Risk Non-Separability without Force of Habit*

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Summary

Internal-habit models entangle two effects: Preferences depends on the level of past consumption (i.e., history dependence); the implied ranking of future consumption risk depends on the level of current consumption (i.e., risk non-separability). This paper proposes, in an axiomatic framework, a utility function that captures risk non-separability without relying on habit formation. We then apply our utility function to a representative agent economy and observe a decline in the volatility of expected returns. This suggests that risk non-separability induces a stable motive for precautionary saving.

1 Introduction

1.1 Outline

The choice problem that involves allocation of resources over time and under risk has been studied based on preferences on temporal consumption lotteries. These lotteries are denoted by $d = (c_t, m(d_\omega))$, where c_t is the current deterministic consumption and $m(d_\omega)$ is the one-step-ahead future consumption risk with a random outcome d_ω. Under this structure, preferences on temporal consumption lotteries d induce the ordering on future consumption risk $m(d_\omega)$, which may well be dependent on the levels of current and past consumption. To address this concern, internal-habit models have been applied extensively. However, a close examination reveals that internal-habit models introduce two distinct features: First, preferences on temporal consumption lotteries d are dependent on the level of past consumption (i.e., *history dependence*). Second, the implied ranking of future consumption risk $m(d_\omega)$ is dependent on the level of current consumption c_t (i.e., *risk non-separability*).

History dependence is captured via a state variable, whereas risk non-separabilty is captured via a choice variable. Hence, it is important to isolate and identify the economic implications of risk non-separability. However, internal-habit models cannot disentangle risk non-separability from history dependence because the latter induces the former. This inability to separate these effects is aggravated by an additional problem: The definition of a habit level and the functional form of the felicity function that defines habit dependence are highly subjective.

Given the above limitation of internal-habit models, the objectives of this paper are threefold: First, we propose, in an axiomatic framework, a utility function that captures risk non-separability without relying on history dependence. We obtain this separation by developing a new form of recursive preference. Second, by adapting the CES aggregator function as proposed by Epstein and Zin [11], we further characterize risk non-separability. Third, to study asset pricing implications of risk non-separability, we apply our utility function to a representative agent economy and examine the mean and volatility of expected returns.

Formally, we adapt the temporal lottery setting as developed by Epstein and Zin [11] and provide axiomatization for the following representation (called *stochastic recursive utility*): The conditional utility of $d = (c_t, m(d_\omega))$ is expressed by the recursive form

$$V(d) \equiv E_m[U(c_t, V(d_\omega))], \tag{1}$$

where $U(c_t, V(d_\omega))$ is an aggregator function that expresses an attitude toward intertemporal substitutability; if $U(c_t, V(d_\omega))$ is time non-additive, (1) introduces risk non-separability. In addition, by treating the future consumption program as a unit, we maintain consistency of decisions over time, while by imposing the independence axiom on distributions m, we generalize the atemporal expected utility specification. However, since $U(c_t, V(d_\omega))$ may not be time additive, our representation includes a larger class of preferences that do not satisfy the atemporal expected utility theory.

To understand the characteristics of risk non-separability, we must compare (1) with the recursive form of a utility function that does not introduce risk non-separability. We consider the following representation (called *recursive utility*) as proposed by Kreps and Porteus [18] and Epstein and Zin [11]: The conditional utility of $d = (c_t, m(d_\omega))$ is expressed by

$$V(d) \equiv U(c_t, E_m[V(d_\omega)]). \tag{2}$$

In this representation, a decision maker (called DM) first finds a certainty equivalent of future risk (i.e., $E_m[V(d_\omega)]$); then the DM considers intertemporal substitution between current consumption and the certainty equivalent. Thus, the ranking of future consumption risk $m(d_\omega)$ is independent of the level of current consumption (called *risk separability*). On the contrary, (1) reverses the order of the aggregation by considering intertemporal substitution before risk aversion (i.e., expected utility). This path-by-path application of intertemporal substitution is the defining property of risk non-separability.

To investigate this concept further, we adopt a version of the CES aggregator function as proposed by Epstein and Zin [11], i.e.,

$$U(c_t, \gamma) = \frac{1}{\alpha} \left\{ c_t^\rho + \beta(\alpha\gamma)^{\frac{\rho}{\alpha}} \right\}^{\frac{\alpha}{\rho}}.$$

The advantage of this aggregator function is that in both (1) and (2), elasticity of intertemporal substitution between current consumption and future utility is expressed by $(1-\rho)^{-1}$, whereas risk aversion is expressed by $1-\alpha$. Thus, any difference between (1) and (2) is the result of risk non-separability.

The CES aggregator function also induces the comparison of *temporal* risk preference in Yaari's [25] sense: The DM is *more* (*less*) *temporal risk averse* under (1) than under (2) if $\alpha > (<)\ \rho$. We expect that the difference in temporal risk aversion affects asset pricing behavior. However, our primary interest is to identify a property that is independent of the attitude toward temporal risk. If such a property exists, it is a distinct characteristic of risk non-separability.

To achieve this objective, we apply our utility function to a version of the Lucas [19] economy. We conduct simulation studies under the consumption process compatible with that of the US economy adapted from Mehra and Prescott [20]. First, we observe that a difference in temporal risk aversion generates a pattern in long-run average expected returns: When the representative agent is less temporal risk averse under (1) than under (2) (i.e., $\alpha < \rho$), (1) generates a lower risk premium and a higher risk-free rate than (2), and vice versa. However, we do not observe any other distinct characteristics of risk non-separability in the long-run average.

On the other hand, we observe distinct a characteristic in the long-run volatility of expected returns: When the level of intertemporal substitution is sufficiently high (i.e., $\rho < 0$), regardless of the attitude toward temporal risk, expected returns generated by our model are less volatile than those generated by Epstein and Zin's [11]. Intuitively, under risk non-separability, *expected* intertemporal substitution determines asset prices. Since an expectation smooths the effect of intertemporal substitution, risk non-separability generates a *stable* motive for precautionary saving, which is translated into low volatility in expected returns.

The paper proceeds as follows: Section 1.2 reviews the related literature. Section 2 derives the representation (1). Section 3 shows the existence of the utility function under the CES aggregator function. Section 4 examines asset pricing implications of risk non-separability. Section 5 concludes the paper.

1.2 Related Literature

As for the order of aggregation between intertemporal substitutability and risk aversion adapted in (1), Klibanoff [14] considers preferences defined over a state space, where an outcome on each state is a lottery over pairs of current consumption and a menu of future acts. He applies Kreps and Porteus's [18] recursive utility on outcomes of each state, then applies Gilboa and Schmeidler's [13] multiple-priors model over states. Although he does not investigate risk non-separability, this effectively results in an order of aggregation similar to that used in (1). To extend the notion of gain/loss asymmetry (i.e., gains are discounted more than losses) to a stochastic setting, Wakai [23] also adapts the order of aggregation used in (1). Wakai's [23] analysis is based on a non-differentiable aggregator function and emphasizes an interpretation of time-and-state-dependent discount factors. On the other hand, based on a differentiable aggregator function, this paper focuses on deriving asset pricing implications of risk non-separability. Moreover, in terms of the domain of preferences, Wakai [23] employs the temporal version of the Anscombe-Aumann [1] framework as developed by Epstein and Schneider [10], whereas this paper employs the temporal lottery framework as developed by Epstein and Zin [11].

Our utility function (1) represents preferences on *late resolution of uncertainty*, where the DM only faces a situation under which today's consumption is deterministic and one-step-ahead risk is resolved tomorrow. This setting captures the nature of the resolution of uncertainty underlying models used in the macro and finance literature. On the other hand, Kreps and Porteus [18] introduce recursive utility as a model that expresses an attitude toward the timing of the resolution of uncertainty.[1] For this reason, the domain of preferences also includes *early resolution of uncertainty* (i.e., a lottery under which uncertainty about future consumption can be resolved today with a given probability). In the same setting, Chew and Epstein [4,5] investigate the connection between the attitude toward temporal resolution and non-expected utility models. Although stochastic recursive utility can be defined on the early resolution of uncertainty as well (see Section 2), it does not offer a comparative

[1] A subjective version of Kreps and Porteus [18] is axiomatized by Skiadas [22] and Klibanoff and Ozdenoren [15].

analysis of the attitude toward temporal resolution. Instead, by restricting attention to late resolution of uncertainty, our model differentiates the attitude toward temporal risk between (1) and (2).

In terms of the attitude toward risk, Chew and Epstein [4,5] and Epstein and Zin [11] show that recursive utility admits broader classes of preferences (for example, *betweenness* as developed by Chew [3] and Dekel [9]). On the contrary, our model admits only the expected utility specification (at least, in the current axiomatic framework).

Epstein and Zin's [11] recursive utility has been applied to the finance literature. For example, Epstein and Zin [12] perform an empirical test of asset returns implied by recursive utility, whereas Weil [24] and Kocherlakota [16] investigate asset returns based on a simulation-based estimation. This paper follows Weil and Kocherlakota and provides simulation-based comparisons of asset pricing implications.

As Constantinides [6] shows, internal-habit models can produce a large equity premium if the felicity function depends on surplus consumption (i.e., current consumption minus the current level of habit). However, since the surplus consumption fluctuates as the consumption process evolves, it tends to generate a volatile risk-free rate process, which is inconsistent with the data. To overcome this difficulty, Campbell and Cochrane [7] introduce an external habit model that explicitly controls the level of precautionary saving. They set parameters so that the precautionary saving effect exactly offsets the volatility of the marginal utility of current consumption; this makes the risk-free rate constant. On the contrary, our utility function (1) generates a stable risk-free rate process without introducing external habit formation. In particular, as opposed to Campbell and Cochrane [7], (1) does not impose any restrictions on the behavior of past consumption. However, similar to Epstein and Zin [11], our model requires a significant level of risk aversion to resolve the equity-premium puzzle. Another limitation is that it allows only single-period dependence because the implied ranking of future consumption risk depends on current consumption but on nothing beyond.

2 Representation

Formally, we follow Epstein and Zin [11] with a slight modification of the consumption space. Time is discrete and its horizon is infinite, denoted by $\mathcal{T} = \{0, 1, ...\}$. For a metric space X, $\mathbb{B}(X)$ denotes the Borel σ-algebra and $M(X)$ denotes a space of Borel probability measures on X endowed with the weak convergence topology; $M^S(X) \subseteq M(X)$ is a space of Borel probability measures of finite support, where S is the maximum number of supports. We also denote m and m^S as elements in $M(X)$ and $M^S(X)$, respectively; when we want to emphasize the support of m, we write m as $m(x_\omega)$. In particular, the probability measure which assigns unit mass to $\{x_\omega\}$ is denoted by $m^1(x_\omega)$.

Let $\bar{b}$ and $\underline{b}$ be given, where $\bar{b} \geq 1 \geq \underline{b} > 0$. For any $\bar{l} > 0$ and $\underline{l} > 0$ such that $\bar{l} > \underline{l} > 0$, define $X(\bar{b}^t; \bar{l}; \underline{b}^t; \underline{l})$ by

$$X(\bar{b}^t; \bar{l}; \underline{b}^t; \underline{l}) \equiv \{x \in \mathbb{R}_+ | \sup \frac{x}{\bar{b}^t} \leq \bar{l} \text{ and } \inf \frac{x}{\underline{b}^t} \geq \underline{l}\}.$$

Then denote a set of deterministic consumption sequences by

$$Y(\bar{b}; \bar{l}; \underline{b}; \underline{l}) \equiv \{y = (c_0, c_1, ...) \in \mathbb{R}_+^\infty | c_t \in X(\bar{b}^t; \bar{l}; \underline{b}^t; \underline{l})\},$$

where $\bar{b}$ and $\underline{b}$ bound growth rates of consumption; $\bar{l}$ and $\underline{l}$ bound consumption levels.[2]

[2] The enumeration of c_t does not correspond to an actual time in the economy; it is an enumeration of a sequence.

We first construct a space of temporal consumption lotteries whose support lies in $Y(\bar{b};\bar{l};\underline{b};\underline{l})$, which will be denoted by $D(\bar{b};\bar{l};\underline{b};\underline{l})$. Let $D_0(\bar{b};\bar{l};\underline{b};\underline{l}) \equiv Y(\bar{b};\bar{l};\underline{b};\underline{l})$ be a space of deterministic consumption sequences, and let $D_t(\bar{b};\bar{l};\underline{b};\underline{l}) \equiv X(\bar{b};\bar{l};\underline{b};\underline{l}) \times M(D_{t-1}(\bar{b};\overline{lb};\underline{b};\underline{lb}))$ be a space of temporal consumption lotteries under which all uncertainty is resolved at or before time t. Then, $D(\bar{b};\bar{l};\underline{b};\underline{l})$ is defined by

$$D(\bar{b};\bar{l};\underline{b};\underline{l}) \equiv \{d = (d_1, d_2, ...) | d_t \in D_t(\bar{b};\bar{l};\underline{b};\underline{l}) \text{ and } d_t = g_t(d_{t+1}) \text{ for all } t \geq 1\},$$

where $g_t(d_{t+1})$ defines the temporal consumption lottery that induces the same uncertainty on $Y(\bar{b};\bar{l};\underline{b};\underline{l})$ as does d_{t+1}, but the uncertainty is resolved one period earlier. Each d_t is embedded into $D(\bar{b};\bar{l};\underline{b};\underline{l})$ by

$$r : d_t \longrightarrow (d_1, d_2, ..., d_{t-1}, d_t, d_t, ...).$$

$D(\bar{b};\bar{l};\underline{b};\underline{l})$ also includes a temporal consumption lottery whose uncertainty is resolved asymptotically (i.e., d such that $d_t \neq d_{t+1}$ for all $t \in \mathcal{T}$).

The domain of a preference relation (i.e., the consumption space) is defined by

$$D(\bar{b};\underline{b}) \equiv \cup_{\bar{l}>0} \cup_{\bar{l}>\underline{l}>0} D(\bar{b};\bar{l};\underline{b};\underline{l}).$$

Each element of this subspace is called a *consumption program* as well as a temporal consumption lottery. $D(\bar{b};\bar{l};\underline{b};\underline{l})$ and $D(\bar{b};\underline{b})$ are connected and separable under a suitable metric; $D(\bar{b};\bar{l};\underline{b};\underline{l})$ is also compact. We denote a collection of probability measures whose support lies in $D(\bar{b};\bar{l};\underline{b};\underline{l})$ for some $\bar{l} > \underline{l} > 0$ by

$$\widehat{M}(D(\bar{b};\underline{b})) \equiv \cup_{\bar{l}>0} \cup_{\bar{l}>\underline{l}>0} M(D(\bar{b};\bar{l};\underline{b};\underline{l})).$$

Appendix A describes $D(\bar{b};\underline{b})$ and $\widehat{M}(D(\bar{b};\underline{b}))$ in detail.

Given these notations, as evident from Theorem 2.2 in Epstein and Zin [11], the following relationship is crucial to develop the representation (see Appendix A):

Theorem 1 (Epstein and Zin [11, P.944]): *$D(\bar{b};\bar{l};\underline{b};\underline{l})$ is homeomorphic to $X(\bar{b};\bar{l};\underline{b};\underline{l}) \times M(D(\bar{b};\overline{lb};\underline{b};\underline{lb}))$, and $D(\bar{b};\underline{b})$ is homeomorphic to $\mathbb{R}_{++} \times \widehat{M}(D(\bar{b};\underline{b}))$.*

We identify an element $d \in D(\bar{b};\underline{b})$ with $(c,m) = (c, m(d_\omega)) \in \mathbb{R}_{++} \times \widehat{M}(D(\bar{b};\underline{b}))$ by $d = (c,m) = (c, m(d_\omega))$ (similarly for $d \in D(\bar{b};\bar{l};\underline{b};\underline{l})$).

We also impose the following crucial assumption on the information structure: The DM is allowed to change her choice only *after* current uncertainty is resolved. Thus, at each time t, the DM's consumption set is always $D(\bar{b};\underline{b})$ (so that current consumption must be deterministic).

For each $t \geq 0$, let $h_t \equiv (c_0, ..., c_{t-1}) \in \mathbb{R}^t_{++}$ be a history of consumption realizations. We define $h_0 \equiv \emptyset$. Then, at each $t \geq 0$ and each history $h_t = (c_0, ..., c_{t-1}) \in \mathbb{R}^t_{++}$, the DM has a preference ordering $\succeq_{h_t}$ on $D(\bar{b};\underline{b})$. We impose axioms on the collection of preference orderings $\{\succeq_{h_t}\} \equiv \{\succeq_{h_t} |$for all $t \in \mathcal{T}$ and for all $h_t \in \mathbb{R}^t_{++}\}$.

To focus on risk non-separability, we first assume the following:

Axiom 1 (History Independence-HI): *For all $t \in \mathcal{T}$ and for all $h_t, h'_t \in \mathbb{R}^t_{++}$, $\succeq_{h_t} = \succeq_{h'_t}$.*

Axiom 2 (Stationarity-ST): *For each $t \in \mathcal{T}$ and for each $h_t \in \mathbb{R}^t_{++}$, there exists $c_t \in \mathbb{R}_{++}$ such that for all d and $d' \in D(\bar{b};\underline{b})$, $d \succeq_{h_t} d'$ if and only if $d \succeq_{(h_t,c_t)} d'$.*

Given HI and ST, we use $\succeq$ to denote a history-independent and stationary preference relation on $D(\bar{b};\underline{b})$ (instead of saying, "for all $t \in \mathcal{T}$ and for all $h_t \in \mathbb{R}_+^{t+1}$, an axiom is satisfied for $\succeq_{h_t}$.").[3]

The next axiom describes a general property of conditional preferences.

