Balancing mother-tongue and common-language education

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Abstract

In many countries, determining the appropriate balance between mother-tongue and common-language education is a crucial policy issue. The ability to use the common language is essential for accessing modern sector jobs, whereas the use of mother tongue skill is often confined to local ethnic jobs. At the same time, mother tongue proficiency provides benefits or pleasure in various aspects beyond the workplace, including child-rearing, health, and communications at home and within the community. Despite the importance of this issue, little is known about the ideal combination of the two types of education in terms of future consumption and welfare, especially when the non-workplace benefits of the mother-tongue skill are taken into account.

This paper develops a simple model to examine these issues both theoretically and numerically. In particular, it analyzes how the desirable educational balance varies with individual wealth and ability, and how the socially-optimal weights on the two types of education differ depending on aggregate conditions of the economy, such as the level and distribution of wealth.

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1 Introduction

Many people around the world live in multilingual environments. In countries such as India, the Philippines, and many in sub-Saharan Africa, people use their mother tongue in daily life and local business, while using the language of the former colonizer as the common language in national business and inter-group communication. In other countries, the mother tongue of the dominant ethnic group serves as the common language, such as Spanish in Bolivia, Mandarin in China, Amharic in Ethiopia, and Vietnamese in Vietnam.

Determining the right balance between mother-tongue and common-language education, as well as selecting the appropriate language of instruction for other subjects, are crucial issues in these countries.[1] The ability to use the common language is essential for accessing many modern sector jobs, whereas the use of mother tongue skill is often confined to local ethnic jobs. However, mother tongue proficiency provides benefits or pleasure in various aspects beyond the workplace, including child-rearing, health, and communications at home and within the community. Moreover, in societies where the common language is the mother tongue of the dominant ethnic group, the language policy can influence economic disparities between subordinate and dominant groups.

Students have little choice between mother-tongue and common-language education in basic education (primary and lower secondary education), because relative weights of the two are mostly determined by the government. Language policies adopted by these countries are varied. In sub-Saharan Africa, former French colonies historically maintained French as the sole language of instruction, while former British colonies conducted mother-tongue education partially. More recently, however, Francophone countries have introduced ethnic languages, and many Anglophone countries have reduced the weight on mother-tongue education (Albaugh, 2007; Heugh, 2011). In societies where the common language is the mother tongue of the dominant group, some countries have greatly increased the emphasis on mother-tongue education, whereas others have maintained or intensified the emphasis on common-language education in recent years.[2]

There is a broad consensus among language and education experts that placing sufficient emphasis on mother-tongue education, at least in primary education, is important for students to develop adequate language and non-language skills (Ball, 2011; Heugh, 2011). In contrast, little is known about the ideal combination of the two types of education in terms of future *consumption and welfare*, especially *when non-workplace benefits of the mother-tongue skill are taken into account.* This paper develops a simple model to examine these issues both theoretically and numerically. In particular, it analyzes how the desirable balance between mother-tongue and common-language education varies across individuals with different levels of wealth and ability, and how the socially-optimal weights on the two types of education differ depending on the economy's conditions, such as the level and distribution of wealth.

Model: The model economy is populated by multiple ethnic groups and has two types of sectors, the *national sector* and group-specific *local sectors*. In the real world, national- (local-) sector jobs correspond to many positions in companies operating nationwide (locally) and jobs involving communications with other ethnic groups (locals).

Working in the national sector requires the common-language skill, while working in the local sector of an ethnic group requires the group's ethnic-language skill. The common language may be the mother tongue of the majority or dominant group, or it may be an external language, such as that of a former colonizer.

[1]The same issues are also relevant to many small nations, including ethnolinguistically homogenous nations, in which the lingua franca in business is English because of strong dependence on international business.

[2]The former group of countries includes Bolivia (Haboud and Limerick, 2017) and Ethiopia (Benson et al., 2012), while the latter group includes China (Gao and Wang, 2017) and Vietnam (Nguyen and Nguyen, 2019).

Each person has a wealth endowment to expend on education for developing both mother-tongue and common-language skills. Individuals are heterogenous in wealth and exogenous non-language ability useful at work. Exogenous ability can be interpreted as early-life cognitive ability realized by the time children begin formal education.[3] They must self-finance education due to the absence of a credit market.[4] The distribution of wealth is such that some people may lack sufficient wealth to afford optimal educational spending.

Individuals *cannot* choose the allocation of spending for developing the two types of skills. This allocation is fixed, reflecting the fact that the government mostly determines the relative weight of common-language education and mother-tongue education, especially in basic education.

The mother-tongue skill is not only valuable in the workplace of the local sector but also yields direct utility to members of the ethnic group. This direct utility reflects the fact that the skill is beneficial or brings pleasure in various aspects beyond the workplace, including child-rearing, health, and communications at home and within the community.

After education, individuals choose a sector to work in, receive earnings, and consume.

Results: The paper examines how the relative weights of the two types of education affect individual consumption, welfare, disparities in these outcomes between different individuals, and social welfare. The main results can be summarized as follows.

First, the standard model in which utility depends solely on goods consumption and thus education serves solely as investment is analyzed. Individual consumption and welfare increase (decrease) with the share of expenditure allocated to mother-tongue education, s, for small (large) s, and these outcomes are maximized at the same level of s for all individuals. This optimal s coincides with a coefficient of the final-goods production function. At this value of s, disparities in consumption and welfare between the poor (those who cannot afford optimal educational expenditure) and the wealthy (those who can), as well as among the wealthy, are minimized, whereas disparities among relatively wealthy individuals with different levels of non-language ability are maximized.

Then, the model in which utility depends directly on ethnic-language skill, as well as on consumption, is examined. In this setting, mother-tongue education have both consumption and investment roles. Individual consumption is maximized at the same level of s as in the standard model. By contrast, the weight on mother-tongue education that maximizes individual welfare is higher than in the standard model and generally differs across individuals. While the optimal s is identical for the poor, it is heterogenous among the wealthy. In particular, the welfare-maximizing weight on mother-tongue education is higher for those with *greater wealth* and lower for those with higher non-language ability, and it *exceeds* the level optimal for wealth-constrained individuals. Also, at the same level of s as in the standard model, consumption and welfare inequalities between the poor and the wealthy and among the wealthy are minimized, whereas inter-ability inequalities in these outcomes among relatively wealthy individuals are maximized. However, unlike in the previous model, this weight is lower than the welfare-maximizing weight, which generally varies across individuals.

Finally, because the welfare-maximizing s differs across individuals, and the properties of the s that maximizes social welfare cannot be derived analytically, the socially optimal s is computed numerically by calibrating the model to the real economy. Under a utilitarian social welfare function and assuming a relatively weak consumption motive for mother-tongue education, the socially

[3]For simplicity, the non-language skill is exogenous, but results remain similar even if a part of the skill is endogenously developed through education.

[4]This is consistent with observations from many developing countries, where students often depend on limited family wealth to cover expenses such as study materials, commuting costs, uniforms, and supplementary education even when public schools do not charge tuition.

optimal s is found to be 0.644. That is, social welfare is maximized when 64.4% of educational spending is allocated to mother-tongue education. This value is greater than the optimal value of 0.503 for the model with investment motive only. Holding wealth inequality fixed, the socially optimal s increases with the average level of wealth, whereas it is relatively insensitive to changes in wealth inequality. As wealth and exogenous ability become more strongly correlated, which would be the case if the influence of family economic conditions on early-childhood rearing environments becomes stronger in the real economy, the optimal *s decreases*. These results suggest that the optimal weight on mother tongue education is *lower* in developing countries where wealth accumulation (for a given productivity level) is lower and the association between family wealth and early-life cognitive ability tends to be stronger. When a social welfare function incorporating inequality aversion is employed, the socially optimal s is lower than under the utilitarian function, and as the degree of inequality aversion increases, the optimal s declines. In other words, as social welfare places greater weight on inequality aversion, it becomes optimal to allocate a larger proportion of educational spending to *common language* education. Moreover, when inequality aversion is high, the optimal s decreases substantially with wealth inequality, and the magnitude of the change is large. That is, when social welfare prioritizes equity, the optimal weight on mother tongue education is *lower* in societies with large wealth inequality.

It should be noted that the model does not account for the potential effects of language choice in education on social capital, political participation, national unity, and public goods provision. Policy implementation in the real world must also consider these effects.

Relations to Yuki (2022, 2024): To the best of the author's knowledge, Yuki (2022, 2024) are the first theoretical studies to examine how the relative emphasis on teaching the common language and the mother tongue affects the skills, net earnings, and consumption of individuals with varying levels of family wealth.[5] The present study extends this framework in important dimensions and conducts a much more complete analysis of the issue.

