Linking Behavioral Economics, Axiomatic Decision Theory and General Equilibrium Theory

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Abstract

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My dissertation links behavioral economics, axiomatic decision theory and general equilibrium theory to analyze issues in financial economics. I investigate two behavioral concepts: *time-variability aversion*, i.e., the aversion to volatility (fluctuation in payoffs over time) and *uncertainty aversion*, i.e., the aversion to uncertainty of state realizations. Chapter 1 develops a new intertemporal choice theory by endogenizing discount factors based on time-variability aversion, and shows that the new model can explain widely noted stylized facts in finance. I find that (1) time-variability aversion can be represented by time-varying discount factors based on very parsimonious axioms; (2) under the assumption of dynamic consistency, time-variability aversion implies *gain/loss asymmetry* in discount factors (3) the gain/loss asymmetry boosts effective risk aversion over states by extreme dislike of losses while maintaining positive average time-discounting. This intertemporal substitution mechanism explains why the risk premium of equity needs to be very high relative to the risk-free rate.

Chapter 2 provides the conditions under which the no-trade theorem of Milgrom & Stokey (1982) holds for an economy of agents whose preferences follow uncertainty aversion

as captured by the multiple prior model of Gilboa and Schmeidler (1989). First, I prove that given the agents' knowledge of the filtration, dynamic consistency and consequentialism imply that a set of ex-ante priors must satisfy the recursive structure. Next, I show that with *perfect anticipation* of ex-post knowledge, the no-trade theorem holds under the economy such that agents follow dynamically consistent multiple prior preferences.

Chapter 3 examines risk-sharing among agents who are uncertainty averse. The main objective is to provide conditions in the exchange economy such that agents' effective priors (and equilibrium consumptions) will be comonotonic and their marginal rates of substitution (weighted by these priors) will be equalized when agents have heterogeneous multiple prior sets. One set of sufficient conditions is for each agent's multiple prior set to be symmetric (or to be defined by a convex capacity) around the center of the simplex.

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Chapter 1

Introduction

1.1 Introduction

My dissertation links behavioral economics, axiomatic decision theory and general equilibrium theory to analyze issues in financial economics. The behavioral issues I investigate are time-variability aversion and uncertainty aversion. The analysis develops new theories and combines them with estimation and calibration.

Chapter 1 develops a new behavioral notion, time-variability aversion, and then applies this idea to a consumption-saving problem to derive implications for asset pricing. Conventionally, risk aversion is regarded as dislike of variations in payoffs of random variables within a period. By contrast, *time-variability* is variation in payoffs over time. In principle, an agent could be averse to such variation even in the absence of risk. For example, Loewenstein & Prelec (1993) show that, in experiments, agents prefer smooth allocations over time even under certainty, and their preferences for smoothing cannot be explained by a time-separable discounted utility representation.

I define time-variability aversion to mean that an agent is averse to mean-preserving spreads of utility over time. To capture this idea, I provide a representation, adapting a method developed in a different context by Gilboa & Schmeidler (1989). In this representation, risk aversion is captured by the concavity of a von Neumann-Morgenstern utility function. Time-variation aversion is captured by the agent selecting a sequence of (normalized) discount factors (from a given set) that minimizes the present discounted value of a given payoff stream. I provide an axiomatization for this representation. More formally, the assignment of discount factors is determined recursively. At each time t, the agent compares present consumption with the discounted present value of future consumption from t+1 onward and then selects the time-t discount factor to minimize the weighted sum of these two values. These recursive preferences are non-time-separable and dynamically consistent by construction (but they differ in form and implication from those used by Epstein & Zin (1989)). Intuitively, this representation exhibits time-variability aversion by allocating a high discount factor when tomorrow's consumption is low (and vice versa).

The derived utility representation is applied to a representative-agent economy. Euler equations show that the marginal rate of substitution is underweighted in good states and overweighted in bad states. This intertemporal substitution mechanism effectively boosts relative risk aversion over tomorrow's consumptions (which also explains the equity premium and risk-free rate puzzles). I also run empirical tests using UK data. The estimates from Euler equations show that the discount factor is lower when consumption growth is positive and higher when consumption growth is negative. Thus, estimated discount factors vary in a manner consistent with time-variability aversion.

Chapters 2 and 3 concern uncertainty aversion as captured by the multiple prior model of Gilboa and Schmeidler (1989). Chapter 2 provides the conditions under which the no-trade theorem of Milgrom & Stokey (1982) holds for an economy of agents whose preferences follow the multiple prior representation. I first investigate individual behavior, and derive the conditions under which agents' preference relations satisfy dynamic consistency with respect to their private information described by the partition of states (or the filtration). The main result is the converse of the proposition in Sarin & Wakker (1998): Given the agents' knowledge of the filtration, dynamic consistency and consequentialism imply that a set of ex-ante priors must satisfy the recursive structure. In addition, each conditional preference must be in the class of multiple prior preferences, and the set of priors must be updated by the Bayes rule point-wise. Second, I examine the maintained assumption of the knowledge of filtrations and study the conditions required for the no-trade theorem to hold. The requirements under which agents stay at the ex-ante Pareto optimal allocations are as follows: (1) All agents have a set of filtrations as their ex-ante knowledge of potential ex-post private information; (2) All agents' preference relations satisfy dynamic consistency and consequentialism with respect to all filtrations in their ex-ante knowledge sets; (3) Ex-post information is one of the filtrations in their ex-ante knowledge set. As opposed to the subjective prior model, agents who follow the multiple prior model need to know the structure of their ex-post information.

Chapter 3 examines risk-sharing among agents who are uncertainty averse, which causes them to behave as though they had multiple priors. Formally, I consider a general equilibrium model of dynamically complete markets. I first consider the case where each agent has the same set of multiple priors, i.e., each agent faces the same uncertainty. Under a weak condition on an aggregate endowment process, I confirm that the previously know result that a convex capacity is a sufficient condition to achieve full insurance, that is, all agents' consumptions are comonotonic (increasing together) with the aggregate endowment and their marginal rates of substitution are equalized. Given the convex capacity, agents's 'effective' prior need to be equalized and the model reduces to the standard common single-prior case. I then consider the case where agents have heterogeneous multiple prior sets. In this case, I provide conditions such that agents' effective priors (and equilibrium consumptions) will be comonotonic and their marginal rates of substitution (weighted by these priors) will be equalized. One set of sufficient conditions is for each agent's multiple prior set to be symmetric (or to be defined by a convex capacity) around the center of the simplex.

Chapter 2

A Model of Consumption Smoothing with an Application to Asset Pricing

2.1 Introduction

Conventionally, risk aversion is regarded as the dislike of variations in payoffs of random variables within a period. By contrast, *time-variability* is variation in payoffs over time. Historically, attitude toward time-variability has gained less attention in economics because a discounted utility representation with concave von Neumann-Morgenstern utility functions already implies a preference for consumption smoothing over time. However, time-preference is highly complex. For example, Loewenstein and Thaler (1989) show that discount rates for gains are much higher than for losses. Loewenstein and Prelec (1993) show in experiments that agents prefer smooth allocations over time even under certainty, and their preferences for smoothing cannot be explained by a time-separable discounted utility representation.

The purpose of this paper is to develop a new behavioral notion, time-variability aversion, and then apply this idea to a consumption-saving problem to derive implications for asset pricing. First we define time-variability aversion to mean that an agent is averse to mean-preserving spreads of utility over time. This idea is captured axiomatically and transformed into a non-time-separable utility representation that separates time-variability aversion from risk aversion. Second, we apply this utility representation under uncertainty, and solve asset pricing equations for a representative-agent economy. The resulting Euler equations are applied to a simple numerical example where our formula can explain the equity-premium and risk-free-rate puzzles.[1] Third, we use UK data to test whether or not

[1] Mehra and Prescott (1985) argue that under the rational expectation hypothesis, the coefficient of the relative risk aversion must be very high to explain the ex-post risk premium in the US stock markets (the

our utility representation is empirically supported.

In the representation, risk aversion is captured by the concavity of a von Neumann-Morgenstern utility function. Time-variation aversion is captured by the agent selecting a sequence of (normalized) discount factors from a given set that minimizes the present discounted value of a given payoff stream. I provide an axiomatization for this representation by adapting a method developed in a different context by Gilboa and Schmeidler (1989). More formally, the assignment of discount factors is determined recursively. At each time t, the agent compares present consumption with the discounted present value of future consumption from t+1 onward and then selects the time-t discount factor to minimize the weighted sum of these two values. These recursive preferences are dynamically consistent by construction. Intuitively, this representation exhibits time-variability aversion by allocating a high discount factor when tomorrow's consumption is low (and vice versa).

To apply this notion under uncertainty, an agent first considers time-variability aversion on a state-by-state basis and then aggregates discounted utility indices on each state with probability weights. Again, this operation is applied recursively, and discount factors depend on *tomorrow's* states. When the derived utility representation is applied to a representative-agent economy, the Euler equations show that the marginal rate of substitution is underweighted in good states, and overweighted in bad states.[2] This intertemporal

equity-premium puzzle). Weil (1989) also points out that under the very high relative risk aversion, the discount factor must be more than one to be consistent with the growth rate in per capita consumption, and covariance between this growth rate and stock returns (the risk-free-rate puzzle).

[2]Our formula involves indeterminacy of asset prices if one of future consumptions is equal to current one.

substitution mechanism effectively boosts relative risk aversion over tomorrow's consumptions and increases the agent's demand for bonds over stocks. This intuition is then applied to a simple numerical example of a two-period economy under which the risk-free rate and first and second moments of the equity premium are matched to those in the empirical data of Campbell, Lo and Mackinlay (1997). For this simple example, the utility representation that incorporates time-variability aversion resolves the equity-premium and risk-free-rate puzzles. To confirm whether time-variability aversion is an observed phenomenon, I also run empirical tests using UK data.[3,4] The estimates from Euler equations show that a discount factor is lower when consumption growth is positive and higher when consumption growth is negative. Thus, estimated discount factors vary in a manner consistent with time-variability aversion.

Historically, there are three lines of attempts to define attitudes toward time-variability. The first approach suggested by Epstein and Zin (1989) is to consider intertemporal substitution by a recursive aggregator function that has present utility and a continuation value as arguments.[5] In their model, an agent first considers risk aversion and then considers

The most general form of asset pricing necessarily involves inequalities to incorporate this indeterminacy. However, in a finite economy, we can focus on consumptions that do not involve any ties. See Section 5-1.

[3]The most rigorous tests must use lifetime consumption data to evaluate the evolution of discount factors.

[4]The reason we select the UK data is that the distribution of per capita consumption growth seems to be close to stationary.

[5]Koopmans (1960) utilizes an aggregator function for a certain consumption stream. Kreps and Porteus (1978) examine issues under uncertainty and derive an aggregator function. Duffie and Epstein (1992) apply

intertemporal substitution. By contrast, in our representation, an agent first considers intertemporal substitution and then considers risk. This reverse ordering requires preference relations to be defined on a slightly enlarged act space.[6]

The second approach is to define utility on differences of consumptions over time: for example, the behavioral models of Kahneman and Tversky (1979) and Loewenstein and Prelec (1992, 1993) and the habit-formation model of Constantinides (1990). These models involve status quo preference with some notion of gain/loss asymmetry. Our utility representation is based only on aversion to fluctuations of payoffs over time but it also captures a notion similar to status quo preference and gain/loss asymmetry without being dependent on a historical habit level. For an axiomatic approach, Gilboa (1989) applies the non-additive prior model of Schmeidler (1989) over time and derives a utility representation that depends on the difference between adjacent consumptions. Shalev (1997) extends the Gilboa's results to incorporate non-symmetric weights to evaluate the gap between adjacent consumptions. Our formula is different in two ways. First, we use a recursive structure so that an agent compares present consumption with a discounted value of all future consumption. Second, our formula guarantees dynamic consistency whereas their models involve dynamic inconsistency.[7]

The third approach is to derive state dependent discount factors under an additively

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the approach by Epstein and Zin (1989) to a continuous time setting.

[6]See Section 2.4 and 2.7.

[7]Sarin and Wakker (1998) and Grant, Kajii and Polak (2000) show that the non-additive prior model cannot be defined under a recursive structure. See Section 2.3.

separable framework. In a discrete-time setting, Epstein (1983) derives a model under which discount factors depend on the level of consumptions up to the current date. In a continuous-time deterministic setting, Uzawa (1968) models a similar utility function. Shi and Epstein (1993) develop time-varying discount factors that depend on a historical habit level. The main departure of our formula from others is to incorporate explicit time-variability aversion over periods, which is a forward looking behavior and generates a non-differentiable shift of discount factors.

In terms of empirical implications, our model shares qualitative features with habit formation, loss aversion and uncertainty aversion: time-variability aversion effectively changes risk aversion over tomorrow's states. However, the main advantage of our model comes from the theoretical aspect: it is based on more parsimonious axioms and the interpretation of empirical results is straight forward. In addition, to distinguish these models, we can find alternative tests. First, for habit formation, we can test whether or not the present utility depends on a habit level. Second, for loss aversion, a desirable test is to investigate whether an agent only considers tomorrow's value or considers all future values. The difference between our model and the uncertainty aversion can be tested by a carefully framed experiment.

The paper proceeds as follows. In Section 2.2, we provide an overview of the paper. In Section 2.3, we axiomatize the notion of time-variability aversion under certainty and derive the utility representation with multiple discount factors. In Section 2.4, we extend the representation with time-variability aversion under uncertainty. In Section 2.5, we derive equilibrium asset pricing equations, and apply them to a simple numerical example to show that our model can explain the equity-premium and risk-free-rate puzzles. In addition, we provide empirical tests of our model using UK data. In Section 2.6, we compare our model with other intertemporal utility functions. In Section 2.7, we provide axioms that derive the utility representation with multiple discount factors under uncertainty. In Section 2.8, we discuss our conclusion and future avenues of research.

2.2 Time-Variability vs. Atemporal Risk

In this section, we define the notion of time-variability aversion and provide an overview of the utility representation we are going to develop. Suppose that an agent faces a decision problem in a two-period economy under certainty. Assume that there is a utility function $U(x_0,x_1)$ that represents the agent's tastes. For example, we then use the discounted utility representation:

$$U(x_0,x_1) = u(x_0) + \delta u(x_1) \tag{2.2.1}$$

This formula express impatience by $0< \delta < 1$, and captures a desire for consumption smoothing by the concavity of $u(.)$. However, as we mentioned in the introduction, intertemporal preferences do not seem to follow a time-separable representation. The limitation becomes clearer once we introduce uncertainty. Suppose that there are S states of nature tomorrow. Under the subjective prior model (or expected utility theory), an agent's preference is expressed by a utility representation:

$$(2.2.2) \qquad \mathrm{E}[U(x_0,x_{1,s})] = \sum_{s=1}^{S} \pi_s U(x_0,x_{1,s})$$

where π_s stands for the prior for state s. Now, if we apply (2.2.1) for (2.2.2):[8]

$$(2.2.3) \qquad \mathrm{E}[U(x_0,x_{1,s})] = u(x_0) + \delta \sum_{s=1}^{S} \pi_s u(x_{1,s})$$

By the standard argument, the preference for consumption smoothing over states is expressed by the concavity of u (atemporal risk aversion), which is identical to the preference for consumption smoothing over time. However, the preference for smoothing over time expresses an attitude toward intertemporal substitution under certainty whereas the preference for smoothing over states expresses an attitude toward atemporal substitution under uncertainty. It is an artifact of the model that these two notions become identical.

In this paper, we return to a formula in (2.2.2). Our representation takes the following form:

$$(2.2.4) \qquad \mathrm{E}[U(x_0,x_{1,s})] = \sum_{s=1}^{S} \pi_s W(u(x_0),u(x_{1,s}))$$

where W is a non-time-separable aggregator function over current and future utilities. Atemporal risk attitude is expressed by characteristics of $u(.)$, and intertemporal attitude toward *time-variability* (by which we mean fluctuation of $u(.)$ over time) is expressed by W. An agent first considers intertemporal substitution and then considers risk. This operation is the reverse of the order in the model suggested by Epstein and Zin (1989).

In the next section, we axiomatically derive a particular form of W as a functional representation of discount factors. In Section 2.4, we discuss the application of W under

[8] In this case, (2.2.1) is considered as a von Neumann-Morgenstern utility function.

uncertainty. From now on, *time-preferences* refers to the structure of W (movement of discount factors) that incorporates time-variability aversion. The attitude toward atemporal risk will be called *risk-preferences.* We use the term *intertemporal preferences* to denote overall preference relations either under certainty or under uncertainty. Intertemporal preferences consist of time-preferences, risk-preferences and subjective priors.

2.3 Multiple Discount Factors under Certainty

2.3.1 Multiple Discount Factors: Examples

In this subsection, we provide a simple example that motivates our particular representation. Suppose that an agent faces a intertemporal decision problem of a two-period economy under certainty. The agent has three choices; a sequence that yields a utility of 2 in each period; a sequence that yield a utility of 1 followed by a utility of 3; and a sequence that yields a utility of 3 followed by a utility of 1:

Sequence 1. $s^1 = (u_0,u_1) = (2,2)$

Sequence 2. $s^2 = (u_0,u_1) = (1,3)$

Sequence 3. $s^3 = (u_0,u_1) = (3,1)$

For any agent with preferences of the form of $u_0 + \delta u_1$, the agent will strongly prefer s^2 or s^3 to s^1 (unless $\delta = 1$ in which case she is indifferent between all three.). However, an agent who is averse to time-variability might prefer s^1 to s^2 or s^3 because s^2 hedges the movement of s^3, and s^1 is a mixture of s^2 and s^3. To capture this notion, suppose that preferences between three sequences are expressed by:

$$s^2 \simeq s^3 \qquad \text{but} \qquad s^1 = \frac{1}{2}s^2 \oplus \frac{1}{2}s^3 \succ s^3$$

One way to express these preference relations is to assume the following representation of discount factors:

$$U(s) = \text{Min}_{\delta \in \Delta}[(1-\delta)u_0 + \delta u_1] \quad \text{with } \Delta = [0.3, 0.7]$$

Then the value of each sequence becomes:

Sequence 1. $\quad U(s^1) = 2.0, \;\; \delta \in [0.3, 0.7].$

Sequence 2. $\quad U(s^2) = 1.6, \;\; \delta = 0.3.$

Sequence 3. $\quad U(s^3) = 1.6, \;\; \delta = 0.7.$

For the sequences 2 and 3 (uneven), the fluctuation of atemporal utilities over time decreases the overall value. By assigning a higher discount factor for $u_t = 1$ and a lower discount factor for $u_{t'} = 3$, an agent shifts relative time-preferences from t' to t, which gives her a strong incentive to move consumptions from $u_t = 3$ to $u_t = 1$. By achieving complete smoothing, an agent can improve her overall utility level. Since this representation involves a set of discount factors, we define this representation as a *multiple discount factors* model.

Note that any strictly concave function of u_1 and u_2 can represent the preference relations in this example. However, our formula has three advantages. First, it is based on very simple axioms, so we can easily understand *why* an agent follows our model. The advantage of an axiomatic approach becomes more evident in the derivation of the representation under uncertainty in Section 2.7. Second, interpretation of time-preferences is direct; we model discount factors themselves. Since our formula becomes a weighted summation of atemporal utilities at an effective selection of discount factors, the departure from the discounted utility model is minimal. Our model shares the tractability of the discounted utility model. Third, in addition to the preference for smoothing, our formula also captures the notion of gain/loss asymmetry. For example, the effective selection of discount factors is 0.3 for the sequence 2 and 0.7 for the sequence 3. If we consider the difference in consumptions to be gains and losses, the non-differentiable shift of discount factors at $u_0 = u_1$ can explain the asymmetric attitude toward gains and losses. This result becomes crucial for explaining asset pricing.

2.3.2 Representation of Intertemporal Preferences

In this subsection, we derive a utility representation with multiple discount factors under certainty. To separate time-variability aversion from risk aversion, we define preference relations over sequences of *consumption lotteries* by adapting the Anscombe-Aumann (1963) framework with a temporal interpretation. Let X be a set of outcomes, and Y be a set of probability distributions over X that satisfies:

$Y = \{y|\ y\colon\ X \to [0,1]$ where y has a finite support.$\}$

For convenience, we call $y \in Y$ a lottery and Y a lottery space. Let $\mathfrak{T} = \{0,1,...,T\}$ be a finite set of periods from 0 to T and Σ be the algebra on $\mathfrak{T}$.[9] Let f be an act where f: $\mathfrak{T} \to Y$, and h be a constant act that assigns identical $y \in Y$ for all $t \in \mathfrak{T}$ denoted as y. Define $\mathfrak{A}$ as a collection of all f, and $\mathfrak{A}_c$ as a collection of all constant acts. We also define the following operation: $[\alpha f \oplus (1-\alpha)g](t)$=$\alpha f(t) + (1-\alpha)g(t)$. In addition, let $f_t = f(t) \in Y$. Now, we assume that the following axioms hold for acts in $\mathfrak{A}$:

Axiom 2.3.1: Weak Order

$\forall f, g, h \in \mathfrak{A}$, (i) $f \succeq g$ or $g \succeq f$ (ii) $f \succeq g$ and $g \succeq h \Rightarrow f \succeq h$.

Axiom 2.3.2: Continuity

$\forall f, g, h \in \mathfrak{A}$ with $f \succ g \succ h$, $\exists 0 < \alpha$, β <1

s.t. $\alpha f \oplus (1-\alpha)h \succ g$ and $g \succ \beta f \oplus (1-\beta)h$.

Axiom 2.3.3: Strict Monotonicity

$\forall f, g \in \mathfrak{A}$ s.t. $f = (y_1,...,y_T)$ and $g = (y_1',...,y_T')$, if $y_t \succeq {y_t}' \forall t \in \mathfrak{T}$ then $f \succeq g$

In addition, if for some t, $y_t \succ {y_t}'$ then $f \succ g$.

Axiom 2.3.4: Nondegeneracy

$\exists f, g \in \mathfrak{A}$ s.t. $f \succ g$.

Axiom 2.3.5: Constant-Independence[10]

$\forall f, g \in \mathfrak{A}$ and $\forall h \in \mathfrak{A}_c$, $\forall \alpha \in (0,1)$, $f \succ g \Leftrightarrow \alpha f \oplus (1-\alpha)h \succ \alpha g \oplus (1-\alpha)h$.

[9] The result may be extended to an infinite horizon by using the extension theorem in Gilboa and Schmeidler (1989).

[10] It is called certainty-independence in Gilboa and Schmeidler (1989).

Axiom 2.3.6: Time-Variability Aversion[11]

$\forall f, g \in \mathfrak{A}$ and $\forall \alpha \in (0,1)$, $f \simeq g \Rightarrow \alpha f \oplus (1-\alpha)g \succeq f$.

The key axioms are Axioms 2.3.5 and 2.3.6. To understand the significance, we compare them with the independence axiom in Anscombe and Aumann (1963) (for all $f, g, h \in \mathfrak{A}$ and for all $\alpha \in (0,1)$, $f \succ g \Leftrightarrow \alpha f \oplus (1-\alpha)h \succ \alpha g \oplus (1-\alpha)h$). Under this axiom, the example in the previous subsection becomes:

$$(1{,}3) \sim (3{,}1) \Rightarrow (2{,}2) \sim (1{,}3) \sim (3{,}1)$$[12]

Clearly, the independence axiom is too strong to admit time-variability aversion. On the other hand, under Axioms 2.3.5:

$$(1{,}3) \sim (3{,}1) \Rightarrow \frac{1}{2}(1{,}3) \oplus \frac{1}{2}(5{,}5) \sim \frac{1}{2}(3{,}1) \oplus \frac{1}{2}(5{,}5) \Rightarrow (3{,}4) \sim (4{,}3)$$

Under this limited independence axiom, the relative difference between (1,3) and (3,1) are not altered among (3,4) and (4,3). Time-variability determines preference ordering, and the shift of a utility level does not change the preference ordering. This feature resembles the characteristics of the reference relations based on differences from a reference point. In addition, time-variability aversion expresses the desire to smooth allocations over time that is analogous to the definition of atemporal risk aversion. Under Axiom 2.3.6 with strict inequality:

[11] It is called uncertainty aversion in Gilboa and Schmeidler (1989).

[12] $0.5(1{,}3) \oplus 0.5(3,1) = (0.5 \cdot 1 + 0.5 \cdot 3, 0.5 \cdot 3 + 0.5 \cdot 1) = (2{,}2)$. All numbers are considered to be utils.

(2,2) $\succ$ (1,3) $\sim$ (3,1)

Clearly, the mixture is better than the original. Hedging the movement of atemporal utility indices over time increases overall utility.

Gilboa and Schmeidler (1989) have proved that the above axioms imply the following representation of preference relations over $\mathfrak{A}$:

Theorem 2.3.1: Adaptation of Gilboa and Schmeidler (1989)[13]

A binary relationship on $\mathfrak{A}$ satisfies Axioms 3-1-1 to 3-1-6 if and only if there exists a non-empty, closed and convex set of finitely additive discount factors on Σ, Δ^0, with $\sum_{t=0}^{T} \delta_t = 1$ and $\delta_\tau > 0$ $\forall 0 \leq \tau \leq T$ such that:

(2.3.1) $\quad \forall f, g \in \mathfrak{A},\ f \succeq g \Leftrightarrow U_0(f) \geq U_0(g)$

where $U_0(f) \equiv \min_{\delta \in \Delta^0} \sum_{t=0}^{T} \delta_t u(f_t)$

Moreover, under these conditions, Δ^0 is unique and u: $Y \rightarrow R$ is a unique up to a positive affine transformation.[14]

Under Axioms 2.3.1 to 2.3.3, the representation becomes $W(u(f_0),...,u(f_T))$, and then Axioms 2.3.5 and 2.3.6 determine the structure of W. Under the representation of (2.3.1), time-variability aversion is captured by the agent selecting discount factors to minimize the weighted sum of atemporal von Neumann-Morgenstern utility indices. Attitude toward risk

[13] We call propositions proved by other authors theorems.

[14] The preference relations over Y is defined by the following way as is defined in monotonicity: $h_t \succeq h'_t \Leftrightarrow h \succeq h'$ s.t. h, $h' \in \mathsf{A}_c$. This relationship is represented by the utility function itself, i.e., $h_t \succeq h'_t \Leftrightarrow \min\sum_{t=1}^{T} \delta_t u(h_t) \geq \min\sum_{t=1}^{T} \delta_t u(h'_t)$, and $u(h_t)$ is defined by $\min\sum_{t=1}^{T} \delta_t u(h_t) = u(h)$.

is expressed by a von Neumann-Morgenstern utility function $u(.)$.[15] In terms of (2.2.4), we derive W for an entire stream of consumption lotteries and (2.3.1) becomes non-time-separable. In fact, time-variability aversion is independent of the structure of $u(.)$, which can be concave or convex. In addition, some point $\widehat{\delta} \in \Delta^0$ can be regarded as a baseline time-preference to calculate the net present value of von Neumann-Morgenstern utility indices in absence of time-variability aversion.

However, if we apply (2.3.1) for more than two-periods, we face dynamic inconsistency. To resolve this difficulty, we need to apply the multiple discount factors recursively. Let $\mathfrak{T}_t$ be a finite set of periods from time t to T and $\mathfrak{T}_{-t}$ be a finite set of periods from time 0 to $t-1$. Define f^t as a function: $f^t : \mathfrak{T}_t \to Y$ and f^{-t} as a function: $f^{-t} : \mathfrak{T}_{-t} \to Y$. If $\mathfrak{T}_{-t}$ is empty, f^t defines an act f and vice versa. Preference relations on $\mathfrak{A}$ conditional on time t is denoted by $\succeq_t$. A collection of all conditional preference relations $\{\succeq_t\}$ on $\mathfrak{A}$ follows additional axioms:

Axiom 2.3.7: Independence of History up to $t-1$

$f = (a^{-t}, f^t)$, $g = (b^{-t}, g^t)$, $f' = (c^{-t}, f^t)$, $g' = (d^{-t}, g^t)$.

Then $f \succeq_t g \Leftrightarrow f' \succeq_t g'$.

Axiom 2.3.8: Dynamic Consistency

$\forall f = (a^{-t}, y_t, f^{t+1}), g = (a^{-t}, y_t, g^{t+1}) \in \mathfrak{A}$, $f \succeq_t g \Longleftrightarrow f \succeq_{t+1} g$.

[15] Note that $u(f_t) = \sum_{s=1}^{S} p_s u(f_{t,s})$. Literally, f_t is a consumption lottery.

Given the above axioms, (2.3.1) needs to be rewritten by the following form:[16,17]

Proposition 2.3.1:

Suppose that the agent's preference relations on $\mathfrak{A}$ satisfy Axioms 2.3.1 to 2.3.6 at time 1 and let U_0 and Δ^0 be as in Theorem 2.3.1. Then a binary relationship $\{\succeq_t\}$ on $\mathfrak{A}$ satisfies Axioms 2.3.7 to 2.3.8 if and only if there exist $\{[\alpha_t,\beta_t]\}_{1\leq t\leq T}$ such that:

(2.3.2) $\forall t, \forall f, g \in \mathfrak{A}$,

$f \succeq_t g \Leftrightarrow U_t(f) \geq U_t(g)$

where $\{U_t(f)\}_{0\leq t\leq T}$ are recursively defined by:

$U_t(f) \equiv \min_{\delta_{t+1}\in[\alpha_{t+1},\beta_{t+1}]}[(1-\delta_{t+1})u(f_t)+\delta_{t+1}U_{t+1}(f)]$

and $U_T(f) \equiv u(f_T)$

(2.3.3) $0 < \alpha_t \leq \beta_t < 1$ $\forall t$ s.t. $1\leq t \leq T$

Moreover:

---

[16] Eichberger and Kelsey (1996) utilize Machina (1989)'s notion for examining a dynamically consistent updating rule for the non-additive prior model of Schmeidler (1989). They show that if agent's preference satisfies strict uncertainty aversion, a dynamically consistent update rule does not produce the conditional preference that confirms the non-additive prior model. Wakai (2001) also show that an identical result holds for the multiple priors model.

[17] Wakai (2001) shows this result in a original formulation of Gilboa and Schmeidler (1989). Epstein and Schneider (2001) recursively use Axioms 2.3.1 to 2.3.6 for conditional preference relations, and derive similar conclusion. Sarin and Wakker (1998) also show that a recursive multiple priors are dynamically consistent. In addition, Wakai (2001) shows that under the assumption of sequential consistency of Sarin and Wakker (1998) and dynamic consistency, the recursive multiple priors is necessary and sufficient to generate consequentialism.

(2.3.4) $[\alpha_t,\beta_t]$ is uniquely defined.

Proof:

See Appendix 2.A:

Given dynamic consistency, $W_t(u(f_t),...,u(f_T))$ becomes $W_t(u(f_t), U_{t+1}(f))$, which is time-dependent and recursive. Dynamic consistency also contributes to one distinct feature: gain/loss asymmetry. More specifically, to avoid time-variability, an agent assigns a higher discount factor for the discounted present value of future utility from t+1 onward when it is lower than the utility of present consumption (and vice versa). An increase from the present utility requires a lower discount factor, and a decrease from the present utility requires a higher discount factor. However, (2.3.2) and loss aversion of Kahneman and Tversky (1979) are different. Formula (2.3.2) considers all future prospects to compare with a present reference level. The loss aversion only compares a future value at t+1 with a present reference level. In addition, formally, Formula (2.3.2) does not assume the existence of a reference point nor gain/loss asymmetry. An agent who is averse to time-variability will smooth consumptions over time by simply comparing two numbers (which makes one number as a reference point).[18] This difference should be clear because u does not include a reference point.

Finally in this subsection, we define time-variability-seeking by reversing the inequality

[18] Gains and losses from a reference point can only be defined by comparing two numbers.

in Axiom 2.3.6:

Axiom 2.3.9: Time-Variability-Seeking

$\forall f, g \in \mathfrak{A}$ and $\forall \alpha \in (0,1)$, $f \simeq g \Rightarrow \alpha f \oplus (1-\alpha) g \preceq f$

Proposition 2.3.2:

A binary relationship $\{\succeq_t\}$ on $\mathfrak{A}$ satisfies Axioms 2.3.1 to 2.3.8 by replacing Axiom 2.3.6 with 2.3.9 if and only if there exist $\{[\alpha_t, \beta_t]\}_{1 \leq t \leq T}$ such that:

(2.3.5) $\forall t, \forall f, g \in \mathfrak{A}$,

$f \succeq_t g \Leftrightarrow U_t(f) \geq U_t(g)$

where $\{U_t(f)\}_{0 \leq t \leq T}$ are recursively defined by:

$U_t(f) \equiv \max_{\delta_{t+1} \in [\alpha_{t+1}, \beta_{t+1}]}[(1-\delta_{t+1})u(f_t) + \delta_{t+1} U_{t+1}(f)]$

and $U_T(f) \equiv u(f_T)$

(2.3.6) $0 < \alpha_t \leq \beta_t < 1$ $\forall t$ s.t. $1 \leq t \leq T$

Moreover:

(2.3.7) $[\alpha_t, \beta_t]$ is uniquely defined.

(2.3.8) u: $Y \rightarrow R$ is a unique up to a positive affine transformation.

Proof:

See Appendix 2.A:

Given the above construction, we consider the discounted utility representation to be time-variability neutral.

2.3.3 Interpretation of Discount Factors

In this subsection, we compare an effective selection of discount factors from (2.3.2) with discount factors in the discounted utility model. First, the discounted utility model is:

$$U_0(f) = \sum_{t=0}^{T} \delta^t u(f_t)$$

On the other hand, Formula (2.3.2) is rewritten by using the effective selection of discount factors for a given consumption stream:

$$\delta^*_{t+1} \in \text{argmim}_{\delta_{t+1} \in [\alpha_{t+1}, \beta_{t+1}]}[(1-\delta_{t+1})u(f_t) + \delta_{t+1}U_{t+1}(f)]$$

and

$$\begin{aligned}
U_0(f) &= [(1-\delta^*_1)u(f_0) + \delta^*_1 U_1(f)] \\
&= (1-\delta^*_1)[u(f_0) + \frac{\delta^*_1}{(1-\delta^*_1)}U_1(f)] \\
&= (1-\delta^*_1)[u(f_0) + \frac{\delta^*_1}{(1-\delta^*_1)}[(1-\delta^*_2)u(f_1) + \delta^*_2 U_2(f)] \\
&= (1-\delta^*_1)[u(f_0) + \frac{\delta^*_1(1-\delta^*_2)}{(1-\delta^*_1)}u(f_1) + \frac{\delta^*_1\delta^*_2}{(1-\delta^*_1)}U_2(f)] \\
&= (1-\delta^*_1)[\widehat{\delta}_0 u(f_0) + \widehat{\delta}_1 u(f_1) + \widehat{\delta}_2 u(f_2) + ... + \widehat{\delta}_T u(f_T)]
\end{aligned}$$

Hence, a normalized discount factor between adjacent time periods becomes:

$(t,t+1)$ $(0 \leq t < T)$: $$[1, \frac{\widehat{\delta}_{t+1}}{\widehat{\delta}_t}] = [1, \frac{\delta^*_{t+1}(1-\delta^*_{t+2})}{(1-\delta^*_{t+1})}] \text{ where } \delta^*_{T+1} \equiv 0$$

If it is normalized at time 0:

at t $(1 \leq t \leq T)$: $$\widehat{\delta}_t = \frac{\delta^*_1 ... \delta^*_t(1-\delta^*_{t+1})}{(1-\delta^*_1)} \text{ where } \delta^*_{T+1} \equiv 0$$

Discount factors in our formulation have three roles. First, it re-normalizes the level of utility from time t+1 onward to a level at time t, which makes the comparison possible. Second, it reflects the agent's base-line time-preference between two dates (roughly $\frac{\widehat{\delta}_{t+1}(1-\widehat{\delta}_{t+2})}{(1-\widehat{\delta}_{t+1})}$ for some $\widehat{\delta}_{t+1} \in [\alpha_{t+1},\beta_{t+1}]$ and $\widehat{\delta}_{t+2} \in [\alpha_{t+2},\beta_{t+2}]$). Third, it expresses time-variability aversion. By the first property, discount factors at each time must add up to one to make $U_t(f_t,...,f_T) = u(f_t)$ if all f_τ are identical for $t \leq \tau \leq T$. $U_t(f_t,...,f_T)$ also summarizes time-variability of future consumption. If there is a fluctuation in $(f_t,...,f_T)$, $U_t(f_t,...,f_T) \leq U_t(\overline{f},...,\overline{f})$ where $\overline{f}$ is the net present value of $(f_t,...,f_T)$ under a base-line time-preference that does not involve time-variability aversion. Clearly, an agent does not prefer time-variability. For this reason, $U_t(f_t,...,f_T)$ can be regarded as a *time-variability-adjusted* present discounted value of future consumption.

2.3.4 Application of (2.3.2) to a Consumption-Saving Problem under Certainty

To analyze the implications of (2.3.2), we restrict our attention to a space of degenerate consumption lotteries. Suppose that an agent faces a two-period decision problem in a partial equilibrium setting. Assume that an agent follows (2.3.2). We consider two alternatives under which the agent's attitude toward risk is different:

Case 1: Time-variability aversion and risk aversion

Max $_{x\in B}\min_{\delta\in[0.2,0.8]}[(1-\delta)u(c_0)+\delta u(c_1)]$ with a concave u

$B = \{(c_0,c_1)|\ p_0c_0 + p_1c_1 = I \text{ and } c_0, c_1 \in R_+\}$

Relative price	$\frac{p_1}{p_0} < \frac{0.2}{0.8}$	$\frac{0.2}{0.8} \leq \frac{p_1}{p_0} \leq \frac{0.8}{0.2}$	$\frac{0.8}{0.2} < \frac{p_1}{p_0}$
Allocations	$c_0 < c_1$	$c_0 = c_1$	$c_0 > c_1$

In this case, for a wide range of relative prices (i.e., interest rates), an agent does not want to move consumptions away from an even allocation. This result reflects gain/loss asymmetry implied in multiple discount factors.

Case 2: Time-variability aversion and risk-seeking

Max $_{x \in B}\min_{\delta \in [0.2,0.8]}[(1-\delta)u(c_0) + \delta u(c_1)]$ with $u(c) = c^2$

$B = \{(c_0,c_1)|\ p_0c_0 + p_1c_1 = I$ and $c_0, c_1 \in R_+\}$

Relative price	$\frac{p_1}{p_0} < \frac{\sqrt{0.8}}{2-\sqrt{0.8}}$	$\frac{\sqrt{0.8}}{2-\sqrt{0.8}} \leq \frac{p_1}{p_0} \leq \frac{2-\sqrt{0.8}}{\sqrt{0.8}}$	$\frac{2-\sqrt{0.8}}{\sqrt{0.8}} < \frac{p_1}{p_0}$
Allocations	$c_0 = 0;\ c_1 = \frac{I}{p_2}$	$c_0 = c_1$	$c_0 = \frac{I}{p_1};\ c_1 = 0$

Note that if an agent is time-variability neutral, a risk-seeking agent always allocates all consumption at one of two periods. However, under very high time-variability aversion implied by a wide range of discount factors, even for the risk-seeking agent, optimal allocations become even for a wide range of relative prices. This example indicates that time-variability aversion is a different notion from atemporal risk aversion. We can also apply a similar construction to the case where an agent is time-variability-seeking. In this case, a risk-averse agent never prefers even allocations.

2.4 Multiple Discount Factors under Uncertainty

2.4.1 Representation of Intertemporal Preferences under Uncertainty

In this subsection, we define the utility representation of multiple discount factors under uncertainty. In the most naive way, we can apply (2.3.2) to an objective probability space of consumption streams. However, this application is not dynamically consistent even though (2.3.2) is dynamically consistent under certainty.[19] To resolve this problem, we need to define preference relations recursively over a state space.

The economy has the following structure. Define $\mathfrak{T} = \{0,1,...,T\}$ as a finite set of periods from 0 to T. At each time after time 0, there is a finite state space $\Omega = \{1,...,S\}$.[20] The entire state space becomes Ω^T, and $\omega^t = (\omega^{t-1},\omega) \in \Omega^t$ stands for a history of state realizations from time 1 to time t. We also define ω^{T-t} to be a path from time $t+1$ to time T so that $\omega^T = (\omega^t,\omega^{T-t})$. In addition, we write ω^T as $(\omega_1,...,\omega_T)$ where $\omega_t \in \Omega$ for $1 \leq t \leq T$. We assume that $\Omega^0 = \{\emptyset\}$, $\omega^0 = \omega_0 = \emptyset$, and $(\omega_1,...,\omega_T) = (\omega_0,\omega_1,...,\omega_T)$. A process $\{x_t\}_{0 \leq t \leq T}$ is a collection of functions x_t such that x_t: $\Omega^t \to R$ at each t. We define $x_t(\omega^t)$ as a value of x_t at ω^t.

As axiomatically derived in Section 2.7, an agent who follows time-variability aversion evaluates a consumption process $\{c_t\}_{0 \leq t \leq T}$ at (t,ω^t) by the following value process

[19] See Appendix 2-B.

[20] In Section 2.7, we derive the utility representation under a more general state setting using a filtration.

$\{V_t(c)\}_{0 \leq t \leq T}$:[21]

$$(2.4.1) \qquad V_t(c)(\omega^t)$$

$$\equiv \mathrm{E}_t[\mathrm{Min}_{\delta_{t+1}(\omega^t,\omega')\in[\alpha_{t+1},\beta_{t+1}]}(1-\delta_{t+1}(\omega^t,\omega'))u(c_t(\omega^t)) + \delta_{t+1}(\omega^t,\omega')V_{t+1}(c)(\omega^t,\omega')]$$

where $\omega' \in \Omega$ and $V_T(c)(\omega^T) \equiv u(c_T(\omega^T))$

with $0<\alpha_t \leq \beta_t < 1$ $\forall t$ s.t. $1\leq t \leq T$

$\mathrm{E}_t[.]$ and $[\alpha_t,\beta_t]$ is uniquely defined, $[\alpha_t,\beta_t]$ are independent of states.

u: $Y \rightarrow R$ is a unique up to a positive affine transformation.

The expectation is based on a subjective prior and α_t and β_t depend only on time. The crucial result is that an agent first considers intertemporal substitution on each tomorrow's state ω' and then aggregate utility indices across states with probability weights. Clearly, the selection of $\delta_{t+1}(\omega^t, \omega')$ depends on tomorrow's state ω'. Also $V_t(c)(\omega^t)$ depends only on a future payoffs of c, which implies history independence. This operation, $V_t(c)(\omega^t)$, is recursively applied. Note that if there are not fluctuations in payoffs over states ω^t at every point of time, (2.4.1) becomes (2.3.2) (i.e., $V_t(c)(\omega^t) = U_t(c)$).

Now we show by a simple two-period example that (2.4.1) captures time-variability aversion. Assume that there are two states in Ω and that (0.5,0.5) is a probability for (state 1, state 2). There are two contracts that pay consumption goods with the following utility at each time and state:

[21] We use an uncertain sequence of consumption lotteries as primitives in the derivation of (2.4.1) in Section 2.7.

Contract A

State\Time	t=0	t=1
ω_2=1	$u_1 = 4$	$u_{2,1} = 5$
ω_2=2	$u_1 = 4$	$u_{2,2} = 3$

Contract B

State\Time	t=0	t=1
ω_2=1	$u_1 = 4$	$u_{2,1} = 4$
ω_2=2	$u_1 = 4$	$u_{2,2} = 4$

We investigate three different preference relations. First, Agent 1 follows the discounted utility model:

$$(2.4.2) \qquad V_0(c)(\omega^0) = A[u_0 + \delta \mathrm{E}[u_{1,\omega}]] \ \text{ with } \delta = 0.9 \text{ and } A = \frac{1}{1.9}$$

Agent 2 follows (2.4.1):

$$(2.4.3) \qquad V_0(c)(\omega^0) = \mathrm{E}[\mathrm{Min}_{\delta\in\Delta}[(1-\delta)u_0 + \delta u_{1,\omega}]] \text{ with } \Delta = [0.3, 0.7]$$

In addition, to show that we need to apply time-variability aversion first before we consider risk, assume that Agent 3 follows:[22]

$$(2.4.4) \qquad V_0(c)(\omega^0) = \mathrm{Min}_{\delta\in\Delta}[(1-\delta)u_0 + \delta \mathrm{E}[u_{1,\omega}]] \text{ with } \Delta = [0.3, 0.7]$$

The difference between (2.4.3) and (2.4.4) is that the order of application of time-variability aversion is reversed. Equation (2.4.4) follows the model of Epstein and Zin (1989).

[22] Although we do not provide a proof, this representation can be axiomatized by a standard recursive argument.

Then, $V_0(c)(\omega^0)$ of Contract A and Contract B becomes:

	Contract A	Contract B
Discounted utility of (2.4.2)	4	4
Time-variability aversion of (2.4.3)	3.8	4
Time-variability aversion of (2.4.4)	4	4

Note that (2.4.2) and (2.4.4) achieve the identical results even though (2.4.4) incorporates time-variability aversion over time because (2.4.4) first aggregates the movement of payoffs over tomorrow's states and only considers time-variability in terms of risk-adjusted average payoffs. This example implies that to capture time-variability aversion more precisely, we need to consider intertemporal substitution before we consider risk. Under this key construction, variable allocations in Contract A decrease overall utility as we see in (2.4.3).

Next, we examine the connection between intertemporal substitution and risk aversion. In terms of (2.2.4), we can write (2.4.1) as:

$$V_t(c)(\omega^t) = \mathrm{E}_t[W_t(u(c_t(\omega^t)), V_{t+1}(c)(\omega^t, \omega'))]$$

At the effective choice of discount factors, W_t becomes linear. Then the concavity of a von Neumann-Morgenstern function captures risk aversion over tomorrow's states. However, the assignment of discount factors changes the effective risk attitude. If $V_{t+1}(c)(\omega^t, \omega')$ distributes over $\omega' \in \Omega$ around the today's $u(c_t(\omega^t))$, $W_t(u(c_t(\omega^t)), V_{t+1}(c)(\omega^t, \omega'))$ effectively generates higher risk aversion over tomorrow's states than u implies. As we saw in the numerical examples above, (2.4.3) has higher effective risk aversion for Contract B than (2.4.2) or (2.4.4) does.

Finally in this subsection, the representation for time-variability-seeking becomes:

(2.4.5) $$V_t(c)(\omega^t)$$

$$\equiv \mathrm{E}_t[\mathrm{Max}_{\delta_{t+1}(\omega^t,\omega')\in[\alpha_{t+1},\beta_{t+1}]}(1-\delta_{t+1}(\omega^t,\omega'))u(c_t(\omega^t))$$

$$+\ \delta_{t+1}(\omega^t,\omega')V_{t+1}(c)(\omega^t,\omega')]$$

where $\omega' \in \Omega$ and $V_T(c)(\omega^T) \equiv u(x_T(\omega^T))$

with $0<\alpha_t \leq \beta_t < 1\ \forall t$ s.t. $1\leq t \leq T$

$\mathrm{E}_t[.]$ and $[\alpha_t,\beta_t]$ is uniquely defined, $[\alpha_t,\beta_t]$ are independent of states.

u: $Y \to R$ is a unique up to a positive affine transformation.

Again, the discounted utility representation is considered to be time-variability neutral.

2.4.2 Interpretation of Discount Factors

In this subsection, we compare the effective selection of discount factors of (2.4.1) with those from the discounted utility model. First, the discounted utility model under uncertainty becomes:

$$V_t(c)(\omega^t) = \mathrm{E}_t[\textstyle\sum_{\tau=t}^{T}\delta^{\tau-1}u(c_\tau(\omega^\tau))]$$

$$= u(c_t(\omega^t)) + \mathrm{E}_t[\delta u(c_{t+1}(\omega^t,\omega')) + \mathrm{E}_{t+1}[\textstyle\sum_{\tau=t+2}^{T}\delta^{\tau-1}u(c_\tau(\omega^\tau))]]$$

$$= u(c_t(\omega^t)) + \delta\mathrm{E}_t[u(c_{t+1}(\omega^t,\omega')) + \mathrm{E}_{t+1}[\textstyle\sum_{\tau=t+2}^{T}\delta^{\tau-1}u(c_\tau(\omega^\tau))]]$$

Under this representation, an average normalized discount factor between adjacent time periods (i.e., $\mathrm{E}_t[\delta]$) is always δ at any point of time and state. On the other hand, (2.4.1) is rewritten by using the effective selection of discount factors for a given consumption stream:

$$\delta^*_{t+1}(\omega^t,\omega')$$
$$\in \arg\min \mathrm{E}_t[\mathrm{Min}_{\delta_{t+1}(\omega^t,\omega')\in[\alpha_{t+1},\beta_{t+1}]}(1-\delta_{t+1}(\omega^t,\omega'))u(c_t(\omega^t))$$
$$+\ \delta_{t+1}(\omega^t,\omega')V_{t+1}(c)(\omega^t,\omega')]$$

Then:

$$V_t(c)(\omega^t)$$
$$= \mathrm{E}_t[(1-\delta^*_{t+1}(\omega^t,\omega'))u(c_t(\omega^t))\ +\ \delta^*_{t+1}(\omega^t,\omega')V_{t+1}(c)(\omega^t,\omega')]$$
$$= \mathrm{E}_t[(1-\delta^*_{t+1}(\omega^t,\omega'))u(c_t(\omega^t))$$
$$+\ \delta^*_{t+1}(\omega^t,\omega')\mathrm{E}_{t+1}[(1\text{-}\delta^*_{t+2}(\omega^t,\omega',\omega''))u(c_{t+1}(\omega^t,\omega'))+\delta^*_{t+2}(\omega^t,\omega',\omega'')V_{t+2}(c)(\omega^t,\omega',\omega'')]]$$
$$= A_t(\omega^t)[u(c_t(\omega^t))$$
$$+\ \mathrm{E}_t[\frac{\delta^*_{t+1}(\omega^t,\omega')A_{t+1}(\omega^t,\omega')}{A_t(\omega^t)}u(c_{t+1}(\omega^t,\omega'))$$
$$+\ \mathrm{E}_{t+1}[\frac{\delta^*_{t+2}(\omega^t,\omega',\omega'')}{A_{t+1}(\omega^t,\omega')}V_{t+2}(c)(\omega^t,\omega',\omega'')]]]$$

where $A_t(\omega^t) = \mathrm{E}_t[(1-\delta^*_{t+1}(\omega^t,\omega'))]$ with $\delta^*_{T+1}(\omega^{T+1}) \equiv 0$ (i.e., $A_T(\omega^T) = 1$).

Hence, an average normalized discount factor (i.e., average time-preference) between adjacent time periods becomes:

$(t,t+1)$ $(0\leq t < T)$ at ω^t: $\quad [1,\mathrm{E}_t[\dfrac{\delta^*_{t+1}(\omega^t,\omega')A_{t+1}(\omega^t,\omega')}{A_t(\omega^t)}]]$

A discount factor at (t,ω^t) normalized at the level of time 0 becomes:

at (t,ω^t) $(1 \leq t \leq T)$: $$\frac{\delta_1^*(\omega^1)...\delta_t^*(\omega^t)A_t(\omega^t)}{A_0(\omega^0)}$$

First, an average normalized discount factor incorporates a global nature of a consumption process from time t onward and it is state dependent. This result contracts with a constant average normalized discount factor δ under the discounted utility model. In the next section, this result plays a crucial role in explaining asset pricing. Second, at each (t,ω^t), a normalized discount factor at the level of time 1 has a similar structure to the one under certainty; however, discount factors are not based on a particular consumption path on ω^t. They reflect the movement of the value process $\{V_t(c)\}$ that incorporates uncertainty implied in the evolution of states.

2.5 Implications for Asset Pricing under Multiple Discount Factors

2.5.1 Asset Pricing Equation

In this subsection, we apply the utility representation of (2.4.1) to a representative-agent economy to derive asset pricing equations. The economy has the same state structure as in the previous section. Let $\mathfrak{D}_{++} \in R_{++}$ be a compact subspace of R with positive elements and $\mathfrak{D}_{+} \in R_{+}$ be a compact subspace of R with non-negative elements. Let $e_t(\omega^t) \in \mathfrak{D}_{++}$ and $c_t(\omega^t) \in \mathfrak{D}_{+}$ be an endowment and consumption for the representative agent at time t on $\omega^t \in \Omega^t$. Assume that there are I assets with zero supply and let $d_t^i(\omega^t)$ and $q_t^i(\omega^t) \in \mathfrak{D}_{+}$ be a dividend and price for asset i at time t on $\omega^t \in \Omega^t$. Let $d_t(\omega^t) = (d_t^1(\omega^t), ..., d_t^I(\omega^t))$ and $q_t(\omega^t) = (q_t^1(\omega^t), ..., q_t^I(\omega^t)) \in \mathfrak{D}_{+}^I$ be a collection of asset dividends and prices at time

t on $\omega^t \in \Omega^t$. Let $\theta_t^i(\omega^t) \in R$ be a holding of asset i, and $\theta_t(\omega^t) = (\theta_t^1(\omega^t), ..., \theta_t^N(\omega^t))$ be a collection of asset holdings at time t on $\omega^t \in \Omega$. Let $\{x_\tau\}_{t \leq \tau \leq T}$ be a conditional process from (t,ω^t) where x_τ is a function such that $x_\tau : \Omega^{\tau - t} \rightarrow R_{++}$. Assume that u is increasing, continuous, concave, and differentiable on $(0,M)$ where M is a large number that exceeds the maximum value of $e_t(\omega^t)$ at all (t,ω^t). In addition, at equilibrium, $\{q_t\}_{0 \leq t \leq T}$ must satisfy rational expectations with $q_T^i(\omega^T) \equiv 0$ $\forall \omega^T \in \Omega^T$ and $i \in I$. The representative agent solves the following problem at each (t,ω^t):

$$V_t(c)(\omega^t) \equiv \mathrm{E}_t[\mathrm{Min}_{\delta_{t+1}(\omega^t,\omega') \in [\alpha_{t+1},\beta_{t+1}]}(1 - \delta_{t+1}(\omega^t,\omega'))u(c_t(\omega^t)) + \delta_{t+1}(\omega^t,\omega')V_{t+1}(c)(\omega^t,\omega')]$$

where $\omega' \in \Omega$ and $V_T(c)(\omega^T) \equiv u(c_T(\omega^T))$

with $0 < \alpha_\tau \leq \beta_\tau < 1$ $\forall \tau$ s.t. $t < \tau \leq T$

s.t.

For all $\tau \geq t$, $\omega^\tau = (\omega^{\tau-1}, \omega) \in \Omega^\tau$

$q_\tau(\omega^\tau)\theta_\tau(\omega^\tau) + c_\tau(\omega^\tau) = \theta_{\tau-1}(\omega^{\tau-1})[q_\tau(\omega^\tau) + d_\tau(\omega^\tau)] + e_\tau(\omega^\tau)$

$\theta_{t-1}(\omega^{t-1}) \equiv 0$ and $\theta_{-1}(\omega^{-1}) \equiv 0$

$q_T^i(\omega^T) \equiv 0$ $\forall \omega^T \in \Omega^T$ and $i \in I$.

$\inf_{i,\tau,\omega^\tau} \theta_\tau^i(\omega^\tau) > -\infty$

Let $\Delta \subset R_{++}^{S^{T+1}}$ be a collection of $\{\delta_t\}_{0 \leq t \leq T}$ where each $\delta_t(\omega^{t-1},\omega) \in [\alpha_t, \beta_t]$ with $0 < \alpha_t \leq \beta_t < 1$ for $1 \leq t \leq T$, and we define $\delta_0(\omega^0) \equiv 1$. We define Δ^* is a subset of Δ such that:

Δ^*

$= \{\{\delta_\tau\}_{0\leq\tau\leq T} \in \Delta |$ at each (t,ω^t) where $0 \leq t \leq T$:

$$\mathrm{E}_t[\mathrm{Min}_{\delta_{t+1}(\omega^t,\omega')\in[\alpha_{t+1},\beta_{t+1}]}(1-\delta_{t+1}(\omega^t,\omega'))u(c_t(\omega^t))$$
$$+\ \delta_{t+1}(\omega^t,\omega')V_{t+1}(c)(\omega^t,\omega')] = V_t(c)(\omega^t)\}$$

This set is a collection of sequences of discount factors under which the value of a consumption process is identical to the optimal value. Now an equilibrium is a price process $\{q_t\}_{0\leq t\leq T}$ such that $\{c_\tau\}_{t\leq\tau\leq T} = \{e_\tau\}_{t\leq\tau\leq T}$ and $\{\theta^i_\tau\}_{t\leq\tau\leq T} = \{0\}$ at all (t,ω^t). Given these notations, an equilibrium asset prices becomes as follows:

Proposition 2.5.1:

$\{q_t\}_{0\leq t\leq T}$ is an equilibrium price process if and only if there exists a discount factor process $\{\delta^*_t\}_{0\leq t\leq T} \in \Delta^*$ such that:

At all (t,ω^t) where $0 \leq t < T$:

(2.5.1) $q^i_t(\omega^t) = \mathrm{E}_t[\dfrac{\delta^*_{t+1}(\omega^t,\omega')A_{t+1}(\omega^t,\omega')}{A_t(\omega^t)}\dfrac{u'(e_{t+1}(\omega^t,\omega'))}{u'(e_t(\omega^t))}(q^i_{t+1}(\omega^t,\omega') + d^i_{t+1}(\omega^t,\omega'))]\forall i$

(2.5.2) $A_t(\omega^t) = \mathrm{E}_t[(1-\delta^*_{t+1}(\omega^t,\omega'))]$ with $\delta^*_{T+1}(\omega^{T+1}) \equiv 0$ $(A_T(\omega^T) = 1)$.

At all (T,ω^T):

(2.5.3) $q^i_T(\omega^T) \equiv 0$ $i \in I$.

Moreover, at almost all endowments, Δ^* is a singleton.

Proof:

See Appendix 2.D:

Note that (2.4.1) is not differentiable if $e_t(\omega^t) = e_{t+1}(\omega^t, \omega')$ at some (t, ω^t) as we saw in the example in Section 2.3.3. In this case, the economy has a continuum of equilibrium prices. By incorporating the possibility of indeterminacy of prices, an asset pricing equation holds under *some* sequence of discount factors $\{\delta_t^*\}_{1\leq t\leq T} \in \Delta^*$.[23] However, in our economy, since time and states are finite, the set of endowments that generates indeterminacy of equilibrium prices has measure zero. Then at almost all endowments, the value function is differentiable under a unique element of $\{\delta_t^*\}_{1\leq t\leq T} \in \Delta^*$. The difference between (2.5.1) and a usual Euler equation under the discounted utility representation is that (2.5.1) has state dependent normalized discount factors, which effectively change the marginal rate of substitution. This discount factors reflect the nature of a consumption sequence from time t onward.

2.5.2 Calibration: Equity-Premium and Risk-Free-Rate Puzzles

In this subsection, we apply equilibrium asset pricing equations to a simple numerical example to show that our model is capable of explaining both the equity-premium and risk-free-rate puzzles.[24] Suppose that the economy consists of two-periods and four states and that an agent's subjective prior is equal to the objective probability. From Table 8.1 of Campbell, Lo and Mackinlay (1997) at P.308:

[23] For more detailed treatment of indeterminacy, see Appendix 2-C.

[24] We can extend this study for a multi-period setting. Under the stationary economy, we expect to see a similar result using a stationary range of normalized discount factors.

Variable	Mean	S.D.	ρ	σ
Consumption growth	0.0172	0.0328	1.000	0.0011
Stock return	0.0601	0.1674	0.4902	0.0027
CP return	0.0183	0.0544	-0.1157	-0.0002
Stock-CP return	0.0418	0.1774	0.4979	0.0029

where CP stands for a commercial paper, S.D. stands for standard deviation, ρ is correlation with consumption growth, and σ is covariance with consumption growth. The heart of the equity-premium and risk-free-rate puzzles is:

1. An agent follows the discounted utility model with an atemporal CRRA utility function.
2. γ needs to be very high ($\gamma > 10$) to explain the equity premium.
3. δ (fixed discount factor) needs to be higher than 1 to explain the equity premium.

First, we define a distribution of consumptions at time 1 to match the mean and standard deviation of the above table. We set consumption growth, MRS (marginal rate of substitution), and the probabilities of state realizations as follows:[25,26]

[25] The consumption growth is calculated by solving two equations: $0.0172 = 0.7x + 0.3y$ and $0.0011 = 0.7x^2 + 0.3y^2 - (0.7x + 0.3y)^2$. Then $x = 0.0172+0.3\sqrt{0.0011/0.7/0.3}$ and $y = 0.0172-0.7\sqrt{0.0011/0.7/0.3}$

[26] We use the results from quarterly UK data from Q3/1975 to Q1/1998 to infer objective probability.