Axiom 3 (Conditional Preference-CP): (i) *For all* d *and* $d' \in D(\bar{b};\underline{b})$, $d \succeq d'$ *or* $d' \succeq d$. (ii) *For all* $d, d', d'' \in D(\bar{b};\underline{b})$, $d \succeq d'$ *and* $d' \succeq d''$ *imply* $d \succeq d''$. (iii) *For each* $d \in D(\bar{b};\underline{b})$, *for any* $\bar{l} > \underline{l} > 0$, $\{d' \in D(\bar{b};\bar{l};\underline{b};\underline{l}) | d' \succeq d\}$ *and* $\{d' \in D(\bar{b};\bar{l};\underline{b};\underline{l}) | d \succeq d'\}$ *are closed in* $D(\bar{b};\bar{l};\underline{b};\underline{l})$.

(i) and (ii) state that $\succeq$ is complete and transitive; (iii) defines continuity. Note that we define continuity only on each compact consumption set $D(\bar{b};\bar{l};\underline{b};\underline{l})$.

For the next three axioms, we need to develop more elaborate notations to describe uncertainty that will be resolved one period ahead. Let $D^S(\bar{b};\bar{l};\underline{b};\underline{l})$ be a collection of all $d = (c, m^S) \in D(\bar{b};\bar{l};\underline{b};\underline{l})$ with $m^S \in M^S(D(\bar{b};\bar{l}\bar{b};\underline{b};\underline{l}\underline{b}))$; $D^S(\bar{b};\underline{b})$ is defined similarly, where $\widehat{M}^S(D(\bar{b};\underline{b})) \equiv \cup_{\bar{l}>0} \cup_{\bar{l}>\underline{l}>0} M^S(D(\bar{b};\bar{l}\bar{b};\underline{b};\underline{l}\underline{b}))$. We also identify an element $d = (c, m^1(d_\omega)) \in D^1(\bar{b};\underline{b})$ with (c, d_ω) by $d = (c, d_\omega)$ (similarly for $d \in D^1(\bar{b};\bar{l};\underline{b};\underline{l})$; see Appendix A). In addition, let $D_c(\bar{b};\bar{l};\underline{b};\underline{l})$ be a collection of all $d = (c, m) \in D(\bar{b};\bar{l};\underline{b};\underline{l})$ that share the same c; $D_c(\bar{b};\underline{b})$ is defined similarly. Given these notations, $D_c^S(\bar{b};\bar{l};\underline{b};\underline{l})$ and $D_c^S(\bar{b};\underline{b})$ have obvious meanings. Then, $\cup_{S=1}^{\infty} D_c^S(\bar{b};\underline{b})$ is a mixture space under the following operation: For each $d = (c, m^S) \in D_c^S(\bar{b};\underline{b})$ and each $d' = (c, m^{S'}) \in D_c^{S'}(\bar{b};\underline{b})$, $\alpha d + (1-\alpha) d'' \equiv (c, \alpha m^S + (1-\alpha) m^{S'}) \in D_c^{S+S'}(\bar{b};\underline{b})$.

The following two axioms extend the atemporal expected utility theory *recursively* to an infinite horizon setting.

Axiom 4 (Independence Axiom-IA): *For all* $d, d', d'' \in \cup_{S=1}^{\infty} D_c^S(\bar{b};\underline{b})$ *and*

[3] HI and ST also imply that preference orderings are independent of irrelevant alternatives.

for any $\alpha \in (0,1]$, $d \succeq d'$ *if and only if* $\alpha d + (1-\alpha)d'' \succeq \alpha d' + (1-\alpha)d''$.

Axiom 5 (Temporal Consistency-TC): *For any* $c \in \mathbb{R}_{++}$ *and for any* $d = (c, d_\omega), d' = (c, d'_\omega) \in D^1_c(\bar{b}; \underline{b})$, $d \succeq d'$ *if and only if* $d_\omega \succeq d'_\omega$.

IA states the scheme of the Independence Axiom on a set of consumption programs that share the same c. We cannot combine c and c', which are different, under associated probabilities because today's consumption is deterministic. TC says that the passage of time does not affect preferences: Given current consumption c_t, among consumption programs that will yield a deterministic d_ω at $t+1$, the DM at t prefers a consumption program that will give her better utility at $t+1$.[4]

So far, we show that for each $c \in \mathbb{R}_{++}$, $\succeq$ on $\cup_{S=1}^\infty D_c(\bar{b}; \underline{b})$ is represented by the expected utility. In particular, the induced time-t ranking of the future program $D(\bar{b}; \underline{b})$ is identical on each $\cup_{S=1}^\infty D_c(\bar{b}; \underline{b})$, i.e., $(c, d_\omega) \succ (c, d'_\omega)$ if and only if $(c', d_\omega) \succ (c', d'_\omega)$. However, this condition alone does not relate each subset of $\succeq$. The next axiom provides such a connection.

Axiom 6 (Monotonicity-MT): *For all* $n > 0$, *for any* $\{\alpha_i\}_{i=1}^n \subseteq (0,1)$ *and any* $\{(c, d_i)\}_{i=1}^n$, $\{(c', d'_i)\}_{i=1}^n \subseteq D^1(\bar{b}; \underline{b})$ *such that* $\sum_1^n \alpha_i = 1$, *if* $(c, d_i) \succeq (c', d'_i)$ *for all* $i \in n$, *then* $(c, \sum_1^n \alpha_i m^1(d_i)) \succeq (c', \sum_1^n \alpha_i m^1(d'_i))$. *The latter ranking is strict if the former ranking is strict for some* $i \in n$.[5]

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[4] Kreps and Porteus [18] define temporal consistency essentially on a set of early resolution of uncertainty $m(d_\omega)$. Here, we define this condition on a set of late resolution of uncertainty d_ω.

[5] MT is similar to the *recursivity* axiom used in Chew and Epstein [5], under which the statement of the axiom is applied to a collection of early resolution of uncertainty $\sum_1^n \alpha_i m^1((c_i, d_i)) \in \widehat{M}(D(\bar{b}; \underline{b}))$.

MT states that a preference ordering satisfies first order stochastic dominance; it also implements risk non-separability on $\{\cup_{S=1}^{\infty} D_c(\bar{b};\underline{b})\}_{c\in\mathbb{R}_{++}}$: $d=(c,m)\succeq d'=(c,m')$ does *not* imply $d''=(c',m)\succeq d'''=(c',m')$.

Clearly from the proof in Appendix B, any probability measures $m\in\widehat{M}(D(\bar{b};\underline{b}))$ implied by $d=(c,m)$ is approximated by a sequence of $\{m_n^{S_n}\}$, where all $m_n^{S_n}$ are in $M(D(\bar{b};\bar{l};\underline{b};\underline{l}))$ for some $\bar{l}>\underline{l}>0$. Then, by continuity (i.e., (iii) of CP), we derive the utility of $d=(c,m)$ as a limit of a convergent sequence of utilities defined on $\{d^n\}=\{(c,m_n^{S_n})\}$.

The final axiom rules out a trivial case.

Axiom 7 (Non–degeneracy-ND): *For each* $\bar{l}>\underline{l}>0$, *there exist* d *and* $d'\in D(\bar{b};\bar{l};\underline{b};\underline{l})$ *such that* $d'\succ d$.

For technical reasons, we must assume non-degeneracy for each $D(\bar{b};\bar{l};\underline{b};\underline{l})$. Our representation is constructed by relating each subset of $\succeq$. It is hard to relate preferences on each subset if some are degenerate and some are non-degenerate.

We now state the result.

Proposition 1: *The following statements are equivalent:*

(i) $\{\succeq_{h_t}\}$ *satisfy Axioms 1 to 7.*

(ii) (a) *For each* $t\in\mathcal{T}$ *and each* $h_t\in\mathbb{R}_+^t$, $\succeq_{h_t}$ *is represented by* $G(.)$, *where*

$$G(d)=E_m\left[Z(c_t,d_\omega))\right], \tag{3}$$

$$Z(c_t,d_\omega)\geq Z(c_t,d'_\omega)\ \textit{if and only if}\ d_\omega\succeq_{h_t}d'_\omega,$$

and for each $\bar{l}>\underline{l}>0$, $Z: X(\bar{b};\bar{l};\underline{b};\underline{l})\times\ D(\bar{b};\overline{lb};\underline{b};\underline{lb})\rightarrow\mathbb{R}$ *is continuous and* m *is a probability measure in* $\widehat{M}(D(\bar{b};\underline{b}))$ *implied by* $d=(c_t,m(d_\omega))$. *Furthermore,* $Z: D^1(\bar{b};\underline{b})\rightarrow\mathbb{R}$ *is unique up to a positive affine transformation.*

Also, for each $\bar{l} > \underline{l} > 0$ *and for each* $c_t \in X(\bar{b}; \bar{l}; \underline{b}; \underline{l})$, *there exist* d *and* $d' \in D^1_{c_t}(\bar{b}; \bar{l}; \underline{b}; \underline{l})$ *such that* $G(d') > G(d)$. *Moreover, for each* $\bar{l} > \underline{l} > 0$, G *is continuous on* $D(\bar{b}; \bar{l}; \underline{b}; \underline{l})$.

(b) $G(.)$ *can be replaced with* $V(.)$, *where*

$$V(d) = G(d) \text{ and } V(d) \equiv E_m\left[U(c_t, G(d_\omega))\right] = E_m\left[U(c_t, V(d_\omega))\right]; \tag{4}$$

$U : \mathbb{R}_{++} \times \mathbb{R} \rightarrow \mathbb{R}$ *is continuous in both arguments, increasing in the second argument, and unique up to a positive affine transformation on the restricted domain of* $\mathbb{R}_{++} \times \mathbb{R}$, *under which for each second argument* $a \in \mathbb{R}$, *there exists* $d_\omega \in D(\bar{b}; \underline{b})$ *such that* $d_\omega \in G^{-1}(a)$ *and* $(c_t, d_\omega) \in D^1(\bar{b}; \bar{l}; \underline{b}; \underline{l})$ *for some* $\bar{l} > \underline{l} > 0$. *Moreover, for each* $\bar{l} > \underline{l} > 0$, V *is continuous on* $D(\bar{b}; \bar{l}; \underline{b}; \underline{l})$.

To derive the recursive utility of the form $V(d) = U(c_t, E_m[V(d_\omega)])$ in our framework, we must replace Axiom 6 (MT) with the *risk separability* axiom (see Chew and Epstein [5]): For all $(c, c') \in \mathbb{R}^2_{++}$ and $(m, m') \in (\widehat{M}(D(\bar{b}; \underline{b})))^2$, $(c, m) \succeq (c, m')$ if and only if $(c', m) \succeq (c', m')$. Hence, Axiom 6 defines risk non-separability and differentiates it from risk separability.

Finally, Kreps and Porteus [18] and Chew and Epstein [5] derive the recursive utility on a collection of *early resolution of uncertainty* (i.e., a lottery $m \in \widehat{M}(D(\bar{b}; \underline{b}))$). To incorporate early resolution into our framework, we must allow the ranking of early resolution to be dependent on c_{t-1}, while we retain Axioms 1 to 7 on $D(\bar{b}; \underline{b})$. Then, we define a preference ordering on $\widehat{M}(D(\bar{b}; \underline{b}))$ as that on $D(\bar{b}; \underline{b}) = \mathbb{R}_+ \times \widehat{M}(D(\bar{b}; \underline{b}))$, where each element is (c_{t-1}, m).[6] By this construction, Proposition 1 holds as it is, and the DM's choice is dynamically consistent.

[6] We need to introduce an additional axiom to define the preference ordering $\succeq_{h_0}$ on $\widehat{M}(D(\bar{b}, \underline{b}))$ at time 0.

3 Applications: Stochastic Recursive Utility

As for the application of a utility function defined by (4), we consider the following form:

$$V(d) = V(c_t, m(d_\omega)) \equiv E_m\left[\frac{1}{\alpha}\left\{c_t^\rho + \beta(\alpha V(d_\omega))^{\frac{\rho}{\alpha}}\right\}^{\frac{\alpha}{\rho}}\right], \tag{5}$$

where $0 \neq \alpha \leq 1$, $0 \neq \rho \leq 1$, $0 < \beta < 1$, and $\{V(d_\omega)\}$ is a random variable that summarizes the distribution of future utility implied by $m \in \widehat{M}(D(\bar{b}; \underline{b}))$. This contrasts with the following form of *recursive utility* as proposed by Epstein and Zin [11]:

$$V(d) = V(c_t, m(d_\omega)) \equiv \frac{1}{\alpha}\left[c_t^\rho + \beta\{\alpha E_m(V(d_\omega))\}^{\frac{\rho}{\alpha}}\right]^{\frac{\alpha}{\rho}}, \tag{6}$$

where $0 \neq \alpha \leq 1$, $0 \neq \rho \leq 1$, and $0 < \beta < 1$. Since (5) resembles (6) and the expectation is applied to an aggregator function $U(c_t, \gamma_\omega) = \dfrac{1}{\alpha}\left\{c_t^\rho + \beta(\alpha\gamma_\omega)^{\frac{\rho}{\alpha}}\right\}^{\frac{\alpha}{\rho}}$, we call (5) *stochastic recursive utility*.

The interpretation of parameters α and ρ is analogous to that of Epstein and Zin [11]. When a consumption program is deterministic, (5) results in a utility function with constant elasticity of substitution $\sigma = (1-\rho)^{-1}$. Thus, we regard ρ as a parameter describing the degree of intertemporal substitutability. On the other hand, (5) is ordinally equivalent to the following function (i.e., $\dfrac{1}{\rho}\left\{\{\alpha \times (5)\}^{\frac{1}{\alpha}}\right\}^\rho$):

$$V(d) = V(c_t, m(d_\omega)) \equiv \frac{1}{\rho}E_m\left[\{c_t^\rho + \beta\rho V(d_\omega)\}^{\frac{\alpha}{\rho}}\right]^{\frac{\rho}{\alpha}}. \tag{7}$$

Consider two functions in the form of (7), V and V^*, where $\rho = \rho^*$ and $\beta = \beta^*$. Suppose that all uncertainty is resolved tomorrow, i.e., $d = (c_t, m(d_\omega))$ and each d_ω is completely deterministic. Then, for this given sequence d, $V \leq V^*$

if and only if $\alpha \leq \alpha^*$.[7] Thus, we interpret α as a parameter describing the degree of risk aversion.

Since (7) is more tractable than (5), for the rest of the paper, we focus on (7). The existence of (7) is proven in Appendix C. Here, we summarize the results.[8]

Proposition 2: *If* (i) $0 < \rho \leq \alpha \leq 1$ *and* $\beta \bar{b}^{\rho} < 1$ *or* (ii) $\alpha \leq \rho < 0$ *and* $\beta \underline{b}^{\rho} < 1$, *there exists a unique solution* V *that satisfies* (7), *where* V *is continuous on* $D(\bar{b}; \bar{l}; \underline{b}; \underline{l})$ *for all* $\bar{l} > \underline{l} > 0$.

To be compatible with Proposition 1, V must be continuous on each $D(\bar{b}; \bar{l}; \underline{b}; \underline{l})$. This requirement restricts the range of parameters α and ρ by $\frac{\alpha}{\rho} \geq 1$.

By following Yaari [25], we also introduce the definition of comparative *temporal* risk aversion. For a given t, let $\widehat{d_t} = (d_1, d_{2,\ldots}) \in D(\bar{b}; \underline{b})$ such that $d_\tau = d_t$ for all $\tau \geq t$. We say that $\succeq^*$ is *more temporal risk averse* than $\succeq$ if for any t and for any $y \in \mathbb{R}_+^\infty$ and $\widehat{d_t} \in D(\bar{b}; \underline{b})$,

$$y \succeq \widehat{d_t} \text{ implies } y \succeq^* \widehat{d_t}.$$

Under this definition, the two preferences agree on the ranking of deterministic consumption sequences, but any temporal consumption lottery disliked by $\succeq$ is disliked by $\succeq^*$ if all uncertainty is resolved at or before time t. Since (5) and (6) agree on the set of deterministic consumption sequences, the DM is more

[7] Compute $\{\alpha \times (5)\}^{\frac{1}{\alpha}}$ first using the following: For a positive random variable x, by Jensen's inequality, for $\alpha \leq \alpha'$, $\alpha \neq 0$, and $\alpha' \neq 0$, $E[x^\alpha]^{\frac{1}{\alpha}} = \{E[(x^{\alpha'})^{\frac{\alpha}{\alpha'}}]\}^{\frac{\alpha'}{\alpha}\frac{1}{\alpha'}} \leq \{E[(x^{\alpha'})]\}^{\frac{\alpha}{\alpha'}\frac{\alpha'}{\alpha}\frac{1}{\alpha'}} = \{E[(x^{\alpha'})]\}^{\frac{1}{\alpha'}}$.

[8] An argument similar to Appendix C shows that Proposition 2 also holds for an ordinally equivalent version of (6), i.e., $\frac{1}{\rho}\left\{\{\alpha \times (6)\}^{\frac{1}{\alpha}}\right\}^{\rho}$.

(less) temporal risk averse under (5) than under (6) if $\alpha > (<) \rho$ (see Appendix C).[9,10,11] Hence, depending on the values of the parameters, the difference in the order of aggregation generates the difference in temporal risk preference.