First, the models of Yuki (2022, 2024) assume that individual utility depends solely on goods consumption, whereas the present model assumes that utility depends on both the mother-tongue skill and goods consumption. This assumption reflects the fact that mother-tongue skill is beneficial or brings pleasure in various domains beyond the workplace, including child-rearing, health, and communications at home and within the community. As mentioned above, the present model with both consumption and investment motives for education yields different results on desirable educational weights from the standard model with an investment motive only. Thus, incorporating the direct utility of mother-tongue skill is crucial when examining the issue.

Second, in Yuki (2022, 2024), individuals are heterogeneous only in wealth, whereas in this paper, they differ in both exogenous non-language ability and wealth. This setting enables the analysis of how the relative weights of the two types of education depends on exogenous ability.

Third, unlike the preceding works, this paper numerically examines how the *socially-optimal* weights on mother-tongue education and common-language education depend on the distributions of wealth and exogenous ability, as well as on the degree of inequality aversion of a social welfare function. It also analytically examines how disparities in individual outcomes vary with the relative weights of the two types of education.

[5] The main differences between Yuki (2022) and Yuki (2024) are as follows. Yuki (2022) analyzes the economy in which ethnic groups are symmetric and the common language is not the native language of any group. This setting is relevant to many sub-Saharan Africa countries where the common language is the language of the former colonizer. By contrast, Yuki (2024) examines the economy in which the common language is the mother tongue of the dominant ethnic group (e.g., Spanish in Bolivia, Mandarin in China, and Amharic in Ethiopia). Similar to the present paper, the human capital production functions in Yuki (2024) exhibit decreasing returns to educational expenditure, while Yuki (2022) assumes linear production functions with an upper bound on expenditure.

Finally, the previous studies assume that the level of mother-tongue skill is *positive even without education* (i.e., part of the skill is acquired at home). Based on this assumption, they show that common-language-only education is optimal for those with little wealth to spend on education, and that when the economy faces unfavorable educational and technological conditions (i.e., low educational effectiveness and sectoral productivities), it is optimal for everyone, whereas under more favorable conditions, balanced bilingual education is desirable, except for the extremely poor. For analytical tractability, the present study instead assumes that the mother-tongue skill is zero without education. Owing to this assumption, unlike Yuki (2022, 2024), we can analytically obtain the relationships between individual outcomes (such as consumption and welfare) and the share of expenditure allocated to mother-tongue education *for any value of that share*.

Organization of the paper: Section 2 reviews the related literature. Section 3 presents the model. Section 4 analyzes the model and presents the analytical results, while Section 5 numerically solves the model and reports the results for the socially optimal weight on mother tongue education. Section 6 concludes. Appendix A provides the proofs of the propositions.

2 Related Literature

Apart from Yuki (2022, 2024), several theoretical works examine related issues. Pool (1991) considers the choice of official languages in a multilingual society in which earnings are exogenous, learning a non-native language is costly, and translations across different official languages are costly and financed by tax. He shows that an efficient and fair choice of official languages exists if appropriate inter-group redistribution is implemented. Lazear (1999) develops a model in which individuals differ in the cost of learning non-native languages, decide whether to master the languages of other groups, are randomly matched, and can produce goods only when the pair can use the same language. He derives several implications of the model and empirically examines them. Ortega and Tangerås (2008) model a society with two language groups in which the dominant group determines the type of schools (bilingual or monolingual in either language) accessible to each group, individuals decide whether to attend school, and goods are produced through bilateral random matching as in Lazear (1999). They show that the dominant group either choose laissez-faire or restricts access to schools using the subordinate group's language, whereas the subordinate group prefers schools using their mother tongue.

Besides addressing different issues, the present work is distinct from these works in several important respects. First, unlike these studies, the mother-tongue skill is not only valuable in the local sector workplace but also yields direct utility to members of the ethnic group. Thus, individuals have both consumption and investment motives for acquiring education. Second, in this paper, individuals within each group are heterogeneous in the amount of wealth available for education and exogenous non-language ability, whereas in the aforementioned works, they are either homogenous (Pool; Ortega and Tangerås) or heterogenous only in the costs of learning non-native languages, which may capture differences in innate ability (Lazear). This work adopts a different setting because it focuses mainly on developing countries in which family wealth is a critical determinant of educational investment even at the basic education level, in contrast to the primary focus on developed countries in the existing studies. Third, unlike the previous works, this paper does not account for effects related to the size of language groups, such as network externalities in language usage. Finally, unlike Ortega and Tangerås (2008), educational institutions are given rather than determined endogenously, and strategic interactions among agents are not considered.

Many studies in education and linguistics examine the effect of education-language policy on students' academic achievement, which is not the focus of the present paper. A general consensus among researchers is that emphasizing mother-tongue education, at least in primary education, is important for skill developmen (Ball, 2011; Heugh, 2011). A small number of works in economics also empirically investigate the effects on educational outcomes. Jain (2017) examines the effect on academic outcomes in South India, where primary education is largely conducted in the official language of a state. By comparing districts in which the official language matched the district's language with those in which it did not, he finds that mismatched districts had lower literacy and college graduation rates. However, after states were reorganized along linguistic lines, the previously mismatched districts caught up with others. Ramachandran (2017) finds that a reform in Ethiopia that introduced mother-tongue instruction in primary education has positive effects on reading skills and years of schooling.

Very few studies empirically examine the effect on labor market outcomes, which are related to the effect on consumption analyzed in this paper. Angrist and Lavy (1997) find that the policy change in Morocco during the 1980s, which replaced French with Arabic as the medium of instruction in post-primary education, substantially reduced returns to schooling. Cappellari and Di Paolo (2018) analyze the effects of the 1983 bilingual-education reform in Catalonia, which significantly increased the weight of Catalan in mandatory education, and find a positive effect on earnings. Chakraborty and Bakshi (2016) find that the policy change in the Indian state of West Bengal, which abolished English education in primary schools, significantly decreased wages.

Finally, the present paper is also related to studies that model both consumption and investment motives for obtaining education, including Lazear (1977), Keane and Wolpin (2001), Romano and Tampieri (2016), and Kaganovich (2025). None of them study the issue examined in this work.

3 Model

3.1 Production

Consider a multilingual society that is populated by multiple ethnic groups and has two types of sectors, the *national sector* and group-specific *local sectors*. The local sector of each group produces final goods specific to their group, using intermediate goods produced by the national sector and the group's labor. Meanwhile, the national sector produces intermediate goods using labor from all ethnic groups.

In the real world, national-sector jobs, which largely overlaps with modern or formal sector jobs in developing countries, correspond to many positions in companies operating nationwide and jobs involving communications with other groups, all requiring proficiency in a common language. Local-sector jobs represent many positions in locally-operating businesses, and jobs involving communications with local customers, such as those in retail, food service, and personal care, necessitating proficiency in the local ethnic language. Modeling the local sectors as those producing group-specific final goods reflects the fact that these services are dominant in the final stage of the production process.

The production function of the local sector of ethnic group i is

$$Y_{iL} = (T_{iL}H_{iL})^{\alpha}(Y_{iN})^{1-\alpha}, \alpha \in (0,1), \tag{1}$$

where H_{iL} is the total human capital of the sector's workers (whose determination is explained later), T_{iL} is the sector's constant total factor productivity (TFP), and Y_{iN} is the amount of intermediate goods used. The production function implies that both human capital and intermediate goods are essential, but substitutable to some extent in the production of the final goods.

The production function of the national sector is

$$Y_N = T_N \textstyle\sum_i H_{iN}, \tag{2}$$

where H_{iN} is the total human capital of group i workers in the sector and T_N is the sector's TFP. Workers from any groups are perfectly substitutable in the production of intermediate goods.

Markets are perfectly competitive. Let the intermediate good be the numeraire. Then, from (2), the wage rate *per unit of human capital* for workers in the national sector is

$$w_N = T_N. \tag{3}$$

Denote the relative price of the final good of group i by P_i and the wage rate per unit of human capital for local-sector workers of group i by w_{iL}. Since the profit of the final-good producer is $P_i Y_{iL} - w_{iL} H_{iL} - Y_{iN}$, from the first-order conditions of the profit-maximization problem,

$$P_i \frac{\partial Y_{iL}}{\partial H_{iL}} = w_{iL} \Leftrightarrow P_i \frac{\alpha Y_{iL}}{H_{iL}} = w_{iL}, \tag{4}$$

$$P_i \frac{\partial Y_{iL}}{\partial Y_{iN}} = 1 \Leftrightarrow P_i \frac{(1-\alpha) Y_{iL}}{Y_{iN}} = 1. \tag{5}$$

From these equations,

$$w_{iL} = \frac{\alpha}{1-\alpha} \frac{Y_{iN}}{H_{iL}}. \tag{6}$$

Because the final goods are group specific, the goods are not traded between the groups. Thus, a group's demand for intermediate goods must equal the amount of intermediate goods produced by the group's workers:[6]

$$Y_{iN} = T_N H_{iN}. \tag{7}$$

By substituting the above equation into (1) and (6), the output of the final goods and the wage rate of local-sector workers can be expressed as functions of H_{iN} and H_{iL}:

$$Y_{iL} = (T_{iL} H_{iL})^{\alpha} (T_N H_{iN})^{1-\alpha}, \tag{8}$$

$$w_{iL} = \frac{\alpha}{1-\alpha} \frac{T_N H_{iN}}{H_{iL}}. \tag{9}$$

From (5), (7), and (8), the relative price of the final good is also expressed as a function of the human capital variables.

$$P_i = \frac{1}{1-\alpha} \left(\frac{T_N H_{iN}}{T_{iL} H_{iL}} \right)^{\alpha}. \tag{10}$$

The equations show that the price variables for group i depend solely on the human capital variables of that group. As will become clear later, this implies that results for an ethnic group can be obtained without considering other groups.

