An Alternative Axiomatization of Intertemporal Utility Smoothing

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October 2012

Abstract

We propose an alternative axiomatization of the model of intertemporal utility smoothing suggested by Wakai (2008) without introducing auxiliary consumption risk.

Keywords: discount factor, utility smoothing, recursive utility

JEL Classification Numbers: D80, D90

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1 Introduction

Intertemporal preferences are defined on the set of deterministic consumption sequences. Koopmans (1960) provides axiomatization for the most popular model, the discounted utility model, on this preference domain. On the other hand, alternative models of intertemporal preferences are often axiomatized by introducing consumption risk because the technique that is related to the expected utility allows us to derive the cardinal utility of a consumption sequence.[1] For example, Wakai (2008) proposes a model that captures the notion of intertemporal utility smoothing, a desire to lower volatility in a utility sequence, by adopting the Anscombe and Aumann (1963) framework with a temporal interpretation: preferences are defined on the set of sequences whose outcome at any period is a lottery defined over a consumption set. However, to derive the cardinal utility of a consumption sequence, Wakai (2008) imposes an unrealistic assumption that the decision maker (DM) consumes lotteries, not the realization of lotteries, at each period. Therefore, the axiomatization based on the Anscombe and Aumann (1963) framework is inconsistent with the sequential nature of the realization of intertemporal consumption risk.[2]

Given the above problem, the objective of this paper is to axiomatize Wakai's (2008) model of intertemporal utility smoothing on the set of deterministic consumption sequences without introducing auxiliary consumption risk. We achieve this goal by deriving a particular form of the aggregator function in the framework of Koopmans' (1960) recursive utility. In particular, we

[1] A well-known example is Epstein (1983), who derives a model of the discount factor that depends on historical consumption.

[2] To model a dislike of utility variations between adjacent periods, Gilboa (1989) and Shalev (1997) adopt the same domain as Wakai (2008).

adopt the method developed by Ghirardato and Marinacci (2001) and Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2003) who derive biseparable utility on a Savage (1954) domain. Our method also simplifies their axiomatic system by focusing on the key idea of intertemporal utility smoothing.

The remainder of the paper presents sets of axioms and representations. We also provide proofs that show the equivalence between these axioms and representations.

2 Representation

We consider an infinite-horizon, discrete-time model, where time varies over $\mathcal{T} = \{0, 1, 2, ...\} = \mathbb{N}$. The axiomatization exhibited below can be easily adapted to a finite-horizon, discrete-time model with a minor modification. The DM consumes a single perishable good at each period $t \in \mathbb{N} = \{0, 1, ...\}$ from a connected and compact set $X = [\underline{x}, \overline{x}] \subset \mathbb{R}_{++}$, where $\overline{x} > \underline{x}$. We denote a set of deterministic consumption sequences by

$$Y \equiv \{(c_0, c_1, ...) \in \mathbb{R}^{\infty} | c_t \in X \text{ for each } t \in \{0, 1, 2, ...\}\},$$

which is endowed with the product topology. Let $\langle c_t \rangle$ be a generic element of Y, where $\langle c_t \rangle = (c_0, c_1, ...)$. Let C be the set of all constant deterministic consumption sequences, where a generic element of C is denoted by $\langle c \rangle^* = (c, c, c, ...)$. The DM faces the same choice set Y at each time t, and the DM's preference ordering on Y, denoted by $\succeq$, is assumed to be complete, transitive, continuous, nondegenerate, and independent of time and the payoff history.

We first assume the following axioms that characterize the recursive utility.

Axiom 1 - Atemporal Preference (AP): *For all* $\langle c_t \rangle, \langle c'_t \rangle \in Y$ *and* $x, x' \in X$, $(x, \langle c_t \rangle) \succeq (x', \langle c_t \rangle)$ *if and only if* $(x, \langle c'_t \rangle) \succeq (x', \langle c'_t \rangle)$.

Axiom 2 - Stationarity (ST): *For all* $(x, \langle c_t \rangle)$ *and* $(x, \langle c'_t \rangle) \in X \times Y$, $(x, \langle c_t \rangle) \succeq (x, \langle c'_t \rangle)$ *if and only if* $\langle c_t \rangle \succeq \langle c'_t \rangle$.