4 Asset Pricing

4.1 Setting

We adapt a version of the Lucas [19] economy with a representative agent. As before, time is discrete and infinite, but for convenience, we rename each period so that $\mathcal{T} = \{1, 2, ...\}$. There are a finite number of states, $\Omega = \{1, ..., S\}$, at each period, which describes an exogenous shock. A shock is generated by a first order Markov process with a stationary transition m. We assume that $m(\omega_t, \omega_{t+1}) > 0$ for all $(\omega_t, \omega_{t+1}) \in \Omega^2$. Let the initial state of shock ω_0 be given, and let $(\Omega^\infty, \mathbb{B}(\Omega^\infty), Q)$ be a probability space generated by a sequence of shocks from ω_0, where $\mathbb{B}(\Omega^\infty)$ is a product Borel σ-algebra on Ω^∞. Ω^t is a collection of all points $\omega^t = (\omega_1, \omega_2, ..., \omega_t)$, and $\mathbb{B}(\Omega^t)$ is a product Borel σ-algebra on Ω^t. Let $\{\mathcal{F}_t\}_{t=1}^\infty$ be a filtration defined on $(\Omega^\infty, \mathbb{B}(\Omega^\infty), Q)$, where $\mathcal{F}_t = (\overline{\pi}^t)^{-1}(\mathbb{B}(\Omega^t))$ and $\overline{\pi}^t$ is a projection map on the first t coordinates; $\mathcal{F}_t(\omega^t) \in \mathcal{F}_t$ is the smallest event that contains $(\overline{\pi}^t)^{-1}(\omega^t)$.

We denote by $\widetilde{x} = \{\widetilde{x}_t\} = \{\widetilde{x}_t\}_1^\infty$ a stochastic process adapted to this

[9] We only consider (5) and (6) that agree on the set of deterministic consumption sequences. Such functions exists for all possible combinations of α and ρ if $\rho > 0$ and $\beta\overline{b}^\rho < 1$ or $\rho < 0$ and $\beta\underline{b}^\rho < 1$ (see Appendix C).

[10] This interpretation is similar to preference for early or late resolution of uncertainty. See Kreps and Porteus [18] and Epstein and Zin [11].

[11] For the parameter values stated in Proposition 2, the definition of temporal risk aversion can be extended to include all $d \in D(\overline{b}; \underline{b})$ (see Appendix C).

filtration, where $\widetilde{x}_t$ is measurable with respect to $\mathcal{F}_t$. We also identify $\widetilde{x}_t$ with a map from $\Omega^t \to \mathbb{R}$ and denote by $x_t(\omega^t)$ the value of this map at ω^t. In addition, $x(\omega^\infty)$ defines a sequence assigned on ω^∞, and $C(\infty;0)$ denotes a space of non-negative adapted stochastic processes.

There is a single perishable consumption good available at each t. The consumption process lies in the space $C(\bar{b};\underline{b}) \equiv \cup_{\bar{l}>0} \cup_{\bar{l}>\underline{l}>0} C(\bar{b};\bar{l};\underline{b};\underline{l})$ for some given $\bar{b}$ and $\underline{b}$ such that $\bar{b} \geq 1 \geq \underline{b} > 0$, where

$$C(\bar{b};\bar{l};\underline{b};\underline{l}) \equiv \{\widetilde{x} | \widetilde{x} \text{ is adapted and } x(\omega^\infty) \in Y(\bar{b};\bar{l};\underline{b};\underline{l}) \text{ for a given } \bar{l} > \underline{l} > 0\}.$$

In addition, at each (t, ω^t), two types of assets are traded in a competitive market. The first one is an infinitely-lived asset that pays a strictly positive consumption good as a dividend; its dividend process and price process are adapted and denoted by $\widetilde{e} = \{\widetilde{e}_t\} \in C(\bar{b};\underline{b})$ and $\widetilde{q}^1 = \{\widetilde{q}^1_t\} \in C(\infty;0)$, respectively. We assume that the net supply of this risky asset is one. The second asset is a short-term risk-free asset that pays one consumption good in the next period. Although each risk-free asset must be treated separately, for convenience, we denote $\widetilde{q}^2 = \{\widetilde{q}^2_t\} \in C(\infty;0)$ as an adapted stochastic process that describes the evolution of all the risk-free assets' prices. We assume that the net supply of each risk-free asset is zero. Furthermore, the consumption good is treated as a numeraire at each time (so that a *present* commodity price always stays at one), and $\widetilde{q}^1$ and $\widetilde{q}^2$ are normalized accordingly. The representative agent is endowed with one unit of the risky asset at the beginning of time 1; however, she is not endowed with any of the risk-free assets.

The plan is represented by $(\widetilde{c}, (\widetilde{\theta}^1, \widetilde{\theta}^2))$, where $\widetilde{c}$ is an adapted consumption process and each $\widetilde{\theta}^i = \{\widetilde{\theta}^i_t\}$ is an adapted asset holding process with $\theta^i_t(\omega^t) \in \mathbb{R}$; we introduce the constant $\widetilde{\theta}^i_0$ that describes an initial portfolio. A *feasible* plan

$(\widetilde{c}, (\widetilde{\theta}^1, \widetilde{\theta}^2))$ must satisfy the following budget constraints:

$$\widetilde{c} \in C(\bar{b}; \underline{b}) \tag{8}$$

$$\widetilde{c}_t + \widetilde{\theta}^1_t \widetilde{q}^1_t + \widetilde{\theta}^2_t \widetilde{q}^2_t \leq \widetilde{\theta}^1_{t-1}(\widetilde{q}^1_t + \widetilde{e}_t) + \widetilde{\theta}^2_{t-1} \text{ for all } t \geq 1 \ , \tag{9}$$

$$\theta^1_t(\omega^t) \in [1-\varepsilon, 1+\varepsilon] \text{ and } \theta^2_t(\omega^t) \in [-\varepsilon, \varepsilon] \text{ for all } t \geq 1 \text{ and } \omega^t \in \Omega^t, \tag{10}$$

$$\widetilde{\theta}^1_0 \equiv 1 \text{ and } \widetilde{\theta}^2_0 \equiv 0, \tag{11}$$

where the second to last condition is a restriction on the trade size with a small number $0 < \varepsilon < 1$.

At the beginning of each period, the representative agent knows all past values, observes a current state of the shock, and receives or pays dividends from her asset holdings. Then, the representative agent plans consumption and investment for available assets for the current and all future periods. Thus, a preference ordering is defined on the space of *conditional* consumption processes emanating from (t, ω^t), denoted by $C(\bar{b}; \underline{b}|\mathcal{F}_t(\omega^t))$; it can be embedded into $D(\bar{b}; \underline{b})$ by the map $\phi_{\omega^t} : C(\bar{b}; \underline{b}|\mathcal{F}_t(\omega^t)) \to D(\bar{b}; \underline{b})$ defined in Appendix D. An element in $C(\bar{b}; \underline{b}|\mathcal{F}_t(\omega^t))$ is written as $\widetilde{c}(\mathcal{F}_t(\omega^t))$. $C(\bar{b}; \bar{l}; \underline{b}; \underline{l}|\mathcal{F}_t(\omega^t))$ is defined similarly.

Given this information structure, at each (t, ω^t), the representative agent takes all price processes as given and maximizes her utility $V(\phi_{\omega^t}(\widetilde{c}(\mathcal{F}_t(\omega^t)))$, defined by (7), by solving the following optimization: At each (t, ω^t),

$$\max_{(\widetilde{c}, (\widetilde{\theta}^1, \widetilde{\theta}^2))} V(\phi_{\omega^t}(\widetilde{c}(\mathcal{F}_t(\omega^t))) \text{ subject to (8), (9), (10), and (11),}$$

where on $\mathcal{F}_t(\omega^t)$, the values of $\{\widetilde{\theta}^1_\tau\}^{t-1}_{\tau=1}$, $\{\widetilde{\theta}^2_\tau\}^{t-1}_{\tau=1}$, and $\{\widetilde{c}_\tau\}^{t-1}_{\tau=1}$ are given by the optimization prior to (t, ω^t). The solution $(\widetilde{c}^*, (\widetilde{\theta}^{1*}, \widetilde{\theta}^{2*}))$ is called a (t, ω^t)-*optimal* allocation.

An *equilibrium* is $(\widetilde{c}^*, (\widetilde{\theta}^{1*}, \widetilde{\theta}^{2*}), (\widetilde{q}^{1*}, \widetilde{q}^{2*}))$ such that (i) $(\widetilde{c}^*, (\widetilde{\theta}^{1*}, \widetilde{\theta}^{2*}))$ is (t, ω^t)-optimal at each (t, ω^t), (ii) $\widetilde{c}^* = \widetilde{e}$, and (iii) $(\widetilde{\theta}^{1*}, \widetilde{\theta}^{2*}) = (\widetilde{1}, \widetilde{0})$. At an equilibrium, the representative agent uses $(\widetilde{q}^{1*}, \widetilde{q}^{2*})$ as the expectations for future prices, and these expectations are in fact fulfilled in the subsequent time periods. Also, since m has full support, if $(\widetilde{c}^{*}, (\widetilde{\theta}^{1*}, \widetilde{\theta}^{2*}))$ is $(1, \omega^1)$-optimal for all $(1, \omega^1)$, then it is (t, ω^t)-optimal at each (t, ω^t). For convenience, let $\widetilde{q}^{*} \equiv (\widetilde{q}^{1*}, \widetilde{q}^{2*})$ and call $\widetilde{q}^{*}$ an equilibrium price process.

We prove the existence of an equilibrium only under the following parameter values, which are used for most applications:[12]

Assumption 1: $\alpha \leq \rho < 0$ and $\beta \underline{b}^{\rho} < 1$.

By Proposition 2, (7) is continuous on each $C(\overline{b}; \overline{l}; \underline{b}; \underline{l} | \mathcal{F}_t(\omega^t))$; it is also increasing and differentiable with respect to each $c_t(\omega^t)$ on each $C(\overline{b}; \overline{l}; \underline{b}; \underline{l} | \mathcal{F}_t(\omega^t))$. It follows from Appendix D that (7) is concave and homogeneous of degree ρ in $\widetilde{c}(\mathcal{F}_t(\omega^t))$.

Using the chain rule of differentiation, the representative agent's marginal rate of substitution between consumption at $t+1$ and at t is given by

$$
\begin{aligned}
& MRS_t^{t+1}(\omega^{t+1}) \qquad (12) \\
&= \beta\Big(\frac{c_{t+1}(\omega^{t+1})}{c_t(\omega^t)}\Big)^{\rho-1}(c_t^{\rho}(\omega^t) + \beta\rho V_{t+1}(\omega^{t+1}))^{\frac{\alpha}{\rho}-1} \\
&\times \Big\{E_Q[(c_{t+1}^{\rho}(\omega^{t+1}) + \beta\rho\widetilde{V}_{t+2})^{\frac{\alpha}{\rho}} | \mathcal{F}_{t+1}(\omega^{t+1})]^{\frac{\rho}{\alpha}-1}\Big\} \\
&\times \frac{\Big\{E_Q[(c_{t+1}^{\rho}(\omega^{t+1}) + \beta\rho\widetilde{V}_{t+2})^{\frac{\alpha}{\rho}-1} | \mathcal{F}_{t+1}(\omega^{t+1})]\Big\}}{\Big\{E_Q[(c_t^{\rho}(\omega^t) + \beta\rho\widetilde{V}_{t+1})^{\frac{\alpha}{\rho}-1} | \mathcal{F}_t(\omega^t)]\Big\}},
\end{aligned}
$$

where $\{\widetilde{V}_t\}$ is a utility process. Let $\{\widetilde{MRS}_{t-1}^{t}\}$ be an MRS process, where $\widetilde{MRS}_0^{1} \equiv 1$. Then, the following proposition derives an equilibrium asset price process $\widetilde{q}^{*}$ (for the proof, see Appendix D):

[12] This is the only range of parameter values on which we are able to show all of the following properties: continuity, differentiability, monotonicity, concavity, and homogeneity.

Proposition 3: *Under Assumption 1, suppose that the following transversality condition holds*:

$$\lim_{t' \to \infty} \sup_{t \geq t'} E_Q[\prod_{s=1}^{t} \widetilde{MRS}_s^{s+1} | \mathcal{F}_1(\omega^1)] \leq 0 \textit{ for each } \mathcal{F}_1(\omega^1). \tag{13}$$

Then, there exists an equilibrium price process $\widetilde{q}^* = (\widetilde{q}^{1*}, \widetilde{q}^{2*})$ *such that*

$$q_t^{1*}(\omega^t) = E_Q\left[\sum_{\tau=t}^{\infty}\left\{\prod_{s=t}^{\tau}\widetilde{MRS}_s^{s+1}\right\}\widetilde{e}_{\tau+1}|\mathcal{F}_t(\omega^t)\right] \tag{14}$$

and

$$q_t^{2*}(\omega^t) = E_Q\left[\widetilde{MRS}_t^{t+1}|\mathcal{F}_t(\omega^t)\right]. \tag{15}$$

The transversality condition (13) is a sufficient condition for (15). In particular, it is satisfied if $E_Q[\widetilde{MRS}_t^{t+1}|\mathcal{F}_t(\omega^t)] < M < 1$ at every (t, ω^t) (i.e., $q_t^{2*}(\omega^t) < M$). Because (7) is homogenous of degree ρ, the transversality condition of (14) is satisfied if $|V(\phi_{\omega^1}(\widetilde{c}(\mathcal{F}_1(\omega^1)))| < \infty$ for each $\mathcal{F}_1(\omega^1)$, which is guaranteed under Assumption 1. Also, an equilibrium price processes $\widetilde{q}^*$ is unique if it constitutes dynamically complete markets.

Similarly, Proposition 3 holds for (6), whereas $\{\widetilde{MRS}_{t-1}^{t}\}$ follows:

$$\begin{aligned} MRS_t^{t+1}(\omega^{t+1}) &= \beta(\frac{c_{t+1}(\omega^{t+1})}{c_t(\omega^t)})^{\rho-1} \\ &\quad \times \{\alpha E_Q[\widetilde{V}_{t+1}|\mathcal{F}_t(\omega^t)]\}^{\frac{\rho}{\alpha}-1} \times \{\alpha V_{t+1}(\omega^{t+1})\}^{\frac{\alpha-\rho}{\alpha}}. \end{aligned} \tag{16}$$

4.2 Simulation Studies

Since the marginal rates of substitution is highly complex, we employ simulation studies to investigate asset pricing implications. For this purpose, we adapt the following assumptions from Mehra and Prescott [20]:

Assumption 2: $\Omega = \{1, 2\}$; $g_1 = 1.054$ and $g_1 = 0.982$; $m(1, 1) = m(2, 2) = 0.43$.

Note that under Assumption 2, (14) and (15) generically constitute dynamically complete markets.

First, we compute differences in long-run average expected returns between (6) and (7), where the long-run average is a weighted sum of short-run expected returns based on the stationary distribution.[13] The results are reported in Tables 1-3. Following Weil [24], to show robustness of our results, the tables in this subsection also report expected returns based on (14)-(15) under parameter values that do not satisfy Assumption 1.[14]

[Insert Tables 1-3 about here]

We observe the following pattern in expected returns and risk premiums: When $\alpha < \rho$, the representative agent is more temporal risk averse under recursive utility than under stochastic recursive utility. Then, under (6), the risk premium is higher, and the risk-free rate is lower (except when $\rho = 1$ and $\alpha = -9$; see the paragraph following Table 6). As Table 3 shows, this effect is analogous to that from the decreasing α at a given ρ. Similarly, when $\alpha > \rho$, we observe the opposite result.[15] Hence, the difference in long-term expected returns between (6) and (7) is explained mainly by the difference in temporal risk aversion; we do not observe any other distinct characteristics of risk non-separability in long-term expected returns.

[13] The stationary distribution m^* is $m^*(1, 1) = m^*(2, 2) = 0.5$.

[14] Although we do not establish the existence for this case, if an equilibrium exists and asset markets are dynamically complete under (14)-(15), then (14)-(15) constitute a unique equilibrium price process.

[15] These patterns are most clearly shown if a growth rate follows *i.i.d.* distribution.

Second, we examine differences in long-run volatility of expected returns between (6) and (7), where the long-run volatility is a standard deviation of short-run expected returns based on the stationary distribution. We do not report a similar table for the risk premium because it reflects the size of the risk premium (i.e., the attitude toward temporal risk).

[Insert Tables 4-5 about here]

In Tables 4-5, we observe a distinct characteristic of risk non-separability: When $\rho < 0$, both risk-free rates and expected returns of the risky asset are less volatile under stochastic recursive utility than under recursive utility regardless of the attitude toward temporal risk (i.e., irrespective of the value of α). Also, the difference in volatility increases as ρ decreases and as the difference between α and ρ increases.

Intuitively, under stochastic recursive utility, asset prices depend on *expected* intertemporal substitution between current consumption and future utility. On the other hand, under recursive utility, asset prices depend on intertemporal substitution between current consumption and the certainty equivalent of one-step ahead risk. Since an expectation smooths the effect of intertemporal substitution, expected returns are less volatile under (7). This smoothing effect strengthens as the level of intertemporal substitution (i.e., $1 - \rho$) increases and as the order of aggregation between intertemporal substitution and risk aversion (i.e., an absolute difference between α and ρ) has increased importance.