Consumption growth

State\Time	t=1
ω_2=1	1.0389
ω_2=2	1.0389
ω_2=3	0.9665
ω_2=4	0.9665

MRS and Probability

State\Variable	MRS	Prob.
ω_2=1	$(1.0389)^{-\gamma}$	0.35
ω_2=2	$(1.0389)^{-\gamma}$	0.35
ω_2=3	$(0.9665)^{-\gamma}$	0.15
ω_2=4	$(0.9665)^{-\gamma}$	0.15

Under this assumption, the consumption growth has (mean, S.D.) = (0.0172,0.0332). In this economy, any asset price is determined by two random variables that span a consumption space. Therefore, we define stock payoffs as follows:

Stock payoffs

State\Time	t=1
ω_2=1	$\theta(1.0389\text{-}\bar{c}) + \bar{c} + \varepsilon$
ω_2=2	$\theta(1.0389\text{-}\bar{c}) + \bar{c} - \varepsilon$
ω_2=3	$\theta(0.9665\text{-}\bar{c}) + \bar{c} + \varepsilon$
ω_2=4	$\theta(0.9665\text{-}\bar{c}) + \bar{c} - \varepsilon$

where $\bar{c}$ is a mean of consumption payoffs and ε is an idiosyncratic error, which is orthogonal to consumption payoffs. Then:

$$\text{Cov}(r_\omega, c_\omega) = \theta\frac{\sigma^{c2}}{P^s}$$
$$\text{E}[r_\omega] = \frac{\bar{c}}{P^s}$$

where r_ω is a gross stock return at state ω, c_ω is a consumption payoff at ω, and P^s is a stock price. From the above equations:

$$\theta = \frac{Cov(r_\omega, c_\omega)}{\sigma^{c2}} \frac{\bar{c}}{E[r_\omega]} = \frac{0.0027}{0.0011}\frac{1.0172}{1.0601} = 2.3552$$

Next, we need to set prices for the stock and the risk-free rate and an idiosyncratic shock ε. The following numbers explain prices, correlation between consumption and stock returns, and variance of stock returns:

Stock price = 0.9594

CP price = 0.9820

$\varepsilon = 0.14$

Now, we investigate whether or not we observe the equity-premium and risk-free-rate puzzles in this economy. Suppose that an agent follows the discounted utility model with an atemporal CRRA utility function. We need to find (δ,γ) to solve the above two prices. Under the discounted utility model, we require (δ,γ) = (13.7, 1.0991) to explain the above prices. To summarize:

γ	δ	CP return	Exp. rtn. of stock	Risk premium
13.7	1.0991	0.0183	0.0602	0.0418

Variable	Mean	S.D.	ρ	σ
Consumption growth	0.0172	0.0332	1.0000	0.0011
Stock return	0.0602	0.1671	0.4973	0.0028
CP return	0.0183	n.a.	n.a.	n.a.
Stock-CP return	0.0418	n.a.	n.a.	n.a.

Although γ =13.7 is lower than $\gamma = 19$ of Campbell, Lo and Mackinlay (1997), we still observe the equity-premium and risk-free-rate puzzles in this economy.

Now we investigate whether or not the model with multiple discount factors can resolve the equity-premium and risk-free-rate puzzles. Since we can always set α_1 and β_1 to match the equity premium and the risk-free rate in the empirical data, appropriate tests are: (1) whether or not $\alpha_1 < \beta_1$; (2) whether or not a low γ and a low *average time-preference* can explain the equity premium and risk-free rate. To find a (α_1, β_1, γ) that explains the empirical moments, we need to solve the following equations that define the risk premium of the consumption asset and the risk-free rate (i.e., Equation (2.5.1)):

$$\frac{\alpha_1[0.7(1.0389)^{-\gamma}\cdot\theta(1.0389-\bar{c})+\bar{c}]+\beta_1[0.3(0.9665)^{-\gamma}\cdot\theta(1.0389-\bar{c})+\bar{c}]}{[0.7(1-\alpha_1)+0.3(1-\beta_1)]}=0.9594$$
$$\frac{\alpha_1[0.7(1.0389)^{-\gamma}]+\beta_1[0.3(0.9665)^{-\gamma}]}{[0.7(1-\alpha_1)+0.3(1-\beta_1)]}=0.9820$$

where 0.9594 in the first equation is a price of stock and 0.9820 is a price of risk-free rate that explains the empirical data. The solutions for these equations are $[\alpha_1, \beta_1]$ = [0.3546,0.8255] under $\gamma = 2$, and with $[\alpha_1, \beta_1]$ = [0.3814,0.7686] under $\gamma = 4$. Then an average time-preference and a risk premium become:

γ	$[\alpha_1, \beta_1]$	$E_0[\frac{\delta_1^*(\omega_0,\omega_2)}{A_0(\omega_0)}]$	CP return	E(r) of stock	Risk premium
2	[0.3546,0.8255]	0.9835	0.0183	0.0602	0.0418
4	[0.3814,0.7686]	0.9902	0.0183	0.0602	0.0418

Note that by construction, these examples with multiple discount factors generate variance, covariance and correlation identical to those in the table for the discounted utility model. Then, the normalized discount factors ($\dfrac{\delta_1^*(\omega_0,\omega_2)}{A_0(\omega_0)}$) become:[27]

Normalized discount factors

State	γ=2	γ=4
ω_2=1	0.7033	0.7591
ω_2=2	0.7033	0.7591
ω_2=3	1.6375	1.5296
ω_2=4	1.6375	1.5296

First, the examples show that $\alpha_1 < \beta_1$. Since an agent assigns a lower discount factor for state 1 and state 2 than for state 3 and state 4, an agent underweights the MRS of state 1 and state 2 and overweights the MRS of state 3 and state 4. This intertemporal substitution mechanism effectively boosts risk aversion over time 1 consumption. To hold assets that are correlated with her consumption, an agent requires more premium than she does under the discounted utility model. This result explains the equity-premium puzzle. In fact, this mechanism expresses 'gain/loss' asymmetry of a future value from a current consumption level.[28,29]

[27] The preference-adjusted prior (see Appendix 2-D) becomes: for γ = 2, (0.249,0.249,0.251,0.251), for γ = 4, (0.266,0.266,0.234,0.234).

[28] For comparison, Benartzi and Thaler (1995) apply a myopic notion of loss aversion and argue that loss aversion that has a recent asset value as a reference point can explain the equity premium puzzle. Barberis and Haung (2000) also apply loss aversion to a individual asset behaviors and argue that it explains excess volatility and some of cross-sectional patterns of asset returns.

[29] Other atemporal models under uncertainty show similar asymmetry. For example, the dual theory of

Second, average normalized discount factors (i.e., average time-preference), $E_0[\frac{\delta_1^*(\omega_0, \omega_2)}{A_0(\omega_0)}]$, becomes less than 1, which resolves the risk-free-rate puzzle. In fact, a range of discount factors is not symmetric. For example, at $\gamma = 2$, $[\alpha_1, \beta_1] = [0.3546, 0.8255]$. Since $\delta = 0.5$ corresponds to unit normalized time-preference, this range implies that on average, an agent weights the future more than she weights the present under certainty. However, under uncertainty, since the probability of positive consumption growth is much higher than the probability of negative consumption growth, an average normalized discount factor becomes less than 1. In addition, for the discounted utility model, if we apply $\gamma = 2$ in the formula (8.2.8) of Campbell, Lo and Mackinlay (1997), it requires $\delta = 1.014$. In our example, we also need $\delta = 1.0127$ for it to have about 1.83% of CP return under $\gamma = 2$. Clearly, without considering the equity-premium puzzle, a smooth function (in this case, the discounted utility model with a CRRA utility of $\gamma = 2$) still shows the risk-free-rate puzzle. The results suggest that there would be a non-differentiable shift in MRS around the present consumption level, which strongly supports our results.

Third, the range of discount factors narrows as γ increases. In addition, an agent becomes less impatient on average, i.e., $E_1[\frac{\delta_2^*(\omega^1, \omega_2)}{A_1(\omega^1)}]$ increases. In fact, as γ increases, (2.4.1) approaches the CRRA model with a fixed discount factor and it converges at $\gamma = 13.7$. This result implies that we need to have lower γ to resolve the equity-premium and

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Yaari (1987) or disappointment aversion of Gul (1991) imply that under a two-state economy, an indifference curve kinks when two consumptions are identical. Segal and Spivak (1990, 1997) connects non-differentiability with an attitude toward risk, and they define the risk attitude at non-differentiable point as first-order risk aversion.

risk-free-rate puzzles together.

2.5.3 Estimation: Simple Test for UK Data

In this subsection, we provide a simple empirical test of (5-1-1) using UK data. We select UK data because the UK most resembles a stable economy, so we can observe positive and negative per capita consumption growths in a relatively short period of data. Our analysis is limited, especially in terms of data length.[30] However, in spite of this limitation, we show below that estimated discount factors from UK data move in a manner consistent with time-variability aversion.

First, assume that the economy follows the same state structure as we define in Section 2.5.1,[31] that (2.5.1) holds with a CRRA utility function, and that the subjective prior is equal to the objective probability. Then a price for asset i is expressed by the following moment condition:

$$q_t^i(\omega^t) = \mathrm{E}_t[\frac{\delta_{t+1}^*(\omega^t,\omega')A_{t+1}(\omega^t,\omega')}{A_t(\omega^t)}\left(\frac{c_{t+1}(\omega^t,\omega')}{c_t(\omega^t)}\right)^{-\gamma}(q_{t+1}^i(\omega^t,\omega')+d_{t+1}^i(\omega^t,\omega'))]$$

where γ is a coefficient of relative risk aversion and $A_t(\omega^t)$ is defined by (2.5.2). In addition, we assume that:

[30] For the most appropriate applications, we should use (2.4.1) for an individual consumptions, and estimates discount factors generation-by-generation. Since it is out of scope of this paper, we only consider aggregated economy.

[31] Again, we can use a more general state space.

$$(2.5.4) \qquad A_t(\omega^t) = A_t(\omega'^t) \text{ for all } \omega^t, \omega'^t \in \Omega^t \text{ at all } t$$

$$(2.5.5) \qquad \frac{\delta^*_{t+1}(\omega^t, \omega') A_{t+1}(\omega^t, \omega')}{A_t(\omega^t)} = \delta^u \text{ if } c_{t+1}(\omega^t, \omega') > c_t(\omega^t)$$

$$(2.5.6) \qquad \frac{\delta^*_{t+1}(\omega^t, \omega') A_{t+1}(\omega^t, \omega')}{A_t(\omega^t)} = \delta^d \text{ if } c_{t+1}(\omega^t, \omega') \leq c_t(\omega^t)$$

$$(2.5.7) \qquad \delta^u > \delta^d$$

These assumptions simplify the estimation of α_t and β_t (in fact, we only need to estimate two normalized discount factors δ^u and δ^d).[32] We justify these assumptions because the UK economy seems to be relatively stable during data periods. Then the above asset pricing equation becomes:

$$(2.5.8) \qquad \mathrm{E}_t[\delta(\omega') \left(\frac{c_{t+1}(\omega^t, \omega')}{c_t(\omega^t)} \right)^{-\gamma} \frac{(q^i_{t+1}(\omega^t, \omega') + d^i_{t+1}(\omega^t, \omega'))}{q^i_t(\omega^t)}] = 1$$

where $\delta(\omega')$ follows (2.5.5) and (2.5.6). Now let $r^i_t(\omega^t)$, $r^f_t(\omega^t)$ and $q^f_t(\omega^t)$ be a return of a stock, a risk-free rate, and a price of risk-free asset respectively. A simple manipulation of (2.5.8) yields the following equation:

$$(2.5.9) \qquad 1 \text{ - } q^f_t(\omega^t) = \frac{r^f_{t+1}(\omega^t)}{1 + r^f_{t+1}(\omega^t)} = \mathrm{E}_t[\delta(\omega') \left(\frac{c_{t+1}(\omega^t, \omega')}{c_t(\omega^t)} \right)^{-\gamma} r^i_{t+1}(\omega^t, \omega')]$$

---

[32] On the other hand, Epstein and Zin (1990) and Bekaert, Hodrick, and Marshall (1997) utilize first-order risk aversion (defined by Segal and Spivak (1990)) under the recursive utility model of Epstein and Zin (1989). The non-differentiability in their models comes from an atemporal non-expected utility model of Yaari (1987). Our model is based on time-variability aversion, and provides much parsimonious logic why we observe non-differentiability in marginal rate of substitution. The results in Sections 2.5.2 and 2.5.3 suggest that our model faces less restriction on a structure of non-differentiability and could explain asset prices better than Epstein and Zin (1990) or Bekaert, Hodrick, and Marshall (1997).

$$\text{(2.5.10)} \qquad \frac{r^f_{t+1}(\omega^t)}{1+r^f_{t+1}(\omega^t)} = \delta(\omega')\left(\frac{c_{t+1}(\omega^t,\omega')}{c_t(\omega^t)}\right)^{-\gamma} r^i_{t+1}(\omega^t,\omega') + \varepsilon^i_{t+1}(\omega^t,\omega')$$

Note that $r^f_{t+1}(\omega^t,\omega')$ depends only on ω^t so that we can write $r^f_{t+1}(\omega^t,\omega') = r^f_{t+1}(\omega^t)$ and take it out of the expectation operator. Equation (2.5.10) is rewritten by using dummy variables dm^u and dm^d where dm^u is one if $\delta(\omega') = \delta^u$ and zero otherwise, and dm^d is one if $\delta(\omega') = \delta^d$ and zero otherwise.

$$\begin{aligned} \text{(2.5.11)} \qquad & \frac{r^f_{t+1}(\omega^t)}{1+r^f_{t+1}(\omega^t)} \\ & = \delta^u \cdot dm^u \left(\frac{c_{t+1}(\omega^t,\omega')}{c_t(\omega^t)}\right)^{-\gamma} r^i_{t+1}(\omega^t,\omega') \\ & \quad +\delta^d \cdot dm^d \left(\frac{c_{t+1}(\omega^t,\omega')}{c_t(\omega^t)}\right)^{-\gamma} + \varepsilon^i_{t+1}(\omega^t,\omega') \end{aligned}$$

Before proceeding to the results, we mention two limitations. First, assumptions (2.5.4) to (2.5.7) are not innocuous. In reality, discount factors can move in a more complicated manner. For more precise investigation, we should model the movement of discount factors more carefully. Second, we do not directly use (2.5.8). Since (2.5.8) generates the regression where a dependent variable is a vector of one (i.e., no variations), it is hard to see a statistical relationship between the movement of the marginal rate of substitution and asset prices. In fact, (2.5.8) always gives us discount factors around one whereas (2.5.11) can result in discount factors different from one.

Now, we apply (2.5.11) to UK data. The data covers a period from the third quarter of 1975 to the first quarter of 1998. (86 data points. Since estimation is extremely sensitive to outliers, we exclude Q1/76, Q1/86, Q1/87, Q4/87 and Q3/90.[33])[34] During this period, there are 61 quarters of positive consumption growths and 25 quarters of negative consumption growth.[35] First, we perform unconditional regressions of (2.5.11). Note that we do not estimate a coefficient of relative risk aversion. Instead, we assume the number and estimate δ^u and δ^d. We also perform two additional tests for the following parameter restrictions:

Restriction 1:	H_0:	$\delta = \delta^u = \delta^d$	H_1:	$\delta^u \neq \delta^d$
Restriction 2:	H_0:	$\delta^u \cdot 0.7 + \delta^d \cdot 0.3 = 0.98$	H_1:	Not H_0

The first restriction tests whether or not two normalized discount factors are statistically different. Under the second restriction, we assume that an average normalized discount factor is 0.98. (The ratio of positive consumption growths to the number of data points is 0.7.)

Now we examine the results.[36]

[33] FT500 industrial returns: Q1/76: 17.25, Q1/86: 18.10, Q1/87: 20.50, Q4/87: -28.40, Q3/90: -18.60.

[34] Consumption is summation of non-durable and service expenditures. As deflators, we use the retail price index. Stock returns are based on FT500 industrial deflated by the retail price index monthly and converted to quarterly figures. For risk-free rates, we use 3 month bank bill at the end of the previous quarter. We subtract inflation rate from 3 month bank bill. Except 3 month bank bill, growth rates are based on arithmetic returns.

[35] From Q2/1956 to Q1/1998, there are 121 positive growth and 47 negative growth. The percentage of negative growth is 28 %, which is close to 30% of data we use for estimation. This result indicates that the UK economy seems to be fairly stable after 1956.

[36] Standard deviations are based on HCSE.

Gamma	2	4	6	8	10	20	21
δ^u	0.2850	0.2908	0.2966	0.3024	0.3081	0.3370	0.3399
δ^d	0.5410	0.5321	0.5231	0.5140	0.5048	0.4579	0.4532
t value for δ^u	2.0716	2.0807	2.0898	2.0987	2.1075	2.1498	2.1538
t value for δ^d	3.0245	3.0197	3.0161	3.0139	3.0131	3.0312	3.0352
R^2	0.1018	0.1020	0.1023	0.1025	0.1028	0.1040	0.1041
δ from Restriction 1	0.3365	0.3420	0.3473	0.3524	0.3571	0.3760	0.3773
Wald for Restriction 1	0.8104	0.7337	0.6590	0.5865	0.5165	0.2152	0.1906
Power of Restriction 1	0.1280	0.1136	0.1011	0.0903	0.0811	0.0553	0.0542
δ^u from Restriction 2	0.8099	0.8285	0.8470	0.8653	0.8835	0.9702	0.9785
δ^d from Restriction 2	1.3951	1.3496	1.3044	1.2597	1.2156	1.0038	0.9836
Wald for Restriction 2	28.217	27.851	27.480	27.105	26.726	24.791	24.595

The regressions show that $\delta^u > \delta^d$, and the gap between them narrows as γ increases. Under Restriction 2, we first see that a range of normalized discount factors is consistent with assumption (2.5.7) although the Wald statistics are quite high.[37] Second, we see that the range of normalized discount factors shrinks as γ goes up, and at $\gamma = 21$, a range vanishes. We consider this point to be an implied relative risk aversion of fixed discount factors. Third, at $\gamma = 2$ and $\gamma = 4$, a range of normalized discount factors roughly corresponds to that of the numerical examples in the previous subsection (note that the previous example

[37] All regressions are very sensitive to outliers. This result clearly implies that our data size is too small to make statistical test valid. Also it is well know that Wald statistics are not reliable in small samples.

uses US data). In addition, R^2 and t values for δ^u and δ^d are nearly constant at any level of γ. This result shows that under time-variability aversion, a model with a low γ does not decrease overall explanatory power relative to a model with a high γ. If we have a normative criteria for the level of γ that is less than 10, we should select (2.4.1) to explain the movement of asset returns. Moreover, under $\delta^u = \delta^d$, δ increases as γ goes up, which is consistent with the results of the numerical examples in the previous subsection.

By contrary, one negative result is that we cannot reject the null hypothesis of $\delta^u = \delta^d$. However, the Wald statistic still decreases as γ increases, which indicates that at the lower level of γ, δ^u and δ^d become more distinguishable. In addition, a data size is too small to conclude whether or not $\delta^u = \delta^d$ is a reasonable assumption. For example, if we test a parameter restriction of δ^u - $\delta^d = 0.25$, the Wald statistic becomes 3.17. To obtain a similar level of the Wald statistic in the opposite direction, we need to set δ^u - $\delta^d =$ -0.75.[38] Clearly, data implies that δ^u - δ^d is more likely to be negative. More precisely, we report the power of the tests assuming that the estimated coefficients are true value. Clearly, the power of the test of Restriction 1 is too low to conclude that $\delta^u = \delta^d$ is statistically supported. The low power is primarily due to the small gap in the estimated coefficients relative to its variance, so we need more data to reduce the variance of the estimates.

Next, we apply GMM to (2.5.11) to test conditional relationships. The instruments we use are a constant, the lag of per capita consumption growth rate, the lag of retail price

[38] The Wald statistic is 3.02.

growth rate, the lag of stock returns, the lag of risk-free rates.[39]

Gamma	2	4	6	8	10	20	21
δ^u	2.1302	2.1965	2.2638	2.3322	2.4017	2.7657	2.8035
δ^d	3.7021	3.6222	3.5434	3.4658	3.3893	3.0215	2.9860
Wald for δ^u	40.269	41.455	42.631	43.797	44.949	50.459	50.983
Wald for δ^d	23.667	23.855	24.057	24.275	24.507	25.862	26.014
Over-identifying restr.	22.703	22.344	21.978	21.605	21.226	19.242	19.036
δ from Restr. 1	2.4198	2.4708	2.5215	2.5716	2.6209	2.8405	2.8585
Wald for Restr. 1	3.3457	2.8837	2.4345	2.0027	1.5933	0.1349	0.0702
Power for Restri. 1	0.9171	0.8222	0.6825	0.5171	0.3572	0.0521	0.0506
δ^u from Restr. 2	0.9057	0.9038	0.9022	0.9008	0.8998	0.9016	0.9025
δ^d from Restr. 2	1.1612	1.1658	1.1699	1.1732	1.1757	1.1712	1.1691
Wald for Restr. 2	26.904	27.396	27.890	28.387	28.888	31.417	31.672

First, this linear GMM is essentially identical to an instrumental variable regression. Since the correlations between explanatory variables and instruments are very low, the estimates are not very reliable.[40] This low correlations explain the large coefficients in the above table.[41] In addition, there are two differences between the estimates in GMM and

[39]Estimates and statistics are based on the optimal weights from the instrumental variable regressions.

[40]Note that the critical values for 5% and 1% of Chi-square distribution for the test of over-identifying restriction are (7.815,11.341) for the GMM.

[41]If an original regression works, there is no need for an instrumental variable regression because it is

the estimates in unconditional regressions. The first difference is that the null hypothesis of Restriction 1 is rejected at a 10% confidence level at γ =2 and γ =4. The second difference is that under Restriction 2, the range of normalized discount factors is narrower than the range under unconditional regressions, and it does not shrink under Restriction 2 as γ increases. However, in general, the normalized discount factors in GMM still capture time-variability aversion.

2.6 Comparison with Other Intertemporal Utility Functions

2.6.1 Recursive Utility, Gilboa (1989) and Shalev (1997)

In this section, we compare the multiple discount factors model with other intertemporal utility functions. We only focus on the utility functions that involve movement of discount factors. First, let $c = (c_0,...,c_T)$ be a stream of consumptions from time 0 to time T.

(2.6.1) $V_0(c) = W(u(c_0),V_1(c_1, ..., c_T))$ (Koopmans: 1960)

(2.6.2) $U(c) = u(c_0) + \sum_1^T u(c_t) \exp(-\sum_{\tau=0}^{t-1} v(c_\tau))$ (Epstein: 1983)

Our formula (2.3.2) shares a structure similar to (2.6.1). The multiple discount factors simply define the way an aggregator function works (although an aggregator function is time-dependent). Epstein (1983) has path-dependent discount factors. However, the level of discount factors depends on the level of historical consumptions, not on the difference

always less efficient than the original regression.

between current and future consumptions. In addition, (2.6.2) is time additive and does not provide gain/loss asymmetry as time-variability aversion suggests.

Gilboa (1989) applies the non-additive prior model of Schmeidler (1989) over a sequence of lotteries and derives a utility function that depends not only on consumption itself but also the difference of adjacent consumptions. Shalev (1997) modifies Gilboa (1989)'s formulation and introduces different weights for positive and negative increments.

(2.6.3) $\quad U(c) = \sum_0^T [\alpha_t u(c_t) + \beta_t |u(c_t) - u(c_{t-1})|]$ with $\beta_0 = 0$ $\quad$ (Gilboa:1989)

(2.6.4) $\quad U(c) = u(c_0) + \sum_0^T \delta_t^+ \max[u(c_t) - u(c_{t-1}), 0] + \delta_t^- \min[u(c_t) - u(c_{t-1}), 0]$

with $\delta_0^+ = \delta_0^- = 0$ $\quad$ (Shalev: 1997)

First, both formulas incorporate a reference point, and Shalev (1997) also captures gain/loss asymmetry over time. However, neither formula satisfies dynamic consistency, i.e., the choice for $(c_1, ..., c_T)$ at time 0 might not be optimal at time 1. This result is due to Sarin and Wakker (1998) and Grant, Kajii and Polak (2000): the rank dependent utility function cannot have a recursive structure. On the other hand, the multiple priors model by Gilboa and Schmeidler (1989) can have dynamically consistent preference relations.[42] Given

[42] Sarin and Wakker (1998) indicate in a simple example that the multiple priors model of Gilboa and Schmeidler (1989) can be dynamically consistent and allow a recursive structure. Epstein and Schneider (2001) prove the existence of recursive multiple priors preference. Wakai (2001) also shows that the ex-ante multiple priors set must be recursive if it satisfies dynamic consistency and independence of irrelevant alternatives. Moreover at each time, the updated preference must be a class of the multiple priors and a set of priors is derived applying Bayes rule point-wise.

this theoretical advantage, we apply the multiple priors model for a stream of consumptions and produce dynamically consistent preference relations.[43,44] The dynamic consistency has a clear implication: an agent compares present consumption and a discounted present value of all future consumption.

Second, our formula automatically assigns a lower discount factors for higher future consumption. On the other hand, (2.6.4) is silent about the magnitude of δ_t^+ and δ_t^-.[45]

2.6.2 Loss Aversion and Habit Formation

In this subsection, we compare formulas (2.3.2) and (2.4.1) with the loss aversion of Tversky and Kahneman (1979,1991) and Loewenstein and Prelec (1992,1993), and the habit formation function of Constantinides (1990). First, prospect theory (Tversky and Kahneman, 1979) provides a static utility representation of risk-preferences based on experimental and behavioral studies, and it has the following form of a utility function on a random variable x:

$$(2.6.5) \qquad u(x) = \sum \pi(p)V(x-r)$$

[43]The usage of the recursive multiple priors began in Epstein and Wang (1994, 1995).

[44]The multiple priors model is in general dynamically inconsistent because revealing more information about states is inconsistent with the original ambiguity unless an agent knows how information is going to be revealed over time, i.e., only when an agent know the filtration of state realizations.

[45]For Shalev (1997) formula to have $\delta_t^+ < \delta_t^-$, it requires more assumptions for agents' behavior. However, if there are only two periods, Shalev (1997) and our model is essentially identical.

where p defines a probability distribution of x, $\pi(p)$ is a probability weight function, r is a reference point, and V is a value function. The summation is over the support of p. V shows gain/loss asymmetry, i.e., loss aversion, by non-differentiable shift of V at the reference point r. The reference point r is a predetermined level that is based on current information so that r represents some *intertemporal* consideration.

To apply (2.6.5) to a consumption-saving problem, there are two approaches. The first approach is to apply loss aversion to gain/loss of asset returns (see Barberis and Huang (2000)). The second approach is to incorporate loss aversion on consumption streams. To be consistent with our motivation of consumption smoothing, we only focus on the second approach. In fact, (2.6.4) of Shalev (1997) is motivated by the second objective. In a more experimental approach, Loewenstein and Prelec (1992,1993) suggest two alternatives:

$$(2.6.6) \qquad u(c) = \textstyle\sum_{t=0}^{T} \phi(t) V(c_t - r)$$

$$(2.6.7) \qquad u(c) = \textstyle\sum_{t=0}^{T} u(c_t) + \rho^+ \sum_{t=0}^{T} \max[d_t, 0] - \rho^- \sum_{t=0}^{T} \min[d_t, 0]$$

where $\phi(t)$ is a function of discount factors (we assume that it is the exponential discounting although they assume hyperbolic discounting.), $0<\rho^+ < \rho^-$ and $d_t = \dfrac{t+1}{T+1} \sum_{\tau=0}^{T} u(c_\tau)$ - $\sum_{\tau=0}^{t} u(c_\tau)$ (difference in cumulative utility). $V(c_t - r)$ in (2.6.6) is a direct application of (2.6.5). On the other hand, our representation of (2.3.2) is based only on aversion to fluctuations of payoffs over time and does not pre-specify the existence of reference points. Gain/loss asymmetry is expressed in (2.3.2) because an agent compares only two numbers at each time due to a recursive structure. In (2.6.7), Loewenstein and Prelec (1993) aim to explain loss aversion based on global characteristics of a consumption sequence. Our formula (2.3.2) shares an idea similar to (2.6.7) because time-variability aversion explains global characteristics. In (2.3.2), an agent compares present consumption and a discounted present value of future consumption and applies an argument similar to loss aversion between them. To this end, we can consider (2.3.2) to be a hybrid of (2.6.6) and (2.6.7). The difference between (2.3.2) and (2.6.7) is that (2.3.2) has a recursive formula whereas (2.6.7) considers differences and base-line preference ($\sum_{t=0}^{T} u(c_t)$) separately and it is not dynamically consistent.

Constantinides (1990) (recently by Campbell and Cochrane (1999,2000)) introduces a utility function that depends on historical consumption, which is a variant of reference-based approaches (the habit formation model). His formula becomes:

$$(2.6.8) \qquad V(c) = \mathrm{E}[\textstyle\sum_{t=0}^{T} \delta^{t-1} u(c_t - x_t)]$$

where $x_0 = 0$ and $x_t = f(c_0, ..., c_{t-1})$. This formula is an application of (2.6.5) over consumptions. The term $u(c_t - x_t)$ represents an idea similar to an existence of reference points on consumption growth in (2.6.5) although $c_t - x_t$ should not be negative under the CRRA u and does not have gain/loss asymmetry in the original work of Constatninides (1990) (there is no loss by definition). In fact, the habit formation model is not quite what its name suggests; it mainly expresses forward-looking concerns. The difference between (2.4.1) and (2.6.8) is that (2.4.1) only focuses on future time-variability and u is history-independent. This feature makes analysis easier. The value function (2.6.8) has history-dependent $u(c_t - x_t)$.

2.6.3 Comparison of Empirical Implications

Our model can explain the equity-premium and risk-free-rate puzzles. This result is not surprising given that our model shares similar qualitative features with habit formation, loss aversion and uncertainty aversion. The main advantage of our model comes from the theoretical aspect: it is based on more parsimonious axioms and the interpretation of empirical results is straight forward. However, to provide more intuitions for our model, we investigate detailed differences in empirical implications among these models.

First, the main difference between our model and the habit formation model is whether u is history dependent or not. We can test this claim directly by designing experimental questions or observing the sequential choices of individuals over time. However, evidence for history dependence itself does not deny the existence of time-variability aversion. Time-variability is an expression of time-preference for consumption smoothing and there are many experimental evidences that support this notion. In addition, by allowing history dependence, we can rewrite (2.3.2) with history dependent u while keeping multiple discount factors. To this end, we can consider the test for history dependence to be a test of u, not the test of time-variability aversion.

Second, the test of loss aversion depends on the interpretation of its concept. The value function (2.3.2) can be categorized in the class of models of loss aversion if we apply a broader interpretation. In fact, gain/loss asymmetry from a reference point is closely related to the preference for smoothing, and we can naively consider (2.3.2) to be an axiomatization of both concepts. In other words, time-variability aversion is a source of loss aversion. However, (2.3.2) has a clear difference from (2.6.6). If we consider (2.6.6) to be a genuine form of loss aversion, we can test whether an agent only considers tomorrow's value or considers all future values to assess gain/loss asymmetry.

Our model derives implications for equilibrium asset prices similar to those of a model with multiple priors.[46,47] The main difference between them is that in the multiple priors model, an agent uses an 'effective' prior that is different from the objective probability. An agent is uncertain about the probability of state realizations and selects the worst possible guess from a choice of priors to evaluate a given consumption stream. This prior overestimates the probability of states that yield lower consumption and underestimates the probability of states that yield higher consumption. This operation makes the *ex-post* equity premium higher relative to the *ex-ante* equity premium because an agent uses a biased prior instead of objective probability. (Positive returns happen more than she expected. The ex-post boosts of equity returns explains the equity-premium puzzle.) On the other hand, in our model with the rational expectation hypothesis, an agent uses the objective probability to calculate asset prices and expected values but still does not like stocks because stocks are correlated with consumption and she is concerned about bad consumption states. *Ex-ante* and *ex-post* expected returns are identical. The interpretation based on preference-adjusted priors defined in Appendix 2.D and 2.E offers a simple convenient analogy between these

[46] Two models are theoretically different. See Appendix 2-E.

[47] In general, our model shares similarity with the models that involve atemporal first-order risk aversion. However, the source of non-differentiability is very different from those in atemporal non-expected utility models.

two models.

Empirically, to test the multiple priors model, we need to estimate the set of priors. We cannot simply consider our results to be the support of the multiple priors because at equilibrium we can always find a set of multiple priors that justify our results (constructing from an equilibrium preference-adjusted prior). In addition, we need to know how an agent updates the multiple priors over time when an agent learns the statistical properties of the economy. It is not reasonable to assume that an agent never learns and keeps the same degree of ambiguity over time.

Finally, to compare with the discounted utility model, we need to conduct experiments under which some questions only involve risk-preferences and other questions involves intertemporal preferences. Then we can investigate the difference between atemporal risk aversion and risk aversion with intertemporal choice.

2.7 Derivation of the Representation of (2.4.1)

As we see in Appendix 2.B, if we apply (2.3.2) to an objective probability space of consumption streams, it is not dynamically consistent. To restore dynamic consistency, a standard approach is to use recursive preferences of Epstein and Zin (1989) with subjective priors. However, under Epstein and Zin (1989), intertemporal substitution is considered after certainty-equivalence is calculated. As we shown in Section 4-1, this approach does not utilize time-variability aversion sufficiently. To reverse the order of aggregation, we need to define preference relations on a slightly bigger act space.

In an actual derivation of the representation, an agent needs to consider the following thought experiment for Contract A and B in Section 2.4.1 at time 0:

1. I know that I am at state 1.
2. However, I also care about payoffs in state 2 even though this state was never realized.
3. To incorporate this concern, I need to weight a utility of payoffs in state 1 and state 2.
4. So even though [4,5] is better than [4,4], I would prefer Contract B.

The reason why we need to consider this thought experiment is to locate a subjective prior outside a utility index of each state. This thought experiment incorporates the fact that the true state is either state 1 or state 2 even when an agent is at time 0. She consider a consumption stream to be certain at each state, and applies time-variability aversion on a state-by-state basis. Then she considers as if only one of them were true and aggregates utility from each stream of consumption with some weights.

Now we derive a utility representation with multiple discount factors under uncertainty. In this section, we use a more general structure of state evolution. There is a finite state space Ω with S elements. Time horizon is finite, and let $\mathfrak{T} = \{1,...,T\}$ be a set of time from 0 to T. Let $\mathfrak{F}_t$ be a filtration at time t, i.e., a partition of Ω that is finer than $\mathfrak{F}_\tau$ $\forall 1 \leq \tau < t$, and any event in $\mathfrak{F}_t$ must be a subset of some event of $\mathfrak{F}_{t-1}$. Assume that $\mathfrak{F}_1$=$\{\Omega\}$ and $\mathfrak{F}_T$ =$\{$1,...,S$\}$ with $\mathfrak{F}_t(\omega)$ denoting an event in $\mathfrak{F}_t$ that contains a state $\omega \in \Omega$. Note that by the nature of filtration, the knowledge of $\mathfrak{F}_t(\omega)$ summarizes all history up to t. Assume that Y is a lottery space defined in Section 3-2. Let f^ω be a state act: f^ω:$\mathfrak{T} \to Y$, i.e., f^ω = $(y_0,...,y_T) \in Y^{T+1}$, and f^t be a time act: f^t:$\Omega \to Y$, i.e., $f^t = (y_1,...,y_S)' \in Y^S$. Let f be an act:f:$(\mathfrak{T},\Omega) \to Y$, i.e., $f = (f^1,...,f^T) = (f^1,...,f^S) \in Y^{S(T+1)}$. Also denote $f^t(\omega)$ $\in Y$ as a lottery in f^t at ω, and $f^\omega(t) \in Y$ as a lottery in f^ω at t, and $f_{t,\omega}$ as a lottery at (t,ω). The primitives that an agent forms preference relations are *forward measurable acts*. A forward measurable act is a function: f: $(\mathfrak{T},\Omega) \to Y$ such that $f = (f^0,...,f^T)$, and f^t is measurable with respect to $\mathfrak{F}_{t+1}$ where $0 \leq t < T$ and f^T is measurable with respect to $\mathfrak{F}_T$. Define a collection of all such functions as $\mathfrak{A}_f$. This space is bigger than the space of acts that are measurable with respect to the filtration $\mathfrak{F}_t$. The forward measurability states that an agent cares how an act assigns lotteries at t on each event $\mathfrak{F}_{t+1}$ in $\mathfrak{F}_t(\omega)$. On the other hand, if an agent is only concerned about preference relations on measurable acts with respect to $\mathfrak{F}_t$, an agent only considers tomorrow's uncertainty, not a path from today to tomorrow on each event $\mathfrak{F}_{t+1}$ in $\mathfrak{F}_t(\omega)$. In other words, under forward measurability, an agent hypothetically constructs preference relations as if she knows which event will occur tomorrow. After forming preference relations on $\mathfrak{A}_f$, an agent uses them to evaluate $\mathfrak{F}_t$-measurable acts.

Now we define subspaces of $\mathfrak{A}_f$. A constant act is a function, $h^t(\omega) = y \; \forall (t,\omega) \in (\mathfrak{T},\Omega)$, that will also be denoted by y. $\mathfrak{A}_c$ is a collection of all constant acts. A certainty act is a function, $h^t(\omega) = y_t \; \forall (t,\omega) \in (\mathfrak{T},\Omega)$, that will also be denoted by y^ω. $\mathfrak{A}_{cty}$ is a collection of all certainty acts. A t-constant act is a certainty act with $f^t = f^\tau \; \forall \tau$ s.t. $t \leq \tau \leq T$. $\mathfrak{A}_{c(t)}$

is a collection of all t-constant acts. A (t,t')-state-constant act is a forward measurable act such that each f^τ of $t \leq \tau \leq T$ is a measurable with respect to $\mathfrak{F}_{t+1}$, and that $f^\omega(t') = f^\omega(\tau)$ $\forall\tau$ s.t. $t \leq t' \leq \tau \leq T$ at $\forall\omega \in \Omega$. $\mathfrak{A}_{cs(t,t')}$ is a collection of all (t,t')-state-constant acts. Clearly, $\mathfrak{A}_c \subset \mathfrak{A}_{cs(t,t')} \subset \mathfrak{A}_f$ and $\mathfrak{A}_c \subset \mathfrak{A}_{c(t)} \subset \mathfrak{A}_{cty} \subset \mathfrak{A}_f$. The following table summarizes allocation of lotteries in each subgroups. Suppose that we are at time 0 for T=2.

Filtration (0 to 2)

$t=0$	$t=1$	$t=2$
$\mathfrak{F}_1$	$\mathfrak{F}_{2,1}$	$\mathfrak{F}_{3,1}$
$\mathfrak{F}_1$	$\mathfrak{F}_{2,1}$	$\mathfrak{F}_{3,2}$
$\mathfrak{F}_1$	$\mathfrak{F}_{2,2}$	$\mathfrak{F}_{3,3}$
$\mathfrak{F}_1$	$\mathfrak{F}_{2,2}$	$\mathfrak{F}_{3,4}$

$\mathfrak{A}_f$

$t=0$	$t=1$	$t=2$
$y_{1,1}$	$y_{2,1}$	$y_{3,1}$
$y_{1,1}$	$y_{2,2}$	$y_{3,2}$
$y_{1,2}$	$y_{2,3}$	$y_{3,3}$
$y_{1,2}$	$y_{2,4}$	$y_{3,4}$

$\mathfrak{A}_c$

$t=0$	$t=1$	$t=2$
y	y	y
y	y	y
y	y	y
y	y	y

$\mathfrak{A}_{cty}$

$t=0$	$t=1$	$t=2$
y_1	y_2	y_3
y_1	y_2	y_3
y_1	y_2	y_3
y_1	y_2	y_3

$\mathfrak{A}_{c(2)}$

$t=0$	$t=1$	$t=2$
y_1	y	y
y_1	y	y
y_1	y	y
y_1	y	y

$\mathfrak{A}_{cs(1,2)}$

$t=0$	$t=1$	$t=2$
$y_{1,1}$	y_1	y_1
$y_{1,1}$	y_1	y_1
$y_{2,2}$	y_2	y_2
$y_{2,2}$	y_2	y_2

At each (t,ω), an agent has her preference relations on $\mathfrak{A}_f$ with $\succeq_{(t,\omega)}$. All conditional preferences in the collection of $\{\succeq_{(t,\omega)}\}\equiv\{\succeq_{(t,\omega)}: (t,\omega)\in(\mathfrak{T},\Omega)\}$ satisfy the following axioms:

Axiom 2.7.1: Outcome Dependence

Conditional preference relations $\succeq_{(t,\omega)}$on $\mathfrak{A}_f$ are based on $\mathfrak{F}_t(\omega)$.

Axiom 2.7.2: Independence of History up to $t-1$ and of Irrelevant Alternatives

$\forall f, q \in \mathfrak{A}$: If $f^{\tau}(\omega') = g^{\tau}(\omega')$ on $\forall \omega' \in \mathfrak{F}_t(\omega)$ for $\forall \tau$ s.t. $t \leq \tau \leq T$, then $f \simeq_{(t,\omega)} g$.

Axiom 2.7.3: Indifference among $\omega' \in \mathfrak{F}_t(\omega)$

$\forall f, q \in \mathfrak{A}$: $\succeq_{(t,\omega)}$is identical to $\succeq_{(t,\omega')}$if $\mathfrak{F}_t(\omega) = \mathfrak{F}_t(\omega')$

Axiom 2.7.4: Dynamic Consistency

For $f, g \in \mathfrak{A}_f$ if $f^t(\omega') = g^t(\omega')$ and $f \succeq_{(t+1,\omega')} g$ on $\forall \omega' \in \mathfrak{F}_t(\omega)$ then $f \succeq_{(t,\omega)} g$.

Axiom 2.7.1 is the key axiom for these preference relations. It states that an agent behaves as if she knows at time t that an event $\mathfrak{F}_{t+1}(\omega)$ will occur tomorrow, but takes into consideration unrealized events in $\mathfrak{F}_t(\omega)$. Axioms 2.7.2 and 2.7.3 state that preference relations only depend on assignment of future lotteries on $\mathfrak{F}_t(\omega)$. Again we consider dynamic consistency as a normative criteria. Now the collection of all preference relations $\{\succeq_{(t,\omega)}\}$ on $\mathfrak{A}_f$ also satisfy axioms below:

Axiom 2.7.5: Weak Order

$\forall f, g, h \in \mathfrak{A}_f$, (i) $f \succeq_{(t,\omega)} g$ or $g \succeq_{(t,\omega)} f$ (ii) $f \succeq_{(t,\omega)} g$ and $g \succeq_{(t,\omega)} h \Rightarrow f \succeq_{(t,\omega)} h$.

Axiom 2.7.6: Continuity

$\forall f, g, h \in \mathfrak{A}_f$ with $f \succ_{(t,\omega)} g \succ_{(t,\omega)} h$, $\exists 0 < \alpha$, β <1

s.t. $\alpha f \oplus (1-\alpha) h \succ_{(t,\omega)} g$ and $g \succ_{(t,\omega)} \beta f \oplus (1-\beta) h$.

Axiom 2.7.7: Nondegeneracy in $\mathfrak{A}_c$

$\exists f, g \in \mathfrak{A}_c$ s.t. $\forall h \in \mathfrak{A}_f$ $f \succeq_{(t,\omega)} h$ and $h \succeq_{(t,\omega)} g$ and $f \succ_{(t,\omega)} g$

Axiom 2.7.8: Strict Monotonicity on Time among $\mathfrak{A}_{cty}$

For $f, g \in \mathfrak{A}_{cty}$ s.t. $f^\omega = (y_t,...,y_T)$ and $g^\omega = (y'_t,...,y'_T)$.

If $y_\tau \succeq_{(t,\omega)} y'_\tau$ $\forall t \leq \tau \leq T$, then $f \succeq_{(t,\omega)} g$.

In addition, if for some τ, $y_\tau \succ_{(t,\omega)} {y_\tau}'$ then $f \succ_{(t,\omega)} g$.

Axiom 2.7.9: Constant-Independence among $\mathfrak{A}_{cty}$

$\forall$ $f, g \in \mathfrak{A}_{cty}$ and $\forall h \in \mathfrak{A}_c$:

$\forall \alpha \in (0,1)$, $f \succ_{(t,\omega)} g \Leftrightarrow \alpha f \oplus (1-\alpha)h \succ_{(t,\omega)} \alpha g \oplus (1-\alpha)h$

Axiom 2.7.10: Time-Variability Aversion among $\mathfrak{A}_{cty}$

$\forall$ $f, g \in \mathfrak{A}_{cty}$ s.t. $f \sim_{(t,\omega)} g$, $\forall \alpha \in (0,1)$, $\alpha p \ \oplus (1-\alpha) f \succeq_{(t,\omega)} g$

Axiom 2.7.11: Weak Independence Axiom among $\mathfrak{A}_{cs(t,t)}$

For $f, g, h \in \mathfrak{A}_{cs(t,t)}$, $f \sim_{(t,\omega)} g \Longleftrightarrow \frac{1}{2}h + \frac{1}{2}f \sim_{(t,\omega)} \frac{1}{2}h + \frac{1}{2}g$

Axiom 2.7.12: Strict Monotonicity on States among $\mathfrak{A}_{cs(t,t+1)}$

For $f, g \in \mathfrak{A}_{cs(t,t+1)}$, if $f^{\omega'} \succeq_{(t,\omega)} g^{\omega'}$ $\forall \omega' \in \mathfrak{F}_t(\omega)$ then $f \succeq_{(t,\omega)} g$

In addition, if for some ω', $f^{\omega'} \succ_{(t,\omega)} g^{\omega'}$ then $f \succ_{(t,\omega)} g$.

Axioms 2.7.5 to 2.7.10 are equivalent to Axioms 2.3.1 to 2.3.6.[48] Axioms 2.7.11 and 2.7.12 ensure the existence of a subjective prior. However, Axiom 2.7.12 is more than a state-independence axiom that derives a subjective prior among $\mathfrak{A}_{cs(t,t)}$ at $\mathfrak{F}_t(\omega)$. An agent needs to consider $\mathfrak{A}_{cs(t,t+1)}$ (which is a bigger act space) to apply time-variability aversion

[48] Note that we only need Axioms 2.7.8 and 2.7.10 for $\succeq_{(1,\omega)}$given Axioms 2.7.2 to 2.7.4.

on a state-by-state basis. This axiom also separates decisions between time t and time $t+1$ from decisions among time $t+1$ onward. An agent does not want to consider the effects of mixing future lotteries (from $t+1$ onward) on different $\mathfrak{F}_{t+1}$ since she knows that at time $t+1$, she would not consider irrelevant alternatives. An agent only wants to consider how she feels if she were at $\mathfrak{F}_{t+1}$ today and then aggregates those utilities.

Given the above axioms, preference relations $\{\succeq_{(t,\omega)}\}$ on $\mathfrak{A}_f$ can be represented by the following formula:

Proposition 2.7.1:

A binary relationship $\{\succeq_{(t,\omega)}\}$ on $\mathfrak{A}_f$ satisfies Axiom 2.7.1 to 2.7.12 if and only if there exists $\{[\alpha_t,\beta_t]\}_{1\leq t\leq T}$ such that:

(2.7.1) $\quad \forall f, g \in \mathfrak{A}_f$,

$f \succeq_{(t,\omega)} g \Leftrightarrow V_{(t,\omega)}(f) \geq V_{(t,\omega)}(g)$

where $\{V_{(t,\omega)}(f)\}_{(t,\omega)\in(\mathfrak{T},\Omega)}$ are recursively defined by:

$$V_{(t,\omega)}(f) \equiv \mathrm{E}_t[\mathrm{Min}_{\delta_{t+1,\omega'}\in[\alpha_{t+1},\beta_{t+1}]}[(1-\delta_{t+1,\omega'})u(f^t(\omega))$$
$$+\delta_{t+1,\omega'}V_{(t+1,\omega')}(f)]\ |\mathfrak{F}_t(\omega)]$$

and $V_{T,\omega}(f) \equiv u(f^T(\omega))$

(2.7.2) $\quad 0 < \alpha_t \leq \beta_t < 1\ \forall t$ s.t. $1 \leq t \leq T$

(2.7.3) $\quad \delta^*_{t+1,\omega'} = \delta^*_{t+1,\omega''}$ at $\omega', \omega'' \in \mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)$

where $\delta^*_{t+1,\omega'} \in \mathrm{argmim}_{\delta_{t+1,\omega'}\in D_t}[(1-\delta_{t+1,\omega'})u(f^t(\omega')) + \delta_{t+1,\omega'}V_{(t+1,\omega')}(f)]$

Moreover,

(2.7.4) $\quad \mathrm{E}_t[.|\mathfrak{F}_t(\omega)]$ and $[\alpha_t,\beta_t]$ is uniquely defined, $[\alpha_t,\beta_t]$ are independent of states.

(2.7.5) u: $Y \to R$ is a unique up to a positive affine transformation.

Proof:

Appendix 2.F:

Note that in an actual situation, only $\mathfrak{F}_t$-measurable acts are available. However, an agent can use the above utility functions for them because a set of $\mathfrak{F}_t$-measurable acts is a subset of $\mathfrak{A}_f$. In addition, a selection of discount factors at time t is based on tomorrow's event $\mathfrak{F}_{t+1}$, not on each state in $\mathfrak{F}_{t+1}$ because preference relations are defined on forward measurable acts. Finally, for time-variability-seeking, we need to reverse the inequality in Axiom 2.7.10. Then the representation has 'max' instead of 'min' as we mentioned in Section 2.3.1 and Appendix 2.A.

2.8 Conclusion and Extensions

In this paper, we axiomatized the behavioral notion of time-variability aversion and then derived the representation. Our model makes the following contributions: (1) time-variability aversion is captured as a separate attitude from risk-aversion, and an axiomatic derivation provides a clear picture of an agent's motives; (2) under uncertainty, intertemporal substitution is considered before risk is considered; (3) the formula is very parsimonious, and at an effective selection of discount factors, Euler equations become very tractable; (4) the multiple discount factors capture the notion of gain/loss asymmetry, which explains asset behavior better than the discounted utility model.

In this paper, we only focus on the behavior of aggregate asset pricing. However, we believe that the most appropriate and interesting application of our utility representation is found in life-time consumption-saving problems. For example, we can investigate either empirically or by simulations the change of time-preferences through generations. For another dimension of research, we can perform an empirical test across countries to examine the differences in time-preferences and consumption-saving behavior.

Appendix 2.A: Proof of Proposition 2.3.1 and 2.3.2

For Proposition 2.3.1, refer to Wakai (2001). For Proposition 2.3.2, reverse all inequalities in the proof of Gilboa and Schmeidler (1989) and apply the same argument as in Proposition 2.3.1.■

Appendix 2.B: Problem of Dynamic Consistency under Uncertainty

We can naively apply (2.3.2) to an objective probability space of consumption streams. However, this application is not dynamically consistent. We illustrate this point by the following example. Suppose that there are three dates and four states. On date 0, an agent does not have any information about states. On date 1, an agent is informed that either (state1,state2) or (state2, state3) happened. On date 2, an agent knows all information about states. The probability of state realization is (0.25,0.25,0.25,0.25). Now an agent assigns a utility index following (2.3.2) with $u(c)$ =c and $[\alpha_t, \beta_t]$=[0.3,0.7] for t =1 and 2, and evaluates the following stream of consumptions:

Contract A

State\Time	0	1	2
1	1.5	2	2
2	1.5	2	1
3	1.5	2	2
4	1.5	2	1

Contract B

State\Time	0	1	2
1	1.5	3	1
2	1.5	3	1
3	1.5	2	2
4	1.5	2	1

Utility

Contract\Time	0	1 at (1,2)
Contract A	1.5050	1.65
Contract B	1.5175	1.60

Clearly, on date 0, an agent chooses Contract B, but after she realizes that she is at (state1, state2), she selects Contract A. This example violates dynamic consistency. Technically speaking, the reason why we face a problem of dynamic consistency is that under a payoff vector approach, an agent forms preference relations on an entire space of acts that do not consider how a state evolves. Since a dynamic consistency is related to the evolution of states, inconsistency emerges over time unless agent's preferences are time additive.

We can learn from this example that we need to use the recursive structure to resolve dynamic consistency. Having a dynamically consistent utility function (2.3.2) on each state is not enough because this function is not time additive.

Appendix 2.C:

The Proof follows Epstein and Wang (1994).

Step 1:

Theorem 2.C.1: (Aubin: 1979. P.118)

(2.C.1) $\mathfrak{P}$ is compact

(2.C.2) $\exists$ a neighborhood U of x s.t. for any $y \in U$:

$p \rightarrow f(y;p)$ is upper semi-continuous

(2.C.3) $\forall p \in \mathfrak{P}$, $y \rightarrow f(y;p)$ is convex and differentiable from the right.

(2.C.4) $g(y) = \sup_p f(y;p)$

(2.C.5) $P_0 = \{p \in \mathfrak{P} | g(x) = f(x;p)\}$

Then

$Dg(x)(y) = \sup_p Df(x;p)(y)$

Let $F(c,\delta) = V_t(c)(\omega^t)$, and $\Delta(t,\omega)$ be a collection of $\{\delta_{t+1}(\omega^t,\omega)\}_{\omega\in\Omega}$ where each $\delta_{t+1}(\omega^t,\omega) \in [\alpha_{t+1},\beta_{t+1}]$ with $0 < \alpha_{t+1} \leq \beta_{t+1} < 1$. It is clear that $\Delta(t,\omega)$ is compact, $F(c,\delta)$ is upper semi-continuous in δ, and $F(c,\delta)$ is *concave* and differentiable at all $\delta \in \Delta(t,\omega)$. By changing *sup* to *inf*, we can derive the right and left derivatives as supergradients instead of subgradients by the right differentiability of u. Then:

$DV_t(c)(\omega^t)(y) = \inf_\delta DF(c;\delta)(y)$ (right)

$DV_t(c)(\omega^t)(y) = \sup_\delta DF(c;\delta)(y)$ (left)

Note that by changing the sign of y, we can use the right differentiability to derive the left derivative.

Step 2:

Utility function (2.4.1) is non-time-separable and, as a result, discount factors depend on future states. To derive asset prices, we need to perturb all points in time and states because discount factors move over time. We construct equilibrium prices by backward induction to address this connection explicitly.

Proposition 2.C.1:

$\{q_t\}_{0\leq t\leq T}$ is an equilibrium price for asset i if and only if at all (t, ω^t), there exits $\{\delta^*_\tau\}_{t+2\leq\tau\leq T} \in \Delta^{**}(t+2, \omega^t, \omega)$ such that:

(2.C.6) $\quad \min_{\delta\in\Delta^*(t+1,\omega^t)}\max_{i\in I}$

$$|\mathrm{E}_t[(1\text{-}\delta_{t+1}(\omega^t,\omega))u'(e_t(\omega^t))(-q^i_t(\omega^t))$$

$$+\delta_{t+1}(\omega^t,\omega)A_{t+1}(\omega^t,\omega)u'(e_{t+1}(\omega^t,\omega))(q^i_{t+1}(\omega^t,\omega)+d^i_{t+1}(\omega^t,\omega))]|\text{=}0$$

where $A_{t+1}(\omega^t,\omega)$ =$\mathrm{E}_{t+1}[(1-\delta^*_{t+2}(\omega^t,\omega,\omega'))]$.

(2.C.7) $\quad \Delta^*(t+1,\omega^t)$ is recursively defined as:

$$\Delta^*(t+1,\omega^t)$$

$$=\{\{\delta_{t+1}(\omega^t,\omega)\}_{\omega\in\Omega}\in[\alpha_{t+1},\beta_{t+1}]^S|$$

$$\mathrm{E}_t[(1-\delta_{t+1}(\omega^t,\omega))u(e_t(\omega^t))+\delta_{t+1}(\omega^t,\omega)V_{t+1}(e)(\omega^t,\omega)]=V_t(e)(\omega^t,\omega)\}.$$

(2.C.8) $\quad \Delta^{**}(t+2,\omega^t,\omega)$

$$=\{\{\delta^*_\tau\}_{t+2\leq\tau\leq T}|\text{ the same }\{q_\tau\}_{t+1\leq\tau\leq T-1}\text{ recursively satisfies (2.C.6)}\}$$

(2.C.9) $\quad q^i_T(\omega^T)=0\ \forall\omega^T\in\Omega^T$ and $i\in I$.