Koopmans (1960) introduced these axioms, which are also a part of assumptions that characterize the discounted utility model (AP is the Postulate (3a) and ST is a combination of Postulate (3b) and Postulate 4). In particular, AP induces the ordering on X, which is independent of a continuation payoff $\langle c_t \rangle$: for $x, x' \in X$, $x \succeq x'$ if and only if $(x, \langle c_t \rangle) \succeq (x', \langle c_t \rangle)$, where $\langle c_t \rangle$ is any element in Y.[3] Furthermore, ST assumes that the passage of time does not alter the preference ordering, which induces a dynamically consistent decision process.

In terms of the relationship between the ordering on X and the ordering on Y, much of the literature assumes the following form of monotonicity.

Monotonicity: *For any* $\langle c_t \rangle, \langle c'_t \rangle \in Y$, *if* $c_t \succeq c'_t$ *for all* $t \in \mathbb{N}$, *then* $\langle c_t \rangle \succeq \langle c'_t \rangle$. *The latter ranking is strict if the former ranking is strict for some* $t \in \mathbb{N}$.

The following result can be easily derived so that we state it without a proof.

Lemma 1: *Given continuity, Axioms 1 and 2 imply monotonicity.*

As shown by Ghirardato and Marinacci (2001, Lemma 29), it follows from continuity and monotonicity that, for each $\langle c_t \rangle \in Y$, there exists $\langle x \rangle^* \in C$

[3] The literature regarding risk and uncertainty defines the induced ordering on the consumption set X as follows: for $x, y \in X$, $x \succeq y$ if and only if $\langle x \rangle^* \succeq \langle y \rangle^*$, where $\langle x \rangle^*$ and $\langle y \rangle^*$ are acts that pays x and y at every state, respectively. This definition lacks a behavioral foundation in an intertemporal setting because consuming x at each period is not identical to consuming x in a single period.

such that $\langle x \rangle^* \simeq \langle c_t \rangle$. We call this x a *constant equivalent of* $\langle c_t \rangle$ and refer to it as $ce(\langle c_t \rangle)$.

We aim to provide an axiomatic foundation for the following model of intertemporal utility smoothing proposed by Wakai (2008)

$$V(c_0, c_1, ...) = \min_{\delta \in [\underline{\delta}, \overline{\delta}]} \left\{ (1-\delta) u(c_0) + \delta V(c_1, c_2, ...) \right\}. \tag{1}$$

Representation (1) is a class of the recursive utility suggested by Koopmans (1960) and captures intertemporal utility smoothing via the following form of recursive gain/loss asymmetry: (i) current utility $u(c_0)$ becomes a reference point to evaluate future utility $V(c_1, c_2, ...)$, where $V(c_1, c_2, ...)$ is defined as the average utility of future periods, and (ii) the difference between future utility $V(c_1, c_2, ...)$ and current utility $u(c_0)$ defines a gain or a loss, and gains are discounted more than losses.

To model the recursive gain/loss asymmetry, we must first derive asymmetric weights for gains versus losses. Thus, we consider a *binary sequence* $\langle x : y \rangle$, which is a consumption sequence $\langle c_t \rangle \in Y$ such that $c_t = x \in X$ for $t = 0$ and $c_t = y \in X$ for $t \geq 1$. Let Y_b be the collection of all binary sequences, each element of which, as shown above, is either increasing or decreasing. Furthermore, for $\langle c_t \rangle, \langle c'_t \rangle \in Y_b$, the *mixture of* $\langle c_t \rangle$ *and* $\langle c'_t \rangle$ is the binary sequence in Y_b, denoted by $\langle \langle c_t \rangle : \langle c'_t \rangle \rangle$, such that, for each $\tau \in \mathbb{N}$, $\langle \langle c_t \rangle : \langle c'_t \rangle \rangle_\tau = ce(\langle c_\tau : c'_\tau \rangle)$. Thus, by monotonicity, for each $\tau \in \mathbb{N}$, $c_\tau \succeq \langle \langle c_t \rangle : \langle c'_t \rangle \rangle_\tau \succeq c'_\tau$ if $c_\tau \succeq c'_\tau$, and $c'_\tau \succeq \langle \langle c_t \rangle : \langle c'_t \rangle \rangle_\tau \succeq c_\tau$ if $c'_\tau \succeq c_\tau$. Moreover, we also state that $\langle c_t \rangle$ and $\langle c'_t \rangle$ are comonotonic if there are no $\tau, \tau' \in \mathbb{N}$ such that $c_\tau \succ c_{\tau'}$ and $c'_{\tau'} \succ c'_\tau$.