On the contrary, the results under $\rho > 0$ seem inconsistent with the above explanation. To investigate these cases further, we report Table 6, the values of which are computed by subtracting the risk-free rate at State 1 from the risk-free rate at State 2.

[Insert Table 6 about here]

Table 6 shows that under stochastic recursive utility, the risk-free rate is higher at State 2 than at State 1, which is consistent with a standard interpretation: When future prospects are not bright, the demand for the risk-free asset increases. On the other hand, under recursive utility, when $\rho = 1$, the risk-free rate is higher at State 1 than at State 2. This effect is generated by a term in the MRS that involves an expectation operator (i.e., $\{\alpha E_Q[\widetilde{V}_{t+1}|\mathcal{F}_t(\omega^t)]\}^{\frac{\rho}{\alpha}-1}$ in (16)). However, as $1-\rho$ increases, this expectation term is dominated by other terms (for example, $(\frac{c_{t+1}(\omega^{t+1})}{c_t(\omega^t)})^{\rho-1}$ in (16)), and the risk-free rate at State 2 surpasses that at State 1. Thus, during this transition, the volatility of the risk-free rates is higher under stochastic recursive utility. A similar effect explains the differences in volatility of the risky asset's expected returns. Hence, we must be cautious when we interpret the results under $\rho > 0$ in all of the simulation studies in this subsection. Note that intertemporal substitution forces risk-free rates to move in the *right* direction.

5 Conclusion

The contributions of this paper are threefold: (1) We axiomatically derive a utility function that allows us to investigate risk non-separability independently of history dependence. (2) We show the existence of the utility function under the CES aggregator function. Based on parameter values, we then define a comparative attitude toward temporal risk. (3) In a representative agent economy, we show two results: When the DM is less temporally risk averse under stochastic recursive utility than under recursive utility, risk non-separability generates a lower risk premium and a higher risk-free rate; risk non-separability induces a stable motive for precautionary saving and decreases the volatility in expected returns.

Appendix A: A Space of Temporal Consumption Lotteries

We follow the construction and notation as defined in Epstein and Zin [11]. First, for $D_0 \equiv \mathbb{R}_+^\infty$, define the distance between any two elements $c, c' \in \mathbb{R}_+^\infty$ by

$$\gamma_0(c, c') \equiv \sum_{t=0}^{\infty} \frac{\overline{\delta}(c_t, c_t')}{2^t},$$

where $\overline{\delta}(c_t, c_t') = \min[\delta(c_t, c_t'), 1]$ with the Euclidian metric $\delta(c_t, c_t')$ on $\mathbb{R}$. This metric generates the Tychonov product topology on $\mathbb{R}_+^\infty$ under which D_0 is connected and separable. Then, for each $t \geq 1$, a metric on D_t is inductively defined as follows: For any two measures $m, m' \in M(D_{t-1})$, define the Prohorov metric $\rho_{t-1}(m, m')$ by

$$\rho_{t-1}(m, m') \equiv \inf\{\varepsilon > 0 | m(B) \leq m'(B^\varepsilon) + \varepsilon \text{ for all Borel sets } B \in \mathbb{B}(D_{t-1})\},$$
$$\text{where } B^\varepsilon \equiv \{y' \in D_{t-1} | \gamma_{t-1}(y, y') < \varepsilon \text{ for some } y \in B\}.$$

This metric induces the weak topology on $M(D_{t-1})$. Then, define a metric on $D_t = \mathbb{R}_+ \times M(D_{t-1})$ by

$$\gamma_t(d_t, d_t') \equiv \overline{\delta}(c_0, c_0') + \frac{1}{2}\rho_{t-1}(m, m'). \tag{A.1}$$

Under ρ_0, $\cup_{S=1}^\infty M^S(D_0)$ is connected because it is path-connected. Since D_0 is separable, it follows from Theorem 6.3 of Parthasarathy [21, P.44] that $\cup_{S=1}^\infty M^S(D_0)$ is dense in $M(D_0)$. Thus, $M(D_0)$ is connected. By Theorem 6.2 of Parthasarathy [21, P.43], $M(D_0)$ is separable because D_0 is separable. Therefore, D_1 is connected and separable. By induction, D_t is connected and separable for all $t \in \mathcal{T}$.

Epstein and Zin [11] defines $f_t : M(D_t) \to M(D_{t-1})$ and $g_t : D_{t+1} \to D_t$ to describe the nature of uncertainty; $g_t(d_{t+1}) = (c_0, f_t(m))$ induces the same uncertainty on $(c_0, c_1, ...)$ as does d_{t+1}, but the uncertainty is resolved one period earlier under the probability measure $f_t(m)$ induced on D_{t-1}. For this construction, first define $f : M(\mathbb{R}_+ \times M(\mathbb{R}_+^\infty)) \to M(\mathbb{R}_+^\infty)$ by

$$f(m)(B) \equiv E_m T_B(.,.),\ B \in \mathbb{B}(\mathbb{R}_+^\infty), \tag{A2}$$

where $T_B(.,.) : \mathbb{R}_+ \times M(\mathbb{R}_+^\infty) \to \mathbb{R}_+$ and $T_B(c, v) \equiv v\{y \in \mathbb{R}_+^\infty | (c, y) \in B\}$.

For each two-stage lottery m, $f(m)$ is the probability measure induced on $\mathbb{R}_+^\infty$ by having all uncertainty resolved at the first stage. Then, define f_t and g_t inductively by

$$\begin{aligned} f_1 &= f, \\ g_t(c_0, m) &= (c_0, f_t(m)) \text{ for } t \geq 1, \\ f_t(m)(B) &= m(g_{t-1}^{-1}(B)) \text{ for all } B \in \mathbb{B}(D_{t-1}) \text{ for all } t \geq 2. \end{aligned}$$

Under this definition, $g_t(d_{t+1}) = d_{t+1}$ if and only if $d_{t+1} \in D_t$.

Now, construct a subspace of the product space $\Pi^{\mathbb{N}\setminus\{0\}} D_t$ by

$$D \equiv \{d = (d_1, d_2, ...) | d_t \in D_t \text{ and } d_t = g_t(d_{t+1}) \text{ for all } t \geq 1\},$$

where a metric on $\Pi^{\mathbb{N}\setminus\{0\}} D_t$ is defined by

$$\gamma(d, d') \equiv \sum_{t=1}^{\infty} \frac{\gamma_t(d_t, d'_t)}{2^t}. \tag{A.3}$$

Also, a metric on $M(D)$ is defined as the Prohorov metric. By Lemmas A1.1 and A1.2 of Epstein and Zin [11], for all $t \geq 1$, g_t is continuous (hence, measurable with respect to $\mathbb{B}(D_{t+1})$). By construction, for all $t \geq 1$, a projection map π_t from $\Pi^{\mathbb{N}\setminus\{0\}} D_t$ to D_t defined by $\pi_t(d_1, ..., d_t, ...) = d_t$ is continuous (hence, measurable with respect to $\mathbb{B}(\Pi^{\mathbb{N}\setminus\{0\}} D_t)$). We also define an injective map from D_t to D by

$$r : d_t \to (d_1, d_2, ..., d_{t-1}, d_t, d_t, ...), \tag{A.4}$$

where $d_i = g_i(d_{i+1})$ for all $i \in [1, t-1]$. Then, under the subspace product topology restricted to D, the image of $\cup_{t=0}^{\infty} D_t$ under r is dense in D (D_0 is recognized as a subspace of D_1).

Lemma A.1: *D and $D(\bar{b}; \underline{b})$ are connected and separable; $D(\bar{b}; \bar{l}; \underline{b}; \underline{l})$ is connected, separable, and compact for each $\bar{l} > \underline{l} > 0$.*

Proof. First, it follows from Theorems 6.4 and 2.6 of Parthasarathy [21, P.45 and P.136, respectively] that $D(\bar{b}; \bar{l}; \underline{b}; \underline{l})$ is compact.

As for separability, by Theorem 2.6 of Parthasarathy [21, P.136], D is a separable metric space. Then, $D(\bar{b}; \bar{l}; \underline{b}; \underline{l})$ and $D(\bar{b}; \underline{b})$ are separable because $D(\bar{b}; \bar{l}; \underline{b}; \underline{l}) \subseteq D(\bar{b}; \underline{b}) \subseteq D$ and D is a metric space and separable (so that D is second countable; a subspace of a second countable space is second countable).

For connectedness, we first define a basis element in D. Let $\cap_{i \in I} \pi_i^{-1}(B_i)$ be a basis element in $\Pi^{\mathbb{N} \backslash \{0\}} D_t$, where B_t is a basis element in D_t and $I = \{t_1, ..., t_N\}$ is a finite set with $t_i > t_{i-1}$. Let $A_{t_1} \equiv B_{t_1}$. Then, by continuity of g_t, $g_{t_2-1}^{-1}(...(g_{t_1}^{-1}(A_{t_1}))...) \cap B_{t_2}$ is open in D_{t_2}. Denote this intersection as A_{t_2}. By induction, A_{t_N} is open in D_{t_N}. Then, $\cap_{i \in I} \pi_i^{-1}(B_i) \cap D$ is $\pi_{t_N}^{-1}(A_{t_N}) \cap D$.[16] This implies that each basis element in D corresponds to $\pi_t^{-1}(A_t) \cap D$ for some t and some open A_t in D_t.

Next, we claim that $r : D_t \to D$ defined by (A.4) is continuous. By (A.1), for each $\tau > t$ and each open $A_\tau \in D_\tau$, $D_t \cap A_\tau$ is open in D_t because $D_t \subset D_\tau$ and $\gamma_\tau(d_t, d_t') = \gamma_t(d_t, d_t')$ for all $d_t, d_t' \in D_t$. Then, the claim follows because $r^{-1}(\pi_\tau^{-1}(A_\tau) \cap D)$ is $D_t \cap A_\tau$ if $\tau \geq t$ and $g_{t-1}^{-1}(...(g_\tau^{-1}(A_\tau))...)$ if $\tau < t$.

Since D_t is connected, $r(D_t)$ is connected for all $t \geq 1$. Then, $\cup_{t=1}^{\infty} r(D_t)$ is connected because $\emptyset \neq r(D_t) \subseteq r(D_{t+1})$. Since $\cup_{t=1}^{\infty} r(D_t)$ is dense in D,

[16] By the definition of g_t, $\pi_t^{-1}(B_t) = \pi_{t+1}^{-1}(g_t^{-1}(B_t))$ on D, and D is closed in $\Pi^{\mathbb{N} \backslash \{0\}} D_t$.

D is connected. Similarly, $D(\bar{b};\bar{l};\underline{b};\underline{l})$ is connected because $r: D_t(\bar{b};\bar{l};\underline{b};\underline{l}) \to D(\bar{b};\bar{l};\underline{b};\underline{l})$ is continuous and $D_t(\bar{b};\bar{l};\underline{b};\underline{l})$ is connected for all $t \geq 0$. Then, $D(\bar{b};\underline{b}) \equiv \cup_{\bar{l}>0} \cup_{\bar{l}>\underline{l}>0} D(\bar{b};\bar{l};\underline{b};\underline{l})$ implies that $D(\bar{b};\underline{b})$ is connected. ■

Under the above construction, Epstein and Zin [11] show that for $d = (d_1, ..., d_t, ...) \in D$ with $d_t = (c_0, m_t)$ and $m_t \in M(D_{t-1})$, there exists a unique $m \in M(D)$ such that

$$m(\pi_t^{-1} B_t) = m_{t+1}(B_t) \text{ for all } B_t \in \mathbb{B}(D_t) \text{ for all } t \geq 1. \tag{A.5}$$

By construction, m describes future uncertainty implied by d. Define a map $\Theta : D \to \mathbb{R}_+ \times M(D)$ by setting $\Theta(d) = (c_0, m)$, which summarizes the structure of d with current consumption and future uncertainty. A metric on $\mathbb{R}_+ \times M(D)$ is defined by

$\gamma((c_0, m), (c_0', m')) \equiv \bar{\delta}(c_0, c_0') + \frac{1}{2}\rho(m, m')$ with the Prohorov metric $\rho(m, m')$.

We also define a map that reverses (A.5) by

$$P_{t+1} : M(D) \to M(D_t), \tag{A.6}$$

where $P_{t+1}m(B_t) \equiv m(\pi_t^{-1} B_t)$ for all $B_t \in \mathbb{B}(D_t)$ for all $t \geq 1$. Then, Theorem 2.2 in Epstein and Zin [11] implies the following:

Theorem 1 (Epstein and Zin [11, P.944]): *Under* Θ, $D(\bar{b};\underline{b})$ *is homeomorphic to* $\mathbb{R}_{++} \times \widehat{M}(D(\bar{b};\underline{b}))$, *where*

$\widehat{M}(D(\bar{b};\underline{b})) \equiv \{m \in M(D(\bar{b};\underline{b})) | f(m_2) \in \cup_{\bar{l}>0} \cup_{\bar{l}>\underline{l}>0} M(Y(\bar{b};\bar{l};\underline{b};\underline{l})), m_2 = P_2 m\}.$

In addition, $D(\bar{b};\bar{l};\underline{b};\underline{l})$ *and* $X(\bar{b};\bar{l};\underline{b};\underline{l}) \times M(D(\bar{b};\overline{lb};\underline{b};\underline{lb}))$ *are homeomorphic.*

Also, by Lemma 6.1 of Parthasarathy [21, P.42], $M^1(D)$ and D are homeomorphic under the map $\Phi : M^1(D) \to D$, where $\Phi(m^1(d_\omega)) = d_\omega$. Then, Lemma A.2 below follows immediately.

Lemma A.2: *$\mathbb{R}_{++} \times M^1(D(\bar{b};\underline{b}))$ is homeomorphic to $\mathbb{R}_{++} \times D(\bar{b};\underline{b})$, and $X(\bar{b};\bar{l};\underline{b};\underline{l}) \times M^1(D(\bar{b};\overline{lb};\underline{b};\underline{lb}))$ is homeomorphic to $X(\bar{b};\bar{l};\underline{b};\underline{l}) \times D(\bar{b};\overline{lb};\underline{b};\underline{lb})$.*

Appendix B: Proof of Proposition 1

As for necessity of the axioms, G in Proposition 1 trivially implies HI, ST, CP, IA, TC, MT, and ND. Since $V(d) = G(d)$ on $D(b)$, V also implies all axioms. The proof of sufficiency is based on Lemmas B.1 to B.4. We assume HI and ST so that the proof is based on a history-independent and stationary $\succeq$.

Lemma B.1. *For each $\bar{l} > \underline{l} > 0$ and each $c \in X(\bar{b};\bar{l};\underline{b};\underline{l})$, a preference relation $\succeq$ on $D_c(\bar{b};\bar{l};\underline{b};\underline{l})$ is represented by (3) with $L^{\bar{l},\underline{l}}(c, d_\omega)$ replacing $Z(c, d_\omega)$, where $L^{\bar{l},\underline{l}}(c, d_\omega)$ is continuous in $d_\omega \in D(\bar{b};\overline{lb};\underline{b};\underline{lb})$ and unique up to a positive affine transformation. Furthermore, $L^{\bar{l},\underline{l}}(c, d_\omega) \geq L^{\bar{l},\underline{l}}(c, d'_\omega)$ if and only if $d_\omega \succeq d'_\omega$. Moreover, there exist $d = (c, d_\omega)$ and $d' = (c, d'_\omega) \in D^1_c(\bar{b};\bar{l};\underline{b};\underline{l})$ such that $L^{\bar{l},\underline{l}}(c, d_\omega) > L^{\bar{l},\underline{l}}(c, d'_\omega)$.*

Proof. Let $c \in \mathbb{R}_{++}$ and $\bar{l} > \underline{l} > 0$ be given. Then, $\cup_{S=1}^{\infty} D^S_c(\bar{b};\bar{l};\underline{b};\underline{l})$ is a mixture space under the operation $+$. Also, $\cup_{S=1}^{\infty} M^S(D(\bar{b};\overline{lb};\underline{b};\underline{lb}))$ is dense in $M(D(\bar{b};\overline{lb};\underline{b};\underline{lb}))$, where $D(\bar{b};\overline{lb};\underline{b};\underline{lb})$ is compact. Thus, under CP and IA, by Theorem 5.24 of Kreps [17, P.68], (3) with $L^{\bar{l},\underline{l}}(c, d_\omega)$ replacing $Z(c_t, d_\omega)$ represents $\succeq$ on $D_c(\bar{b};\bar{l};\underline{b};\underline{l})$, where $L^{\bar{l},\underline{l}}(c, d_\omega)$ is continuous in $d_\omega \in D(\bar{b};\overline{lb};\underline{b};\underline{lb})$ and unique up to a positive affine transformation. By TC, $L^{\bar{l},\underline{l}}(c, d_\omega) \geq L^{\bar{l},\underline{l}}(c, d'_\omega)$ if and only if $d_\omega \succeq d'_\omega$. Also, by ND, there exist d_ω and $d'_\omega \in D(\bar{b};\overline{lb};\underline{b};\underline{lb})$ such that $d_\omega \succ d'_\omega$. Thus, by TC, there exist $d = (c, d_\omega)$ and $d' = (c, d'_\omega) \in D^1_c(\bar{b};\bar{l};\underline{b};\underline{l})$ such that $L^{\bar{l},\underline{l}}(c, d_\omega) > L^{\bar{l},\underline{l}}(c, d'_\omega)$. ■

Lemma B.2. *For each $\bar{l} > \underline{l} > 0$, a preference relation $\succeq$ on $D(\bar{b};\bar{l};\underline{b};\underline{l})$*

is represented by (3) *with* $Z^{\bar{l},\underline{l}} : X(\bar{b};\bar{l};\underline{b};\underline{l})\times\ D(\bar{b};\overline{lb};\underline{b};\underline{lb}) \to \mathbb{R}$ *replacing* Z, *where* $Z^{\bar{l},\underline{l}}$ *satisfies the conditions stated in Proposition* 1-(ii)-(a).