3.2 Education and Preference

The national sector requires the common-language skill, while the local sector of an ethnic group requires the group's ethnic-language skill. The common language may be the mother tongue of the majority or historically dominant group, or it may be the language of the former colonizer.

The assumption that only the common (local) language is used in the national (local) sector would exaggerate reality, yet it captures the fact that the language essential for tasks varies depending on occupations and sectors in multilingual societies. Using survey data from China, Dovì (2019) shows that proficiency in Mandarin is not statistically related to employment probabilities in rural areas, but it is strongly associated with employment probabilities in urban areas, where modern-sector jobs are concentrated. Azam, Chin, and Prakash (2013) argue that proficiency in

[6]Note, however, that a portion of the intermediate goods used for producing the final goods of group i, Y_{iN}, are produced by other groups. The proportion of Y_{iN} produced by group i is $\frac{H_{iN}}{\sum_j H_{jN}}$.

English, which serves as a lingua franca in India, is essential for many management and technical jobs in the modern sector. Hellerstein and Neumark (2008) discover that proficiency in English, rather than education level, explains a large part of (establishment-level) workplace segregation between Hispanics and whites in the U.S. Further, they find that (one-digit) occupation can account for Hispanic-white segregation to a similar extent as English proficiency, owing to a large overlap in the distributions of occupations and English skill among Hispanics.

Each individual has a wealth endowment a to be spent on education for developing both mother-tongue and common-language skills, which are distinct unless the common language is the mother tongue of the group. Individuals are heterogenous in wealth a. Let e denote the amount of educational spending. An individual with wealth a can spend at most $e = a$ on education due to the absence of a credit market to finance education.[7] This is consistent with observations from many developing countries, where students often depend on limited family wealth to cover expenses such as study materials, commuting costs, uniforms, and supplementary education even when public schools do not charge tuition.

Individuals *cannot* choose the allocation of spending for developing the two types of skills. It is fixed, reflecting the fact that the government mostly determines the relative weight of common-language education and mother-tongue education in primary and lower-secondary education.

The language-skill production functions of group i individuals for the local and national sectors are respectively given by[8]

$$h_{iL} = (\delta_L s e)^\gamma, \quad \gamma \in (0,1), s \in [0,1], \tag{11}$$

$$h_{iN} = [\delta_N (1-s) e]^\gamma, \tag{12}$$

unless the group's mother tongue is the common language, in which case the production function is

$$h_i \equiv h_{iL} = h_{iN} = (e)^\gamma. \tag{13}$$

In (11) and (12), $s \in [0,1]$ is the share of e allocated to developing the mother-tongue skill, and δ_N (δ_L) is the effectiveness of common-language (mother-tongue) education for skill development.[9]

As mentioned above, the local (national) sector values the ethnic-language (common-language) skill. Thus, the human capital of an individual in the local (national) sector is given by $\widetilde{h_{iL}} = \omega h_{iL}$ ($\widetilde{h_{iN}} = \omega h_{iN}$), where ω is exogenous non-language ability. Individuals are heterogenous in ω. Exogenous ability can be interpreted as early-life cognitive ability realized by the time children begin formal education. For simplicity, the non-language skill is exogenous, but the results remain similar even if a part of the skill is endogenously developed through education.[10]

The mother-tongue skill is not only valuable in the workplace of the local sector but also yields utility to members of the group. Individual utilities when choosing to work in the local sector and

[7] Introducing a government that partially finances education complicates the analysis but would not affect the results qualitatively.

[8] In Yuki (2024), the skill production functions are given by $h_{iL} = (\bar{l} + \delta_L s e)^\gamma$ and $h_{iN} = [\delta_N (1-s) e]^\gamma$, where $\bar{l} > 0$ is a constant. This formulation implies that the level of mother-tongue skill is positive without education, reflecting the fact that the skill is developed partly at home. For analytical tractability and to align with the present paper's focus, the current model assumes $\bar{l} = 0$.

[9] $\delta_N < \delta_L$ would be reasonable considering the higher cost-effectiveness of mother-tongue education in skill development (Vaillancourt and Grin, 2000), although the results do not depend on it.

[10] Suppose that a fixed proportion $q \in [0,1]$ of educational expenditure is allocated to language education, with the remaining portion allocated to non-language education. Then, the language-skill production functions (11) and (12) are modified to $h_{iL} = (\delta_L s q e)^\gamma$ and $h_{iN} = [\delta_N (1-s) q e]^\gamma$, respectively. Assuming that language and non-language skills are complementary (i.e., each stimulates the development of the other), the human capital production functions become $\widetilde{h_{iL}} = \omega (\delta_L s q e)^\gamma [(1-q)e]^{\gamma_n}$ and $\widetilde{h_{iN}} = \omega [\delta_N (1-s) q e]^\gamma [(1-q)e]^{\gamma_n}$, where $\gamma_n, \gamma + \gamma_n \in (0,1)$. Under this specification, the results are qualitatively unchanged.

in the national sector are respectively given by

$$u_{iL} = (c_{iL})^{\theta}(h_{iL})^{1-\theta}, \ \theta \in (0,1), \tag{14}$$

$$u_{iN} = (c_{iN})^{\theta}(h_{iL})^{1-\theta}, \tag{15}$$

where c_{iL} (c_{iN}) denotes goods consumption when working in the local (national) sector.

The setting that individuals derive utility directly from the mother-tongue skill reflects the fact that the skill provide benefits or pleasure beyond the workplace in various aspects, including child-rearing, health, and communations at home and within the community.

After making an educational choice, each person chooses a sector to work in and receives earnings, which, together with the remaining wealth $a-e$, are spent on final goods for consumption.

4 Analytical Results

How does the share of educational expenditure allocated to developing the mother-tongue skill, s, affect economic outcomes such as educational expenditure, consumption, and welfare for individuals who differ in wealth a and non-language ability ω? How does it influence disparities in these outcomes among the heterogenous population? This section examines these questions. Since results hold for any ethnic group whose naive language is not the common language, thereafter, variables are presented without subscript i for simplicity.

4.1 Investment motive only

First, these issues are examined for the standard model in which utility depends only on consumption, i.e., $\theta = 1$ in the utilities (14) and (15), and thus individuals pursue education purely as an investment to increase their earnings.

The consumption (=utility) maximization problem when an individual chooses the local sector is

$$\max_{e} \left[\frac{w_L}{P}\omega(\delta_L se)^{\gamma} + a - e\right] \quad \text{s.t. } e \leq a. \tag{16}$$

From the first-order condition, $\gamma\frac{w_L}{P}\omega(\delta_L s)^{\gamma}e^{\gamma-1} - 1 \geq 0$, educational expenditure is given by

$$e_L(\omega, a) = \begin{cases} a & \text{for } a < e^{\sharp}_L(\omega), \\ e^{\sharp}_L(\omega) \equiv \left[\gamma\frac{w_L}{P}\omega(\delta_L s)^{\gamma}\right]^{\frac{1}{1-\gamma}} & \text{for } a \geq e^{\sharp}_L(\omega). \end{cases} \tag{17}$$

If the level of wealth a is so low that an individual cannot afford the optimal educational expenditure, $e^{\sharp}_L(\omega)$, she spends entire wealth on education. Otherwise, educational expenditure equals $e^{\sharp}_L(\omega)$ and does not depend on a. $e^{\sharp}_L(\omega)$ increases with ω because non-language skill positively affects the marginal return to language education. By substituting the above equation into (16), consumption is expressed as

$$c_L(\omega, a) = \begin{cases} \frac{w_L}{P}\omega(\delta_L sa)^{\gamma} & \text{for } a < e^{\sharp}_L(\omega), \\ (\frac{1}{\gamma} - 1)e^{\sharp}_L(\omega) + a & \text{for } a \geq e^{\sharp}_L(\omega). \end{cases} \tag{18}$$

Consumption increases with a for those with $a \geq e^{\sharp}_L(\omega)$ as well, because wealth in excess of $e^{\sharp}_L(\omega)$ is spent on consumption.

