Two-Fund Separation under Homogeneous Ambiguity

Katsutoshi Wakai*

The State University of New York at Buffalo

April, 2004

*Department of Economics, The State University of New York at Buffalo, 427 Fronczak Hall, Buffalo, NY, 14260; E-mail: kwakai@buffalo.edu; Tel: (716)645-2121 ext 430; Fax: (716)645-2127.
Abstract

This paper derives two-fund separation for an economy with agents of multiple-priors utility: Given homogeneous sets of priors and linear risk tolerance felicity functions, an interior equilibrium is Pareto efficient and all agents hold a combination of the riskless security and the market portfolio (i.e., aggregate endowment). This result permits the construction of a representative agent whose preferences follow the multiple-priors model with a utilitarian von-Neumann-Morgenstern utility function.

*Journal of Economic Literature* Classification Numbers: D52, D81, G11

Key Words: fund-separation, uncertainty aversion, multiple priors, risk sharing, representative agent, linear risk tolerance, aggregation, HARA utility function
1 Introduction

Two-fund separation is a property under which an optimal consumption allocation (or monetary outcome) over states consists of a combination of two mutual funds. Tobin [18] shows that if a riskless security is available, two-fund separation holds under the mean-variance framework of Markowitz [15]. Cass and Stiglitz [4] generalize this property and show that if there is a riskless security, two-fund separation holds for general security returns if and only if an agent’s von-Neumann-Morgenstern (vNM) utility function satisfies the linear risk tolerance (LRT) condition. Wilson [19] examines a group decision problem among agents and finds that if agents share homogeneous beliefs and an LRT family of vNM utility functions, the optimal risk sharing rule satisfies two-fund separation. Rubinstein [17] applies Wilson’s [19] findings to the context of financial markets and confirms that under the same set of assumptions, if a riskless security exists, two-fund separation holds at equilibrium.

The assumptions maintained throughout the above studies are that (a) agents’ preferences follow an expected utility model with an identical subjective prior (or objective probability)
(i.e., a common-prior assumption) and (b) for a two-period model, the vNM utility function is time-additive. Given the recent development of non-expected utility theory, the purpose of the paper is to revisit the common-prior assumption. In particular, while maintaining the LRT and time-additivity assumptions, we investigate an optimal risk-sharing rule under an economy where agents’ preferences follow the multiple-priors model as developed by Gilboa and Schmeidler [11].

The result is a generalization of two-fund separation under the common-prior assumption: Given some regularity conditions for security markets, if agents share an identical set of priors (i.e., homogeneous ambiguity) under the LRT and time-additivity assumptions, an interior equilibrium is Pareto efficient and two-fund separation holds; an agent’s equilibrium consumption consists only of a combination of the riskless security and the market portfolio (i.e., aggregate endowment). To prove this result, we first demonstrate that Pareto efficient allocations are identical to those of the common-prior model so that two-fund separation holds. The crucial condition for this equivalence is that all agents “effectively” use an identical prior (not necessarily unique) to evaluate Pareto efficient allocations. Surprisingly, we find that the LRT and time-additivity assumptions themselves are sufficient for this condition to hold; apart from non-emptyness, closedness, and convexity, we do not require additional conditions on the structure of the set of priors. Furthermore, we utilize the above results and construct a representative agent whose preferences follow the multiple-priors model with a utilitarian vNM utility function. This finding is also surprising because
homogeneous ambiguity alone, even under complete markets, does not necessarily lead to the construction of a representative agent.

Related literature mainly focuses on a structure of sets of priors that derives commonality in equilibrium allocations and an effective selection of priors. For example, Epstein and Wang [10] and Chateauneuf, Dana, and Tallon [5] show that under homogeneous sets of priors based on a convex capacity and given complete markets, all agents share an identical prior (not necessarily unique) at equilibrium, and equilibrium allocations are those of the common-prior model. On the other hand, we focus on a structure of vNM utility functions under a general but homogeneous set of priors and establish a clear analogy with the previous work on optimal risk sharing and fund separation in general equilibrium and finance literature. However, in terms of optimal allocations of consumption, our result is also identical to that of the common-prior model: Ambiguity can affect equilibrium allocations, in particular, if it is heterogeneous among agents; see Liu [13], Epstein [7], Epstein and Miao [8], and Chateauneuf, Dana, and Tallon [5].

The remainder of the paper is organized as follows. In Section 2, we describe the model and provide results. Section 3 concludes the paper. All proofs are collected in Appendices.

2 Model and Results

We adapt the standard two-period finance economy under uncertainty. There is a single perishable consumption good at \( t = 0 \) and over \( \Omega = \{1, \ldots, S\} \) at \( t = 1 \), where \( S \) is finite;
we denote $x_0$ as consumption at $t = 0$ and $x_s$ as consumption at state $s$ at $t = 1$ so that $x = (x_0, x_1, ..., x_S)'$ is a consumption bundle. There are $H$ (finite) agents who have the endowment $e^h = (e^h_0, e^h_1, ..., e^h_S)' \in \mathbb{R}^{S+1}_{++}$. The aggregate endowment for the economy is defined as $e = (e_0, e_1, ..., e_S)' = \sum_1^H e^h$. There are $K$ securities that pay a consumption good as a dividend at each state $s$; vectors of payments of each security at $t = 1$, $d^h = (d^h_1, ..., d^h_S)' \in \mathbb{R}^{S}_{+}$, are linearly independent of each other. Let $d^h = (d^h_1, ..., d^h_K) \in \mathbb{R}^K_+$ be a collection of dividends at $s$ from $K$ securities and $D = [d^1, ..., d^K] \in \mathbb{R}^{S \times K}_{+}$ be a collection of vectors of dividend payments of all securities (i.e., $D$ is a payoff matrix). Also denote $\langle D \rangle$ as a space spanned by $[d^1, ..., d^K]$. At $t = 0$, the available securities are traded in competitive markets with prices $q = (q^1, q^2, ..., q^K)' \in \mathbb{R}^K$, where $q^k$ is a price of asset $k$. We assume that each agent has zero endowment of shares for $K$ securities so that the total supply of these securities is zero. Also, we normalize all commodity prices to 1.