Proof. Let $\bar{l} > \underline{l} > 0$ be given. We want to construct $Z^{\bar{l},\underline{l}}(c, d_\omega)$ that covers all $c \in X(\bar{b};\bar{l};\underline{b};\underline{l})$ by connecting each $L^{\bar{l},\underline{l}}(c, d_\omega)$ defined in Lemma B.1. For this purpose, consider a continuous function $W(d)$ that represents $\succeq$ on $D(\bar{b};\bar{l};\underline{b};\underline{l})$; the existence of such a function follows from Debreu [8] under CP defined on a connected and separable $D(\bar{b};\bar{l};\underline{b};\underline{l})$. Given $D^1(\bar{b};\bar{l};\underline{b};\underline{l}) \subseteq D(\bar{b};\bar{l};\underline{b};\underline{l})$, by Lemma A.2, define a continuous $W^1 : X(\bar{b};\bar{l};\underline{b};\underline{l}) \times D(\bar{b};\overline{lb};\underline{b};\underline{lb}) \to \mathbb{R}$ by $W^1(c, d_\omega) \equiv W(d)$ on $d \in D^1(\bar{b};\bar{l};\underline{b};\underline{l})$; $W^1(.,.)$ is also identified as a function from $D^1(\bar{b};\bar{l};\underline{b};\underline{l})$ to $\mathbb{R}$.

For each $c \in X(\bar{b};\bar{l};\underline{b};\underline{l})$, $D^1_c(\bar{b};\bar{l};\underline{b};\underline{l})$ is connected and compact. Hence, the image of $D^1_c(\bar{b};\bar{l};\underline{b};\underline{l})$ under $W^1(.,.)$ is a connected and compact interval in $\mathbb{R}$, denoted by $I(c)$. By Lemma B.1, $I(c) = [a, b]$, where $a \neq b$. Since $X(\bar{b};\bar{l};\underline{b};\underline{l}) \times D(\bar{b};\overline{lb};\underline{b};\underline{lb})$ is connected and compact, $[\underline{a}, \bar{b}] \equiv \cup_{c\in X(\bar{b};\bar{l};\underline{b};\underline{l})} I(c)$ is connected and compact. For each $I(c) = [a, b]$, define $OI(c)$ by (i) $(a, b) \subseteq OI(c)$, (ii) if $a = \underline{a}$, then $a \in OI(c)$; otherwise $a \notin OI(c)$, and (iii) if $b = \bar{b}$, then $b \in OI(c)$; otherwise $b \notin OI(c)$. Let $\mathcal{R} = \{OI(c)|c \in X(\bar{b};\bar{l};\underline{b};\underline{l})\}$.

(Step 1) $\mathcal{R}$ is an open cover of $[\underline{a}, \bar{b}]$; $\mathcal{S} = \{OI(c^n)\}_{n=0}^N$ is a finite open subcover of a simple chain with $OI(c^0) = [\underline{a}, b)$ and $OI(c^N) = (a, \bar{b}]$.

Given that $[\underline{a}, \bar{b}]$ is connected and compact, it suffices to show that for any $a \in (\underline{a}, \bar{b})$, there exists $c \in X(\bar{b};\bar{l};\underline{b};\underline{l})$ such that $a \in OI(c)$. Suppose that $a \in (\underline{a}, \bar{b})$ does not belong to any element of $\mathcal{R}$. Let $\mathbb{C}_1 \equiv \{c \in X(\bar{b};\bar{l};\underline{b};\underline{l})|b' \leq a$ for $I(c) = [a', b']\}$ and $\mathbb{C}_2 \equiv \{c \in X(\bar{b};\bar{l};\underline{b};\underline{l})|a' \geq a$ for $I(c) = [a', b']\}$. Then, since $a \neq b$ for each $I(c) = [a, b]$, $\mathbb{C}_1$ and $\mathbb{C}_2$ are non-empty and disjoint. Also, $W^1(\mathbb{C}_1 \cup \mathbb{C}_2 \times D(\bar{b};\overline{lb};\underline{b};\underline{lb})) = [\underline{a}, \bar{b}]$ because $\mathbb{C}_1 \cup \mathbb{C}_2 = X(\bar{b};\bar{l};\underline{b};\underline{l})$.

Furthermore, $\mathbb{C}_1 \times D(\bar{b};\overline{lb};\underline{b};\underline{lb}) \supseteq (W^1)^{-1}([\underline{a},a))$ and $\mathbb{C}_2 \times D(\bar{b};\overline{lb};\underline{b};\underline{lb}) \supseteq (W^1)^{-1}((a,\bar{b}])$. Thus, $\mathbb{C}_1 \times D(\bar{b};\overline{lb};\underline{b};\underline{lb})\backslash(W^1)^{-1}(\{a\}) = (W^1)^{-1}([\underline{a},a))$ and $\mathbb{C}_2 \times D(\bar{b};\overline{lb};\underline{b};\underline{lb})\backslash(W^1)^{-1}(\{a\}) = (W^1)^{-1}((a,\bar{b}])$; they are both open because W^1 is continuous. Also, $a \neq b$ for each $I(c) = [a,b]$ so that $\pi_1(\mathbb{C}_1 \times D(\bar{b};\overline{lb};\underline{b};\underline{lb})\backslash(W^1)^{-1}(\{a\})) = \mathbb{C}_1$ and $\pi_1(\mathbb{C}_2 \times D(\bar{b};\overline{lb};\underline{b};\underline{lb})\backslash(W^1)^{-1}(\{a\})) = \mathbb{C}_2$, where π_1 is a projection map. Since π_1 is an open map, $\mathbb{C}_1$ and $\mathbb{C}_2$ are open. This contradicts the connectedness of $X(\bar{b};\bar{l};\underline{b};\underline{l})$. □

(Step 2) There exists a function $Z^{\bar{l},\underline{l}} : X(\bar{b};\bar{l};\underline{b};\underline{l}) \times D(\bar{b};\overline{lb};\underline{b};\underline{lb}) \to \mathbb{R}$, unique up to a positive affine transformation, such that $Z^{\bar{l},\underline{l}}(c,d_\omega)$ replaces $L^{\bar{l},\underline{l}}(c,d_\omega)$ in Lemma B.1, where $Z^{\bar{l},\underline{l}}(c,d_\omega)$ satisfies the conditions stated in Lemma B.1. Moreover, $(c,d_\omega) \succeq (c',d'_\omega)$ if and only if $Z^{\bar{l},\underline{l}}(c,d_\omega) \geq Z^{\bar{l},\underline{l}}(c',d'_\omega)$.

Let $Z^{\bar{l},\underline{l}}(c^0,d_\omega) \equiv \alpha(c^0)L^{\bar{l},\underline{l}}(c^0,d_\omega) + \beta(c^0)$ on $D^1_{c^0}(\bar{b};\bar{l};\underline{b};\underline{l})$, where $\alpha(c^0) \equiv 1$ and $\beta(c^0) \equiv 0$. By construction, $OI(c^0) \cap OI(c^1)$ is non-empty and contains more than two elements. Then, it follows from Lemma B.1, MT, and (ii) of CP (transitivity) that there exist real numbers $\alpha(c^1) > 0$ and $\beta(c^1)$ such that for any $(c^0,d_\omega),(c^1,d'_\omega) \in X(\bar{b};\bar{l};\underline{b};\underline{l}) \times D(\bar{b};\overline{lb};\underline{b};\underline{lb})$, $(c^0,d_\omega) \succeq (c^1,d'_\omega)$ if and only if $Z^{\bar{l},\underline{l}}(c^0,d_\omega) \geq Z^{\bar{l},\underline{l}}(c^1,d'_\omega)$, where

$$Z^{\bar{l},\underline{l}}(c^1,d_\omega) = \alpha(c^1)L^{\bar{l},\underline{l}}(c^1,d'_\omega) + \beta(c^1). \tag{B.1}$$

Hence, define $Z^{\bar{l},\underline{l}}(c^1,d_\omega)$ on $D^1_{c^1}(\bar{b};\bar{l};\underline{b};\underline{l})$ by (B.1). Since $\mathcal{S}$ is a simple chain, repeat the same construction until we exhaust all elements in $\{0,...,N\}$. This defines a function $Z^{\bar{l},\underline{l}}(c^i,d_\omega)$ on $D^1_{c^i}(\bar{b};\bar{l};\underline{b};\underline{l})$ for $\{c^i\}_{i=0}^N$.

Consider $c \notin \{c^i\}_{i=0}^N$. Let $\{OI(c^i)\}_{i=\underline{k}}^{\bar{k}}$ be a subcollection of $\{OI(c^i)\}_{i=0}^N$ such that (i) $OI(c) \subseteq \cup_{i=\underline{k}}^{\bar{k}} OI(c^i)$, (ii) $OI(c) \cap OI(c^{\underline{k}}) \neq \emptyset$, and (iii) $OI(c) \cap (OI(c^{\bar{k}})\backslash OI(c^{\bar{k}-1})) \neq \emptyset$ if $\bar{k} > \underline{k}$. Then, it follows from Lemma B.1, MT, and (ii) of CP (transitivity) that there exist real numbers $\alpha(c) > 0$ and $\beta(c)$ such that for any $(c^i, d_\omega), (c, d'_\omega) \in X(\bar{b}; \bar{l}; \underline{b}; \underline{l}) \times D(\bar{b}; \overline{lb}; \underline{b}; \underline{lb})$ for some $i \in \{\underline{k}, ..., \overline{k}\}$, $(c^i, d_\omega) \succeq (c, d'_\omega)$ if and only if $Z^{\bar{l},\underline{l}}(c^i, d_\omega) \geq Z^{\bar{l},\underline{l}}(c, d'_\omega)$, where

$$Z^{\bar{l},\underline{l}}(c, d_\omega) = \alpha(c) L^{\bar{l},\underline{l}}(c, d'_\omega) + \beta(c). \tag{B.2}$$

Define a function $Z^{\bar{l},\underline{l}}(c, d_\omega)$ on $D^1_c(\bar{b}; \bar{l}; \underline{b}; \underline{l})$ by (B.2). This proves the existence of a function $Z^{\bar{l},\underline{l}}(c, d_\omega)$ on $D^1(\bar{b}; \bar{l}; \underline{b}; \underline{l})$, which is unique up to a positive affine transformation. By construction, $(c, d_\omega) \succeq (c', d'_\omega)$ if and only if $Z^{\bar{l},\underline{l}}(c, d_\omega) \geq Z^{\bar{l},\underline{l}}(c', d'_\omega)$. □

(Step 3) $Z^{\bar{l},\underline{l}}$ is continuous on $X(\bar{b}; \bar{l}; \underline{b}; \underline{l}) \times D(\bar{b}; \overline{lb}; \underline{b}; \underline{lb})$.

Consider $(c, d_\omega) \in D^1(\bar{b}; \bar{l}; \underline{b}; \underline{l})$, where $W^1(c, d_\omega) \in (\underline{a}, \bar{b})$. By (Step 1), $W^1(c, d_\omega) \in int(I(c'))$ for some c'. Thus, there exists (c', d'_ω) such that $(c, d_\omega) \simeq (c', d'_\omega)$. Suppose that for some sequence $\{(c^n, d^n_\omega)\} \subset D^1(\bar{b}; \bar{l}; \underline{b}; \underline{l})$ converging to (c, d_ω), $Z^{\bar{l},\underline{l}}(c, d_\omega) > \liminf_n Z^{\bar{l},\underline{l}}(c^n, d^n_\omega)$. Since $Z^{\bar{l},\underline{l}}(c', .)$ is continuous on a connected space $D(\bar{b}; \overline{lb}; \underline{b}; \underline{lb})$, there exists $(c', d''_\omega) \in \{c'\} \times D(\bar{b}; \overline{lb}; \underline{b}; \underline{lb})$ such that $Z^{\bar{l},\underline{l}}(c', d'_\omega) > Z^{\bar{l},\underline{l}}(c', d''_\omega) > \liminf_n Z^{\bar{l},\underline{l}}(c^n, d^n_\omega)$. Then, for each $n \geq 0$, there exists $k(n) \geq n$ such that (i) $Z^{\bar{l},\underline{l}}(c', d''_\omega) > Z^{\bar{l},\underline{l}}(c^{k(n)}, d^{k(n)}_\omega)$ and (ii) $k(n) \geq k(n-1)$ if $n \geq 1$. Then, an infinite subsequence $\{(c^{k(n)}, d^{k(n)}_\omega)\}$ is in the set $\{d''' \in D^1(\bar{b}; \bar{l}; \underline{b}; \underline{l}) \;|(c', d''_\omega) \succeq d'''\}$. By (iii) of CP (continuity), $(c', d''_\omega) \succeq (c,\ d_\omega)$, which contradicts $Z^{\bar{l},\underline{l}}(c, d_\omega) = Z^{\bar{l},\underline{l}}(c', d'_\omega) > Z^{\bar{l},\underline{l}}(c', d''_\omega)$. Similarly, $\limsup_n Z^{\bar{l},\underline{l}}(c^n, d^n_\omega) > Z^{\bar{l},\underline{l}}(c, d_\omega)$ will lead to the contradiction of (ii) of CP. Hence, $\liminf_n Z^{\bar{l},\underline{l}}(c^n,\ d^n_\omega) = \limsup_n Z^{\bar{l},\underline{l}}(c^n, d^n_\omega)$.

At $Z^{\bar{l},\underline{l}}(c, d_\omega) = \underline{a}$ or $Z^{\bar{l},\underline{l}}(c, d_\omega) = \bar{b}$, the proof of continuity is similar (consider only $\limsup_n Z^{\bar{l},\underline{l}}(c^n,\ d^n_\omega)$ or $\liminf_n Z^{\bar{l},\underline{l}}(c^n, d^n_\omega)$). □

(Step 4) For all $d, d' \in D(\bar{b}; \bar{l}; \underline{b}; \underline{l})$, $d \succeq d'$ if and only if $G^{\bar{l},\underline{l}}(d) \geq G^{\bar{l},\underline{l}}(d')$, where $G^{\bar{l},\underline{l}}(d) = G^{\bar{l},\underline{l}}(c, m(d_\omega)) \equiv E_m\left[Z^{\bar{l},\underline{l}}(c, d_\omega)\right]$. Moreover, $G^{\bar{l},\underline{l}}(d)$ is continuous in $d = (c, m(d_\omega)) \in D(\bar{b}; \bar{l}; \underline{b}; \underline{l})$.

By Lemma B.1 and connectedness of $D(\bar{b};\bar{l};\underline{b};\underline{l})$, for any $d=(c,m)\in D_c(\bar{b};\bar{l};\underline{b};\underline{l})$, there exists $(c,d_\omega)\in D_c^1(\bar{b};\bar{l};\underline{b};\underline{l})$ such that $(c,d_\omega)\simeq d$. Then it follows from (Step 2) and (ii) of CP (transitivity) that for all $d,d'\in D(\bar{b};\bar{l};\underline{b};\underline{l})$, $d\succeq d'$ if and only if $G^{\bar{l},\underline{l}}(d)\geq G^{\bar{l},\underline{l}}(d')$.

For continuity, let A be an open set in $\mathbb{R}$. Without loss of generality, assume that $(G^{\bar{l},\underline{l}})^{-1}(A)\neq\emptyset$ and $(c,m(d_\omega))\in(G^{\bar{l},\underline{l}})^{-1}(A)$. Then, there exists $\varepsilon>0$ such that an open ball $B_\varepsilon(G^{\bar{l},\underline{l}}(c,m(d_\omega)))\subseteq A$. Since $X(\bar{b};\bar{l};\underline{b};\underline{l})\times D(\bar{b};\overline{lb};\underline{b};\underline{lb})$ is a compact metric space, $Z^{\bar{l},\underline{l}}(.,.)$ is uniformly continuous on $X(\bar{b};\bar{l};\underline{b};\underline{l})\times D(\bar{b};\overline{lb};\underline{b};\underline{lb})$. Thus, there exists $\widetilde{\varepsilon}$ such that for each $d'_\omega\in D(\bar{b};\overline{lb};\underline{b};\underline{lb})$, $|Z^{\bar{l},\underline{l}}(c',d'_\omega)-Z^{\bar{l},\underline{l}}(c,d'_\omega)|<\dfrac{\varepsilon}{2}$ for any $c'\in B_{\widetilde{\varepsilon}}(c)$. This implies that for each $m'\in M(D(\bar{b};\overline{lb};\underline{b};\underline{lb}))$, $|E_{m'}\left[Z^{\bar{l},\underline{l}}(c',d_\omega)\right]-E_{m'}\left[Z^{\bar{l},\underline{l}}(c,d_\omega)\right]|\leq\dfrac{\varepsilon}{2}$ for any $c'\in B_{\widetilde{\varepsilon}}(c)$. In addition, for a given $c\in X(\bar{b};\bar{l};\underline{b};\underline{l})$, $Z^{\bar{l},\underline{l}}(c,.)$ is continuous and bounded on a separable and compact metric space $D(\bar{b};\overline{lb};\underline{b};\underline{lb})$. By the weak topology, there exists $\widehat{\varepsilon}$ such that $|E_{m'}\left[Z^{\bar{l},\underline{l}}(c,d_\omega)\right]-E_m\left[Z^{\bar{l},\underline{l}}(c,d_\omega)\right]|<\dfrac{\varepsilon}{2}$ for any $m'\in B_{\widehat{\varepsilon}}(m)$. Thus, by the triangular inequality, for any $(c',m'(d_\omega))\in B_{\widetilde{\varepsilon}}(c)\times B_{\widehat{\varepsilon}}(m)$,

$$
\begin{aligned}
&|E_{m'}\left[Z^{\bar{l},\underline{l}}(c',d_\omega)\right]-E_m\left[Z^{\bar{l},\underline{l}}(c,d_\omega)\right]|\\
&=|E_{m'}\left[Z^{\bar{l},\underline{l}}(c',d_\omega)\right]-E_{m'}\left[Z^{\bar{l},\underline{l}}(c,d_\omega)\right]+E_{m'}\left[Z^{\bar{l},\underline{l}}(c,d_\omega)\right]-E_m\left[Z^{\bar{l},\underline{l}}(c,d_\omega)\right]|\\
&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,
\end{aligned}
$$

which implies that $G^{\bar{l},\underline{l}}(B_{\widetilde{\varepsilon}}(c)\times B_{\widehat{\varepsilon}}(m))\subseteq B_\varepsilon(G^{\bar{l},\underline{l}}(c,m(d_\omega)))\subseteq A$. Hence, $G^{\bar{l},\underline{l}}(c,m(d_\omega))$ is continuous in $d=(c,m(d_\omega))\in D(\bar{b};\bar{l};\underline{b};\underline{l})$. ■

Lemma B.3. *A preference relation* $\succeq$ *on* $D(\bar{b};\underline{b})$ *is represented by* (3).