Recursive gain/loss asymmetry implies that comonotonic consumption sequences in Y_b are evaluated under the same decision weight. To capture this idea, we adopt the following version of the independence axiom on Y_b from Ghirardato and Marinacci (2001), which is suitably modified to fit our framework, where $\{x, y\} \succeq z$ stands for $x \succeq z$ and $y \succeq z$.

Axiom 3 - Comonotonic Independence for Binary Consumption Sequences (CI): *For all* $\langle x : y \rangle$, $\langle x' : y' \rangle$, $\langle x'' : y'' \rangle \in Y_b$ *that are pairwise comonotonic, if* $\{x, x'\} \succeq x''$ *and* $\{y, y'\} \succeq y''$ *(or* $x'' \succeq \{x, x'\}$ *and* $y'' \succeq \{y, y'\}$*), then* $\langle x : y \rangle \succeq \langle x' : y' \rangle$ *implies* $\langle \langle x : y \rangle : \langle x'' : y'' \rangle \rangle \succeq \langle \langle x' : y' \rangle : \langle x'' : y'' \rangle \rangle$ *and* $\langle \langle x'' : y'' \rangle : \langle x : y \rangle \rangle \succeq \langle \langle x'' : y'' \rangle : \langle x' : y' \rangle \rangle$.[4]

CI states that among the comonotonic consumption sequences in Y_b satisfying the stated condition, the mixture operation does not alter the preference ordering. The required condition is that the mixture must be taken with a dominated (or dominating) consumption sequence because such an operation guarantees that $\langle x : y \rangle$, $\langle x' : y' \rangle$, and a mixture of $\langle x : y \rangle$ or $\langle x' : y' \rangle$ with $\langle x'' : y'' \rangle$ are all pairwise comonotonic.

The above axiom leads to the following lemma.

Lemma 2: *Assume that* $\succeq$ *satisfies Axioms 1 and 2. Then the following statements are equivalent.*

(i) $\succeq$ *satisfies Axiom 3 on* Y_b.

(ii) *There exists a continuous and nontrivial function* $u : X \to \mathbb{R}$, *real numbers* $\underline{\delta}$ *and* $\overline{\delta}$ *satisfying* $0 < \underline{\delta}, \overline{\delta} < 1$ *such that* $\succeq$ *on* Y_b *is represented by* $F : Y_b \to \mathbb{R}$, *where*

$$F(\langle x : y \rangle) \equiv \begin{cases} (1-\overline{\delta})u(x) + \overline{\delta}u(y) & \text{if } x \succeq y \\ (1-\underline{\delta})u(x) + \underline{\delta}u(y) & \text{if } x \preceq y \end{cases}. \tag{2}$$

Moreover, $\underline{\delta}$ *and* $\overline{\delta}$ *are unique, and* u *is unique up to a positive affine transformation.*

[4] This is a simplified version of the Binary Comonotonic Act Independence axiom used in Ghirardato and Marinacci (2001).

Note that (2) does not define a relationship between $\underline{\delta}$ and $\overline{\delta}$.

Proof. Let $\Sigma = \{\varnothing, \{0\}, \mathbb{N}\backslash\{0\}, \mathbb{N}\}$. In Ghirardato and Marinacci (2001), a binary sequence $\langle x : y \rangle$ corresponds to the bet on A, and a mixture $\langle \langle c_t \rangle : \langle c'_t \rangle \rangle$ corresponds to the statewise A-mixture of $\langle c_t \rangle$ and $\langle c'_t \rangle$, where $A \in \{\{0\}, \mathbb{N}\backslash\{0\}\}$. Furthermore, because monotonicity holds on a strict ordering, all nonempty subsets in Σ satisfy their definitions of essential events. Then (2) follows from Theorem 11 of Ghirardato and Marinacci (2001), which is a class of the Choquet expected utility defined on Σ with a unique set function $\rho : \Sigma_\Phi \to [0, 1]$ satisfying $\rho(\varnothing) = 0, \rho(\mathbb{N}) = 1$, $\rho(\{0\}) = (1 - \overline{\delta})$, and $\rho(\mathbb{N}\backslash\{0\}) = \underline{\delta}$. ■