At $t = 0$, each agent plans current investment in the available securities and consumption at $t = 0$ and $t = 1$, denoted by $(c^h, \theta^h)$ with $c^h \in (\mathbb{X}^h)^{S+1}$ and $\theta^h = (\theta^{h,1}, \theta^{h,2}, ..., \theta^{h,K})' \in \mathbb{R}^K$, where $\theta^{h,k}$ is a net holding of asset $k$ and $\mathbb{X}^h \subseteq \mathbb{R}_+$ is a consumption set of each element of $c^h$. After planning, each agent trades $\theta^h$ shares of securities and consumes $c^h_0$. At $t = 1$, state $s$ is realized. Then each agent receives or pays dividends and consumes $c^h_s$. The budget set for $h \in H$ is given by

$$B(c^h, q) \equiv \{(x^h, \theta^h) \in (\mathbb{X}^h)^{S+1} \times \mathbb{R}^K | x^h_0 + q \cdot \theta^h \leq e^h_0 \text{ at } t = 0, x^h_s \leq \theta^h \cdot d^h_s + e^h_s \text{ for all } s \in \Omega\}.$$
assume that agents’ preferences follow a version of the multiple-priors model as developed by Gilboa and Schmeidler [11], denote by \( V^h(x) \) \(^6,7\)

\[
V^h(x) \equiv \min_{p \in \Delta^h} \sum_{s=1}^{S} p_s \left[ u^h(x_0) + \delta^h u^h(x_s) \right] = u^h(x_0) + \min_{p \in \Delta^h} \sum_{s=1}^{S} p_s \delta^h u^h(x_s), \tag{1}
\]

where \( \Delta^h \subset \mathbb{R}^{S+} \) is a non-empty, closed, and convex set of priors and \( \delta^h \in (0, 1) \) is a discount factor. We assume that a felicity function \( u^h(.) \) is twice continuously differentiable in its domain. We also denote \( Q^h(x) \) as a set of all priors in \( \Delta^h \) such that

\[
Q^h(x) \equiv \{ p \in \Delta^h | V^h(x) = u^h(x_0) + \sum_{s=1}^{S} p_s \delta^h u^h(x_s) \}.
\]

For the economy defined by \( \mathcal{E}((V^h(.), e^h)_{h \in H}, D) \), an equilibrium is \((q, (c^h, \theta^h)_{h \in H}) \in \mathbb{R}^K \times (\mathbb{R}^{S+} \times \mathbb{R}^K)^H \) such that

\[
(c^h, \theta^h) \in B(e^h, q) \text{ for all } h \in H,
\]

if \((x^h, \eta^h) \in B(e^h, q)\), then \(V^h(x^h) \leq V^h(c^h)\) for all \( h \in H \),

\[
\sum_{h=1}^{H} c^h_0 = e_0 \text{ and } \sum_{h=1}^{H} c^h_s = e_s \text{ for all } s \in \Omega, \text{ and}
\]

\[
\sum_{h=1}^{H} \theta^h = 0.
\]

\(^6\)Our result holds if we use \( u^h_0(x_0) + \delta^h u^h_1(x_s) \) as a vNM utility function, where we only require the LRT condition in \( u^h_1(.) \), as long as (a) \((u^h_0(.)')' > 0 \) and \((u^h_0(.)')'' < 0 \) in the domain \( D^h_0 \) that covers \((0, e_0] \) and (b) the program (8) has a unique interior solution for any utility weight \((\mu^h)_{h \in H} \in \mathbb{R}^{H+}_+ \) for \( i = 0 \) (see Appendix A). Since the discounted utility \( u^h(x_0) + \delta^h u^h(x_s) \) is used in most applications, we assume \( u^h_0(.) = u^h_1(.) \) to avoid complication.

\(^7\)This form of a utility function can be derived axiomatically. See Epstein and Schneider [9].
Now, the following are the main assumptions of our study:

**Assumption 1:**

(i) Each $u^h$ satisfies the LRT condition: $-\frac{(u^h(x))'}{(u^h(x))''} = \alpha^h + \beta^h x$ for all $x \in \{x \in \mathbb{R} | \alpha^h + \beta^h x > 0\} \equiv D^h$ with $(\alpha^h, \beta^h) \in \mathbb{R}_+ \times \mathbb{R}$, where $D^h$ is a domain of $u^h(x)$.

(ii) For all $h \in H$, $(u^h(x))' > 0$ and $(u^h(x))'' < 0$ for all $x \in D^h$.

(iii) For all $h \in H$ and for $i \in [0, S]$, $(0, e_i) \subseteq D^h$.