Proof. Let $\bar{l}'$ and $\underline{l}'$ be given, where $\bar{l}'>\underline{l}'>0$. Consider $\bar{l}>0$ and $\underline{l}>0$ with $\bar{l}\geq\bar{l}'>\underline{l}'\geq\underline{l}$. Let $\eta(\bar{l}',\underline{l}'|\bar{l}',\underline{l}')\equiv 1$ and $\theta(\bar{l}',\underline{l}'|\bar{l}',\underline{l}')\equiv 0$. It follows from Lemma B.2, MT, and (ii) of CP (transitivity) that there exist real numbers $\eta(\bar{l},\underline{l}|\bar{l}',\underline{l}') > 0$ and $\theta(\bar{l},\underline{l}|\bar{l}',\underline{l}')$ such that for $(c, d_\omega) \in D^1(\bar{b};\bar{l};\underline{b};\underline{l})$ and $(c', d'_\omega) \in D^1(\bar{b};\bar{l}';\underline{b};\underline{l}')$, $(c, d_\omega) \succeq (c', d'_\omega)$ if and only if $Z(c, d_\omega) \geq Z(c', d'_\omega)$, where

$$Z(c, d_\omega) \equiv \eta(\bar{l},\underline{l}|\bar{l}',\underline{l}')Z^{\bar{l},\underline{l}}(c, d_\omega) + \theta(\bar{l},\underline{l}|\bar{l}',\underline{l}'). \tag{B.3}$$

Define Z on $\cup_{\bar{l}\geq\bar{l}'} \cup_{\underline{l}'\geq\underline{l}>0} D^1(\bar{b};\bar{l};\underline{b};\underline{l})$ by (B.3). Again by MT and (ii) of CP (transitivity), for $(c, d_\omega), (c', d'_\omega) \in \cup_{\bar{l}\geq\bar{l}'} \cup_{\underline{l}'\geq\underline{l}>0} D^1(\bar{b};\bar{l};\underline{b};\underline{l})$, $(c, d_\omega) \succeq (c', d'_\omega)$ if and only if $Z(c, d_\omega) \geq Z(c', d'_\omega)$. This proves the existence of Z that represents $\succeq$ on $D^1(\bar{b};\underline{b})$ because each $D^1(\bar{b};\bar{l};\underline{b};\underline{l})$ is a subset of $\cup_{\bar{l}\geq\bar{l}'} \cup_{\underline{l}'\geq\underline{l}>0} D^1(\bar{b};\bar{l};\underline{b};\underline{l})$. Other properties of Z stated in Proposition 1-(ii)-(a) is inherited from Lemma B.2.

For each $d = (c, m(d_\omega)) \in D(\bar{b};\underline{b})$, define $G(d) \equiv E_m[Z(c, d_\omega)]$. Since m is in $M(D(\bar{b};\overline{lb};\underline{b};\underline{lb}))$ for some $\bar{l} > \underline{l} > 0$, by (iii) of CP (continuity), we consider only a sequence of $\{(c^n, m^{S_n})\}$ that weakly converges to $(c, m) \in D(\bar{b};\underline{b})$, where each element $(c^n, m^{S_n}) \in D(\bar{b};\bar{l};\underline{b};\underline{l})$ for the same $\bar{l} > \underline{l} > 0$. Then, given the construction of Z above, it follows from Lemma B.2 and (ii) of CP (transitivity) that $G(d) = E_m[Z(c, d_\omega)]$ represents $\succeq$ on $D(\bar{b};\underline{b})$. Finally, by Lemma B.2, $G(d)$ is continuous on $D(\bar{b};\bar{l};\underline{b};\underline{l})$ for each $\bar{l} > \underline{l} > 0$. ■

Lemma B.4. *A preference relation $\succeq$ on $D(\bar{b};\underline{b})$ is represented by* (4).

Proof. First, define $U(c, \gamma_\omega) \equiv Z(c, d_\omega)$, where $\gamma_\omega \equiv G(d_\omega)$. By TC, $U(c, \gamma_\omega)$ is increasing in γ_ω and well-defined (i.e., if $(c, d_\omega) \simeq (c, d'_\omega)$, then $Z(c, d_\omega) = Z(c, d'_\omega)$ and $G(d_\omega) = G(d'_\omega)$). Let $J^{\bar{l},\underline{l}} \equiv X(\bar{b};\bar{l};\underline{b};\underline{l}) \times G(D(\bar{b};\overline{lb};\underline{b};\underline{lb})) \subseteq \mathbb{R}^2$, where $G(D(\bar{b};\overline{lb};\underline{b};\underline{lb}))$ is connected and compact (because $D(\bar{b};\overline{lb};\underline{b};\underline{lb})$ is connected and compact and $G(d)$ is continuous in $d \in D(\bar{b};\overline{lb};\underline{b};\underline{lb})$).

(Step 1) U is continuous on the domain $\cup_{\bar{l}>0} \cup_{\bar{l}>\underline{l}>0} J^{\bar{l},\underline{l}}$.

Consider a sequence $\{(c^n, \gamma_\omega^n)\}$ in $\cup_{\bar{l}>0} \cup_{\bar{l}>\underline{l}>0} J^{\bar{l},\underline{l}}$ that converges to $(c, \gamma_\omega) \in \cup_{\bar{l}>0} \cup_{\bar{l}>\underline{l}>0} J^{\bar{l},\underline{l}}$. Then, there exist $\bar{l}' > \underline{l}' > 0$ and $N > 0$ such that $(c^n, \gamma^n) \in J^{\bar{l}',\underline{l}'}$ for all $n \geq N$ (this implies $(c, \gamma_\omega) \in J^{\bar{l}',\underline{l}'}$). Without loss of generality, let $N \equiv 0$.

Assume that there exist $(c, d_\omega), (c', d'_\omega) \in D^1(\bar{b}; \bar{l}'; \underline{b}; \underline{l}')$ such that $U(c', G(d'_\omega)) < U(c, \gamma_\omega) < U(c, G(d_\omega))$ (the proofs of the other cases are similar). Suppose that $\liminf_n U(c^n, \gamma_\omega^n) < U(c, \gamma_\omega)$. Let $\alpha \in \mathbb{R}$ be such that $U(c, \gamma_\omega) > \alpha > \liminf_n U(c^n, \gamma_\omega^n)$. Then, for each $n \geq 0$, there exists $k(n) \geq n$ such that $\alpha > U(c^{k(n)}, \gamma_\omega^{k(n)})$ and $k(n{+}1) \geq k(n)$. Let $\{(c^{k(n)}, d_\omega^{k(n)})\} \subset D^1(\bar{b}; \bar{l}'; \underline{b}; \underline{l}')$ be a corresponding sequence, where $d_\omega^{k(n)} \in G^{-1}(\gamma_\omega^{k(n)})$. Also, $(c, d_\omega) \in D^1(\bar{b}; \bar{l}'; \underline{b}; \underline{l}')$ is a consumption program such that $d_\omega \in G^{-1}(\gamma_\omega)$.

$Z(.,.)$ is continuous on a connected space $X(\bar{b}; \bar{l}'; \underline{b}; \underline{l}') \times D(\bar{b}; \bar{l}'\bar{b}; \underline{b}; \underline{l}'\underline{b})$. Thus, there exists $(\widetilde{c}, \widetilde{d}_\omega) \in D^1(\bar{b}; \bar{l}'; \underline{b}; \underline{l}')$ such that $Z(c, d_\omega) > Z(\widetilde{c}, \widetilde{d}_\omega) > \alpha > Z(c^{k(n)}, d_\omega^{k(n)})$ for each $k(n)$. By Lemma B.3, $(c, d_\omega) \succ (\widetilde{c}, \widetilde{d}_\omega) \succ (c^{k(n)}, d_\omega^{k(n)})$. Since $\{(c^{k(n)}, d_\omega^{k(n)})\}$ is in a compact space $X(\bar{b}; \bar{l}'; \underline{b}; \underline{l}') \times D(\bar{b}; \bar{l}'\bar{b}; \underline{b}; \underline{l}'\underline{b})$, there exists a subsequence $\{(c^{k_j(n)}, d_\omega^{k_j(n)})\}$ that converges to some $(c, \widehat{d}_\omega) \in X(\bar{b}; \bar{l}'; \underline{b}; \underline{l}') \times D(\bar{b}; \bar{l}'\bar{b}; \underline{b}; \underline{l}'\underline{b})$. Given that $\{(c^{k(n)}, \gamma_\omega^{k(n)})\}$ converges to (c, γ_ω), by continuity of G on $D(\bar{b}; \bar{l}\bar{b}; \underline{b}; \underline{l}\underline{b})$, $G(\widehat{d}_\omega) = \gamma_\omega$. Then, by TC, $(c, \widehat{d}_\omega) \simeq (c, d_\omega)$. However, an infinite subsequence $\{(c^{k_j(n)}, d_\omega^{k_j(n)})\}$ is in the set $\{(c', d'_\omega) \in D^1(\bar{b}; \bar{l}'; \underline{b}; \underline{l}') \,|\, (\widetilde{c}, \widetilde{d}_\omega) \succeq (c'_t, d'_\omega)\}$. By (iii) of CP (continuity), $(\widetilde{c}, \widetilde{d}_\omega) \succeq (c, \widehat{d}_\omega)$, which contradicts $(c, \widehat{d}_\omega) \simeq (c, d_\omega) \succ (\widetilde{c}, \widetilde{d}_\omega)$. Similarly, $\limsup_n U(c^n, \gamma_\omega^n) > U(c, \gamma_\omega)$ leads to a contradiction of (iii) of CP. Thus, $\liminf_n U(c^n, \gamma_\omega^n) = \limsup_n U(c^n, \gamma_\omega^n)$.
□

By construction, $U(.,.)$ is unique up to a positive affine transformation on the domain $\cup_{\bar{l}>0} \cup_{\bar{l}>\underline{l}>0} J^{\bar{l},\underline{l}}$. Extend $U(.,.)$ continuously to $\mathbb{R}_{++} \times \mathbb{R}$ as an increasing function in the second argument. Finally, define $V(d) \equiv E_m[U(c, G(d_\omega))]$.

Clearly, $V(d) = E_m[U(c, V(d_\omega))] = E_m[Z(c, d_\omega)] = G(d)$ if $d \in D(\bar{b}; \underline{b})$. Then, by Lemma B.3, $V(d)$ is continuous on $D(\bar{b}; \bar{l}; \underline{b}; \underline{l})$ for each $\bar{l} > \underline{l} > 0$. ■

Appendix C: Proof of Proposition 2

The proof follows Appendix 3 of Epstein and Zin [11] with a slight modification. Let $S(D(\bar{b}; \underline{b}))$ be the set of all functions v from $D(\bar{b}; \underline{b})$ into $\mathbb{R}$ such that v is continuous on each $D(\bar{b}; \bar{l}; \underline{b}; \underline{l})$. For a strictly positive $h \in S(D(\bar{b}; \underline{b}))$, $h-$norm is defined for $v \in S(D(\bar{b}; \underline{b}))$ by $\| v \|_h \equiv \sup \frac{|v(d)|}{h(d)}$. Then, we define the following subspace of $S(D(\bar{b}; \underline{b}))$:

$$S_h(D(\bar{b}; \underline{b})) \equiv \{v \in S(D(\bar{b}; \underline{b})) | \| v \|_h \equiv \sup \frac{|v(d)|}{h(d)} < \infty\}. \tag{C.1}$$

$S_h(D(\bar{b}; \underline{b}))$ is a complete metric space under the norm $\| v \|_h$. Let $S^+(D(\bar{b}; \underline{b}))$ and $S^-(D(\bar{b}; \underline{b}))$ be a collection of all functions in $S(D(\bar{b}; \underline{b}))$ whose values are in $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively; we also define complete metric spaces $S_h^+(D(\bar{b}; \underline{b})) \subset S^+(D(\bar{b}; \underline{b}))$ and $S_h^-(D(\bar{b}; \underline{b})) \subset S^-(D(\bar{b}; \underline{b}))$ based on (C.1). A transformation $T : S_h(D(\bar{b}; \underline{b})) \to S_h(D(\bar{b}; \underline{b}))$ is a strict contraction if $\| Tv - T\varphi \|_h \leq \Theta \| v - \varphi \|_h$ with $0 < \Theta < 1$.

(Case 1): $\alpha \leq \rho < 0$ and $\beta \underline{b}^\rho < 1$.

First, we derive the following version of the weighted contraction mapping theorem, which modifies the original theorem developed by Boyd [2].

Weighted Contraction Mapping Theorem II (WCMT-II): Let $T : S_h^-(D(\bar{b}; \underline{b})) \to S^-(D(\bar{b}; \underline{b}))$ be such that (i) $u \leq v \Rightarrow T(u) \leq T(v)$, (ii) $T(0) \in S_h^-(D(\bar{b}; \underline{b}))$, and (iii) $T(u + Ah) \geq T(u) + \Theta Ah$ for some constant $0 < \Theta < 1$ and for all $A < 0$. Then, T has a unique fixed point v^* in $S_h^-(D(\bar{b}; \underline{b}))$. Moreover, $T^N(0) \to v^*$ in $S_h^-(D(\bar{b}; \underline{b}))$.

Proof. This proof follows P. 332 of Boyd [2]. For all $\zeta, \psi \in S_h^-(D(\bar{b};\underline{b}))$, $\zeta - \psi \in S_h(D(\bar{b};\underline{b}))$ and $|\zeta - \psi| = |(-\psi) - (-\zeta)| \leq \| \zeta - \psi \|_h \cdot h$. So, $-\zeta \leq -\psi + \| \zeta - \psi \|_h \cdot h$ and $-\psi \leq -\zeta + \| \zeta - \psi \|_h \cdot h$. Multiplying both sides by (-1) yields $\zeta \geq \psi - \| \zeta - \psi \|_h \cdot h$ and $\psi \geq \zeta - \| \zeta - \psi \|_h \cdot h$. Then, (i) and (iii) yield $T(\zeta) \geq T(\psi) - \Theta \| \zeta - \psi \|_h \cdot h$ and $T(\psi) \geq T(\zeta) - \Theta \| \zeta - \psi \|_h \cdot h$. Multiplying both sides (-1) leads to $-T(\zeta) \leq -T(\psi) + \Theta \| \zeta - \psi \|_h \cdot h$ and $-T(\psi) \leq -T(\zeta) + \Theta \| \zeta - \psi \|_h \cdot h$, which implies that $|(-T(\psi)) - (-T(\zeta))| = |T(\zeta) - T(\psi)| \leq \Theta \| \zeta - \psi \|_h \cdot h$. Thus, $\| T(\zeta) - T(\psi) \|_h \leq \Theta \| \zeta - \psi \|_h$.