Proof:

Necessity:

Suppose that $\{q_t\}_{0\leq t\leq T}$ is an equilibrium price with $\{c_t\}_{0\leq t\leq T}=\{e_t\}_{0\leq t\leq T}$ and $\{\theta^i_t\}_{0\leq t\leq T}$ $=\{0\}$. By assumption, $q^i_T(\omega^T)=0\ \forall\omega^T\in\Omega^T$ and $i\in I$. In addition, under the rational expectation, asset prices are dynamically consistent and determined recursively. At $(T-1,\omega^{T-1})$, consider the perturbation of the optimal policy by:

(2.C.10) $\quad c_{T-1}(\omega^{T-1}) = e_{T-1}(\omega^{T-1}) - \xi(\Lambda \cdot q_{T-1}(\omega^{T-1}))$ and $\theta_{T-1} = \xi\Lambda$

(2.C.11) $\quad c_T(\omega^{T-1}, \omega) = e_T(\omega^{T-1}, \omega) + \xi\Lambda \cdot d_T(\omega^{T-1}, \omega)$ on $\omega \in \Omega$

(2.C.12) $\quad \Lambda \in R^I$ and $\xi \in R$

This perturbation must make the representative agent worse off.

$$0 \in \arg\max_{\xi}\{V_t(e + \xi(0, ...0, -(\Lambda \cdot q_{T-1}(\omega^{T-1})), \{\Lambda \cdot d_T(\omega^{T-1}, \omega)\}_{\omega\in\Omega}))(\omega^t)\}$$

Apply Step 1 and define $\Delta^*(T-1, \omega^{T-1}) \subseteq R^S_{++}$ as:

(2.C.13) $\quad \Delta^*(T, \omega^{T-1})$

$$= \{\{\delta_T(\omega^{T-1}, \omega)\}_{\omega\in\Omega} \in [\alpha_T, \beta_T]^S |$$

$$\mathrm{E}_{T-1}[(1 - \delta_T(\omega^{T-1}, \omega))u(e_{T-1}(\omega^{T-1}))$$

$$+\delta_T(\omega^{T-1}, \omega)V_T(e)(\omega^{T-1}, \omega)] = V_{T-1}(e)(\omega^{T-1})\}$$

Then the first-order conditions become:

From a right derivative:

$\mathrm{Min}_{\delta\in\Delta^*(T,\omega^{T-1})}$

$\mathrm{E}_{T-1}[(1 - \delta_T(\omega^{T-1}, \omega))u'(e_{T-1}(\omega^{T-1}))(-\Lambda \cdot q_{T-1}(\omega^{T-1}))$

$+\delta_T(\omega^{T-1}, \omega)u'(e_T(\omega^{T-1}, \omega))(\Lambda \cdot d_T(\omega^{T-1}, \omega))] \leq 0$

From a left derivative:

$\mathrm{Max}_{\delta\in\Delta^*(T,\omega^{T-1})}$

$\mathrm{E}_{T-1}[(1 - \delta_T(\omega^{T-1}, \omega))u'(e_{T-1}(\omega^{T-1}))(-\Lambda \cdot q_{T-1}(\omega^{T-1}))$

$+\delta_T(\omega^{T-1},\omega)u'(e_T(\omega^{T-1},\omega))(\Lambda \cdot d_T(\omega^{T-1},\omega))] \geq 0$

Now, let $\Lambda' = -\Lambda$. Then the left derivative becomes:

-Min$_{\delta \in \Delta^*(T,\omega^{T-1})}$

$\mathrm{E}_{T-1}[(1-\delta_T(\omega^{T-1},\omega))u'(e_{T-1}(\omega^{T-1}))(-\Lambda' \cdot q_{T-1}(\omega^{T-1}))$

$+\delta_T(\omega^{T-1},\omega)u'(e_T(\omega^{T-1},\omega))(\Lambda' \cdot d_T(\omega^{T-1},\omega))] \geq 0$

$\Leftrightarrow$

Min$_{\delta \in \Delta^*(T,\omega^{T-1})}$

$\mathrm{E}_{T-1}[(1-\delta_T(\omega^{T-1},\omega))u'(e_{T-1}(\omega^{T-1}))(-\Lambda' \cdot q_{T-1}(\omega^{T-1}))$

$+\delta_T(\omega^{T-1},\omega)u'(e_T(\omega^{T-1},\omega))(\Lambda' \cdot d_T(\omega^{T-1},\omega))] \leq 0$

Combining two inequalities:

(2.C.14) Min$_{\delta \in \Delta^*(T,\omega^{T-1})}$

$\mathrm{E}_{T-1}[(1-\delta_T(\omega^{T-1},\omega))u'(e_{T-1}(\omega^{T-1}))(-\Lambda \cdot q_{T-1}(\omega^{T-1}))$

$+\delta_T(\omega^{T-1},\omega)u'(e_T(\omega^{T-1},\omega))(\Lambda \cdot d_T(\omega^{T-1},\omega))] \leq 0$

$\forall \Lambda \in R^N$

Rewrite (2.C.14):

(2.C.15) $\sup_{\Lambda \in R^I} \min_{\delta \in \Delta^*(T,\omega^{T-1})} G(\Lambda,\delta) \leq 0$

Since $G(.,\delta)$ is linearly homogeneous, (2.C.15) is equivalent to:

(2.C.16) $\max_{\Lambda \in \gamma} \min_{\delta \in \Delta^*(T,\omega^{T-1})} G(\Lambda,\delta) \leq 0$

where γ is the convex hull of {$\pm i$th coordinate vector: i=1,...,N}. Since γ and $\Delta^*(T,\omega^{T-1})$ are convex and compact, and $G(\Lambda,\delta)$ is linear in both arguments, applying Minmax theorem, there exists δ^* and Λ^* such that:

$$\max_{\Lambda\in\gamma}\min_{\delta\in\Delta^*(T,\omega^{T-1})}G(\Lambda,\delta) = G(\Lambda^*,\delta^*) = \min_{\delta\in\Delta^*(T,\omega^{T-1})}\max_{\Lambda\in\gamma}G(\Lambda,\delta)$$

Now suppose that $G(\Lambda^*,\delta^*) < 0$. Then by the fact that $\Lambda \in \gamma \Leftrightarrow$ -$\Lambda \in \gamma$, $G(-\Lambda^*,\delta^*) > 0$. Then $\max_{\Lambda\in\gamma}G(\Lambda,\delta^*) \geq G(-\Lambda^*,\delta^*) > 0$. Clearly, (δ^*,Λ^*) is not a solution of $\min_{\delta\in\Delta^*(T,\omega^{T-1})}\max_{\Lambda\in\gamma}G(\Lambda,\delta)$, which violates minmax theorem. Since $G(.,\delta)$ is linearly homogeneous, and $\Lambda \in \gamma \Leftrightarrow$ -$\Lambda \in \gamma$, for all $\Lambda' \in \gamma$:

$$\text{(2.C.17)} \qquad G(\Lambda',\delta^*) = \min_{\delta\in\Delta^*(T,\omega^{T-1})}\max_{\Lambda\in\gamma}G(\Lambda,\delta) = 0$$

Since for each δ, $G(\Lambda,\delta)$ is liner, max{$G(\Lambda,\delta)$ under $\Lambda \in \gamma$} is attained on the set of extreme points of γ. Therefore:

$$\text{(2.C.18)} \qquad \min_{\delta\in\Delta^*(T,\omega^{T-1})}\max_{i\in I}$$
$$|\mathrm{E}_{T-1}[(1\text{-}\delta_T(\omega^{T-1},\omega))u'(e_{T-1}(\omega^{T-1}))(-q^i_{T-1}(\omega^{T-1}))$$
$$+\delta_T(\omega^{T-1},\omega)u'(e_T(\omega^{T-1},\omega))(d^i_T(\omega^{T-1},\omega)]|=0$$

Let $\delta^* \in \Delta^*(T,\omega^{T-1})$ be an optional choice of discount factors. Define $\Delta^{**}(T,\omega^{T-1})$ as:

$$\text{(2.C.19)} \qquad \Delta^{**}(T,\omega^{T-1}) = \{\delta \in \Delta^*(T,\omega^{T-1})|\ G(\Lambda,\delta) = G(\Lambda,\delta^*)$$
$$\text{under identical } q^i_{T-1}(\omega^{T-1})\}$$

Now consider the perturbation of the optimal policy at $(T-2,\omega^{T-2})$ by:

(2.C.20) $\quad c_{T-2}(\omega^{T-2}) = e_{T-2}(\omega^{T-2}) - \xi(\Lambda \cdot q_{T-2}(\omega^{T-2}))$ and $\theta_{T-2} = \xi\Lambda$

(2.C.21) $\quad c_{T-1}(\omega^{T-2},\omega) = e_{T-1}(\omega^{T-2},\omega) + \xi\Lambda \cdot (q_{T-1}(\omega^{T-2},\omega) + d_{T-1}(\omega^{T-2},\omega))$

(2.C.22) $\quad \Lambda \in R^I$ and $\xi \in R$

This perturbation must make the representative agent worse off.

$$0 \in \arg\max_{\xi}\{V_{t,\omega}(e + \xi(0,...0,-(\Lambda \cdot q_{T-2}(\omega^{T-2})),$$
$$\{\Lambda \cdot (q_{T-1}(\omega^{T-2},\omega) + d_{T-1}(\omega^{T-2},\omega))\}_{\omega\in\Omega}, 0))\}$$

Apply Step 1 and define $\Delta^*(T-1,\omega^{T-2}) \subseteq R^S_{++}$ as:

(2.C.23) $\quad \Delta^*(T-1,\omega^{T-2})$

$$= \{\{\delta_{T-1}(\omega^{T-2},\omega)\}_{\omega\in\Omega} \in [\alpha_{T-1},\beta_{T-1}]^S|$$
$$\mathrm{E}_{T-2}[(1-\delta_{T-1}(\omega^{T-2},\omega))u(e_{T-2}(\omega^{T-2}))$$
$$+ \delta_{T-1}(\omega^{T-2},\omega)V_{T-1}(e)(\omega^{T-2},\omega)] = V_{T-2}(e)(\omega^{T-2})\}$$

Applying the same argument for time T-1, we conclude that under the rational expectations, there exits $\delta^*_T(\omega^{T-2},\omega,\omega') \in \Delta^{**}(T,\omega^{T-2},\omega)$ under which the following equation holds:

(2.C.24) $\quad \min_{\delta\in\Delta^*(T-1,\omega^{T-2})}\max_{i\in I}$

$$|\mathrm{E}_{T-2}[(1\text{-}\delta_{T-1}(\omega^{T-2},\omega))u'(e_{T-2}(\omega^{T-2}))(-q^i_{T-2}(\omega^{T-2}))$$
$$+\delta_{T-1}(\omega^{T-2},\omega)(1-\delta^*_T(\omega^{T-2},\omega,\omega'))$$
$$\cdot u'(e_{T-1}(\omega^{T-2},\omega))(q^i_{T-1}(\omega^{T-2},\omega) + d^i_{T-1}(\omega^{T-2},\omega))]|{=}0$$

Let $A_{T-1}(\omega^{T-2},\omega) = \mathrm{E}_{T-1}[(1-\delta^*_T(\omega^{T-2},\omega,\omega'))]$. Then:

(2.C.25) $\min_{\delta\in\Delta^*(T-1,\omega^{T-2})}\max_{i\in I}$

$$|\mathrm{E}_{T-2}[(1\text{-}\delta_{T-1}(\omega^{T-2},\omega))u'(e_{T-2}(\omega^{T-2}))(-q^i_{T-2}(\omega^{T-2}))$$

$$+\delta_{T-1}(\omega^{T-2},\omega)A_{T-1}(\omega^{T-2},\omega)$$

$$\cdot u'(e_{T-1}(\omega^{T-2},\omega))(q^i_{T-1}(\omega^{T-2},\omega)+d^i_{T-1}(\omega^{T-2},\omega))]|\text{=}0$$

Define $\Delta^{**}(T-1:T,\omega^{T-2}) \subseteq R^{S^2}_{++}$

(2.C.26) $\Delta^{**}(T-1:T,\omega^{T-2})$

$$=\{\delta\in\Delta^*(T-1,\omega^{T-2})\times\{\Delta^{**}(T,\omega^{T-2},\omega)\}_{\omega\in\Omega}|$$

$$\text{the same } \{q_t\}_{T-2\le t\le T-1} \text{ satisfies (2.C.18) and (2.C.25)}\}$$

The identical procedures is used to derive:

(2.C.27) $\min_{\delta\in\Delta^*(t+1,\omega^t)}\max_{i\in I}$

$$|\mathrm{E}_t[(1\text{-}\delta_{t+1}(\omega^t,\omega))u'(e_t(\omega^t))(-q^i_t(\omega^t))$$

$$+\delta_{t+1}(\omega^t,\omega)A_{t+1}(\omega^t,\omega)u'(e_{t+1}(\omega^t,\omega))(q^i_{t+1}(\omega^t,\omega)+d^i_{t+1}(\omega^t,\omega))]|\text{=}0$$

where $t\le\tau<T$ with $A_{t+1}(\omega^t,\omega)$ $=\mathrm{E}_{t+1}[(1-\delta^*_{t+2}(\omega^t,\omega,\omega'))]$, and $\Delta^*(t+1,\omega^t)$ is recursively defined as:

(2.C.28) $\Delta^*(t+1,\omega^t)$

$$=\{\{\delta_{t+1}(\omega^t,\omega)\}_{\omega\in\Omega}\in[\alpha_{t+1},\beta_{t+1}]^S|$$

$$\mathrm{E}_t[(1-\delta_{t+1}(\omega^t,\omega))u(e_t(\omega^t))+\delta_{t+1}(\omega^t,\omega)V_{t+1}(e)(\omega^t,\omega)]=V_t(e)(\omega^t,\omega)\}$$

Also $\Delta^{**}(t+2:T,\omega^{t+1})$ is used to obtain (2.C.28) where:

(2.C.29) $\Delta^{**}(t{+}2, \omega^{t+1}) = \{\{\delta_\tau\}_{t+2\leq\tau\leq T} \mid$ the same $\{q_\tau\}_{t+1\leq\tau\leq T-1}$ recursively satisfies (2.C.27)$\}$

Then we obtain (2.C.6) to (2.C.9).

Sufficiency:

Suppose that (2.C.6) to (2.C.9) hold. Then with $\theta_{t-1}(\omega^{t-1}) \equiv 0$ $\forall\omega^{t-1} \in \Omega^{t-1}$ and θ_τ $\forall\tau$ s.t. $t \leq \tau \leq T$:

(2.C.30) $\quad e_\tau(\omega^\tau) - c_\tau(\omega^\tau) - \theta_\tau(\omega^\tau) \cdot q_\tau(\omega^\tau) = -\theta_{\tau-1}(\omega^{\tau-1})(q_\tau(\omega^\tau) + d_\tau(\omega^\tau))$

(2.C.31) $\quad$ There is some $\widehat{\delta} \in \Delta^*(\tau+1, \omega^\tau)$ such that

$$\mathrm{E}_\tau[A_\tau(\omega^\tau)u'(e_\tau(\omega^\tau))(-q_\tau(\omega^\tau)) \cdot \theta_t(\omega)]$$

$$\leq -\mathrm{E}_\tau[\widehat{\delta}_{\tau+1}(\omega^\tau,\omega)A_{\tau+1}(\omega^{\tau+1},\omega)$$

$$\cdot u'(e_{\tau+1}(\omega^\tau,\omega))(q^i_{\tau+1}(\omega^\tau,\omega) + d^i_{\tau+1}(\omega^\tau,\omega)) \cdot \theta_\tau(\omega^\tau)]$$

with $A_\tau(\omega^\tau) = \mathrm{E}_\tau[1 - \widehat{\delta}_{\tau+1}(\omega^\tau,\omega)]$.

(2.C.32) $\quad u(c_\tau(\omega^\tau)) - u(e_\tau(\omega^\tau)) \leq -u'(e_\tau(\omega^\tau))(e_\tau(\omega^\tau) - c_\tau(\omega^\tau))$

where (2.C.31) is from (2.C.6) (i.e., there exists $\widehat{\delta} \in \Delta^*(\tau+1, \omega^\tau)$ that satisfies (2.C.16) at (τ, ω^τ)) and (2.C.32) is from the concavity of u. First, observe that:

$$\mathrm{E}_t[(1\text{-}\widehat{\delta}_{t+1}(\omega^t,\omega))u(c_t(\omega^t)) + \widehat{\delta}_{t+1}(\omega^t,\omega)V_{t+1}(c)(\omega^t,\omega))] \geq V_t(c)(\omega^t)$$

Then:

$V_t(c)(\omega^t) - V_t(e)(\omega^t)$

$\leq \mathrm{E}_t[A_t(\omega^t)(-u'(e_t(\omega^t))(e_t(\omega^t) - c_t(\omega^t))) + \widehat{\delta}_{t+1}(\omega^t, \omega)(V_{t+1}(c)(\omega^t, \omega) - V_{t+1}(e)(\omega^t, \omega))]$

$= \mathrm{E}_t[A_t(\omega^t)(-u'(e_t(\omega^t))\theta_t(\omega^t) \cdot q_t(\omega^t) + \widehat{\delta}_{t+1}(\omega^t, \omega)(V_{t+1}(c)(\omega^t, \omega) - V_{t+1}(e)(\omega^t, \omega))]$

$\leq \mathrm{E}_t[\widehat{\delta}_{t+1}(\omega^t, \omega)A_{t+1}(\omega^{t+1}, \omega)(-u'(e_{t+1}(\omega^t, \omega))\theta_t(\omega^t) \cdot (q^i_{t+1}(\omega^t, \omega) + d^i_{t+1}(\omega^t, \omega)))$

$+\widehat{\delta}_{t+1}(\omega^t, \omega)(V_{t+1}(c)(\omega^t, \omega) - V_{t+1}(e)(\omega^t, \omega))]$

$= \mathrm{E}_t[\widehat{\delta}_{t+1}(\omega^t, \omega)A_{t+1}(\omega^{t+1}, \omega)u'(e_{t+1}(\omega^t, \omega))$

$\cdot(e_{t+1}(\omega^t, \omega) - c_{t+1}(\omega^t, \omega) - \theta_{t+1}(\omega^t, \omega) \cdot q_{t+1}(\omega^t, \omega))$

$+\widehat{\delta}_{t+1}(\omega^t, \omega)(V_{t+1}(c)(\omega^t, \omega) - V_{t+1}(e)(\omega^t, \omega))]$

$\leq \mathrm{E}_t[\widehat{\delta}_{t+1}(\omega^t, \omega)[A_{t+1}(\omega^{t+1}, \omega)(-u'(e_{t+1}(\omega^t, \omega))\theta_{t+1}(\omega^t, \omega) \cdot q_{t+1}(\omega^t, \omega))$

$+(V_{t+2}(c)(\omega^t, \omega, \omega') - V_{t+2}(e)(\omega^t, \omega, \omega'))]]$

...

$= \mathrm{E}_t[\widehat{\delta}_{t+1}(\omega^t, \omega)\widehat{\delta}_{t+2}(\omega^t, \omega, \omega')...\widehat{\delta}_T(\omega^t, \omega^{T-t})$

$[(-u'(e_T(\omega^t, \omega^{T-t}))\theta_{T-1}(\omega^t, \omega^{T-t}) \cdot (q^i_T(\omega^t, \omega^{T-t}) + d^i_T(\omega^t, \omega^{T-t})))$

$+(u(c_T(\omega^t, \omega^{T-t})) - u(e_T(\omega^t, \omega^{T-t})))]]$

$= \mathrm{E}_t[\widehat{\delta}_{t+1}(\omega^t, \omega)\widehat{\delta}_{t+2}(\omega^t, \omega, \omega')...\widehat{\delta}_T(\omega^t, \omega^{T-t})$

$[(u'(e_T(\omega^t, \omega^{T-t}))(e_T(\omega^t, \omega^{T-t}) - c_T(\omega^t, \omega^{T-t}))) + (u(c_T(\omega^t, \omega^{T-t})) - u(e_T(\omega^t, \omega^{T-t})))]]$

≤ 0

Hence, $\{e_t\}_{0 \leq t \leq T}$ is optimal if asset prices follow (2.C.6) to (2.C.9).■

To gain more insight from (2.C.6), we provide an additional proposition that describes the property of equilibrium prices. Let $\Delta \subset R^{S^{T+1}}_{++}$ be a collection of $\{\delta_t\}_{0 \leq t \leq T}$ where each $\delta_t(\omega^{t-1}, \omega) \in [\alpha_t, \beta_t]$ with $0 < \alpha_t \leq \beta_t < 1$ for $0 < t \leq T$, and we define $\delta_0(\omega_0) = 1$.

Proposition 2.C.2:

$\{q_t\}_{0\leq t\leq T}$ is an equilibrium price for asset i if and only if $\exists\delta \in \Delta^*$ such that at all (t,ω^t):

(2.C.33) $\mathrm{E}_t[(1\text{-}\delta_{t+1}(\omega^t,\omega))(-q_t^i(\omega^t))$

$$+\delta_{t+1}(\omega^t,\omega)A_{t+1}(\omega^t,\omega)\frac{u'(e_{t+1}(\omega^t,\omega))}{u'(e_t(\omega^t))}(q_{t+1}^i(\omega^t,\omega)+d_{t+1}^i(\omega^t,\omega))] = 0 \ \ \forall i.$$

(2.C.34) The set of equilibrium prices are closed.

(2.C.35) $A_t(\omega^t) = \mathrm{E}_\tau[1-\delta_{t+1}(\omega^t,\omega)]$.

(2.C.36) Δ^*

$$= \{\delta \in \Delta|\text{ at all }(t,\omega^t)\text{ s.t. }0\leq t<T:$$

$$\mathrm{E}_t[(1-\delta_{t+1}(\omega^t,\omega))u(e_t(\omega^t))+\delta_{t+1}(\omega^t,\omega)V_{t+1}(e)(\omega^t,\omega)] = V_t(e)(\omega^t)$$

$$\text{and }\delta_0(\omega_0)\equiv 1.\}.$$

Proof:

From Proposition 2.C.1, there exist discount factors $\delta^* \in \Delta^* \subseteq R_{++}^{ST+1}$ that achieve minmax points of (2.C.6) from 1 to $T-1$. Since u is increasing, we can divide (2.C.6) by $u'(e_t(\omega^t))$. Now take a sequence of equilibrium prices. For each points, they must satisfy (2.C.6) to (2.C.9). By the continuity of $G(\Lambda,\delta)$ and compactness of Δ and γ, the limiting point of (δ,Λ) is also in Δ and γ (because a range is compact). Hence, a set of equilibrium prices and a set of corresponding Δ^* is closed.■

Equality (2.C.6) is a global characteristic that shows that asset prices simultaneously satisfy the same conditions under identical discount factors. Equality (2.C.33) restates the property of equilibrium prices implied by (2.C.6), and it shows that at any equilibrium, there is a sequence of discount factors $\delta \in \Delta^*$ that justify all asset prices. Given (2.C.33), we can write asset prices by Euler equations with some discount factors. The closure property of (2.C.35) is clear given compactness of Δ and continuity of u.

Appendix 2.D:

If there are no ties between $u(e_t(\omega^t))$ and $V_{t+1}(c)(\omega^t,\omega)$, a selection of discount factors is unique. Then $V_t(c)(\omega^t)$ is differentiable at e and by the nature of representative agent model, asset prices are uniquely determined. From (2.D.33):

$$\begin{aligned}&\mathrm{E}_t[(1\text{-}\delta_{t+1}(\omega^t,\omega))(-q_t^i(\omega^t))\\ &\quad+\delta_{t+1}(\omega^t,\omega)A_{t+1}(\omega^t,\omega)\frac{u'(e_{t+1}(\omega^t,\omega))}{u'(e_t(\omega^t))}(q_{t+1}^i(\omega^t,\omega)+d_{t+1}^i(\omega^t,\omega))]=0 \quad \forall i.\end{aligned}$$

Then:

$$\begin{aligned}(2.D.1)\qquad &A_t(\omega^t)(-q_t^i(\omega^t))\\ &+\mathrm{E}_t[\delta_{t+1}(\omega^t,\omega)A_{t+1}(\omega^t,\omega)\frac{u'(e_{t+1}(\omega^t,\omega))}{u'(e_t(\omega^t))}(q_{t+1}^i(\omega^t,\omega)+d_{t+1}^i(\omega^t,\omega))]=0\end{aligned}$$

$$(2.D.2)\qquad A_t(\omega^t)=\mathrm{E}_t[(1-\delta_{t+1}(\omega^t,\omega))]$$

By simple manipulation:

$$(2.D.3)\qquad A_t(\omega^t)(-q_t^i(\omega^t))\ +\ B_t(\omega^t)\mathrm{E}_t[\frac{u'(e_{t+1}(\omega^t,\omega))}{u'(e_t(\omega^t))}(q_{t+1}^i(\omega^t,\omega)+d_{t+1}^i(\omega^t,\omega))]=0$$

$$(2.D.4)\qquad B_t(\omega^t)=\mathrm{E}_t[\delta_{t+1}(\omega^t,\omega)A_{t+1}(\omega^t,\omega)]$$

$$(2.D.5)\qquad \pi_{t,\delta}(\omega^t,\omega)=\pi(\omega^t,\omega)\frac{\delta_{t+1}(\omega^t,\omega)A_{t+1}(\omega^t,\omega)}{B_t(\omega^t)}$$

where $\pi(\omega^t, \omega)$ is the probability of state ω at time $t+1$ on the path from ω^t. Hence:

$$(2.D.6) \qquad q_t^i(\omega^t) = \mathrm{E}_t[\frac{\delta_{t+1}(\omega^t,\omega)A_{t+1}(\omega^t,\omega)}{A_t(\omega^t)}\frac{u'(e_{t+1}(\omega^t,\omega))}{u'(e_t(\omega^t))}(q_{t+1}^i(\omega^t,\omega)+d_{t+1}^i(\omega^t,\omega))]$$

$$(2.D.7) \qquad q_t^i(\omega^t) = \frac{B_t(\omega^t)}{A_t(\omega^t)}\mathrm{E}_{t,\delta}[\frac{u'(e_{t+1}(\omega^t,\omega))}{u'(e_t(\omega^t))}(q_{t+1}^i(\omega^t,\omega)+d_{t+1}^i(\omega^t,\omega))]$$

$$(2.D.8) \qquad A_t(\omega^t) = \mathrm{E}_t[(1-\delta_{t+1}(\omega^t,\omega))]$$

$$(2.D.9) \qquad B_t(\omega^t) = \mathrm{E}_t[\delta_{t+1}(\omega^t,\omega)A_{t+1}(\omega^t,\omega)]$$

$$(2.D.10) \qquad \pi_{t,\delta}(\omega^t,\omega) = \pi(\omega^t,\omega)\frac{\delta_{t+1}(\omega^t,\omega)A_{t+1}(\omega^t,\omega)}{B_t(\omega^t)}$$

In (2.D.7), equilibrium prices are determined as if the representative agent used a 'preference-adjusted prior' $\pi_{t,\delta}(\omega^t,\omega)$ with 'preference-adjusted average normalized discount factors' $\frac{B_t(\omega^t)}{A_t(\omega^t)}$. Then (2.D.7) becomes a standard Euler equation. However, note that the representative agent never uses this preference-adjusted priors for her decision making. The equilibrium asset pricing simply implies that the usage of preference-adjusted priors can be justified at equilibrium. Equation (2.D.7) is the just mathematically equivalent way to write Equation (2.D.6).■

Appendix 2.E

For this comparison, we focus on a two-period economy with two states. Let consumptions be (c_0,$c_{1,1}$,$c_{1,2}$), and u be increasing. We define the intertemporal version of the multiple priors model of Gilboa and Schmeidler (1989) from Epstein and Schneider (2001):

$$(2.E.1) \qquad V(c) = A[u(c_0) + \beta \mathrm{Min}_{p\in P}[pu(c_{1,1}) + (1-p)u(c_{1,2})]]]$$

where P is a convex and closed set of priors for state 1. Now we define the multiple discount factors of (2.4.1) under the priors of state realizations (0.5,0.5) as follows:

$$\text{(2.E.2)} \qquad V(c) = 0.5[\text{Min}_{\delta\in[\alpha,\beta]}\delta u(c_0) + (1-\delta)u(c_{1,1})]$$
$$+0.5[\text{Min}_{\delta\in[\alpha,\beta]}\delta u(c_0) + (1-\delta)u(c_{1,2})]]$$
$$\text{with } \alpha = 0.3 \text{ and } \beta = 0.7$$

First we investigate two cases:

Case 1: $c_0 < c_{1,1}$ and $c_0 > c_{1,2}$

$$V(c) = 0.50[u(c_0) + \frac{0.5}{0.5}(\frac{0.3\cdot 0.5}{0.5}u(c_{1,1}) + \frac{0.7\cdot 0.5}{0.5}u(c_{1,2}))]$$

Case 2: $c_0 > c_{1,1}$ and $c_0 < c_{1,2}$

$$V(c) = 0.50[u(c_0) + \frac{0.5}{0.5}(\frac{0.7\cdot 0.5}{0.5}u(c_{1,1}) + \frac{0.3\cdot 0.5}{0.5}u(c_{1,2}))]$$

The above choices of discount factors can be justified by the multiple priors model with $A = 0.5$, $\beta = 1$ and $P = [0.3,0.7]$. For example, we can consider $[\frac{0.3\cdot 0.5}{0.5}, \frac{0.7\cdot 0.5}{0.5}]$ to be a 'preference-adjusted prior' for (state 1 and state 2) under Case 1. However, as the next choice shows, (2.E.2) cannot be represented by (2.E.1).

Case 3: $c_0 < c_{1,1}$ and $c_0 < c_{1,2}$ and $c_{1,1} > c_{1,2}$

$$\text{(2.E.3)} \qquad V(c) = 0.70[u(c_0) + \frac{0.3}{0.7}(0.5u(c_{1,1}) + 0.5u(c_{1,2}))]$$

First, the multiple priors model selects the prior of (0.3,0.7) instead of (0.5,0.5). The selection of (0.5,0.5) cannot be justified by $P = [0.3,0.7]$. Second, β moves from 1 to $\frac{0.3}{0.7}$ and A moves from 0.5 to 0.7. This movement of the time-preference is absent in (2.E.1).

To conclude that (2.E.2) cannot be represented by (2.E.1), we need to assume some functional form in u, and show that (2.E.3) cannot be written as (2.E.1). Assume that u is risk neutral, and that $(c_0,c_{1,1},c_{1,2}) = (1,3,2)$. (2.E.3) can be rewritten as:

(2.E.4) $$V(c) \simeq 0.5[u(c_0) + 0.54(0.3u(c_{1,1}) + 0.7u(c_{1,2}))]$$

or

(2.E.5) $$V(c) \simeq 0.5[u(c_0) + 1(-0.75u(c_{1,1}) + 1.75u(c_{1,2}))]$$

Clearly, the prior of (0.3,0.7) cannot justify the preference-adjusted discount factor of 1, and the preference-adjusted discount factor of 1 cannot justify the preference-adjusted prior of (0.3,0.7).

The above examples show that the recursive multiple priors model by Epstein and Schneider (2001) and our formula (2.4.1) are different. Since our model is based on intertemporal substitution (i.e., time-variability aversion), an agent does not consider the set of priors over states. 'Preference-adjusted priors' offer mere convenience to express the qualitative feature of time-variability aversion. In fact, the multiple discount factors model is more parsimonious than the multiple priors model because the degree of freedom to change the priors are much higher than the degree of freedom to change discount factors over time.

Appendix 2.F

Sufficiency:

Step 1:

There exits a closed and convex set of discount factors, $\Delta_{(t,\omega)}$,

with $\delta_\tau > 0$ *and* $\sum_{\tau=t}^{T} \delta_\tau = 1$ *s.t.*

For $f, g \in \mathfrak{A}_{cty}$, $f \succeq_{(t,\omega)} g \Leftrightarrow$

$W_{(t,\omega)}(f) = W_{(t,\omega)}(f^t, f^{t+1}, \ldots f^T) = W_{(t,\omega)}(f^t(\omega), \ldots, f^T(\omega))$

$= Min_{\delta \in \Delta_{(t,\omega)}}[\sum_{\tau=t}^{T} \delta_\tau u_{(t,\omega)}(f^\tau(\omega))]$

$\geq Min_{\delta \in \Delta_{(t,\omega)}}[\sum_{\tau=t}^{T} \delta_\tau u_{(t,\omega)}(g^\tau(\omega))] = W_{(t,\omega)}(g)$

Moreover, $\Delta_{(t,\omega)}$ *is a unique, and* $u_{(t,\omega)}$ *is unique up to a positive affine transformation.*

From Axioms 2.7.1 to 2.7.3 and 2.7.5 to 2.7.10, there is a utility function that represents preference relations on $\mathfrak{A}_c$ at (t,ω), which is a von Neuman-Morgenstern utility function on Y. Denote this representation as $u_{(t,\omega)}(y)$ for $y \in Y$. Since there is no uncertainty here, by Axioms 2.7.8 to 2.7.10, we obtain Theorem 2.3.1 over time from t to T. By Axiom 2.8.7 (strict monotonicity), $\delta_\tau > 0$.

Step 2:

At $0 \leq t < T$, *assignment of discount factors must follow* $\Delta_{(t,\omega)} = D_{(t,\omega)} \otimes \Lambda_{(t,\omega)}$ *s.t.*

For $f \in \mathfrak{A}_{cty}$:

$W_{(t,\omega)}(f)= Min_{\delta_{t+1}\in D_{(t,\omega)}}[(1-\delta_{t+1})u_{(t,\omega)}(f^t(\omega))$

$+\delta_{t+1}\min_{\lambda\in\Lambda_{(t,\omega)}}[\sum_{\tau=t+1}^{T}\lambda_\tau u_{(t,\omega)}(f^\tau(\omega))]$

For $f\in\mathfrak{A}_{c(t+1)}$:

$W_{(t,\omega)}(f)=W_{(t,\omega)}(f^t(\omega),f^{t+1}(\omega))=W_{(t,\omega)}(f^t,f^{t+1},\ldots f^T)$

$= Min_{\delta_{t+1}\in D_{(t,\omega)}}[(1-\delta_{t+1})u_{(t,\omega)}(f^t(\omega))+\delta_{t+1}u_{(t,\omega)}(f^{t+1}(\omega))]$

where $D_t = [\alpha_{t+1},\beta_{t+1}]$ *with* $0<\alpha_{t+1}\leq\beta_{t+1}<1$.

At T, $W_{(T,\omega)}(f) = u_{(T,\omega)}(f^T(\omega))$

The first result is from Proposition 2.3.1 applying between t and t+1 under Axiom 2.7.4. Given this result, the second result is immediate. In fact, for $f\in\mathfrak{A}_{c(t+1)}$, assignment of discount factors among lotteries from t+1 to T is indeterminate. Even without dynamic consistency, assignment of discount factors only depends on $f^t(\omega)$ and $f^{t+1}(\omega)$ because all future consumptions are same from t+1 onward, and minimizing a weighted sum under multiple discount factors for an entire sequence is achieved if and only if we allocation the lowest discount factors on $u_{(t,\omega)}(f^t(\omega))$ when $u_{(t,\omega)}(f^t(\omega))>u_{(t,\omega)}(f^{t+1}(\omega))$ and vice versa. We do not consider the shape of $\Lambda_{(t,\omega)}$ at this point. The last result is also immediate from Step 1.

From Step 3 to Step 10, we assume that $0\leq t<T$.

Step 3:

For $f,g\in\mathfrak{A}_{cs(t,t)}$,

There exits a weight on an event $\mathfrak{F}_{t+1}$ *s.t.* $\{\mu(\mathfrak{F}_{t+1}|\mathfrak{F}_t(\omega))\}$ *with* $\mu(\mathfrak{F}_{t+1}|\mathfrak{F}_t(\omega)) > 0$ $\forall \mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)$ *and* $\sum_{\mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)} \mu(\mathfrak{F}_{t+1}|\mathfrak{F}_t(\omega)) = 1$:

$f \succeq_{(t,\omega)} g \Leftrightarrow$

$U_{(t,\omega)}(f) = \sum_{\mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)} u_{(t,\omega)}(f^t(\omega'))\mu(\mathfrak{F}_{t+1}|\mathfrak{F}_t(\omega))$

$\geq U_{(t,\omega)}(g) = \sum_{\mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)} u_{(t,\omega)}(g^t(\omega'))\mu(\mathfrak{F}_{t+1}|\mathfrak{F}_t(\omega))$

where for each $\mathfrak{F}_{t+1}$, *we select one of* $f^t(\omega')$ *and* $g^t(\omega')$ *on* $\omega' \in \mathfrak{F}_{t+1}$.

Moreover,

$\mu(\mathfrak{F}_{t+1}|\mathfrak{F}_t(\omega))$ *is unique, and* $u_{(t,\omega)}$ *is unique up to a positive affine transformation.*

First, $\mathfrak{A}_{cs(t,t)}$ is a mixture space. From Axioms 2.7.5 to 2.7.8, and 2.7.11, preference relation on $\mathfrak{A}_{cs(t,t)}$ is represented by (Kreps:1988):

$f \succeq_{(t,\omega)} g$

$\Leftrightarrow U_{(t,\omega)}(f) \geq U_{(t,\omega)}(g)$ where $U_{(t,\omega)}(f) = \sum_{\mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)} F_{\mathfrak{F}_{t+1}}(f)$.

Now Axiom 2.7.12, $F_{\mathfrak{F}_{t+1}}(f)$ must represent the preference relation of $f \in \mathfrak{A}_{c(t)} \subset \mathfrak{A}_{cty}$. By the linearity of $F_{\mathfrak{F}_{t+1}}(f)$, and Step 1:

$F_{\mathfrak{F}_{t+1}}(f) = a_{\mathfrak{F}_{t+1}} u_{(t,\omega)}(f^{t+1}(\omega')) + b_{\mathfrak{F}_{t+1}}$ where we select one of $f^{t+1}(\omega')$ on $\omega' \in \mathfrak{F}_{t+1}$.

Note that $u_{(t,\omega)}$ is unique up to a positive affine transformation. Following Kreps (1988), this proves the existence of a unique subjective prior μ over $\mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)$ s.t. $\mu(\mathfrak{F}_{t+1}|\mathfrak{F}_t(\omega))$ $= \dfrac{a_{\mathfrak{F}_{t+1}}}{\sum_{\mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)} a_{\mathfrak{F}_{t+1}}}$, $\mu(\mathfrak{F}_{t+1}|\mathfrak{F}_t(\omega)) > 0$ $\forall \mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)$ and $\sum_{\mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)} \mu(\mathfrak{F}_{t+1}|\mathfrak{F}_t(\omega)) = 1$. Then the above formula becomes:

$U_{(t,\omega)}(f) = \sum_{\mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)} u_{(t,\omega)}(f^t(\omega'))\mu(\mathfrak{F}_{t+1}|\mathfrak{F}_t(\omega))$ where for each $\mathfrak{F}_{t+1}$, we select one of $f^t(\omega')$ on $\omega' \in \mathfrak{F}_{t+1}$ (because $f^t(\omega') = f^t(\omega'')$ on $\omega', \omega'' \in \mathfrak{F}_{t+1}$).

Non-negativity of $\mu(\mathfrak{F}_{t+1}|\mathfrak{F}_t(\omega))$ is from Axiom 2.7.12. Note that $\mu(\mathfrak{F}_{t+1}|\mathfrak{F}_t(\omega))$ is a weight for $\mathfrak{F}_{t+1}$, not for ω' s.t. $\omega' \in \mathfrak{F}_{t+1}$.

Step 4:

$\forall\ f \in \mathfrak{A}_f$, $\exists g \in \mathfrak{A}_c$ *s.t.* $f \simeq_{(t,\omega)} g\ \forall 1 \leq t \leq T$ *and* $\omega \in \Omega$.

Axioms 2.7.5 to 2.7.7, for any $f \in \mathfrak{A}_f$, there is a constant act that achieves the same utility on $\succeq_{(t,\omega)}$.

Step 5:

For any $f \in \mathfrak{A}_f$, *there is a* $g \in \mathfrak{A}_{cs(t,t+1)}$*:*

(i) $f^t(\omega')$=$g^t(\omega')$ *on* $\omega' \in \mathfrak{F}_t(\omega)$

(ii) $g^{\omega''}(\tau) = g^{\omega''}(\tau')$ *on* $\omega'' \in \mathfrak{F}_{t+1}(\omega') \subset \mathfrak{F}_t(\omega)$ *for all* τ, τ' s.t. $t < \tau, \tau' \leq T$

(iii) $f \simeq_{(t+1,\omega)} g$

Then $f \simeq_{(t,\omega)} g$. *Call this* g *as* $G_{(t,\omega)}(f)$.

By applying Step 4 at time t+1 and Axiom 2.7.4 (dynamic consistency), we can replace all lotteries in f from t+1 to T on an event $\mathfrak{F}_{t+1}$ to an identical lottery.

Step 6:

For any $f, g \in \mathfrak{A}_f$, $f \succeq_{(t,\omega)} g \Leftrightarrow G_{(t,\omega)}(f) \succeq_{(t,\omega)} G_{(t,\omega)}(g)$

By Axiom 2.7.5 (transitivity).

Step 7:

For any $f \in \mathfrak{A}_f$, *there is a* $g \in \mathfrak{A}_{cs(t,t)}$ *s.t.* $f \simeq_{(t,\omega)} g$. *Call this* g *as* $K_{(t,\omega)}(f)$.

Under Axiom 2.7.12, any act in $\mathfrak{A}_{cs(t,t+1)}$ is represented by an act in $\mathfrak{A}_{cs(t,t)}$ by replacing each state act to a constant state act with $g^{\omega'}$ that is calculated by the formula in Step 2. Using the results in Step 6:

$$g_\tau^{\omega'} = (1-\delta_{(t+1,\omega')})G_{(t,\omega)}(f)_t^{\omega'} + \delta_{(t+1,\omega')}G_{(t,\omega)}(f)_{t+1}^{\omega'} \ \forall t \leq \tau \leq T$$

where $\delta_{(t+1,\omega')} = \text{argmin}_{\delta_{t+1,\omega'} \in D_{(t,\omega)}}[(1-\delta_{t+1,\omega'})u_{(t,\omega)}(G_{(t,\omega)}(f)_t^{\omega'})$
$+\delta_{t+1,\omega'}u_{(t,\omega)}(G_{(t,\omega)}(f)_{t+1}^{\omega'})]$

Since $K_{(t,\omega)}(f) \in \mathfrak{A}_{cs(t,t)}$, by applying formula $U_{(t,\omega)}(.)$ in Step 3:

$$\begin{aligned}
&U_{(t,\omega)}(K_{(t,\omega)}(f)) \\
&= \textstyle\sum_{\mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)} u_{(t,\omega)}(K_{(t,\omega)}(f)^t(\omega'))\mu(\mathfrak{F}_{t+1}|\mathfrak{F}_t(\omega)) \\
&= \textstyle\sum_{\mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)} u_{(t,\omega)}((1-\delta_{(t+1,\omega')})G_{(t,\omega)}(f)_t^{\omega'} + \delta_{(t+1,\omega')}G_{(t,\omega)}(f)_{t+1}^{\omega'})\mu(\mathfrak{F}_{t+1}|\mathfrak{F}_t(\omega)) \\
&= \textstyle\sum_{\mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)}[(1-\delta_{(t+1,\omega')})u_{(t,\omega)}(G_{(t,\omega)}(f)_t^{\omega'})+\delta_{(t+1,\omega')}u_{(t,\omega)}(G_{(t,\omega)}(f)_{t+1}^{\omega'})]\mu(\mathfrak{F}_{t+1}|\mathfrak{F}_t(\omega)) \\
&= \textstyle\sum_{\mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)}[\min_{\delta_{t+1,\omega'} \in D_{(t,\omega)}}[(1-\delta_{t+1,\omega'})u_{(t,\omega)}(G_{(t,\omega)}(f)_t^{\omega'}) \\
&\qquad +\delta_{t+1,\omega'}u_{(t,\omega)}(G_{(t,\omega)}(f)_{t+1}^{\omega'})]\ ]\mu(\mathfrak{F}_{t+1}|\mathfrak{F}_t(\omega))
\end{aligned}$$

$= \sum_{\mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)} [\min_{\delta_{t+1,\omega'} \in D_{(t,\omega)}} [(1 - \delta_{t+1,\omega'}) u_{(t,\omega)}(f^t(\omega'))$

$+\delta_{t+1,\omega'} u_{(t,\omega)}(G_{(t,\omega)}(f)^{\omega'}_{t+1})] \,] \mu(\mathfrak{F}_{t+1} | \mathfrak{F}_t(\omega))$

Step 8:

For any $f, g \in \mathfrak{A}_f$, $f \succeq_{(t,\omega)} g \Leftrightarrow K_{(t,\omega)}(f) \succeq_{(t,\omega)} K_{(t,\omega)}(g) \Leftrightarrow U_{(t,\omega)}(K_{(t,\omega)}(f)) \geq U_{(t,\omega)}(K_{(t,\omega)}(f))$ *where* $U_{(t,\omega)}(.)$ *is defined in Step 3.*

By transitivity (Axiom 2.7.5),

$f \succeq_{(t,\omega)} g \Leftrightarrow G_{(t,\omega)}(f) \succeq_{(t,\omega)} G_{(t,\omega)}(g) \Leftrightarrow K_{(t,\omega)}(f) \succeq_{(t,\omega)} K_{(t,\omega)}(g)$

$\Leftrightarrow U_{(t,\omega)}(K_{(t,\omega)}(f)) \geq U_{(t,\omega)}(K_{(t,\omega)}(f))$

Step 9:

For any $f \in \mathfrak{A}_f$, *there is a* $g \in \mathfrak{A}_c$*:*

At all (τ,ω'') *s.t.* $t \leq \tau \leq T$ *and* $\omega'' \in \mathfrak{F}_t(\omega)$:

$g^\tau(\omega'') = \sum_{\mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)} [(1 - \delta_{(t+1,\omega')}) G_{(t,\omega)}(f)^{\omega'}_t + \delta_{(t+1,\omega')} G_{(t,\omega)}(f)^{\omega'}_{t+1}] \mu(\mathfrak{F}_{t+1} | \mathfrak{F}_t(\omega)) \in Y$

where for each $\mathfrak{F}_{t+1}$, *we select one of* $G_{(t,\omega)}(f)^{\omega'}_t$ *and* $G_{(t,\omega)}(f)^{\omega'}_{t+1}$ *on* $\omega' \in \mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)$.

And $\delta_{(t+1,\omega')}$ *is an effective discount factor from* $W_{(t,\omega)}(.)$ *in Step 2 applying to* $G_{(t,\omega)}(f)$ *at* ω'. *Then* $f \simeq_{(t,\omega)} g$. *Call this* g *as* $C_{(t,\omega)}(f)$ *and* $g^\tau(\omega')$ *as* $y_{(t,\omega)}(f)$.

Since $g \in \mathfrak{A}_c$, apply $u_{(t,\omega)}$ over $y_{(t,\omega)}(f)$:

$u_{(t,\omega)}(y_{(t,\omega)}(f))$

$= u_{(t,\omega)}(\sum_{\mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)} [(1 - \delta_{(t+1,\omega')}) G_{(t,\omega)}(f)^{\omega'}_t + \delta_{(t+1,\omega')} G_{(t,\omega)}(f)^{\omega'}_{t+1}] \mu(\mathfrak{F}_{t+1} | \mathfrak{F}_t(\omega)))$

$= \sum_{\mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)} [(1-\delta_{(t+1,\omega')}) u_{(t,\omega)}(G_{(t,\omega)}(f)^{\omega'}_t) + \delta_{(t+1,\omega')} u_{(t,\omega)}(G_{(t,\omega)}(f)^{\omega'}_{t+1})] \mu(\mathfrak{F}_{t+1} | \mathfrak{F}_t(\omega))$

$= \sum_{\mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)} \min_{\delta_{t+1,\omega'} \in D_{(t,\omega)}} [(1 - \delta_{t+1,\omega'}) u_{(t,\omega)}(f^t(\omega'))$

$+ \delta_{t+1,\omega'} u_{(t,\omega)}(G_{(t,\omega)}(f)^{\omega'}_{t+1})]] \mu(\mathfrak{F}_{t+1} | \mathfrak{F}_t(\omega))$

$= V_{(t,\omega)}(K_{(t,\omega)}(f))$

$= V_{(t,\omega)}(C_{(t,\omega)}(f))$

Hence, $g \equiv C_{(t,\omega)}(f) \simeq_{(t,\omega)} f$.

Note that $W_{(t,\omega)}(C_{(t,\omega)}(f)) = u_{(t,\omega)}(y_{(t,\omega)}(f))$. Define $u_{(t,\omega)}(y_{(t,\omega)}(f)) = Z_{(t,\omega)}(f)$.

Then $Z_{(t,\omega)}(f) = W_{(t,\omega)}(C_{(t,\omega)}(f))$.

Step 10:

For any $f, g \in \mathfrak{A}_f$,

$f \succeq_{(t,\omega)} g \Leftrightarrow C_{(t,\omega)}(f) \succeq_{(t,\omega)} C_{(t,\omega)}(g)$

$\Leftrightarrow W_{(t,\omega)}(C_{(t,\omega)}(f)) \geq W_{(t,\omega)}(C_{(t,\omega)}(g))$

$\Leftrightarrow Z_{(t,\omega)}(f) \geq Z_{(t,\omega)}(g)$

where $W_{(t,\omega)}(.)$ *is defined in Step 2 and* $Z_{(t,\omega)}(.)$ *is defined in Step 9.*

The first equality follows from Axiom 2.7.5 (transitivity), and the second equality follow from Step 2. The third equality follows from Step 9.

Step 11:

$\{\succeq_{(t,\omega)}\}$ *on* $\mathfrak{A}_f$ *satisfies Axioms 2.7.1 to 2.7.12 if and only if:*

There exists $D_t = [\alpha_{t+1}, \beta_{t+1}]$ *with* $0 < \alpha_{t+1} \leq \beta_{t+1} < 1$ $\forall t$ *with* $0 \leq t < T$ *s.t.*

For any $f, g \in \mathfrak{A}_f$, $f \succeq_{(t,\omega)} g \Leftrightarrow V_{(t,\omega)}(f) \geq V_{(t,\omega)}(g)$

where $\{V_{(t,\omega)}(f)\}_{(t,\omega)\in(\mathfrak{T},\Omega)}$ *is recursively defined by*:

$V_{(t,\omega)}(f) \equiv \mathrm{E}_t[Min_{\delta_{t+1,\omega'}\in D_t}[(1-\delta_{t+1,\omega'})u(f^t(\omega')) + \delta_{t+1,\omega'}V_{(t+1,\omega')}(f)]|\mathfrak{F}_t(\omega)]]$

and

$V_{(T,\omega)}(f) \equiv u(f^T(\omega))$

and

$\delta^*_{t+1,\omega'} = \delta^*_{t+1,\omega''}$ *at* $\omega', \omega'' \in \mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)$

where $\delta^*_{t+1,\omega'} \in \mathrm{argmim}_{\delta_{t+1,\omega'}\in D_t}[(1-\delta_{t+1,\omega'})u(f^t(\omega')) + \delta_{t+1,\omega'}V_{(t+1,\omega')}(f)]$

Moreover,

D_t *is unique and only depends on* t, *and* u *is unique up to a positive affine transformation.*

To satisfy Step 1 and Axiom 2.7.4 (dynamic consistency) for certain acts at time 0, $u_{(0,\omega)} = u_{(t,\omega')}$ on all $(t,\omega') \in (\mathfrak{T},\Omega)$. In addition, $D_{(t,\omega)}$ only depends on t otherwise it violates Axiom 2.7.4 at time t-1 among certain acts. Also, D_t must follow Proposition 3.1.1 otherwise it violates Axiom 2.7.4 at some (t-1,ω) among certain acts.

Define $Z_{(0,\omega)}(f) = V_{(0,\omega)}(f)$. From Step 9:

$$V_{(0,\omega)}(f) = Z_{(0,\omega)}(C_{(1,\omega)}) = u(y_{(0,\omega)}(f))$$
$$= u(\textstyle\sum_{\mathfrak{F}_1\subseteq\mathfrak{F}_0(\omega)}[(1-\delta_{(1,\omega')})G_{(0,\omega)}(f)_0^{\omega'} + \delta_{(1,\omega')}G_{(0,\omega)}(f)_1^{\omega'}]\mu(\mathfrak{F}_1|\mathfrak{F}_0(\omega)))$$
$$= \textstyle\sum_{\mathfrak{F}_1\subseteq\mathfrak{F}_0(\omega)}\min_{\delta_{1,\omega'}\in D_0}[(1-\delta_{1,\omega'})u(f^0(\omega')) + \delta_{1,\omega'}u(G_{(0,\omega)}(f)_1^{\omega'})]]\mu(\mathfrak{F}_1|\mathfrak{F}_0(\omega))$$

Clearly, from Step 9, $G_{(0,\omega)}(f) \simeq_{(1,\omega')} C_{(1,\omega')}(f)$ and $G_{(0,\omega)}(f) \in \mathfrak{A}_{cs(0,1)}$, we can use an act that assigns $G_{(0,\omega)}(f)^1(\omega')$ from 1 to T on $\omega'' \in \mathfrak{F}_1$ as $C_{(1,\omega')}(f)$ at $\mathfrak{F}_1$. Then $y_{(1,\omega')}(f)$ $= G_{(0,\omega)}(f)^1(\omega')$ Since $Z(C_{(1,\omega')}(f)) = V_{(1,\omega')}(f) = u(y_{(1,\omega')}(f)) = u(G_0(f)^1(\omega'))$:

$V_{(0,\omega)}(f)$

$= \sum_{\mathfrak{F}_1 \subseteq \mathfrak{F}_0(\omega)} \min_{\delta_{1,\omega'} \in D_0}[(1-\delta_{1,\omega'})u(f^0(\omega')) + \delta_{1,\omega'}V_{(1,\omega')}(f)]]\mu(\mathfrak{F}_1|\mathfrak{F}_0(\omega))$

Applying the same procedure for $V_{(1,\omega')}(f)$:

$V_{(1,\omega')}(f)$

$= \sum_{\mathfrak{F}_2 \subseteq \mathfrak{F}_1(\omega')} \min_{\delta_{2,\omega''} \in D_1}[(1-\delta_{2,\omega''})u(f^1(\omega'')) + \delta_{2,\omega''}V_{(2,\omega'')}(f)]]\mu(\mathfrak{F}_2|\mathfrak{F}_1(\omega'))$

Then at any (t, ω) with $0 \leq t < T$:

$V_{(t,\omega)}(f)$

$= \sum_{\mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)} \min_{\delta_{t+1,\omega'} \in D_t}[(1-\delta_{t+1,\omega'})u(f^t(\omega')) + \delta_{t+1,\omega'}V_{(t+1,\omega')}(f)]]\mu(\mathfrak{F}_{t+1}|\mathfrak{F}_t(\omega))$

and at T, from Step 2:

$V_{(T,\omega)}(f) = u(f^T(\omega))$

Applying the argument in a reverse order, $V_{(t,\omega)}(f)$ is defined recursively from time T by keep replacing $V_{(t,\omega)}(f)$ by $y_{(t,\omega)}(f)$. Next, we define $\mu(\omega)$ by:

$\mu(\omega) = \mu(\mathfrak{F}_1|\mathfrak{F}_0(\omega))\mu(\mathfrak{F}_2|\mathfrak{F}_1(\omega))...\mu(\mathfrak{F}_T|\mathfrak{F}_{T-1}(\omega))$

Then, $\mu(\omega)$ satisfies:

(i) $\sum_{\omega \in \Omega} \mu(\omega) = 1$ and $\mu(\omega) > 0 \forall \omega \in \Omega$

$$\text{(ii)} \qquad \mu(\mathfrak{F}_{t+1}|\mathfrak{F}_t(\omega)) = \frac{\sum_{\omega' \in \mathfrak{F}_{t+1}} \mu(\omega')}{\sum_{\omega' \in \mathfrak{F}_t(\omega)} \mu(\omega')}$$

This satisfies the condition for $\mu(\omega)$ to be probability measure with conditional probability of (ii). By Step 3, this measure is unique. Let $\mathrm{E}_t[|\mathfrak{F}_t(\omega)]$ be a conditional probability operator on $\mathfrak{F}_t(\omega)$. Under Axiom 2.7.3, $V_{(t+1,\omega')}(f) = V_{(t+1,\omega'')}(f)$ for all $\omega', \omega'' \in \mathfrak{F}_{t+1}$. However, note that an effective choice of discount factors at time t depends on $\mathfrak{F}_{t+1}$, not on each $\omega' \in \mathfrak{F}_{t+1}$. Then we can use $\mu(\omega)$ to write $V_{(t,\omega)}(f)$ as:

$$V_{(t,\omega)}(f) = \mathrm{E}_t[\min_{\delta_{t+1,\omega'} \in D_t}[(1-\delta_{t+1,\omega'})u(f^t(\omega')) + \delta_{t+1,\omega'}V_{(t+1,\omega')}(f)]|\mathfrak{F}_t(\omega))]$$

$$\text{and } \delta^*_{t+1,\omega'} = \delta^*_{t+1,\omega''} \text{ at } \omega', \omega'' \in \mathfrak{F}_{t+1} \subseteq \mathfrak{F}_t(\omega)$$

$$\text{where } \delta^*_{t+1,\omega'} \in \mathrm{argmim}_{\delta_{t+1,\omega'} \in D_t}[(1-\delta_{t+1,\omega'})u(f^t(\omega')) + \delta_{t+1,\omega'}V_{(t+1,\omega')}(f)]$$

Necessity:

Axioms 2.7.1 to 2.7.6 are immediate. By the nature of a von Neuman Morgenstern utility function u, 2.7.7 is satisfied. For certainty acts, from Theorem 2.3.1, the formula Satisfies Axioms 2.7.8 to 2.7.10. By the linearity of weights, Axiom 2.7.11 is satisfied. Finally, since the formula has state-by-state application of multiple discount factors of $W_{(t,\omega')}$, Axiom 2.7.12 is satisfied.■

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Chapter 3

Conditions for Dynamic Consistency and No-Trade Theorem under Multiple Priors

3.1 Introduction

The no trade theorem of Milgrom-Stokey (1982) was followed by large literature that investigates when this theorem holds.[1] Dow-Madrigal-Werlang (1990) show that state separability is an essential element for this theorem to hold, and devised a counter example using the non-additive prior model of Schmeidler (1989). On the other hand, in a financial setting, Epstein-Wang (1994,1995) apply the multiple priors model of Gilboa-Schmeidler (1989) to the recursive utility framework, which guarantees dynamic consistency without considering a specific updating rule. Recently, Sarin-Wakker (1998) show that under the multiple priors model, if a set of priors confirms a recursive structure, preference relations satisfy sequential consistency, consequentialism, and dynamic consistency. Clearly, the multiple priors model is well-behaved compared with other non-expected utility models. The purpose of this paper is to determine under what conditions the no trade theorem holds for the economy of agents with multiple priors.

The crucial condition comes from the argument of Dow-Madrigal-Werlang (1990): state separability. The beauty of the no trade theorem is that agents never speculate on any ex-post knowledge. The result of Dow-Madrigal-Werlang (1990) implies that the most general form of the no trade theorem does not hold under the multiple priors model. However, the result of Sarin-Wakker (1998) gives us hope that under some ex-ante knowledge about ex-post partitions, we might restore the no trade theorem. The main assumption of the no trade theorem of Milgrom-Stokey (1982) is that agents follow the subjective prior model

[1] For example, Geanakoplos (1989) and Morris (1994).

using the Bayes' update rule. The intuition is that if the multiple priors model satisfies the similar conditional property to the Bayes' rule over a number of ex-post partitions, we should expect the no trade theorem to hold over those partitions.

In order to confirm this intuition, we must first closely investigate the property of individual behavior. In other words, under what conditions does an agent behave dynamically consistently? We are interested in the evolution of preference relations and the conditions on ex-ante multiple priors set that generates dynamically consistent conditional preference over time. For this analysis, Sarin-Wakker (1998) find the sufficient conditions for an ex-ante multiple priors set to guarantee dynamic consistency, consequentialism, and sequential consistency. However, they define sequential consistency by a folding back operation and do not derive a update rule from the original preference relations. Instead, they construct the multiple priors model that satisfies the recursive structure of multiple priors. On the other hand, in this paper, we directly derive conditional preference relations with some update rule for a set of priors given dynamic consistency. In other words, we derive the necessary conditions for the multiple priors model to satisfy dynamic consistency, consequentialism, and sequential consistency. It turns out that dynamic consistency and sequential consistency (or consequentialism) implies the recursive structure of an ex-ante multiple priors set. The result implies that an agent can use a conditional update rule that is similar to the Bayes' rule, especially the maximum likelihood rule proposed by Gilboa-Schmeidler (1993). This approach clarifies the connection between dynamic consistency and conditional preference relations, which is the *Bayes' rule under the multiple priors.*

Given this result, we extend the notion of dynamic consistency over a number of ex-post partitions. The essential assumption of the no trade theorem is that agents can have any ex-post partitions as their private information. (In this paper, we treat initial private signals as one of the ex-post partitions.) By repeatedly applying conditions for dynamic consistency over multiple ex-post partitions, we can derive the conditions where an ex-ante knowledge incorporates all of those ex-post partitions. Clearly, the more ex-post partitions an agent anticipates, the more restrictions are necessary for an ex-ante multiple priors set to guarantee dynamic consistency. In the limit of this operation, the multiple priors model must converge to the subjective prior model in order to guarantee dynamic consistency. In other words, if we consider the multiple priors model as a normative standard, ignorance about ex-post partitions forces an agent to have a single prior. This is an alternative view of the relationship between the multiple priors model and the subjective prior model.

Finally, we apply the conditions derived above to a multiple agents setting. As expected, we show that dynamic consistency of individual preference becomes the sufficient condition for the no trade theorem to hold. Most importantly, agents must know which ex-post partitions are possible to be realized. However, agents do not need to know exactly which ex-post partition is going to be realized. In other words, agents must know all contingencies of possible evolutions of ex-post partitions. This condition is certainly a strong one. However, we can allow differences in private information, which is the central objective of the no trade result. For the multiple priors model, agents must be slightly more sophisticated than agents with subjective priors. This view supports the idea that the subjective prior model is a limiting form of the multiple priors model under ultimate ignorance about ex-post partitions. As long as agents are sophisticated enough, they can form their multiple priors.

The paper proceed as follows: in section 3.2, we first show the limitation of the multiple priors model regarding state separability. Then we prove the main propositions that show that dynamic consistency and sequential consistency (or consequentialism) imply the recursive structure of multiple priors. Next, in section 3.3, we formally define the knowledge structure, and extend the recursive multiple priors to incorporate multiple ex-post partitions. We also investigate the limiting nature of the multiple priors model. Finally, in section 3.4, we prove the no trade theorem under the multiple priors model.

3.2 Consistency for Individual Preference

In this section, we derive the conditions under which the multiple priors model guarantees consistency for a dynamic choice problem. Let Ω be a finite state of nature (N_0 states) and Σ be the algebra on Ω. For any element ω of Ω, we have a set of outcomes denoted by X^K=$X \times X \times ... \times X$ that has finite elements in each X (N_X elements).[2] There are $T+1$ sessions of trades, i.e., one ex-ante trade and T ex-post trades. An agent observes an ex-post partition (or event) $P_{t,i}(\omega)$ at each t where $\{P_t\} = \{P_{t,1},...,P_{t,N_t}\}$ and N_t is the number of events in $\{P_t\}$. In this section, we assume that an agent knows an evolution of ex-post partitions $\{P_t\}_1^T$ at $t=0$. We defer a formal analysis of ex-ante and ex-post knowledge until the next section.

[2] The same results hold if we use simple probability distributions over an arbitrary set of X.

Now we define preference relations. An act $f_{t,i}$ at $\omega \in P_{t,i}$ is defined as a function $f_{t,i}$: $P_{t,i} \to \Delta(X^K)$ where $\Delta(X^K)$ is the set of probability distributions over X^K. Also $N_{t,i}$ be the number of states in $P_{t,i}$. Define $F_{t,i}$ as a set of possible functions at $P_{t,i}$ and let $A_{t,i} \subseteq F_{t,i}$ be a set of choices available at $P_{t,i}$ from which an agent can select the optimal act. Let $F^c_{t,i}$ be a space of acts over $P^c_{t,i}$ (complement of $P_{t,i}$) and $A^c_{t,i}$ and $f^c_{t,i}$ are defined respectively. In particular, $F_0 = \times F_{t,i}$, so any permutation of $f_{t,i}$ is possible. In addition, let $\overline{F}_{t,i}$ be a collection of conditional acts that assigns the identical element of $\Delta(X^K)$ for each $\omega \in R_{t,i}$. We call these acts constant acts. Define $\overline{F}^c_{t,i}$ and $\overline{F}_0$ respectively. Finally, let $f(\omega)$ be a element of $\Delta(X^K)$ that is assigned on $\omega \in \Omega$.

The preference is all based on the information we have learned up to t. First, we assume that the following axioms hold for acts in F_0 at $t = 0$:

Axiom 3.2.1: Weak Order

$\forall f, g, h \in F_0$, (i) $f \succeq g$ or $f \succeq g$ (ii) $f \succeq g$ and $g \succeq h \Rightarrow f \succeq h$

Axiom 3.2.2: Continuity

$\forall f, g, h \in F_0$ with $f \succ g \succ h$, $\exists 0 < \alpha$, β <1

s.t. $\alpha f \oplus (1-\alpha)h \succ g$ and $g \succ \beta f \oplus (1-\beta)h$.

Axiom 3.2.3: Monotonicity

$\forall f, g \in F_0$, if $f(\omega) \succeq g'(\omega) \forall \omega \in \Omega \Rightarrow f \succeq g$

where $h(\omega) \succeq h'(\omega)$ iff $h \succeq h'$ s.t. $h, h' \in \overline{F}_0$

Axiom 3.2.4: Nondegeneracy

$\exists f, g \in F_0$ s.t. $f \succ g$

Axiom 3.2.5: **Certainty-Independence**

$\forall f, g \in F_0$ and $\forall h \in \overline{F}_0$, $\forall \alpha \in (0,1)$, $f \succ g$ iff $\alpha f \oplus (1-\alpha)h \succ \alpha g \oplus (1-\alpha)h$

Axiom 3.2.6: **Uncertainty Aversion**

$\forall f, g \in F_0$ and $\forall \alpha \in (0,1)$, $f \simeq g \Rightarrow \alpha f \oplus (1-\alpha) g \succeq f$

Gilboa-Schmeidler (1989) show that the above axioms imply the following representation of preference relations on F_0:

Theorem 3.2.1: (Gilboa-Schmeidler :1989)[3]

A binary relationship on F_0 satisfies Axioms 3.2.1 to 3.2.6 if and only if it is represented by the following formula:

$\forall f, g \in F_0$, $f \succeq g$ iff $\min_{p \in C_0} \int u \circ f dp \geq \min_{p \in C_0} \int u \circ f dp$

where C_0 is a unique non-empty, closed and convex set of finitely additive probability measures on Σ, and u is a unique up to a positive affine transformation, which confirms the VNM expected utility from.[4]

Note that $u \circ f = (..., u(f(\omega)) ,...)$, i.e., a vector of utility. Given Axiom 3.2.1 to 3.2.6, among constant acts $h \in \overline{F}_0$, the independence axiom and the expected utility theorem hold. In other words, and $u(f(\omega)) = \sum_1^{N_X^K} p_s u(x_1, ..., x_K)$.

[3] We call propositions proved by other authors theorems.

[4] The preference relations over $\Delta(X^K)$ is defined by the following way as follows: $h(\omega) \succeq h'(\omega)$ iff $h \succeq h'$ s.t. h, $h' \in \overline{F}_0$. (This definition is used in the definition of monotonicity.) This binary relationship is represented by the utility function itself, i.e., $h(\omega) \succeq h'(\omega)$ iff $\min \int u \circ h(\omega) dp \geq \min \int u \circ h' dp$, and $u(h(\omega))$ is defined by $\min \int u \circ h dp = u(h(\omega))$.

Throughout this paper, we assume the following three conditions:

Assumption 3.2.1: Nondeluded Partitions[5]

An agent has nondeluded partitions, i.e., $\omega \in P_{t,i}(\omega)$[6]

Assumption 3.2.2: Full Support

Let π_0 be a prior from C_0. Then $\pi_0(\omega) > 0 \forall \omega \in \Omega, \forall \pi_0 \in C_0$

Assumption 3.2.3: Complete Markets

All permutations of feasible ex-post acts $f_{t,i} \in A_{t,i}$ is in an ex-ante feasible set, i.e., $A_0 = \times A_{t,i}$

Now, we define conditional preference relations, dynamic consistency, and related notions as follows:

Definition 3.2.1: Conditional Preference on $F_{t,i}$ Given an $a \in F^c_{t,i}$

Let $f_0 = (f_{t,i}, a)$ and $g_0 = (g_{t,i}, a)$ where $f_{t,i}, g_{t,i} \in F_{t,i}$, and $a \in F^c_{t,i}$. A preference relation on $f_{t,i}, g_{t,i}$ given a is called a conditional preference at $P_{t,i}$ given a and written by $\succeq_{P_{t,i}(a)}$.