Similarly, from the first-order condition for the maximization problem when an individual chooses the national sector, $\gamma\frac{T_N}{P}\omega[\delta_N(1-s)]^{\gamma}e^{\gamma-1} - 1 \geq 0$,

$$e_N(\omega, a) = \begin{cases} a & \text{for } a < e^{\sharp}_N(\omega), \\ e^{\sharp}_N(\omega) \equiv \left\{\gamma\frac{T_N}{P}\omega[\delta_N(1-s)]^{\gamma}\right\}^{\frac{1}{1-\gamma}} & \text{for } a \geq e^{\sharp}_N(\omega). \end{cases} \tag{19}$$

In this case, consumption is given by

$$c_N(\omega, a) = \begin{cases} \frac{T_N}{P}\omega[\delta_N(1-s)a]^\gamma & \text{for } a < e_N^\sharp(\omega), \\ (\frac{1}{\gamma}-1)e_N^\sharp(\omega) + a & \text{for } a \geq e_N^\sharp(\omega). \end{cases} \quad (20)$$

Individuals do not have comparative advantages in a particular sector (since non-language skill ω is the same for both sectors), and goods from both sectors are essential in final good production. Hence, individuals must be indifferent between the sectors. That is,

$$\begin{aligned} & c_L(\omega, a) = c_N(\omega, a) \\ \Leftrightarrow\ & e_L(\omega, a) = e_N(\omega, a) \text{ (from (18) and (20))} \\ \Leftrightarrow\ & w_L(\delta_L s)^\gamma = T_N[\delta_N(1-s)]^\gamma \text{ (from (17) and (19))} \\ \Leftrightarrow\ & \frac{\alpha}{1-\alpha}\frac{T_N H_N}{H_L}(\delta_L s)^\gamma = T_N[\delta_N(1-s)]^\gamma \text{ (from (9))} \\ \Leftrightarrow\ & \frac{H_N}{H_L} = \frac{1-\alpha}{\alpha}\left(\frac{\delta_N}{\delta_L}\frac{1-s}{s}\right)^\gamma. \end{aligned} \quad (21)$$

By substituting the above equation into (10), the relative price of the final good equals

$$P = \frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\left(\frac{T_N}{T_L}\right)^\alpha\left(\frac{\delta_N}{\delta_L}\frac{1-s}{s}\right)^{\alpha\gamma}. \quad (22)$$

By substituting this equation into (19) and (20),

$$e(\omega, a) = \begin{cases} a & \text{for } a < e^\sharp(\omega), \\ e^\sharp(\omega) \equiv \left(\gamma(\alpha T_L)^\alpha\left[(1-\alpha)T_N\right]^{1-\alpha}\omega\left\{(\delta_L s)^\alpha[\delta_N(1-s)]^{1-\alpha}\right\}^\gamma\right)^{\frac{1}{1-\gamma}} & \text{for } a \geq e^\sharp(\omega). \end{cases} \quad (23)$$

$$c(\omega, a) = \begin{cases} (\alpha T_L)^\alpha\left[(1-\alpha)T_N\right]^{1-\alpha}\omega\left\{(\delta_L s)^\alpha[\delta_N(1-s)]^{1-\alpha}a\right\}^\gamma & \text{for } a < e^\sharp(\omega), \\ (\frac{1}{\gamma}-1)e^\sharp(\omega) + a & \text{for } a \geq e^\sharp(\omega). \end{cases} \quad (24)$$

Note that $e^\sharp(\omega)$ increases (decreases) with s for $s < (>)\alpha$.

These equations reveal the relationship between the weight on mother-tongue education and educational expenditure, consumption, and welfare.

Proposition 1 *In an economy where education serves only as an investment, $e(\omega, a)$ for $a \geq e^\sharp(\omega)$ and $c(\omega, a)$—and thus welfare—increase (decrease) with s for $s < (>)\alpha$, attaining the maximum at $s = \alpha$.*

The proposition shows that educational expenditure for those with sufficient wealth to afford the optimal level $e^\sharp(\omega)$, and consumption (and thus welfare) for everyone, increase (decrease) with the share of expenditure allocated to mother-tongue education, s, when $s < (>)\alpha$. Hence, these outcomes are maximized at $s = \alpha$, which corresponds to the income share of local-sector workers in final-good production (see (1)).

These results can be explained as follows. Given the real wage rate, a greater emphasis on mother-tongue education raises $e_L^\sharp(\omega)$ and $c_L(\omega, a)$ for local-sector workers (see (17) and (18)), whereas it lowers the corresponding variables for national-sector workers (see (19) and (20)). In contrast, increasing the weight on mother-tongue education reduces the total human capital of national-sector workers relative to local-sector workers $\frac{H_N}{H_L}$ (see (21)), leading to a lower (higher) real wage rate for local-sector (national-sector) workers and negatively (positively) affecting their educational expenditure and consumption. That is, the direct effect of increasing s and the indirect effect through the wage rate work in opposite directions.[11] The magnitude of the direct effect is large when the weight on mother-tongue education is close to 0 for local-sector workers (1 for national-sector workers) due to diminishing returns in skill production, while that of the indirect effect is large when the weight is close to 0 or 1, i.e., when the human capital ratio $\frac{H_N}{H_L}$ is unbalanced. Consequently, when the weight is relatively small (large), an increase in the weight has a positive (negative) effect on the outcomes.[12] Hence, a balanced allocation of the education budget between mother-tongue and common-language education is optimal for everyone.

The previous proposition shows that $s = \alpha$ maximizes consumption and thus welfare for all. But how does s affect disparities in educational expenditure, consumption, and welfare among individuals who differ in wealth and exogenous non-language ability? Here, disparities in outcomes between two individuals are measured by the *ratio* of their respective outcomes. The next proposition examines disparities across individuals with different wealth levels for a given level of non-language skill.

Proposition 2 *Consider individuals with the same ω in an economy where education serves only as an investment.*

(*i*) *Educational disparities between the poor, i.e., those with $a < e^{\sharp}(\omega)$, and the wealthy, i.e., those with $a \geq e^{\sharp}(\omega)$, increase (decrease) with s for $s < (>) \alpha$. Disparities within each group—among the poor and among the wealthy—do not depend on s.*

(*ii*) *Consumption (thus welfare) inequality between the poor and the wealthy, as well as among the wealthy, decreases (increases) with s for $s < (>) \alpha$. Inequality among the poor is unaffected by s.*

For a given level of non-language skill ω, educational disparities between the poor (those with $a < e^{\sharp}(\omega)$) and the wealthy (those with $a \geq e^{\sharp}(\omega)$) increase (decrease) with s for $s < (>) \alpha$, while disparities within each group do not depend on s.[13] By contrast, consumption (and thus welfare) inequality between the poor and the wealthy, as well as among the wealthy, decreases (increases) with s for $s < (>) \alpha$, whereas inequality among the poor is unaffected by s.

Thus, the weight on mother-tongue education that maximizes consumption and welfare for every individual, i.e., $s = \alpha$, also minimizes disparities in these outcomes between the poor and the wealthy, and among the wealthy.

The results can be understood as follows. $s = \alpha$ maximizes $e^{\sharp}(\omega)$ (Proposition 1) and earnings net of educational expenditure for the wealthy, which equals $(\frac{1}{\gamma} - 1)e^{\sharp}(\omega)$ from (24). Thus, the proportion of consumption financed by net earnings—identical among the wealthy—rather than heterogeneous wealth is highest at $s = \alpha$.[14] Hence, consumption (and thus welfare) disparities among wealthy individuals become lowest at this value of s. In contrast, disparities in educational expenditure and net earnings between the poor (who spend $e = a$) and the wealthy (who spend $e = e^{\sharp}(\omega)$) are greatest at $s = \alpha$.[15] However, consumption (and thus welfare) inequality is *minimized*

[11]Because individuals are indifferent between the sectors, the total effect on the outcomes is the same regardless of sectoral choice.

[12]When s is close to 0, the positive direct effect outweighs the large negative indirect effect for local-sector workers, and when s is close to 1, the negative direct effect outweighs the large positive indirect effect for national-sector workers.

[13]Since $e^{\sharp}(\omega)$ increases with ω, the threshold wealth $a = e^{\sharp}(\omega)$ dividing the poor and the wealthy increases with non-language ability.

[14]This is because the return to educational expenditure exceeds the return to wealth (which equals 1).

[15]The ratio of net earnings for a wealthy individual to a poor individual is equal to $\frac{(\frac{1}{\gamma}-1)(\gamma\Psi)^{\frac{1}{1-\gamma}}}{\Psi(a_l)^{\gamma}-a_l}$, where $\Psi \equiv (\alpha T_L)^{\alpha}\left[(1-\alpha)T_N\right]^{1-\alpha}\omega\left\{(\delta_L s)^{\alpha}[\delta_N(1-s)]^{1-\alpha}\right\}^{\gamma}$ and a_l is the wealth of a poor individual. The ratio increases (decreases) with s for $s < (>) \alpha$.

at such s. This is because the consumption of the poor is more sensitive to changes in s than that of the wealthy. Since the wealth-scarce poor finance a greater share of their consumption from net earnings rather than wealth, their consumption increases at a higher rate as s approaches α.