(iv) For all $h \in H$, $X^h \equiv D^h \cap \mathbb{R}_+$.

(v) $\beta^h = \beta^{h'}$, $\Delta^h = \Delta^{h'} = \Delta$ for all $h, h' \in H$.

(vi) $\tilde{1} = (1, 1, ..., 1)' \in \langle D \rangle$ and $\tilde{e}^h = (e^h_1, ..., e^h_S)' \in \langle D \rangle$ for all $h \in H$.

(iii) ensures that the LRT condition (i) is satisfied at an interior equilibrium, and (vi) facilitates Pareto efficient trades. (v) is the condition that defines “homogeneity” among agents’ degrees of risk aversion and uncertainty aversion. In particular, we generalize the

---

8 $\alpha^h > 0$ is required for $(u^h(x))'' < 0$ to hold when $\beta = 0$.

9 This condition implies that $\alpha^h > 0$ if $\beta < 0$, and $\alpha^h \geq 0$ if $\beta > 0$.

10 Although we naively define homogeneity of an attitude toward uncertainty among agents as in (v), we cannot say that all agents are equally uncertainty averse unless they share the same felicity functions $u^h$ and the same discount factor $\delta^h$. 
common-prior assumption to the assumption of a common set of priors. Note that \( \Delta \) does not require any additional conditions apart from non-emptiness, closedness, and convexity.

By integrating the LRT equation in (i) of Assumption 1, we obtain the HARA family of felicity functions that are defined on the domain of \( \alpha^h + \beta x > 0 \) (see Rubinstein [17]):

\[
    u^h(x) = \frac{\alpha^h}{\beta(1 - \frac{1}{\beta^2})} \quad \text{if } \beta \neq 0 \text{ and } \beta \neq 1, \\
    u^h(x) = -\alpha^h e^{-\frac{x}{\alpha^h}} \quad \text{if } \beta = 0 \text{ and } \alpha^h > 0, \quad \text{and} \\
    u^h(x) = \ln(\alpha^h + x) \quad \text{if } \beta = 1.
\]

As derived in Gilboa and Schmeidler [11], any positive affine transformation of the above function can represent preferences by the form described in (1). The domain of \( u^h(.) \) is summarized as

- \( \mathbb{D}^h \equiv \mathbb{R} \) for \( \beta = 0 \),
- \( \mathbb{D}^h \equiv \{ x \in \mathbb{R} | x > -\frac{\alpha^h}{\beta} \} \) for \( \beta > 0 \), and
- \( \mathbb{D}^h \equiv \{ x \in \mathbb{R} | x < -\frac{\alpha^h}{\beta} \} \) for \( \beta < 0 \).

Clearly, for all cases, \( (u^h(x))' > 0 \) and \( (u^h(x))'' < 0 \) for all \( x \in \mathbb{D}^h \). Also, \( \sup_{x \in \mathbb{D}^h} (u^h(x))' = \infty \) and \( \inf_{x \in \mathbb{D}^h} (u^h(x))' = 0 \). By (iv) of Assumption 1, the consumption space is defined as
follows:\footnote{For $\beta > 0$ and $\alpha^h = 0$, $(u^h(0))'$ is undefined; for $\beta < 0$, $(u^h(-\frac{\alpha^h}{\beta}))''$ is undefined for some $\beta < 0.$}

\[ X^h \equiv \mathbb{R}_+ \text{ for } \beta \geq 0 \text{ and } \alpha^h > 0, \]

\[ X^h \equiv \mathbb{R}_{++} \text{ for } \beta > 0 \text{ and } \alpha^h = 0, \] and

\[ X^h \equiv [0, -\frac{\alpha^h}{\beta}) \text{ for } \beta < 0. \]

Note that if $\beta > 0$ and $\alpha^h = 0$, an indifference curve that goes through a strictly positive consumption bundle does not intersect the boundary of the non-negative orthant $\mathbb{R}^{S+1}_{++}$. Therefore, the boundary condition for an agent’s optimization holds.

Next, we want to show that an equilibrium exists for the economy $E((V^h(\cdot), e^h)_{h \in H}, D)$ under Assumption 1. Let $E^h \equiv (\mathbb{D}^h)^{S+1}$ be a domain of $V^h(\cdot)$. For $\beta \geq 0$, $V^h(\cdot)$ is continuous, strictly concave, and strongly monotone on $E^h$ for all $h \in H$. Hence, given $(e^h)_{h \in H} \in (\mathbb{R}^{S+1}_{++})^H$, an equilibrium exists (see Magill and Quinzii [14, Proposition 10.5]). For $\beta < 0$, define a new function $\tilde{u}^h(\cdot)$ such that

\[ \tilde{u}^h(x) = u^h(x) \text{ for } x \in (-\infty, a), \]

\[ = v^h(x) \text{ for } x \in [a, \infty), \]

where $v^h(x)$ is continuous, strictly increasing, and strictly concave and $a \in (\max_{i \in [0,S]} e_s, -\frac{\alpha^h}{\beta})$.