[5] From Geanakoplos (1989)

[6] We always assume nondeluded partitions; otherwise, an agent's behavior is too irrational to be described. The problem is aggravated if we consider the interaction of agents because it is very hard to build some rational consensus if agent's partition is not nondeluded. In other words, we cannot apply the logic of common knowledge.

Definition 3.2.2: Consequentialism

A preference relation $\succeq_{P_{t,i}(a)}$ satisfies consequentialism if it is independent of $a \in F^c_{t,i}$. We call it $\succeq_{P_{t,i}}$.

Definition 3.2.3: Sequential Consistency

Suppose that $\succeq$ satisfies the multiple priors model. Then $\succeq$ satisfies sequential consistency if $\succeq_{P_{t,i}(a)}$ satisfies the multiple priors model $\forall P_{t,i}$ $\forall a \in F^c_{t,i}$.

Definition 3.2.4: Ex-post Dynamic Consistency (or simply, Dynamic Consistency)[7]

Let $f_0 = (f_{t,i},a)$ and $g_0 = (g_{t,i},a)$ where $f_{t,i}, g_{t,i} \in F_{t,i}$, and $a \in F^c_{t,i}$.

A preference relation on F_0 satisfies ex-post dynamic consistency if:

$\forall f_0, g_0 \in F_0$, $f_0 \succeq g_0$ if and only if $f_{t,i} \succeq_{P_{t,i}(a)} g_i \; \forall \; P_{t,i}$, $\forall a \in F^c_{t,i}$.

Definition 3.2.5: Monotonicity on Events[8]

Suppose that agent's preference relations on $F_{t,i}$ satisfies consequentialism.

A preference relation on F_0 satisfies monotonicity on events if:

At $\forall t$, for $f_0, g_0 \in F_0$, if $f_{t,i} \succeq_{p_{t,i}} g_{t,i} \; \forall P_{t,i} \Rightarrow f_0 \succeq g_0$

First, for convenience, whenever we apply ex-post dynamic consistency, we refer to it

[7] Dynamic consistency could be defined for a subset of A_0; however, this partial ordering does not yield attractive characteristics for applications.

[8] Monotonicity on events is a necessary condition of the preference that satisfies dynamic consistency and consequentialism. Under this condition, each event in a partition behaves as if it were a state.

as *dynamic consistency* unless we think that it is confusing. (In the next section, we will define ax-ante dynamic consistency.) In addition, note that if Axioms 3.2.1 to 3.2.6 are satisfied for conditional preference relations, it is represents by the multiple priors model.

Sequential consistency defined here is a similar condition introduce by Sarin-Wakker (1998) but we define it on a conditional preference given an $a \in F^c_{t,i}$.[9] As far as the notion of dynamic consistency is concerned, our notion of dynamic consistency is the one introduced by Machina (1989) where consistency is defined over each conditional preference relations given an $a \in F^c_{t,i}$ and does not guarantee consequentialism. In fact, Machina's notion requires that the original preference relations are used for any stage of choice. Eichberger-Kelsey (1996) utilize Machina's notion for examining a dynamically consistent updating rule for the non-additive prior model of Schmeidler (1989). They show that if an agent's preference relation satisfies strict uncertainty aversion, a dynamically consistent update rule does not produce the conditional preference that confirms the non-additive prior model, i.e., sequential consistency is violated. Now we apply their result to the multiple priors model:

Proposition 3.2.1:

Suppose agent's preference relations confirm the multiple priors model with Assumption

[9] Sarin-Wakker (1998) define sequential consistency as follows: a preference relation $\succeq$ satisfies sequential consistency if $\succeq$ confirms the multiple priors model when $\succeq_{P_{t,i}}$ satisfies the multiple priors model $\forall P_{t,i}$. This definition is based on backward induction or a holding back operation. Clearly, consequentialism is assumed. Note that we define sequential consistency in a forward looking manner: if the multiple priors model holds for $\succeq$, then $\succeq_{P_{t,i(a)}}$ must confirm the multiple priors model. In other words, we consider a preference update from the original one over time. Note that our definition does not assume consequentialism.

3.2.1 to 3.2.3, and with the following additional condition:

(Strict uncertainty aversion: Eichberger-Kelsey (1996))

$\forall f_0, g_0 \in F_0$ that does not assign elements in $\Delta(X^K)$ with same utility for all $\omega \in \Omega$. $\forall \lambda \in (0,1)$, $f_0 \succeq g_0 \Rightarrow \lambda\ f_0 \oplus (1-\lambda)\ g_0 \succ g_0$.

Then a dynamically consistent conditional preference $\succeq_{P_{t,i}(a)}$ does not confirm the multiple prior model (i.e., sequential consistency is violate) if $1 < N_{t,i} < N_0$.[10]

Proof:

Suppose that $N_0 > N_{t,i} > 1$ is the cardinality of $P_{t,i}$. Let $f_0 = (f_{t,i}, a)$ be an act in F_0 where $f_{t,i}$ does not assign the same element from $\Delta(X^K)$ on all $\omega \in P_{t,i}$. By monotonicity and continuity on F_0, $\exists \overline{f}_0$ that assigns the same element from $\Delta(X^K)$ on each $\omega \in P_{t,i}$ and assign a for $P^c_{t,i}$, and $f_0 \simeq \overline{f}_0$. W.O.L.G., assume that $a \neq \overline{f}_{t,i}$. Then by definition of a dynamically consistent conditional preference, $f_{t,i} \simeq_{P_{t,i}(a)} \overline{f}_{t,i}$. Now, by strict uncertainty aversion, $\lambda\ f_0 \oplus (1-\lambda)\ \overline{f} \succ \overline{f}$, which implies that $\lambda\ f_{t,i} \oplus (1-\lambda)\ \overline{f}_{t,i} \succ_{P_{t,i}(a)} \overline{f}_{t,i} = \lambda\ \overline{f}_{t,i} \oplus (1-\lambda)\ \overline{f}_{t,i}$. However, since $\overline{f}_{t,i}$ assigns the identical element on $\omega \in P_{t,i}$, this inequality violates certainty-independence on $P_{t,i}$. Since certainty-independence is a necessary condition for the multiple priors model, $\succeq_{P_{t,i}(a)}$ cannot be represented by the multiple priors model. ■

Proposition 3.2.1 is a discouraging result for dynamic consistency of the multiple priors

[10] We need to have at least two states in $P_{t,i}$; otherwise, the argument does not have any bite. If $N_{t,i} = 1$, by dynamic consistency and monotonicity on F_0, it is obvious that f_0 that assigns an element in $\Delta(X^K)$ with a higher utility on $P_{t,i}$ achieves a higher value. In other words, there is a single prior over $P_{t,i}$, which is a point mass.

model. Since under a set of multiple priors with a strictly concave utility function, we can easily observe the preference with strict uncertainty aversion, the above result implies that the multiple priors model may not satisfy sequential consistency in general. We observe a similar result for consequentialism:

Proposition 3.2.2:

Suppose agent's preference relations confirm the multiple priors model with Assumption 3.2.1 to 3.2.3 and with strict uncertainty aversion. Then a dynamically consistent conditional preference $\succeq_{P_{t,i}(a)}$ does not confirm consequentialism if $1 < N_{t,i} < N_0$.

Proof:

Suppose that $N_0 > N_{t,i} > 1$ is the cardinality of $P_{t,i}$, and that agent's preference satisfies dynamic consistency and consequentialism. Let f_0 and $\overline{f}_0$ be an act in F_0 in the proof of Proposition 3.2.1. Clearly, $f_{t,i} \simeq_{P_{t,i}(a)} \overline{f}_{t,i}$, and $\lambda\ f_{t,i} \oplus (1-\lambda)\ \overline{f}_{t,i} \succ_{P_{t,i}(a)} \overline{f}_{t,i}$. Now let f_0' $= (f_{t,i},b)$ and $\overline{f}_0' = (\overline{f}_{t,i},b)$ where b assigns the same element as in $\overline{f}_{t,i}$ from $\Delta(X^K)$ on each $\omega \in R_{t,i}^c$. Suppose that consequentialism holds. Then $f_{t,i} \simeq_{P_{t,i}(b)} \overline{f}_{t,i}$ and $\lambda\ f_{t,i} \oplus (1-\lambda)$ $\overline{f}_{t,i} \succ_{P_{t,i}(b)} \overline{f}_{t,i}$. Dynamic consistency implies that $f_0' \simeq \overline{f}_0'$ and $\lambda\ f_0' \oplus (1-\lambda)\ \overline{f}_0' \succ \overline{f}_0' = \lambda$ $\overline{f}_0' \oplus (1-\lambda)\ \overline{f}_0'$, which contradicts certainty-independence on F_0. ■

Although Proposition 3.2.1 and Proposition 3.2.2 show that dynamic consistency without strict uncertainty aversion might produce the violation of sequential consistency or consequentialism,[11] it is not constructive to investigate general conditions for dynamic con-

[11] Sequential consistency is a conditional property whereas strict uncertainty aversion is an aggregate

sistency because dynamically consistent preference relations always exist under Machina's notion. In a normative sense, we want to restrict our attention to the same family of preferences with more strict notion of state separation under dynamic decision, and investigate the conditions to ensure dynamic consistency. The multiple priors model loses tractability if we only assume sequential consistency or consequentialism. Fortunately, as we will see later, under the dynamically consistent multiple priors model, these two notions are equivalent.

Before exploring this relationship, we need to define more notations. Let π_0 be a prior from C_0 and $\pi_0(P_{t,i})$ be $\sum_{\omega \in P_{t,i}} \pi_0(\omega)$ and $\pi_0(P^c_{t,i})$ be $\sum_{\omega \in P^c_{t,i}} \pi_0(\omega)$ where $P^c_{t,i}$ is the complement of $P_{t,i}$. We also define $\pi_{0,t,i} = (\pi_0(\omega_{k+1}), \dots, \pi_0(\omega_{k+1+I_{t,i}}))$ as the corresponding entry of probabilities over $\omega \in P_{t,i}$ under π_0 where $I_{t,i}$ is the cardinality of $P_{t,i}$ and $k = \sum_{j=1}^{i-1} I_{t,j}$. For a fixed π_0, the intersection between C_0 and a hyperplane $\{\pi'_0 | \sum_{\omega \in P_{t,i}} \pi_0(\omega) = \pi_0(P_{t,i})\}$ (or a line if $P_{t,i}$ has a single element) forms a non-empty, closed and convex set. Note that this set is identical among π_0 and π'_0 as long as $\pi_0(P^c_{t,i}) = \pi'_0(P^c_{t,i})$. Hence without loss of generality, we define the collection of $\pi_{0,t,i}$ in this intersection as $C_0(P_i|\pi_0(P^c_{t,i}))$, which is conditional on $\pi_0(P^c_{t,i})$, not on π_{0,t,i^c} that is defined over $\omega \in P^c_{t,i}$ by the same way as for $\pi_{0,t,i}$.

First, the following proposition derives the conditions on C_0 that satisfies dynamic consistency and sequential consistency. Then the next proposition shows the equivalence of sequential consistency and consequentialism under dynamic consistency.

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property. Hence, from the violation of strict uncertainty aversion, we cannot infer the violation of sequential consistency.

Proposition 3.2.3: Necessary and Sufficient Conditions on C_0 to Guarantee Dynamic Consistency and Sequential Consistency

Suppose that agent's preference relations confirm the multiple priors model with C_0 that satisfies Assumptions 3.2.1 to 3.2.3. Then given ex-post partitions $\{P_t\}_1^T$, dynamic consistency and sequential consistency are satisfied if and only if the following conditions are satisfied:

(3.2.1) $C_0(P_{t,i}|\pi_0(P_{t,i}^c))/\pi_0(P_{t,i}) = C_0(P_{t,i}|\pi'_0(P_{t,i}^c))/\pi'_0(P_{t,i}) \; \forall \pi'_0(P_{t,i}^c) \neq \pi_0(P_{t,i}^c)$

$\forall 1 \leq t \leq T$, $\forall 1 \leq i \leq N_t$

(3.2.2) $\exists \overline{C_t}$ that is a non-empty, closed and convex set of probability measures over $(P_{t,1}, ..., P_{t,N_t})$

(3.2.3) $\succeq_{P_{t,i}(a)}$ is represented by the multiple priors model with $C_0(P_{t,i}|\pi_0(P_{t,i}^c))/\pi_0(P_{t,i})$ and $u_{t,i(a)}(.)$, where $u_{t,i(a)}(.)$ is a positive affine transformation of the original $u(.)$ and π_0 is the optimal prior for $f_0 = (f_{t,i}, a) \in F_0$ under $\succeq$.

(The conditional update is the Bayes' rule under multiple priors.)[12]

Proof:

See Appendix 3.A. ■

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[12] For example, time preference is incorporated through discount factors. At $t = 0$, $u = \delta^T \widehat{u}$, and at $t = \tau$, $u = \delta^{T-\tau}\widehat{u}$.

Proposition 3.2.4: **Dynamic Consistency and Consequentialism $\Leftrightarrow$ (3.2.1), (3.2.2), (3.2.3)**

Suppose that agent's preference relations confirm the multiple priors model with C_0 that satisfies Assumptions 3.2.1 to 3.2.3. Then given ex-post partitions $\{P_t\}_1^T$, dynamic consistency and consequentialism are satisfied if and only if (3.2.1), (3.2.2) and (3.2.3) are satisfied.

Proof:

See Appendix 3.B.■

Conditions (3.2.1) to (3.2.3) imply that an agent must use the Bayes' rule for updating her multiple priors set over time. This Bayes' rule is defined as follows: given the optimal prior π_0 for an act f_0, collect priors $\pi'_0 \in C_0$ that achieve $\pi_0(P_{t,i}) = \pi'_0(P_{t,i})$. Then use the elements in π'_0 over $P_{t,i}$ as the elements in the multiple priors set at $P_{t,i}$. Finally normalize it by $\pi_0(P_{t,i})$. This operation gives us $C_0(P_{t,i}|\pi_0(P^c_{t,i}))/\pi_0(P_{t,i})$. In other words, we confirms the intuition of Esptein-Breton (1993): dynamically consistent beliefs must be Bayesian. (Epstein-Berton's result is restricted to a subclass of the non-additive prior model.)

From Proposition 3.2.3 and Proposition 3.2.4, it is also clear that under dynamic consistency, sequential consistency and consequentialism are equivalent. In fact, it is quite surprising that a weakly conditional notion of sequential consistency and dynamic consistency guarantee a strongly unconditional notion of consequentialism, and that consequentialism itself forces the preference relations to satisfy sequential consistency when dynamic consistency is assumed. In addition, Conditions (3.2.1), (3.2.2), and (3.2.3) summarize the nature of consequentialism. Under dynamic consistency and consequentialism, monotonicity on events holds, and each event $P_{t,i}$ becomes a new state under $\overline{C}_t$. Hence, the multiple priors set C_0 is defined by the recursive operation over other multiple priors sets, which is essentially the structure Sarin-Wakker (1998) apply. We define this structure as the recursive multiple priors set.

Now we summarize our results:

Corollary 3.2.1:

Suppose that under given $\{P_t\}_1^T$, agent's preference relations satisfy the dynamically consistent multiple priors model with Assumptions 3.2.1 to 3.2.3. The following conditions are equivalent:

(1) consequentialism

(2) sequential consistency

(3) Conditions (3.2.1), (3.2.2), and (3.2.3)

In fact, Condition (3.2.3) implies that any conditional updating $C_0(P_{t,i}|\pi_0(P_{t,i}^c))/\pi_0(P_{t,i})$ works because it is identical for all possible $\pi_0(P_{t,i})$. Especially, this condition implies that an agent can use the following update rule proposed by Gilboa-Schmeidler (1993):

Definition 3.2.6: Maximum Likelihood Rule (Gilboa-Schmeidler: 1993)

$C_{t,i} = \{\pi_0 | \; \pi_0(P_{t,i}) = \max_{\pi_0' \in C_0} \pi_0'(P_{t,i})\}$. This set is equivalent to $C_0(P_{t,i}|\pi_0(P_{t,i}^c))$ where $\pi_0(P_{t,i}^c)$ is derived from the optimal prior for an act f that assigns the highest element $\overline{f}(\omega)$

of $\Delta(X^K)$ to a state $\omega \in P^c_{t,i}$ and assigns $g(\omega) \in \Delta(X^K)$ s.t. $g(\omega) \prec \overline{f}(\omega)$ to a state $\omega \in P_{t,i}$. In other words, this update rule produces the most pessimistic view given the realization of $P_{t,i}$.

The above update rule produces consequentialism. In fact, our result can be restated for an agent who uses the maximum likelihood rule:

Corollary 3.2.2:

Suppose that agent's preference relations confirm the multiple priors model with C_0 and Assumptions 3.2.1 to 3.2.3, and that the agent updates her multiple priors set by the maximum likelihood rule with the identical utility function over time, i.e., the preference relations satisfy sequential consistency and consequentialism. Given $\{P_t\}_1^T$, agent's preference relations satisfy dynamic consistency if and only if Conditions (3.2.1) and (3.2.2) are satisfied.

This corollary relates our results to the one by Eichberger-Kelsey (1996) where they show the maximum likelihood rule does not always generate dynamically consistent behavior. Here, we derive the necessary and sufficient conditions for this update rule to produce dynamically consistent behavior. The original C_0 must confirm the recursive nature of Conditions (3.2.1) and (3.2.2). In some sense, an agent must specify how to form conditional preferences ex-ante. If the agent keeps sequential consistency and consequentialism with some conditional update rule as a normative objective, dynamic consistency is satisfied only under the recursive multiple priors set.

In a nutshell, Proposition 3.2.3 and Proposition 3.2.4 provide a quite reasonable formulation of conditional property on C_0, which essentially requires the recursive structure of multiple priors sets. This result confirms the findings by Sarin-Wakker (1998) that under the multiple priors model an agent can stay in the same family of representation over the course of history as long as she has a recursive multiple priors set. Their proposition is essentially equivalent to the sufficiency of our Propositions 3.2.3 and 3.2.4, i.e., Conditions (3.2.1), (3.2.2), and (3.2.3) imply dynamic consistency, consequentialism, and sequential consistency. Our main results here are a converse of their proposition.

Once we assume dynamic consistency, the necessary conditions for sequential consistency and consequentialism are Conditions (3.2.1), (3.2.2), and (3.2.3), where C_0 is a recursive multiple priors set, and utility functions for a updated preference must be within a positive affine transformation of the original utility function. In other words, we derive the structure of the original multiple priors set that satisfies dynamic consistency, consequentialism, and sequential consistency. In fact, the proposition of Sarin-Wakker (1998) is based on backward induction or a holding back operation under the recursive preference. Here, we consider a updating scheme from the original preference, and construct conditional preference relations by forward looking behavior. Note again that dynamic consistency itself does not guarantee consequentialism nor sequential consistency as Proposition 3.2.1 and Proposition 3.2.2 suggest. We need to assume a recursive multiple priors set in order to achieve these two properties, and it is the necessary conditions under dynamically consistent preference relations.

Now we understand the necessary and sufficient conditions for a agent with multiple priors to behave dynamically consistently. The next question is whether the non-degenerate multiple priors set C_0 that satisfies Conditions (3.2.1) and (3.2.2) exists. The answer is "yes" but not always.[13] The recursive multiple priors set must satisfy the following tight structure. Note that for the subjective prior model, Conditions (3.2.1) and (3.2.2) are automatically satisfied.

Proposition 3.2.5: Existence of C_0

An ex-ante multiple priors set C_0 that satisfies Conditions (3.2.1) and (3.2.2) exists if the following conditions hold for ex-post partitions:

Let $\widetilde{P}$ be the finest partitions constructed by $\forall\{P_{t,i}\}$ s.t. $1 \leq t \leq T$, $\forall 1 \leq i \leq N_t$

i.e., $P_{t,i} \supseteq \widetilde{P}_m$ or $\widetilde{P}_m \setminus P_{t,i} = \emptyset$.

(3.2.4) $\exists$ a set of multiple priors $\widetilde{C}_m$ over states in a event $\widetilde{P}_m$ where $\widetilde{C}_m$ is non-empty, closed, and convex.

(3.2.5) $\exists$ a set of multiple priors $\widetilde{C}$ over events $\widetilde{P}_m$ where $\widetilde{C}$ is non-empty, closed, and convex.

Let $\{R_j\}$ be a meet of $\{P_{t,i}\}$.

(3.2.6) For a meet R_j in which there are no overlaps among $\{P_{t,i}\}$, $\{P_{t,i}\}$ can be rearranged to form $\{\widehat{P}_\gamma\}_1^\Gamma$ that is a non-increasing sequence of partitions and each $\widehat{P}_{\gamma,j}$ corresponds to some $P_{t,i}$ except $\{\widehat{P}_\Gamma\}_1^J = \{\widetilde{P}_m\}_k^{k+J}$

[13] Given that the subjective prior model is a subset of the multiple priors model, this answer is "always". We will see the connection between these models in the next section.

where $R_j = \cup_k^{k+l} \widetilde{P}_m$. If $\widehat{P}_{\gamma,i} = \cup_{k'}^{k'+l} \widehat{P}_{\gamma+1,j}$ with $l > 1$, then $\exists$ a set of multiple priors $\widetilde{C}_{\gamma,i}$ over events $\{\widehat{P}_{\gamma+1,j}\}_{k'}^{k'+l}$ where $\widetilde{C}_{\gamma,i}$ is non-empty, closed, and convex.

(3.2.7) Suppose that at $\omega \in P_{t,i}(\omega)$, there are $P_{t',j}$ and $P_{t',j+1}$ *s.t.* $P_{t,i} \cap P_{t',j} \neq \emptyset$ and $P_{t,i} \cap P_{t',j+1} \neq \emptyset$. Let R_j be the meet among all $\{P_{t,i}(\omega)\}_{t=1}^{T}$ at ω, and π be any prior from $\widetilde{C}$. Then $\pi(\widetilde{P}_m)/\pi(\widetilde{P}_{m'})$ is fixed between any two events in $\{\widetilde{P}_m\}_k^{k+l}$ where $R_j = \cup_k^{k+l} \widetilde{P}_m$.

Proof:

See Appendix 3.C.■

This proposition imposes restrictions on an ex-ante multiple priors set C_0. A sufficient condition for C_0 to satisfy dynamic consistency is a recursive structure over the finest partition with an adjustment (3.2.7).[14] The most interesting observation here is that if ex-post partitions are not nested, then we must have a fixed ratio of probability over the events in the meet that includes non-nested events even though we can have multiple priors over states within each event. Clearly, this construction indicates the connection between the multiple priors model and the subjective prior model. This observation is formalized once we define the connection between ex-ante and ex-post partitions in the next section.

[14] A necessary condition permits slightly more movement in $\widetilde{C}$ under Condition (3.2.7). However, it is hard to state it explicitly as a proposition.

3.3 Ex-ante and Ex-post Knowledge

In this section, we formalize the relationship between ex-ante knowledge and ex-post knowledge. First we introduce ex-ante partitions and also formally define ex-post partitions.

1. Q_t is an ex-ante partition of Ω that summarizes the ex-ante knowledge of information process over T given the knowledge up to t with a generic element $Q_t(s,j,m)$ for $0 \leq t \leq s \leq T$, $1 \leq i \leq I_{s,m}$, and $1 \leq m \leq M_{t(s)}$ where where $I_{s,m}$ is the cardinality of $\{Q_t(s,.,m)\}$ and $M_{t(s)}$ is the cardinality of conjectures in $\{Q_t(s,1,.)\}$. For each s s.t. $t \leq s \leq T$, an agent conjectures all possible ex-post partitions. In other words, $Q_t(s,i,m)$ is the i-th event of the m-th conjecture about ex-post partitions at time s when an agent is at time t[15].

2. P_t is an ex-post partition of Ω that summarizes ex-post knowledge about information available up to time t where $1 \leq t \leq T$. A generic element is $P_{t,i}$ where the subscript i of $P_{t,i}$ stands for the i-th event in P_t. Note that an agent learns not only $P_{t,i}(\omega)$ but

[15] An agent must form beliefs how ex-post partitions evolve. When an ex-post partition $P_{t,i}$ is realized, she needs to reform her beliefs at $P_{t,i}$. There are three ways she can do this. If she has perfect memory, she only forms partitions over $\omega \in P_{t,i}$ and the rest of states forms another partition. If she forgets everything or is not confident of what she has learned at all, she must form beliefs about all partitions over $\omega \in \Omega$. If she has a partial memory or is not perfectly confident of what she has learned, she can form partitions that includes states in $P_{t,i}$, but not necessarily over $\omega \in \Omega$. For this case, she must categorize the states that is not included for these partitions as one alternative.

also all other $\{P_{t,i}\}$ in P_t. P_1 is considered to be a private signal.[16][17]

3. $P_0 = \emptyset$, $Q_0(0, 1, 1) = \Omega$, $M_{0(0)} = 1$, $Q_t(t, ., 1) = P_t$.

Information processes are as follows: at $t = 0$, an agent forms an ex-ante partition $Q_0(s, i, m)$ and trades. At $t = 1$, the agent receives a private information partition P_1 and trades. After trades, the agent forms an ex-ante partition $Q_1(s, i, m)$. At $t = 2$, the agent forms an ex-post partition P_2 based on the information from trades and prices at $t = 1$, and additional private information and trades, and so on.

Given the above notations, the definitions of conditional preference, consequentialism, sequential consistency, monotonicity on events are defined over ex-ante partitions by the identical constructions in Definitions 3.2.1 to 3.2.5 (simply replacing $P_{t,i}$ with $Q_{t(s,i,m)}$). Here we only define ex-ante dynamic consistency:

Definition 3.3.1: Ex-ante Dynamic Consistency at time t

Suppose that an agent is at time t. Let $f_0 = (f_{s,i}, a)$ and $g_0 = (g_{s,i}, a)$ where $f_{s,i}, g_{s,i} \in F_{s,i}$, and $a \in F^c_{s,i}$. Preference relations on F_0 satisfy ex-ante dynamic consistency if:

$\forall f_0, g_0 \in F_0$, $f_0 \succeq g_0$ if and only if $f_{s,i} \succeq_{W_{s,i}(a)} g_{s,i}$ $\forall$ $W_{s,i} = Q_{t(s,i,m)}$

$\forall 1 \leq t \leq s \leq T, \forall\ 1 \leq m \leq M_{t(s)}$.

[16] If an agent only learns $P^h_{t,i}(\omega)$, she cannot infer the meet of $\{P^h_{t,i}(\omega)\}$. In this case, the agent either believes her partition for sure or ignores her private information completely and stays in the ex-ante knowledge.

[17] Usually, a signal is called an interim partition (or ex-ante partition in Milgrom-Stokey (1982)). For details of ex-ante, interim, and ex-post relationship, refer Morris (1994).

Note that ex-ante dynamic consistency is a notion of consistency among ex-ante conjectures. Now we define the following two structures of ex-ante knowledge:

Definition 3.3.2: Perfect Anticipation[18]

An agent has perfect anticipation if $\forall t \leq s \leq T$ and $\forall P_s$, $\exists m$ s.t. $P_{s,i} = Q_t(s,i,m)$ $\forall i$ s.t. $1 \leq i \leq I_{s,m}$. In other words, the agent with perfect anticipation has a correct guess at an ex-post partition P_s in an ex-ante partition Q_t.

Definition 3.3.3: Ex-ante Sophisticated

Without loss of generality, $t' > t$. An agent has an ex-ante sophisticated partition under the following condition: If $\exists m$ s.t. $Q_{t'}(t',.,1) = Q_t(t',.,m)$, then $\forall s$ s.t. $t' \leq s \leq T$ and $\forall m'$ s.t. $1 \leq m' \leq M_{t'(s)}$, $\exists m$ s.t. $Q_t(s,.,m) = Q_{t'}(s,.,m')$. In other words, if the agent has a correct guess at an ex-post partition at t', then all conjectures in $Q_{t'}(s,i,m)$ become a subset of $Q_t(s,i,m)$.

Definition 3.3.4: Ex-post Sophisticated

An agent has ex-post sophisticated knowledge if she does not revise C_0 as long as ex-post partitions are within ex-ante partitions, i.e., her knowledge satisfies perfect anticipation.

Clearly, under perfect anticipation, an agent has a right conjecture, but does not know which one is the true one. Ex-ante sophistication is an innocuous consistency requirement

[18] This definition includes *perfect knowledge*: an agent has perfect knowledge if $Q_{t,T}(s,i,1) = P_{s,i}$ for all t, i, and s s.t. $t \leq s \leq T$ and $M_{t,T(s)} = 1$. In other words, the agent with perfect knowledge knows ex-ante how much information is available, and how it is processed.

for the ex-ante knowledge structure. Ex-post sophistication is a strong assumption. This assumption eliminates the case where an agent changes her taste as she knows more about the truth as long as her knowledge satisfies perfect anticipation. In other words, an agent must incorporate ex-ante how to change her taste over time within perfectly anticipated ex-post partitions, and she must forms C_0 to satisfy ex-ante dynamic consistency. Given this consistency, it is very irrational for an agent to change the multiple priors set upon receiving an anticipated ex-post partition.

In the most general case, if we do not assume perfect anticipation, P_t can be any partition of Ω over time, and there is no consistency between ex-ante and ex-post partitions. Clearly, this situation is problematic because the original preference must be based on an ex-ante partition Q_0 and a revelation of inconsistent partition P_t with Q_0 can lead to a correction of the original preference relations, which makes our analysis invalid. In fact, this is precisely the point that Milgrom-Stokey (1982) focus on, and their no trade theorem holds without assumption of perfect anticipation, i.e., it does not require any relationship between ex-ante and ex-post partitions. However, as we will see later, the multiple priors model requires a strong tie between ex-ante and ex-post partitions for dynamic consistency to hold. The crucial intuition is that the original preference relations have a stronger connection to an ex-ante partition Q_0 than the subjective prior model does. The structure of a set of multiple priors hinges on ex-ante knowledge in Q_0 about P_t, and if an agent has inconsistent initial beliefs, sequential consistency no longer holds or an updated preference relations become dynamically inconsistent. C_0 is only valid at $t = 0$, and an agent can use a different C_0

for calculating conditional preference relations at different times. Even worse, the utility function could change, or the agent could have a different family of preferences. Under this condition, it is clear that we do not observe any consistency. We at least need minimum rationality or connection between ex-ante knowledge and ex-post knowledge and rationality on the agent's behavior.

Now we formalize the relationship between ex-ante and ex-post partitions. So far, we have only shown the conditions for dynamic consistency under ex-post partitions. However, an agent should form her beliefs or preference based on her ex-ante knowledge at the beginning; in other words, the structure of C_0 must reflect the ex-ante beliefs about the evolution of ex-post partitions. Conditions (3.2.1) and (3.2.2) are a mere coincidence if we ignore this relationship. In order to behave dynamically consistently over time, an agent must form the preference relations that takes into account her own hypothetical behavior in the future. Otherwise, it is illogical to assume that dynamic consistency is one of agent's ex-post objectives. Hence we first assume the following knowledge and rationality:

Assumption 3.3.1:

Preference relations on F_0 must satisfy ex-ante dynamic consistency, ex-ante sophistication, and ex-post sophistication.

Assumption 3.3.1 is to ensure internal consistency and is innocuous. Given this assumption, the following proposition summarizes how much an agent should know about the evolution of ex-post partitions at the beginning.

Proposition 3.3.1: Informational Requirement for Dynamic Consistency

Under Assumptions 3.2.1 to 3.2.3 and 3.3.1, an ex-ante partition Q_0 must satisfy Conditions (3.2.1) and (3.2.2) for the conjectures about ex-post partitions. Then the agent's behavior satisfies ex-post dynamic consistency if the agent's ex-ante partitions satisfy perfect anticipation.

Proof:

If ex-ante partitions satisfy perfect anticipation and ex-ante dynamic consistency under Conditions (3.2.1) and (3.2.2) for the conjectures about ex-post partitions, by definition, ex-post partitions must satisfy Conditions (3.2.1) and (3.2.2), which implies that ex-post dynamic consistency holds if an agent does not change her taste by increasingly or decreasingly being informed over time. By Assumption 3.3.1, this never happens. ■

This result is very intuitive. Clearly, an agent must anticipate all possible evolutions of ex-post partitions at the beginning, and this conjecture must be right. It does not necessarily require perfect knowledge about ex-post partitions, but rather requires the knowledge of all possible evolutions of ex-post partitions for avoiding the risk of missing some information or possible contingency. Then the agent forms the C_0 that satisfies Conditions (3.2.1), (3.2.2), (3.2.4), (3.2.5), (3.2.6), and (3.2.7), i.e., dynamic consistency over all possible evolutions of her conjectured ex-post partitions. On the other hand, the C_0 might be able to justify additional ex-post partitions as a mere coincidence. In this case, this ex-post partition would have been incorporated into the ex-ante partitions given that the agent would not have changed the C_0 by the presence of this knowledge. However, it is most likely that she would have change her preference if an agent had anticipated $\{P_t\}$.[19]

Clearly, we need much stronger conditions for dynamic consistency here than under the subjective prior model. In general, dynamic consistency is not a robust notion under the multiple priors model. We state it in the following corollary:

Corollary 3.3.1: Dynamic Inconsistency for the Multiple Priors Model

There is no set C_0 with multiple elements that satisfies Conditions (3.2.1) and (3.2.2) for all evolutions of ex-post partitions.

It is clear from Proposition 3.3.1 that we need to impose Conditions (3.2.1) and (3.2.2) on conjectured ex-post partitions in Q_0 in order to guarantee dynamic consistency. As we add more ex-post partitions to Q_0, we eventually reach the finest partition in which each event has a single state. Under this partition, Proposition 3.2.5 shows that an agent must have a single prior to satisfy the sufficient condition for dynamic consistency.[20] Clearly, these observations lead us to have an alternate view about the relationship between the subjective prior model and the multiple priors model.

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[19] It it especially true for the states in which an agent assigns a single prior because the Bayes' rule works over those states. However, if an agent faces an unexpected ex-post partition, she would change an ex-ante partition Q_t, which would change C_0 or C_{t+1}. This behavior is not internally consistent.

[20] We can easily construct partitions under which a single prior becomes the necessary condition. For example, partition $\{P_t^1\}$ has $P_{t,i}^1=\{\omega_{2i-1},\omega_{2i}\}$, $P_{t,1}^2=\{\omega_1\}$, $P_{t,i}^2=\{\omega_{2i},\omega_{2i+1}\}$.

Corollary 3.3.2: Information and Multiple Priors

Suppose that Assumptions 3.2.1 to 3.2.3 and 3.3.1, and Axioms 3.2.1 to 3.2.6 hold. Then:

1. If an agent has perfect anticipation, the agent forms the multiple priors C_0
2. If agent's perfectly anticipated knowledge becomes increasingly uninformative, i.e., Q_0 includes more ex-post partitions, the multiple priors set C_0 shrinks, and it will converge to a single point when Q_0 induces the finest partition in which each event has a single state ω.

This view is quite informative. Given the multiple priors model as a normative standard, if accuracy of information is decreased and an agent tries to form the finest perfectly anticipated ex-ante partitions, eventually, the agent is forced to have a single prior. We generally consider the agent with a multiple priors set to be more naive than the agent with a subjective prior because the multiple priors are formed on some uncertainty. Here, we argue that the most uninformative form of a set of multiple priors is a subjective prior. Clearly, the interpretation of ex-ante knowledge structure gives us a new insight into the connection between the multiple priors model and the subjective prior model. In other words, the multiple priors model is well-behaved compared with other non-expected utility models because of this limit structure.

3.4 Consistency under Equilibrium

In the previous section, we showed the conditions under which an agent with multiple priors behaves dynamically consistently. The next question is under what conditions equilibrium allocations satisfy dynamic consistency, i.e., the no trade theorem holds. After Milgrom-Stokey (1982), a large literature developed on this topic. Instead of examining a complex setting, we want to focus on a simplified version of Milgrom-Stokey (1982) and investigate the conditions on a set of multiple priors to guarantee the no trade theorem. Specifically, a state space is restricted to payoff-relevant ones so that we ignore concordant conditions[21].

The critical assumptions on the utility function and information structure used in Milgrom-Stokey (1982) are:

(3.4.1) Individuals follow the subjective prior model.

(3.4.2) Individuals use the Bayes' rule to update their prior.

(3.4.3) Individuals receive private information in a form of partition at the beginning (ex-post partition at $t = 1$), and they learn ex-post partitions through trades. Ex-post partitions are not necessarily in ex-ante knowledge. (The number of trading sessions are also arbitrary.)

Condition (3.4.2) is hidden or taken as given in most of the literature; however, it is a very critical assumption. It guarantees that we can use the Fubini theorem (the Bayes' rule) to

[21] We can include payoff irrelevant states. For this case, agents must have concordant sets of multiple priors for the no trade theorem to hold.

update the prior. Under the Fubini theorem, we can apply the iterated expectation, which is the core of the proof of Milgrom-Stokey (1982). The beauty of the no trade theorem is that it holds under any ex-post partitions, and the ex-ante knowledge about ex-post partitions are totally irrelevant. In fact, Dow-Madrigal-Werlang (1990) show that the state additivity of utility functions is a crucial assumption for the no trade theorem to hold under all possible private information (ex-post partitions). They use the non-additive prior model by Schmeidler (1989) to show a counter example for the no trade theorem. For the following development, we state the result of Dow-Madrigal-Werlang (1990):

Definition 3.4.1:

A utility function U for $f \in F_0$ has a state additive structure if $U(f) = \sum W(f(\omega), \omega)$ $\omega \in \Omega$

Theorem 3.4.1: (Dow-Madrigal-Werlang: 1990)

The no trade theorem of Milgrom-Stokey (1982) holds if and only if all agents' preference relations satisfy a state additive structure.

Note first that Conditions (3.4.1) and (3.4.2) guarantee that an agent has a state additive utility function. The essence of state additivity is that individual behavior becomes dynamically consistent under any ex-post partitions. However, under the multiple priors model, the utility of an act $f_0 \in F_0$ cannot be expressed by the state additive structure unless C_0 is a singleton, which is the subjective prior model. Since the multiple priors model violates this condition, we need to assume that agents must know something about ex-post partitions. In other words, the finest partition in Q_0 formed from conjectured ex-post partitions must have more than one element on some event in order for C_0 not to be a singleton.

Now given these results, we can conjecture that under a dynamically consist multiple priors set C_0, equilibrium allocations stay the same after receiving private information. We prove this results now:

We follow the basic setting in the previous section with the following minor extensions:

- Number of agents: H
- Endowment: $e^h : \Omega \to \Delta(X^K)$
- Utility function: $u^h : \Delta(X^K) \to R$ where u^h is state independent
- Informational partition: $Q^h_{t,T}$ (ex-ante); P^h_t (ex-post)
- Multiple prior set: C^h_0 (initial);

  $C^h_{t,i}$ for $P^h_{t,i}$ (updated after receiving information)
- Private signal: P^h_1 with $P^h_{1,i}(\omega)$
- Feasible Trade: $e^h + \theta^h \geq 0$ and $\sum_1^h \theta^h \leq 0$

Note that the definition of feasible trade implies that the assumption of complete markets, i.e., agents can span any consumptions. Now Proposition 3.4.1 summarizes our intuition:

Proposition 3.4.1: No Trade Theorem under Multiple Priors with Perfect Anticipation

Suppose that all traders have a concave Bernoulli utility function, that their preference relations follow the multiple priors model that satisfies the conditions of Proposition 3.2.3, that they have perfectly anticipated ex-ante knowledge $Q_0(s,i,m)$ with Assumption 3.3.1, that initial allocation e=$(e^1,...,e^H)$ is ex-ante Pareto-optimal, and that each trader h observes the private information conveyed by the partition P_1^h. Suppose that it is common knowledge at ω that all agents are rational and behave dynamically consistently, that they do not trade unless they can weakly improve their utility, and that the ex-post market must clear. Then, there is no trade that ex-post Pareto dominates e.

Proof:

The proof is almost the repetition of Milgrom-Stokey (1982). Let $R(\omega)$ be the meet of $(P_{t,i}^1(\omega),...,P_{t,i}^H(\omega))$ for $1\leq t\leq T$. By assumption, it is common knowledge at ω that θ is a feasible trade and mutually acceptable. In other words, whenever agents agree to trades, it must be weakly preferred than a null trade on all ex-post partitions, i.e., θ^h must satisfy $e_{t,i}^h+\theta_{t,i}^h\succeq e_{t,i}^h$ on $P_{t,i}^h$ $\forall P_{t,i}^h$.[22][23] If $\theta^h(\omega)\neq 0$ on $\omega\in P_{t,i}^h$ wherever they agree to trade, then

[22]This result comes from the common knowledge assumption. In other words, the common knowledge forces agents to coordinate a certain action on a certain event. Otherwise, there might be some trade under which an agent with the finest $P^h(\omega)$ exploits the opportunity by selling some endowment at $P^{h'}(\omega)\backslash P^h(\omega)$ in exchange of the endowment at $P^h(\omega)$, but both of them feel happy ex-post.

[23]If markets are incomplete, it is not always possible to construct trades under which $e_{t,i}^h+\theta_{t,i}^h\succeq e_{t,i}^h$.

the minimum agreement among agents about possible states is obtained under the meet of $(P^1_{t,i}(\omega), ..., P^H_{t,i}(\omega))$, which is $R(\omega)$.[24] Then $\forall\omega' \in R(\omega)$ and $\forall h$ at $P^h_{t,i}(\omega)$, by sequential consistency:

$$\text{(3.4.4)} \qquad \min_{\pi_{t,i}\in C^h_{t,i}} \int (u \circ (e^h + \theta^h))d\pi_{t,i} \geq \min_{\pi_{t,i}\in C^h_{t,i}} \int (u \circ e^h) d\pi_{t,i}$$

Suppose that $\exists h$ s.t. (3.4.4) holds strictly at some $P^h_{t,i}(\omega')$, and let θ^* the feasible trade defined by:

$$\theta^{*h} \equiv \theta^h 1_{R(\omega)} \ \forall h$$

where $1_{R(\omega)} = 1$ for $\omega \in R(\omega)$, and $1_{R(\omega)} = 0$ otherwise. Since θ^h is feasible, under the assumption of complete markets, the restriction of θ^h to $R(\omega)$ is also feasible. When an agent evaluates θ^{*h} ex-ante:

$$\min_{\pi_0\in C^h_0} \int (u \circ (e^h + \theta^{*h}))d\pi_0$$
$$= \min_{\pi\in \overline{C_t}^h} \int [\sum \min_{\pi_{t,i}\in C^h_{t,i}} \int (u \circ (e^h_{t,i} + \theta^h_{t,i} 1_{R(\omega)}))d\pi_{t,i}] d\pi$$
$$= \min_{\pi\in \overline{C_t}^h} \int [\sum_{P_{t,i}\subseteq R(\omega)} \min_{\pi_{t,i}\in C^h_{t,i}} \int (u \circ (e^h_{t,i} + \theta^h_{t,i}) 1_{R(\omega)}) d\pi_{t,i}$$
$$+ \sum_{P_{t,i}\nsubseteqq R(\omega)} \min_{\pi_{t,i}\in C^h_{t,i}} \int (u \circ e^h_{t,i}) 1_{R^c(\omega)} d\pi_{t,i}] d\pi$$
$$= \min_{\pi\in \overline{C_t}^h} \int [1_{R(\omega)} \sum_{P_{t,i}\subset R(\omega)} \text{mim}_{\pi_{t,i}\in C^h_{t,i}} \int (u \circ (e^h_{t,i} + \theta^h_{t,i}))d\pi_{t,i}$$

This implies that agents cannot infer from trades that disadvantageous trades imply beneficial trades for others, which in turn leads to speculation. In other words, there is no way to distinguish speculative trades from rational transactions on the meet $R(\omega)$.

[24] Agents do not need to know each others' partitions. They only need to know that there is a meet, an they carry out hypothetical calculation as in the proof in their head.

$+ 1_{R^c(\omega)} \sum_{P_{t,i} \nsubseteq R(\omega)} \text{mim}_{\pi_{t,i} \in C^h_{t,i}} \int (u \circ e^h_{t,i}) d\pi_{t,i}] d\pi$

$\geq \min_{\pi \in \overline{C_t}^h} \int [1_{R(\omega)} \sum_{P_{t,i} \subset R(\omega)} \text{mim}_{\pi_{t,i} \in C^h_{t,i}} \int (u \circ e^h_{t,i}) d\pi_{t,i}$

$+ 1_{R^c(\omega)} \sum_{P_{t,i} \nsubseteq R(\omega)} \text{mim}_{\pi_{t,i} \in C^h_{t,i}} \int (u \circ e^h_{t,i}) d\pi_{t,i}] d\pi$

$= \min_{\pi_0 \in C^h_0} \int (u \circ e^h) d\pi_0$

where $R^c(\omega)$ denotes the complement of $R(\omega)$, the second equality follows $R(\omega) \supseteq P^h_{t,i}(\omega')$ for $\omega \in R(\omega)$, and the fourth inequality is due to (3.4.4) and monotonicity on events by dynamic consistency and consequentialism. In fact, all equalities and inequalities are based on the recursive multiple priors, which are from perfect anticipation of ex-post partitions and ex-ante dynamic consistency. Since for h, the above inequality is strict, θ^{*h} ex-ante Pareto dominates a null trade, which contradicts that e is ex-ante Pareto optimal. Hence there are no feasible trades on $R(\omega)$ that strictly Pareto improve e. Since the above result holds for any meet including Ω itself, we conclude that there are no feasible trades that strictly ex-post Pareto improve e. In addition, if all agents are strictly risk-averse, there is no other equilibrium that assigns the identical utility for all agents with different allocations (local uniqueness). Therefore, there is no trade that agents agree on.■

As in Milgrom-Stokey (1982), the critical assumption is that agents only agree to trade when trades improve their utility on $P^h_{t,i}$. The common knowledge of this result implicitly discloses the meet $R(\omega)$. In fact, there is a trade to move to another ex-ante optimal allocation with identical utility as long as a subset of individuals have quasi-concave utility functions. However, the main result of Proposition 3.4.1 is that there are no trades that strictly Pareto improve the original allocation.

From the results of the previous section, it is clear that the no trade theorem does not hold under some ex-post partitions that are outside an ex-ante partition Q_0. We need ex-ante knowledge of ex-post partitions. In addition, we need to assume dynamic consistency for individual behavior, which is a hidden assumption in Milgrom-Stokey (1982) under rational expectations. Under perfect anticipation, $P^h_{t,i}$ behaves as if it were a single state, and the existence of $\overline{C}_t$ ensures the monotonicity of $\{f_{t,i}\}$ over $\{P^h_{t,i}\}$.

3.5 Conclusion

In this paper, we investigate the conditions under which the economy of agents with multiple priors demonstrates the no trade theorem. Our key intuition is that an agent must behave dynamically consistently over a number of ex-post partitions. Then we first show that the conditions for dynamic consistency. The main result is that dynamic consistency and sequential consistency (or consequentialism) implies the recursive multiple priors set under which an agent must use a conditional update rule. We extend this result for multiple ex-post partitions, and show that we can construct the multiple priors model as long as the finest partition has some event in which there is more than a single state. Then we proved the no trade theorem for the multiple priors model. The crucial assumption is individual consistency under a dynamic choice problem and perfect anticipation of ex-post partitions. Given this restriction on a knowledge structure, differences on private information do not lead to speculative trades.

Appendix 3.A: (Proof of Proposition 3.2.3)

Sufficiency: (3.2.1), (3.2.2), and (3.2.3) $\Rightarrow$ dynamic consistency, sequential consistency

First, note that Condition (3.2.3) implies sequential consistency. So we need to show dynamic consistency. Given (3.2.1) and (3.2.2), let an act $f_0 = (f_{t,i}, a)$ and an act $g_0 = (g_{t,i}, a)$ s.t. $f_0, g_0 \in F_0$ and $f_{t,i} \succeq_{P_{t,i(a)}} g_{t,i}$. Define $\overline{\pi} \in \overline{C_t}$ as the optimal prior for f_0. Then:

$\min_{\pi_0 \in C_0} \int (u \circ f) d\pi_0$

$= \min_{\pi \in \overline{C_t}} \int [\min_{\pi_{t,i} \in C_{t,i}} \int (u \circ f_{t,i}) d\pi_{t,i} + \sum_{j \neq i} \min_{\pi_{t,j} \in C_{t,j}} \int (u \circ a) d\pi_{t,j}] d\pi$

$\geq \int [\min_{\pi_{t,i} \in C_{t,i}} \int (u \circ g_{t,i}) d\pi_{t,i} + \sum_{j \neq i} \min_{\pi_{t,j} \in C_{t,j}} \int (u \circ a) d\pi_{t,j}] d\overline{\pi}$

$\geq \min_{\pi \in \overline{C_t}} \int [\min_{\pi_{t,i} \in C_{t,i}} \int (u \circ g_{t,i}) d\pi_{t,i} + \sum_{j \neq i} \min_{\pi_{t,j} \in C_{t,j}} \int (u \circ a) d\pi_{t,j}] d\pi$

$= \min_{\pi_0 \in C_0} \int (u \circ g) d\pi_0$

where the first and the last equalities are from (3.2.1) and (3.2.2), and the second inequality is from (3.2.3): $\min_{\pi_{t,i} \in C_{t,i}} \int (u_{t,i} \circ f_{t,i}) d\pi_{t,i} \geq \min_{\pi_{t,i} \in C_{t,i}} \int (u_{t,i} \circ g_{t,i}) d\pi_{t,i} \Leftrightarrow \min_{\pi_{t,i} \in C_{t,i}} \int (u \circ f_{t,i}) d\pi_{t,i} \geq \min_{\pi_{t,i} \in C_{t,i}} \int (u \circ g_{t,i}) d\pi_{t,i}$. Clearly, $f_0 \succeq g_0$.

Conversely, suppose $f_0 \succeq g_0$. Then:

$\min_{\pi_0 \in C_0} \int (u \circ f) d\pi_0$

$= \min_{\pi \in \overline{C_t}} \int [\min_{\pi_{t,i} \in C_{t,i}} \int (u \circ f_{t,i}) d\pi_{t,i} + \sum_{j \neq i} \min_{\pi_{t,j} \in C_{t,j}} \int (u \circ a) d\pi_{t,j}] d\pi$

$\geq \min_{\pi \in \overline{C_t}} \int [\min_{\pi_{t,i} \in C_{t,i}} \int (u \circ g_{t,i}) d\pi_{t,i} + \sum_{j \neq i} \min_{\pi_{t,j} \in C_{t,j}} \int (u \circ a) d\pi_{t,j}] d\pi$

$= \min_{\pi_0 \in C_0} \int (u \circ g) d\pi_0$

By monotonicity (it hold over $P_{t,i}$ because of the existence of $\overline{C_t}$), this only hold when $\min_{\pi_{t,i}\in C_{t,i}} \int(u \circ f_{t,i})d\pi_{t,i} \geq \min_{\pi_{t,i}\in C_{t,i}} \int(u \circ g_{t,i})d\pi_{t,i}$. By (3.2.3), $u_{t,i(a)} = \alpha u + \beta$, which implies $f_{t,i} \succeq_{P_{t,i(a)}} g_{t,i}$.

Necessity: *dynamic consistency, sequential consistency* $\Rightarrow$ (3.2.1), (3.2.2), and (3.2.3)

(Step 1) $u_{t,i(a)}(.)$ is a positive affine transformation of $u(.)$

By sequential consistency, there is a utility function for each conditional preference given $a \in F^c_{t,i}$. Let $u_{t,i(a)}(.)$ be a utility function given a. Suppose that $u_{t,i(a)}(.)$ is not a positive affine transformation of $u(.)$. Then indifference sets over $\Delta(X^K)$ are different somewhere. W.L.O.G., for $x, y \in \Delta(X^K)$, $x \succ y$ on $u_{t,i(a)}(.)$ but $x \simeq y$ on $u(.)$. Let $f_{t,i}$ be a conditional act that assigns x for each $\omega \in P_{t,i}$, and $g_{t,i}$ be a conditional act that assigns y for each $\omega \in P_{t,i}$. Let an act $f_0 = (f_{t,i}, a)$ and an act $g_0 = (g_{t,i}, a)$. Then by monotonicity under conditional acts (sequential consistency) and unconditional acts on F_0, $f_{t,i} \succ_{P_{t,i(a)}} g_{t,i}$ but $f \simeq g$, which contradicts dynamic consistency. Hence, $u_{t,i(a)}(.)$ is an positive affine transformation of $u(.)$. From now on, W.L.O.G., we assume that $u_{t,i(a)}(.) = u(.)$.

(Step 2) Condition (3.2.1)

$N_{t,i}$ =1

(i) For $P_{t,i}$ that has a single state, Assumption 3.2.2 (full support) implies $C_0(P_{t,i}|\pi_0(P^c_{t,i}))$ /$\pi_0(P_{t,i})$ =1 under any π_0.[25]

[25] Dynamic consistency and monotonicity on F_0 also imply that conditional preference over acts in $F_{t,i}$ is

$N_{t,i}$ >1

(ii) First, we will show that conditional preferences are identical among acts in F_0 that assign an identical probability distribution from $\Delta(X^K)$ on $\omega \in P^c_{t,i}$, i.e., among a generic element $f_0 = (f_{t,i}$,$a)$. Let $\overline{F}_{0,(a)}$ be a set of acts that assigns a on $P^c_{t,i}$. By dynamic consistency, for f_0 and $g_0 \in \overline{F}_{0,(a)}$, $f_0 \succeq g_0$ iff $f_{t,i} \succeq_{Pt,i(a)} g_{t,i}$. Now let $\overline{F}_0$ be a set of acts that assign an identical element from $\Delta(X^K)$ on $\omega \in \Omega$, and $\overline{F}_{t,i}$ be a set of conditional acts of $F_{t,i}$ that assign an identical element on $\omega \in P_{t,i}$. Then by certainty-independence on F_0, if $\forall h \in \overline{F}_0$, $\forall \alpha \in (0,1)$, $f \succ g$ iff $\alpha f \oplus (1-\alpha)h \succ \alpha g \oplus (1-\alpha)h$. Then by dynamic consistency, $\forall h_{t,i} \in \overline{F}_{t,i}$, $\forall \alpha \in (0,1)$, $f_{t,i} \succ_{Pt,i(a)} g_{t,i}$ iff $\alpha f_{t,i} \oplus (1-\alpha)h_{t,i} \succ_{Pt,i(a)} \alpha g_{t,i} \oplus (1-\alpha)h_{t,i}$. Note first that it is obvious that conditional preference relations $\succeq_{Pt,i(a)}$ and $\succeq_{Pt,i(b)}$are identical if $u(a(\omega)) = u(b(\omega))$ because the utility for an act f_0 is based on the weighted sum of $u(f_0(\omega))$ so that the exact shape of a distribution of $f_0(\omega)$ does no matter. (Hence dynamic consistency implies $\succeq_{Pt,i(a)}$ and $\succeq_{Pt,i(b)}$ are identical if $u(a(\omega)) = u(b(\omega))$.)

Now, suppose that $\succeq_{Pt,i(a)}$and $\succeq_{Pt,i(b)}$are different, and $u(a(\omega)) > u(b(\omega))$. Suppose also that b is not a element in $\Delta(X^K)$ that yields a minimum utility. Then indifference sets under $\succeq_{Pt,i(a)}$ and $\succeq_{Pt,i(b)}$ are different somewhere. W.L.O.G., for $f_{t,i}, g_{t,i} \in F_{t,i}$, $f_{t,i} \succ_{Pt,i(a)} g_{t,i}$ but $f_{t,i} \simeq_{Pt,i(b)} g_{t,i}$. By continuity and monotonicity on F_0, there is a constant act $h_c \in \overline{F}_0$ that assigns $c(\omega)$ on $\omega \in \Omega$ s.t. $u(a(\omega)) > u(b(\omega)) > u(c(\omega))$ and $u(\lambda a(\omega) + (1-\lambda)c(\omega))$ $= u(b(\omega))$ with some $\lambda \in (0,1)$. By $f_{t,i} \succ_{Pt,i(a)} g_{t,i}$, dynamic consistency, and certainty-independence on F_0, $\lambda f \oplus (1-\lambda)h_c \succ \lambda g \oplus (1-\lambda)h_c$. By dynamic consistency and the

identical to the preference of probability distribution over $\Delta(X^K)$, which is independent of $a \in F^c_{t,i}$.

result in the previous paragraph, $\lambda f_{t,i} \oplus (1-\lambda)h_{t,i} \succ_{Pt,i(b)} \lambda g_{t,i} \oplus (1-\lambda)h_{t,i}$. However, this inequality contradicts the assumption of $f_{t,i} \simeq_{Pt,i(b)} g_{t,i}$. By the same argument, if a is not an element in $\Delta(X^K)$ that yields a maximum utility, it leads to a contradiction. Hence, If $\succeq_{Pt,i(a)}$ and $\succeq_{Pt,i(b)}$are not identical, $a(\omega)$ must be a maximum element in $\Delta(X^K)$, $b(\omega)$ must be a minimum element in $\Delta(X^K)$, and all others give identical preference relations. However, by continuity of preference and closeness of C_0, this is impossible. Therefore, a conditional preference over $\overline{F}_{0,(a)}$ is independent of a where a is a constant act over $P^c_{t,i}$.

(iii) Let $C_{t,i,c}$ be the multiple priors set that represents the conditional preference for acts in F_0 that assign an identical element from $\Delta(X^K)$ on $\omega \in P^c_{t,i}$, i.e., acts with a generic element $f_0 = (f_{t,i}$,$a)$. Suppose that under some $\pi_0(P_{t,i})$, $C_{t,i,c} \neq C_0(P_{t,i}|\pi_0(P^c_{t,i}))/\pi_0(P_{t,i})$. Let $C_{t,i} = C_0(P_{t,i}|\pi_0(P^c_{t,i}))/\pi_0(P_{t,i})$ for short. First we assume that A=$C_{t,i}\backslash C_{t,i,c} \neq \emptyset$. Note that $C_{t,i}$ and $C_{t,i,c}$ *does not* depend on $a \in \overline{F}^c_{t,i}$ where $\overline{F}^c_{t,i}$ is a set of acts that assigns an identical element from $\Delta(X^K)$ on $\omega \in P^c_{t,i}$. Let $\widetilde{\pi}_{t,i}$ be a boundary point of $C_{t,i,c}$ that satisfies $\widetilde{\pi}_{t,i} \in \partial A$ and $\forall\varepsilon$ s.t. $\exists 0 < \varepsilon$, $B(\varepsilon, \widetilde{\pi}_{t,i})$ includes points in A and $C_{t,i,c}$. Then there is a sequence of $\pi^n_{t,i} \in A$ that converges to $\widetilde{\pi}_{t,i}$. In other words, $\widetilde{\pi}_{t,i}$ is the element of $C_{t,i,c}$ that faces A. By applying the supporting hyperplane theorem, $\exists$ a sequence of α^n with norm one that satisfies $\alpha^n \cdot \pi_{t,i} > \alpha^n \cdot \pi^n_{t,i}$ if $\pi_{t,i} \in C_{t,i,c}$. By finiteness of state space, there is a subsequence $(\pi^{n^k}_{t,i}, \alpha^{n^k})$ s.t. $\alpha^{n^k} \to \alpha$. By continuity of linear function, $\alpha \cdot \pi_{t,i} \geq \alpha \cdot \widetilde{\pi}_{t,i}$ if $\pi_{t,i} \in C_{t,i,c}$. By a positive affine transformation, we can define a support of the original utility $u(.)$ of $\Delta(X^K)$ as [-L,L] with $L \gg \max|\alpha(\omega)|$ where α is a vector used for a support function for $C_{t,i,c}$.

Let an act $f_0 = (f_{t,i},a)$ s.t. $\alpha = u \circ f_{t,i}$. Clearly, $u \circ f_{t,i}$ serves as a vector for the support function of $C_{t,i,c}$ at $\widetilde{\pi}_{t,i}$, i.e.:

(A) $\quad \int u \circ f_{t,i} d\pi_{t,i} \geq \int u \circ f_{t,i} d\widetilde{\pi}_{t,i}$ where $\pi_{t,i} \in C_{t,i,c}$

For the above subsequence of $\pi^{n^k}_{t,i} \in A$ that converges to $\widetilde{\pi}_{t,i}$, $\int u \circ f_{t,i} d\widetilde{\pi}_{t,i} > \int u \circ f_{t,i} d\pi^{n^k}_{t,i}$. By finiteness of the state space, $\exists \widetilde{\pi}'_{t,i} \in C_{t,i}$ s.t. $\widetilde{\pi}'_{t,i} = \text{argmin} \int u \circ f_{t,i} d\pi_{t,i}$ where $\pi_{t,i} \in C_{t,i}$. The existence of the element $\pi^{n^k}_{t,i}$ in A implies:[26]

$\int u \circ f_{t,i} d\pi_{t,i} \geq \int u \circ f_{t,i} d\widetilde{\pi}'_{t,i}$ where $\pi_{t,i} \in C_{t,i}$ in particular, $\pi_{t,i} = \pi^{n^k}_{t,i}$.

