y_2)(2y_1 - 2y_2\alpha - t - k + \alpha t + \alpha k)}{t(-1 + \alpha)^2(-2y_2 + t + k)} < 0. \] (55)

Since \( ATC_{z=\hat{x}} \) has been excluded in the analysis of \( 0 \leq z \leq \hat{x} \), the optimal location of the vendor is at \( z = 0 \) which minimizes ATC.

**Proposition 5.** The optimal location regulation is at \( z^*_g = 0 \), and then the average transportation cost is minimized as \( ATC_{z=0} = (2\alpha y_2 + (\alpha - 1)(t + k))/(8\alpha) \).

### 2.3 Discussions

All the above results can be summarized as in the following table. As shown in Table 1, both in the one-group and two-group models, the vendor location \( z \), the price of composite good \( p \), and the right border of the city \( (x^R) \) are independent to the shopping cost in the market equilibrium, because there is no \( k \). This is due to the right border of the current model being identical to the AMM model (see Figure 2) and thus the (right side) land rent has a smaller slope (flatter) than the AMM model. This result can be seen as that the vendor pursues the lowest land rent to maximize its profit. While for both the one-group and two-group models, the left border of the city \( (x^L) \), the vendor’s profit \( \pi \), and the average transportation cost \( (ATC) \) are smaller than the AMM model, because the unit shopping cost is positive, \( k > 0 \). In other words, the vendor at the right border pulls the city to the right side and makes an asymmetric (and smaller) city.
Table 1: The results of this paper

<table>
<thead>
<tr>
<th>Status</th>
<th>1-group Equilibrium</th>
<th>1-group Optimum</th>
<th>2-group Equilibrium</th>
<th>2-group Optimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
<td>$\frac{y}{2t}$</td>
<td>0</td>
<td>$\frac{y}{2t}$</td>
<td>0</td>
</tr>
<tr>
<td>$p$</td>
<td>$\frac{y}{2t}$</td>
<td>$\frac{y}{2} + \frac{k}{4} + \frac{k}{2}$</td>
<td>$\frac{y}{2t}$</td>
<td>$\frac{2y}{3} + \frac{t+k}{3a}$</td>
</tr>
<tr>
<td>$\pi^R$</td>
<td>$\frac{y}{2t}$</td>
<td>$\frac{y}{2} - \frac{t-k}{4(t+k)}$</td>
<td>$\frac{y}{2t}$</td>
<td>$\frac{2y}{3} - \frac{t-k}{4a(t+k)}$</td>
</tr>
<tr>
<td>$\pi^L$</td>
<td>$\frac{y}{2(t+k)}$</td>
<td>$\frac{(2y-t-k)^2}{8(t+k)}$</td>
<td>$\frac{y}{2(t+k)}$</td>
<td>$\frac{(y_k)^2}{2(t+k)}$</td>
</tr>
<tr>
<td>$ATC$</td>
<td>$\frac{y(t+k)}{4t}$</td>
<td>$\frac{y}{4} - \frac{t}{8} - \frac{k}{8}$</td>
<td>$\frac{(t+k)y_2}{4t}$</td>
<td>$\frac{2y_2 - t - k}{8a}$</td>
</tr>
</tbody>
</table>

Note: $\% = \frac{4y^2\alpha^2 + t^2 + 2tk + k^2 - 8y_1\alpha t - 8y_1\alpha k + 4y_2\alpha t + 4y_2\alpha k}{8\alpha^2(t+k)}$.

Under an optimal location regulation, the price of the composite good is higher than the equilibrium price without regulation.\(^\text{10}\) The city size is smaller than the size without regulation. The vendor’s profit is reduced under the location regulation, while the average transportation cost is reduced under the location regulation.

In short, the simplified AMM model is a special case of the current model and many interesting results are obtained in this one-monopoly vendor model. If there are two vendors in the AMM model, then the city size, prices, profits, as well as average transportation cost may be very different to the current model.\(^\text{11}\) This will be left for future research.

3 Conclusions

This paper develops a simplified AMM model with a monopoly vendor, where not only is the location of the monopolist different, but also the bid rent is different to the traditional AMM model. The major findings are that the vendor will locate at one of the city border in both the one-group and two-group models. The land rent pattern is symmetric to the central business district (CBD) in the AMM model, while an asymmetric land rent configuration emerges when

\(^\text{10}\)One can plug $k = 0$ into Table 1 for further comparison between our model and the AMM model.

\(^\text{11}\)Spatial duopoly models such as Hotelling (1929), d’Aspremont et al. (1979), and Economides (1986) do not consider the land rent pattern. Instead, in those models, price undercutting is very important for the existence of a price-location equilibrium, which may not be the major role of an urban issue. Fujita and Thisse (1986) analyze the land rent pattern in the Hotelling model, but the price of the composite good is assumed to be fixed, which is also far away from the current paper.
Figure 5: Equilibrium land rent patterns

a monopoly vendor is introduced. In order to minimize the average transportation cost, the government may enact a location regulation policy to force the vendor to choose a location at CBD. These results are striking in the literature of urban configuration.
Figure 6: The poor live in the inner city when $k = 0$, while no poor live in the city when $k > 0$

References