In addition, $u^h(a) = v^h(a)$ and $(u^h(a))' = \lim_{b \to 0^+} \frac{v^h(a + b) - v^h(a)}{b}$. Then, given $(e^h)_{h \in H} \in (\mathbb{R}^{S+1}_{++})^H$, an equilibrium exists under $X^h \equiv \mathbb{R}_+$. Note that at an equilibrium, $c^h_i \in [0, e_i]$ for
all \( i \in [0, S] \) and for all \( h \in H \). Hence, an equilibrium of the extended model defined above is an equilibrium of the original economy \( \mathcal{E}((V^h(\cdot), e^h)_{h \in H}, D) \).\(^{12}\)

An allocation \((x^h)_{h \in H}\) satisfies two-fund separation (or money separation) provided that there exists \((\gamma^h, \lambda^h)_{h \in H} \in (\mathbb{R} \times \mathbb{R}^+)^H\) such that

\[
\sum_{h=1}^H \gamma^h = 0, \quad \sum_{h=1}^H \lambda^h = 1, \quad \text{and} \quad x^h_s = \gamma^h + \lambda^h e_s \quad \text{for all} \quad s \in \Omega \text{ and for all} \quad h \in H,
\]

which means that each agent holds a positive proportion of the aggregate endowment (i.e., the market portfolio) and either purchases or shorts the riskless security.\(^{13,14}\)

To show that two-fund separation holds at an equilibrium, we first characterize the property of Pareto efficient allocations that are feasible in \( \mathbb{E}^1 \times \cdots \times \mathbb{E}^H \) (i.e., the combined domain of all \( V^h(\cdot) \)). Consider the following program for any utility weight \((\mu^h)_{h \in H} \in \mathbb{R}^{+H} \):

\[
\max_{(x^h)_{h \in H}} \sum_{h=1}^H \mu^h V^h(x^h) \quad \text{such that} \quad \sum_{h=1}^H x^h_i = e_i \quad \text{for all} \quad i \in [0, S]. \tag{5}
\]

Then, we denote a corresponding part of the Pareto frontier, \( \mathbb{PF} \), as the set given by

\[
\mathbb{PF} \equiv \{(x^h)_{h \in H} \mid (x^h)_{h \in H} \text{ is a solution of the program (5) for some } (\mu^h)_{h \in H} \in \mathbb{R}^{+H}\}.
\]

Note that in the program (5), we consider allocations in \( \mathbb{E}^1 \times \cdots \times \mathbb{E}^H \) that may include

\(^{12}\)It is because \( u^h \) is strictly increasing and strictly concave in \( X^h \equiv B^h \cap \mathbb{R}_+ \). The converse is also true.

\(^{13}\)Since one of the mutual funds is the riskless security, it is often called “money separation.”

\(^{14}\)If there is no aggregate uncertainty, this property is equivalent to one-fund separation. Then, our result is a special case of Chateauneuf, Dana, and Tallon [5] and Billot, Chateauneuf, Gilboa, and Tallon [2].
negative consumption; however, in an agent’s consumption set \((X^h)^{S+1}\), we only allow non-negative consumption.

The main result of this paper is that Pareto efficient allocations associated with strictly positive utility weights are identical to those of the common-prior model (see Appendix A).

**Proposition 1:** For each \((\mu^h)_{h \in H} \in \mathbb{R}^H_{++}\), there exists a unique solution of the program (5). Moreover, for any \((x^h)_{h \in H} \in \mathbb{P}\), two-fund separation holds and \(Q^h(x^h) = Q^{h'}(x^{h'})\) for all \(h, h' \in H\). In addition, \(\frac{\delta^h(u^h(x^h))'}{(u^h(x^h_0))'}\) is identical among all \(h \in H\) for all \(s \in \Omega\).\(^{15}\)

The crucial condition is that at a Pareto efficient allocation, \(Q^h(x^h) = Q^{h'}(x^{h'})\) for all \(h, h' \in H\). Then, any prior \(p \in Q^h(x^h)\) can serve as an effective selection of priors for all agents. Therefore, \((x^h)_{h \in H}\) must be identical to that of the common-prior model under \(p\).\(^{16}\)

To satisfy \(Q^h(x^h) = Q^{h'}(x^{h'})\) for all \(h, h' \in H\) at a Pareto efficient allocation, not only must the ratios of marginal utilities \(\frac{\delta^h(u^h(x^h))'}{(u^h(x^h_0))'}\) be equalized among agents, but the ratios of utilities \(\frac{u^h(x^h_s)}{u^h(x^h_0)}\) between any two states must also be equalized among agents. Surprisingly, we find that under the LRT condition (i) of Assumption 1, we can equalize the two ratios simultaneously. Clearly, this condition where both ratios are equalized is non-trivial, and it is easy to see that under a general felicity function \(u^h\), homogeneous ambiguity alone does not guarantee that \(Q^h(x^h) = Q^{h'}(x^{h'})\) holds for all \(h, h' \in H\) at a Pareto efficient allocation. This

\(^{15}\)Proposition 1 parallels the result by Magill and Quinzii [14, Proposition 16.13].

\(^{16}\)Two-fund separation requires the condition that \(\cap_{h' = 1}^H Q^{h'}(x^h) \neq \emptyset\) holds for all \((x^h)_{h \in H} \in \mathbb{P}\).
implies that in general, a set of all Pareto efficient allocations from homogeneous ambiguity
is not identical to that of the common-prior model. Furthermore, to equate ratios of utilities,
the existence of the riskless security is essential even for the case of quadratic felicity functions
(i.e., $\beta = -1$).\[17\]

Given Proposition 1, the following conclusion is a simple extension of the result by Magill
and Quinzii [14, Proposition 16.14], which shows that two-fund separation holds under the
common-prior assumption (i.e., when $\Delta$ is a singleton set).