Clearly,

(B) $\quad \int u \circ f_{t,i} d\widetilde{\pi}_{t,i} > \int u \circ f_{t,i} d\pi^{n^k}_{t,i} \geq \int u \circ f_{t,i} \widetilde{\pi}'_{t,i}$

Now define $\overline{f_{t,i}}$ as a act that assigns the identical probability distribution from $\Delta(X^K)$ on $\omega \in P_{t,i}$ and satisfies $\int u \circ f_{t,i} d\pi^{n^k}_{t,i} = \int u \circ \overline{f_{t,i}} d\pi^{n^k}_{t,i}$. Since $\overline{f_{t,i}}$ is constant over $P_{t,i}$, $\int u \circ \overline{f_{t,i}} d\pi^{n^k}_{t,i} = \int u \circ \overline{f_{t,i}} d\widetilde{\pi}_{t,i} = \int u \circ \overline{f_{t,i}} d\widetilde{\pi}'_{t,i} = \int u \circ \overline{f_{t,i}} d\mu$ where $\mu \in \Delta(P_{t,i})$. Hence:

(C) $\quad \int u \circ \overline{f_{t,i}} d\pi^{n^k}_{t,i} \geq \int u \circ f_{t,i} \widetilde{\pi}'_{t,i}$

Note that from the separating hyperplane theorem and (B), it is clear that $\alpha \nparallel 1$. By sequential consistency, given a, the preference must satisfy the multiple priors model within $f_{t,j} \in F_{t,i}$. This implies that given a, $\theta u \circ f_{t,i}$ and $\theta u \circ \overline{f_{t,i}}$ must have the same preference order as in $u \circ f_{t,i}$ and $u \circ \overline{f_{t,i}}$ as long as $\max|\theta u \circ f_{t,i}| < L$ and $\max|\theta u \circ f_{t,i}| < L$. Let

[26] In fact, $\widetilde{\pi}'_{t,i}$ is in A. Otherwise, $\widetilde{\pi}'_{t,i} \in C_{t,i} \cap C_{t,i,c}$. Then $\int u \circ f_{t,i} d\widetilde{\pi}'_{t,i} > \int u \circ f_{t,i} d\pi^{n^k}_{t,i}$.

$f'_{t,i}$ be an act that satisfies $\theta u \circ f_{t,i} = u \circ f'_{t,i}$, and $\overline{f}'_{t,i}$ be an act that satisfies $\theta u \circ \overline{f}_{t,i} = u \circ \overline{f}'_{t,i}$. Since $\succeq_{Pt,i(a)}$ is independent of a as long as a is constant over $\omega \in P_{t,i}$, W.O.L.G., $\exists\theta$ s.t. $\overline{f}'_{t,i}(\omega) = a(\omega)$ where $|a(\omega)| << L$. Let f'_0 be an act in $F_{0,(a)}$ with $f'_{t,i}$ for $P_{t,i}$ and with a for $P^c_{t,i}$, and g'_0 be an act in F_0 with $\overline{f}'_{t,i}$ for $P_{t,i}$ and with a for $P^c_{t,i}$. Then by dynamic consistency, $f'_0 \succ g'_0$. Let $\pi_0 = (\widetilde{\pi}'_{0,t,i}, \pi^c_{0,t,i})$ where $\widetilde{\pi}'_{0,t,i} = \widetilde{\pi}'_{t,i} \cdot \pi_0(P_{t,i})$. Then by (C), $\int u \circ g'_0 d\pi_0 \geq \int u \circ f'_0 d\pi_0$. Since g'_0 is a constant act, $\int u \circ g'_0 d\pi_0 = \min_{\pi \in C_0} \int u \circ g'_0 d\pi \geq \min_{\pi \in C_0} \int u \circ f'_0 d\pi$, which implies $g'_0 \succeq f'_0$. This is a contradiction.

(iv) We know that there is no $C_{t,i}$ s.t. A $= C_{t,i} \backslash C_{t,i,c} \neq \emptyset$. Now, suppose that $\exists C_{t,i}$ s.t. A$=C_{t,i,c} \backslash C_{t,i} \neq \emptyset$. By repeating the same argument as in (iii), $\exists$ $f_{t,i}$ and $\overline{f}_{t,i}$, s.t. $\min_{\pi_{t,i} \in C_{t,i}} \int u \circ f_{t,i} d\pi_{t,i} > \int u \circ \overline{f_{t,i}} d\pi'_{t,i} \geq \min_{\pi_{t,i} \in C_{t,i,c}} \int u \circ f_{t,i} d\pi_{t,i}$ where $\overline{f_{t,i}} \in \overline{F}^c_{t,i}$ and $\pi'_{t,i} \in A$. Note that by construction, $f_{t,i}$ is not a constant act. Again by the same operation as in (iii), define $f'_{t,i}$ and $\overline{f}'_{t,i}$. Suppose that π_0 is the optimal prior for $f'_0 = (f'_{t,i}, a)$. Then since $a \in F^c_{t,i}$ is a constant, any probability distribution over $\{a(\omega)\}$ will yield the same integral over $\{a(\omega)\}$. Hence, if $\pi_0 = (\widetilde{\pi}_{t,i}, \pi^c_{t,i})$ is the optimal prior for f'_0, then $\widetilde{\pi}_{t,i}$ $= \text{argmin}_{\pi_{t,i} \in C_{t,i}} \int u \circ f'_{t,i} d\pi_{t,i}$. Also by construction, $\int u \circ g'_0 d\pi_0 = \min_{\pi \in C_0} \int u \circ g'_0 d\pi$. Now by assumption, A=$C_{t,i,c} \backslash C_{t,i} \neq \emptyset$, which implies that $\overline{f}'_{t,i} \succeq f'_{t,i}$. By dynamic consistency, $g'_0 \succeq f'_0$. However, by construction, $f'_0 \succ g'_0$, which is a contradiction. Clearly the preference order becomes inconsistent, so $C_{t,i,c} = C_{t,i}$. Hence, if π_0 is an optimal prior for $f'_{t,i} = \theta f_{t,i}$, then $C_{t,i} = C_{t,i,c}$.

Next, since $a \in \overline{F}^c_{t,i}$ is a constant act and $\overline{f}_{t,i} \in \overline{F}_{t,i}$ is a constant act, if $\theta\overline{f}_{t,i}(\omega) > a(\omega)$, $\pi_0(P_{t,i})$ assigns the lowest probability over $P_{t,i}$, and if $\theta\overline{f}_{t,i}(\omega) < a(\omega)$, $\pi_0(P_{t,i})$ assigns the highest probability over $P_{t,i}$. Let $\pi_0^L(P_{t,i})$ be the lowest $\pi_0(P_{t,i})$ and $\pi_0^H(P_{t,i})$ be the highest $\pi_0(P_{t,i})$. Take a sequence of acts $\theta f_{t,i}^n$ that converges to $\theta \overline{f}_{t,i}$, where $f_{t,i}^n$ does not assign the identical elements from $\Delta(X^K)$. For this sequence, by continuity, $\pi_0^n(P_{t,i})$ must converge to $\pi_0^L(P_{t,i})$ if $\theta\overline{f}_{t,i}(\omega) > a(\omega)$, and to $\pi_0^H(P_{t,i})$ if $\theta\overline{f}_{t,i}(\omega) < a(\omega)$, where π_0^n is the optimal prior for f_0^n= $(\theta f_{t,i}^n$,$a)$. Since $C_{t,i,c} = C_{t,i}$ at all $\pi_0^n(P_{t,i})$, again by continuity of the preference and closeness of C_0, $C_{t,i,c} = C_{t,i}$ at $\theta\overline{f}_{t,i}(\omega) > a(\omega)$ or $\theta\overline{f}_{t,i}(\omega) < a(\omega)$.

Now by assumption, there is some $C_{t,i}$ s.t. A=$C_{t,i,c}\backslash C_{t,i} \neq \emptyset$ at π_0 where $\pi_0(P_{t,i})$ does not assign the highest or lowest probability on $P_{t,i}$. Then there is γ s.t. $\gamma\pi_0^L(P_{t,i})\oplus(1-\gamma)\pi_0^H(P_{t,i})$ =$\pi_0(P_{t,i})$. However, since $C_{t,i,c} = C_{t,i}$ at π_0^L and π_0^H, $\exists \pi_{t,i} \in C_{t,i}$ at π_0 s.t. $\pi_{t,i}^L = \pi_{t,i}^H \neq \pi_{t,i}$. Let $\widetilde{\pi}_0^L = (\widetilde{\pi}_{t,i}^L$,$\pi_{t,i}^{c,L})$, $\widetilde{\pi}_0^H = (\widetilde{\pi}_{t,i}^H$,$\pi_{t,i}^{c,H})$, and $\pi_0 = (\widetilde{\pi}_{t,i}$,$\pi_{t,i}^c)$, where $\widetilde{\pi}_{t,i}^L = \pi_{t,i}^L \cdot \pi_0^L(P_{t,i})$, $\widetilde{\pi}_{t,i}^H = \pi_{t,i}^H \cdot \pi_0^H(P_{t,i})$, and $\widetilde{\pi}_{t,i} = \pi_{t,i} \cdot \pi_0(P_{t,i})$. Then, $\gamma\widetilde{\pi}_{t,i}^L \oplus (1-\gamma)\widetilde{\pi}_{t,i}^H \neq \widetilde{\pi}_{t,i}$, which contradicts the convexity of C_0. Hence, there is no $C_{t,i}$ s.t. A $= C_{t,i,c}\backslash C_{t,i} \neq \emptyset$. Therefore, $C_{t,i,c} = C_0(P_{t,i}|\pi_0(P_{t,i}^c))/\pi_0(P_{t,i})$ $\forall \pi_0 \in C_0$.

(v) For the case such that $C_{t,i} \cap C_{t,i,c} = \emptyset$, let α^n be a sequence that separates these two sets and converges to a boundary point of $C_{t,i,c}$. Then we can use (iii) to show that it contradicts dynamic consistency.

(Step 3) Condition (3.2.2)

Next, we will show Condition (3.2.2). Given (3.2.1), any $\pi_0 \in C_0$ is defined by ($\pi_{0,t,1}$,...,π_{0,t,N_t}) where N_t is the cardinality of P_t. For $\pi_0, \pi_0' \in C_0$ and $\alpha \in$ (0,1), $\alpha\pi_0$ + (1-α)π_0' = (...,$\alpha\pi_0(P_{t,i})q_{t,i}$ + (1-α)$\pi_0'(P_{t,i})q_{t,i}'$,...) $\in C_0$, where $q_{i,t} = \pi_{0,t,i}/\pi_0(P_{t,i}) \in$

$C_0(P_{t,i}|\pi_0(P^c_{t,i}))/\pi_0(P_{t,i})$. Now, $\alpha\pi_0(P_{t,i})q_{t,i}$ + (1-α)$\pi'_0(P_{t,i})q'_{t,i} = [\alpha\pi_0(P_{t,i})+(1-\alpha)\pi'_0(P_{t,i})]$ $[\theta_{t,i}q_{t,i}$ + (1-$\theta_{t,i})q'_{t,i}]$ where $\theta_{t,i} = \alpha\pi_0(P_{t,i})/[\alpha\pi_0(P_{t,i})+(1-\alpha)\pi'_0(P_{t,i})]$. Clearly, $\theta_{t,i} \in (0,1)$ and by convexity of $C_0(P_{t,i}|\pi_0(P^c_{t,i}))$ / $\pi_0(P_{t,i})$, $[\theta_{t,i}q_{t,i}$ + (1-$\theta_{t,i})q'_{t,i}]\in$ $C_0(P_{t,i}|\pi_0(P^c_{t,i}))$ / $\pi_0(P_{t,i})$. Define $\overline{C}_t$ as the collection of $[\pi_0(P_{t,1})$,...,$\pi_0(P_{t,N_t})]$. The non-emptiness is obvious. Suppose not convex. Then by the above calculation, $\exists\pi_0, \pi'_0 \in C_0$ and $\alpha \in$ (0,1), $\alpha\pi_0$ + (1-α)$\pi'_0 \notin C_0$, which violates convexity of C_0. By the same logic, if $\overline{C}_t$ is not closed, it violates closeness of C_0.

(Step 4) $C_0(P_{t,i}|\pi_0(P^c_{t,i}))/\pi_0(P_{t,i})$ = the multiple priors set for $P_{t,i}$ at π_0 for given $a \in F^c_{t,i}$

We have shown that given $a \in \overline{F}^c_{t,i}$, $C_{t,i,c} = C_0(P_{t,i}|\pi_0(P^c_{t,i}))/\pi_0(P_{t,i})$ serves as the multiple priors set for $f_{t,i} \in F_{t,i}$. Finally, we need to show that a conditional preference on $F_{t,i}$ given $a \in F^c_{t,i}$ has $C_0(P_{t,i}|\pi_0(P^c_{t,i}))/\pi_0(P_{t,i})$ as the multiple priors set. Note that a is not a constant act any more.

Suppose that $C_0(P_{t,i}|\pi_0(P^c_{t,i}))/\pi_0(P_{t,i})$ is not a multiple priors set for a conditional preference on $F_{t,i}$ given $a \in F^c_{t,i}$ but $a \notin \overline{F}^c_{t,i}$. $\exists$ $f_{t,i}, g_{t,i}$ s.t. $f_{t,i} \succ_{P_{t,i(a)}} g_{t,i}$ but min$\int u \circ f_{t,i} d\pi_{t,i}$ = min$\int u \circ g_{t,i} d\pi_{t,i}$ under $C_0(P_{t,i}|\pi_0(P^c_{t,i}))$ / $\pi_0(P_{t,i})$. By dynamic consistency, $f_0 \succ g_0$. Then by Conditions (3.2.1) and (3.2.2):

$\min_{\pi_0 \in C_0} \int (u \circ f) d\pi_0$

$= \min_{\pi \in \overline{C_t}} \int [\min_{\pi_{t,i} \in C_{t,i}} \int (u \circ f_{t,i}) d\pi_{t,i} + \sum_{j \neq i} \min_{\pi_{t,j} \in C_{t,j}} \int (u \circ a) d\pi_{t,j}] d\pi$

$= \min_{\pi \in \overline{C_t}} \int [\min_{\pi_{t,i} \in C_{t,i}} \int (u \circ g_{t,i}) d\pi_{t,i} + \sum_{j \neq i} \min_{\pi_{t,j} \in C_{t,j}} \int (u \circ a) d\pi_{t,j}] d\pi$

$= \min_{\pi_0 \in C_0} \int (u \circ g) d\pi_0$

This implies $f_0 \simeq g_0$, which is a contradiction.$\blacksquare$

Appendix 3.B: (Proof of Proposition 3.2.4)

Sufficiency: *(3.2.1), (3.2.2), and (3.2.3) $\Rightarrow$ dynamic consistency, consequentialism*

Condition (3.2.1), (3.2.2), and (3.2.3) implies consequentialism because of the recursive structure of multiple priors. The proof of dynamic consistency is identical to the one in Appendix 3.A.

Necessity: *dynamic consistency, consequentialism* $\Rightarrow$ (3.2.1), (3.2.2), and (3.2.3)

(Step 1) dynamic consistency and consequentialism implies sequential consistency

Let f_0 be an act in F_0 with $f_0 = (f_{t,i}, a)$ where $a \in F^c_{t,i}$, and let $F_{0,(a)}$ be a set of acts that assigns a on $P^c_{t,i}$. Let $h_0 \in \overline{F}_0$, where $\overline{F}_0$ is a set of acts that assigns an identical element from $\Delta(X^K)$ on $\omega \in \Omega$. Assume f_0 and g_0 are in $F_{0,(a)}$. Then by certainty independence on F_0, for $\forall h \in \overline{F}_0$ and $\forall \alpha \in (0,1)$, if $f_0 \succ g_0$ then $\alpha f_0 \oplus (1-\alpha)h_0 \succ \alpha g_0 \oplus (1-\alpha)h_0$. Let $b(\omega) = \alpha a(\omega) \oplus (1-\alpha)h(\omega)$. By dynamic consistency, $\forall h_{t,i} \in \overline{F}_{t,i}$ where $h_{t,i}$ has an identical element from $\Delta(X^K)$ on $P_{t,i}$, if $f_{t,i} \succ_{Pt,i(b)} g_{t,i}$, then $\alpha f_{t,i} \oplus (1-\alpha)h_{t,i} \succ_{Pt,i(b)} \alpha g_{t,i} \oplus (1-\alpha)h_{t,i}$. By consequentialism, $\alpha f_{t,i} \oplus (1-\alpha)h_{t,i} \succ_{Pt,i(a)} \alpha g_{t,i} \oplus (1-\alpha)h_{t,i}$ $\forall a \in F^c_{t,i}$. Conversely, by certainty independence on F_0, for $\forall h \in \overline{F}_0$ and $\forall \alpha \in (0,1)$, if $\alpha f_0 \oplus (1-\alpha)h_0 \succ \alpha g_0 \oplus (1-\alpha)h_0$, then $f_0 \succ g_0$. By dynamic consistency, for $\forall h_{t,i} \in \overline{F}_{t,i}$, if $\alpha f_{t,i} \oplus (1-\alpha)h_{t,i} \succ_{Pt,i(b)} \alpha g_{t,i} \oplus (1-\alpha)h_{t,i}$, then $f_{t,i} \succ_{Pt,i(b)} g_{t,i}$. By consequentialism,

$f_{t,i} \succ_{Pt,i(a)} g_{t,i}$ $\forall a \in F^c_{t,i}$. This implies that certainty-independence holds under $\succ_{Pt,i}$. Also by uncertainty aversion, $\forall f_0, g_0 \in F_{0,(a)}$ and $\forall \alpha \in (0,1)$, $f \simeq g \Rightarrow \alpha f \oplus (1-\alpha)g \succeq f$. Again, by dynamic consistency and consequentialism, $\forall f_{t,i}, g_{t,i} \in F_{t,i}$ and $\forall \alpha \in (0,1)$, $f_{t,i} \simeq_{Pt,i(a)} g_{t,i} \Rightarrow \alpha f_{t,i} \oplus (1-\alpha)g \succeq_{Pt,i(a)} f$, which implies that uncertainty aversion holds under $\succeq_{Pt,i(a)}$. Other Axioms also hold by the same construction. Hence, the conditional preference $\succeq_{Pt,i(a)}$ is represented by the multiple priors model. By consequentialism, this preference is independent of elements on $F^c_{t,i}$, so we write it as $\succeq_{Pt,i}$. Clearly, $u_{t,i(a)}(.)$ can be different up to a positive affine transformation.

(Step 2) $\quad u_{t,i(a)}(.)$ is an positive affine transformation of $u(.)$

Suppose that $u_{t,i(a)}(.)$ is not an positive affine transformation of $u(.)$. Then indifference sets over $\Delta(X^K)$ are different somewhere. W.L.O.G., for $x, y \in \Delta(X^K)$, $x \succ y$ on $u(.)$ but $x \simeq y$ on $u_{t,i(a)}(.)$. Let $f_{t,i}$ be a conditional act that assigns x for each $\omega \in P_{t,i}$, and $g_{t,i}$ be a conditional act that assigns y for each $\omega \in P_{t,i}$. Let an act $f_0 = (f_{t,i}, a)$ and an act $g_0 = (g_{t,i}, a)$. Then by monotonicity on F_0, $f_0 \succ g_0$. By dynamic consistency, $f_{t,i} \succ_{P_{t,i(a)}} g_{t,i}$, which is a contradiction. Hence, $u_{t,i(a)}(.)$ is an positive affine transformation of $u(.)$. From now on, W.L.O.G., we assume that $u_{t,i}(.) = u(.)$.

(Step 3) $\quad$ Condition (3.2.1)

(i) $\quad$ By the same reason in the proof of Proposition 3.2.4, we only need to prove for $P_{t,i}$ that includes more than one state.

(ii)-(iii) $\quad$ These are identical to (iii)-(iv) of (Step 2) of the proof of Proposition 3.2.3.

(Step 4) Same as in (Step 3) of the proof of Proposition 3.2.3 ■

Appendix 3.C: (Proof of Proposition 3.2.5)

For any ex-post partition $P_{t,i}$, $P_{t,i} = \cup_k^{k+l} \widetilde{P}_m$ with $l \geq 1$. If there are no overlap between $P_{t,i}$ and $P_{t',j}$, by Condition (3.2.4) and (3.2.6), there is a multiple priors set $C_{t,i}$ defined by elements $\prod_{\tau=0}^{\Gamma} \pi_{\gamma+\tau,i'}(\widehat{P}_{\gamma,i'}(\omega))$ at $\omega \in P_{t,i}$ where $P_{t,i} = \widehat{P}_{\gamma,i'}$ and $\pi_{\gamma+\tau,i'}(\widehat{P}_{\gamma,i'}(\omega)) \in \widetilde{C}_{\gamma+\tau,i'}$. Clearly, this set is non-empty, closed, and convex, which implies Condition (3.2.1).

Condition (3.2.4) and (3.2.6) implies that if there is an overlap between $P_{t,i}$ and $P_{t',j}$ at ω, all $C_{t,i}$ is a singleton for $P_{t,i} \subseteq R_j$ where R_j is the meet of $\{P_{t,i}(\omega)\}$ and $R_j = \cup_k^{k+l} \widetilde{P}_m$. In other words, $\pi \in \widetilde{C}$ treats R_j as a single event, and within R_j, a prior is fixed over $\{\widetilde{P}_m\}_k^{k+l}$. Hence within R_j, any combinations of $\{\widetilde{P}_m\}$ justify Condition (3.2.1).

Finally, Condition (3.2.5) implies that there is a multiple priors set $\widehat{C}$ over $\{R_j\}$. From $\widehat{C}$, we can form $\overline{C}_t$ over $\{P_{t,i}\}$ by the following calculation:

- For $P_{t,i} \subseteq R_j$ that includes an overlap, $\pi_0(P_{t,i}) = \widehat{\pi}_j \sum_k^{k+l} \pi_m$ where $P_{t,i} = \cup_k^{k+l} \widetilde{P}_m$ and $\widehat{\pi}_j \in \widehat{C}$.

- For $P_{t,i} \subseteq R_j$ that does not include an overlap, $\pi_0(P_{t,i}) = \widehat{\pi}_j \prod_{\tau=0}^{\gamma} \pi_{\tau,i'}(\widehat{P}_{\tau,i'}(\omega))$ where $\widehat{\pi}_j \in \widehat{C}$ and $\pi_{\tau,i'}(\widehat{P}_{\tau,i'}(\omega)) \in \widetilde{C}_{\tau,i'}$.

This $\overline{C}_t$ is defined over $\{P_{t,i}\}$, which satisfies Condition (3.2.2). ■

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Chapter 4

Aggregation of Agents with Multiple Priors and Homogeneous Equilibrium Behavior

4.1 Introduction

Ellsberg (1961) suggests through his famous paradox that under the presence of uncertainty, information is too vague to be represented by probabilities. Gilboa-Schmeidler (1989) capture this behavioral intuition through a utility representation in which aversion to uncertainty is expressed as if an agent selected the most pessimistic prior from a give set to evaluate an uncertain outcome (i.e., the multiple prior model). The implications of this new concept are investigated under many settings. For example, Epstein-Wang (1994) demonstrate the possibility under which we can utilize a familiar construction of the representative agent in a dynamic general equilibrium setting even though agents follow the multiple prior model. In their example, all agents share the same 'effective' prior at equilibrium regardless of their initial endowment, and this prior is the most pessimistic prior in the set used to evaluate an aggregate endowment process. In a different example, Ozdenoren (2000) examines the effects of the aggregation of uncertainty-averse agents in an auction model. He shows that under the regularity condition, uncertainty-averse agents bid higher than non-uncertainty-averse agents because they have a pessimistic view toward the behavior of others. The implications from the above two examples are particularly interesting because uncertainty aversion creates a distinct *bias* in agents' behavior at equilibrium.

In this paper, we are interested in deriving conditions under which agents' risk-sharing becomes homogeneous at equilibrium. More precisely, we apply the multiple prior model to a multiple-agents economy, and investigate the possibility that agents select priors with *similar bias* and share *similar equilibrium allocations*. The paper examine two cases: (1)

the case where each agent has the same set of multiple priors, i.e., each agent faces the same uncertainty; (2) the case where agents have heterogeneous multiple prior sets.

In order to investigate the question above, we first examine the single agent economy as a benchmark case. We focus on a special structure of the multiple prior set under which an effective prior depends only on the order of consumptions (i.e., the convex-capacity). Under the convex-capacity, we relax the assumption of a stationary endowment process used in Epstein-Wang (1994). We show that if an endowment grows in a reasonably stable way, the agent selects the most pessimistic prior with respect to tomorrow's endowment, not with respect to the continuation value of the future endowment. Then, we extend this condition to the multiple-agents economy where each agent faces the same uncertainty. Under similar conditions on the aggregate endowment to those in the benchmark case, we confirm the previously known result that the convex capacity is a sufficient condition to achieve full insurance, that is, all agents' consumptions are comonotonic (increasing together) with the aggregate endowment and their marginal rates of substitution are equalized. The effective priors are identical among agents, which justifies the construction of the representative agent. The existence of the representative agent reduces the complex economy to the one where all individuals behave as if they were expected utility maximizers with the common subjective prior. In other words, agents have *globally* optimal consumptions with respect to the *common* prior. Clearly the aggregation forces agents to agree on their beliefs, and the original heterogeneity in endowments and beliefs must disappear at equilibrium.

We then consider the case where agents have heterogeneous multiple prior sets. In this case, we provide conditions such that agents' effective priors (and equilibrium consumptions) will be comonotonic and their marginal rates of substitution (weighted by these priors) will be equalized. More specifically, we derive the structure of commonality among agents' multiple-priors set. One set of sufficient conditions is for each agent's multiple prior set to be symmetric (or to be defined by a convex capacity) around the center of the simplex. Intuitively, by locating multiple prior sets around the center, we can avoid heterogeneous tastes regarding states, whereas under the nesting feature, agents share similar uncertainty. In addition, the multiple prior model introduces *local risk aversion* at the allocations where all consumptions are identical (Segal and Spivak (1990) call this risk attitude first-order risk aversion). Given a sufficient commonality among agents' multiple-priors sets, this risk characteristic forces all agents to behave similarly. Moreover, under heterogeneous multiple prior sets, all agents have *locally* optimal consumptions relative to their most pessimistic priors. This result contrasts with that of the homogeneous convex-capacity case where consumptions are *globally* optimal with respect to the most pessimistic prior. We can also show that for the nested multiple prior sets, the more uncertainty averse the agent is, the less volatile her/his consumption over states.

We then examine how the equilibrium prices evolve. In a single-agent model, Epstein-Wang (1994) show that there is a continuum of equilibrium prices if there are multiple choices of effective priors. Under a multiple-agents economy, we need a restriction on each individual endowment to generate a continuum of equilibrium prices. One set of sufficient conditions is that each agent has identical endowments over at least two states that are also identical among agents.

Finally, we compare our results with other models. Chateauneuf-Dana-Tallon (2000) show a similar risk-sharing property under a two-period economy where agents have convex-capacity. Our model has two differences. First, we examines a dynamic economy. Second, our result includes the analysis of a general multiple prior set. In terms of the relationship between a degree of uncertainty and volatility of optimal allocations, Liu (1998) examines a special case of convex-capacity and concludes that if an agent becomes more uncertainty-averse, she/he can bear more volatility in equilibrium allocations. We can easily show that Liu's result is a special case of our results under heterogeneous multiple prior sets.

We organize the paper as follows: First, we define the economy in Section 4.2 mostly following the notations and formulations of Epstein-Wang (1994, 1995). In Section 4.3, we examine the single-agent economy and construct the benchmark case where the agent possesses the same pessimism over time. In Section 4.4, we extend the results of Section 4.3 and derive the conditions under which full insurance is achieved. In Section 4.5, we derive similar conditions for heterogeneous multiple prior sets. In Section 4.6, we examine the possibility that a continuum of equilibrium prices exists. Most of the proofs are in appendices.

4.2 Stochastic Exchange Economy with Uncertainty Aversion

4.2.1 Intertemporal Utility Function and Structure of Beliefs

We use the dynamic utility structure for the discrete states and finite-horizon economy. It is defined by:

$$(4.2.1) \qquad V(c) = E[\textstyle\sum_1^T u_t(c_t)]$$

The problem here is how to define the expectation operator. In the rational expectations model, we assume that all agents have the homogenous knowledge about the true "objective probability" of the evolution of economy, and use this probability law to calculate the above utility. On the other hand, in the Savage model, this probability is derived as the subjective probability measure.

Gilboa-Schmeidler (1989) axiomatize the notion of uncertainty aversion. In their multiple-priors model, agents behave as if they had a preference over acts which is equivalent to the minium expected value with respect to the closed and convex probability set. In other words, applying their idea[1], (4.2.1) would be written as:

$$(4.2.2) \qquad V(c) = \inf_{m \in \mathfrak{P}} E[\textstyle\sum_1^T u_t(c_t)]$$

[1] In Gilboa-Schmeidler (1989), the utility function and the multpile-priors set jointly represent the agent's preference over uncertain outcome. Here we intentionally make a argument that when the agent become uncertain about the future payoffs, she/he makes the closed and convex multiple-priors set around her/his subjective single prior which is used under non-uncertain situation. We give the rationale behind this assumption in Section 4.5.

where $\mathfrak{P}$ is a closed and convex set of priors[2]

However, (4.2.2) does not generally derive the dynamically consistent choice behavior (Epstein-Wang 1994 p.293-294)[3]. Since we want to investigate the dynamic consumption/investment behavior, (4.2.2) is essentially intractable. Hence, we need to impose the more structure on the set of priors, which requires that each conditional distribution at t also has the closed and convex set of priors:

(4.2.3) $V(c)= u_1(c_1) + \inf_{m\in\mathfrak{P}_1(\omega^1)} E_1[u_2(c_2)+\inf_{m\in\mathfrak{P}_2(\omega^1)} E_2[u_3(c_3)$

$+...\inf_{m\in\mathfrak{P}_{T-1}(\omega^{T-1})} E_{T-1}[u_T(c_T)]...]]$

where $\mathfrak{P}_t(\omega^t)$ is a closed and convex set of priors at t for the history ω^t [4]

Or more concisely,

(4.2.4) $V(c)= u_1(c_1) + \inf_{m\in\mathfrak{P}_1(\omega^1)} E_1[V(c_2,...c_T)]$

where $\mathfrak{P}_1(\omega^1)$ is a closed and convex set of priors at t=1

for the history ω^1; $V(c_2,...c_T)$ is defined as (4.2.3) from t=2,T[5]

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[2] In Appendix 4.A, we show that (4.2.2) is a special case of the dynamic version of Gilboa-Schmeidler (1989).

[3] Precise argument of dynamic consistency is found in Appendix 4.A.

[4] In Appendix 4.A, we show that (4.2.3) is a special case of the dynamic version of Gilboa-Schmeidler (1989). Also in the same section, we show that (4.2.3) delivers the dynamically consistent evolution of multiple-priors sets, i.e., the agent's preference confirms the dynamically consistent multiple-priors model.

[5] In this finite-horizen model, (4.2.3) and (4.2.4) are a equivalent formulation. The proof is in Appendix 4.A.

In the next subsection, we show that (4.2.3) and (4.2.4) deliver the dynamically consistent behavior.

Now, we define the above argument more precisely. The set of discrete states is Ω with discrete topology $\mathfrak{D}(\Omega)$ (N: number of states), and the evolution of state spaces is defined as Ω^T with discrete topology $\mathfrak{D}(\Omega^T)$ (T is finite). Denote $\omega^t = (\omega_1,\omega_2,...,\omega_t) \in \Omega^t$. We also use $(\omega_{1,s},\omega_{2,s},...,\omega_{t,s})$ to specify the particular realization of states over time. Note that we only focus on the finite states and finite-time horizon model because the behavior of (4.2.4) becomes very discontinuous in the continuous states case (or in the infinite-time horizon)[6]. Since the continuous states can be approximated by finite states, the only conceptual limit seem to be the finite time horizon. However, given the utility for the distance future is decreasing, after sufficiently large T, we would neglect the rest of the time. In order words, we avoid the continuous states and infinite-time horizon model because we want to allow more general endowment evolution and derive clear intuition on the aggregated behavior of agents with multiple-priors without considering the discontinuous tail behavior. (On the other hand, the focus of Epstein-Wang (1995) is exactly the discontinuous tail behavior of multiple-priors model as its defining characteristics; see Appendix 4.F)

Define $\mathfrak{M}(\Omega)$ as the space of all probability measures over $\mathfrak{D}(\Omega)$, and assign non-zero probability for each element of $\mathfrak{D}(\Omega)$. Assume that each agent has the probability kernel

[6] In Appendix 4.F, we summarize the results from Epstein-Wang (1995), which shows the discontinous behavior of the continuous states model.

correspondence $\mathfrak{P}_t^h$: $\Omega^t \to \mathfrak{M}(\Omega)$ for each t[7]. We assume that there is no objective probability law, which is essentially equivalent to saying that there is no learning in this model. However, by the non-time-homogenous structure of prior sets can imply that agents actually behave as if they learned something over time even though there is no reference to the objective realization of states.

For notational convenience, we define the following integral: For a set $\mathfrak{P} \in \mathfrak{M}(\Omega)$, and for any bounded measurable function f: $\Omega \to R$,

$$(4.2.5) \qquad \int_\Omega f d\mathfrak{P} \equiv \min\{\int_\Omega f dm : m \in \mathfrak{P}\}$$

where the minimum exist because of the finite state space (by Weirstrause theorem: the minimum over a compact set $\in R^N$exists). We call this value as an "expected value" for the function f as the extension of standard terminology. We also use E[f] as the short hand notation of (4.2.5).

The consumption process $\{X_t\}$ is the adapted process over $\mathfrak{D}(\Omega^T)$. In other words, X_t is $\mathfrak{D}(\Omega^t)$-measurable for all t. Hence, the consumption processes form t to T is in the complete normed vector space:

$$\mathfrak{X}_t = \{{}^\tau X=\{X_\tau\}\text{: } \{X_\tau\} \text{ is an adapted, real valued process}$$
$$\text{s.t. } X_\tau(\omega^\tau)\geq 0 \ \forall\tau=t,T \ , \omega^\tau \in \Omega^\tau, \|X\| = \sup_\tau \sup_{\omega^\tau}|X_\tau(\omega^\tau)| <\infty \ \}$$

Utilities over $\mathfrak{X}_t$ are defined with a probability corresponds and a vNM instantaneous utility functions u_t^h: $R_+ \to R_+$ assumed to be continuous, increasing, strictly concave, and

[7]The name "kernel" is justified by the convolution like formula of integration for each ω^t.

normalized to be $u_t^h(0)=0$[8]. The assumption of strictly concavity of u_t^h is for the expositional purposes for the case of identical multiple priors, but it is crucial for the case of heterogenous sets of multiple priors.[9] In fact, as we see in section 4.4.3, for the economy of risk-neutral agents, the presence of uncertainty changes the equilibrium behavior drastically, which is another reason why we want to separate the strictly concave utility from the weakly concave case.

For each ${}^t c$ in $\mathfrak{X}_\mathrm{t}$, a utility process $\{V_t^h({}^t c)\}_1^T$ is defined as the unique element of $\mathfrak{X}_\mathrm{t}$ s.t.: $\forall$t $\geq$1 and ω^t in Ω^t :

$$(4.2.6) \qquad V_t^h({}^t c;\omega^t)=u_t^h(c_t(\omega^t))+\int V_{t+1}^h(c;\omega^t,\omega)d\mathfrak{P}^h(\omega^t;\omega)$$

We can consider $V_t^h(c;\omega^t)$ to be the utility of the continuation consumption process ${}^t c\equiv \{c_t\}=(c_t,..,c_T)$ conditional on the history ω^t. Given of equivalence of (4.2.3) and (4.2.4) (i.e. (4.2.6)), we can uniquely define $V_t^h(c;\omega^t)$ by backward induction.

4.2.2 The Structure of the Economy

We adapt the standard stochastic exchange economy. There is a single perishable consumption good over $\mathfrak{D}(\Omega^T)$ and there are H agents who have their endowment process $e^h =$

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[8] $u_t(0)=0$ assumption is used to guarantee that the utility process $\{\mathrm{V}_t\}_1^T$ is bounded, i.e. $\mathrm{V}_t < \infty$. Under this assumption, the utility process is in fact in X.

[9] For the identical prior case, the reader can easily verify the most of the results holds under minor modification. We will mention the general concave utility case at footnotes whenever it seems helpful.

$\{e_t^h(\omega^t)\} \in \mathfrak{X}_t$. The aggregate endowment for the whole economy is defined as $e = \sum_1^H e^h$. For simplicity, assume that all endowments are positive:

$$(4.2.7) \qquad e_t^h(\omega^t)>0 \ \text{ for } \forall\omega^t \in \Omega^t$$

There are K securities ($N \leq K < \infty$), where the dividend process for the ith security is $d_i=\{d_{i,t}\}\in \mathfrak{X}_t$. Particularly, we assume that these K securities can span all possible consumption in $\mathfrak{X}_t$, i.e. the asset markets are dynamically complete. In each period, the available securities are trades in a competitive market, and they have prices $q_i = \{q_{i,t}\}\in \mathfrak{X}_t$, where the consumption good is treated as a numeraire at each ω^t. Let $q_t=(q_{1,t},...,q_{K,t})$ and $q=\{q_t\}\in \mathfrak{X}_t^K$. We assume that each agent has zero endowment of shares for K securities so that the total supply of these securities is zero.

At the beginning of each period, each agent plans consumption and investment for available securities for the current and all future periods by $({}^tc^h,{}^t\theta^h)$, where ${}^tc^h \in \mathfrak{X}_t$ and ${}^t\theta^h= \{\theta_\tau^h\}\in \mathfrak{X}_t^K$ with $\theta_\tau^h=(\theta_{1,\tau}^h,..,\theta_{K,\tau}^h)$. We call $({}^tc^h,{}^t\theta^h)$ as (t,ω^t)-*feasible* if it satisfies the following budget constraints:

$$(4.2.8) \qquad q_\tau \bullet \theta_\tau^h + c_\tau = \theta_{\tau-1}^h \bullet [q_\tau + d_\tau] + e_\tau^h \ \forall\tau \geq t$$

$$\theta_0^h(\omega^0) = 0$$

$$\inf_{i,\tau,\omega^\tau} \ \theta_{i,\tau}(\omega^\tau) > - \infty$$

The third inequality is restriction on short sale, which guarantees the existence of equilibrium. Now, agents maximize their utility value $V_t^h({}^tc;\omega^t)$ by solving the following optimization: For each t, (4.2.9) $\quad \text{Max}_{(c,\theta)}\ V_t^h({}^t c;\omega^t)= u_t^h(c_t(\omega^t))+\int V_{t+1}^h({}^{t+1}c;\omega^t,\omega)\,d\mathfrak{P}^h(\omega^t;\omega)$

s.t. $\quad$ (4.2.8)

The solution for this optimization achieves (t,ω^t)-*optimal* allocation $({}^t c^h, {}^t\theta^h)$.

Finally, an *equilibrium* is a price process $\{q_t\}_1^T$ and allocation $\{(c_t^h,\theta_t^h)\}_1^T$ such that: $\forall(t,\omega^t)$ s.t. $1\leq t \leq T$,

(4.2.10) $\quad ({}^t c_t^h, {}^t\theta_t^h)$ is (t,ω^t)-*optimal* for all agents

$$\sum_1^H c_t^h(\omega^t) = e_t(\omega^t)$$

$$\sum_1^H \theta_t^h(\omega^t) = 0$$

At an equilibrium, agents use q as the expectations for future prices and these prices are in fact fulfilled in the subsequent time periods. As we show in Appendix 4.B, the consumption c^h is dynamically consistent, in other words the (t,ω^t)-optimal consumption plan remains optimal for later dates.

As opposed to Epstein-Wang (1995), this economy has the following property. An Arrow-Debrue complete markets equilibrium is generically implemented by a dynamic equilibrium by randomly picking N securities from the K asset pool (Kreps (1982)). More strongly, it is easily seen that a dynamic equilibrium always corresponds to an Arrow-Debreu counterpart by generating Arrow-Debrue securities from dynamic trading. Hence, by examining the Arrow-Debrue equilibrium, we can investigate the property of the corresponding dynamic equilibrium. The proof of the existence of equilibrium for an Arrow-Debreu economy is given in Appendix 4.C.

4.2.3 Special Case

For the rest of this paper, we focus mainly on the specific structure of multiple-priors sets. From Schmeidler (1989), define the multiple-priors set $\mathfrak{P}$ from the non-additive prior v which satisfies the following properties:

(4.2.11) (i) $v(\emptyset) = 0$ and $v(\Omega) = 1$

(ii) For A,B$\in \mathfrak{D}(\Omega)$ s.t. A$\subset$B, v(A)$\leq v$(B)

(iii) v is the convex capacity:

s.t. A,B$\in \mathfrak{D}(\Omega)$; v(A)+v(B)$\leq v$(A$\cap$B)+v(A$\cup$B)

(iv) $\mathfrak{P} = \{m \in \mathfrak{M}(\Omega)\text{: } m(\text{A}) \geq v(\text{A})\}$ (core)

The resulting $\mathfrak{P}$ has very convenient property. It has the Choquet integral formulation:

(4.2.12) $\min_{p\in\mathfrak{P}} \int u(x)dP = \int u(x)dv = \sum_{i=1}^{N}(u_i - u_{i+1})v(\cup_{j=1}^{i} s_j)$ [10]

$= \sum_{i=1}^{N} u_i(v(\cup_{j=1}^{i} s_j) - v(\cup_{j=1}^{i-1} s_j))$

$= \sum_{i=1}^{N} u_i p_i$

where $u_1 > u_2 > ... > u_N \geq 0$, $u_{N+1} = 0 = v(\cup_{j=1}^{0} s_j)$ and s_i corresponds to the state of u_i. We call this $\mathfrak{P}$ the core of convex capacity v or *capacity-based* $\mathfrak{P}$ in short. It is apparent from the definition (iv) and the above expression that the identical prior is used to calculate the expected value among consumptions with the same strong order of utilities. In fact, by

[10] As we mention in Appendix 4.A, the non-additive prior model can be derived on degenerated lotteries on R for each state.

the continuity of preference, the weak order of u at any point does not change the above calculation, so the same prior can be used. More specifically, we say that the utility vector u and u' are *comonotonic* if:

$$(4.2.13) \qquad [u(\omega)\text{-}u(\omega')][u'(\omega)\text{-}u'(\omega')] \geq 0 \ \forall \omega, \omega' \in \Omega$$

In other words, among comonotonic consumptions, there is a single prior for the expectation operator. Later in Section 4.4, we show that the uniqueness of prior among comonotonic consumptions is essential for the existence of the *dynamic* representative agent. From the definition of (4.2.12), it is also apparent that switching the utility of two consecutive states in the utility order only changes the probability of these two states. More specifically, in (4.2.12), if we have $\widetilde{u} = (\ \widetilde{u}_1,...\ \widetilde{u}_N\ )$ where $\widetilde{u}_i = u_{i+1}$, $\widetilde{u}_{i+1} = u_i$, $\widetilde{u}_j = u_j \ \forall j \neq i, i+1$.

$$\sum_{i=1}^N \widetilde{u}_i(v(\cup_{j=1}^i s_j) - v(\cup_{j=1}^{i-1} s_j)) = \sum_{i=1}^N \widetilde{u}_i \widetilde{p}_i$$

Clearly, $\widetilde{p} = (\widetilde{p}_1,...,\widetilde{p}_N)$ is different only at $\widetilde{p}_i$ and $\widetilde{p}_{i+1}$.[11]

In addition, for the multiple-priors model, the following inequality holds (which we need in Section 4.4 and Section 4.5). For u and u', if $\mathfrak{P}$ is a closed and convex set:

$$(4.2.14) \qquad \int (u+u')d\mathfrak{P} \geq \int u d\mathfrak{P} + \int u' d\mathfrak{P}$$

and equality holds when u and u' are comonotonic if $\mathfrak{P}$ is capacity-based.

[11] In fact, any permutation of utilities for k consecutive states in the utility order changes only the probability of thoes states.

Now, given this capacity-based $\mathfrak{P}$, we can rewrite the agents' problem for consumption/investment decision in a more tractable formula.

$$\begin{aligned}
&\text{Max}_{(c,\theta)}\ V_t^h({}^t c;\omega^t) = u_t^h(c_t(\omega^t)) + \int V_{t+1}^h({}^{t+1}c;\omega^t,\omega)\, d\mathfrak{P}^h(\omega^t;\omega) \\
&= \text{Max}_{(c_m,\theta_m)_{m=1,M}} \text{Max}_{(c,\theta)\in(c,\theta)_m} \\
&V_t^h({}^t c;\omega^t) = u_t^h(c_t(\omega^t)) + \int V_{t+1}^h({}^{t+1}c;\omega^t,\omega)\, d\mathfrak{P}^h(\omega^t;\omega) \\
&\text{where among } (c,\theta)_m\ ,\ V_{t+1}^h({}^{t+1}c;\omega^t,\omega) \text{ becomes comonotonic}
\end{aligned} \tag{4.2.15}$$

In (4.2.15), agents first divide the (t,ω^t)-feasible allocation into M parts where in each partition agents behave as if they were subjective prior optimizers for the choice of $V_t^h({}^t c;\omega^t,\omega)$ with the fixed prior, and solve the local optimization. Then they choose $(c,\theta)_m$ that achieves the highest value from these local maxima. This interpretation will be particularly important for the interpretation of equilibrium allocations and prices in the later section. The proof of this statement is found in Appendix 4.D.

4.2.4 Utility Supergradients and Asset Prices

Finally in this subsection, we state the results about differentiability, which will be used in Section 4.5.3 and Section 4.6. Since the formula by Gilboa-Schmeidler (1989) is pointwise minimum, the differentiability does not necessarily follow. However, by utilizing the results from Aubin (1978), we can define the left and right derivative for the utility process $\{V_t^h({}^t c;\omega^t)\}$. We just restate the results from Epstein-Wang (1994) in a single-period model without current consumptions. Note that the similar result holds for the T-periods model.

Lemma 4.2.1:

Assume one period economy without the current consumptions. Let $\{x^h\} = \{x_2^h(\omega_2)\}$ be positive. Define the convex-valued, compact-valued correspondence Q^h: $\Omega \to M(\Omega)$ by:

(4.2.16) $\quad Q^h(\omega_1)=\{m\in P^h(\omega_1) \mid V^h(x) = \int u_2^h(x_2^h(\omega_2))dm = \int u_2^h(x_2^h(\omega_2))dP^h(\omega_1,\omega)\}$

Then the one-side derivative of $V^h(x)$ at x and in the direction $h = \widetilde{h_2}$ where $\widetilde{h_2} \in R^N$, and $\xi \in R$ are given by

(4.2.17)
$$\frac{d}{d\xi}V(x+\xi h)|_{0^+} = \min_m\{\int u'(x_2(\omega_2))\widetilde{h_2}dm : m \in Q^h(\omega_1)\}$$
$$\frac{d}{d\xi}V(x+\xi h)|_{0^-} = \max_m\{\int u'(x_2(\omega_2))\widetilde{h_2}dm : m \in Q^h(\omega_1)\}$$

In addition, at equilibrium, take the perturbation in the budget set: $\widetilde{h_2}$ s.t. $h_2(\omega_2)=\Delta$, $h_2(\omega_2')=-\Delta q(\omega_2)/q(\omega_2')$ where $q(\omega_2)$ is the state price at ω_2. Then:

(4.2.18) $\quad \exists m \in Q^h(\omega_1)$ s.t. $\forall \omega_2, \omega_2' \in \Omega \quad \int\{\frac{u'(x_2(\omega_2))}{u'(x_2(\omega_2'))} - \frac{q(\omega_2)}{q(\omega_2')}\}dm = 0$

The proof of (4.2.16) and (4.2.17) is in Appendix 4.E, which is just an application of Aubin (1979)'s result. The equation for the state price ratio (4.2.18) is from Epstein-Wang (1994: p.297).

The natural interpretation of Lemma 4.2.1 is as follows: Assume that there are two states and agents have the sets of capacity-based multiple priors. Then the indifference curve has a kink at $x_2(\omega_2) = x_2(\omega_2')$. At this kink point, the derivative cannot be defined. If $\widetilde{h_2}$ = (1,-1), then $\frac{d}{d\xi}V(x+\xi h)|_{0^+}$ defines the flattest tangent line and $\frac{d}{d\xi}V(x+\xi h)|_{0^-}$ defines the steepest tangent line. From (4.2.18), if at equilibrium $x_2(\omega_2) = x_2(\omega_2')$, we can conclude that $\frac{d}{d\xi}V(x+\xi h)|_{0^+} = \frac{\underline{p(\omega_2)}}{\overline{p(\omega_2')}} \leq \frac{q(\omega_2)}{q(\omega_2')} \leq = \frac{\overline{p(\omega_2)}}{\underline{p(\omega_2')}} = \frac{d}{d\xi}V(x+\xi h)|_{0^-}$, where $[\underline{p(\omega_2)}, \overline{p(\omega_2')}]$ is the optimal choice of prior for $x_2(\omega_2) > x_2(\omega_2')$, and $[\overline{p(\omega_2)}, \underline{p(\omega_2')}]$ is for

$x_2(\omega_2) < x_2(\omega_2')$.

4.3 Single Agent Economy

4.3.1 Background

For continuous states and infinite-time horizon setting, Epstein-Wang (1994) apply the multiple-priors model to a Lucus representative agent economy. They derive the existence of equilibrium with recursive utility and the several distinct features of the multiplicity of priors as opposed to the single-prior model. They justify the use of the representative agent model by examining the possibility of its construction from heterogenous agents with identical sets of priors that are capacity-based. In this paper, we want to extend their results and investigate the more general conditions where multiple agents behave in a similar fashion. Before proceeding in this direction, it is critically important to derive a benchmark case, i.e. a single agent economy. This setting is traditionally called a representative-agent economy. However, since we want to *construct* the representative agent from multiple-agents economy later, we reserve the terminology "representative agent" for this artificial object, and define this economy as a "single-agent economy". In this section, we want to find the conditions where the single agent behaves *similarly* over time. In the following sections of multiple agents economies, we examine what conditions for the single agent economy must be altered or narrowed. Note that the single agent economy is defined as H=1.

4.3.2 General Order Property of Utility Process

First, we would like to investigate when the single agent behaves as if she/he had the same or similar prior over time or when the agent's behavior shows similar pessimism throughout time. In a one-period model, the agent's utility is define by the most pessimistic prior over the tomorrow's endowment distribution over Ω. Naturally, we can conjecture that the agent simply behaves as if she/he follows the most pessimistic prior with respect to the endowment process $\{e_t\}$ over time. However, this conjecture does not really capture the evolution of time, i.e. the connection of today's endowment and tomorrow's endowment, and tomorrow's endowment and the following day's endowment, and so on. In fact, this connection is the essence of dynamic decision making. Now we show the example that our simple conjecture is false:

First, we assume that there are two states $\Omega = (\omega_1,\omega_2)$ and three dates. Assume that u_t is identical over time, and $\mathfrak{P}_1(\omega_1) = \mathfrak{P}_2(\omega_1, \omega_{2,1}) = \mathfrak{P}_2(\omega_1, \omega_{2,2})$, where they are all capacity-bases. Endowment process is given as follows:

At t=1, $[\omega_1] = [e_1]$

At t=2, $\begin{bmatrix} (\omega_1, \omega_{2,1}) \\ (\omega_1, \omega_{2,2}) \end{bmatrix} = \begin{bmatrix} e_{1,1} \\ e_{1,2} \end{bmatrix}$

At t=3, $\begin{bmatrix} (\omega_1, \omega_{2,1}, \omega_{3,1}) \\ (\omega_1, \omega_{2,1}, \omega_{3,2}) \end{bmatrix} = \begin{bmatrix} e_{1,1,1} \\ e_{1,1,2} \end{bmatrix}, \begin{bmatrix} (\omega_1, \omega_{2,2}, \omega_{3,1}) \\ (\omega_1, \omega_{2,2}, \omega_{3,2}) \end{bmatrix} = \begin{bmatrix} e_{1,2,1} \\ e_{1,2,2} \end{bmatrix}$

The utility process $\{\mathrm{V}_t({}^t e)\}$ is defined:

At t=1, $[\omega_1] = \left[u(e_1) + \int V_{1,\bullet} dP(\bullet)\right]$

At t=2, $\begin{bmatrix} (\omega_1, \omega_{2,1}) \\ \\ (\omega_1, \omega_{2,2}) \end{bmatrix} = \begin{bmatrix} u(e_{1,1}) + \int u(e_{1,1,\bullet}) dP(\omega_1, \omega_{2,1}, \bullet) \\ \\ u(e_{1,2}) + \int u(e_{1,2,\bullet}) dP(\omega_1, \omega_{2,2}, \bullet) \end{bmatrix} = \begin{bmatrix} V_{1,1} \\ \\ V_{1,2} \end{bmatrix}$

At t=3, $\begin{bmatrix} (\omega_1, \omega_{2,1}, \omega_{3,1}) \\ \\ (\omega_1, \omega_{2,1}, \omega_{3,2}) \end{bmatrix} = \begin{bmatrix} u(e_{1,1,1}) \\ \\ u(e_{1,1,2}) \end{bmatrix}, \begin{bmatrix} (\omega_1, \omega_{2,2}, \omega_{3,1}) \\ \\ (\omega_1, \omega_{2,2}, \omega_{3,2}) \end{bmatrix} = \begin{bmatrix} u(e_{1,2,1}) \\ \\ u(e_{1,2,2}) \end{bmatrix}$

First, suppose that $e_{1,1,1} > e_{1,1,2}$ and $e_{1,2,1} < e_{1,2,2}$. Then the agent uses a different prior at $(\omega_1, \omega_{2,1})$ from that at $(\omega_1, \omega_{2,2})$ to calculate the expected value for the endowment process over Ω. Obviously this result implies that the agent behaves *very differently* at t=2.

Next, suppose that the endowments have the same order at t=3, i.e., $e_{1,1,1} > e_{1,1,2}$ and $e_{1,2,1} > e_{1,2,2}$. The agent's utility is defined by the identical prior at $(\omega_1, \omega_{2,1})$ and $(\omega_1, \omega_{2,2})$. However, if $V_{1,1} < V_{1,2}$, her/his utility at t=1 must be based on the different prior from that at t=2. In this case, the agent changes the direction of pessimism over time. It happens even though $e_{1,1} > e_{1,2}$ because $e_{1,1,1} > e_{1,1,2}$ and $e_{1,2,1} > e_{1,2,2}$ does not guarantee $\int u(e_{1,1,\bullet}) dP(\omega_1, \omega_{2,1}, \bullet) > \int u(e_{1,2,\bullet}) dP(\omega_1, \omega_{2,2}, \bullet)$. In fact, the changing prior is the essential difference between a single prior economy and a multiple-priors one.

From the above example, in order to have the same pessimism over Ω and t, we need to have the identical order of the endowment process over Ω at any history of ω^t, and the utility process $\{V_t({}^t e; \omega^{t-1}, \omega_t)\}$ follows the comonotonic movement with the endowment process $\{e_t(\omega^{t-1}, \omega_t)\}$ over Ω $\forall \omega^{t-1}$ $T \geq t > 1$. Now we are ready to formalize this intuition:

Proposition 4.3.1:

In a single agent economy, under the following conditions, the agent behaves as if she/he had the same prior $\forall T > t \geq 1$, which is the most pessimistic prior over $\{\mathrm{e}_t(\omega^t)\}$. In other words, the utility process $\{\mathrm{V}_t({}^t\mathrm{e};\omega^{t-1}, \omega_t)\}$ becomes comonotonic with the endowment process $\{\mathrm{e}_t(\omega^{t-1}, \omega_t)\}$ over Ω $\forall \omega^{t-1}$ T$\geq$t$>$1.

(4.3.1) $\mathrm{e}_t(\omega^{t-1},\omega) \neq \mathrm{e}_t(\omega^{t-1},\omega')$ $\omega, \omega' \in \Omega$ (strong order of endowment)

(4.3.2) $\mathrm{e}_t(\omega^{t-1},\omega) > \mathrm{e}_t(\omega^{t-1},\omega') \Rightarrow \mathrm{e}_{t'}(\omega^{t'-1},\omega) > \mathrm{e}_{t'}(\omega^{t'-1},\omega')$

$\forall T \geq t, t' > 1, \omega, \omega' \in \Omega$,$\omega^{t-1} \in \Omega^{t-1}$, $\omega^{t'-1} \in \Omega^{t'-1}$

(comonotonic order of endowments over Ω for all $\{\mathrm{e}_t(\omega^t)\}$)

(4.3.3) $\mathrm{e}_t(\omega^{t-1},\omega_t) > \mathrm{e}_t(\omega^{t-1},\omega_t') \Rightarrow$

$\mathrm{E}_t[\mathrm{V}_{t+1}({}^{t+1}\mathrm{e};\omega^{t-1},\omega_t,\omega_{t+1})] \geqq \mathrm{E}_t[\mathrm{V}_{t+1}({}^{t+1}\mathrm{e}';\omega^{t-1},\omega_t',\omega_{t+1})]$

$\Rightarrow \mathrm{V}_t({}^t\mathrm{e};\omega^{t-1},\omega_t) > \mathrm{V}_t({}^t\mathrm{e}';\omega^{t-1},\omega_t')$

$\forall T > t > 1, \omega_t, \omega_t', \omega_{t+1} \in \Omega$, $\omega^{t-1} \in \Omega^{t-1}$

(comonotonic order of the endowment $\{\mathrm{e}_t(\omega^{t-1},\omega_t)\}$ and $\{\mathrm{V}_t({}^t\mathrm{e};\omega^{t-1},\omega)\}$)

(4.3.4) The agent has an identical capacity-based multiple-priors set over Ω $\forall \omega^t$:

$P_t = P_{t'}$ $\forall t, t'$ s.t. $1 \leq t, t' \leq$ T (independent prior set)

Proof:

The agent chooses her/his endowment as an optimal consumption plan. By the backward induction, at T-1, its expected utility is:

$$V_{T-1}({}^{T-1}e;\omega^{T-1}) = u_{T-1}(e_{T-1}(\omega^{T-1})) + \int u_t(e_T(\omega^{T-1},\omega_T))dP(\omega^{T-1},\omega_T)$$

By Condition (4.3.1) and increasing u_t, $V_{T-1}({}^{T-1}e;\omega^{T-1})$ is defined by the most pessimistic prior over $\{e_T(\omega^{T-1},\omega_T)\}$. Then, at T-2:

$$V_{T-2}({}^{T-2}e;\omega^{T-2}) = u_{T-2}(e_{T-2}(\omega^{T-2})) + \int V_{T-1}({}^{T-1}e;\omega^{T-2},\omega_{T-1})\,dP(\omega^{T-2},\omega_{T-1})$$

$$= u_{T-2}(e_{T-2}(\omega^{T-2})) + \int \{u_{T-1}(e_{T-1}(\omega^{T-2},\omega_{T-1}))$$

$$+ \int u_T(e_T(\omega^{T-2},\omega_{T-1},\omega_T'))\,dP(\omega^{T-2},\omega_{T-1},\omega_T')\}\,dP(\omega^{T-2},\omega_{T-1})$$

$$= u_{T-2}(e_{T-2}(\omega^{T-2})) + \int \{u_{T-1}(e_{T-1}(\omega^{T-2},\omega_{T-1}))$$

$$+ E_{T-1}[V_T({}^{T}e;\omega^{T-1},\omega_T)]\}\,dP(\omega^{T-2},\omega_{T-1})$$

By Condition (4.3.3) and increasing u_t, $V_{T-2}({}^{T-2}e;\omega^{T-2})$ is defined by the most pessimistic prior over $\{e_{T-1}(\omega^{T-2},\omega_{T-1})\}$. By mathematical induction, $V_t({}^{t}e;\omega^t)$ is defined by the most pessimistic prior over $\{e_{t+1}(\omega^t,\omega_{t+1})\}$ $\forall T > t \geq 1$. Finally, by Condition (4.3.2) and (4.3.4), this defining prior is identical over time. ■

Condition (4.3.3), i.e., the comonotonicity of the endowment process $\{e_t(\omega^{t-1},\omega_t)\}$ and the utility process $\{V_t({}^{t}e;\omega^{t-1},\omega_t)\}$ over Ω is crucial. If they are not comonotonic, the agent could potentially choose her/his defining prior which is not most pessimistic over $\{e_t\}$. Conditions (4.3.1) and (4.3.4) are used to guarantee the uniqueness of this defining prior over Ω, whereas Conditions (4.3.2) and (4.3.4) ensure the uniqueness of this prior over time.

4.3.3 Sufficient Conditions for the Order Property of Utility Process

Condition (4.3.3) is very intuitive. However, since it is defined by the expected value of the utility process, we cannot see the direct connection to the endowment process. In this section, we want to derive the conditions for the endowment process $\{e_t\}$ to guarantee (4.3.3), which are the sufficient conditions that do not involve the restrictions on the utility functions. We can easily guess the situation for two-periods case. In the above example, apparently if $e_{1,1,s} = e_{1,2,s}\ \forall s = 1$ and 2, then $\int u(e_{1,1,\bullet})dP(\omega_1, \omega_1, \bullet) = \int u(e_{1,2,\bullet})dP(\omega_1, \omega_2, \bullet)$. Clearly, the endowment at t=2 is the only variable that the agent must consider to assess her/his utility process. The next guess is what conditions make $\int u(e_{1,1,\bullet})dP(\omega_1, \omega_1, \bullet) \geq \int u(e_{1,2,\bullet})dP(\omega_1, \omega_2, \bullet)$. First, this inequality is satisfied when $e_{1,1,1} > e_{1,2,1}$ and $e_{1,1,2} > e_{1,2,2}$. Second, if the endowment distribution of t=3 at (ω_1, ω_2) is the *mean-preserving-spread* of the endowment distribution of t=3 at (ω_1, ω_1), then this $\{e_t\}$ satisfies the above inequality with respect to the *identical* prior for both side of inequality.