Now, given the cardinal utility defined by (2), for $x, y \in X$, consider $z \in X$ that satisfies

$$u(z) = \frac{1}{2}u(x) + \frac{1}{2}u(y). \tag{3}$$

The existence of such z is guaranteed because X is connected and u is continuous. Furthermore, Ghirardato et al. (2003) show that

$$\begin{aligned} E(x, y) &\equiv \{z' \in X \,|z' \simeq z, \text{ where } z \text{ satisfies (3)}\} \\ &= \{z' \in X \,|x \succeq z' \succeq y \text{ and } \langle x : y \rangle \simeq \langle ce(\langle x : z' \rangle) : ce(\langle z' : y \rangle) \rangle\}. \end{aligned}$$

Thus, we can elicit the equivalent class of z in X, denoted by $E(x, y)$, without referring to the utility function u. Denote by $\frac{1}{2}x \oplus \frac{1}{2}y \in X$ an element in $E(x, y)$, and for $\langle c_t \rangle, \langle c'_t \rangle \in Y_b$, define $\left\langle \frac{1}{2} \langle c_t \rangle \oplus \frac{1}{2} \langle c'_t \rangle \right\rangle \in Y_b$ by

$$\left\langle \frac{1}{2} \langle c_t \rangle \oplus \frac{1}{2} \langle c'_t \rangle \right\rangle_\tau \equiv \frac{1}{2} c_\tau \oplus \frac{1}{2} c'_\tau \text{ for all } \tau \in \mathbb{N}.$$

The next axiom assumes that the DM is averse to the volatility involved in utility sequences.

Axiom 4 - Time-Variability Aversion (TVA): *For any* $\langle c_t \rangle, \langle c'_t \rangle \in Y_b$, *if* $\langle c_t \rangle \simeq \langle c'_t \rangle$, *then* $\left\langle \frac{1}{2} \langle c_t \rangle \oplus \frac{1}{2} \langle c'_t \rangle \right\rangle \succeq \langle c_t \rangle$.

TVA makes an indifference curve convex in the utility domain, regardless of the functional form of u. Thus, TVA defines *utility* smoothing, which leads to $\underline{\delta} \leq \overline{\delta}$.

The following is a stationary and infinite-horizon version of Wakai's (2008) model of utility smoothing.

Proposition 1: *The following statements are equivalent.*

(i) $\succeq$ *satisfies Axioms 1 to 4 on* Y*.*

(ii) *There exists a continuous and nontrivial function* $u : X \to \mathbb{R}$*, a set* $[\underline{\delta}, \overline{\delta}] \subset \mathbb{R}$ *satisfying* $0 < \underline{\delta} \leq \overline{\delta} < 1$*, and a nonempty, weak*-closed, and convex set* $\mathbb{D}$*, each element of which,* $b \in \mathbb{D}$*, is a strictly positive discount function* $b : \mathbb{N} \to \mathbb{R}_{++}$*, satisfying* $\sum_{t=0}^{\infty} b_t = 1$ *such that:* $\succeq$ *on* Y *is represented by* $V : Y \to \mathbb{R}$*, where*

$$V(c_0, c_1, ...) \equiv \min_{b \in \mathbb{D}} \left\{ \sum_{t=0}^{\infty} b_t u(c_t) \right\} = \min_{\delta \in [\underline{\delta}, \overline{\delta}]} \left\{ (1-\delta) u(c_0) + \delta V(c_1, c_2, ...) \right\}. \quad (4)$$

Moreover, $\underline{\delta}$*,* $\overline{\delta}$*, and* $\mathbb{D}$ *are unique, and* u *is unique up to a positive affine transformation. Furthermore,* V *is continuous on* Y*, and* $\mathbb{D}$ *is recursively constructed from* $[\underline{\delta}, \overline{\delta}]$*, as shown in Wakai* (2008).