**Proposition 2:** Suppose that the economy $\mathcal{E}((V^h(\cdot), e^h)_{h \in H}, D)$ satisfies Assumption 1. Let
$(q, (c^h, \theta^h)_{h \in H})$ be an equilibrium. If $(c^h)_{h \in H} \in (\mathbb{R}^{S+1}_+)^H$, then $(c^h)_{h \in H}$ is Pareto efficient (with
respect to $\mathbb{E}^1 \times ... \times \mathbb{E}^H$) and two fund separation holds. Moreover, $Q^h(c^h) = Q^{h'}(c^{h'})$ for all
$h, h' \in H$, and $q^k = \sum_{s=1}^S p_s \frac{\delta^h(u^h(c^h_s))'}{(u^h(c^h_0))')} d^k_s$ for all $k \in K$ under some $p = (p_1, ..., p_S)' \in Q^h(c^h)$.

In Appendix B, we show that an interior equilibrium corresponds to some Pareto efficient
allocation associated with a strictly positive utility weight. In fact, if we let $X^h = D^h$ (i.e.,
allowing negative consumption) under Assumption 1, all equilibria are Pareto efficient and
two-fund separation holds. Asset pricing equations are due to Epstein and Wang [10]. In
order to ensure that asset pricing equations hold at a Pareto efficient allocation, we must
have $Q^h(c^h) = Q^{h'}(c^{h'})$ for all $h, h' \in H$. Note that although $Q^h(c^h)$ can contain more than

\[17\] Under the common-prior model with quadratic felicity functions, two-fund separation can be obtained
without the riskless security.
one prior, it does not imply that there is indeterminacy in asset pricing.\footnote{See Epstein and Wang \cite{10} for details.}

The proofs of Propositions 1 and 2 also lead to the following corollary. Here, an interior equilibrium price $q$ corresponds to an equilibrium price in the representative-agent economy where the agent’s preferences follow the multiple-priors model with a utilitarian vNM utility function.

**Corollary 1:** Suppose that the economy $\mathcal{E}((V^h(\cdot), e^h)_{h \in H}, D)$ satisfies Assumption 1 and $\delta^h = \delta^{h'} = \delta$ for all $h, h' \in H$.\footnote{For the case of $\delta^h \neq \delta^{h'}$, in order for a representative agent to exist, we must use a vNM utility function of the form $u^h_0(x_0) + \delta^h u^h_1(x_s)$, where we permit $u^h_0(\cdot) \neq u^h_1(\cdot)$.} Let $(q, (e^h, \theta^h)_{h \in H})$ be an equilibrium. If $(e^h)_{h \in H} \in (\mathbb{R}^{S+1})^H$, then the representative-agent economy $\mathcal{E}(V^*(\cdot), e, D)$ has an equilibrium $(q, e, 0)$, where $V^*(x)$ is defined as follows:

$$V^*(x) \equiv \min_{p \in \Delta} \sum_{s=1}^{S} p_s \left( u^*(x_0) + \delta u^*(x_s) \right),$$

$$u^*(x_i) \equiv \max \left\{ \sum_{h=1}^{H} \mu^h u^h(x^h_i) \text{ such that } \sum_{h=1}^{H} x^h_i = x_i \right\} \text{ for all } i \in [0, S],$$

$$\mu^h \equiv \frac{1}{(u^h(c^h_0))'} \text{ for all } h \in H.$$

**Proof:** By construction, $u^*(x_i)$ is continuous, increasing, and strictly concave. The proof of Proposition 1 implies that $u^*(x_i)$ is differentiable and $(u^*(e_i))' = \mu^h(u^h(c^h_i))'$; it also implies that $Q^*(e) = Q^h(c^h)$ holds for any $h \in H$ because the ratios of utilities between any two
states are equalized among agents at an interior equilibrium \((c^h)_{h \in H}\). By Epstein and Wang [10], the representative-agent economy \(E(V^\ast(.), e, D)\) has an equilibrium \((q, e, 0)\). □

Again, the condition that \(Q^h(x^h) = Q^{h'}(x^{h'})\) holds for all \(h, h' \in H\) and for all \((x^h)_{h \in H} \in \mathbb{F}\) is crucial, because it ensures that \(Q^\ast(e) = Q^h(c^h)\) holds at an interior equilibrium \((c^h)_{h \in H}\). Clearly, if \(\cap_{h'=1}^H Q^{h'}(c^h) = \emptyset\), we cannot construct a representative agent.

## 3 Conclusion

We have derived two-fund separation for an economy where agents’ preferences follow the multiple-priors model. In order for two-fund separation to hold, we only require the LRT condition; we do not require any additional conditions on the structure of the set of priors as long as it is homogeneous among agents.