These intuitions carry over to the case of more than two periods. In a general time horizon, however, we must think that all subtrees satisfy the above intuition and the aggregation of subtrees by backward induction still maintains the similar structure over time. Now the comparison of endowments becomes multi-dimensional because each subtree can be nested into another subtree. The next example captures this multiple connection of endowment distribution:

Suppose we add one more dates to the above example. We defined the evolution as follows:

$$\text{At t=4,} \quad \begin{bmatrix} (\omega_1, \omega_{2,1}, \omega_{3,1}, \omega_{4,1}) \\ \\ (\omega_1, \omega_{2,1}, \omega_{3,1}, \omega_{4,2}) \end{bmatrix} = \begin{bmatrix} e_{1,1,1,1} \\ \\ e_{1,1,1,2} \end{bmatrix}$$

$$\begin{bmatrix} (\omega_1, \omega_{2,1}, \omega_{3,2}, \omega_{4,1}) \\ \\ (\omega_1, \omega_{2,1}, \omega_{3,2}, \omega_{4,2}) \end{bmatrix} = \begin{bmatrix} e_{1,1,2,1} \\ \\ e_{1,1,2,2} \end{bmatrix}$$
$$\begin{bmatrix} (\omega_1, \omega_{2,2}, \omega_{3,1}, \omega_{4,1}) \\ \\ (\omega_1, \omega_{2,2}, \omega_{3,1}, \omega_{4,2}) \end{bmatrix} = \begin{bmatrix} e_{1,2,1,1} \\ \\ e_{1,2,1,2} \end{bmatrix}$$
$$\begin{bmatrix} (\omega_1, \omega_{2,2}, \omega_{3,2}, \omega_{4,1}) \\ \\ (\omega_1, \omega_{2,2}, \omega_{3,2}, \omega_{4,2}) \end{bmatrix} = \begin{bmatrix} e_{1,2,2,1} \\ \\ e_{1,2,2,2} \end{bmatrix}$$

The utility process $\{\mathrm{V}_t(e)\}$ is defined:

At t=1, $[\omega_1] = \left[u(e_1) + \int V_{1,\bullet} dP(\bullet)\right]$

At t=2, $\begin{bmatrix} (\omega_1, \omega_{2,1}) \\ \\ (\omega_1, \omega_{2,2}) \end{bmatrix} = \begin{bmatrix} u(e_{1,1}) + \int V_{1,1,\cdot} dP(\omega_1, \omega_{2,1}, \bullet) \\ \\ u(e_{1,2}) + \int V_{1,2,\cdot} dP(\omega_1, \omega_{2,2}, \bullet) \end{bmatrix} = \begin{bmatrix} V_{1,1} \\ \\ V_{1,2} \end{bmatrix}$

At t=3, $\begin{bmatrix} (\omega_1, \omega_{2,1}, \omega_{3,1}) \\ \\ (\omega_1, \omega_{2,1}, \omega_{3,2}) \end{bmatrix} = \begin{bmatrix} u(e_{1,1,1}) + \int u(e_{1,1,1,\bullet}) dP(\omega_1, \omega_{2,1}, \omega_{3,1}\bullet) \\ \\ u(e_{1,1,2}) + \int u(e_{1,1,2,\bullet}) dP(\omega_1, \omega_{2,1}, \omega_{3,2}\bullet) \end{bmatrix} = \begin{bmatrix} V_{1,1,1} \\ \\ V_{1,1,2} \end{bmatrix}$

$$\begin{bmatrix} (\omega_1, \omega_{2,2}, \omega_{3,1}) \\ \\ (\omega_1, \omega_{2,2}, \omega_{3,2}) \end{bmatrix} = \begin{bmatrix} u(e_{1,2,1}) + \int u(e_{1,2,1,\bullet}) dP(\omega_1, \omega_{2,2}, \omega_{3,1}\bullet) \\ \\ u(e_{1,2,2}) + \int u(e_{1,2,2,\bullet}) dP(\omega_1, \omega_{2,2}, \omega_{3,2}\bullet) \end{bmatrix} = \begin{bmatrix} V_{1,2,1} \\ \\ V_{1,2,2} \end{bmatrix}$$

At t=4, $\begin{bmatrix} (\omega_1, \omega_{2,1}, \omega_{3,1}, \omega_{4,1}) \\ \\ (\omega_1, \omega_{2,1}, \omega_{3,1}, \omega_{4,2}) \end{bmatrix} = \begin{bmatrix} u(e_{1,1,1,1}) \\ \\ u(e_{1,1,1,2}) \end{bmatrix}$

$$\begin{bmatrix} (\omega_1, \omega_{2,1}, \omega_{3,2}, \omega_{4,1}) \\ \\ (\omega_1, \omega_{2,1}, \omega_{3,2}, \omega_{4,2}) \end{bmatrix} = \begin{bmatrix} u(e_{1,1,2,1}) \\ \\ u(e_{1,1,2,2}) \end{bmatrix}$$
$$\begin{bmatrix} (\omega_1, \omega_{2,2}, \omega_{3,1}, \omega_{4,1}) \\ \\ (\omega_1, \omega_{2,2}, \omega_{3,1}, \omega_{4,2}) \end{bmatrix} = \begin{bmatrix} u(e_{1,2,1,1}) \\ \\ u(e_{1,2,1,2}) \end{bmatrix}$$

$$\begin{bmatrix} (\omega_1, \omega_{2,2}, \omega_{3,2}, \omega_{4,1}) \\ \\ (\omega_1, \omega_{2,2}, \omega_{3,2}, \omega_{4,2}) \end{bmatrix} = \begin{bmatrix} u(e_{1,2,2,1}) \\ \\ u(e_{1,2,2,2}) \end{bmatrix}$$

Now we can easily see that the relationship between $e_{1,1,1,1}$ and $e_{1,2,1,1}$ and between $e_{1,1,1,2}$ and $e_{1,2,1,2}$ must be defined in order to have $V_{1,1,1} \geq V_{1,2,1}$. A similar consideration is required for the mean-preserving-spread case.

To summarize, we propose three sufficient conditions:

Proposition 4.3.2:

Under time-state-homogenous multiple-priors set (4.3.4), the endowment process $\{\mathrm{e}_t\}$ that follows (4.3.1) and (4.3.2) guarantees (4.3.3) if it satisfies any one of the following conditions:

(4.3.5) Markov structure:

$\mathrm{e}_t(\omega^t) = \mathrm{e}_t(\omega_t) \ \forall T \geq t \geqq 1$

(4.3.6) State monotonic:

$\mathrm{e}_t(\omega^{t-1}, \omega_t) \geq \mathrm{e}_t(\omega'^{t-1}, \omega_t)$

if $\mathrm{e}_\tau(\omega^\tau) > \mathrm{e}_\tau(\omega'^\tau)$ for some τ: T$\geq$ t$>\tau>$1

where ω^{t-1} and ω'^{t-1} are identical except at τ

(4.3.7) Mean-preserving-spread:

$\int_0^x G(e_t(\omega'^{t-1}, \omega_t)) de_t(\omega'^{t-1}, \omega_t) \geq \int_0^x F(e_t(\omega^{t-1}, \omega_t)) de_t(\omega^{t-1}, \omega_t)$

if $e_\tau(\omega^\tau) > e_\tau(\omega'^\tau)$ for some τ: T$\geq$ t$>\tau>$1

where ω^{t-1} and ω'^{t-1} are identical except at τ

F is the distribution function of $\{e_t(\omega^{t-1}, \omega_t)\}$ at ω^{t-1}, and

G is the distribution function of $\{e_t(\omega'^{t-1}, \omega_t)\}$ at ω'^{t-1}

The distribution functions are based on the identical most pessimistic prior

Proof: Appendix 4.G:

Condition (4.3.5) says that the coherent endowment process generates constant evolution of the utility process over ω^t, which does not alter the comonotonicity of $\{e_t(\omega^{t-1}, \omega_t)\}$ and $\{V_t({}^t e; \omega^{t-1}, \omega_t)\}$ over Ω. Condition (4.3.6) is more general. It implies that some positive tilt toward favorite direction would not harm the comonotonicity of the endowment and utility process. Condition (4.3.7) is essentially equivalent to the notion of second order stochastic dominance. Since we assume the concave utility, some endowment processes which satisfy (4.3.7) produce exactly the same result as that of (4.3.6), not through the direct dominance over non-stochastic endowment numbers, but through the integration with the utility functions.

The above conditions are sufficiently general. For example, the binomial approximation of Brownian motion satisfies (4.3.6), whereas it is easy to produce the martingale process $\{e_t\}$ which satisfies some of the conditions. Note that both of them are defined on the most pessimistic prior over $e_t(\omega^t)$.

4.3.4 Time and State Heterogenous Prior Set

Now, consider another generalization over the structure of uncertainty. So far, we have assumed the independent capacity-based multiple-priors case. The essence of Proposition 4.3.1

and Proposition 4.2.2 carries over if we make $\mathfrak{P}$ a time-heterogeneous-i.i.d. process because all defining conditions are based on the order property of $\{V_t({}^t e)\}$ within time. Moreover, if we make $\mathfrak{P}$ narrower for the state where the current endowment is higher, clearly (4.3.3) is preserved under (4.3.1), (4.3.2) and (4.3.6). In other words, $\mathfrak{P}$ can follow Markov structure at any t so long as appropriate nesting is taken because $\int \mathrm{V}_{t+1}({}^t e;\omega^t,\omega)dP(\omega^t;\omega) \geq \int \mathrm{V}^h_{t+1}({}^t e;\omega^t,\omega)d\widetilde{P}(\omega^t;\omega)$ if $\widetilde{\mathfrak{P}}(\omega^t;\omega) \supseteq \mathfrak{P}(\omega^t;\omega)$. In addition, the order of the utility process within time over $\omega^t \in \Omega^t$ that is implied by Proposition 4.3.2 stay same because (4.3.5) and (4.3.6) guarantee the history wise dominance[12]. Finally, we need to adjust the definition of the mean-preserving spread process because of heterogeneity of $\mathfrak{P}_t(\omega_t)$ over Ω. We summarize above intuitions in Corollary 4.3.1 without the proof. (The proof is essentially the repetition of Proposition 4.3.1 and Proposition 4.3.2.)

Corollary 4.3.1:

In Proposition 4.3.1, if we replace Condition (4.3.4) with (4.3.8) and (4.3.9), and replace Condition (4.3.3) with and one of (4.3.5), (4.3.6), (4.3.10) with (4.3.11), the agent behaves as if she/he had the most pessimistic prior over $\{\mathrm{e}_t\}$ $\forall T > t \geq 1$. In other words, the utility process $\{\mathrm{V}_t({}^t\mathrm{e};\omega^{t-1},\omega_t)\}$ becomes comonotonic with the endowment process $\{\mathrm{e}_t(\omega^{t-1},\omega_t)\}$ over Ω $\forall\omega^{t-1}$ T$\geq$t>1. Note that the direction of pessimism is constant over time.

(4.3.8) At each t, $P_t(\omega^t) \subseteq P_t(\omega'^t)$ if $e_t(\omega^{t-1},\omega_t) > e_t(\omega^{t-1},\omega'_t)$

where P_t is either capacity-based or a general multiple-priors set

(4.3.9) $P_t(\omega^{t-1},\omega_t) = P_t(\omega'^{t-1},\omega_t)$ (time-heterogenous Markov)

[12] For more detail, see Appendix 4.G.

(4.3.10) Mean-preserving-spread:

$\int_0^x G(e_t(\omega'^{t-1},\omega_t))de_t(\omega'^{t-1},\omega_t) \geq \int_0^x F(e_t(\omega^{t-1},\omega_t))de_t(\omega^{t-1},\omega_t)$

if $e_\tau(\omega^\tau) > e_\tau(\omega'^\tau)$ for some τ: T$\geq$ t>τ>1

F is the distribution function of $\{e_t(\omega^{t-1},\omega_t)\}$ at ω^{t-1}, and

G is the distribution function of $\{e_t(\omega'^{t-1},\omega_t)\}$ at ω'^{t-1}

The distribution functions are based on the most pessimistic prior

conditional on ω_{t-1}

(4.3.11) ε-open neighborhood around (4.3.5), (4.3.6) or (4.3.10) with the norm on $D(\Omega^T)$

Condition (4.3.8) defines the appropriate nesting of multiple-priors sets. We can interpret this condition as if the agent became less uncertain about the future if the *good* state were realized so that the expected value increased. Condition (4.3.9) preserves the order relationship between $\{V_t({}^te;\omega^{t-1},\omega_t)\}$ and $\{V_t({}^te';\omega'^{t-1},\omega_t)\}$ over Ω, which is essential for the dynamic ordering of $\{V_t({}^te)\}$ process. The mean-preserving-spread is now redefined by the conditional distribution condition (4.3.10) instead of the unique prior. (because of (4.3.8) and (4.3.9), conditioning is taken by the time and the current state, not by the whole history.) Now F and G are adjusted accordingly to incorporate the underlying probability change. Condition (4.3.11) is just ε-perturbation of the defining endowment process. If ε is small enough, the distortion of the static order of $E_t[V_{t+1}({}^{t+1}e;\omega^t,\omega_{t+1})|\omega_t]$ stays within the range of the gap among $\{e_t(\omega^{t-1},\omega_t)\}$. Then the $\{V_t({}^te;\omega^{t-1},\omega_t)\}$ becomes comonotonic with $\{e_t(\omega^{t-1},\omega_t)\}$ over Ω $\forall\omega^{t-1}$ T$\geq t$>1.

Finally, we want to state Corollary 4.3.2 without proof. This Corollary does not directly related to the objective of this paper. However, in the later sections, it becomes useful. It states the conditions where the single agent utility process becomes comonotonic with the aggregate endowment process so that the agent selects the most pessimistic prior with respect to the aggregate endowment at each ω^t. The difference of Corollary 4.3.2 and Corollary 4.3.1 is Condition (4.3.2). Here, we do not assume that the order of the aggregate endowment process over Ω is identical over time, which implies that the direction of pessimism over Ω can change over time. Given this change, in order to ensure that the agent chooses the most pessimistic prior over Ω, we need to make $\{V_t({}^te)\}$ constant over all the history of ω^t. This implies that the single agent only focus on the order of $\{e_t(\omega^{t-1}, \omega_t)\}$ in order to decide the prior used to evaluate the $\{V_t(\omega^t)\}$ process. In other words, the continuation value of the future endowment does not alter the order of utility process at ω^t, and the multiple-periods decision making becomes the repetition of a single period's one. Apparently under this condition, the single agent always chooses the most pessimistic prior only with respect to the aggregate endowment process.

Corollary 4.3.2:

In a single agent economy, under (4.3.12) and (4.3.13) with (4.3.14), the agent behaves as if she/he had the most pessimistic prior over $\{\mathrm{e}_t\}$ $\forall T > t \geq 1$. In other words, the utility process $\{\mathrm{V}_t({}^t\mathrm{e};\omega^{t-1}, \omega_t)\}$ becomes comonotonic with the endowment process $\{\mathrm{e}_t(\omega^{t-1}, \omega_t)\}$ over Ω $\forall\omega^{t-1}$ T$\geq$t$>$1. Note that the direction of pessimism is not necessarily constant over time.

(4.3.12) $e_t(\omega^{t-1}, \omega) \neq e_t(\omega^{t-1}, \omega')\ \omega, \omega' \in \Omega$

(strong order of the endowment)

(4.3.13) Time-heterogenous Markov structure:

$e_t(\omega^t) = e_t(\omega_t) \forall T \geq t \geqq 1$

(4.3.14) The agent has either a time-heterogeneous capacity-based multiple-priors set

or a time-heterogeneous general multiple-priors set

over Ω within time $\forall T > t \geq 1$ with:

$P_{\mathrm{t}}(\omega^{t-1}, \omega_t) = P_t(\omega'^{t-1}, \omega'_t)$ (i.i.d. prior set within time)

(4.3.15) ε-open neighborhood around (4.3.12) and (4.3.13) with the norm on $D(\Omega^T)$

Conditions (4.3.12) is for the uniqueness of the prior selection under the capacity-based multiple-priors set. Condition (4.3.13) without (4.3.2) implies that the order of the endowment over Ω can change over time. Finally, Condition (4.3.14) makes the utility process constant. Again, Condition (4.3.15) is just ε-perturbation of the defining endowment process.

4.4 Multiple Agents Economy with the Identical Capacity-Based Multiple-Priors Sets

4.4.1 Background

Given the results for the single agent economy, we now investigate the conditions where multiple agents behave similarly over time, i.e., they behave as if they had an identical prior over time, which is the most pessimistic prior over the aggregate endowment process. In other words, we would like to see the consistent behavior among all agents.

This analysis is closely related to constructing a representative agent. In fact, Epstein-Wang (1994) construct the *dynamic representative agent* under the multiple-agents economy with uncertainty aversion, where the dynamic representative agent summarizes the multiple-agents economy under which all agents behave as if they had the identical prior over time. In this paper, we have a different motivation, i.e., deriving the conditions for agents to have homogeneous behavior. However, if all agents share the same uncertainty, these two motivations become almost identical. In fact, if we can construct the dynamic representative agent, all agents must have identical prior, although it may not share the same pessimism over time. Here, instead of deriving all possible conditions for the existence of the dynamic representative agent, we focus on our main goal of agents' consistent behavior, which is the subset of the dynamic representative agent case. In section 4.4.5. we argue that in fact, our restriction is very natural and constructive, and the restriction to the coherent aggregate endowment process captures most of the intuitions and ideas for the dynamic representative agent economy.

Naturally, we can expect that the conditions under which the single-agent behaves similarly over time are applied to the multiple-agents case, but these conditions would be narrower because each agent now has an arbitrary endowment process. Surprisingly, under the dynamically complete markets, we can allow the heterogeneous utility function and endowment process for each individual in order for the dynamic representative agent to exist. The equilibrium is of full risk sharing where all agent have comonotonic consumption with respect to the aggregate endowment process. However, not all of conditions for the aggregate endowment of the single agent economy deliver the above results. In order to make all agents move similarly, the aggregate endowment must evolve coherently. We will see the result in section 4.4.3.

4.4.2 Definition of the Representative Agent

Before proceeding to the main proposition of this section, first we want to define the notion of the dynamic representative agent. In complete markets, the standard Pareto optimality results imply that there are weights α^h such that the weighted sum of individual utility functions becomes the social welfare function for the representative agent, and the solution of this linear function corresponds to a competitive equilibrium allocation with some endowment process. More formally, in the Arrow-Debreu complete markets economy, there is a social welfare function:

$$\text{(4.4.1)} \qquad \mathrm{V}_1(e) = \mathrm{Max}_{(c^1,\ldots,c^H)} \sum \alpha^h \mathrm{V}_1^h(c^h)$$
$$\text{s.t.} \quad \sum c^h = e$$

The single-agent economy with this utility function and aggregate endowment process produces the identical allocation for a multiple-agents economy with some individual endowment process.[13] We call this agent as a *static representative agent*. The name *static*

[13] For general concave utility functions, there is a case where the range of α is small, i.e., not all α is

is used because in general, the utility function (4.4.1) does not evolve in a dynamically consistent way. In other words, (4.4.1) is defined at the beginning of the economy and all subsequent allocations are predetermined before any uncertainty is resolve in the later dates. This feature is well-know "Walras auctioneer". More precisely, the auctioneer has (4.4.1) as the objective function and he decides everything at the beginning. There is no sense of dynamics here. Mathematically, this intuition means that allocations of (4.4.1) for t=2 do not coincide with the solution of $\mathrm{V}_2(e)$=Max$\sum \alpha \mathrm{V}_2^h(c^h)$. The utility weight α only makes sense at the beginning, not in the later dates.

On the other hand, if we can find the recursive function V such that:

$$
\begin{aligned}
(4.4.2) \qquad \mathrm{V}_1(e) &= \mathrm{Max}_{(c^1,\ldots,c^H)}\{\textstyle\sum \alpha u_1^h(c^h) + \sum \alpha \int V_2^h(c^h) dP^h(\omega_1)\} \\
&= \mathrm{Max}_{(c^1,\ldots,c^H)}\{\textstyle\sum \alpha u_1^h(c^h) + \int \sum \alpha V_2^h(c^h) dP^h(\omega_1)\} \\
&= \mathrm{Max}_{(c^1,\ldots,c^H)}\{u_1(e_1) + \textstyle\int V_2(e) dP(\omega_1)\} \\
\text{s.t. } & \textstyle\sum c^h = e \\
\text{where } & u_1(e) = \textstyle\sum \alpha u_1^h(c^h),\ V_2(e) = \sum \alpha V_2^h(c^h)
\end{aligned}
$$

then it is clear that V satisfies dynamic consistency. For this reason, we call (4.4.2) the *dynamic representative agent*. For this dynamic representative agent, the utility weight α solves the optimization at any point of history, i.e., the solution of $\{\mathrm{c}^h\}_2^T$ from V_1 is equivalent to the solution of $\{\mathrm{c}^h\}_2^T$ from V_2. For the common subjective prior model where $\mathfrak{P}$ is a singleton, Constantinides (1982) shows that we can construct the dynamic representative

feasible. Suppose T=2, H=2, $\mathrm{u}^1=\mathrm{k}^1\mathrm{x}$, $\mathrm{u}^2=\mathrm{k}^2\mathrm{x}$. Then only $(\alpha^1,\alpha^2) = (1/\mathrm{k}^1,1/\mathrm{k}^2)$ solves (4.4.1) meaningfully. Otherwise, one agent must consume everything.

agent by maximizing $\sum \alpha u_t^h(c_t^h(\omega_t))$ for each ω^t.[14] Epstein-Wang (1994) use his argument to prove the existence of the dynamic representative agent for the identical capacity-based multiple-priors case.[15]

Clearly, (4.4.2) requires the integration of V at each period. In other words, we must have a single common prior for all V^h. It is precisely why we start the analysis of homogeneous capacity-based multiple-priors model under which there is possibility that all agents behave as if they had the identical prior over time. The capacity-based assumption is critical because the common multiple-priors set does not guarantee that agents select the identical prior among comonotonic consumptions.

4.4.3 Single Period Economy

It is very informative to investigate the equilibrium properties for a single period economy before we move to the dynamic setting. It captures most of fundamental issues in equilibrium, and later we consider the dynamic connection of single period economies.

Now assume that Condition (4.3.1) ($e(\omega) \neq e(\omega')$ $\omega, \omega' \in \Omega$: strong order of endowment) holds, $\mathfrak{P}$ is capacity-based, and there is no consumption at t=1.

First, we want to show that the equilibrium consumptions are comonotonic among all

[14] If some agents have different subjective priors, it is not optimal to use the solution of max $\sum \alpha^h u_t^h(c_t^h(\omega^t))$ because the weights must be adjusted to take the difference of priors into account. The proof of the non-existence of the dynamic representative agent with heterogeneous single priors is found in Appendix 4.H.

[15] In Section 4.4.5, we mention some sufficient conditions where the dynamic representative agent exist.

agents. By the argument of Section 4.4.2, there is a static representative agent for this economy where (4.4.1) holds with the utility weights α^h. By Duffie (1996), the Pareto optimal allocation must solve (4.4.1). In order words, any Arrow-Debreu equilibrium must solve (4.4.1). Following Constantinides (1982), define the optimization:

$$(4.4.3) \qquad u(e) = \text{Max } \{\sum \alpha^h u^h(x^h) : \sum x^h = e \}$$

Let the optimal allocation vector be $c^h(e)= \{c^h(e(\omega))$ for the solution of (4.4.3) at each $\omega \in \Omega$ }. Now we argue that $c^h(e)$ is increasing in e. From F.O.C. of (4.4.3), $\nabla u(e)=(\alpha^1 u'^1(c^1)$,..., $\alpha^H u'^H(c^H))$ //1. If $e(\omega_i)>e(\omega_j)$ then $\exists\, h$ s.t. $c^h(e(\omega_i)) > c^h(e(\omega_j))$, so $u'^h(c^h(e(\omega_i))) < u'^h(c^h(e(\omega_j)))$, which implies $u'^{h'}(c^{h'}(e(\omega_i))) < u'^{h'}(c^{h'}(e(\omega_j)))$ $\forall h'$ by the strictly concave utility functions[16].

Now we want to prove that this allocations maximize (4.4.1). For any other feasible allocations x, by (4.2.14):

$$\begin{aligned}(4.4.4) \qquad \sum \alpha^h \int u^h(x^h) dP^h(\omega) &\leq \int \sum \alpha^h u^h(x^h) d\overline{P}(\omega) \\ &\leq \int \sum \alpha^h u^h(x^h) dP(\omega) \\ &\leq \int \sum \alpha^h u^h(c^h(e(\omega)) dP(\omega) \\ &= \int u(e) dP(\omega)\end{aligned}$$

with strictly inequality for non-comonotonic allocations, where $\overline{P}(\omega)$ is the optimal prior that minimizes $\int \sum \alpha^h u^h(x^h) d\overline{P}(\omega)$, and $P(\omega)$ is the most pessimistic prior with respect to the aggregate endowment. Since $c^h(e)$ achieves the highest value among the comonotonic

[16] In Appenxid 4.I, we show that under general concave utility functions, $x^h(e)$ becomes non-decreasing.

consumptions, it is the optimal solution for given α. Hence, all agents' consumption must be comonotonic with each other.[17] In other words, the aggregate endowment order is the sufficient statistic to summarize the behavior of individual consumptions.

We also offer a direct proof. First we assume that all agents have the identical most pessimistic prior over the aggregate endowment. Then we can prove that resulting allocations are all comonotonic to the aggregate endowment and confirm the selection of the most pessimist prior. Next, assume that there is another equilibrium where consumptions are not comonotonic with each other. Let α be for this allocation. Then, by (4.2.14), we know that this α does not support non-comonotonic allocations. (Note that we do not need (4.4.3). Under any common single prior, we know that the full risk-sharing is the only solution, which dominated the non-comonotonic allocations.) This result implies that non-comonotonic allocations are not Pareto optimal, which contradicts the assumption. Therefore, only equilibria that solve (4.4.1) are comonotonic ones.

To confirm the argument above, we show now that agents behave as if each were a single prior optimizer with the identical most pessimistic prior. Suppose that all agents have the identical most pessimistic prior. Standard F.O.C.s imply that all agents must have comonotonic order of consumptions because there is always someone who must consume more at the state where aggregate endowment is higher. More precisely, if $e(\omega_i)>e(\omega_j)$ then $\exists$ h s.t. $c^h(e(\omega_i)) > c^h(e(\omega_j))$ By F.O.C.:

[17] If all u^h is globally strictly concave, for given α^h, there is a single equilibrium allocation. Otherwise, they may be multiple equilibrium allocations.

$$\frac{p(\omega_i)u'^h(c^h(\omega_i))}{p(\omega_j)u'^h(c^h(\omega_j))} = \frac{u'^{h'}(c^{h'}(\omega_i))}{u'^{h'}(c^{h'}(\omega_j))} = \frac{u'^{h'}(c^{h'}(\omega_i))}{u'^{h'}(c^{h'}(\omega_j))} = \frac{p(\omega_i)u'^{h'}(c^{h'}(\omega_i))}{p(\omega_j)u'^{h'}(c^{h'}(\omega_j))}$$

so by the uniqueness of state prices under complete asset markets, all other agents must have $c^{h'}(e(\omega_i)) > c^{h'}(e(\omega_j))$. This fact is already implied by (4.4.3). The maximization at each state without probability weights only makes sense if its solutions are globally optimal with respect to the identical prior. In terms of the efficiency of allocations, given the above observations, we know that equilibrium allocations are *full risk sharing*, i.e., the consumption order is strongly comonotonic[18]. Clearly, the economy is observationally equivalent to the one with a common subjective prior, where this single prior is the most pessimistic one with respect to the aggregate endowment. In order words, we effectively reduce the multiple agents economy with identical capacity-based multiple-priors sets to the multiple agents economy with the common subjective prior. However, there is a clear distinction between them. For the case of the common subjective prior model, we "assume" the common prior, and cannot define the pessimism unless there is an objective probability law, whereas for the case of the identical capacity-based multiple-priors model, we "derive" the common prior from the aggregation of agents, and the optimal prior is the most pessimistic one among the agents' priors. Hence the pessimism is clearly defined without any reference to the objective probability law. In other words, the pessimism is *internal* concept among agents' beliefs, and at equilibrium, all agents share the common pessimism, i.e., the aggregation of uncertainty averse agents forces them to have homogeneous bias.

Now we formally state the above result as Lemma 4.4.1.

[18] The analogy holds for general concave utility functions by weak order.

Lemma 4.4.1:

In a multiple-agents economy, under (4.4.5) with (4.4.6), all agents behave as if they had the same prior, which is the most pessimistic prior over {e(ω)}, regardless of their initial endowment. Moreover, the consumption is "interior" or strongly comonotonic with the aggregate endowment, which means that there are no ties among the agents's consumptions, and the equilibrium allocations are globally optimal with respect to the most pessimistic prior.

(4.4.5) $e(\omega) \neq e(\omega')\ \omega, \omega' \in \Omega$

(strong order of the aggregate endowment)

(4.4.6) All agents have identical capacity-based multiple-priors sets over Ω:

$P^h = P^{h'}$

By the property of the optimal value function, $u(e)$ is continuous, increasing, strictly concave in e[19]. With the maximal value with respect to α at each state, given the fixed prior, clearly (4.4.1) achieves the optimal value with u replacing $\sum \alpha^h u^h$. Now the economy has the artificial single agent at t=2, which is the condition for the existence of the dynamic representative agent. By now, it is clear that in the dynamic setting, we can anticipate the presence of the dynamic representative agent because the equilibrium allocations are comonotonic everywhere and the identical priors are chosen by all individuals over time. The presence of u implies that the artificial single agent represents the economy by (4.4.2).

Finally, we want to investigate the analogy between risk aversion and uncertainty aver-

[19] Strict concavity follows because all agents have a strictly concave utility.

sion. Note first that a risk-averse agents' indifference curve is convex. Now, assume that agents become uncertainty averse over their original subjective prior, i.e., $\overline{p}^h \in \text{int}(\mathfrak{P}^h)$ where $\overline{p}^h$ is the original subjective prior. Define the indifference curve $u^h(c^h)$=$u^h(\overline{c}^h)$ where $\overline{c}^h$=($\overline{c}$,...,$\overline{c}$). By (4.2.5), at any point of the original indifference curve except $\overline{c}^h$, the new indifference curve must lie strictly on the interior of the upper contour set defined by the original indifference curve, i.e.:

$$\int u^h(\overline{c}^h)d\overline{p}^h = \min \int u^h(\overline{c}^h)dP$$

$$\int u^h(c^h)d\overline{p}^h > \min \int u^h(c^h)dP$$

$$\int u^h(c^h)d\overline{p}^h = \min \int u^h(kc^h)dP \text{ where } k > 1$$

So under uncertainty aversion, the original consumption c^h that gives the same utility before does not achieve the level we need. Hence we need more consumptions in order for the level of utility to stay constant, i.e., we need more consumptions on the array of c^h, i.e. kc^h where k>1.

This feature is the essence of uncertainty aversion.[20] Agents behave as if they progressively became more risk-averse. The term *progressive* is used because of the following reason. Suppose that an agent has prior $\overline{p}^h$ at $\overline{c}^h$. By ε-trades that gives the same utility as $u^h(\overline{c}^h)$, this agents chooses a different prior. Next, from this new allocation, the agent

[20]It is well known that the expected utility maximizer will take ε risky position over $\overline{c}$ as long as it is actuarily favorable. In other words, they are locally risk neutral. However, agents with uncertainty aversion do not necessarily take this position because the indifference curve moves inward around $\overline{c}$, i.e., they become locally risk averse in some range of state prices.

trades another ε to the more distant direction from $\overline{c}^h$. Now the prior which is optimal for the first trade is no longer optimal, and the more pessimistic prior must be used. In other words, each time agents move away from the even allocations, the prior moves in the direction that makes the indifference curve more inward bending. For the case of capacity-based multiple-priors set, this progressive change only happens when the consumption order is changed. For the more general multiple-priors case, this progressive change can happen virtually for all movement.

This similarity between risk aversion and uncertainty aversion is most evident for the case of risk-neutral agents. Suppose all agents are risk-neutral with a common single prior, and agents have non-comonotonic initial endowment. By F.O.C. of the individual optimization, all state prices must be equal to their state probability.[21] Given these state prices, the optimal value of the agent's utility is fixed (u^h=a^h+$\mathrm{b}^h\mathrm{W}^h$: W^h=$q\cdot e$=$p\cdot e$). It is obvious that initial endowment is one of the equilibria. In fact, there is a continuum of equilibria which is not comonotonic. Here agents can trade the Arrow-Debreu asset at the price that is the state probability. As long as markets clear, any points on the budget line are optimal. However, things will change drastically once we introduce uncertainty over risk-neutral agents. Since it is possible to have a comonotonic order of consumptions for all agents, all other non-comonotonic allocations are dominated by (4.4.4). Clearly, by introduction of uncertainty, suddenly, every agent must have comonotonic consumption. This result resembles the case for the strictly concave utility functions. Under identical capacity-based

[21] More precisely, the state price vector is parallel to the state probability vector.

multiple-priors sets, agents are still risk neutral within the consumptions of the same order. However, they behave as if they became *risk averse* for the different order of consumptions. Now summarize the above findings:

Lemma 4.4.2:

Under the presence of uncertainty, risk averse or risk neutral agents behave as if they became progressively risk averse as they move their allocations away from the even allocation.

4.4.4 Dynamic Setting

Now, we are ready for the extension of the results of Section 4.3. In Section 4.4.2, we show that if all agents share the identical prior at equilibrium, the dynamic representative agent exists, and the dynamic representative agent must behave consistently over time if all agents have the identical pessimistic prior over time. Clearly, an agent in a single agent economy must behave similarly over time if the dynamic representative agent needs to behave consistently. Therefore, in order to investigate the conditions for all agents to behave homogeneously, we can restrict our attention to the conditions for the single agent economy, and examine which conditions are valid for the multiple agents case.

The difficulty is how to aggregate individuals and derive their behavior under the conditions of the single-agent economy. From Section 4.4.3, we know that for any equilibrium of the multiple-agents economy, there is a static representative agent. Conversely, for any static representative agent equilibrium must corresponds to the equilibrium of the multiple-agents economy with some individual endowment processes. Therefore, by examining the static representative agent economy, we effectively investigate the multiple agents economy.[22] In other words, as long as the allocations c^h solve (4.4.1), they must solve individual optimization (4.2.9) with some endowment processes. Clearly, when all agents behave homogeneously, there is a dynamic representative agent, which is a subset of the static representative agent economy. Now the central question becomes: Under which conditions of the single agent economy does the dynamic representative agent exit?[23] The answer for this question is given in Proposition 4.4.1 and Corollary 4.4.1.

In the dynamic setting, we have to consider two-dimensional heterogeneity. One is within time, the other is across time. In order to have identical prior selection, the aggregate endowment must have similar structures within and across time. For the single agent case, these *similarities* are summarized in Proposition 4.3.1 and Proposition 4.3.2. Here, we only focus on the sufficient conditions in Proposition 4.3.2 and combine both propositions to

[22] For some equilibria under the static representative agent economy, $e^h(\omega) = 0$ for some ω. Since we assume $e^h > 0$, the equilibrium set of the static representative agent economy would be bigger than that of multiple agents one.

[23] The dynamic representative agent economy is still the multiple agents economy. Although the allocation property is identical to that of the single agent case, the equilibrium price evolution would be different. (Pareto optimality is nothing to do with equilibrium prices.) The equilibrium prices must be agreed among agents in the dynamic representative agent economy, whereas in the single agent case, the only one person decides them.

have Proposition 4.3.1, where multiple agents with the identical capacity-based multiple-priors sets behave as if they had the identical single prior over time. The only difference between the single agent economy and the multiple agents one is that we are no longer able to have Condition (4.3.7) (mean-preserving-spread) because this condition is concerned with the single agent endowment distribution, whereas here, we have H agents and their consumption distribution does not necessarily confirm (4.3.7) even though the aggregate endowment does. Although we could develop conditions like (4.3.7), it must depend on the form of the utility functions or individual endowment processes. We consider it to be too restrictive because we want to derive the conditions only on the aggregate endowment process.

Now first state the main result for multiple agents economy with the identical capacity-based multiple-priors sets:

Proposition 4.4.1: (Extension of Epstein-Wang:1994)

In a multiple-agents economy, under (4.4.7), (4.4.8), (4.4.11) with any one of (4.4.9) or (4.4.10), all agents behave as if they had the same prior $\forall T > t \geq 1$, which is the identical most pessimistic prior over $\{\mathrm{e}_t(\omega^t)\}$, regardless of their initial endowment. In other words, the utility process $\{\mathrm{V}_t^h(\mathrm{c}^h;\omega^{t-1},\omega_t)\}$ becomes comonotonic with the aggregate endowment process $\{\mathrm{e}_t(\omega^{t-1},\omega_t)\}$ over Ω at $\forall\omega^{t-1}$ T$\geq$t$>$1. Moreover, the consumption process is "interior" or strongly comonotonic with the aggregate endowment process, which means that there are no ties among the next period's consumptions emerging from the same node. Note that the direction of pessimism is constant over time.

(4.4.7) $\mathrm{e}_t(\omega^{t-1}, \omega) \neq e_t(\omega^{t-1}, \omega')\ \omega, \omega' \in \Omega$

(strong order of the aggregate endowment)

(4.4.8) $e_t(\omega^{t-1}, \omega) > e_t(\omega^{t-1}, \omega') \Rightarrow e_{t'}(\omega^{t'-1}, \omega) > e_{t'}(\omega^{t'-1}, \omega')$

$\forall T \geq t, t' > 1, \omega, \omega' \in \Omega\ , \omega^{t-1} \in \Omega^{t-1},\ \omega^{t'-1} \in \Omega^{t'-1}$

(comonotonic order of aggregate endowments over Ω for all $\mathrm{e}_t(\omega^t)$)

(4.4.9) Markov structure (aggregate endowment):

$e_t(\omega^t) = e_t(\omega_t) \forall T \geq t \geqq 1$

(4.4.10) State monotonic (aggregate endowment):

$e_t(\omega^{t-1}, \omega_t) \geq e_t(\omega'^{t-1}, \omega_t)$

if $e_\tau(\omega^\tau) > e_\tau(\omega'^\tau)$ for some τ: T$\geq$ t>τ>1

where ω^{t-1} and ω'^{t-1} are identical except at τ

(4.4.11) All agents have identical capacity-based multiple-priors sets

over Ω $\forall \omega^t$: $P_t^h = P_{t'}^h$ (independent prior set)

$\forall h, h' \in H\ P_t^h = P_t^{h'}$ (identical prior set among agents)

Proof:

The proof is the extension of Epstein-Wang (1994). We only utilize the property of Pareto optimality of the Arrow-Debreu equilibrium, in other words, equation (4.4.1). Since (4.4.10) includes (4.4.9) as the special case, we only need to prove the case of (4.4.7), (4.4.8), (4.4.10) and (4.4.11).

First, given α, apply (4.4.3) for each ω^t to get $u_t(e_t(\omega^t))$, and call the solution for

$u_t(e_t(\omega^t))$[24]as $c_t(e_t(\omega^t)) = (c_t^h(e_t(\omega^t)), \ldots ,c_t^h(e_t(\omega^t)))$. Then from Section 4.4.3, $c_t^h(e_t(\omega^t))$ is an increasing function of $e_t(\omega^t)$. It is apparent that $\{c_t^h(\omega^{t-1}, \omega_t)\}$ is comonotonic with $\{e_t(\omega^{t-1}, \omega_t)\}$ over Ω at $\forall\omega^{t-1}$ T≥t>1, and $\{c_t^h(\omega^{t-1}, \omega_t)\}$ satisfies the same properties as those of the aggregate endowment process, especially (4.4.10). Hence, for $\forall h$, their consumptions ensure (4.3.3) of Proposition 4.3.1, and all agents behave as if they had the identical prior over time.

Now, we need to show that $c_t(e_t(\omega^t))$ Pareto dominates other allocations, especially non-comomotonic ones by using (4.4.1). For any other feasible allocations $\{x_t(\omega^t)\}$, by (4.4.4) at ω^{t-1} t>1, define $G_{t-1}^h(x_t^h(\omega^{t-1}, \bullet))$, $G_{t-1}(x_t(\omega^{t-1}, \bullet))$, $G_{t-1}^h(c_t^h(e_t(\omega^{t-1}, \bullet)))$ and $G_{t-1}(c_t(e_t(\omega^{t-1}, \bullet)))$:

$$
\begin{aligned}
G_{t-1}(x_t(\omega^{t-1}, \bullet)) &= \sum \alpha^h G_{t-1}^h(x_t^h(\omega^{t-1},\bullet)) \\
&= \sum \alpha^h \int u_t^h(x_t^h(\omega^{t-1}, \omega_t))dP^h(\omega^{t-1}, \omega_t) \\
&\leq \int \sum \alpha^h u_t^h(x_t^h(\omega^{t-1}, \omega_t))dP(\omega^{t-1}, \omega_t) && \text{by (4.4.11)} \\
&\leq \int \sum \alpha^h u_t^h(c_t^h(e_t(\omega^{t-1}, \omega_t)))dP(\omega^{t-1}, \omega_t) && \text{by (4.4.3)} \\
&= \int u_t(e_t(\omega^{t-1}, \omega_t))dP(\omega^{t-1}, \omega_t) && \text{by (4.4.3)} \\
&= \sum \alpha^h \int u_t^h(c_t^h(e_t(\omega^{t-1}, \omega_t)))dP(\omega^{t-1}, \omega_t) && \text{by the argument above} \\
&= \sum \alpha^h G_t^h(c_t^h(e_t(\omega^{t-1}, \omega_t))) \\
&= G_{t-1}(c_t(e_t(\omega^{t-1}, \bullet)))
\end{aligned}
$$

where $P^h(\omega^{t-1}, \omega_t)$ is the optimal prior selection at ω^{t-1} when agent h follows the allocations $\{x_t^h(\omega^t)\}$, and $P(\omega^{t-1}, \omega_t)$ is the most pessimistic prior over $\{e_t(\omega^{t-1}, \omega_t)\}$. Since

[24] u_t only depends on time, not on the state.

$u_t(e_t(\omega^{t-1}, \omega_t))$ is increasing, strictly concave, and continuous, at $t>2$, by (4.4.7), (4.4.8) and (4.4.10), $\{G_{t-1}\ (c_t(e_t\ (\omega^{t-2}, \omega_{t-1}, \bullet)))\}$ is comonotonic with $\{e_{t-1}(\omega^{t-2}, \omega_{t-1})\}$ over Ω at $\forall\omega^{t-2}$. From the above results, $G_{t-1}(c_t(e_t(\omega^{t-2}, \omega_{t-1}, \bullet))) \geq G_{t-1}(x_t(\omega^{t-2}, \omega_{t-1}, \bullet))$ at $\forall\omega^{t-2}$ with strict inequality for non-comonotonic consumptions. Hence,

$$
\begin{aligned}
G_{t-2}(x_t(\omega^{t-2}, \bullet)) &= \sum \alpha^h G^h_{t-2}(x^h_t(\omega^{t-2}, \bullet)) \\
&= \sum \alpha^h \int G^h_{t-1}(x^h_t(\omega^{t-2}, \omega_{t-1}, \bullet))dP^h(\omega^{t-2}, \omega_{t-1}) \\
&\leq \int \sum \alpha^h G^h_{t-1}(x^h_t(\omega^{t-2}, \omega_{t-1}, \bullet))d\widetilde{P}(\omega^{t-2}, \omega_{t-1}) \\
&= \int G_{t-1}(x_t(\omega^{t-2}, \omega_{t-1}, \bullet))d\widetilde{P}(\omega^{t-2}, \omega_{t-1}) \\
&\leq \int G_{t-1}(x_t(\omega^{t-2}, \omega_{t-1}, \bullet))dP(\omega^{t-2}, \omega_{t-1}) \\
&\leq \int G_{t-1}(c_t(e_t(\omega^{t-2}, \omega_{t-1}, \bullet)))dP(\omega^{t-2}, \omega_{t-1}) \\
&= \int \sum \alpha^h G^h_{t-1}(c^h_t(e_t(\omega^{t-2}, \omega_{t-1}, \bullet)))dP(\omega^{t-2}, \omega_{t-1}) \\
&= \sum \alpha^h \int G^h_{t-1}(c^h_t(e_t(\omega^{t-2}, \omega_{t-1}, \bullet)))dP(\omega^{t-2}, \omega_{t-1}) \\
&= \sum \alpha^h G^h_{t-2}(c^h_t(e_t(\omega^{t-2}, \bullet))) \\
&= G_{t-2}(c_t(e_t(\omega^{t-2}, \bullet)))
\end{aligned}
$$

where $P^h(\omega^{t-2}, \omega_{t-1})$ is the optimal prior selection at ω^{t-2} when agent h follows the allocations $\{x^h_t(\omega^t)\}$, $P(\omega^{t-2}, \omega_{t-1})$ is the most pessimistic prior for the aggregate endowment process $\{e_{t-1}(\omega^{t-2}, \omega_{t-1})\}$ over Ω at ω^{t-2}, and $\widetilde{P}(\omega^{t-2}, \omega_{t-1})$ is the optimal prior selection at ω^{t-2} which gives the most pessimistic value for $\{G_{t-1}(x_t(\omega^{t-2}, \omega_{t-1}, \bullet)))\}$. Repeat the argument above up to t-k=1, where k is the number of above operation, then $G_1(x_t(\omega_1, \bullet)) \leq G_1(c_t(e_t(\omega_1, \bullet)))$ with the strict inequality for non-comonotonic consumptions.[25] Now, applying the same exercise for $\forall t$ s.t. $T \geq t > 1$, and combining all inequalities, $\sum_1^T G_1(c_t(e_t(\omega_1, \bullet))) \geq \sum_1^T G_1(x_t(\omega_1, \bullet))$. Therefore, $\sum \alpha^h E^h[\sum u_t^h(c_t^h(e_t(\omega^t)))] \geq \sum \alpha^h \widetilde{E}^h[\sum u_t^h(x_t^h(\omega^t))]$ with strict inequality for non-comonotonic $\{x_t(\omega^t)\}$. Since the above inequality holds for all possible choice of α^h which solves (4.4.1), all Arrow-Debreu equilibria must have comonotonic consumptions for $\forall h$ and agents behave as if they had the identical most pessimistic prior over $\{e_t(\omega^t)\}$ $\forall t$.■

The results are very intuitive. Since the solution of (4.4.1) is comonotonic with $\{e_t(\omega^t)\}$, all individual allocations satisfy the same conditions as those of the aggregate endowment. It implies that effectively, all agents face the identical situation of Proposition 4.3.1 and Proposition 4.3.2. Apparently, under these conditions, all agents must choose the identical most pessimistic prior over time. Pareto domination over other allocations is just the repeated application of the single period results.

Next, we want to confirm the similar results to Corollary 4.3.1 without proof. For the case of the identical capacity-based multiple-priors sets, the generalization of the structure of uncertainty does not distort homogeneous behavior among agents:

Corollary 4.4.1:

In Proposition 3, if we replace Condition (4.4.11) with (4.4.12) and (4.4.13), and add Condition (4.4.14), all agents agent behave as if they had the identical time-state heterogeneous most pessimistic prior over $\{e_t\}$ $\forall T > t \geq 1$, regardless of their initial endowment. In

[25] By (4.4.10), the pointwise domination of non-stochastic consumptions implies $\{G_{t-j}(c_t(e_t(\omega^{t-j-1}, \omega_{t-j}, \cdot)))\}$ is comonotonic with $\{e_{t-j}(\omega^{t-j-1}, \omega_{t-j}, \cdot)\}$.

other words, the utility process $\{\mathrm{V}_t^h(\mathrm{c}^h;\omega^{t-1},\omega_t)\}$ becomes comonotonic with the aggregate endowment process $\{\mathrm{e}_t(\omega^{t-1},\omega_t)\}$ over Ω at $\forall\omega^{t-1}$ T≥t>1. Moreover, the consumption process is "interior" or strongly comonotonic with the aggregate endowment process, which means that there are no ties among the next period's consumptions emerging from the same node. Note that the direction of pessimism is consistent over time.

(4.4.12) All agents have identical capacity-based multiple-priors sets over Ω $\forall\omega^t$ and at each t, $P_t(\omega^t) \subseteq P_t(\omega'^t)$ if $e_t(\omega^{t-1},\omega_t) > e_t(\omega^{t-1},\omega'_t)$

(4.4.13) $P_{\mathrm{t}}(\omega^{t-1},\omega_t) = P_t(\omega'^{t-1},\omega_t)$ (time-heterogenous Markov)

(4.4.14) ε-open neighborhood around (4.4.9), (4.4.10) with the norm on $D(\Omega^T)$

This result is very natural because from (4.4.1) and (4.4.3), we know that all consumptions hold the same property as those of the aggregate endowment. This property implies that each agent effectively faces Corollary 4.3.1. Pareto dominations of other allocations are essentially identical to the above argument of Proposition 4.4.1, where the comonotonic order of the utility process is preserved under (4.4.12) and (4.4.13). Condition (4.4.14) is again just ε-perturbation of the defining endowment process. If ε is small enough, the distortion of static order of $E_t\{V_{t+1}^h(c^h;\omega^t,\omega_{t+1})\}$ stays within the range of the gap among $\{c_t^h(e_t(\omega^{t-1},\omega_t))\}$. Then the $\{V_t^h(c^h;\omega^{t-1},\omega_t)\}$ becomes comonotonic with $\{c_t^h(e_t(\omega^{t-1},\omega_t))\}$ over Ω at $\forall\omega^{t-1}$ T≥t>1.

Finally in this subsection, we formally state the background properties of equilibrium:

Corollary 4.4.2:

In the economy of Proposition 4.4.1 and Corollary 4.4.1, the equilibrium is globally optimal with respect to the most pessimistic prior, which means that all agents behave as if they were subjective prior maximizers. Since all agents have identical priors, the equilibrium behavior is essentially observationally equivalent to that of the common subjective prior model. In other words, there is a dynamic representative agent.

4.4.5 Sufficient Conditions for the Existence of the Dynamic Representative Agent

In this final subsection, we want to investigate the sufficient conditions for the dynamic representative agent to exit. It is obvious from Section 4.4.3 that in order to have the dynamic representative agent, all agents must have identical prior at each ω^t. For the case of consistent pessimism, we derive the sufficient conditions in Proposition 4.4.1 and Corollary 4.4.1. Here, we want to briefly investigate the other case where the aggregate endowment process does not evolve coherently, i.e., Condition (4.4.8) does not hold. In other words, we want to examine the case where agents have the most pessimistic prior with respect to the aggregate endowment even though the pessimism is not similar over time.

The problem here is evident. As we see in (4.4.2), in order for all agents to have comonotonic consumptions, the utility process $\{V_t^h(c^h;\omega^{t-1},\omega)\}$ must be comonotonic with other agents' utility processes over Ω at each ω^{t-1}. Now, suppose that agents' utility processes $\{V_t^h(c^h;\omega^{t-1},\omega)\}$ are strictly comonotonic so that prior selection is identical at $\forall\omega^{t-1}$ t>1. Then from (4.2.15), we can differentiate this process locally, and for all agents,

$$\begin{aligned}\frac{Sp(\omega^{t-1},\omega_t)}{Sp(\omega^{t-1},\omega_t')} &= \frac{p(\omega^{t-1},\omega_t)u_t'^h(c_t^h(e_t(\omega^{t-1},\omega_t)))}{p(\omega^{t-1},\omega_t')u_t'^h(c_t^h(e_t(\omega^{t-1},\omega_t')))} \\ &= \frac{p(\omega^{t-1},\omega_t)u'^{h'}(c_t^{h'}(e_t(\omega^{t-1},\omega_t)))}{p(\omega^{t-1},\omega_t')u'^{h'}(c_t^{h'}(e_t(\omega^{t-1},\omega_t')))}\end{aligned}$$

Clearly, this equality only holds when all agents' consumption process $\{c_t^h(e^t\ (\omega^{t-1},\omega))\}$ are comonotonic. In order to clear markets, the only possibility is that all agents must have comonotonic consumptions with respect to the aggregate endowment process. In this situation, we can apply (4.4.3) and, by the same argument as in the proof of Proposition 4.4.1, we can construct the dynamic representative agent.

Now when does this construction work? Apparently, if we apply (4.4.3) for each ω^t, the agents consumption process $\{c_t^h(e^t(\omega^{t-1},\omega))\}$ will be comonotonic to the aggregate endowment process $\{e^t(\omega^{t-1},\omega)\}$. However, the converse of the above construction does not work in general. In other words, comonotonic consumptions with respect to the aggregate endowment process do not necessarily produce the comonotonic utility process simply because the utility process is the summation of the continuation value of the future consumptions and the utility of present consumption. If these two numbers are not comonotonic, it is highly likely that the prior selection does not confirm the most pessimistic prior with respect to the present consumption.

In Section 4.3 and 4.4, we focus on the well-ordered aggregate consumption process to avoid this problem. In general, if the order property (4.4.8) does not hold, we are not sure that the consumption process is comonotonic with the utility process. Moreover, the conditions for the existence of the dynamic representative agent become contingent on the number of agents, their utility functions and endowment processes, and the aggregate endowment process. The mixture of these properties does not lead to the constructive argument, rather leads to find coincidence. However, there is a case that only requires conditions on the aggregate endowment process. If we have a simple Markov structure within time, all continuation value becomes constant.[26] Therefore, agents only use their current consumptions to decide their priors. To summarize this intuition without proof:

Corollary 4.4.3:

In a multiple-agents economy, under (4.4.15) and (4.4.16) with (4.4.17), there is a dynamic representative agent, where all agents behave as if they had the identical time-heterogeneous most pessimistic prior over $\{e_t\}$ $\forall T > t \geq 1$, regardless of their initial endowment. In other words, the utility process $\{\mathrm{V}_t^h(\mathrm{c}^h;\omega^{t-1},\omega_t)\}$ becomes comonotonic with the endowment process $\{e_t(\omega^{t-1},\omega_t)\}$ over Ω at $\forall\omega^{t-1}$ T≥t>1. Moreover, the consumption process is "interior" or strongly comonotonic with the aggregate endowment process, which means that there are no ties among the next period's consumptions emerging from the same node. Note that the direction of pessimism is not necessarily constant over time.

(4.4.15) $e_t(\omega^{t-1},\omega) \neq e_t(\omega^{t-1},\omega')\omega,\omega' \in \Omega$

(strong order of the aggregate endowment)

(4.4.16) Markov structure (aggregate endowment):

$e_t(\omega^t) = e_t(\omega_t)\forall T \geq t \geqq 1$

(4.4.17) All agents have identical capacity-based multiple-priors sets

[26] For a general concave utility, the continuation value does not necessarily become constant. However, non-constant continuation value is Pareto dominated by the constant continuation one.

over Ω within time $\forall T > t \geq 1$:

$P_{\mathrm{t}}(\omega^{t-1}, \omega_t) = P_t(\omega'^{t-1}, \omega'_t)$ (i.i.d. prior set within time)

$\forall h, h' \in H \ P_t^h(\omega^{t-1}) = P_t^{h'}(\omega^{t-1})$ (identical prior set among agents)

(4.4.18) ε-open neighborhood around (4.4.15) and (4.4.16) with the norm on $D(\Omega^T)$

We omit the proof because it is the simple repetition of that of Proposition 4.4.1. Condition (4.4.15) ensures the strong comonotonicity of consumptions. Condition (4.4.16) without (4.4.8) implies that the order of the aggregate consumption over Ω can change over time. Finally, Condition (4.4.17) makes the utility process constant. Again, Condition (4.4.18) is just ε-perturbation of the defining endowment process.

4.5 Multiple Agents Economy with Heterogeneous Multiple Prior Sets

4.5.1 Background

Up to now, we have focused on agents with homogeneous uncertainty. In this section, we introduce heterogeneous prior sets, and ask similar questions: Is it possible for all agents to behave similarly at the equilibrium? More precisely, can we derive the conditions under which all agents behave as if they had the most pessimistic prior over the aggregate endowment process at each ω^t? We would like to answer these questions progressively in the following subsections.

This task is difficult unless the agents share some "common" characteristics. For example, in the single prior economy, if agents have different priors, we can anticipate the different consumption order over Ω among agents even though their prior probabilities are comonotonic with each other. The similar results is expected if agents have multiple-priors sets which do not have any common element. In this case, the difference of the prior sets is most likely to be priced or reflected into the allocation order. In order to agree with Arrow-Debreu security prices, it would be better to have the different order of consumptions because their priors show sufficiently heterogeneous preferences over states. However, simply having common elements in their prior sets is not sufficient to avoid this dispersion. It turns out that the order of the prior probability over Ω which minimizes the expected value of allocation x must be comonotonic among agents $\forall x$. Or more strongly, the optimal prior probability must be ordered oppositely to the allocation, which implies that all prior sets are around the center of probability simplex.

The reader may wonder why we suddenly need the strong conditions. In fact, the reason is rather simple. We need to utilize the property of the Arrow-Debreu equilibrium, i.e., state prices. In order to have a clear Pareto domination with state prices, the probability order is essential. In fact, the same situation is applied for the identical capacity-based multiple-priors sets. The reason why we can move the prior sets to the non-center position is that the equilibrium allocation is globally optimal with respect to the *identical* most pessimistic prior. In other words, there is a dynamic representative agent, and the prior probability does not matter for deriving optimal allocations for give α, so that the equilibrium allocations from the multiple-priors set around the center of the probability simplex represents all other cases. However, under heterogeneous multiple-priors sets, we cannot utilize the Pareto domination of (4.4.4). Even if the equilibrium allocations were globally optimal with respect to the most pessimistic prior for all agents, it would not guarantee that under the translated multiple-priors sets, we could achieve the same allocations. We cannot summarize the economy with heterogeneous multiple-priors sets by the representative case, i.e., there is no dynamic representative agent.

The main results of this section are Lemma 4.5.1 and Proposition 4.5.1. The distinct feature of a stochastic exchange economy with heterogeneous uncertainty aversion is that agents still maintain homogeneous order of consumptions. This result implies that the introduction of heterogeneity over the identical capacity-based multiple-priors set around the center of probability simplex does not distort *similarity* among agents at equilibrium, which shows the robustness of Proposition 4.4.1 in Section 4.4. Clearly this robustness feature is the fundamental difference from the common subjective prior model, where a sufficiently large perturbation of the prior usually distorts the comonotonicity of consumptions. In addition, as opposed to the identical capacity-based multiple-priors case, the equilibrium allocations are not observationally equivalent to those of the heterogeneous single prior model. In fact, we cannot observe weakly comonotonic order of consumptions in the latter case. Since these results are critical for this section, we will examine them thoroughly in Section 4.5.3.

Another objective of this section is to compare different attitudes toward uncertainty among agents. Since by definition, the identical capacity-based multiple-priors set does not offer any heterogeneity in the attitude of uncertainty, we also pay special attention to this analysis. The implication is simple. The more uncertainty averse the agent becomes, the less volatile the consumption over Ω, which is very natural because the agent behaves as if she/he became more *risk averse* as she/he becomes more uncertainty averse.

In this section, first we define the three different type of commonality among agents' multiple-priors sets in Section 4.5.2. Then in Section 4.5.3, we investigate thoroughly a single-period model and derive intuitions for the economy with heterogeneous multiple-priors agents. This result is extended to the dynamic setting and we derive the main result of this paper in Section 4.5.4. We also briefly discuss the similar results to Corollary 4.5.1 in Section 4.5.4.

4.5.2 Definition of Commonality among Heterogeneous Multiple-Priors Sets

In order to compare the difference of a certain property, we need to assume some commonality. For example, in order to compare the different attitude toward risk, we assume that the different curvature of *concave* utility functions, different expected utility with respect to the *same* money lotteries, etc. In Section 4.4, similar behavior is observed for agents with the identical capacity-based multiple-priors sets because their attitude toward uncertainty is homogeneous. In order to see similar results, we need to introduce some common properties among heterogeneous multiple-priors sets. First, we want to define the *commonality* in the capacity-based multiple-priors sets.

First define the commonality among capacity-based multiple-priors sets:

Definition 4.5.1:

Agents have the *translationally homogeneous capacity-based multiple-priors set* ($THCB$) each other if their prior sets satisfy the following conditions:

(4.5.1) $\cap_1^H \mathfrak{P}^h \subset \varepsilon \subset \mathfrak{P}^h \ \forall h$ where ε is non-empty open set

(4.5.2) $\overline{p} \in int(\cap_1^H P^h)$ where $\overline{p}$ is the center of probability simplex

Let p^h be the prior which minimizes the expected utility for x^h

(4.5.3) $p^h = \overline{p} + \widetilde{p}^h \ \forall p^h \in \mathfrak{P}^h$

(4.5.4) $\widetilde{p}^h = (\widetilde{p}^h_{\omega_1}, ..., \widetilde{p}^h_{\omega_N})$

where $\widetilde{p}^h$ and allocation $x^h = (x^h_{\omega_1}, ..., x^h_{\omega_N})$ are weakly oppositely comonotonic:

if $x^h_{\omega_{n(1)}} > ... > x^h_{\omega_{n(N)}}$, then $\widetilde{p}^h_{\omega_{n(1)}} \leq ... \leq \widetilde{p}^h_{\omega_{n(N)}}$

(n(1),..., n(N)) is the correspondence between the allocation order and states

In a word, THCB are the heterogenous capacity-based multiple-priors sets which are located closely together around the center of probability simplex although their shape would be different. After decomposing each prior to the center of probability simplex and residual, the optimal selection of residual becomes *weakly oppositely* comonotonic to the order of the allocation. Note that this feature, i.e., Condition (4.5.4) is not guaranteed by the existence of the center of probability simplex in the strictly interior of $\mathfrak{P}^h$.

The second commonality we investigate is *comonotonically homogeneous uncertainty aversion*, which has the following property:

Definition 4.5.2:

Agents are *comonotonically homogeneous uncertainty averse* (CHUA) with each other if their prior sets satisfy the following conditions:

(4.5.1) $\cap_1^H \mathfrak{P}^h \subset \varepsilon \subset \mathfrak{P}^h \ \forall h$ where ε is non-empty open set

(4.5.2) $\overline{p} \in int(\cap_1^H P^h)$ where $\overline{p}$ is the center of probability simplex

(4.5.5) $\mathfrak{P}^h \subseteq \mathfrak{P}^{h'} \forall h, h'$

(4.5.6) $\mathfrak{P}^h$ is symmetric:

$\forall p^h \in \mathfrak{P}^h$: $p^h = \overline{p} + \widetilde{p}^h$, $\exists \widehat{p}^h \in \mathfrak{P}^h$ s.t. $\widehat{p}^h = \overline{p}$ - $\widetilde{p}^h$

This case is very simple. All symmetric prior sets are nested, and the center of symmetry must be the center of the probability simplex. It is as if agents had the same single prior (the center of the probability simplex) and heterogenous uncertainty aversion. Under this prior sets, any reorder of the allocation gives the same utility, which is the reason why we call it as *comonotonically homogeneous*. In other words, agents behave as if they did not care about the name of states. They only cares about the order of consumptions, and how it is ordered does not change agents preferences. In other words, agents do not have preference over states. Technical relationship between CHUA and the preference over acts is found in Appendix 4.J.