The following proposition examines disparities across individuals with different levels of non-language ability for a given level of wealth. Remember that disparities in outcomes between two individuals are measured by the ratio of their respective outcomes.

Proposition 3 *Consider individuals with the same a in an economy where education serves only as an investment.*

(i) Educational disparities between individuals at a middle level of wealth ($a \in (e^{\sharp}(\underline{\omega}), e^{\sharp}(\overline{\omega}))$) with different levels of non-language ability tend to decrease (increase) with s for $s < (>) \alpha$.[16] *Inter-ability disparities among the poorer ($a \leq e^{\sharp}(\underline{\omega})$) and among the wealthier ($a \geq e^{\sharp}(\overline{\omega})$) do not depend on s.*

(ii) Inter-ability consumption (and thus welfare) inequality among those with a not-low level of wealth ($a > e^{\sharp}(\underline{\omega})$) tends to increase (decrease) with s for $s < (>) \alpha$.[17] *Inter-ability inequality among the poorer ($a \leq e^{\sharp}(\underline{\omega})$) is not affected by s.*

In the middle range of the wealth distribution, educational disparities between individuals with equal wealth but differing levels of non-language ability tend to decrease (increase) with s for $s < (>) \alpha$, while inter-ability disparities among the poorer and the wealthier segments of the population do not depend on s. A different pattern emerges for consumption and welfare inequality. Among those with sufficient wealth, inter-ability inequality tends to increase (decrease) with s for $s < (>) \alpha$, whereas inequality among the poorer segment is not affected by s.

Thus, the weight on mother-tongue education that maximizes individual consumption and welfare, i.e., $s = \alpha$, leads to the highest inter-ability inequality in these outcomes among the non-poor.

The results can be interpreted in the following way. As mentioned above, $s = \alpha$ maximizes the optimal educational expenditure $e^{\sharp}(\omega)$, where $e^{\sharp}(\omega)$ is increasing in ω. Hence, educational disparities among individuals in the middle of the wealth distribution, specifically, between those who can afford $e^{\sharp}(\omega)$ due to relatively *low* ability ω_l (those with $a \geq e^{\sharp}(\omega_l)$) and those who cannot due to relatively *high* ability ω_h (those with $a < e^{\sharp}(\omega_h)$) is lowest at $s = \alpha$. By contrast, inter-ability consumption and welfare inequality among those with sufficient wealth tends to be *maximized* at such s. This occurs because higher-ability individuals derive a greater share of their consumption from net earnings, making their consumption more responsive to changes in s. As s approaches α, their consumption therefore increases at a faster rate, widening the disparity.

Summary: The main results for the standard model, in which education is purely an investment, are summarized as follows. Individual consumption and welfare increase (decrease) with the share of expenditure allocated to mother-tongue education, s, when $s < (>) \alpha$. Hence, consumption and welfare for everyone are maximized at $s = \alpha$, which corresponds to the income share of local-sector workers in final-good production. At this value of s, disparities in consumption

[16] At this level of a, educational expenditure of an individual with the highest ability $\overline{\omega}$ is constrained by wealth, while that of an individual with the lowest ability $\underline{\omega}$ is not. Educational disparities vary with s between higher-ability, wealth-constrained individuals (those with $a < e^{\sharp}(\omega_h)$) and lower-ability, unconstrained ones (those with $a > e^{\sharp}(\omega_l)$), where $\omega_h > \omega_l$. In contrast, disparities between those who are both unconstrained or both constrained are not affected by s.

[17] At this level of a, the educational expenditure of an individual with the lowest ability $\underline{\omega}$ is not constrained by wealth. Consumption inequality varies with s between higher-ability, wealth-constrained individuals and lower-ability, unconstrained ones, as well as among unconstrained individuals, while inequality between those who are both constrained is unaffected by s.

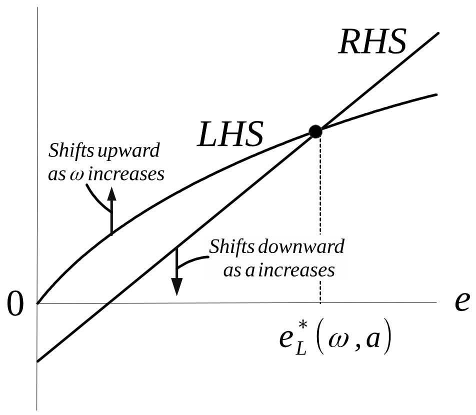


Figure 1: Determination of $e_L^*(\omega, a)$ and its relation with ω and a

and welfare between the poor (those who cannot afford optimal educational expenditure) and the wealthy (those who can), and among the wealthy are lowest, but disparities between individuals with sufficient wealth and different levels of non-language ability are highest.

4.2 Both consumption and investment motives

Now, the model in which utility depends directly on ethnic-language skill as well as consumption is examined.

The utility maximization problem when an individual chooses the local sector is

$$\begin{aligned} \max_e\{(c_L)^\theta (h_L)^{1-\theta}\} &= \max_e \left\{ \left[\frac{w_L}{P}\omega(\delta_L se)^\gamma + a - e\right]^\theta (\delta_L se)^{\gamma(1-\theta)} \right\}, \\ \text{s.t.} \quad e &\leq a. \end{aligned} \tag{25}$$

Denote educational expenditure when the constraint $e \leq a$ does not bind by $e_L^*(\omega, a)$. From the first-order condition of the maximization problem, $e_L^*(\omega, a)$ is e satisfying the following equation.

$$\frac{\theta}{\frac{w_L}{P}\omega(\delta_L se)^\gamma + a - e}\left[\gamma \frac{w_L}{P}\omega(\delta_L s)^\gamma (e)^{\gamma-1} - 1\right] + \frac{\gamma(1-\theta)}{e} = 0, \tag{26}$$

$$\Leftrightarrow \gamma \frac{w_L}{P}\omega(\delta_L se)^\gamma = -\gamma(1-\theta)a + [\theta + \gamma(1-\theta)]e. \tag{27}$$

$e_L(\omega, a) = a$ for $a < e_L^*(\omega, a)$, and $e_L(\omega, a) = e_L^*(\omega, a)$ for $a \geq e_L^*(\omega, a)$, where $e_L^*(\omega, a)$ increases with ω and a (see Figure 1).

The optimal expenditure $e_L^*(\omega, a)$ is greater than that in the previous model, $e^\sharp(\omega)$, which maximizes consumption and is given by e such that the first term of (26) equals 0. This is because increased expenditure also adds to utility by raising the mother-tongue skill in the present model. Unlike the previous model, $e_L^*(\omega, a)$ increases with wealth. While spending more than $e^\sharp(\omega)$ reduces consumption, it also enhances the ethnic-language skill. For wealthier individuals, the marginal utility loss from reduced consumption is smaller, since the baseline level of consumption is higher due to their wealth (see the first term of (26)). Hence, they spend more on education to gain utility from increased mother-tongue proficiency.

The maximization problem when an individual chooses the national sector is given by

$$\max_e\{(c_N)^\theta(h_N)^{1-\theta}\} \;=\; \max_e\left(\left\{\frac{T_N}{P}\omega[\delta_N(1-s)e]^\gamma + a - e\right\}^\theta(\delta_L se)^{\gamma(1-\theta)}\right), \tag{28}$$
$$\text{s.t.}\quad e \;\leq\; a.$$

From the first-order condition of the problem, $e^*_N(\omega,a)$ is e satisfying the following equation.

$$\gamma\frac{T_N}{P}\omega[\delta_N(1-s)e]^\gamma = -\gamma(1-\theta)a + [\theta+\gamma(1-\theta)]\,e. \tag{29}$$

Similar to $e_L(\omega,a)$, $e_N(\omega,a) = a$ for $a < e^*_N(\omega,a)$, and $e_N(\omega,a) = e^*_N(\omega,a)$ for $a \geq e^*_N(\omega,a)$, where $e^*_N(\omega,a)$ is increasing in ω and a.

As in the previous model, all individuals are indifferent between the two sectors, i.e., $u_L = u_N$, which is formally proved in Appendix A. This is because individuals do not have comparative advantages in a particular sector and goods from both sectors are essential in final good production. The proof shows that $e_L(\omega,a) = e_N(\omega,a) \Leftrightarrow c_L(\omega,a) = c_N(\omega,a)$ holds for any ω and a, thus $\frac{H_N}{H_L} = \frac{1-\alpha}{\alpha}\left(\frac{\delta_N}{\delta_L}\frac{1-s}{s}\right)^\gamma$ from (21) and $P = \frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\left(\frac{T_N}{T_L}\right)^\alpha\left(\frac{\delta_N}{\delta_L}\frac{1-s}{s}\right)^{\alpha\gamma}$ from (22), as before.