By setting $\psi = 0$ (the function assigns zero for any d in $D(\bar{b};\underline{b})$), we have $\| T(\zeta) \|_h - \| T(0) \|_h \leq \| T(\zeta) - T(0) \|_h \leq \Theta \| \zeta \|_h$. Thus, (ii) implies that $\| T(\zeta) \|_h \leq \Theta \| \zeta \|_h + \| T(0) \|_h < \infty$ and $T : S_h^-(D(\bar{b};\underline{b})) \to S_h^-(D(\bar{b};\underline{b}))$ (i.e., T is bounded). Since $0 < \Theta < 1$, T is a strict contraction on $S_h^-(D(\bar{b};\underline{b}))$. Hence, by the Contraction Mapping Theorem, it has a unique fixed point. ■

Proof. We use the following strictly positive $h \in S^+(D(\bar{b};\underline{b}))$ as a weighting function: For $d = (c_0, m(d_\omega))$ with $m_1 = f(P_2 m)$,

$$h(d) \equiv [1 + c_0^\alpha + E_{m_1} \sum_1^\infty \lambda^t (\frac{\widetilde{c}_t}{\underline{b}^t})^\alpha]^{\frac{\rho}{\alpha}}, \tag{C.2}$$

where $\widetilde{c}_t$ denotes a random payment at period t and $\lambda \in (0,1)$ will be defined below (f and P_2 are defined by (A.2) and (A.6), respectively; m_1 is an atemporal probability measure on future consumption sequences induced by m). $h(d)$ is finite for each $d \in D(\bar{b};\underline{b})$ because the summation is finite for each $y \in Y(\bar{b};\bar{l};\underline{b};\underline{l})$ and $m_1 \in \cup_{\bar{l}>0} \cup_{\bar{l}>\underline{l}>0} M(Y(\bar{b};\bar{l};\underline{b};\underline{l}))$.

Note that $h \in L^{\frac{\alpha}{\rho}}(m)$ for any $m \in \widehat{M}(D(\bar{b};\underline{b}))$. Then, for any $v \in S_h^-(D(\bar{b};\underline{b}))$, $|v(d)| \leq \| v \|_h h(d)$, so $E_m[|v(d_\omega)|^{\frac{\alpha}{\rho}}]^{\frac{\rho}{\alpha}} \leq \| v \|_h E_m[h(d_\omega)^{\frac{\alpha}{\rho}}]^{\frac{\rho}{\alpha}} < \infty$. Also, for any $c_0 \in \mathbb{R}_{++}$, c_0^ρ is a constant function on $D(\bar{b};\underline{b})$. It follows that $c_0^\rho + \beta\rho v \in L^{\frac{\alpha}{\rho}}(m)$ for any $m \in \widehat{M}(D(\bar{b};\underline{b}))$. Then, define the following transformation T: For each $v \in S_h^-(D(\bar{b};\underline{b}))$ and each $(c_0, m(d_\omega)) \in D(\bar{b};\underline{b})$,

$$T(v)(c_0, m(d_\omega)) \equiv \frac{1}{\rho} E_m[(c_0^\rho + \beta\rho v(d_\omega))^{\frac{\alpha}{\rho}}]^{\frac{\rho}{\alpha}}. \tag{C.3}$$

Then, it follows from (Step 4) of Lemma B.2 that $T(v)$ is continuous on each $D(\bar{b};\bar{l};\underline{b};\underline{l})$ so that $T : S_h^-(D(\bar{b};\underline{b})) \to S^-(D(\bar{b};\underline{b}))$.

We want to show that T satisfies WCMT-II. Conditions (i) and (ii) follow immediately. For (iii), let $v \in S_h^-(D(\bar{b};\underline{b}))$ and $A \in \mathbb{R}_{--}$. Then,

$$\begin{aligned} T(v + Ah)(c_0, m(d_\omega)) &= \frac{1}{\rho} E_m[(c_0^\rho + \beta\rho(v(d_\omega) + Ah(d_\omega)))^{\frac{\alpha}{\rho}}]^{\frac{\rho}{\alpha}} \\ &\geq \frac{1}{\rho} E_m[(c_0^\rho + \beta\rho v(d_\omega)))^{\frac{\alpha}{\rho}}]^{\frac{\rho}{\alpha}} + \frac{1}{\rho} E_m[(\beta\rho Ah(d_\omega))^{\frac{\alpha}{\rho}}]^{\frac{\rho}{\alpha}} \\ &= T(v)(c_0, m(d_\omega)) + \beta A E_m[h(d_\omega)^{\frac{\alpha}{\rho}}]^{\frac{\rho}{\alpha}} \\ &\geq T(v)(c_0, m(d_\omega)) + \beta \frac{\underline{b}^\rho}{\lambda^{\frac{\rho}{\alpha}}} A E_m[h(c_0, d_\omega)^{\frac{\alpha}{\rho}}]^{\frac{\rho}{\alpha}} \\ &= T(v)(c_0, m(d_\omega)) + \beta \frac{\underline{b}^\rho}{\lambda^{\frac{\rho}{\alpha}}} Ah(c_0, m(d_\omega)), \end{aligned}$$

where the second line follows by Minkowski's inequality with $\frac{\alpha}{\rho} \geq 1$, the fourth line follows from $\frac{\underline{b}^\alpha}{\lambda} > 1$ (so that $h(d_\omega)^{\frac{\alpha}{\rho}} \leq \frac{\underline{b}^\alpha}{\lambda} h(c_0, d_\omega)^{\frac{\alpha}{\rho}}$), and the last line follows from the property that $h(c_0, .)^{\frac{\alpha}{\rho}}$ satisfies the independence axiom on $\widehat{M}(D(\bar{b};\underline{b}))$. Hence, (iii) of WCMT-II is satisfied with $\Theta = \beta\frac{\underline{b}^\rho}{\lambda^{\frac{\rho}{\alpha}}}$ if λ is any real number $\beta\underline{b}^\rho < \lambda^{\frac{\rho}{\alpha}}$. Given $\lambda \in (0, 1)$, this condition leads to $\beta\underline{b}^\rho < 1$.

By WCMT-II, $T^N(0) \to V$ in the $\| \,.\, \|_h$ topology on $S_h^-(D(\bar{b};\underline{b}))$. Moreover, since $D(\bar{b};\bar{l};\underline{b};\underline{l})$ is compact, $\max\{|V(d)||d \in D(\bar{b};\bar{l};\underline{b};\underline{l})\}<\infty$. ∎

(Case 2): $0 < \rho \leq \alpha \leq 1$ and $\beta\bar{b}^\rho < 1$.

We use the following theorem applied in Appendix 3 of Epstein and Zin [11], which is adapted from Boyd [2].

Weighted Contraction Mapping Theorem (WCMT): Let $T : S_h^+(D(\bar{b};\underline{b})) \to S^+(D(\bar{b};\underline{b}))$ be such that (i) $u \leq v \Rightarrow T(u) \leq T(v)$, (ii) $T(0) \in S_h^+(D(\bar{b};\underline{b}))$, and (iii) $T(u + Ah) \leq T(u) + \Theta Ah$ for some constant $0 < \Theta < 1$ and for all $A > 0$. Then, T has a unique fixed point v^* in $S_h^+(D(\bar{b};\underline{b}))$. Moreover, $T^N(0) \to v^*$ in $S_h^+(D(\bar{b};\underline{b}))$.

Proof: We replace (C.2) with the following strictly positive $h \in S^+(D(\bar{b};\underline{b}))$: For $d = (c_0, m(d_\omega))$ with $m_1 = f(P_2 m)$,

$$h(d) \equiv [1 + c_0^\alpha + E_{m_1} \sum_1^\infty \lambda^t (\frac{\widetilde{c}_t}{\bar{b}^t})^\alpha]^{\frac{\rho}{\alpha}}.$$

Then, $h(d)$ is finite for each $d \in D(\bar{b};\underline{b})$. Define the transformation $T : S_h^+(D(\bar{b};\underline{b}))) \to S^+(D(\bar{b};\underline{b}))$ by (C.3).

We want to show that T satisfies WCMT. Conditions (i) and (ii) follow immediately. For (iii), by a similar argument to that used in (Case 1), for $v \in S_h^+(D(\bar{b};\underline{b}))$ and $A \in \mathbb{R}_{++}$,

$$T(v + Ah)(c_0, m(d_\omega)) \leq Tv(c_0, m(d_\omega)) + \beta \frac{\bar{b}^\rho}{\lambda^{\frac{\rho}{\alpha}}} Ah(c_0, m(d_\omega)).$$

Note that $\rho > 0$ and $A > 0$ reverse the direction of the inequality. Then (iii) is satisfied if $\beta \bar{b}^\rho < 1$.

By WCMT, $T^N(0) \to V$ in the $\| \ . \ \|_h$ topology on $S_h^+(D(\bar{b};\underline{b}))$. It also follows that $\max\{|V(d)||d \in D(\bar{b};\bar{l};\underline{b};\underline{l})\}<\infty$. ■

Finally, we characterize comparative temporal risk aversion based on parameter values α and ρ. For (Case 1), let $\widehat{V} = \frac{1}{\alpha}(\rho V)^{\frac{\alpha}{\rho}}$, where V is derived above. Similarly, let $\widetilde{V} = \frac{1}{\alpha}(\rho V')^{\frac{\alpha}{\rho}}$, where V' is a limit of

$$T'(v)(c_0, m(d_\omega)) \equiv \frac{c_0^\rho}{\rho} + \beta \frac{1}{\rho}\{E_m[(\rho v(d_\omega)]^{\frac{\alpha}{\rho}}\}^{\frac{\rho}{\alpha}};$$

Under this $T'(v)$, Proposition 2 holds for $\frac{1}{\rho}\left\{\{\alpha \times (6)\}^{\frac{1}{\alpha}}\right\}^{\rho}$. By definition, $\widehat{V}$ satisfies (5) and $\widetilde{V}$ satisfies (6). For each $d = (d_1, d_2, ...)$, define a sequence $\{\widehat{d_n}\}$ such that $\widehat{d_n} = (d_1, .., d_n, d_n, ...)$. By construction, $\{\widehat{d_n}\}$ converges to d, and $\widehat{V}$ and $\widetilde{V}$ agree on the set of deterministic consumption sequences. Then, by Jensen's inequality, $\widetilde{V}(\widehat{d_1}) \leq \widehat{V}(\widehat{d_1})$ and

$$\begin{aligned}\widetilde{V}(\widehat{d_2}) = \frac{1}{\alpha}\left[c_0^{\rho} + \beta\{\alpha E_m(\widetilde{V}(\widehat{d}_{\omega,1}))\}^{\frac{\rho}{\alpha}}\right]^{\frac{\alpha}{\rho}} &\leq \frac{1}{\alpha}\left[c_0^{\rho} + \beta\{\alpha E_m(\widehat{V}(\widehat{d}_{\omega,1}))\}^{\frac{\rho}{\alpha}}\right]^{\frac{\alpha}{\rho}} \\ &\leq E_m\left[\frac{1}{\alpha}\left\{c_0^{\rho} + \beta(\alpha\widehat{V}(\widehat{d}_{\omega,1}))^{\frac{\rho}{\alpha}}\right\}^{\frac{\alpha}{\rho}}\right] \\ &= \widehat{V}(\widehat{d_2}).\end{aligned}$$

It follows from this process that $\widetilde{V}(d) \leq \widehat{V}(d)$ because $\widetilde{V}$ and $\widehat{V}$ are continuous on each $D(\bar{b}; \bar{l}; \underline{b}; \underline{l})$. (Case 2) follows by a similar argument.

For other possible combination of parameter values α and ρ, it is enough to construct $\widehat{V}$ and $\widetilde{V}$ that satisfy (5) and (6), respectively, and agree on the set of deterministic consumption sequences. Then it follows from the above argument that $\widetilde{V}(\widehat{d_t}) \geq (\leq)\widehat{V}(\widehat{d_t})$ for all t if if $\alpha > (<)\ \rho$. We apply the argument similar to Appendix 3 of Epstein and Zin [11].

For $\rho > 0$, $\alpha < \rho$, and $\beta\bar{b}^{\rho} < 1$, consider $T^N(V^*)$ and $T'^N(V^*)$, where V^* is the limit of $T^N(0)$ at $\alpha = \rho$. Then $\{T^N(V^*)\}$ and $\{T'^N(V^*)\}$ are decreasing sequences bounded below by 0. Let $V \equiv \lim T^N(V^*)$ and let $V' \equiv \lim T'^N(V^*)$. V and V' are upper semicontinuous on each $D(\bar{b}; \bar{l}; \underline{b}; \underline{l})$ because each $T^N(V^*)$ and $T'^N(V^*)$ is continuous on $D(\bar{b}; \bar{l}; \underline{b}; \underline{l})$; they are also bounded on each $D(\bar{b}; \bar{l}; \underline{b}; \underline{l})$. By construction, V and V' agree on the set of deterministic consumption sequences. Finally, let $\widehat{V} \equiv \frac{1}{\alpha}(\rho V)^{\frac{\alpha}{\rho}}$ and let $\widetilde{V} \equiv \frac{1}{\alpha}(\rho V')^{\frac{\alpha}{\rho}}$.

For $\rho < 0$, $\alpha > \rho$, and $\beta\underline{b}^{\rho} < 1$, consider $T^N(0)$ and $T'^N(0)$. Then $\{T^N(0)\}$ and $\{T'^N(0)\}$ are decreasing sequences bounded below by V^*, where V^* is the limit of $T^N(0)$ at $\alpha = \rho$. Let $V \equiv \lim T^N(0)$ and let $V' \equiv \lim T'^N(0)$. By construction, V and V' are bounded and upper semicontinuous on each $D(\bar{b}; \bar{l}; \underline{b}; \underline{l})$, and V and V' agree on the set of deterministic consumption sequences. Finally, let $\widehat{V} \equiv \frac{1}{\alpha}(\rho V)^{\frac{\alpha}{\rho}}$ and let $\widetilde{V} \equiv \frac{1}{\alpha}(\rho V')^{\frac{\alpha}{\rho}}$.

Appendix D: Proof of Proposition 3

First, following Appendix 4 of Epstein and Zin [11], we construct an embedding map $\phi_{\omega^t} : C(\bar{b}; \underline{b} | \mathcal{F}_t(\omega^t)) \to D(b; \gamma)$ by defining m_t inductively. At each (t, ω^t), define $m_1(.|\mathcal{F}_t(\omega^t))$ as follows: for $B \in \mathbb{B}(D_0(\bar{b}; \underline{b}))$,

$$m_1(B|\mathcal{F}_t(\omega^t)) \equiv P(\{\omega^\infty \in \Omega^\infty | \widetilde{c}(\omega^\infty|\omega^{t+1}) \in B\}|\mathcal{F}_t(\omega^t)),$$

where $\widetilde{c}(\omega^\infty|\omega^{t+1}) \in D_0(\bar{b}; \underline{b}) \equiv \cup_{\bar{l}>0} \cup_{\bar{l}>\underline{l}>0} Y(\bar{b}; \bar{l}; \underline{b}; \underline{l})$ is the continuation of an adapted stochastic process $\widetilde{c} \in C(\bar{b}; \underline{b})$ from time $t+1$ onward assigned on a state ω^∞. Then, $m_1 \in \cup_{\bar{l}>0} \cup_{\bar{l}>\underline{l}>0} M(Y(\bar{b}; \bar{l}; \underline{b}; \underline{l}))$. Suppose that we have constructed $m_\tau(.|\mathcal{F}_t(\omega^t)) \in \cup_{\bar{l}>0} \cup_{\bar{l}>\underline{l}>0} M(D_{\tau-1}(\bar{b}; \underline{b}))$ for some $\tau \geq 1$ for all (t, ω^t). Then, define $m_{\tau+1}(.|\mathcal{F}_t(\omega^t))$ as follows: for $B \in \mathbb{B}(D_\tau(\bar{b}; \underline{b}))$,

$$m_{\tau+1}(B|\mathcal{F}_t(\omega^t)) \equiv P(\{\omega^\infty \in \Omega^\infty | (c_{t+1}(\omega^{t+1}), m_\tau(.|\mathcal{F}_{t+1}(\omega^{t+1}))) \in B\}|\mathcal{F}_t(\omega^t)).$$

By induction, the map $\phi_{\omega^t}(\widetilde{c}(\mathcal{F}_t(\omega^t)))$ is constructed by

$$\phi_{\omega^t}(\widetilde{c}(\mathcal{F}_t(\omega^t))) \equiv (d_1, d_2, ...) = ((c_t(\omega^t), m_1(.|\mathcal{F}_t(\omega^t))), (c_t(\omega^t), m_2(.|\mathcal{F}_t(\omega^t))), ...).$$

Lemma D.1: *Under Assumption 1, $V(\phi_{\omega^t}(\widetilde{c}(\mathcal{F}_t(\omega^t))))$ is concave and homogeneous of degree ρ in $\widetilde{c}(\mathcal{F}_t(\omega^t))$.*

Proof. We prove this claim by induction. For this purpose, we identify $\phi_{\omega^t}(\widetilde{c}(\mathcal{F}_t(\omega^t)))$ with $d_{\omega^t} = (c_t(\omega^t), m(\omega_t, .)(d_{\omega^{t+1}}))$, where m is a Markov transition matrix. By (C.3), define the sequence $\{v^n\}$ of non-positive real-valued functions on $C(\bar{b};\underline{b}|\mathcal{F}_t(\omega^t))$ inductively by

$$v^1(d_{\omega^t}) \equiv T0(d_{\omega^t}) = \frac{c_t^\rho(\omega^t)}{\rho},$$
$$v^n(d_{\omega^t}) \equiv T^n 0(d_{\omega^t}) = \frac{1}{\rho}E_{m(\omega_t,.)}[(c_t^\rho(\omega^t) + \beta\rho v^{n-1}(d_{\omega^{t+1}}))^{\frac{\alpha}{\rho}}]^{\frac{\rho}{\alpha}} \text{ for } n \geq 2.$$

Note that $V(d)$ is defined as a limit of $T^N(0)$, and $V(d)$ and each $T^N(0)$ are continuous on each $C(\bar{b};\bar{l};\underline{b};\underline{l}|\mathcal{F}_t(\omega^t))$ (called Property 1).