Third commonality is used only for the two-states case.

Definition 4.5.3:

Agents have *nested multiple-priors sets* (NP) each other if their prior sets satisfy the following conditions:

(4.5.1) $\quad \cap_1^H \mathfrak{P}^h \subset \varepsilon \subset \mathfrak{P}^h \ \forall h$ where ε is non-empty open set

(4.5.5) $\quad \mathfrak{P}^h \subseteq \mathfrak{P}^{h'} \forall h, h'$

For the two-states case, there is only one degree of freedom for probability assignment. For this reason, we only require (NP) to have all agents behave similarly.

4.5.3 Single Period Economy

Now we examine a single period economy and gain most of intuitions for the heterogenous multiple-priors sets case. First we state Lemma 4, which proves the comonotonic equilibrium consumptions among agents for three different multiple-priors settings.

Lemma 4.5.1:

In a multiple-agents economy, under multiple states with (4.5.7) and any one of (4.5.8) or (4.5.9), or under two states with (4.5.7) and (4.5.10), each agent behaves as if she/he had the most pessimistic prior over $\{\mathrm{e}(\omega)\}$ which is heterogeneous among agents, regardless of her/his initial endowment. Moreover, the consumption is weakly comonotonic with the aggregate endowment, which means that there could be ties among the agent's consumptions, and the equilibrium allocations are locally optimal with respect to the most pessimistic prior. Under (4.5.8) and (4.5.9), state prices are strictly oppositely comonotonic with the aggregate endowment.

(4.5.7) $\quad \mathrm{e}(\omega) \neq \mathrm{e}(\omega') \ \omega, \omega' \in \Omega$

(strong order of the aggregate endowment)

(4.5.8) Translationally homogeneous capacity-based multiple-priors set

(4.5.9) Comonotonically homogeneous uncertainty aversion

(4.5.10) Nested multiple-priors sets

Proof: Appendix 4.K

Lemma 4.5.1 shows that under the conditions stated, agents with heterogenous multiple-priors sets behave as if they had the most pessimistic prior with respect to the aggregate endowment tomorrow. Since the results are analogous to the case of previous section, we call this economy a *semi-dynamic representative agent* economy. We want to emphasize that this allocation is *locally optimal with respect to the most pessimistic prior* as opposite to the case of the identical capacity-based multiple-priors sets, where we obtain the globally optimal solutions relative to the identical most pessimistic prior.

For the case of multiple states (N>2), the proof heavily relies on Condition (4.5.18) or Condition (4.5.9). The basic intuition of these conditions is that agents seem to care only the order of consumptions, not on which state they have a higher or lower consumption. In other words, the relative importance of the state is irrelevant here. Under these conditions, it is better for all agents to have the same consumption order as the aggregate endowment process because it is most easily implemented and the reorder of this allocation gives very close or identical utility. Other combinations of consumptions inevitably involves the disagreement of the prior probability order, which makes it harder for the prices of Arrow-Debreu securities to be matched among agents. In fact, Condition (4.5.4) or Condition (4.5.6) ensures that the order of prior probability is oppositely comonotonic to the allocation. This condition and strict concavity of utility functions imply that state prices must be oppositely comonotonic to the allocation. Conversely, it is clear from the proof of Lemma 4.5.1 that given state prices, agents optimal consumptions must be oppositely comonotonic to the order of state prices. Now at equilibrium, agents must agree on state prices. Given the above individual behavior, all consumptions are inevitably comonotonic. In other words, the budget set induced by state prices touch the same side of indifference curve for all agents. Note that this prior probability order property (4.5.4) or (4.5.6) holds only when the prior sets are located around the center. Having the center as an interior point does not guarantee these conditions.

For two-states case, the prior set does not need to be located around the center of the probability simplex because there is only one degree of freedom for the probability determination. By Condition (4.5.10), all p^h can be written as: $p^h = \overline{p} + \widetilde{p}^h$ s.t. $\exists \overline{p} = (\overline{p}_1,\overline{p}_2) \in \mathfrak{P}^h \; \forall h$ where $\widetilde{p}_1^h < 0$ and $\widetilde{p}_2^h > 0$ when $c^h(\omega_1) > c^h(\omega_2)$, and $\widetilde{p}_1^h > 0$ and $\widetilde{p}_2^h < 0$ when $c^h(\omega_1) < c^h(\omega_2)$. In other words, one of $\widetilde{p}^h$ is positive and the other is negative. Then for the agent with $c^h(\omega_1) \neq c^h(\omega_2)$, F.O.C of (4.2.15) implies:

$$\frac{SP(\omega_1)}{SP(\omega_2)} = \frac{\overline{p}_1 + \widetilde{p}_1^h}{\overline{p}_2 + \widetilde{p}_2^h} \frac{u'^h(c^h(\omega_1))}{u'^h(c^h(\omega_2))}$$

Clearly, if $c^h(\omega_1) > c^h(\omega_2)$ and $c^{h\prime}(\omega_1) < c^{h'}(\omega_2)$, the state price ratio does not match. The same logic does not work for the multiple states case (N>2), because if $\overline{p}$ does not have identical numbers for all states, we cannot conclude that the state price ratios are different. In other words, there is more than one degree of freedom for the prior probability determination, and the increase of indeterminacy shadows the relationship among probability ratios.

For example, assume that agents have THCB and strongly ordered consumptions. Suppose that $\exists h, h'$ s.t. $c^h(\omega_i) > c^h(\omega_{i+1})$ and $c^{h'}(\omega_i) < c^{h'}(\omega_{i+1})$. F.O.C. of (4.2.15) implies:

$$\frac{SP(\omega_i)}{SP(\omega_{i+1})} = \frac{\overline{p}_i + \widetilde{p}_i^h}{\overline{p}_{i+1} + \widetilde{p}_{i+1}^h} \frac{u'^h(c^h(\omega_i))}{u'^h(c^h(\omega_{i+1}))} = \frac{\overline{p}_i + \widetilde{p}_i^{h'}}{\overline{p}_{i+1} + \widetilde{p}_{i+1}^{h'}} \frac{u'^{h'}(c^{h'}(\omega_i))}{u'^{h'}(c^{h'}(\omega_{i+1}))}$$

We only know that $\widetilde{p}_i^h < \widetilde{p}_{i+1}^h$ and $\widetilde{p}_i^{h'} > \widetilde{p}_{i+1}^{h'}$, and this condition is not enough to show $\dfrac{\overline{p}_i + \widetilde{p}_i^h}{\overline{p}_{i+1} + \widetilde{p}_{i+1}^h} < \dfrac{\overline{p}_i + \widetilde{p}_i^{h'}}{\overline{p}_{i+1} + \widetilde{p}_{i+1}^{h'}}$ unless $\overline{p}_i = \overline{p}_{i+1}$.

The above argument must hold for the identical capacity-based multiple-priors sets case. Now we consider why we can move the identical capacity-based multiple-priors sets to the non-center position. First, note that for the capacity-based multiple-priors set, we can move the prior set to the center of the probability simplex where it satisfies the condition (4.5.4). Let the original prior be $p^h = \overline{p} + \widetilde{p}^h + p'$ and the new prior be $\widehat{p}^h = \overline{p} + \widetilde{p}^h$ where $\overline{p}$ is the center of probability simplex. Then by Lemma 4.4.1 and Lemma 4.5.1, all agents must have strongly comonotonic consumptions. The difference between the identical capacity-based multiple-priors case and heterogenous multiple-priors sets is that by Lemma 4.4.1, the former achieve the global optimum under the most pessimistic prior. We restate (4.4.2). For any other allocation x^h, the optimal c^h satisfies:

$$\begin{aligned}
\sum \alpha^h \int u^h(x^h) d\widehat{p}^h &= \sum \alpha^h \int u^h(x^h) d(\overline{p} + \widetilde{p}^h) \\
&\leq \int \sum \alpha^h u^h(x^h) d(\overline{p}+\widetilde{p}) \\
&\leq \int \sum \alpha^h u^h(c^h) d(\overline{p} + \widetilde{p})
\end{aligned}$$

where $\overline{p}+\widetilde{p}$ is the most pessimistic prior with respect to the aggregate endowment. Now we translate this prior back to the original location.

$$\begin{aligned}\sum \alpha^h \int u^h(x^h)dp^h &= \sum \alpha^h \int u^h(x^h)d(\,\overline{p} + \widetilde{p}^h + p')\\ &= \sum \alpha^h \{\int u^h(x^h)d(\,\overline{p} + \widetilde{p}^h) + \int u^h(x^h)dp'\}\\ &\le \int \{\sum \alpha^h u^h(x^h)d(\,\overline{p} + \widetilde{p}) + \int \sum \alpha^h u^h(x^h)dp'\}\\ &\le \int \sum \alpha^h u^h(c^h)d(\,\overline{p} + \widetilde{p}) \ + \int \sum \alpha^h u^h(c^h)dp'\\ &= \int \sum \alpha^h u^h(c^h)d(\overline{p} + \widetilde{p} + p')\end{aligned}$$

The second last inequality holds because the consumption c^h is strongly comonotonic and globally optimal with respect to *identical* priors: $p + \widetilde{p}$ and p'. From F.O.C. of (4.2.15):

$$\frac{\overline{p}_i + \widetilde{p}_i}{\overline{p}_{i+1} + \widetilde{p}_{i+1}} \frac{u'^h(c^h(\omega_i))}{u'^h(c^h(\omega_{i+1}))} = \frac{\overline{p}_i + \widetilde{p}_i}{\overline{p}_{i+1} + \widetilde{p}_{i+1}} \frac{u'^{h'}(c^{h'}(\omega_i))}{u'^{h'}(c^{h'}(\omega_{i+1}))} \Rightarrow$$
$$\frac{u'^h(c^h(\omega_i))}{u'^h(c^h(\omega_{i+1}))} = \frac{u'^{h'}(c^{h'}(\omega_i))}{u'^{h'}(c^{h'}(\omega_{i+1}))} \Rightarrow$$
$$\frac{\overline{p}_i + p'_i + \widetilde{p}_i}{\overline{p}_{i+1} + p'_{i+1} + \widetilde{p}_{i+1}} \frac{u'^h(c^h(\omega_i))}{u'^h(c^h(\omega_{i+1}))} = \frac{\overline{p}_i + p'_i + \widetilde{p}_i}{\overline{p}_{i+1} + p'_{i+1} + \widetilde{p}_{i+1}} \frac{u'^{h'}(c^{h'}(\omega_i))}{u'^{h'}(c^{h'}(\omega_{i+1}))}$$

In other words, for the same α^h, the same c^h is optimal, i.e., the allocations are independent of priors. Of course, the endowment and state prices for the new and original equilibrium allocations are different, but for the same α^h, the same allocations must be globally optimal with respect to the most pessimistic prior for each case. In other words, a single multiple-priors set represents all other translated multiple-priors sets.

From the above results, it must be clear why we cannot move the heterogeneous multiple-priors sets away from the center of probability simplex. Suppose that each agent has the strongly comonotonic consumptions and they are globally optimal with respect to the most pessimistic prior. However, since every agent has the *different* prior, the above calculations do not hold. In other words, even though allocations are globally optimal, for any movement of the heterogeneous multiple-priors sets, we must reconsider whether agents have comonotonic consumptions. In general, we only have locally optimal consumptions to which we cannot apply the above argument at all. Clearly, for the heterogeneous multiple-priors sets case, there is no way for the single location of multiple-priors sets to *represent* other translated ones.

Next, we want to investigate the difference between the heterogeneous subjective prior model and heterogeneous multiple-priors one. The critical assumption of Condition (4.5.2) and (4.5.9) is that the state prices must be oppositely comonotonic to the consumptions, and at $\overline{c}^h = (\overline{c}^h, ..., \overline{c}^h)$, the indifference curve kinks inwards by shifting the prior probability order. However for the single subjective prior model, even if agents have the same order of priors, by moving consumptions slightly away from $\overline{c}^h$, we can still maintain the state prices order. In other words, uncertainty aversion makes the indifference curve kinked at $\overline{c}^h$, whereas the expected utility maximizer with the single subjective prior does not have a kink in her/his indifference curve.

More precisely, uncertainty averse agents are *locally risk averse* at $\overline{c}^h$.[27] Condition (4.5.2) and Condition (4.5.9) ensure that at $\overline{c}^h$, the right and left derivatives between two state prices become $[\frac{p_i^h}{p_j^h}, \frac{p_j^h}{p_i^h}]$ by (4.2.17) where $\frac{p_i^h}{p_j^h} < 1 < \frac{p_j^h}{p_i^h}$. Clearly, if $\frac{SP(\omega_i)}{SP(\omega_j)} > 1$, the budget hyperplane must touch where $c^h(\omega_i) \leq c^h(\omega_j)$ $\forall h$ and vice versa. On the other hand,

[27] For the capacity-based multiple-priors set, at any $c^h(\omega_i) = c^h(\omega_j)$, the indifference curve has a kink because the capacity-based case assumes *comonotonic* independence instead of *certainty* independence. In other words, there are finitely many discountinuous probability shifts among strongly ordered consumptions (not smooth change as in the general multple-priors case).

with the single subjective prior, every agent becomes *locally risk neutral* at $\overline{c}^h$. Moreover, the probability to judge the *actuarially fairness*[28] is not identical.[29] This heterogeneous judgement implies that at $\overline{c}^h$, given state prices, some assets are actuarially favorable for one agent and unfavorable for another agent, which makes their consumptions non-comonotonic with each other. The above argument becomes even clearer if we assume that there are two hypothetical trades. First we would trade assets and achieve $\overline{c}^h$ that is in the budget set, then we would take a risky position over $\overline{c}^h$. Clearly, the actuarial judgement at $\overline{c}^h$ determines the order of consumptions, and homogeneity of this judgement is essential for comonotonic consumptions. Note that when all agents have the identical prior, the probability to assess *actuarially fairness* is identical. Hence for any asset, all agents agree whether they are actuarially favorable. This is the reason why agents have a full risk-sharing allocation for the identical prior case.

The above result is particularly interesting. Condition (4.5.2) and (4.5.9) can be interpreted as if agents became heterogeneously uncertainty averse over the common capacity-based multiple-priors set that is located at the center of the probability simplex. The introduction of heterogeneity does not distort the homogeneous equilibrium behavior among agents. This is a clear distinction from the common subjective prior model, where a sufficiently large perturbation of the prior probability usually results in non-comonotonic con-

[28] If the Arrow-Debreu price is equal to the state probability, the asset prices become actuarily fair, i.e., the expected return is identical to the acquisition cost of the asset.

[29] If the expected return is greater than one, it is actuarily favorable.

sumptions. In other words, uncertainty aversion induces more commonality among agents' behavior.

Finally, we want to examine the effect of a different level of uncertainty.[30] Consider the two-states case with Condition (4.5.10), and assume that there are two agents with identical utility functions and endowments but with different prior set, $\mathfrak{P}^{h'} \subset \mathfrak{P}^h$. For these agents, F.O.C. of (4.2.15) with non-binding constraint implies:

$$\frac{SP(\omega_1)}{SP(\omega_2)} = \frac{\overline{p}_1 + \widetilde{p}_1^h}{\overline{p}_2 + \widetilde{p}_2^h} \frac{u'^h(c^h(\omega_1))}{u'^h(c^h(\omega_2))} = \frac{\overline{p}_1 + \widetilde{p}_1^{h'}}{\overline{p}_2 + \widetilde{p}_2^{h'}} \frac{u'^{h'}(c^{h'}(\omega_1))}{u'^{h'}(c^{h'}(\omega_2))}$$

with $\widetilde{p}_1^h$ =- $\widetilde{p}_2^h$, $\widetilde{p}_1^{h'}$ =- $\widetilde{p}_2^{h'}$, and $\widetilde{p}_1^{h'} < \widetilde{p}_1^h < 0$, $0 < \widetilde{p}_2^h < \widetilde{p}_2^{h'}$ when $e(\omega_1) > e(\omega_2)$, $c^h(\omega_1)$ $> c^h(\omega_2)$, and $c^{h'}(\omega_1) > c^{h'}(\omega_2)$. This condition implies:

$$\frac{u'^h(c^h(\omega_1))}{u'^h(c^h(\omega_2))} < \frac{u'^{h'}(c^{h'}(\omega_1))}{u'^{h'}(c^{h'}(\omega_2))}$$

Under the identical budget set and consumption order, in order to achieve this inequality, $c^{h'}(\omega_1)$ and $c^{h'}(\omega_2)$ must be closer than $c^h(\omega_1)$ and $c^h(\omega_2)$. Therefore, the more uncertainty averse the agent becomes, the less volatile the consumption. The reader can verify by using a specific function that essentially the same results hold for the multiple states case with Condition (4.5.9): nested and symmetric priors. The above result confirms that uncertainty aversion magnifies the effects of risk aversion in Lemma 4.4.2. (The identical prior with more concave $u^{h'}$ produces the same results as in the above case.) As in Lemma 4.4.2, the uncertainty aversion redefine the utility functions $V(x)$ by (4.2.2). The new utility function becomes *globally* more concave than the original function, and two important properties of

[30] In Appendix 4.L, we define the *more-uncertainty-averse-than* relation.

expected utility are also preserved: translation invariance and homogeneity of the preference over acts. Clearly the argument for the risk aversion imply the same results. The only clear distinction between risk aversion and uncertainty aversion is the local attitude of actuarial judgement at $\overline{c}^h$, which is shown in our results as comonotonic consumptions among agents.

4.5.4 Dynamic Setting

Now, we are ready for the extension of the results of Section 4.3 and Section 4.4.4. In Section 4.5.3, we show the conditions for heterogenous multiple-priors sets to produce the comonotonic consumptions among agents. Here, we keep these conditions and consider the dynamic linkage of state evolution, and seek the answer for the same question as in Section 4.4.4: under what conditions does each agent behave as if she/he had the most pessimistic prior to the aggregate endowment process?

As opposed to the identical capacity-based multiple-priors sets case, heterogeneous multiple-priors do not produce the dynamic representative agent by the argument in Section 4.5.3. Moreover importantly, as in Section 4.5.3, the equilibrium consumptions are generally locally optimal with respect to the most pessimistic prior. The second result is extremely crucial because as we will see in the proof later, without global optimality, we cannot employ the same logic of Pareto domination by (4.4.4). This result forces us only to utilize the results for the competitive equilibrium in Lemma 4.5.1. Now we face two fundamental problems for the linkage of the dynamic evolution of states.

The first problem is the relative order among the aggregate endowment within time.

Given the strong order [Condition (4.4.7)] and the comonotonic order of the aggregate endowment process over time [Condition (4.4.8)], we now investigate the state monotonic Condition (4.4.10). In the proof of Proposition 4.4.1, the critical condition is that $c_t^h(e_t(\omega^t))$ is increasing in $e_t(\omega^t)$. This condition no longer holds for the case of the heterogeneous multiple-priors sets because we now have the local optima with respect to the most pessimistic prior. In other words, we cannot use the construction of Constantinides (1982) (4.4.3). Define the similar optimization as:

$$(4.5.11) \qquad u(e) = \text{Max } \{\textstyle\sum \alpha^h p^h(\omega^{t-1}, \omega_t) u_t^h(c_t^h) : \sum c_t^h = e_t(\omega^t) \}$$

where $p^h(\omega^{t-1}, \omega_t)$ is the conditional probability from the most pessimistic prior over Ω at ω^{t-1} for agent h. It is clear that this solution $c_t^h(e_t(\omega^t))$ does not necessarily increase in $e_t(\omega^t)$ because the probability $p^h(\omega^{t-1}, \omega_t)$ shifts according to the movement of $e_t(\omega^t)$. In fact, the solution from (4.5.11) corresponds to the solution of the heterogeneous subjective prior model, where the agent's subjective prior is the most pessimistic one relative to the aggregate endowment. It is clear that the solutions $\{c_t^h(e_t(\omega^t))\}$ are not necessarily comonotonic with each other, which implies that $\{c_t^h(e_t(\omega^t))\}$ from heterogeneous multiple-priors model are not globally optimal in general. Under this result, it is very hard to verify the implication of Condition (4.4.10). This condition implies that there is some ω^{t-1} and ω'^{t-1}, where at ω^{t-1} the aggregate endowment process over Ω next period is monotonically greater than that from ω'^{t-1}. In other words, the utility frontier shifts outwards. However, even though we fix the utility weights α, since the allocations are only locally optimal, it is possible that $\exists h$ s.t. $\int u_t^h(c_t^h(e_t(\omega^{t-1}, \omega_t)))\, dP^h(\omega^{t-1}, \omega_t) > \int u_t^h(c_t^h(e_t(\omega'^{t-1}, \omega_t)))\, dP^h(\omega'^{t-1}, \omega_t)$ whereas $\exists h'$ s.t.

$\int u_t^{h'}(c_t^{h'}(e_t(\omega^{t-1},\omega_t)))\ dP^{h'}(\omega^{t-1},\omega_t) < \int u_t^{h'}(c_t^{h'}(e_t(\omega'^{t-1},\omega_t)))\ dP^{h'}(\omega'^{t-1},\omega_t)$. Hence, we cannot compare the equilibrium allocations $\{c_t^h(e_t(\omega^{t-1},\omega_t))\}$ with $\{c_t^h(e_t(\omega'^{t-1},\omega_t))\}$. This is simply the restatement of the fact that the same α does not necessarily guarantee that the separating hyperplane touches the homogeneous side of the utility frontier, where all agents have $\int u_t^h(c_t^h(e_t(\omega^{t-1},\omega_t)))\ dP^h(\omega^{t-1},\omega_t) > \int u_t^h(c_t^h(e_t(\omega'^{t-1},\omega_t)))\ dP^h(\omega'^{t-1},\omega_t)$ if $\{e_t(\omega^{t-1},\omega_t)\} \geq \{e_t(\omega'^{t-1},\omega_t)\}$.

The second problem is the relative order of the aggregate endowment over time. The above argument clearly indicates that even though $\{c_t^h(e_t(\omega^t))\}$ might achieve the global optimum with respect to the most pessimistic prior at $\forall\omega^t$, it does not necessarily guarantee that $\{c_{t-1}^h(e_{t-1}(\omega^{t-1}))\}$ achieves the globally optimum with respect to the most pessimistic prior at $\forall\omega^{t-1}$. The consumptions must be globally optimal with respect to the most pessimistic prior over time, otherwise it is most likely that the prior over the utility process $\{V_t^h(c;\omega^{t-1},\omega_t)\}$ shifts over time.

Clearly from the above observation, we can no longer utilize Condition (4.4.10). How about (4.4.9)? It turns out to be fine. Since the same α and the same distribution of the aggregate endowment over Ω necessarily ensure the identical solution because of the strict concavity of u^h,[31] we can effectively make the expected value of the utility vector: $\int u_t^h(c_t^h(e_t(\omega^{t-1},\omega_t)))\ dP^h(\omega^{t-1},\omega_t)$ constant $\forall\ \omega^{t-1}$, $\forall 1 < t \leq T$. Now we are ready to state Proposition 4.5.1, which is the main result of this paper:

[31] The strict concavity of u^h implies the strict concavity of the utility frontier.

Proposition 4.5.1:

In a multiple-agents economy with (4.5.12), (4.5.13), (4.5.14) and (4.5.15), under multiple states (N>2) with any one of (4.5.6) or (4.5.17), or under two states with (4.5.18), each agent behaves as if she/he had the most pessimistic prior over $\{\mathrm{e}_t(\omega^t)\}$ $\forall T > t \geq 1$, with constant pessimism over time, regardless of her/his initial endowment. In other words, the utility process $\{\mathrm{V}_t^h(\omega^{t-1}, \omega_t)\}$ and consumption process $\{\mathrm{c}_t^h(\omega^{t-1}, \omega_t)\}$ become weakly comonotonic with the aggregate endowment process $\{\mathrm{e}_t(\omega^{t-1}, \omega_t)\}$ over Ω $\forall \omega^{t-1}$ T≥t>1, and under multiple states (N>2), state prices are strictly oppositely comonotonic with the aggregate endowment process $\{\mathrm{e}_t(\omega^{t-1}, \omega_t)\}$ over Ω $\forall \omega^{t-1}$ T≥t>1.

(4.5.12) $e_t(\omega^{t-1}, \omega) \neq e_t(\omega^{t-1}, \omega')$ $\omega, \omega' \in \Omega$

(strong order of the aggregate endowment)

(4.5.13) $e_t(\omega^{t-1}, \omega) > e_t(\omega^{t-1}, \omega') \Rightarrow e_{t'}(\omega^{t'-1}, \omega) > e_{t'}(\omega^{t'-1}, \omega')$

$\forall T \geq t, t' > 1, \omega, \omega' \in \Omega$,$\omega^{t-1} \in \Omega^{t-1}$, $\omega^{t'-1} \in \Omega^{t'-1}$

(comonotonic order of aggregate endowments over Ω for all $e_t(\omega^t)$)

(4.5.14) Markov structure (aggregate endowment):

$e_t(\omega^t) = e_t(\omega_t) \forall T \geq t \geqq 1$

(4.5.15) All agents have time-homogeneous i.i.d. multiple prior set

over Ω $\forall \omega^t$: $P_t^h = P_{t'}^h$ (independent prior set)

(4.5.16) Translationally homogeneous capacity-based prior set

(4.5.17) Comonotonically homogeneous uncertainty aversion

(4.5.18) Nested prior sets

Proof:

We utilize the property of Pareto optimality of the Arrow-Debrue equilibrium and Lemma 4.5.1. First, apply Lemma 4.5.1 over $\omega_t \in \Omega$ at the history ω^{t-1} to get the allocation $\{c_t(e_t(\omega^{t-1}, \omega_t))\} = (\{c_t^h(e_t\ (\omega^{t-1},\ \omega_t))\}\ ,...,\ \{c_t^h(e_t(\omega^{t-1}, \omega_t))\})$. We know that $\{c_t(e_t(\omega^{t-1}, , \omega_t))\}$ is the Arrow-Debreu equilibrium for this segregated economy (one period without consumption at ω^{t-1}). For any other feasible allocations $\{x_t(\omega^t)\}$ at ω^{t-1}, define $G_{t-1}^h(x_t^h(\omega^{t-1}, \bullet))$, $G_{t-1}(x_t(\omega^{t-1}, \bullet))$, $G_{t-1}^h(c_t^h(e_t(\omega^{t-1}, \bullet)))$, and $G_{t-1}(c_t(e_t(\omega^{t-1}, \bullet)))$:

$$
\begin{aligned}
G_{t-1}(x_t(\omega^{t-1}, \bullet)) &= \sum \alpha^h G_{t-1}^h(x_t^h(\omega^{t-1}, \bullet)) \\
&= \sum \alpha^h \int u_t^h(x_t^h(\omega^{t-1}, \omega_t)) dP^h(\omega^{t-1}, \omega_t) \\
&\leq \sum \alpha^h \int u_t^h(c_t^h(e_t(\omega^{t-1}, \omega_t))) d\widehat{P}^h(\omega^{t-1}, \omega_t) \text{ (By Lemma 4.5.1)} \\
&= \sum \alpha^h G_{t-1}^h(c_t^h(e_t(\omega^{t-1}, \bullet))) \\
&= G_{t-1}(c_t(e_t(\omega^{t-1}, \bullet)))
\end{aligned}
$$

where $P^h(\omega^{t-1}, \omega_t)$ is the optimal prior selection at ω^{t-1} when agent h follows the allocation $\{x_t^h(\omega^t)\}$, and $\widehat{P}^h(\omega^{t-1}, \omega_t)$ is the most pessimistic prior with respect to $\{e_t(\omega^{t-1}, \omega_t)\}$ for agent h. By strict concavity of u^h, as we see in Appendix 4.M, $G_{t-1}^h(x_t^h(\omega^{t-1}, \cdot))$ becomes strictly concave in $\{x_t^h(\omega^{t-1}, \omega_t)\}$. This implies that the utility frontier $UF = (G_{t-1}^1(x_t^1(\omega^{t-1}, \cdot)),\ ...\ , G_{t-1}^H(x_t^H(\omega^{t-1}, \cdot)))$ is strictly concave. Therefore, for any given α, there is only one tangent point on the UF, i.e., $G_{t-1}^h(c_t^h(e_t(\omega^{t-1}, \cdot)))$ that is uniquely determined. By (4.5.14), it is obvious that $G_{t-1}^h(c_t^h(e_t(\omega^{t-1}, \cdot)))$ are constant $\forall\ \omega^{t-1}$.

From the above results, $\mathrm{G}_{t-1}(c_t(e_t(\omega^{t-2},\omega_{t-1},\bullet))) \geq \mathrm{G}_{t-1}(x_t(\omega^{t-2},\omega_{t-1},\bullet))\ \forall\omega^{t-2}$ with strict inequality for non-comonotonic allocations. Hence,

$$
\begin{aligned}
G_{t-2}(x_t(\omega^{t-2},\bullet)) &= \textstyle\sum \alpha^h G^h_{t-2}(x^h_t(\omega^{t-2},\bullet)) \\
&= \textstyle\sum \alpha^h \int G^h_{t-1}(x^h_t(\omega^{t-2},\omega_{t-1},\bullet))dP^h(\omega^{t-2},\omega_{t-1}) \\
&\leq \textstyle\sum \alpha^h \int G^h_{t-1}(x^h_t(\omega^{t-2},\omega_{t-1},\bullet))d\overline{P} \\
&= \textstyle\int \sum \alpha^h G^h_{t-1}(x^h_t(\omega^{t-2},\omega_{t-1},\bullet))d\overline{P} \\
&= \textstyle\int G_{t-1}(x_t(\omega^{t-2},\omega_{t-1},\bullet))d\overline{P} \\
&\leq \textstyle\int G_{t-1}(c_t(e_t(\omega^{t-2},\omega_{t-1},\bullet)))d\overline{P} \\
&= \textstyle\int \sum \alpha^h G^h_{t-1}(c^h_t(e_t(\omega^{t-2},\omega_{t-1},\bullet)))d\overline{P} \\
&= \textstyle\sum \alpha^h \int G^h_{t-1}(c^h_t(e_t(\omega^{t-2},\omega_{t-1},\bullet)))d\overline{P} \\
\text{(A)} \qquad &= \textstyle\sum \alpha^h \int G^h_{t-1}(c^h_t(e_t(\omega^{t-2},\omega_{t-1},\bullet)))d\widehat{P}^h(\omega^{t-2},\omega_{t-1}) \\
&= \textstyle\sum \alpha^h G^h_{t-2}(x^h_t(\omega^{t-2},\bullet)) \\
&= \mathrm{G}_{t-2}(c_t(e_t(\omega^{t-2},\bullet)))
\end{aligned}
$$

where $P^h(\omega^{t-2},\omega_{t-1})$ is the optimal prior choice for the allocation process $\{x^h_t(\omega^t)\}$ at ω^{t-2}, $\overline{P}$ is a strictly interior point that satisfies Condition (4.5.1) ($\overline{P} \in int(\cap_1^H P^h)$), and $\widehat{P}^h(\omega^{t-2},\omega_{t-1})$ is the optimal prior choice for the allocation process $\{c^h_t(e_t(\omega^t))\}$ at ω^{t-2}. Note that equation (A) holds because $\{G^h_{t-1}(c^h_t(e_t(\omega^{t-2},\omega_{t-1},\bullet)))\}$ are constant over $\omega_{t-1} \in \Omega\ \forall\omega^{t-2}$. Repeat the argument above up to t-k=1, where k is the number of above operation, then $G_1(x_t(\omega_1,\bullet)) \leq G_1(c_t(e_t(\omega_1,\bullet)))$ with strict inequality for non-comonotonic allocations. Now, we can apply the same exercise for $\forall t$ s.t. T$\geq t>$1. Combining all inequalities, $\sum_1^T \mathrm{G}_1(c_t(e_t(\omega_1,\bullet))) \geq \sum_1^T G_1(x_t(\omega_1,\bullet))$. Therefore, $\sum \alpha^h \widehat{E}^h[\sum u^h_t(c^h_t(e_t(\omega^t)))]$

$\geq \sum \alpha^h E^h[\sum u_t^h(x_t^h(\omega^t))]$ with strict inequality for non-comonotonic $\{x_t(\omega^t)\}$.

Now $\{\sum_t^T \mathrm{G}_t(c_\tau(e_\tau(\omega^{t-1},\bullet)))\} \geq \{\sum_t^T \mathrm{G}_t(x_\tau(\omega^{t-1},\bullet))\}$ and $\{\sum_t^T \mathrm{G}_t(c_\tau(e_\tau(\omega^{t-1},\bullet)))\}$ are constant over ω_t at $\forall\omega^{t-1}$. Hence $\{\sum_t^T \mathrm{G}_t(c_\tau(e_\tau(\omega^{t-1},\bullet)))\}$ and $\{c_t(e_t(\omega^{t-1},\omega_t))\}$ become comonotonic, i.e., $\{V_t^h(\omega^{t-1},\omega_t)\}$ and $\{e_t(\omega^{t-1},\omega_t)\}$ become comonotonic over $\omega_t \in \Omega$ at $\forall\omega^{t-1}$ $T \geq t > 1$. Since the above inequality holds for all possible choice of α^h which solves (4.4.1), at any Arrow-Debreu equilibrium, each agent must have comonotonic consumption and utility process, and the agent behaves as if they had the most pessimistic prior over $\{e_t(\omega^t)\}$ $\forall t$. Finally, from Lemma 4.5.1, for multiple-states case (N>2), state prices are strongly oppositely comonotonic to $\{e_t(\omega^t)\}$ $\forall t$.■

The critical assumption is the strict concavity of utility functions, which ensures that the separating hyperplane touches at the single point on the utility frontier. This results makes $\int u_t^h(c_t^h(e_t(\omega^{t-1},\omega_t)))dP^h(\omega^{t-1},\omega_t)$ identical for all ω^{t-1}. Given this uniqueness and Markov assumption of Condition (4.5.4), we repeat the similar Pareto domination argument as in the proof of Proposition 4.4.1. However, it is not clear that $\mathrm{G}_{t-1}(x_t(\omega^{t-1},\bullet))$ $\leq \mathrm{G}_{t-1}(c_t(e_t(\omega^{t-1},\bullet)))$ at $\forall\omega^{t-1}$ implies that $\mathrm{G}_{t-2}(x_t(\omega^{t-2},\bullet)) \leq \mathrm{G}_{t-2}(c_t(e_t(\omega^{t-2},\bullet)))$ because each agent has a different prior. Condition (4.5.1) ($\exists$ strictly interior points for intersection of all agents multiple-priors sets) ensures that it is in fact the case. Now we essentially neutralize the effects of the dynamic connection of consumption process, so the problem becomes the repetition of the single period optimization.

As for the extension of the structure of uncertainty, we face the similar difficulty as in Proposition 4.5.1. We cannot compare the equilibrium allocations over $\omega_t \in \Omega$ at ω^{t-1} and

${\omega'}^{t-1}$ if $\mathfrak{P}_{t-1}(\omega^{t-1}) \subseteq \mathfrak{P}_{t-1}(\omega'^{t-1})$ at $e_{t-1}(\omega^{t-1}) > e_{t-1}(\omega'^{t-1})$. Since the prior probability changes, it would be harder for $c_t^h(e_t(\omega^t))$ to be increasing in $e_t(\omega^t)$ even though allocations were globally optimal with respect to the most pessimistic prior. Similarly, ε-perturbation does not hold, either. However, the prior set can vary over time because of the time separable utility structure, Markov endowment process (4.5.14), and order property of (4.5.13). We can generalize the structure of uncertainty as Corollary 6 without proof:

Corollary 4.5.1:

In Proposition 4.5.1, if we replace Condition (4.5.15) with (4.5.19), each agent behaves as if she/he had the most pessimistic prior over $\{\mathrm{e}_t\}$ $\forall T > t \geq 1$, with constant pessimism over time, regardless of her/his initial endowment. In other words, the utility process $\{\mathrm{V}_t^h(\omega^{t-1}, \omega_t)\}$ and consumption process $\{\mathrm{c}_t^h(\omega^{t-1}, \omega_t)\}$ become weakly comonotonic with the aggregate endowment process $\{\mathrm{e}_t(\omega^{t-1}, \omega_t)\}$ over Ω $\forall\omega^{t-1}$ T≥t>1, and under multiple states (N>2), state prices are strictly oppositely comonotonic with the aggregate endowment process $\{\mathrm{e}_t(\omega^{t-1}, \omega_t)\}$ over Ω $\forall\omega^{t-1}$ T≥t>1.

(4.5.19) $\quad P_{\mathrm{t}}^h(\omega^{t-1}, \omega_t) = P_t^h(\omega'^{t-1}, \omega'_t)$ (time-heterogenous i.i.d.)

Finally, Corollary 4.4.3 is replaced by Corollary 4.5.2 for the heterogeneous multiple-priors sets case. This Corollary is obvious from the proof of Proposition 4.5.1, which hinges on the assumption of separation of the optimization over time. Under very stationary evolution of aggregate endowments and prior sets, optimal consumptions become comonotonic with each other.

Corollary 4.5.2:

In a multiple-agents economy under (4.5.12), (4.5.14) and (4.5.19) under multiple states (N>2) with any one of (4.5.16) or (4.5.17), or under two states with (4.5.18), each agent behaves as if she/he had the most pessimistic prior over $\{\mathrm{e}_t\}$ $\forall T > t \geq 1$, regardless of her/his initial endowment. In other words, the utility process $\{\mathrm{V}_t^h(\omega^{t-1}, \omega_t)\}$ and consumption process $\{\mathrm{c}_t^h(\omega^{t-1}, \omega_t)\}$ become weakly comonotonic with the aggregate endowment process $\{\mathrm{e}_t(\omega^{t-1}, \omega_t)\}$ over Ω $\forall\omega^{t-1}$ T$\geq$t>1, and under multiple states (N>2), state prices are strictly oppositely comonotonic with the aggregate endowment process $\{\mathrm{e}_t(\omega^{t-1}, \omega_t)\}$ over Ω $\forall\omega^{t-1}$ T$\geq$t>1. Note that the direction of pessimist is not necessarily consistent over time.

4.6 Continuum of Equilibrium Prices

4.6.1 Single Agent Economy (Epstein-Wang: 1994)

In this section, we investigate the possibility of the existence of continuum of equilibria for multiple-agents economy. In this area of research, first Dow-Werlang (1992) have shown that from a riskless position there is a range of prices where an agent does not take any risky investment. This result hinges on the assumption that the initial allocations are riskless. As we saw in Section 4.2.4, if an allocation admits the multiple choices of optimal priors, the indifference curve becomes non-differentiable. The range of price Dow-Werlang (1992) prove essentially captures the differences of the right and left derivatives at the initial riskless allocation.

The argument is extended considerably by Epstein-Wang (1994). They show by the representative agent Lucus model that there is a continuum of equilibrium prices when there are multiple choices of optimal priors. In fact, in order to prove the continuum of equilibria, it is sufficient to examine a single period model because once the prices become a continuum over one period sometime in the future, there will be a continuum of prices today. The connection between single-period price indeterminacy and multiple period price indeterminacy is thoroughly investigated by Epstein-Wang (1994), so we avoid repetition.

Formally, we summarize results of Epstein-Wang (1994) for the single agent case as Lemma 4.6.1:

Lemma 4.6.1: (Epstein-Wang:1994)

For a single-agent economy, under the following conditions, there is a continuum of equilibrium prices for asset i from τ=1 to τ=t:

(a) $Q_{t-1}(\omega^{t-1})$[32] has multiple elements of priors

(b) $p, p' \in Q_{t-1}(\omega^{t-1})$, $p(\omega^{t-1}, \omega_{t,s}) \neq p'(\omega^{t-1}, \omega_{t,s})$
and $p(\omega^{t-1}, \omega_{t,s'}) \neq p'(\omega^{t-1}, \omega_{t,s'})$

(c) $d_{i,t}(\omega^{t-1}, \omega_{t,s}) \neq d_{i,t}(\omega^{t-1}, \omega_{t,s'})$

This condition is very intuitive. If the multiple elements of priors achieve the same value, the left and right derivatives are different. This implies that Arrow-Debreu security prices at (ω^{t-1},$\omega_{t,s}$) and (ω^{t-1},$\omega_{t,s'}$) become indeterminate. The price that falls within this range

[32] Q(ω) is defined at Section 4.2.4.

essentially supports the equilibrium allocations. Clearly if asset i pays different dividends over these states, its price becomes the continuum. Note that this analysis critically depends on the *fixed* consumption process $\{e_t\}$. In addition, Condition (a) is necessary. We must have the multiple elements in the convex-compact set $\mathrm{Q}_{t-1}(\omega^{t-1})$, which is only possible when the multiple-priors set has a flat boundary somewhere. For the general multiple-priors set with a very smooth boundary, we cannot observe the continuum of equilibrium unless all aggregate endowments are identical over Ω.

4.6.2 Multiple Agents Economy

From the argument in the previous subsection, it is clear that for the general multiple-priors set, we cannot construct the universal conditions where the continuum of equilibria exist. Especially, it is hard to derive the conditions that satisfies Condition (a) in Lemma 4.6.1. Even though agents share the similar structure in their multiple-priors sets (similar flat boundary somewhere), the different individual wealth does not guarantee that all agents' consumptions confirm Condition (a) at equilibrium. The only situation where we are sure that there is the continuum of equilibria is that in which the aggregate endowments are identical over Ω.[33] However, this case corresponds to the riskless economy, and we are not interested in this case.[34]

On the contrary, as for the capacity-based multiple-priors set, because of the structure

[33] From the proof of Lemma 4.5.1, we know that if $e(\omega) = e(\omega')$, then $c^h(e(\omega)) = c^h(e(\omega'))$ for all agents.

[34] In fact, this is the case for the sunspot equilibria.

on the prior sets from Section 4.2.3, Condition (b) in Lemma 4.6.1 are equivalent to the following condition:

(b)′ $\quad e_t(\omega^{t-1}, \omega_{t,s}) = e_t(\omega^{t-1}, \omega_{t,s+i})$ $i=1,k$ where $k>1$

Now the condition is expressed by the endowment, and we hope that we can derive the condition on the endowment for the multiple agents economy to produce the continuum of equilibria.[35] Note that Condition (b)′ guarantees Condition (a) in Lemma 4.6.1.

First, from the argument for Lemma 4.4.1 and the proof of Lemma 4.5.1, under a single period economy, it is clear that for any α, we have even consumptions $c_2^h(e_2(\omega_{2,s}))$ $= c_2^h(e_2(\omega_{2,s+1}))$ over the states where $e_2(\omega_{2,s}) = e_2(\omega_{2,s+1})$. Note that we assume the strong order property (4.4.7) or (4.5.11) for other states. Now suppose that we fix these allocations, i.e., fix α. By changing the prior over the states where $e_2(\omega_{2,s}) = e_2(\omega_{2,s+1})$, we could potentially generate a continuum of equilibrium prices as in the single agent economy as long as F.O.C. of (4.2.18) holds.[36] However, there is a critical difference from the single agent economy. The Arrow-Debreu equilibrium is the combination of allocations and equilibrium prices. Here we fix the allocations. In general, for different prices, agents' endowments achieve different levels of wealth, which implies that the equilibrium allocations

[35] Epstein-Wang (1994) offers very heuristic justification for the existence of a continuum of equilibrium under multiple-agents case.

[36] From Section 4.2.3, we know that only the probability of consective states change. The similar construction is not possible for the general multple-priors set because the elements in $Q^{t-1}(\omega^{t-1})$ can have different probability over the states where the aggregate endowment is not identical.

will change. In order to investigate the possibility of a continuum of equilibria, we must show that there is an infinite combination of (c^h, q), which is virtually impossible to confirm for the general endowment structure. The only hope is that we can find the conditions where equilibrium allocations always stays the same so that we can apply the single agent argument. It is now clear that we need the conditions for the individual endowment to guarantee that changing the prices among states where $e_2(\omega_{2,s}) = e_2(\omega_{2,s+1})$ does not change the budget set or the set of feasible allocations. It is only possible when $e_2^h(\omega_{2,s}) = e_2^h(\omega_{2,s+1})$ $\forall h$. Under this endowment, now we need to confirm F.O.C.s in order to have the identical optimal consumptions and continuum of equilibrium prices. In fact, for the capacity-based multiple-priors set, either homogeneous or heterogeneous with THCB, we can confirm that both the constant wealth level and F.O.C.s and a continuum of equilibria exits for the economy of Proposition 4.4.1 and Proposition 4.5.1.[37] We summarize the results as Proposition 4.6.1.

Proposition 4.6.1:

For a multiple-agents economy of Proposition 4.4.1 with the identical capacity-based multiple-priors set or Proposition 4.5.1 with THCB,[38] under the following conditions, there is a continuum of equilibrium prices for asset i from τ=1 to τ=t:

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[37] The results hold for the economy of Corollary 4.4.1, 4.4.2, 4.4.3, 4.5.1, and 4.5.2 with capacity-based multiple-priors sets.

[38] For two-states case, the conditions for a continuum of equilibria is to have identical endowment over all Ω, which is a riskless economy.

(a) $Q_{t-1}^{h}(\omega^{t-1})^{39}$ has multiple elements of priors

(b)′ $e_t^h(\omega^{t-1}, \omega_{t,s}) = e_t^h(\omega^{t-1}, \omega_{t,s+i})\ i = 1, k\ where\ k \geq 1 \forall h$

(c) $d_{i,t}(\omega^{t-1}, \omega_{t,s}) \neq d_{i,t}(\omega^{t-1}, \omega_{t,,s+j})$ for some j s.t. 1$\leq$ j $\leq k$

Proof: **Appendix 4.M**

The only difference between Lemma 4.6.1 and Proposition 4.6.1 is (b)′, which requires the degenerate individual endowment (without uncertainty over two states). This condition indicates that the existence of the continuum of equilibrium prices seemingly trivial events if there are large number of agents.

4.7 Conclusion

We have constructed the conditions on the aggregate endowment process and the structure of uncertainty that result in all agents behaving as if they had the most pessimistic prior with respect to the aggregate endowment process regardless of their initial endowment. In other words, agents have the similar *bias* for their prior selection and consumption decision, and if they share the structure of uncertainty, the dynamic representative agent exists, where we can reduce the economy with uncertainty aversion to the one with the common subjective prior. Clearly our results are in line with Ozdenoren (2000). In addition, as opposed to Esptein-Wang (1994), we also show that the existence of continuum of equilibrium prices are non-generic.

[39] Q(ω) is defined at Section 4.2.4.

4.8 Extension

We have investigated the conditions where all agents behave homogeneously, and also gained insight into how uncertainty is shared among agents. However, we did not examine the difference between the heterogeneous subjective prior model and the heterogeneous multiple-priors model thoroughly. The natural extension of this work is to examine the case where agents' prior sets are not located around the center but share common elements. We expect that there is less possibility that all agents have similar consumptions. However, the intuition from the results in this paper suggests that the local risk aversion at $\overline{c}^h$ would reduce the dispersion or volatility of consumptions among agents. This question becomes clearer when we allow the situations where multiple-priors sets do not share common elements, which corresponds to the case of true generalization of the heterogeneous subjective prior case. Through further investigation of this comparison, we hope that the nature of aggregation of agents with uncertainty aversions becomes better understood. This comparison would also complement the results in this paper, and lead to the clear understanding of the impact of uncertainty aversion on asset price volatility. Finally, we would like to extend the analysis to the incomplete markets. Here, we would expect that we need conditions for individual endowments and asset structures in order for all agents share the similar bias.

Appendix 4.A: Representation Theorem of Uncertainty Aversion under Multiple-Periods Economy

Gilboa and Schmeidler (1989) axiomatize the notion of uncertainty aversion into the decision process with multiple-priors. In this model, an agent has a closed and convex set of priors instead of a single subjective prior, and the preference over acts are defined as the minimum of the expected utility among the given set of priors. Schmeidler (1989) also shows that the connection between the multiple-priors model and the non-additive prior model. In fact, if the set of priors coincide with the core of the capacity, then the expected utility defined by the Choquet integral is equivalent to the minimum of the expected utility among the priors in the core of the capacity. Here we just state the decision rule under the multiple-priors by Gilboa and Schmeidler (1989).

Theorem 4.A.1:

With C-independence[40] and uncertainty aversion[41], an agent behaves as if she/he had a set of probability measure that is closed and convex, and his preference over acts are determined by the minimax criterion:

f $\succeq$ g iff $mim_{p\in\mathfrak{P}} \int u(f)dp \geq mim_{p\in\mathfrak{P}} \int u(g)dp$

where P is a set of probability measure that is closed and convex.

[40] C-independence says that taking the convex combination between acts and a constant act does not change the order of preference over acts.

[41] For an act f and g, and $\alpha \in$ (0,1), $f \simeq g$ implies αf + (1-α)$g \geq f$.

Now, we extend this representation theorem into the multiple-periods setting. By appropriate adjustment of lottery space and C-independence axiom, their result is extended to the dynamic setting. Let $\mathfrak{L} = (L_1,L_2,...,L_T)$ be the lotteries over $\Omega^T = (\Omega \times \Omega \times ... \times \Omega)$ where elements of $\mathfrak{L}$ satisfy the measurability requirement for the evolution of state on $(\Omega \times \Omega \times ... \times \Omega)$. Define an act as correspondence from $\Omega^T \rightarrow \mathfrak{L}$. Let L_{cT} be the acts which have same sublotteries for all elements in L_t. Then Theorem 4.A.1 holds over the whole structure of economy:

Theorem 4.A.2:

In the dynamic case, with CT-independence[42] and uncertainty aversion, an agent behaves as if she/he had a set of probability measure that is closed and convex, and his preference over acts are determined by the minimax criterion:

f $\succeq$ g iff $\min_{p\in\mathfrak{P}} \int u(f)dp \geq min_{p\in\mathfrak{P}} \int u(g)dp$

where P is a set of probability measure that is closed and convex.

The set of priors are defined over whole histories, rather than each conditional distribution over states at each time. Here, the utility function is defined for a lottery, which is not necessarily time-separable.

For the construction of the set of multiple-priors modeled over a single-period, Gilboa-Schmeidler (1989) assume that there is the set of probability distributions at each state over X, where X has only finite elements. Instead of their formula, here we assume that

[42] CT-independence means that taking the convex combination between acts and L_{cT} does not alter the order of preference.

$X = R$ but there is only degenerated probability distributions over X. However, our structure guarantees the continuity of preference, and it suffices to show that Theorem 4.A.1 and Theorem 4.A.2 hold[43].

Now, we need to consider the issue of dynamic consistency. The first issue is how to update the multiple-priors set. As Eichberger and Kelsey (1996) address, under the formulation of Theorem 4.A.2, after the initial date, an agent's preference no longer confirms the multiple-priors model in general. In order to avoid this inconsistency, we directly impose the conditions that makes the final date's prior set closed and convex regardless of where we calculate it, i.e., the set of conditional distributions is closed and convex at any point of history. Under this condition, the agent becomes everywhere uncertainty averse, i.e., the agent has the dynamically consistent multiple-priors.

The second issue is the dynamic consistency of the optimal choice. Machina (1989) proposes that the notion of the dynamic consistency for non-expected utility models. He argues that an agent must incorporate the states which did not happen in the past and will not reach in the following history in order to evaluate the future lotteries. In this case, the original choice stays optimal. This notion essentially changes the dynamic problem to the static one. However, the interest of our study is precisely the dynamic behavior over time. Hence, we want to impose the stronger notion of dynamic consistency, i.e., the future utility does not depend on the past and unreachable future. This condition separates the decision over time, which is a standard notion of the dynamic consistency. For this purpose, we want

[43] The same argumene holds for the non-additive probability measure by Schmeidler (1989).

to impose the structure (4.2.2), time-separability of the utility function over consumptions.

Now we are ready to show that this prior structure (4.2.3) is the special case of the general formula (4.2.2).

Proposition 4.A.1:

(4.2.3) V(c)= $u_1(c_1)$ + $\inf_{m\in\mathfrak{P}_1(\omega^1)}$$E_1$[$u_2(c_2)$+$\inf_{m\in\mathfrak{P}_2(\omega^2)}$$E_2$[$u_3(c_3)$

+...$\inf_{m\in\mathfrak{P}_{T-1}(\omega^{T-1})}$$E_T$[$u_T(c_T)$]...]]

where $P_t(\omega^t)$ is a closed and convex set of priors at t for the history ω^t

confirms (4.2.2):

1. Probability distribution at the final date is closed and convex by multiplying each conditional prior probability from t to T.

2. Given the allocation, the set of optimal choice of priors will not change over time. In other words, the conditional distribution which is originally optimal at t=1 must be optimal at t>1. Hence, the backward induction defined by (4.2.4) is equivalent to (4.2.3).

Proof:

(a) *Closed and convex multiple prior set at T*

Let $p_t(\omega^t, \omega_{t+1})$ is the prior probability for ω_{t+1} over Ω at ω^t. From any $\tau = t$ to T-1, we can calculate the probability distribution from ω^t onward over $\omega^T \in \{(\omega^t, ..., \omega_T)\}$. Let

$p_T(\omega^t)$ be the probability of each history and $P_T(\omega^t)$ is the set of priors at the final date from ω^t. Then:

$$p_T(\omega^t) = p_t(\omega^t, \omega_{t+1}) p_{t+1}(\omega^{t+1}, \omega_{t+2}) ... p_{T-1}(\omega^{T-1}, \omega_T)$$

$$\widetilde{p}_T(\omega^t) = \widetilde{p}_t(\omega^t, \omega_{t+1}) \widetilde{p}_{t+1}(\omega^{t+1}, \omega_{t+2}) ... \widetilde{p}_{T-1}(\omega^{T-1}, \omega_T)$$

Now, take a convex combination of $p_T(\omega^t)$ and $\widetilde{p}_T(\omega^t)$

$$\alpha_t p_T(\omega^t) + (1\text{-}\alpha_t) \widetilde{p}_T(\omega^t)$$

$$= \{\alpha_t p_t(\omega^t, \omega_{t+1}) + (1\text{-}\alpha_t) \widetilde{p}_t(\omega^t, \omega_{t+1})\}$$

$$\cdot \{\alpha_{t+1} p_{t+1}(\omega^{t+1}, \omega_{t+2}) ... p_{T-1}(\omega^{T-1}, \omega_T) + (1\text{-}\alpha_{t+1}) \widetilde{p}_{t+1}(\omega^{t+1}, \omega_{t+2}) ... \widetilde{p}_{T-1}(\omega^{T-1}, \omega_T)\}$$

where $\alpha_{t+1} = \alpha_t p_t(\omega^t, \omega_{t+1}) / \{\alpha_t p_t(\omega^t, \omega_{t+1}) + (1\text{-}\alpha_t) \widetilde{p}_t(\omega^t, \omega_{t+1})\}$

Now repeat the same calculation:

$$\alpha_t p_T(\omega^t) + (1\text{-}\alpha_t) \widetilde{p}_T(\omega^t)$$

$$= \{\alpha_t p_t(\omega^t, \omega_{t+1}) + (1\text{-}\alpha_t) \widetilde{p}_t(\omega^t, \omega_{t+1})\} \{\alpha_{t+1} p_{t+1}(\omega^{t+1}, \omega_{t+2}) + (1\text{-}\alpha_{t+1}) \widetilde{p}_{t+1}(\omega^{t+1}, \omega_{t+2})\} \cdots$$

$$\cdot \{\alpha_{T-1} p_{T-1}(\omega^{T-1}, \omega_T) + (1\text{-}\alpha_{T-1}) \widetilde{p}_{T-1}(\omega^{T-1}, \omega_T)\}$$

Now, by assumption, $\alpha_\tau p_\tau(\omega^\tau, \omega_{\tau+1}) + (1\text{-}\alpha_\tau) \widetilde{p}_\tau(\omega^\tau, \omega_{\tau+1}) \in \mathfrak{P}_\tau(\omega^\tau, \omega_{\tau+1})$ $\forall \tau$ s.t. $t \leq \tau \leq T$. Hence $\alpha_t p_T(\omega^t) + (1\text{-}\alpha_t) \widetilde{p}_T(\omega^t) \in \mathfrak{P}_T(\omega^t)$, which implies $\mathfrak{P}_T(\omega^t)$ is convex. Since each $\mathfrak{P}_\tau(\omega^\tau, \omega_{\tau+1})$ is closed, the above calculation confirms that $\mathfrak{P}_T(\omega^t)$ follows the same property. In fact, it is clear that the probability distribution over any subtrees are closed and convex.

(b) *Invariance of the optimal selection of priors over time*

First fix the allocation. Let Q(x;ω^1) be the set of priors which gives the lowest value of (4.2.3). Suppose the prior is changed at ω^t s.t. t>1. Then it implies that at t>1, $\min_{\mathfrak{P}t(\omega^t)}[\mathrm{E}^{p'_t(\omega^t)}(\sum_t^T u(x_t))] < \mathrm{E}^{p_t(\omega^t)}(\sum_t^T u(x_t))$, where $p_t(\omega^t) \in$ Q(x;ω^t) is the optimal choice of prior at the beginning, and $p'_t \notin$ Q(x;ω^t) is the revised prior at t. Now, use the original prior from τ= 1 to t-1:

$$\mathrm{E}^{p_{1,t-1}}[\textstyle\sum_1^{t-1} u(x_t) + \mathrm{E}^{p'_t(\omega^t)}(\sum_t^T u(x'_t(\omega^t)))]]$$
$$< \mathrm{E}^{p_{1,t-1}}[\textstyle\sum_1^{t-1} u(x_t) + \mathrm{E}^{p_t(\omega^t)}(\sum_t^T u(x_t(\omega^t)))]]$$

The new selection of prior p'_t achieves smaller expected utility, which contradicts that p_t is optimal choice at the beginning. This result implies the equivalence of (4.2.3) and (4.2.4).■

Appendix 4.B: Dynamic Consistency of (4.2.9)

Let $\{x_t\}$ be the optimal consumption chosen at t =1 for τ=1 to τ=T by (4.2.9). Suppose that there is another feasible allocation $\{x'_t\}$ which has the identical evolution except one history after ω'^t, where it gives higher utility. In other words, $\min_{\mathfrak{P}t(\omega'^t)}[\mathrm{E}^{p'_t(\omega'^t)}(\sum_t^T u(x'_t(\omega'^t)))]$ $> \min_{\mathfrak{P}t(\omega'^t)}[\mathrm{E}^{p_t(\omega'^t)}(\sum_t^T u(x_t(\omega'^t)))]$, where $p_t(\omega'^t)$ is the optimal choice of prior for $\{x_t\}$ at the beginning, and $p'_t(\omega'^t)$ is the optimal prior for $\{x'_t\}$ at ω'^t. Assume that the agent revises her/his consumption process at ω'^t. Then using the identical consumption for other history,

$$\min_{P_{1,t-1}} \mathrm{E}^{p'_{1,t-1}}[\textstyle\sum_1^{t-1} u(x_t) + \mathrm{E}^{p'_t}(\sum_t^T u(x'_t(\omega^t)))]]$$
$$> \mathrm{E}^{p'_{1,t-1}}[\textstyle\sum_1^{t-1} u(x_t) + \mathrm{E}^{p_t}[\sum_{t=1}^T u(x_t(\omega^t))]]$$
$$\geq \min_{P_{1,t-1}} \mathrm{E}^{p_{1,t-1}}[\textstyle\sum_1^{t-1} u(x_t) + \mathrm{E}^{p_t}(\sum_t^T u(x_t(\omega^t)))]]$$

where $p'_{1,t-1}$ is the optimal choice of prior at the beginning for $\{x'_t\}$ given p'_t is fixed, where $p'_t(\omega^t) = p_t(\omega^t)$ for $\omega^t \neq \omega'^t$, and $p_{1,t-1}$ is the optimal choice of prior at the beginning for $\{x_t\}$ given p_t is fixed. The second inequality holds because of $\min_{Pt}[\mathrm{E}^{p'_t}(\sum_t^T u(x'_t(\omega'^t)))]$ $> \min_{Pt}[\mathrm{E}^{p_t}(\sum_t^T u(x_t(\omega'^t)))]$ and the equivalence of other evolution. By Appendix 4.A, we know that the utility process does not alter the prior selection over time, in other words, the optimal prior at t>1 is also optimal at t=1. Therefore, the above selection of prior is optimal for $\{x'_t\}$ and it gives higher utility at the beginning, which violates the optimality of $\{x_t\}$. By repeating the same construction, the above inequality holds for any $\{x'_t\}$ that gives higher utility at an arbitrary point. Hence allocations are dynamically consistent, ex-ante and ex-post efficient, and backward induction must work.■

Appendix 4.C: Proof of the Existence of an Arrow-Debreu Equilibrium under Uncertainty Aversion

It is sufficient to show that the preference relation is convex. This condition is satisfied if its upper contour set is convex. Define C=$\{y\in \mathfrak{X}_1 | y\succeq z\}$. Let x,$y\in C$. Then $\alpha \int u(x)d\mathfrak{P}$ + (1-α)$\int u(y)d\mathfrak{P} \leq \int[\alpha u(x) + (1-\alpha)u(y)]d\mathfrak{P} \leq \int u(\alpha x + (1-\alpha)y)d\mathfrak{P}$.

In addition, we also want to show that the optimal priors for any allocation is on the boundary of $\mathfrak{P}$. Suppose not. Then there is π s.t. $\pi \cdot 1 = 0$, and it assigns the number which has the opposite order to the allocation. Then if p is the prior the agent chooses, $p + \varepsilon\pi \in \mathfrak{P}$. So $\int u(x)dpa > \int u(x)d(\ p + \varepsilon\pi)$, which contradicts.■

Appendix 4.D: Proof of (4.2.15)

From Section 4.2.3, for the case of capacity-based multiple priors, we can rewrite the agent's optimization problems as follows:

Proposition 4.D.1.