Hence, educational expenditure is given by

$$e(\omega,a) = \begin{cases} a & \text{for } a < a^{**}(\omega), \\ e^*(\omega,a) & \text{for } a \geq a^{**}(\omega), \end{cases} \tag{30}$$

where

$$e^*(\omega,a) \text{ is } e \text{ satisfying } (\alpha T_L)^\alpha[(1-\alpha)T_N]^{1-\alpha}\,\omega\{(\delta_L s)^\alpha[\delta_N(1-s)]^{1-\alpha}e\}^\gamma = \frac{1}{\gamma}\{-\gamma(1-\theta)a + [\theta+\gamma(1-\theta)]e\}, \tag{31}$$

$$\text{and } a^{**}(\omega) \equiv \left(\frac{\gamma}{\theta}(\alpha T_L)^\alpha[(1-\alpha)T_N]^{1-\alpha}\,\omega\{(\delta_L s)^\alpha[\delta_N(1-s)]^{1-\alpha}\}^\gamma\right)^{\frac{1}{1-\gamma}} \text{ is } a \text{ satisfying } a = e^*(\omega,a). \tag{32}$$

In the above equation, $e^*(\omega,a)$ increases with ω and a, and increases (decreases) with s for $s <(>) \alpha$. Thus, as in the model where individuals have only the investment motive for education, the optimal expenditure $e^*(\omega,a)$ is maximized at $s = \alpha$.[18]

Consumption and utitlity are given by

$$c(\omega,a) \;=\; \begin{cases} \widehat{y}(\omega)(a)^\gamma & \text{for } a < a^{**}(\omega), \\ \widehat{y}(\omega)(e^*(\omega,a))^\gamma + a - e^*(\omega,a) & \text{for } a \geq a^{**}(\omega). \end{cases} \tag{33}$$

$$u(\omega,a) \;=\; \begin{cases} [\widehat{y}(\omega)(a)^\gamma]^\theta(\delta_L sa)^{\gamma(1-\theta)} & \text{for } a < a^{**}(\omega), \\ [\widehat{y}(\omega)(e^*(\omega,a))^\gamma + a - e^*(\omega,a)]^\theta(\delta_L se^*(\omega,a))^{\gamma(1-\theta)} & \text{for } a \geq a^{**}(\omega), \end{cases} \tag{34}$$

$$\text{where } \widehat{y}(\omega) \;\equiv\; (\alpha T_L)^\alpha[(1-\alpha)T_N]^{1-\alpha}\,\omega\{(\delta_L s)^\alpha[\delta_N(1-s)]^{1-\alpha}\}^\gamma. \tag{35}$$

As in the previous model, individual consumption is maximized at $s = \alpha$.[19] What about individual utility, which now depends directly on both consumption and the ethnic-language skill? The next proposition addresses this question. It also reports the above results on educational expenditure and consumption.

Proposition 4 *Consider an economy where education has both consumption and investment roles.*

[18]The result can be understood from (26) for local-sector workers. Since $e^*(\omega,a) > e^\sharp(\omega)$, additional education at $e = e^*(\omega,a)$ reduces consumption but improves the ethnic-language skill. Because $\frac{w_L}{P}\omega(\delta_L s)^\gamma$ is maximized at $s = \alpha$, the marginal utility loss from reduced consumption (the first term of (26)) is smallest at this s, owing to the largest consumption (the denominator of the term) and the smallest negative economic returns to education (the numerator). Hence, they spend most on education at $s = \alpha$ to gain utility from improved mother-tongue proficiency.

[19]When $a \geq a^{**}(\omega)$, this is because $\widehat{y}(\omega)(e^*(\omega,a))^\gamma + a - e^*(\omega,a) = \theta\left[(\frac{1}{\gamma}-1)e^*(\omega,a) + a\right]$ from (31).

(*i*) *As in the economy where education serves only as an investment,* $e(\omega, a)$ *for* $a \geq a^{**}(\omega)$ *and* $c(\omega, a)$ *increase (decrease) with* s *for* $s < (>)\alpha$*, and is maximized at* $s = \alpha$*.*

(*ii*) *When* $a < a^{**}(\omega)$ *(where* $a^{**\prime}(\omega) > 0$*),* $u(\omega, a)$ *increases (decreases) with* s *for* $s < (>) 1-(1-\alpha)\theta$*, and is maximized at* $s = 1-(1-\alpha)\theta > \alpha$*.*

(*iii*) *When* $a \geq a^{**}(\omega)$*,* $u(\omega, a)$ *increases (decreases) with* s *for* $s < (>) s^*(\omega, a) \in [1-(1-\alpha)\theta, 1)$ *(when* $a > a^{**}(\omega), s^*(\omega, a) > 1-(1-\alpha)\theta$*), and is maximized at* $s = s^*(\omega, a)$*, where* $s^*(\omega, a)$ *increases with* a *and decreases with* ω*.*

The proposition shows that when wealth is small enough that educational expenditure is constrained by wealth, i.e., $a < a^{**}(\omega)$ (where $a^{**\prime}(\omega) > 0$), individual welfare increases (decreases) with s for $s < (>) 1 - (1-\alpha)\theta$, and is maximized at $s = 1 - (1-\alpha)\theta$. This value exceeds α, the welfare-maximizing level in the model where education serves only as an investment. By contrast, when wealth is sufficient to allow the optimal expenditure, i.e., $a \geq a^{**}(\omega)$, welfare increases (decreases) with s for $s < (>) s^*(\omega, a) \in [1 - (1-\alpha)\theta, 1)$ (when $a > a^{**}(\omega), s^*(\omega, a) > 1-(1-\alpha)\theta$), and is maximized at $s = s^*(\omega, a)$, where $s^*(\omega, a)$ increases with a and decreases with ω. That is, the welfare-maximizing weight on mother-tongue education is greater for wealthier individuals and smaller for those with higher non-language ability, and it exceeds the level optimal for wealth-constrained individuals.

The results for wealth-constrained individuals (those with $a < a^{**}(\omega)$ and thus $e(\omega, a) = a$) can be explained as follows. An increase in s affects individual welfare through two channels. It alters earnings by changing the wage rate and human capital, thereby influencing consumption. It also directly enhances the mother-tongue skill. Consumption is maximized at $s = \alpha$, the same level as in the model with only the investment motive, while the ethnic-language skill is maximized at $s = 1$. Because the weight on consumption (the mother-tongue skill) in utility is θ $(1-\theta)$, the welfare-maximizing weight on ethnic-language education is given by $\theta\alpha + (1-\theta) * 1$.

For wealthier individuals (with $a \geq a^{**}(\omega)$), an increase in s affects welfare through four channels. Since their educational expenditure $e(\omega, a) = e^*(\omega, a)$ depends on s, the change in s alters (i) earnings, and thus consumption, and (ii) the mother-tongue skill, through its effect on educational expenditure. The remaining channels are identical to those for wealth-constrained individuals: for a given level of educational expenditure, an increase in s affects consumption via changes in the wage rate and human capital, and directly increases the ethnic-language skill. Because educational expenditure is chosen to maximize utility, the two effects operating through expenditure cancel out from the envelop theorem. Consequently, the remaining two channels for a given expenditure determine welfare, as in the case of constrained individuals.

However, the welfare-maximizing weight on mother-tongue education is higher than for the wealth-constrained, and unlike these individuals, the optimal weight increases with wealth and decreases with non-language ability. The intuition is as follows. Increasing s above α has a negative effect on welfare by reducing earnings and thus consumption, while it has a positive effect on welfare through increased mother-tongue skill. Because they finance part of their consumption from wealth net of educational expenditure—a source unaffected by changes in s—the marginal utility loss from reduced consumption is smaller than for those who rely entirely on earnings, i.e., the wealth-constrained. Moreover, since individuals with greater wealth (non-language ability) finance a larger (smaller) share of their consumption from wealth rather than earnings, the marginal utility loss is smaller (greater). Hence, the welfare-maximizing s is greater than for the wealth-constrained, increases with a, and decreases with ω.

The next proposition examines disparities in educational expenditure, consumption, and welfare across individuals with different wealth levels for a given level of non-language skill. Remember that disparities in outcomes between two individuals are measured by the ratio of their respective outcomes.

Proposition 5 *Consider individuals with the same ω in an economy where education has both consumption and investment roles.*

(*i*) *Educational disparities between the poor, i.e., those with $a < a^{**}(\omega)$, and the wealthy, i.e., those with $a \geq a^{**}(\omega)$, increase (decrease) with s for $s < (>) \alpha$. Disparities among the wealthy decrease (increase) with s for $s < (>) \alpha$, while disparities among the poor do not depend on s.*

(*ii*) *Consumption and welfare inequalities between the poor and the wealthy and among the wealthy decrease (increase) with s for $s < (>) \alpha$. Inequalities among the poor are unaffected by s.*

As in the model with only the investment motive for education, for a given level of non-language skill ω, educational disparities between the poor (those with $a < a^{**}(\omega)$) and the wealthy (those with $a \geq a^{**}(\omega)$) increase (decrease) with s for $s < (>) \alpha$, while disparities within the poor do not depend on s. By contrast, unlike in the previous model, disparities within the wealthy *decrease* (increase) with s for $s < (>) \alpha$. Also consistent with the previous model, consumption and welfare inequalities between the poor and the wealthy, as well as among the wealthy, decrease (increase) with s for $s < (>) \alpha$, whereas inequalities among the poor are unaffected by s.