Proof. Necessity of the axioms is routine. The proof of sufficiency is divided into three steps.

(Step 1)**:** *The real numbers* $\underline{\delta}, \overline{\delta}$ *derived in Lemma 2 satisfy* $0 < \underline{\delta} \leq \overline{\delta} < 1$ *and* (2) *is rewritten as follows: for all* $\langle c_t \rangle \in Y_b$

$$F(\langle c_t \rangle) = \min_{\delta \in [\underline{\delta}, \overline{\delta}]} [(1-\delta) u(c_0) + \delta u(c_1)]. \quad (5)$$

Representation (2) is a class of the Choquet expected utility. Thus, given TA, the conclusion follows from Schmeidler (1989). □

Given (5), define $V : Y \to \mathbb{R}$ by

$$V(\langle c_t \rangle) \equiv u(ce(\langle c_t \rangle)). \tag{6}$$

Then, it is clear that $V(\langle c_t \rangle) = F(\langle c_t \rangle)$ for $\langle c_t \rangle \in Y_b$.

(Step 2)**:** *For all* $\langle c_t \rangle \in Y$,

$$V(\langle c_t \rangle) = \min_{\delta \in [\underline{\delta}, \bar{\delta}]} [(1-\delta)u(c_0) + \delta V((c_1, c_2, ...))]. \tag{7}$$

By ST, $\langle c_t \rangle \simeq \langle c_0 : ce((c_1, c_2, ...)) \rangle$. Then (5) implies that

$$V(\langle c_t \rangle) = F(\langle c_0 : ce((c_1, c_2, ...)) \rangle) = \min_{\delta \in [\underline{\delta}, \bar{\delta}]} [(1-\delta)u(c_0) + \delta u(ce(c_1, c_2, ...))].$$

The conclusion follows from (6). □

Let Σ be the $\sigma-$algebra that consists of all subsets of $\mathbb{N}$. Let $\mathbb{B}$ be a collection of all bounded and real-valued functions on $\mathbb{N}$, where we endow $\mathbb{B}$ with the sup norm. By construction, each element in $\mathbb{B}$ is Σ-measurable. Furthermore, the dual space of $\mathbb{B}$ is denoted by $\mathbb{B}^*$, on which we use the weak* topology. Given (5), by following Wakai (2008), we construct $\mathbb{D} \subset \mathbb{B}^*$ as follows: let $\{\delta_t\}_1^\infty$ be a sequence of single-period discount factors, where $\delta_t \in [\underline{\delta}, \bar{\delta}]$ for each $t \geq 1$. From $\{\delta_t\}_1^\infty$, define a sequence $\{\gamma_t\}_0^\infty$ by

$$\gamma_0 \equiv 1 \text{ and } \gamma_t \equiv \delta_t \gamma_{t-1} \text{ for } t > 0.$$

Construct $b \in \mathbb{B}^*$ from $\{\delta_t\}_1^\infty$ and $\{\gamma_t\}_0^\infty$ as follows:

$$b(a) \equiv \sum_{\tau=0}^{\infty} b_\tau a_\tau \text{ for each } a \in \mathbb{B}, \text{ where } b_s \equiv \gamma_s - \gamma_{s+1} \text{ for } s \geq 0. \tag{8}$$

Define a nonempty set $\mathbb{D} \subset \mathbb{B}^*$ by

$$\mathbb{D} \equiv \{b \in \mathbb{B}^* | b \text{ satisfies (8) for some admissible } \{\delta_t\}_1^\infty \text{ and } \{\gamma_t\}_0^\infty\}.$$

Each element of $\mathbb{D}$ is a strictly positive discount function $b : \mathbb{N} \to \mathbb{R}_{++}$ such that $\sum_{\tau=0}^{\infty} b_\tau = \sum_{\tau=0}^{\infty}(\gamma_\tau - \gamma_{\tau+1}) = \gamma_0 = 1$; b is also a discrete and countably additive weighting function on Σ. In addition, $\mathbb{D}$ is closed and compact in $\mathbb{B}^*$.