In the capacity-based multiple-priors model, the agent selects the (t,ω^t)-optimal allocation $\{(c_t^h, \theta_t^h)\}$ by the following optimization:

$$\mathrm{Max}_{(c_m,\theta_m)_{m=1,M}} \mathrm{Max}_{(c,\theta)\in(c,\theta)_m} \mathrm{V}_t^h({}^t\mathrm{c};\omega^t) = \mathrm{u}_t^h(\mathrm{c}_t(\omega^t)) + \int \mathrm{V}_{t+1}^h({}^{t+1}\mathrm{c};\omega^t,\omega)\mathrm{d}P^h(\omega^t;\omega)$$

where among $(c, \theta)_m$, $\mathrm{V}_{t+1}^h(\mathrm{c};\omega^t,\omega)$ becomes comonotonic.

The solution of this optimization is "Max of the local Maxes".

Proof:

If the solution is interior, it is obvious because the prior is uniquely determined. We only need to show that a corner solution $\{c_t\}$ is optimal for any sub-optimization which includes this allocation in the feasible set. From Section 4.2.3, by the property of the Choquet integral, the prior probability change only on the states which have equal consumptions.

Hence, $\min_{\mathfrak{P}^t}[\mathrm{E}^{p_t}(\sum_t^T u(c_t))]$ obtains the identical value under any priors that correspond to the sub-optimization that includes $\{c_t\}$ in the feasible set. This result implies that at the corner solutions, we achieve the same solution among the sub-optimizations that includes $\{c_t\}$ in the feasible set, and this solution dominates others in every subdivision. Therefore, the optimal priors that justifies $\{c_t\}$ are multiple (in fact continuum from the argument in Section 4.2.4).■

Appendix 4.E: Proof of Lemma 4.1.1

Follow Aubin (1979: p.118):

Theorem 4.E.1:

(i) $\mathfrak{P}$ is compact

(ii) $\exists$ a neighborhood U of x s.t. for any $y \in U$:

$p \rightarrow f(y;p)$ is upper semi-continuous

(iii) $\forall p \in \mathfrak{P}$, $y \rightarrow f(y;p)$ is convex and differentiable from the right.

(iv) $\mathrm{g}(y)=\sup_p f(y;p)$

(v) $P_0 = \{p \in \mathfrak{P} |\ g(x)=f(x;p)\}$

Then

$Dg(x)(y)=\sup_p Df(x;p)(y)$

Here, our model satisfy (i)-(iii) by $f(y;p)=\int u(y)dp$ (general integral). By changing *sup* to *inf*, we can derive the right and left derivative as supergradients instead of subgradients by the right differentiability of u (in fact, u is differentiable):

$Dg(x)(y)=\inf_{P_0} Df(x;p)(y)$ (right) where $g(y)=\min_{P_0} \int u(y)dp$

$Dg(x)(y)=\sup_{P_0} Df(x;p)(y)$ (left) where $g(y)=\min_{P_0} \int u(y)dp$

Note that by changing the sign of y, we can use the right differentiability to derive the left derivative.

Appendix 4.F: Continuous States v.s. Discrete States

Epstein-Wang (1994) use very smooth evolution of endowments and sets of multiple-priors to avoid the potential discontinuity of V at the limit. In the second paper (1995), they show the existence of equilibrium of the general multiple-priors model under a continuous states and infinite horizon economy. In this paper, we avoid the continuous states and infinite-time horizon model because we want to allow more general evolution of endowments and prior sets and derive a clear intuition on the aggregated behavior of agents with multiple priors without considering the limit behavior of V. However, it is helpful to know the difficulty in the continuous states case, which gives us another intuition behind the multiple-priors model. As Bewley (1972) points, in the continuous states case, we need some smooth condition for preference to guarantee clear representation of price behavior. In the multiple-priors model, it turns out that this assumption is violated if the optimal choice of prior does not move continuously at the limit. In other words, the tail behavior of multiple-priors model is potentially very discontinuous. The following three points are clear distinctions between the model with a single prior and the model with multiple priors.

(a) *Non-measurability of utility process* $\{V_t({}^tc)\}$

First, we show for the finite state case, (4.2.2) $\{\ V_t({}^tc)=\inf_{m\in\mathfrak{P}} E[\sum_t^T u_\tau(c_\tau)]\ \}$ can be defined as minimum. Clearly, there are only finite $u_t(c_t(\omega^t))$ and $V_t({}^tc)$ is continuous with respect to m. By Weierstrass's Theorem, over compact $\mathfrak{P} \subset \Delta \subset R^{N^T}$, the minimum exists.

For the infinite state space, define the integral for each m_i: $x_i(\omega^t) = \int \sum_t^T u_\tau(c_\tau)dm_i$. If x_i converge in Cauchy, then minimum is defined as $\underline{x}$. This is possible if $m \in$ *ba(*Ω^{T-t+1}*)*[44], where *ba* is the space of finitely additive signed measure over Ω^{T-t+1} , i.e., the dual space of L^∞ , which is a complete normed vector space. However, here we only use the countably additive probability measure $\mathfrak{P}$, which is not a complete space. Now we can only calculate infimum from uncountable number of $x_i(\omega^t)$. Then the set A=$\{\omega^t | \inf\{x_i(\omega^t)\} < \alpha\}$=$\{\omega^t | \cup \{\ x_i(\omega^t)<\alpha\}\}$ is not necessarily Borel set because the intersection is uncountable.

(b) *Arrow Debreu equilibrium may not be supported as the dynamic equilibrium*

Let x be any allocation process over continuous states and infinite-time horizon. Then we can write:

$$v(\omega^t)\cdot x(\omega^t) = \int x(\omega^t)dv(\omega^t) = \sum_t^\infty \int x_\tau(\omega^{\tau-1},\omega_\tau)dv_\tau(\omega^{\tau-1},\omega_\tau)$$

where $v(\omega^t)\in$ $ba(\mathfrak{T}\times\ \Omega^\infty, \sum^{ad}), \sum^{ad}$ is σ algebra on $\mathfrak{T}\times\ \Omega^\infty$ generated by adapted processes, and $ba(\mathfrak{T}\times\ \Omega^\infty, \sum^{ad})$ is finitely additive signed measure over $\sum^{ad}$. Let $\omega^0 = \emptyset$

[44] The details of infinite comodities economy should be refered to Bewley (1972), Gilles (1989), Stokey-Lucus (1989).

and $\mathfrak{T}$ be the time set = {1,2,...}.[45]

Evolution of this process does not necessarily imply the dynamically consistent behavior, i.e., generally there is no $\exists\widetilde{v}_t(\omega^t)$ s.t. $v(\omega^t)\cdot x(\omega^t) = \int x(\omega^t)dv(\omega^t) = \int [\int x(\omega^t,\omega_{t+1})\, dv_{t+1}(\omega^t,\omega_{t+1})]\, d\widetilde{v}_t(\omega^t)$. In other words, the Fubini theorem does not hold because the monotone convergent theorem fails[46]. Note that an Arrow-Debreu equilibrium is the element v with consumption c s.t. $v\cdot c \geq v\cdot x \Rightarrow \mathrm{V}(c) \geq \mathrm{V}(x)$. However, from the above result, v does not necessarily support the dynamic consistency.

(c) *Conditions for the existence of risk neutral measure*

Now, we assume dynamic consistency. The following Epstein-Wang (1995), first define:

$\widehat{Q_t}(\omega^t) = \{\pi \in P^{ba} |\ V^*(c) = \sum_t^\infty \int u_\tau(c)d\pi_\tau$ where $V^*(c)$ is the optimal value$\}$

where P^{ba} is the $\sigma(ba, \mathfrak{D})$ closure of P in ba, c is the optimal consumption.

Then Epstein-Wang (1995) show that $\forall \pi \in \widehat{Q_t}(\omega^t)$, we can define $\pi' \in ba$ s.t. $d\pi' = u'(c)d\pi$. Now following Epstein-Wang (1994,1995), for some $\pi \in \widehat{Q_t}(\omega^t)$, the standard F.O.C. must hold for all assets. In other words, given dividend process $\{d_{i,t}\}$, the asset price $q_{i,t}$ $\forall i$:

$$q_{i,t} = \int \frac{u'(c(\omega^t,\omega_{t+1}))}{u'(c(\omega^t))}(q_{i,t+1}(\omega^t,\omega_{t+1}) + d_{i,t+1}(\omega^t,\omega_{t+1}))d\pi_t(\omega^t,\omega_{t+1})$$

[45] See Kandori (1988).

[46] Although the product measure of v can be defined in a usual way, the limit of integral is not identical to the integral of limit. As in Bewley (1972), v contains the purely finitely additive componets, which prevents the usage of the montone convergent theorem.

$$= \int (q_{i,t+1}(\omega^t,\omega_{t+1}) + d_{i,t+1}(\omega^t,\omega_{t+1}))d\pi_t'(\omega^t,\omega_{t+1})$$

$$q_{i,t}^* = \int (q_{i,t+1}^*(\omega^t,\omega_{t+1}) + d_{i,t+1}^*(\omega^t,\omega_{t+1}))d\pi_t''(\omega^t,\omega_{t+1})$$

where $d\pi_t' = \dfrac{u'(c(\omega^t,\omega_{t+1}))}{u'(c(\omega^t))}d\pi_t$, $d\pi_t'' = \dfrac{1}{q_{1,t}}d\pi_t'$, $q_{i,t}^* = \dfrac{q_{i,t}}{q_{1,t}}$,
$q_{i,t+1}^*(\omega^t,\omega_{t+1}) = \dfrac{q_{i,t+1}}{q_{1,t+1}+d_{1,t+1}}$, $d_{i,t+1}^*(\omega^t,\omega_{t+1}) = \dfrac{d_{i,t+1}}{q_{1,t+1}+d_{1,t+1}}$

In order for π_t'' to be *probability measure*, π_t must be countably additive measure instead of finitely additive signed measure. Epstein-Wang (1995) show that if P is continuous at certainty, the charge in π_t disappear, which implies that π_t'' will be probability measure.

P is continuous at certainty if $P(\mathrm{A}_n)\nearrow 1$ $\forall$An $\nearrow \Omega$[47]

Appendix 4.G: Proof of Proposition 4.3.2

For (4.3.6), at T-2, the expected utility is:

$$\begin{aligned}V_{T-2}({}^{T-2}e;\omega^{T-2}) &= u_{T-2}(e_{T-2}(\omega^{T-2})) + \int V_{T-1}({}^{T-1}e;\omega^{T-2},\omega_{T-1})\,dP(\omega^{T-2},\omega_{T-1})\\ &= u_{T-2}(e_{T-2}(\omega^{T-2})) + \int\{u_{T-1}(e_{T-1}(\omega^{T-2},\omega_{T-1}))\\ &+ \int u_T(e_T(\omega^{T-2},\omega_{T-1},\omega_T'))\,dP(\omega^{T-2},\omega_{T-1},\omega_T')\}\,dP(\omega^{T-2},\omega_{T-1})\end{aligned}$$

By assumption, at T-1, $\omega^{T-1} = (\omega^{T-2},\omega_{T-1})$, and the only difference among $\{\omega^{T-1}\}$ is the realization of ω_{T-1}. Now by (4.3.1), (4.3.2), (4.3.4) and (4.3.6), if $e_{T-1}(\omega^{T-2},\omega_{T-1})$ $> e_{T-1}(\omega^{T-2},\omega_{T-1}')$:

[47] $\pi = \phi + \psi$, where ψ is purely finitely additive. Then $\exists$ B_n decending s.t. $\psi(\Omega\backslash B_n) \to 0$, $\phi(B_n) \to 0$. In a different way, $\psi(\Omega\backslash A_n) = \psi(B_n) \nrightarrow 0$.

(4.G.1) $\int u_T(e_T(\omega^{T-2},\omega_{T-1},\omega'_T))\,dP(\omega^{T-2},\omega_{T-1},\omega'_T)$

$\geq \int u_T(e_T(\omega^{T-2},\omega'_{T-1},\omega'_T))\,dP(\omega^{T-2},\omega'_{T-1},\omega'_T)$

Since by (4.3.1), (4.3.2), and (4.3.4), the integral is defined by the identical prior for both sides of equations, the pointwise domination of endowments by (4.3.6) implies the above inequality.[48] Clearly, the above inequality is (4.3.3). Hence the utility process $\{V_{T-1}({}^{T-1}e;\omega^{T-2},\omega_{T-1})\}$ and the endowment process $\{e_{T-1}(\omega^{T-2},\omega_{T-1})\}$ becomes comonotonic over $\omega_{T-1}\in\Omega$ at ω^{T-2}, and the most pessimistic prior over $\{e_{T-1}(\omega^{T-2},\omega_{T-1})\}$ is chosen.

Now, at T-3, we can group $\{\omega^T\}$ and $\{\omega^{T-1}\}$ by the realization of ω_{T-2}. Then by (4.3.6), $\{e_T(\omega^{T-3},\omega_{T-2},\omega_{T-1},\omega_T)\}\geq\{e_T(\omega^{T-3},\omega'_{T-2},\omega_{T-1},\omega_T)\}$[49] and $\{e_{T-1}(\omega^{T-3},\omega_{T-2},\omega_{T-1})\}\geq\{e_{T-1}(\omega^{T-3},\omega'_{T-2},\omega_{T-1})\}$ if $e_{T-2}(\omega^{T-3},\omega_{T-2})>e_{T-2}(\omega^{T-3},\omega'_{T-2})$. Then by (4.3.1), (4.3.2), (4.3.6):

(4.G.2) $\int u_T(e_T(\omega^{T-3},\omega_{T-2},\omega_{T-1},\omega'_T))\,dP(\omega^{T-3},\omega_{T-2},\omega_{T-1},\omega'_T)$

$\geq \int u_T(e_T(\omega^{T-3},\omega'_{T-2},\omega_{T-1},\omega'_T))\,dP(\omega^{T-3},\omega'_{T-2},\omega_{T-1},\omega'_T)$

and

(4.G.3) $\int u_{T-1}(e_{T-1}(\omega^{T-3},\omega_{T-2},\omega'_{T-1}))\,dP(\omega^{T-3},\omega_{T-2},\omega'_{T-1})$

[48] Under time-state heterogeneous prior conditions (4.3.8) and (4.3.9), the above inequality is still sustained.

[49] $\{x(\omega)\}\geq\{y(\omega)\}$ means that $x(\omega)\geq y(\omega)\ \forall\omega$.

$$\geq \int u_{T-1}(e_{T-1}(\omega^{T-3}, \omega'_{T-2}, \omega'_{T-1}))dP(\omega^{T-3}, \omega'_{T-2}, \omega'_{T-1})^{50}.$$

Here by backward induction with the result for T-2, $P(\omega^{T-3}, \omega_{T-2}, \omega_{T-1})$ is the most pessimistic prior over $\{e_{T-1}(\omega^{T-3}, \omega_{T-2}, \omega_{T-1})\}$ s.t. $P(\omega^{T-3}, \omega_{T-2}, \omega_{T-1}) = P(\omega^{T-3}, \omega'_{T-2}, \omega_{T-1})$. In other words, From T-2 to T-1, we use the identical prior for integration. Given this prior, the above pointwise domination of $\{\omega'_{T-2}\}$ by $\{\omega_{T-2}\}$ implies:

$$\int\{u_{T-1}(e_{T-1}(\omega^{T-3}, \omega_{T-2}, \omega_{T-1}))$$
$$+ \int u_T(e_T(\omega^{T-3}, \omega_{T-2}, \omega_{T-1}, \omega'_T))dP(\omega^{T-3}, \omega_{T-2}, \omega_{T-1}, \omega'_T)\}dP(\omega^{T-3}, \omega_{T-2}, \omega_{T-1})$$
$$\geq \int u_{T-1}(e_{T-1}(\omega^{T-3}, \omega'_{T-2}, \omega_{T-1}))$$
$$+ \int u_T(e_T(\omega^{T-3}, \omega'_{T-2}, \omega_{T-1}, \omega'_T))dP(\omega^{T-3}, \omega'_{T-2}, \omega_{T-1}, \omega'_T)\}dP(\omega^{T-3}, \omega'_{T-2}, \omega_{T-1})^{51}$$

This inequality is $\mathrm{E}_{T-2}[\mathrm{V}_{T-1}({}^{T-1}e;\omega^{T-3},\omega_{T-2},\omega_{T-1})] \geq \mathrm{E}_{T-2}[\mathrm{V}_{T-1}({}^{T-1}e';\omega^{T-3},\omega'_{T-2},\omega_{T-1})]$, which implies (4.3.3). Applying the same argument for all t: T>t≥1, we verify (4.3.3).

(4.3.5) is the special case of (4.3.6), which makes all utility process constant over $\{\omega^t\}$[52].

(4.3.7) directly defines (4.G.1), (4.G.2), and (4.G.3)[53], and the same argument holds for all t. (4.3.3) is evident.■

[50] (4.G.2) and (4.G.3) holds under (4.3.8) and (4.3.9).

[51] This inequality holds under (4.3.8) and (4.3.9) because $\mathsf{P}(\omega^{T-3},\omega_{T-2},\omega_{T-1}) \subseteq \mathsf{P}(\omega^{T-3},\omega_{T-2},\omega_{T-1})$.

[52] Under (4.3.8) and (4.3.9), the utility process no longer constant over $\{\omega^t\}$.

[53] (4.G.2) and (4.G.3) hold under (4.3.8) and (4.3.9).

Appendix 4.H: Proof of Non-existence of the Dynamic Representative Agent under Different Subjective Priors

Assume that agents solve the optimization of (4.2.1) with non-identical subjective priors. There is a representative agent $V_1(e)$ at t=1 defined by (4.4.1).

(4.4.1) $$V_1(e)=\text{Max}_{(c^1,\ldots,c^H)} \sum \alpha^h V_1^h(c^h)$$

$$\text{s.t.} \quad \sum c^h = e$$

From Section 4.3, we utilize the similar construction of (4.4.3) by changing the utility weights at each ω^t:

(4.H.1) $$u_t(e_t(\omega^t)) = \text{Max}\ \{\sum \beta_t^h(\omega^t) u^h(c^h e_t(\omega^t)) : \sum c_t^h(e_t(\omega^t)) = e_t(\omega^t)\ \}$$

where $\beta^h = \alpha^h \times p_{t-1}^h(\omega^{t-1}, \omega_t)$. In other words, the utility weight is a multiple of the original utility weight α^h and the subjective probability of the state. This allocation dominates other allocations by a similar argument in Section 4.3, so they are optimal for given α. (The sum of the pointwise maxima of the fixed weight must be the maximum of the whole structure.) Now, rewrite (4.4.1) as (4.H.2):

(4.H.2) $$V_1(e)=\text{Max}_{(c^1,\ldots,c^H)}\{\sum \alpha^h u_1^h(c_1^h(\omega_1)) + \sum \alpha^h \int V_2^h(c_2^h(\omega_1,\omega))dP^h(\omega_1,\omega)\}$$

$$\text{s.t.} \quad \sum c_t^h = e_t\ \forall T \geq t \geq 1$$

Define $V_2(e)$:

(4.H.3) $$V_2(e)=\text{Max}_{(c^2,\ldots,c^H)}\{\sum \alpha^h u_2^h(c_2^h(\omega^2)) + \sum \alpha^h \int V_3^h(c_3^h(\omega^2,\omega)))dP^h(\omega^2,\omega)\}$$

$$= \sum \alpha^h V_2^h(c^h)$$

$$\text{s.t.} \quad \sum c_t^h = e_t \ \forall T \geq t \geq 2$$

Applying (4.H.1), we can obtain the allocations $\{\widetilde{c}_t\}_2^T$ from (4.H.3). However, allocations $\{\widetilde{c}_2\}$ does not deliver the optimal allocations $\{c_2\}$ of V_1 at $t = 2$ because $\{c_2\}$ need the probability weight, whereas $\{\widetilde{c}_2\}$ only use α^h. Hence, (4.4.1) cannot be written in the recursive formula, so it does not confirms the dynamic consistency.■

Appendix 4.I: Proof of Non-decreasing Function of $x^h(e)$.

From F.O.C. of (4.4.3), $\nabla u(e)$=$(\alpha^1 u'^1(c^1)$,...,$\alpha^H u'^H(c^H))//1$. Let $e(\omega_i)$>$e(\omega_j)$. Then $\exists$ h s.t. $c^h(e(\omega_i)) > c^h(e(\omega_j))$ and $u'^h(c^h(e(\omega_i))) \leq u'^h(c^h(e(\omega_j)))$, which implies $u'^{h'}(c^{h'}(e(\omega_i))) \leq u'^{h'}(c^{h'}(e(\omega_j))) \ \forall h'$. The only concern is the case of $u'^h(c^h(e(\omega_i))) = u'^h(c^h(e(\omega_j)))$. In this case, all agents with the strictly concave u^h have a constant $c^h(e(\omega_i))$ $= c^h(e(\omega_j))$, and among risk neutral agents (at least locally around $c^h(e(\omega_i))$ and $c^h(e(\omega_j))$), the solution $c^h(e(\omega_i))$ becomes indeterminate because infinite combinations of consumptions could deliver the same aggregate utility $\sum \alpha^h u^h(c^h(e))$.[54] However, it is always possible to make x s.t. $c^h(e(\omega_i)) \geq c^h(e(\omega_j))$ among them. By the same argument in Section 4.4.3, by (4.2.14), all allocations which is not comonotonic is strictly dominated by the non-decreasing allocations. Hence, we can only restrict our attention to the case of non-decreasing $c^h(e)$.

Appendix 4.J: Comonotonically Homogeneous Uncertainty Aversion

[54] For risk neutral agents (at least locally) with $\alpha^h = 1/k^h$, where u^h=a+$k^h x^h$, $\alpha^h u'^h(c^h) = x^h$.

First we define comonotonically homogeneous uncertainty aversion.

Definition 4.J.1:

Preference of acts follows *comonotonically homogeneous uncertainty aversion* (CHUA):

(a) $\succeq$ is represented by the multiple-priors model

(b) f $\approx$ g if $g(s)$ is the reorder of state lotteries of $f(s)$

Given this definition, we prove the following Proposition:

Proposition 4.J.1:

An agent has CHUA iff their multiple-priors set is symmetric, where the center of symmetry must be the center of the probability simplex.

Proof:

Sufficiency is obvious. We prove the necessity for two different cases in order to derive more intuitions for the capacity-based multiple-priors set:

(a) *Capacity-based multiple-priors set*

Step 1) *The center of probability simplex is in the multiple-priors set*

Suppose not. Then there is a state mim$p(s) =\ \upsilon(s) > \frac{1}{N}$ and mim$p(s') = \upsilon(s') < \frac{1}{N}$. Consider two acts: $f(s) \succ f(s')\ = f(s'')$ for $\forall s', s'' \neq$ s, $g(s)$=$f(s')$, $g(s')$=$f(s)$ and $g(s'')$ $= f(s'')$ for $\forall s'' \neq s, s'$. In other words, g is the reorder of f. Now clearly, $\int u \circ f dP >$ $\int u \circ g dP$, which violates the assumption.

Step 2) *Optimal prior probability and state lottery preference are oppositely comonotonic*

Suppose not. Take an act f where state lotteries have the strong order[55] and $f(s) \succ f(s+1)$, $p(s) > p(s+1)$. Now change the order of these two state lotteries. By the property of the Choquet integral in Section 4.2.3, only $p(s)$ and $p(s+1)$ are adjusted to $\widetilde{p}(s)$ and $\widetilde{p}(s+1)$. In order to have the identical expected utility for this new reordered act, apparently, $p(s) = \widetilde{p}(s+1)$ and $p(s+1) = \widetilde{p}(s)$. However, by using this new prior, the original act can have lower utility, which contradicts the optimal selection of p at the beginning.

Step 3) *Prior set is symmetric*

Suppose not. Then define N-1 step acts:

Act 1: $u(f(1)) > u(f(2))$, and $u(f(2)) = u(f(\mathrm{s}))$ where s=[2,N]

Act 2: $u(f(2)) > u(f(3))$, and $u(f(2)) = u(f(\mathrm{s}))$ where s=[1,2], $u(f(3)) = u(f(\mathrm{s}))$ where s=[3,N]

...

Act N-1: $u(f(\text{N-1})) > u(f(\mathrm{N}))$, and $u(f(\text{N-1})) = u(f(\mathrm{s}))$ where s=[1,N-1]

For ith step act, $\int u \circ f dP = u(f(1))\sum_1^i p(s) + u(f(i{+}1))\sum_{i+1}^N p(s)$. In order to match the value of this integral for any permutation of any step acts, p must follow the same permutation. Hence the prior set is symmetric at the center of probability simplex. (If the

[55] This act exists because of the non-degeneracy and continuity of f(s).

center of symmetry is not the center of probability simplex, we cannot have this permutation property.)

(b) General multiple prior case

Step 1) *The center of probability simplex is in the prior set*: Same as above

Step 2) *Optimal prior probability and state lottery preference are oppositely comonotonic*

Suppose not. $\exists$ act f s.t. $f(s) \succ f(s+1)$, $p(s) > p(s+1)$. Define act g which is just the reorder of these two state lotteries of act f. Then $\int u \circ f dp = \int u \circ g dp'$ where p is the optimal prior for f and p' is the optimal prior for g. By the definition of multiple-priors set:

$\int u \circ g dp \geq \int u \circ g dp' = \int u \circ f dp.$

$\int (u \circ g - u \circ f) dp \geq 0$

$(u(g(s)) - u(f(s)))p(s) + (u(g(s+1)) - u(f(s+1)))p(s+1) \geq 0$

$(u(f(s)) - u(f(s+1)))(p(s+1)$ - $p(s)\ ) \geq 0$ $(\because u(g(s)) = u(f(s')),\ u(g(s')) = u(f(s)))$

Hence $p(s+1) \geq p(s)$, which contradicts the assumption.

Step 3) *Prior set is symmetric*

Suppose not. $\exists$ act f with the optimal prior p. By Step 1 and Step 2, we can rewrite p $= \overline{p} + \widetilde{p}$ where $\overline{p}$ is the center of probability simplex, and $\widetilde{p}{\cdot}1 = 0$ and satisfies the property: if $f(s_{n(1)}) > \ldots > f(s_{n(N)})$ then $\widetilde{p}s_{n(1)} \leq \ldots \leq \widetilde{p}s_{n(N)}$. For the reordered act g of act f with optimal $p' = \overline{p} + \widetilde{p}'$. Let $\widetilde{p}_g$ be the permutation of $\widetilde{p}$ associated with the reorder.

Then, $\int u \circ f dp = \int u \circ g dp'$ implies $\widetilde{p}' = \widetilde{p}_g + l$ where $l \cdot (u \circ g) = 0$. By assumption (not symmetric), $p'' = \overline{p} + k\widetilde{p}_g \notin \mathfrak{P}$ if $k = 1$. Also if $k > 1$, then $\int u \circ f dp > \int u \circ g dp'$. Hence $k < 1$. Now by supporting hyperplane theorem, $\exists\ \alpha$ s.t. $\alpha \cdot p'' \leq \alpha \cdot p'''$ where $p''' \in \mathfrak{P}$. Since the affine transformation of the utility function does not change the representation of preference, we can take $\alpha = u \circ h$. Take reorder of h (opposite permutation of f to g), and call it h'. Then $\int u \circ h dp'' > \int u \circ h' dp$ (p is not necessarily the optimal prior for h'), which contradicts the assumption.■

Appendix 4.K: Proof of Lemma 4.5.1:

(a) *Translationally homogeneous capacity-based multiple-priors set* (4.5.8)

We define the agents' optimization problem by (4.2.15). The reader can easily verify by investigating its bordered Hessian[56] that F.O.C. is necessary and sufficient.

First, arrange the optimal consumptions of agent h by $c^h(\omega_{n^h(1)}) \geq ... \geq c^h(\omega_{n^h(N)})$. Now suppose that there is an agent h whose $c^h(\omega_{n^h(i)}) > c^h(\omega_{n^h(i+1)})$ at $e(\omega_{n^h(i)}) < e(\omega_{n^h(i+1)})$[57]. Then by optimality conditions with inequality constraints (Constraints are defined over the utility order: $... \geq u^h(c^h(\omega_{n^h(i)})) \geq u^h(c^h(\omega_{n^h(i+1)})) \geq ...$), and from Condition (4.5.8):

$$\frac{SP(\omega_{n^h(i)})}{SP(\omega_{n^h(i+1)})} = \frac{\overline{p} + \widetilde{p}^h_{n^h(i)} - \phi^h_{i-1}}{\overline{p} + \widetilde{p}^h_{n^h(i+1)} + \phi^h_i} \frac{u'^h(c^h(\omega_{n^h(i)}))}{u'^h(c^h(\omega_{n^h(i+1)}))} < 1$$

Where $p^h_{n^h(i)}$ stands for the agent h's prior probability of the state $\omega_{n^h(i)}$ if this state

[56] In fact, since the Hessian of Lagurangian is negative definite, the second order condition is satisfied.

[57] $e(\omega_{n^h(i)}) \leq e(\omega_{n^h(j+1)})$ does not change the conclusion.

utility is i-th position. $SP(\omega_{n^h(i)})$ is the equilibrium state price for the state $\omega_{n^h(i)}$, and ϕ^h_{i-1} is the Lagrange multiplier for the inequality constraint $u^h(c^h(\omega_{n^h(i-1)}))\geq u^h(c^h(\omega_{n^h(i)}))$.

Now by Condition (4.5.7), $\exists$ h' s.t. $c^{h'}(\omega_{n^h(i)}) < c^{h'}(\omega_{n^h(i+1)})$[58]. Arrange the optimal consumptions for this agents, $c^{h'}(\omega_{n^{h'}(1)}) \geq ... \geq c^{h'}(\omega_{n^{h'}(N)})$. Now, let $n^{h\prime}(k)=n^h(i)$, and $n^{h\prime}(m)=n^h(i+1)$. By assumption, $c^{h'}(\omega_{n^{h'}(m)}) > c^{h'}(\omega_{n^{h'}(k)})$. Rearrange the consumptions so that these consumptions locate as close as possible. Then, the order of consumption becomes: $...\geq c^{h'}(\omega_{n^{h'}(m)}) > .. > c^{h'}(\omega_{n^{h'}(k)}) \geq ...$ Hence, the optimality conditions and Condition (4.5.8) imply:

$$\frac{SP(\omega_{n^h(i)})}{SP(\omega_{n^h(i+1)})} = \frac{\overline{p}+\widetilde{p}^{h'}_{n^{h'}(k)}+\phi^{h'}_k}{\overline{p}+\widetilde{p}^{h'}_{n^{h'}(m)}-\phi^{h'}_{m-1}}\frac{u'^{h'}(c^{h'}(\omega_{n^{h'}(k)}))}{u'^{h'}(c^{h}(\omega_{n^{h'}(m)}))} > 1$$

This state price ratio does not match with the state price ratio of agent h, which contradicts the optimality. Hence, $\forall h$, $c^h(\omega_i) \geq c^h(\omega_j)$ if $e(\omega_i) > e(\omega_j)$, i.e., the consumption order is comonotonic to the order of the aggregate endowment $\forall h$. This consumption order can justify the selection of the most pessimistic prior over the aggregate endowment $\forall h$.

(b) *Comonotonically homogeneous uncertainty aversion* (4.5.9)

First, by Condition (4.5.7), arrange the aggregate consumption by $e(\omega_1) >...> e(\omega_N)$. Then $\forall i$ s.t. $1 \leq i <$ N, there is an agent h whose $c^h(\omega_i) > c^h(\omega_{i+1})$. Suppose the state price $SP(\omega_i)$ is higher than the state price $SP(\omega_{i+1})$, i.e., $SP(\omega_i) > SP(\omega_{i+1})$. Then by selling the goods at ω_i and buying the goods at ω_{i+1}, agent h can achieve the new allocation $\widetilde{c}^h(\omega_i) = c^h(\omega_{i+1})$, $\widetilde{c}^h(\omega_{i+1}) = c^h(\omega_i)$, and $\widetilde{c}^h(\omega_j) = c^h(\omega_j)$ for $\forall j \neq i, i+1$ with additional

[58] For the case of $e(\omega_{n^h(i)}) \leq e(\omega_{n^h(i+1)})$, by the assumption of $c^h(\omega_{n^h(i)}) > c^h(\omega_{n^h(i+1)})$, the same argument holds.

commodity left at ω_i. Now without commodity left at ω_i, the new allocation have the same utility as the original allocation because of the symmetry of prior set (4.5.9). Then by distributing additional commodities over all states while keeping the utility ratio constant, the new allocation has a higher utility than the original allocation, which contradicts the optimality of c^h. Hence $SP(\omega_i) \leq SP(\omega_{i+1})$. By repeating the same argument, state prices must be weakly oppositely comonotonic to the aggregate endowment:

$$e(\omega_1) > e(\omega_2) > ... > e(\omega_N) \Rightarrow SP(\omega_1) \leq SP(\omega_2) \leq ... \leq SP(\omega_N)$$

Now suppose that all state prices are identical. Then agents can make all consumptions identical, i.e., $\widetilde{c}^h(\omega_i) = \frac{1}{N}\sum_1^N c^h(\omega_j)\ \forall i$, and this allocation is in the budget set. By the strict concavity of utility function, the new allocation strictly dominates the optimal consumption for all h with any prior in $\mathfrak{P}^h$, so all agents follow the same procedures. However, by (4.5.7), this allocation does not clear markets. Since at equilibrium, markets must clear, agents do not choose the optimal allocation under the budget set at the beginning, which violates the optimality of the Arrow-Debreu equilibrium. So $\exists\ SP(\omega_i) < SP(\omega_{i+1})$. Now, suppose $\exists h'$ s.t. $c^{h'}(\omega_k) < c^{h'}(\omega_m)$ where $k \leq i$ and $m \geq i{+}1$. Then by selling the goods at ω_m and buying the goods at ω_k, agent h' can achieve the new allocation $\widetilde{c}^{h'}(\omega_k) = c^{h'}(\omega_m)$, $\widetilde{c}^{h'}(\omega_m) = c^{h'}(\omega_k)$, and $\widetilde{c}^{h'}(\omega_j) = c^{h'}(\omega_j)$ for $\forall j \neq k, m$ with additional commodity left at ω_m. Again by distributing additional commodities over all states while keeping the utility ratio constant, this new allocation has a higher utility because of the symmetry of prior set (4.5.9), which contradicts the optimality of $c^{h'}$. Hence, $c^h(\omega_k) \geq c^h(\omega_m)\ \forall h$ where $k \leq i$ and $m \geq i{+}1$.

Now suppose that there are r (r<N-1) strict inequalities for the state prices. Define $r(i)$ as the state where $SP(\omega_{r(i)}) < SP(\omega_{r(i)+1})$ and $r(i) < r(j)$ if i < j. Clearly from the above argument, $c^h(\omega_k) \geq c^h(\omega_m)$ $\forall h$ where $k \leq r(i)$ and $m \geq r(j)$ s.t. $r(i) < r(j)$, $SP(\omega_{r(i)}) < SP(\omega_{r(j)})$. Now assume that $\exists r(i)$ s.t. $SP(\omega_{r(i-1)+s}) = SP(\omega_{r(i)})$ for s=1 to $r(i) - r(i-1)$. Following the construction in the previous paragraph: $\tilde{c}^h(\omega_k) = \frac{1}{r(i)-r(i-1)} \sum_1^{r(i)-r(i-1)} c^h(\omega_{r(i-1)+s})$ for k s.t. $r(i\text{-}1)<k\leq r(i)$, and keeping $c^h(\omega)$ for other ω as it is. Clearly $\tilde{c}^h$ is the budget set. Then by strictly concavity of the utility function, the new allocation strictly dominates the optimal c^h $\forall h$ with any prior in $\mathfrak{P}^h$. However, this allocation does not clear markets, which implies that agents do not choose the optimal allocation under the budget set at the beginning. This violates the optimality of the Arrow-Debreu equilibrium. Hence, $\exists \omega_k$ s.t. $SP(\omega_k) < SP(\omega_{k+1})$ for $r(i\text{-}1)<k<r(i)$.

By repeating this argument, Condition (4.5.7) induces $SP(\omega_1) < SP(\omega_2) < ... < SP(\omega_N)$, and this implies $c^h(\omega_1) \geq c^h(\omega_2) \geq ... \geq c^h(\omega_N)$ $\forall h$. Hence the consumption order is comonotonic to the order of the aggregate endowment $\forall h$, and this consumption order can justify the selection of the most pessimistic prior over the aggregate endowment $\forall h$.

In case of $e(\omega_i) = e(\omega_{i+1})$, if we assume that $\exists h$ s.t. $c^h(\omega_i) > c^h(\omega_{i+1})$, by the argument above, $SP(\omega_i) \leq SP(\omega_j)$. However, we must have h s.t. $c^h(\omega_i) > c^h(\omega_{i+1})$, which implies $SP(\omega_i) = SP(\omega_j)$. Under these prices, $c^h(\omega_i) \neq c^h(\omega_{i+1})$ is not optimal, which contradicts the optimality of c^h. By the same reason, the assumption that $\exists h$ s.t. $c^h(\omega_i) < c^h(\omega_{i+1})$ contradicts the optimality of c^h. Hence, $c^h(\omega_i) = c^h(\omega_{i+1})$ $\forall h$. Now the relationship between

$SP(\omega_k)$ and $SP(\omega_{k+1})$ is not clear. However, from the argument above, that among the states where $e(\omega_i) > e(\omega_j)$, $SP(\omega_i) \leq SP(\omega_j)$. Keeping $c^h(\omega_i) = c^h(\omega_{i+1})$ $\forall h$ and repeat the above construction of dominating allocations, we conclude that $c^h(\omega_1) \geq c^h(\omega_2) \geq ...$ $\geq c^h(\omega_N)$ $\forall h$ with equality when $e(\omega_i) = e(\omega_j)$.

(c) *Two states with Nested multiple-priors sets* (4.5.10)

For two state case, we can rewrite the agent optimization problem as (4.2.15). Following the proof in (a), arrange the aggregate endowment by $e(\omega_1) > e(\omega_2)$. Now by Condition (4.5.10), all p^h can be written as: $p^h = \overline{p} + \widetilde{p}^h$ s.t. $\exists \overline{p} = (\overline{p}_1, \overline{p}_2) \in \cap \mathfrak{P}^h \in \mathfrak{P}^h$ $\forall h$ where $\widetilde{p}_1^h$ <0 and $\widetilde{p}_2^h >0$ when $c^h(\omega_1) > c^h(\omega_2)$, and $\widetilde{p}_1^h >0$ and $\widetilde{p}_2^h <0$ when $c^h(\omega_1) < c^h(\omega_2)$. Now suppose that there is an agent h whose $c^h(\omega_1) < c^h(\omega_2)$[59]. Then by optimality conditions with inequality constraints (Constraints are defined over the utility order: $u^h(c^h(\omega_1)) \geq$ $u^h(c^h(\omega_2))$:

$$\frac{SP(\omega_1)}{SP(\omega_2)} = \frac{\overline{p}_1 + \widetilde{p}_1^h}{\overline{p}_2 + \widetilde{p}_2^h} \frac{u'^h(c^h(\omega_1))}{u'^h(c^h(\omega_2))} \qquad \text{(Not binding)}$$

Now by Condition (5.3.1), $\exists$ h' s.t. $c^{h'}(\omega_1) > c^{h'}(\omega_2)$. The optimality conditions and Condition (4.5.10) imply:

$$\frac{SP(\omega_1)}{SP(\omega_2)} = \frac{\overline{p}_1 + \widetilde{p}_1^{h'}}{\overline{p}_2 + \widetilde{p}_2^{h'}} \frac{u'^{h'}(c^{h'}(\omega_1))}{u'^{h'}(c^{h'}(\omega_2))} \qquad \text{(Not binding)}$$

By the above construction, we know that $\frac{\overline{p}_1 + \widetilde{p}_1^h}{\overline{p}_2 + \widetilde{p}_2^h} > \frac{\overline{p}_1 + \widetilde{p}_1^{h'}}{\overline{p}_2 + \widetilde{p}_2^{h'}}$ and $\frac{u'^h(c^h(\omega_1))}{u'^h(c^h(\omega_2))} > \frac{u'^{h'}(c^{h'}(\omega_1))}{u'^{h'}(c^h(\omega_2))}$. Clearly, these state price ratios do not match each other, which contradicts. Therefore, $c^h(\omega_1) \geq c^h(\omega_2)$ $\forall h$, i.e. the consumption order is comonotonic

[59] $e(\omega_1) \leq e(\omega_2)$ does not change the conclusion.

to the order of the aggregate endowment $\forall h$. This consumption order can justify the selection of the most pessimistic prior over the aggregate endowment $\forall h$. ■

Appendix 4.L: Definition of ***more-uncertainty-averse-than*** relation

Definition 4.L.1:

Agent h' is more uncertainty averse than agent h if:

(a) $u^h = u^{h'}$

(b) $\mathfrak{P}^h \subset \mathfrak{P}^{h'}$

Proposition 4.L.1:

Under (a), (b) and the following condition are equivalent:

(c) $C^{h'}(f,u) < C^h(f,u)\ \forall f$

where $C^h(f,u)$ is the certainty equivalent of f for agent h.

Proof:

Suppose that (b) holds. For non-constant act f, $\int u \circ f dP^{h'} < \int u \circ f dP^h$, $\int u \circ \overline{g}^{h'} dP^{h'} = \int u \circ f dP^{h'}$, and $\int u \circ \overline{g}^h dP^h = \int u \circ f dP^h$, where $\overline{g}^h$ and $\overline{g}^{h'}$ are constant degenerated acts, and certainty equivalent of f for agent h and agent h'. Now suppose that (c) and $\neg$(b). Since $\mathfrak{P}^h$ and $\mathfrak{P}^{h'}$are closed and convex sets with $\mathfrak{P}^h \backslash \mathfrak{P}^{h'} \neq \phi$, by separating hyperplane theorem, $\exists f$ s.t. $\int u \circ f dP^{h'} > \int u \circ f dP^h$, which contradicts the assumption.■

Appendix 4.M: Strict Concavity of $\mathbf{G}^h_{t-1}(\mathbf{x}^h_t(\omega^{t-1},\cdot))$

$$\alpha G_{t-1}^h(x_t^h(\omega^{t-1},\cdot)) + (1\text{-}\alpha) G_{t-1}^h(\widetilde{x}_t^h(\omega^{t-1},\cdot))$$

$$= \int \alpha u_t^h(x_t^h(\omega^{t-1},\cdot)) dP^h(\omega^{t-1},\cdot) + \int (1-\alpha) u_t^h(\widetilde{x}_t^h(\omega^{t-1},\cdot)) d\widetilde{P}^h(\omega^{t-1},\cdot)$$

$$\leq \int [\alpha u_t^h(x_t^h(\omega^{t-1},\cdot)) + (1-\alpha) u_t^h(\widetilde{x}_t^h(\omega^{t-1},\cdot))]\ dP'^h(\omega^{t-1},\cdot)$$

$$\leq \int [\alpha u_t^h(x_t^h(\omega^{t-1},\cdot)) + (1-\alpha) u_t^h(\widetilde{x}_t^h(\omega^{t-1},\cdot))]\ dP''^h(\omega^{t-1},\cdot)$$

$$< \int u_t^h(\alpha x_t^h(\omega^{t-1},\cdot)+(1\text{-}\alpha)\widetilde{x}_t^h(\omega^{t-1},\cdot))\ dP''^h(\omega^{t-1},\cdot)$$

$$= G_{t-1}^h(\alpha x_t^h(\omega^{t-1},\cdot)+(1\text{-}\alpha)\widetilde{x}_t^h(\omega^{t-1},\cdot))$$

where $P(\omega^{t-1},\cdot)$, $\widetilde{P}(\omega^{t-1},\cdot)$, $P'(\omega^{t-1},\cdot)$, and $P''(\omega^{t-1},\cdot)$ are the optimal prior for $\{u_t^h(x_t^h(\omega^{t-1},\cdot))\}$, $\{u_t^h(\widetilde{x}_t^h(\omega^{t-1},\cdot))\}$, $\{[\alpha u_t^h(x_t^h(\omega^{t-1},\cdot)) + (1-\alpha) u_t^h(\widetilde{x}_t^h(\omega^{t-1},\cdot))]\}$, $\{u_t^h(\alpha x_t^h(\omega^{t-1},\cdot)+(1\text{-}\alpha)\widetilde{x}_t^h(\omega^{t-1},\cdot))\}$■

Appendix 4.N: Proof of Proposition 4.6.1

(a) *Identical capacity-based multiple-priors sets*

For the identical capacity-based multiple-priors sets, agents' optimization can be written as (4.2.15). Given the results from Lemma 4.4.1 we know that all agents have comonotonic consumptions to the aggregate endowment process. Now assume a single period economy with $e_2(\omega_{2,s}){=}e_2(\omega_{2,s+1})$. Other aggregate endowments confirm the strong order of Conditions (4.4.7). It is clear from the argument of Lemma 4.4.1 that $c_2^h(\omega_{2,s}) = c_2^h(\omega_{2,s+1})\ \forall h$. Suppose that all equilibrium allocations are fixed. From F.O.C. of (4.2.15):

$$\frac{SP(\omega_{2,s-1})}{SP(\omega_{2,s})} = \frac{p_{2,s-1}}{p_{2,s}+\phi_s^h}\frac{u'^h(c_2^h(\omega_{2,s-1}))}{u'^h(c_2^h(\omega_{2,s}))} = \frac{p_{2,s-1}}{p_{2,s}+\phi_s^{h'}}\frac{u'^{h'}(c_2^{h'}(\omega_{2,s-1}))}{u'^{h'}(c_2^{h'}(\omega_{2,s}))}$$

$$\frac{SP(\omega_{2,s})}{SP(\omega_{2,s+1})} = \frac{p_{2,s}+\phi_s^h}{p_{2,s+1}-\phi_s^h}\frac{u'^h(c_2^h(\omega_{2,s}))}{u'^h(c_2^h(\omega_{2,s+1}))} = \frac{p_{2,s}+\phi_s^{h'}}{p_{2,s+1}-\phi_s^{h'}}\frac{u'^{h'}(c_2^{h'}(\omega_{2,s}))}{u'^{h'}(c_2^{h'}(\omega_{2,s+1}))}$$
$$\frac{SP(\omega_{2,s+1})}{SP(\omega_{2,s+2})} = \frac{p_{2,s+1}-\phi_s^h}{p_{2,s+2}}\frac{u'^h(c_2^h(\omega_{2,s+1}))}{u'^h(c_2^h(\omega_{2,s+2}))} = \frac{p_{2,s+1}-\phi_s^{h'}}{p_{2,s+2}}\frac{u'^{h'}(c_2^{h'}(\omega_{2,s+1}))}{u'^{h'}(c_2^{h'}(\omega_{2,s+2}))}$$

where ϕ_s^h is the Lagrange multiplier from the constraints $u_2^h(c_2^h(\omega_{2,s})) \geq u_2^h(c_2^h(\omega_{2,s+1}))$. Note that from the second equation, $\frac{p_{2,s}+\phi_s^h}{p_{2,s+1}-\phi_s^h} = \frac{p_{2,s}+\phi_s^{h'}}{p_{2,s+1}-\phi_s^{h'}}$, which implies $\phi_s^h = \phi_s^{h'}$. In order to keep the wealth level constant,

$$\widetilde{SP}(\omega^1,\omega_{2,s})e_2^h(\omega^1,\omega_{2,s}) + \widetilde{SP}(\omega^1,\omega_{2,s+1})e_2^h(\omega^1,\omega_{2,s+1})$$
$$= SP(\omega^1,\omega_{2,s})e_2^h(\omega^1,\omega_{2,s}) + SP(\omega^1,\omega_{2,s+1})e_2^h(\omega^1,\omega_{2,s+1})$$

where SP is the original state price and $\widetilde{SP}$ is the new state price. Note that we keep other state prices unchanged. The above equality implies:

$$k(p_{2,s}+\phi_s^h)u'^h(c_2^h(\omega_{2,s}))e_1^h(\omega^1,\omega_{2,s}) + k(p_{2,s+1}-\phi_s^h)u'^h(c_2^h(\omega_{2,s+1}))e_2^h(\omega^1,\omega_{2,s+1})$$
$$= ku'^h(c_1^h(\omega_{2,s}))[(p_{2,s}+\phi_s^h)+(p_{2,s+1}-\phi_s^h)]e_2^h(\omega^1,\omega_{2,s})$$
$$= ku'^h(c_1^h(\omega_{2,s}))[p_{2,s}+p_{2,s+1}]e_2^h(\omega^1,\omega_{2,s})$$
$$= ku'^h(c_1^h(\omega_{2,s}))[(p_{2,s}+\widetilde{\phi_s^h})+(p_{2,s+1}-\widetilde{\phi_s^h})]e_2^h(\omega^1,\omega_{2,s})$$
$$= k(p_{2,s}+\widetilde{\phi_s^h})u'^h(c_2^h(\omega_{2,s}))e_2^h(\omega^1,\omega_{2,s}) + k(p_{2,s+1}-\widetilde{\phi_s^h})u'^h(c_2^h(\omega_{2,s+1}))e_2^h(\omega^1,\omega_{2,s+1})$$

where ϕ_s^h is the original Lagrange multiplier and $\widetilde{\phi_s^h}$ is the new one[60], k is a common factor which was cancelled by taking state price ratio. Clearly, the wealth level stays the same, and the original consumptions are feasible and satisfy F.O.C.s. Since ϕ_s^h, $\widetilde{\phi_s^h} \in [0,$

[60] In fact, from the second equation, we know that $\widetilde{\phi}_s^h = \widetilde{\phi}_s^{h'}$.

$|p_{2,s+1}-p_{2,s}|]$, we prove the existence of a continuum of equilibrium prices for the fixed optimal allocation $\{c_2^h(\omega_{2,s})\}$.

For the general case where $e_2(\omega_{2,s})$=$e_2(\omega_{2,s+j})$ for j=1 to J, from F.O.C. of (4.2.15):

$$\begin{aligned}\frac{SP(\omega_{2,s+k})}{SP(\omega_{2,s+k+1})} &= \frac{p_{2,s+k}-\phi^h_{s+k-1}+\phi^h_{s+k}}{p_{2,s+k+1}-\phi^h_{s+k}+\phi^h_{s+k+1}}\frac{u'^h(c_2^h(\omega_{2,s+k}))}{u'^h(c_2^h(\omega_{2,s+k+1}))}\\ &= \frac{p_{2,s+k}-\phi^{h'}_{s+k-1}+\phi^{h'}_{s+k}}{p_{2,s+k+1}-\phi^{h'}_{s+k}+\phi^{h'}_{s+k+1}}\frac{u'^{h'}(c_2^{h'}(\omega_{2,s+k}))}{u'^{h'}(c_2^{h'}(\omega_{2,s+k+1}))}\end{aligned}$$

Clearly $\sum_0^J(-\phi^h_{s+k-1}+\phi^h_{s+k})=0$ $(\because \phi^h_{s-1}=0, \phi^h_{s+J}=0)$. Now take $\phi^h_{s+k}=\phi^{h'}_{s+k}$[61]. Then the original consumptions are optimal for this selection, and since we can take $\phi^h_{s+k}\in[0, \max_{i,j}|p_{2,s+i}-p_{2,s+j}|]$, there is a continuum of equilibria for the fixes optimal allocations.

(b) *Heterogeneous capacity-based multiple-priors sets (THCB)*

For heterogeneous capacity-based multiple-priors sets, again from the argument in Section 4.6.2, we must consider two issues: constant wealth level and F.O.C.s. Clearly for the former condition, by the same reason in the identical capacity-based multiple-priors set, we need $e_2^h(\omega_{2,s})=e_2^h(\omega_{2,s+1})$ $\forall h$ for the states where state prices are going to change. $e_2(\omega_{2,s})$ $=e_2(\omega_{2,s+1})$ also implies $c_2^h(\omega_{2,s})=c_2^h(\omega_{2,s+1})$ $\forall h$. Now we need to investigate F.O.C.s. From (4.2.18):

$$\begin{aligned}\frac{SP(\omega_{2,s-1})}{SP(\omega_{2,s})} &= \frac{p^h_{2,s-1}}{p^h_{2,s}}\frac{u'^h(c_2^h(\omega_{2,s-1}))}{u'^h(c_2^h(\omega_{2,s}))} = \frac{p^{h'}_{2,s-1}}{p^{h'}_{2,s}}\frac{u'^{h'}(c_2^{h'}(\omega_{2,s-1}))}{u'^{h'}(c_2^{h'}(\omega_{2,s}))}\\ \frac{SP(\omega_{2,s})}{SP(\omega_{2,s+1})} &= \frac{p^h_{2,s}}{p^h_{2,s+1}}\frac{u'^h(c_2^h(\omega_{2,s}))}{u'^h(c_2^h(\omega_{2,s+1}))} = \frac{p^{h'}_{2,s}}{p^{h'}_{2,s+1}}\frac{u'^{h'}(c_2^{h'}(\omega_{2,s}))}{u'^{h'}(c_2^{h'}(\omega_{2,s+1}))}\\ \frac{SP(\omega_{2,s})}{SP(\omega_{2,s+1})} &= \frac{p^h_{2,s+1}}{p^h_{2,s+2}}\frac{u'^h(c_2^h(\omega_{2,s+1}))}{u'^h(c_2^h(\omega_{2,s+2}))} = \frac{p^{h'}_{2,s+1}}{p^{h'}_{2,s+2}}\frac{u'^{h'}(c_2^{h'}(\omega_{2,s+1}))}{u'^{h'}(c_2^{h'}(\omega_{2,s+2}))}\end{aligned}$$

[61] Agents can choose any prior as long as it clears the markets.

Fixed allocations and probability for other states. From the second equation:

$$\text{(4.N.1)} \qquad \frac{p^h_{2,s}}{p^h_{2,s+1}} = \frac{p^{h'}_{2,s}}{p^{h'}_{2,s+1}} \Rightarrow p^{h'}_{2,s} = p^h_{2,s}\frac{p^{h'}_{2,s+1}}{p^h_{2,s+1}} \text{ and } p^{h'}_{2,s+1} = p^h_{2,s+1}\frac{p^{h'}_{2,s}}{p^h_{2,s}}$$

Now move the probability between these states slightly.[62] Then the new state prices must satisfy the same F.O.C.s as above. Define $[\widetilde{p}^h_{2,s}, \widetilde{p}^h_{2,s+1}] = [p^h_{2,s} + \varepsilon^h, p^h_{2,s+1} - \varepsilon^h]$ be the new probability for agent h and $[\widetilde{p}^{h'}_{2,s}, \widetilde{p}^{h'}_{2,s+1}] = [p^{h'}_{2,s} + \varepsilon^{h'}, p^{h'}_{2,s+1} - \varepsilon^{h'}]$ be the new probability for agent h'. From the second equality,

$$\text{(4.N.2)} \qquad \frac{\widetilde{p}^h_{2,s}}{\widetilde{p}^h_{2,s+1}} = \frac{\widetilde{p}^{h'}_{2,s}}{\widetilde{p}^{h'}_{2,s+1}} \Rightarrow \widetilde{p}^{h'}_{2,s} = \widetilde{p}^h_{2,s}\frac{\widetilde{p}^{h'}_{2,s+1}}{\widetilde{p}^h_{2,s+1}} \text{ and } \widetilde{p}^{h'}_{2,s+1} = \widetilde{p}^h_{2,s+1}\frac{\widetilde{p}^{h'}_{2,s}}{\widetilde{p}^h_{2,s}}$$

Now in order for the first equation of F.O.C.s to hold, from (4.N.1) and (4.N.2):

$$\frac{\widetilde{p}^{h'}_{2,s+1}}{\widetilde{p}^h_{2,s+1}} = \frac{p^{h'}_{2,s+1}}{p^h_{2,s+1}} \Rightarrow \frac{p^{h'}_{2,s+1}}{p^h_{2,s+1}} = \frac{p^{h'}_{2,s+1} - \varepsilon^{h'}}{p^h_{2,s+1} - \varepsilon^h} \Rightarrow \varepsilon^{h'} = \varepsilon^h \frac{p^{h'}_{2,s+1}}{p^h_{2,s+1}}$$

Clearly this construction is possible for all other agents h'. Then in order for the third equation of F.O.C.s to hold, from (4.A.1) and (4.A.2):

$$\frac{\widetilde{p}^{h'}_{2,s}}{\widetilde{p}^h_{2,s}} = \frac{p^{h'}_{2,s}}{p^h_{2,s}} \Rightarrow \frac{p^{h'}_{2,s}}{p^h_{2,s}} = \frac{p^{h'}_{2,s} + \varepsilon^{h'}}{p^h_{2,s} + \varepsilon^h} \Rightarrow \varepsilon^{h'} = \varepsilon^h \frac{p^{h'}_{2,s}}{p^h_{2,s}}$$

Since $\dfrac{p^{h'}_{2,s+1}}{p^h_{2,s+1}} = \dfrac{p^{h'}_{2,s}}{p^h_{2,s}}$ from (4.N.1), the same probability which satisfies the first equation of F.O.C.s solves the third equation of F.O.C.s. Hence there is continuum of the new probability assignment which satisfy the original F.O.C.s. In addition, since $\widetilde{p}^h_{2,s} + \widetilde{p}^h_{2,s+1} = p^h_{2,s} + \varepsilon^h + p^h_{2,s+1} - \varepsilon^h = p^h_{2,s} + p^h_{2,s+1}$, the original consumptions are feasible and on the budget line. Hence we prove a continuum of equilibrium prices for the fixed allocations.

[62] Again, from Section 4.2.3, only the probabilities of consective states (identical consumptions) change.

For the general case where $e_2(\omega^1,\omega_{2,s})=e_2(\omega^1,\omega_{2,s+j})$ for j=1 to J. From F.O.C. of (4.2.18):

$$\frac{SP(\omega_{2,s})}{SP(\omega_{2,s+j})} = \frac{p^h_{2,s}}{p^h_{2,s+j}}\frac{u'^h(c^h_2(\omega_{2,s}))}{u'^h(c^h_2(\omega_{2,s+j}))} = \frac{p^{h'}_{2,s}}{p^{h'}_{2,s+j}}\frac{u'^{h'}(c^{h'}_2(\omega_{2,s}))}{u'^{h'}(c^{h'}_2(\omega_{2,s+j}))}$$

Clearly,

$$\text{(4.N.3)} \qquad \frac{p^h_{2,s}}{p^h_{2,s+j}} = \frac{p^{h'}_{2,s}}{p^{h'}_{2,s+j}} \Rightarrow p^{h'}_{2,s} = p^h_{2,s}\frac{p^{h'}_{2,s+j}}{p^h_{2,s+j}} \text{ and } p^{h'}_{2,s+j} = p^h_{2,s+j}\frac{p^{h'}_{2,s}}{p^h_{2,s}}$$

Fix the equilibrium allocations and the probabilities of the states where the aggregate endowments are not same. Define $[\widetilde{p}^h_{2,s}, \widetilde{p}^h_{2,s+j}] = [p^h_{2,s} + \varepsilon^h_s, p^h_{2,s+j} + \varepsilon^h_{s+j}]$ to be the new probability for agent h and $[\widetilde{p}^{h'}_{2,s}, \widetilde{p}^{h'}_{2,s+j}] = [p^{h'}_{2,s} + \varepsilon^{h'}_s, p^{h'}_{2,s+1} + \varepsilon^{h'}_{s+j}]$ to be the new probability for agent h'. By the same construction as (4.N.2):

$$\text{(4.N.4)} \qquad \frac{\widetilde{p}^h_{2,s}}{\widetilde{p}^h_{2,s+j}} = \frac{\widetilde{p}^{h'}_{2,s}}{\widetilde{p}^{h'}_{2,s+j}} \Rightarrow \widetilde{p}^{h'}_{2,s} = \widetilde{p}^h_{2,s}\frac{\widetilde{p}^{h'}_{2,s+j}}{\widetilde{p}^h_{2,s+j}} \text{ and } \widetilde{p}^{h'}_{2,s+j} = \widetilde{p}^h_{2,s+j}\frac{\widetilde{p}^{h'}_{2,s}}{\widetilde{p}^h_{2,s}}$$

From (4.N.3) and (4.N.4):

$$\frac{\widetilde{p}^{h'}_{2,s+j}}{\widetilde{p}^h_{2,s+j}} = \frac{p^{h'}_{2,s+j}}{p^h_{2,s+j}} \Rightarrow \frac{p^{h'}_{2,s+j}}{p^h_{2,s+j}} = \frac{p^{h'}_{2,s+j} - \varepsilon^{h'}_{s+j}}{p^h_{2,s+j} - \varepsilon^h_{s+j}} \Rightarrow \varepsilon^{h'}_{s+j} = \varepsilon^h_{s+j}\frac{p^{h'}_{2,s+j}}{p^h_{2,s+j}}$$

Combining all F.O.C.s, $\varepsilon^{h'} = \varepsilon^h \circledast \eta^{h\prime}$ where ε^h and $\varepsilon^{h'}$are the perturbations of the prior probabilities and $\circledast$ defines the element-wise multiplication. Clearly $\eta^{h\prime} = [\frac{p^{h'}_{2,s}}{p^h_{2,s}}, \frac{p^{h'}_{2,s+1}}{p^h_{2,s+1}}, ..., \frac{p^{h'}_{2,s+J}}{p^h_{2,s+J}}]$. Repeated application of (4.N.3), $\frac{p^{h'}_{2,s}}{p^h_{2,s}} = \frac{p^{h'}_{2,s+j}}{p^h_{2,s+j}}$ $\forall j = 1, J$ and $\forall h$. This result implies that $\varepsilon^{h'} = \varepsilon^h \times \frac{p^{h'}_{2,s}}{p^h_{2,s}}$, where $\varepsilon^h \cdot 1 = \varepsilon^{h'} \cdot 1 = 0$. Hence we can define the probability perturbations for each agent which satisfy the original F.O.C.s, and the original consumptions are feasible and on the budget line. Hence we prove a continuum of equilibrium prices for the fixed allocations. ■

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