Thus, the weight on mother-tongue education that maximizes consumption for every individual, i.e., $s = \alpha$, also minimizes consumption and welfare inequalities between the poor and the wealthy, and among the wealthy. However, unlike the previous model, this weight is lower than the welfare-maximizing weight, which is heterogenous among individuals.

The result that educational disparities within the wealthy are smallest at $s = \alpha$ can be explained as follows. Among the wealthy, educational expenditure increases with wealth due to due to its consumption value, but less than proportionally because of decreasing returns to expenditure in skill production. Since $e^*(\omega, a) > e^\sharp(\omega)$, where consumption is maximized, the marginal utility loss from reduced consumption arises at the margin. This marginal utility loss is most similar across different wealth levels when $s = \alpha$, because earnings are maximized and thus total income available for consumption is most equalized at this level of s. This leads to the lowest educational disparities among the wealthy.

This also accounts for the lowest consumption and welfare inequalities among the wealthy at $s = \alpha$, because the proportion of consumption financed by net earnings—more similar across individuals than wealth—is highest at this level. Consumption inequality between the poor and the wealthy is minimized at $s = \alpha$, because, as explained for the corresponding result on the previous model, the consumption of the poor is more sensitive to changes in s than that of the wealthy. Welfare inequality between them is also lowest at this level, as the effect of s on consumption inequality dominates its effect on mother-tongue-skill inequality in the utility function.

The next proposition examines disparities across individuals with different levels of non-language ability for a given level of wealth. In the proposition, $\overline{\omega}$ ($\underline{\omega}$) denotes the highest (lowest) level of non-language ability in the distribution.

Proposition 6 *Consider individuals with the same a in an economy where education has both consumption and investment roles.*

(*i*) *Educational disparities between individuals at a middle level of wealth ($a \in (a^{**}(\underline{\omega}), a^{**}(\overline{\omega}))$) with different levels of non-language ability tend to decrease (increase) with s for $s < (>) \alpha$. Inter-ability disparities among the poorer ($a \leq a^{**}(\underline{\omega})$) do not depend on s, while among the wealthier ($a \geq a^{**}(\overline{\omega})$), they increase (decrease) with s for $s < (>) \alpha$.*

(*ii*) *Inter-ability consumption and welfare inequalities among those with a not-low level of wealth* ($a > a^{**}(\underline{\omega})$) *tend to increase* (*decrease*) *with* s *for* $s < (>)\alpha$. *Inter-ability inequalities among the poorer* ($a \leq a^{**}(\underline{\omega})$) *are unaffected by* s.

As in the model with only the investment motive, in the middle range of the wealth distribution, educational disparities between individuals with equal wealth but different levels of non-language ability tend to decrease (increase) with s for $s < (>)\alpha$, while inter-ability disparities among the poorer are unaffected by s. Unlike in the previous model, inter-ability disparities among the wealthier *increase* (decrease) with s for $s < (>)\alpha$. As before, among those with sufficient wealth, inter-ability consumption and welfare inequalities tend to increase (decrease) with s for $s < (>)\alpha$, whereas inequalities among the poorer segment are unchanged by s.

Thus, the weight on mother-tongue education that maximizes individual consumption, i.e., $s = \alpha$, generates the highest inter-ability consumption and welfare inequalities among the non-poor. Unlike in the previous model, however, this weight is lower than the welfare-maximizing weight, which varies across individuals.

The result that inter-ability educational disparities among the wealthier (those with $a \geq a^{**}(\overline{\omega})$) are greatest at $s = \alpha$ can be understood as follows. For these individuals, educational spending equals $e^*(\omega, a)$. Since $e^*(\omega, a) > e^\sharp(\omega)$, where consumption is maximized, the marginal utility loss from reduced consumption arises at the margin. This marginal utility loss is most dissimilar across ability levels when $s = \alpha$, because earnings are maximized and thus total income available for consumption is most unequal at this level of s. This leads to the highest educational disparities.

Inter-ability consumption inequality among the non-poor tends to be highest at the earnings-maximizing level of s, i.e., $s = \alpha$, since higher-ability individuals derive a greater share of their consumption from net earnings, making their consumption more responsive to changes in s. Inter-ability welfare inequality is also greatest at this level, because the effect of s on consumption inequality outweighs its effect on mother-tongue-skill inequality, which is lowest at $s = \alpha$ in the middle range of the wealth distribution, in the utility function.

Summary: The main results for the model in which utility depends on both ethnic-language skill and consumption are summarized as follows. As in the standard model, individual consumption is maximized at $s = \alpha$. By contrast, the weight on mother-tongue education that maximizes individual welfare is higher than in the standard model and generally differs among individuals. While the optimal s is identical for the poor (those who cannot afford optimal educational expenditure), it is heterogenous among the wealthy. In particular, the welfare-maximizing weight on mother-tongue education is higher for those with greater wealth and lower for those with higher non-language ability, and it exceeds the level optimal for wealth-constrained individuals.

Also as in the standard model, consumption and welfare inequalities between the poor and the wealthy, as well as among the wealthy, are minimized at $s = \alpha$, while inter-ability inequalities in these outcomes among relatively wealthy individuals are maximized at this level of s. However, unlike in the previous model, this weight is lower than the welfare-maximizing weight, which generally varies among individuals.

5 Socially-optimal weight on mother tongue education

While Proposition 4 examined the weight on mother-tongue education s that maximizes individual welfare, this section investigates the weight that maximizes social welfare. The proposition shows that the optimal weight differs depending on wealth and non-language ability. Due to this heterogeneity, it is generally not possible to obtain analytical results on the socially optimal weight.

Therefore, this section solves for the optimal s numerically, assuming a and ω follow log normal distributions, and examines how it depends on the distributional parameters of the two variables.

Two types of social welfare functions are employed. The first is the utilitarian social welfare function, which is the sum of individual utilities. This function does not account for inequality in welfare across individuals. However, Propositions 5 and 6 show that welfare inequality between two individuals with different wealth or different levels of non-language ability depends on s. Hence, to incorporate inequality aversion in the evaluation of social welfare, the following CES function is also considered:

$$W = \left(\frac{1}{N}\sum\nolimits_i \left\{N_i \int [u_i(\omega, a)]^{1-\sigma} dF_i(\omega, a)\right\}\right)^{\frac{1}{1-\sigma}}, \ \sigma \in [0, 1), \tag{36}$$

where $u_i(\omega, a)$ is the utility function of an individual with ω and a from ethnic group i, $F_i(\omega, a)$ is the joint distribution function of ω and a for group i, N_i is the population size of group i, and N is the total population. A larger value of σ implies a stronger degree of inequality aversion. When $\sigma = 0$, the function coincides with the utilitarian function.

Under these social welfare functions, the value of s that maximizes the social welfare of one group also maximizes the social welfare of the whole society, because an individual's utility depends solely on the value of s for her group. Accordingly, the following analysis focuses on a single ethnic group.

5.1 Calibration

The model parameters are set as follows. The parameter α represents the weight on human capital of local-sector workers in the sector's production function, (1). From (3) and (9), α equals the ratio of local-sector workers' earnings to total earnings. The earnings ratio is calculated using the dataset on sectoral labor-income shares created by Oishi and Paul (2018), together with sectoral value-added data from the GGDC 10-Sector Database (Timmer, de Vries, and de Vries, 2015). In the real economy, local-sector jobs correspond to jobs requiring proficiency in the local ethnic language. Accordingly, the earnings of local-sector workers are computed using data for Agriculture (including agriculture, hunting, forestry, and fishing), Trade services (including wholesale and retail trade, hotels, and restaurants), and Personal Services (including community, social, and personal services) in these datasets. The value for α is set to 0.5033, the average for eight developing countries.[20]

The national sector's TFP is normalized to one, i.e., $T_N = 1$, while the local sector's TFP, T_L, is calculated as the ratio of the weighted average TFP of the above three sectors (using sectoral value added shares as weights) to the weighted average TFP of the remaining sectors, based on the sectoral dataset of Inklaar, Marapin, and Gräler (2024).[21] The value for T_L is set to 0.7511, the average across country–year observations.

The parameter γ determines the degree of diminishing returns to educational expenditure in the language-skill production functions, (11) – (13). It equals the elasticity of earnings with respect

[20] The eight countries, for which paired data on labor-income shares and value added for these sectors are available, are Egypt, Ethiopia, Ghana, Indonesia, Thailand, Argentina, Mexico, and Peru.