---

\(^{20}\)Since both a solution and a Lagrange multiplier of the program (8) are differentiable with respect to \(x_i\) in an open neighborhood in \(\mathbb{R}_{++}\) that covers \((0, e_i]\), the optimal value function \(u^\ast\) is differentiable at \(e_i\) for all \(i \in [0, S]\). See Appendix A.
Appendix A: Proof of Proposition 1

For $p = (p_1, ..., p_S)' \in \Delta$, we define $V^h_p(x)$ as

$$V^h_p(x) \equiv u^h(x_0) + \sum_{s=1}^{S} p_s \delta^h u^h(x_s).$$

Let $(\mu^h)_{h \in H} \in \mathbb{R}^{H}_{++}$ be a utility weight. For some $\bar{p} = (\bar{p}_1, ..., \bar{p}_S)' \in \Delta$, consider the following program:

$$\max_{(x^h \in E^h)_{h \in H}} \sum_{h=1}^{H} \mu^h V^h_{\bar{p}}(x^h), \tag{6}$$

such that $\sum_{h=1}^{H} x^h_i = e_i$ for all $i \in [0, S]. \tag{7}$

First, we want to show that a unique solution of the program (6) exists. Since $V^h_{\bar{p}}(x^h)$ is strictly concave and twice continuously differentiable on $E^h$, the first order conditions are necessary and sufficient for $(\pi^h)_{h \in H}$ to be a unique solution of the program. By solving the first order conditions, for all $h \in H$ and for all $i \in [0, S]$,

$$\bar{x}^h_i = \beta e_i + \sum_{h' \in H} \alpha^h \ln(\mu^h \rho^h_i) - \beta \sum_{h' \in H} \alpha^{h'} \ln(\mu^{h'} \rho^{h'}_i) + \frac{\alpha^h}{\beta} \frac{\alpha^{h'}}{\beta}$$

for $\beta \neq 0$, and

$$\bar{x}^h_i = \alpha^h \ln(\mu^h \rho^h_i) - \frac{\alpha^h}{\beta} \frac{\alpha^{h'}}{\beta} \ln(\mu^{h'} \rho^{h'}_i) + \frac{\alpha^h}{\beta} \frac{\alpha^{h'}}{\beta} e_i$$

for $\beta = 0$,

where $\rho^h_i = 1$ if $i = 0$ and $\rho^h_i = \delta^h$ otherwise. By the definition of $\pi^h$, $\pi^h \in E^h$ for all $h \in H$ (so that it is an interior solution; in particular, for $\beta < 0$, by (iii) of Assumption 1, $\beta e_i + \alpha^h > 0$, which implies that $\beta e_i + \sum_{h' \in H} \alpha^{h'} > 0$). Hence, a unique solution $(\pi^h)_{h \in H}$
exists. Note that for all \( h \in H \), the coefficient of \( e_s \) is positive and constant for all \( s \in \Omega \); the remaining term is also constant for all \( s \in \Omega \). Hence, by the feasibility condition (Eq. (7)), there exists \((\gamma^h, \lambda^h)_{h \in H} \in (\mathbb{R} \times \mathbb{R}_{++})^H\) such that \( \sum_{h=1}^H \gamma^h = 0 \), \( \sum_{h=1}^H \lambda^h = 1 \), and \( \overline{x}^h_s = \gamma^h + \lambda^h e_s \) for all \( s \in \Omega \) and for all \( h \in H \) (note that since we do not impose non-negativity constraints on \( \overline{x}^h \), \( \overline{x}^h_s \) can be negative). Moreover, \((\overline{x}^h)_{h \in H}\) is a solution for any fixed \( \overline{p} \in \Delta \) because for all \( i \in [0, S] \), \((\overline{x}^h_i)_{h \in H}\) is an argument maximum of\(^{21}\)

\[
\max_{(x^h_i \in \mathbb{D})_{h \in H}} \sum_{h=1}^H \mu^h \rho^h_i u^h(x^h_i) \text{ such that } \sum_{h=1}^H x^h_i = e_i.
\] (8)

In addition, by the first order conditions, all agents’ marginal rates of substitution between every pair of goods must be equalized. For example, assume that \( \beta = 0 \) (case (3); we will investigate other cases at the end of the proof). Then, for some fixed \( \overline{p} \in \Delta \),

\[
\frac{\overline{p}^h \delta^h(u^h(\overline{x}^h_s))'}{(u^h(\overline{x}^h_0))')} = \frac{\overline{p}^h \delta^h(e^{-\frac{1}{\alpha^h} \pi^h_s}(-\frac{1}{\alpha^h}))}{(u^h(\overline{x}^h_0))')} = \frac{\overline{p}^h \delta^h(e^{-\frac{1}{\alpha^h} \pi^h_s'})}{(u^h(\overline{x}^h_0))')} = \overline{p}^h R^h_s.
\] (9)

\(^{21}\)From Constantinides [6].
Eq. (9) implies that $e^{-\frac{1}{\alpha h} \bar{x}_h^s} = \frac{R_s}{\delta h}(u^h(\bar{x}_0^h))'$. Hence,

$$Q^h(\bar{x}^h)$$

$$\equiv \{ p \in \Delta | V^h(\bar{x}^h) = u^h(\bar{x}_0^h) + \sum_{s=1}^{S} p_s \delta h u^h(\bar{x}_s^h) \}$$

$$= \{ p \in \Delta | \min_{p \in \Delta} \sum_{s=1}^{S} p_s u^h(\bar{x}_s^h) = \sum_{s=1}^{S} p_s u^h(\bar{x}_s^h) \}$$

$$= \{ p \in \Delta | \min_{p \in \Delta} \sum_{s=1}^{S} p_s (-\alpha h e^{-\frac{1}{\alpha h} \bar{x}_s^h}) = \sum_{s=1}^{S} p_s (-\alpha h e^{-\frac{1}{\alpha h} \bar{x}_s^h}) \}$$