(Step 3)**:** *For all* $\langle c_t \rangle \in Y$,

$$V(\langle c_t \rangle) = \min_{b \in \mathbb{D}} \left\{ \sum_{t=0}^{\infty} b_t u(c_t) \right\}. \tag{9}$$

Moreover, V *is continuous on* Y.

First, for all $\langle c_t \rangle \in Y_b$, (7) implies (9). For $\langle c_t \rangle \in Y$, consider two sequences, $\{\langle \overline{c}_t^n \rangle\}_1^\infty$ and $\{\langle \underline{c}_t^n \rangle\}_1^\infty$, such that for each $n \geq 1$

$$\overline{c}_t^n = c_t \text{ for } t \leq n \text{ and } \overline{c}_t^n = x^u \text{ for } t > n, \text{ and}$$
$$\underline{c}_t^n = c_t \text{ for } t \leq n \text{ and } \underline{c}_t^n = x^l \text{ for } t > n,$$

where $x^u \in \arg\max_{x \in X} u(x)$ and $x^l \in \arg\min_{x \in X} u(x)$. By monotonicity, for each $n \geq 1$

$$V(\langle \overline{c}_t^n \rangle) \geq V(\langle c_t \rangle) \geq V(\langle \underline{c}_t^n \rangle). \tag{10}$$

Furthermore, it follows from monotonicity that $V(\langle \overline{c}_t^n \rangle)$ is weakly decreasing and $V(\langle \underline{c}_t^n \rangle)$ is weakly increasing. As both sequences are bounded, each sequence converges. Let $\overline{V}$ and $\underline{V}$ be the limits of $V(\langle \overline{c}_t^n \rangle)$ and $V(\langle \underline{c}_t^n \rangle)$, respectively. Moreover, the repeated application of (7) implies that, for each $n \geq 1$,

$$\begin{aligned} V(\langle \overline{c}_t^n \rangle) = \min_{b \in \mathbb{D}} \left\{ \sum_{t=0}^{\infty} b_t u(\overline{c}_t^n) \right\} &\geq \min_{b \in \mathbb{D}} \left\{ \sum_{t=0}^{\infty} b_t u(c_t) \right\} \\ &\geq \min_{b \in \mathbb{D}} \left\{ \sum_{t=0}^{\infty} b_t u(\underline{c}_t^n) \right\} = V(\langle \underline{c}_t^n \rangle). \end{aligned} \tag{11}$$

Now, we claim that $\overline{V} = \underline{V}$ so that (10) and (11) imply (9). For each $n \geq 1$, let $\underline{b}^n$ be an element in $\mathbb{D}$ such that

$$\underline{b}^n \in \arg\min_{b\in\mathbb{D}} \left\{ \sum_{t=0}^{\infty} b_t u(\underline{c}_t^n) \right\},$$

and let $\{\underline{\gamma}_t^n\}_0^\infty$ be the sequence that defines $\underline{b}^n$, as shown in (8). By construction, for any $\varepsilon > 0$, there exists $T > 0$ such that for each $t \geq T$

$$\underline{\gamma}_t^T \left| u(x^u) - u(x^l) \right| < \varepsilon. \tag{12}$$

Moreover, for each $n \geq 1$,

$$V(\langle \overline{c}_t^n \rangle) \leq \sum_{t=0}^{\infty} \underline{b}_t^n u(\overline{c}_t^n). \tag{13}$$

Then, given (12) and (13), (8) and (11) imply that, for any $\varepsilon > 0$, there exists $T > 0$ such that

$$\left| V(\langle \overline{c}_t^T \rangle) - V(\langle \underline{c}_t^T \rangle) \right| \leq \left| \sum_{t=0}^{\infty} \underline{b}_t^T u(\overline{c}_t^T) - V(\langle \underline{c}_t^T \rangle) \right| = \underline{\gamma}_{T+1}^T \left| u(x^u) - u(x^l) \right| < \varepsilon,$$

which proves the claim. Note that the boundedness of X is crucial for this conclusion.

Finally, because X is bounded and Y is adopted with the product topology, it can be shown by a standard $\varepsilon - \delta$ argument that V, defined by (9), is continuous on Y. ■

Acknowledgements

Financial support from the Japanese government in the form of research grant, Grant-in-Aid for Scientific Research (C) (23530219), is gratefully acknowledged.

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