As for homogeneity, $T^N(0)(kd_{\omega^t}) = k^\rho T^N(0)(d_{\omega^t})$ for $k > 0$. Thus, the conclusion follows from Property 1. Note that $kd_{\omega^t} \in C(\bar{b};\underline{b}|\mathcal{F}_t(\omega^t))$ if $d_{\omega^t} \in C(\bar{b};\underline{b}|\mathcal{F}_t(\omega^t))$.

For concavity, at each (t,ω^t), $v^1(d_{\omega^t})$ is concave in d_{ω^t}. Assume that for a given $n \geq 2$, at each (t,ω^t), $v^{n-1}(d_{\omega^t})$ is concave in d_{ω^t}. Then for v^n,

$$\begin{aligned}
& v^n(\frac{1}{2}d_{\omega^t} + \frac{1}{2}d'_{\omega^t}) \\
& \equiv \frac{1}{\rho}E_{m(\omega_t,.)}[((\frac{1}{2}c_t(\omega^t) + \frac{1}{2}c'_t(\omega^t))^\rho + \beta\rho v^{n-1}(\frac{1}{2}d_{\omega^{t+1}} + \frac{1}{2}d'_{\omega^{t+1}}))^{\frac{\alpha}{\rho}}]^{\frac{\rho}{\alpha}} \\
& \geq \frac{1}{\rho}E_{m(\omega_t,.)}[(\frac{1}{2}c_t^\rho(\omega^t) + \frac{1}{2}c_t'^\rho(\omega^t) + \beta\rho v^{n-1}(\frac{1}{2}d_{\omega^{t+1}} + \frac{1}{2}d'_{\omega^{t+1}}))^{\frac{\alpha}{\rho}}]^{\frac{\rho}{\alpha}} \\
& \geq \frac{1}{\rho}E_{m(\omega_t,.)}[(\frac{1}{2}c_t^\rho(\omega^t) + \frac{1}{2}c_t'^\rho(\omega^t) + \beta\rho(\frac{1}{2}v^{n-1}(d_{\omega^{t+1}}) + \frac{1}{2}v^{n-1}(d'_{\omega^{t+1}})))^{\frac{\alpha}{\rho}}]^{\frac{\rho}{\alpha}} \\
& \geq \frac{1}{\rho}E_{m(\omega_t,.)}[(\frac{1}{2}c_t^\rho(\omega^t) + \beta\rho\frac{1}{2}v^{n-1}(d_{\omega^{t+1}})))^{\frac{\alpha}{\rho}}]^{\frac{\rho}{\alpha}} \\
& + \frac{1}{\rho}E_{m(\omega_t,.)}[(\frac{1}{2}c_t'^\rho(\omega^t) + \beta\rho\frac{1}{2}v^{n-1}(d'_{\omega^{t+1}}))^{\frac{\alpha}{\rho}}]^{\frac{\rho}{\alpha}} \\
& = \frac{1}{2}v^n(d_{\omega^t}) + \frac{1}{2}v^n(d'_{\omega^t}),
\end{aligned}$$

where the third line follows from the convexity of $c_t^\rho(\omega^t)$, the fourth line follows from the concavity of v^{n-1}, and the fifth line follows from Minkowski's inequality with $\dfrac{\alpha}{\rho} \geq 1$. Hence, $v^n(d_{\omega^t})$ is concave in d_{ω^t}. By induction, the conclusion follows from Property 1. ■

Proof of Proposition 3. It suffices to consider the optimization problem at each $(1, \omega^1)$. We examine a critical point of the following Lagrangian problem:

$$
\begin{aligned}
L(\widetilde{c}, \widetilde{\theta}^1, \widetilde{\theta}^2, \widetilde{\lambda}) &= V(\phi_{\omega^1}(\widetilde{c}(\mathcal{F}_1(\omega^1))) \\
&\quad - E_Q\left[\sum_{t=1}^{\infty} \widetilde{\lambda}_t(\widetilde{c}_t + \widetilde{\theta}_t^1 \widetilde{q}_t^1 + \widetilde{\theta}_t^2 \widetilde{q}_t^1 - \widetilde{\theta}_{t-1}^1(\widetilde{q}_t^1 + \widetilde{d}_t^1) - \widetilde{\theta}_{\tau-1}^2) | \mathcal{F}_1(\omega^1)\right],
\end{aligned}
$$

where $\widetilde{c} \in C(\bar{b}; \underline{b})$, $\theta_t^1(\omega^t) \in [1-\varepsilon, 1+\varepsilon]$, and $\theta_t^2(\omega^t) \in [-\varepsilon, \varepsilon]$. The budget constraint (9) is binding for all $t \geq 1$ because V is increasing. Also, the equilibrium allocation $\{\widetilde{c}^*, (\widetilde{\theta}^{1*}, \widetilde{\theta}^{2*})\} = \{\widetilde{e}, (\widetilde{1}, \widetilde{0})\}$ is an interior point of (8) and (10) at each (t, ω^t). Thus, by Lemmas D.1, for asset price processes to satisfy the first order conditions at the equilibrium allocation, they must follow

$$
q_t^{1*}(\omega^t) = E_Q[\widetilde{MRS}_t^{t+1}(\widetilde{q}_{t+1}^{1*} + \widetilde{e}_{t+1}) | \mathcal{F}_t(\omega^t)], \tag{D.1}
$$

$$
q_t^{2*}(\omega^t) = E_Q[\widetilde{MRS}_t^{t+1} | \mathcal{F}_t(\omega^t)], \tag{D.2}
$$

where $MRS_t^{t+1}(\omega^{t+1}) = \dfrac{MU_{t+1}(\omega^{t+1})}{MU_t(\omega^t)}$ and $\{\widetilde{MU}_t\}$ is the marginal utility process defined by

$$
\begin{aligned}
MU_t(\omega^t) &= \beta^{t-1} c_t^{\rho-1}(\omega^t) \left\{ \prod_{\tau=1}^{t} E_Q[(c_\tau^\rho(\omega^\tau) + \beta\rho\widetilde{V}(d_{\omega^{t+1}}))^{\frac{\alpha}{\rho}} | \mathcal{F}_\tau(\omega^\tau)]^{\frac{\rho}{\alpha}-1} \right\} \\
&\quad \times \left\{ \prod_{\tau=1}^{t-1} (c_\tau^\rho(\omega^\tau) + \beta\rho V(d_{\omega^{t+1}}))^{\frac{\alpha}{\rho}-1} \right\} \\
&\quad \times E_Q[(c_t^\rho(\omega^t) + \beta\rho\widetilde{V}(d_{\omega^{t+1}}))^{\frac{\alpha}{\rho}-1} | \mathcal{F}_t(\omega^t)].
\end{aligned}
$$

Note that we let $\prod_{\tau=1}^{0} x_\tau \equiv 1$. Clearly, (15) satisfies (D.2). If $q_t^{1*}(\omega^t)$ defined by (14) is finite at each (t, ω^t), it satisfies (D.1).

Given the concavity of (7), to prove that (14) and (15) form equilibrium prices, we must show that (i) $q_t^{1*}(\omega^t)$ is finite under (14) and (ii) the following transversality conditions are satisfied: for any feasible $(\widetilde{\theta}^1, \widetilde{\theta}^2)$,

$$\limsup E_Q[-\widetilde{\lambda}_t \widetilde{q}_t^{1*}(\widetilde{\theta}_t^1 - 1)|\mathcal{F}_1(\omega^1)] = \limsup E_Q[-\widetilde{MU}_t \widetilde{q}_t^{1*}(\widetilde{\theta}_t^1 - 1)|\mathcal{F}_1(\omega^1)] \leq 0, \tag{D.3}$$

$$\limsup E_Q[-\widetilde{\lambda}_t \widetilde{q}_t^{2*}(\widetilde{\theta}_t^2 - 0)|\mathcal{F}_1(\omega^1)] = \limsup E_Q[-\widetilde{MU}_t \widetilde{q}_t^{2*}(\widetilde{\theta}_t^2 - 0)|\mathcal{F}_1(\omega^1)] \leq 0. \tag{D.4}$$

First, for (D.3), define $W(c_t(\omega^t), \{a_{t+1}(\omega^{t+1})\}) \equiv U(\phi_{\omega^t}(\widetilde{c}(\mathcal{F}_t(\omega^t))$ by

$$W(c_t(\omega^t), \{a_{t+1}(\omega^{t+1})\}) = \frac{1}{\rho} E_Q[(c_t^\rho(\omega^t) + \beta a_{t+1}^\rho(\omega^{t+1}))^{\frac{\alpha}{\rho}}|\mathcal{F}_t(\omega^t)]^{\frac{\rho}{\alpha}},$$

where $a_{t+1}(\omega^{t+1}) = (\rho V(d_{\omega^{t+1}}))^{\frac{1}{\rho}}$. It is easy to see that W is homogeneous of degree ρ in $(c_t(\omega^t), \{a_{t+1}(\omega^{t+1})\})$. Then since Ω is finite, by Euler's theorem,

$$\begin{aligned}
\rho W(c_t(\omega^t), \{a_{t+1}(\omega^{t+1})\}) = {} & E_Q[(c_t^\rho(\omega^t) + \beta a_{t+1}^\rho(\omega^{t+1}))^{\frac{\alpha}{\rho}}|\mathcal{F}_t(\omega^t)]^{\frac{\rho}{\alpha}-1} \\
& \times E_Q[(c_t^\rho(\omega^t) + \beta a_{t+1}^\rho(\omega^{t+1}))^{\frac{\alpha}{\rho}-1}|\mathcal{F}_t(\omega^t)] c_t^\rho(\omega^t) \\
& + \beta E_Q[(c_t^\rho(\omega^t) + \beta a_{t+1}^\rho(\omega^{t+1}))^{\frac{\alpha}{\rho}}|\mathcal{F}_t(\omega^t)]^{\frac{\rho}{\alpha}-1} \\
& \times E_Q[(c_t^\rho(\omega^t) + \beta a_{t+1}^\rho(\omega^{t+1}))^{\frac{\alpha}{\rho}-1} a_{t+1}^\rho(\omega^{t+1})|\mathcal{F}_t(\omega^t)].
\end{aligned}$$

By applying this formula at $\widetilde{e}$ recursively from time 1, for $n \geq 1$,

$$\begin{aligned}
\rho W(e_1(\omega^1), \{a_2(\omega^2)\}) = {} & MU_1(\omega^1) e_1(\omega^1) \\
& + MU_1(\omega^1) E_Q\left[\sum_{\tau=1}^{n} \left\{\prod_{s=1}^{\tau} \widetilde{MRS}_s^{s+1}\right\} \widetilde{e}_{\tau+1}|\mathcal{F}_1(\omega^1)\right] \\
& + MU_1(\omega^1) \beta E_Q\left[\left\{\prod_{s=1}^{n} \widetilde{MRS}_s^{s+1}\right\} Z_{n+2}(\omega^{n+2})|\mathcal{F}_1(\omega^1)\right],
\end{aligned} \tag{D.5}$$

where

$$Z_{n+2}(\omega^{n+2}) = \frac{(c_{n+1}^\rho(\omega^{n+1}) + \beta a_{n+2}^\rho(\omega^{n+2}))^{\frac{\alpha}{\rho}-1} a_{n+2}^\rho(\omega^{n+2})}{E_Q[(c_{n+1}^\rho(\omega^n) + \beta a_{n+2}^\rho(\omega^{n+2}))^{\frac{\alpha}{\rho}-1}|\mathcal{F}_{n+1}(\omega^{n+1})] c_{n+1}^\rho(\omega^{n+1})}.$$

Each term of (D.5) is non-negative, and the second term on the right hand side is an increasing sequence. Also, by Assumption 1, $0 < \rho W(e_1(\omega^1), \{a_2(\omega^2)\}) < \infty$ at $\widetilde{e}$. Hence, the second term converges. By $\theta_t^1 \in [1-\epsilon, 1+\varepsilon]$, for $t \geq 1$,

$$E_Q[-\widetilde{MU}_t \widetilde{q}_t^{1*}(\widetilde{\theta}_t^1 - 1)|\mathcal{F}_1(\omega^1)] \leq \epsilon MU_1(\omega^1) E_Q\left[\sum_{\tau=t}^{\infty}\left\{\prod_{s=1}^{\tau}\widetilde{MRS}_s^{s+1}\right\}\widetilde{e}_{\tau+1}|\mathcal{F}_1(\omega^1)\right].$$

The right hand side is the tail end of the second term of (D.5) multiplied by ϵ so that it converges to zero. Thus, (D.3) follows.

Second, as for (D.4), by $\theta_t^2 \in [-\epsilon, \varepsilon]$, for $t \geq 1$,

$$E_Q[-\widetilde{MU}_t \widetilde{q}_t^{2*}(\widetilde{\theta}_t^2 - 0)|\mathcal{F}_1(\omega^1)] \leq \epsilon MU_1 E_Q[\prod_{s=1}^{t}\widetilde{MRS}_s^{s+1}|\mathcal{F}_1(\omega^1)].$$

(D.4) is satisfied if $\limsup E_Q[\prod_{s=1}^{t}\widetilde{MRS}_s^{s+1}|\mathcal{F}_1(\omega^1)] \leq 0$, which is the condition stated in Proposition 3 (i.e., (13)).

Finally, since the second term of (D.5) converges and m has full support, $q_t^{1*}(\omega^t)$ defined by (14) is finite at each (t, ω^t). ■

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Table 1. Difference in Long-run Average:

Risk-free Rate ((7)-(6)); Unit (%)

$\rho \backslash \alpha$	1.0	0.5	-0.5	-1.0	-4.0	-9.0
1.0	0.0000	0.0001	0.0004	0.0006	0.0014	-0.0008
0.5	-0.0006	0.0000	0.0012	0.0018	0.0053	0.0091
-0.5	-0.0052	-0.0034	0.0000	0.0017	0.0116	0.0275
-1.0	-0.0096	-0.0072	-0.0024	0.0000	0.0137	0.0359
-4.0	-0.0840	-0.0746	-0.0566	-0.0479	0.0000	0.0701
-9.0	-0.5513	-0.5157	-0.4472	-0.4143	-0.2352	0.0000

Table 2. Difference in Long-run Average:

Risk Premium ((7)-(6)); Unit (%)

$\rho \backslash \alpha$	1.0	0.5	-0.5	-1.0	-4.0	-9.0
1.0	0.0000	-0.0001	-0.0004	-0.0006	-0.0012	0.0015
0.5	0.0006	0.0000	-0.0012	-0.0018	-0.0053	-0.0088
-0.5	0.0051	0.0034	0.0000	-0.0017	-0.0116	-0.0277
-1.0	0.0095	0.0071	0.0023	0.0000	-0.0136	-0.0362
-4.0	0.0749	0.0668	0.0509	0.0433	0.0000	-0.0669
-9.0	0.4015	0.3770	0.3295	0.3065	0.1788	0.0000

Table 3. Long-run Average at $\rho = -1$; Unit (%)

		1.0	0.5	-0.5	-1.0	-4.0	-9.0
Risk-free Rate	(7)	5.7077	5.6210	5.4480	5.3616	4.8475	4.0215
Risk-free Rate	(6)	5.7173	5.6282	5.4503	5.3616	4.8339	3.9856
Risk Premium	(7)	0.0439	0.1054	0.2282	0.2895	0.6530	1.2302
Risk Premium	(6)	0.0345	0.0984	0.2259	0.2895	0.6666	1.2665

Table 4. Difference in Long-run Volatility:

Risk-free Rate ((7)-(6)); Unit (%)

$\rho\backslash\alpha$	1.0	0.5	-0.5	-1.0	-4.0	-9.0
1.0	0.0000	0.0000	-0.0004	-0.0007	-0.0054	-0.0219
0.5	0.0001	0.0000	0.0001	0.0003	0.0032	0.0137
-0.5	-0.0003	-0.0001	0.0000	0.0000	-0.0014	-0.0084
-1.0	-0.0010	-0.0005	0.0000	0.0000	-0.0024	-0.0163
-4.0	-0.0183	-0.0144	-0.0078	-0.0052	0.0000	-0.0306
-9.0	-0.1443	-0.1292	-0.1011	-0.0881	-0.0274	0.0000

Table 5. Difference in Long-run Volatility:

Expected return of Risky Asset ((7)-(6)); Unit (%)

$\rho\backslash\alpha$	1.0	0.5	-0.5	-1.0	-4.0	-9.0
1.0	0.0000	0.0000	0.0003	0.0009	0.0070	0.0281
0.5	0.0001	0.0000	0.0001	0.0003	0.0029	0.0129
-0.5	-0.0003	-0.0001	0.0000	-0.0001	-0.0020	-0.0108
-1.0	-0.0009	-0.0004	0.0000	0.0000	-0.0031	-0.0195
-4.0	-0.0165	-0.0124	-0.0059	-0.0034	0.0000	-0.0378
-9.0	-0.1219	-0.1066	-0.0788	-0.0661	-0.0102	0.0000

Table 6. Difference in Risk-free Rates (State 2-State 1); Unit (%)

	$\rho\backslash\alpha$	1.0	0.5	-0.5	-1.0	-4.0	-9.0
(7)	1.0	0.0000	0.0000	0.0001	0.0002	0.0017	0.0062
(6)	1.0	0.0000	0.0000	-0.0008	-0.0017	-0.0124	-0.0499
(7)	-1.0	2.0919	2.0913	2.0873	2.0839	2.0438	1.9081
(6)	-1.0	2.0938	2.0924	2.0874	2.0839	2.0486	1.9407