$$= \{ p \in \Delta | \min_{p \in \Delta} \sum_{s=1}^{S} p_s \left[ -\alpha h R_s \frac{\delta h}{\delta h}(u^h(\bar{x}_0^h))' \right] = \sum_{s=1}^{S} p_s \left[ -\alpha h R_s \frac{\delta h}{\delta h}(u^h(\bar{x}_0^h))' \right] \}$$

$$= \{ p \in \Delta | \max_{p \in \Delta} \sum_{s=1}^{S} p_s R_s = \sum_{s=1}^{S} p_s R_s \}$$

$$= Q'^h(\bar{x}'^h)$$

Therefore, any $\bar{p} \in \cap_{h=1}^{H} Q'^h(\bar{x}^h) = Q^h(\bar{x}^h)$ can serve as an effective selection of a prior for $V^h(\bar{x}^h)$ for all $h \in H$. Also, $(\bar{x}^h)_{h \in H}$ is a unique solution of the program (6). Thus, for any $(x^h)_{h \in H} \in \mathbb{E}^1 \times \ldots \times \mathbb{E}^H$ with $(x^h)_{h \in H} \neq (\bar{x}^h)_{h \in H}$ that satisfies Eq. (7),

$$\sum_{h=1}^{H} \alpha^h V^h(x^h) \leq \sum_{h=1}^{H} \alpha^h V^h_p(x^h) < \sum_{h=1}^{H} \alpha^h V^h_p(\bar{x}^h) = \sum_{h=1}^{H} \alpha^h V^h(\bar{x}^h). \quad (10)$$

Hence, $(\bar{x}^h)_{h \in H}$ is a unique solution of the program (5) under $(\mu^h)_{h \in H} \in \mathbb{R}^{H+}$, which implies that for any $(\mu^h)_{h \in H} \in \mathbb{R}^{H+}$, there exists a unique solution of the program (5) that is identical to a unique solution of the program (6). Therefore, for any allocation $(x^h)_{h \in H} \in \mathbb{P} \mathbb{F}$, two-fund separation holds and $Q^h(x^h) = Q'^h(x'^h)$ for all $h, h' \in H$. Also, by construction, $\frac{\delta^h(u^h(c^h_s))'}{(u^h(c^h_0))')}$ is identical among all $h \in H$ for all $s \in \Omega$. Note that this proof requires the
existence of a unique solution of the program (6) for all \((\mu^h)_{h \in H} \in \mathbb{R}^H_{++}\), which is not the case for the common-prior model.

Finally, we want to confirm cases (2) and (4). It suffices to show that \(Q^h(x^h) = Q^{h'}(x^{h'})\) for all \(h, h' \in H\). For case (2), assume that \(\beta \neq 0\) and \(\beta \neq 1\). Then, Eq. (9) is rewritten as

\[
\frac{p_s \delta^h (u^h(x^h_s)'}}{(u^h(x_0^h)')} = \frac{p_s \delta^h (\alpha^h + \beta x^h_s)^{-\frac{1}{\beta}}}{(u^h(x_0^h)')} = \frac{p_s \delta^{h'} (\alpha^{h'} + \beta x^{h'}_s)^{-\frac{1}{\beta}}}{(u^{h'}(x_0^{h'})')} \equiv p_s R_s. \tag{11}
\]

Eq. (11) implies that \(\alpha^h + \beta x^h = \left(\frac{R_s(u^h(x_0^h))'}{\delta^h}\right)^{-\beta}\). Therefore,

\[
\frac{(\alpha^h + \beta x^h)^{1-\frac{1}{\beta}}}{\beta(1-\frac{1}{\beta})} = (R_s)^{-\beta(1-\frac{1}{\beta})} \left[\left(\delta^{-1}(u^h(x^h_s)')\right)^{-1}\right]^{-\beta(1-\frac{1}{\beta})} \text{ for all } s \in \Omega.
\]

For case (4), assume that \(\beta = 1\). Then, Eq. (9) is rewritten as

\[
\frac{p_s \delta^h (u^h(x^h_s)')}{(u^h(x_0^h)')} = \frac{p_s \delta^h \frac{1}{\alpha^h + x^h_s}}{(u^h(x_0^h)')} = \frac{p_s \delta^{h'} \frac{1}{\alpha^{h'} + x^{h'}_s}}{(u^{h'}(x_0^{h'})')} \equiv p_s R_s. \tag{12}
\]

Eq. (12) implies that \(\alpha^h + \delta^h = \frac{\delta^h}{R_s(u^h(x_0^h))'}\). Therefore,

\[
\ln(\alpha^h + \delta^h) = -\ln(R_s) - \ln((u^h(x_0^h)')) + \ln(\delta^h) \text{ for all } s \in \Omega.
\]

Clearly, for both cases, \(Q^h(x^h) = Q^{h'}(x^{h'})\) for all \(h, h' \in H\).
4 Appendix B: Proof of Proposition 2

Let \((q, (c^h, \theta^h)_{h \in H})\) be an equilibrium. Denote a set of all \(D\)-feasible allocations by the following:

\[
F_D \equiv \{(x^h)_{h \in H} \in (\mathbb{X}^1)^{S+1} \times ... \times (\mathbb{X}^H)^{S+1} | \\
\sum_{h=1}^{H} x^h - e \leq 0 \text{ and } (x^1_h, ..., x^S_h) - (e^1_h, ..., e^S_h) \in \langle D \rangle \text{ for all } h \in H\}.
\]

Also define a set of all \(Dom\)-feasible allocations as follows:

\[
F_{Dom} \equiv \{(x^h)_{h \in H} \in \mathbb{E}^1 \times ... \times \mathbb{E}^H | \\
\sum_{h=1}^{H} x^h - e \leq 0 \text{ and } (x^1_h, ..., x^S_h) - (e^1_h, ..., e^S_h) \in \langle D \rangle \text{ for all } h \in H\}.
\]

Notice that by (iv) of Assumption 1, \(F_D \subseteq F_{Dom}\).

By Magill and Quinzii [14, Proposition 12.3], \((c^h)_{h \in H}\) is constrained Pareto efficient with respect to \(F_D\) (i.e., there does not exist \((x^h)_{h \in H} \in F_D\) such that \(V^h(x^h) \geq V^h(c^h)\) for all \(h \in H\) with a strict inequality for some \(h \in H\)).\(^{22}\) Suppose that there exists \((x^h)_{h \in H} \notin F_D\) but \((x^h)_{h \in H} \in F_{Dom}\) such that \(V^h(x^h) \geq V^h(c^h)\) for all \(h \in H\) with a strict inequality.

---

\(^{22}\)For \(\beta < 0\), \(F_D\) only contains \(x^h_i \in [0, e_i]\) for all \(i \in [0, S]\) and for all \(h \in H\). Hence, an equilibrium allocation \((c^h)_{h \in H}\) based on the extended function \(\tilde{u}^h(.)\) under the domain of \(\mathbb{X}^h \equiv \mathbb{R}_+\) satisfies the constrained Pareto efficiency with respect to \(F_D\).
for some $h \in H$. For a small $\varepsilon > 0$, let $\tilde{x}^h \equiv \varepsilon x^h + (1 - \varepsilon)c^h$ such that $(\tilde{x}^h)_{h \in H} \in \mathbb{F}_D$ (because $0 < c_i^h < e_i$ for all $i \in [0, S]$ and for all $h \in H$).\textsuperscript{23} By strict concavity of $V^h(\cdot)$, $V^h(\varepsilon x^h + (1 - \varepsilon)c^h) > \varepsilon V^h(x^h) + (1 - \varepsilon)V^h(c^h) \geq V^h(c^h)$ for all $h \in H$ with a strict second inequality for some $h \in H$. This contradicts the assumption that $(c^h)_{h \in H}$ is constrained Pareto efficient with respect to $\mathbb{F}_D$.\textsuperscript{24} Hence, $(c^h)_{h \in H}$ is constrained Pareto efficient with respect to $\mathbb{F}_{Dom}$.

Clearly, a Pareto frontier contains $\mathbb{PF}$, where feasibility is defined with respect to $\mathbb{E}^1 \times \ldots \times \mathbb{E}^H$. Also, $\mathbb{PF} \subset \mathbb{F}_{Dom}$ under (vi) of Assumption 1. Therefore, $(c^h)_{h \in H}$ is constrained Pareto efficient for a domain that includes $\mathbb{PF}$. Moreover, let $(x^h)_{h \in H} \in \mathbb{E}^1 \times \ldots \times \mathbb{E}^H$ be a solution of the program (5) under $(\mu^h)_{h \in H} \in \mathbb{R}^H_+$ with some $\alpha^h = 0$ (if any solution exists). Given (ii) and (iii) of Assumption 1, $x^h \notin \mathbb{R}^{S+1}_{++}$ so that $(c^h)_{h \in H} \in \mathbb{R}^H_{++}$ and $(x^h)_{h \in H}$ cannot be Pareto ranked. Hence, $(c^h)_{h \in H}$ is a Pareto efficient allocation associated with some $(\mu^h)_{h \in H} \in \mathbb{R}^H_{++}$.

By Proposition 1, $(c^h)_{h \in H}$ satisfies two-fund separation, which is a generalization of the result by Magill and Quinzii [14, Proposition 16.14] to a case of homogeneous ambiguity.

In terms of asset pricing, we have already established that $q$ is a vector of equilibrium prices under which market clearing conditions are satisfied. In addition, by Epstein and Wang [10], at an interior equilibrium, the first order conditions of an agent’s optimization

\textsuperscript{23}This argument does not hold for a boundary equilibrium, which involves either $c_i^h = 0$ or $c_i^h = e_i$ for some $i \in [0, S]$.

\textsuperscript{24}If $\mathbb{F}_{Dom} \setminus \mathbb{F}_D \neq \emptyset$, some $x_i^h$ must be negative.
imply that there exists \( p = (p_1, ..., p_S) \) such that \( q^h = \frac{\delta^h(u^h(c^h_s)^{\prime})}{u^h(c^h_0)^{\prime}} d_s^h \) for all \( k \in K \). Also, by Proposition 1, \( Q^h(c^h) = Q^{h'}(c^{h'}) \) for all \( h, h' \in H \), and \( \frac{\delta^h(u^h(c^h_s))}{u^h(c^h_0)^{\prime}} \) is identical among all \( h \in H \) for all \( s \in \Omega \). Then, an equilibrium price \( q \) satisfies the first order conditions for all \( h \in H \) under \( p \). Hence, \( q \) defined by the above equation supports \((c^h)_{h \in H}\) as an equilibrium. ■
References