[21] Inklaar, Marapin, and Gräler (2024) compute sectoral TFPs relative to the U.S. for 84 countries for the years 2005, 2011, and 2017, using four methods. In the present study, the ratio of the weighted average TFP of the three sectors to that of the remaining sectors is calculated using their dataset and their fourth method, which uses sectoral value-added PPPs (purchasing power parities) and sectoral employment data, and estimates country- and sector-specific factor shares. Unlike Inklaar, Marapin, and Gräler (2024), capital stock rather than capital services data are used and samples without sectoral employment are excluded.

to educational expenditure. Jackson, Johnson, and Persico (2016) find that a 10% increase in per-pupil spending on U.S. public K-12 education leads to an average 7.74% rise in adult wages. Based on this result, γ is set to 0.774.

The parameter δ_N (δ_L) in the language-skill production functions represents the effectiveness of common-language (mother-tongue) education in skill development. δ_L is normalized to 1. Empirical studies (e.g., Alkateb-Chami, 2024) suggest that mother-tongue instruction is more effective, implying $\delta_N < 1$.[22] In particular, Bender et al. (2005), a World Bank study of Mali, find that French-only education is about 27% more costly than mother-tongue instruction for students to complete six years of primary education, largely due to higher repetition and drop-out rates of the former. They also report similar findings for Guatemala. Based on their finding, δ_N is set to $1/1.27 = 0.787$.

As for θ, the weight on consumption in the utility function, there is no empirical evidence to guide its calibration. Accordingly, the value is set to 0.8, which implies that the consumption motive for mother-tongue education, which increases with $1-\theta$, is fairly weak. Sensitivity analysis is conducted to assess how varying this parameter affects the main results.

The distributions of a and ω are assumed to be log normal. When a variable X is log-normally distributed, its Gini coefficient is fully determined by the mean and variance of the variable. Specifically, $Gini = 2\Phi\Big(\sqrt{\frac{1}{2}\ln(1+\frac{Var(X)}{[E(X)]^2})}\Big) - 1$, where Φ is the distribution function of standard normal distribution. Accordingly, the distribution of a is fixed by setting its mean and *Gini coefficient*. In the real economy, a corresponds to wealth for households with school-age children. For the U.S., Kuhn and Ríos-Rull (2025) report a Gini coefficient of 0.8 for wealth net of debt among married households with children. They also find that the net-wealth Gini for households whose heads are in child-rearing age groups ranges from 0.77 (ages 46−50) to 0.82 (ages 26−30 and ages 31−35). For Japan, Kitao and Yamada (2025) report a Gini of 0.647 for financial wealth among household with children aged 16 or younger. Davies et al. (2011) present estimates of the net-wealth Gini for 25 developed and developing countries and show that the Gini is highest for the U.S. and lowest for Japan. Hence, in the baseline economy, the Gini of a is set to 0.72, a value approximately midway between the number for Japan and those for the U.S.

The mean of a is set to 0.0051 so that the fraction of individuals unable to afford optimal education, i.e. those with $a < a^{**}(\omega)$, is around 0.45 in the baseline economy. This target value motivated by evidence from PISA, a major international test for 15-year-old students. In PISA 2022, in which 37 OECD countries and 44 partner countries and economies participated, on average across OECD countries, the percentage of students without basic reading skills (below Level 2) was 26%, while in most non-OECD developing countries, more than half of students failed to attain basic reading proficiency, with the cross-country average exceeding 60% (OECD, 2023). Assuming that 15 percentage points of low-performing students reflect non-economic factors unrelated to family economic conditions (which implies that 26%−15% =11% of students perfom badly due to economic reasons), the share of students who cannot attain basic reading skills for economic reasons in developing countries is roughly 60%−15% =45%.

The exogenous ability ω roughly corresponds to early-life cognitive ability realized by the time children begin formal education. Restuccia and Urrutia (2004) develop a dynamic macroeconomic model of intergenerational human capital transmission featuring a human capital production function in which innate ability and educational expenditure enter multiplicatively and innate ability

[22] Alkateb-Chami (2024), based on cross-country data from 56 countries, finds a strong positive relation between an indicator of poor basic literacy (the proportion of children unable to read and understand a simple text by age ten) and a measure of misalignment between students' home language and the language of instruction at school, after controlling for per capita income.

is log-normally distributed, as in the present model. Calibrating the model to the U.S. economy, they find that the standard deviation of the log of innate ability is 0.48, with its mean set to 0. Seshadri and Yuki (2004) construct a dynamic model with a similar human capital production function, but with different settings for other parts of the model. They also calibrate the model to the U.S. economy and obtain a similar number for the standard deviation, 0.45, despite using different calibration targets. Hence, the standard deviation of log ω is set at the midpoint of the two values, 0.465, with its mean equal 0 (the Gini of ω is about 0.258).

Finally, calibration of the correlation coefficient between a and ω relies on evidence from the psychological literature. Dilworth-Bart (2012) finds that, among preschool children, the correlation coefficient between household income and a composite index of verbal and non-verbal working memory, which can be interpreted as a measure of early cognitive endowment, is 0.33. This value is used in the baseline economy.

5.2 Baseline economy

Figure 2 presents the results for the baseline economy when the utilitarian social welfare function is employed. Panel (a) shows that social welfare increases (decreases) with s for small (large) s, and is maximized at $s = 0.644$. That is, social welfare is highest when 64.4% of educational spending is allocated to mother-tongue education. This value exceeds the optimal s in the model with the investment motive only, which equals $\alpha = 0.503$ from Proposition 1. It is also greater than the value maximizing individual welfare for wealth-constrained individuals, i.e., those with $a < a^{**}(\omega)$, which equals $1 - (1-\alpha)\theta = 0.603$ from Proposition 4 (ii). This is because the welfare-maximizing values of s for unconstrained individuals, $s^*(\omega, a)$, are larger (Proposition 4 (iii)).

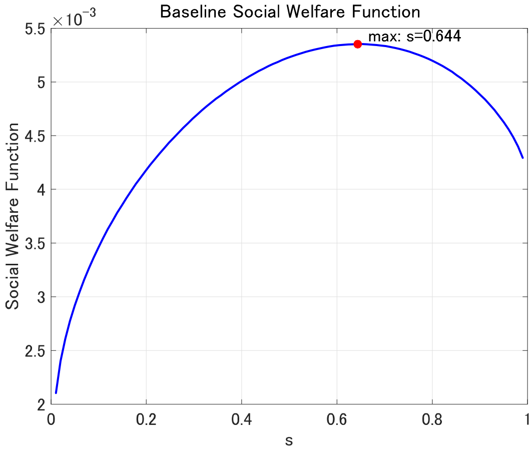
Panel (b) shows that the proportion of wealth-constrained individuals is close to the target value of 0.45 when s is in the intermediate range. This proportion is low for both small and large values of s, because educational spending for unconstrained individuals, $e^*(\omega, a)$, is low in these ranges, as implied by Proposition 4 (i).

Individual utility depends directly on consumption and mother-tongue skill, h_L. Panel (c) illustrates that the group's average consumption is maximized at $s = \alpha = 0.503$ from Proposition 4 (i), while Panel (d) shows that average mother-tongue skill is highest at $s = 0.792$.[23] The socially-optimal value of 0.644 lies between these two values.

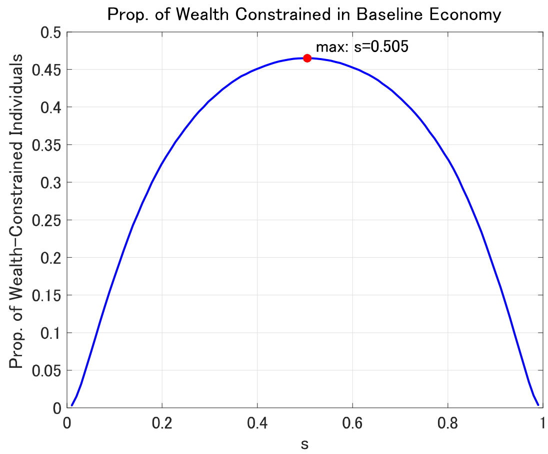
5.3 Numerical experiments

To examine how the distributions of a and ω affect the socially optimal s, three numerical experiments are conducted. First, to assess the effect of the average level of wealth on the optimal s, the mean of a is varied with its Gini coefficient fixed. According to the equation for the Gini coefficient in Section 5.1, this is achieved by changing the mean and standard deviation of the variable proportionately. Second, to evaluate the effect of wealth inequality, the variance (and thus the Gini coefficient) of a is varied with its mean held constant. Third, the correlation coefficient between a and ω is varied. This does not affect the mean, variance, or Gini coeffieint of either variable. As mentioned in Section 5.1, in the real economy, a corresponds to wealth for households with school-age children,.and ω roughly represents early-life cognitive ability realized by the time children begin formal education. Thus, the correlation between these variables would differ across countries, reflecting the extent to which early-childhood rearing environments are shaped by family economic conditions.

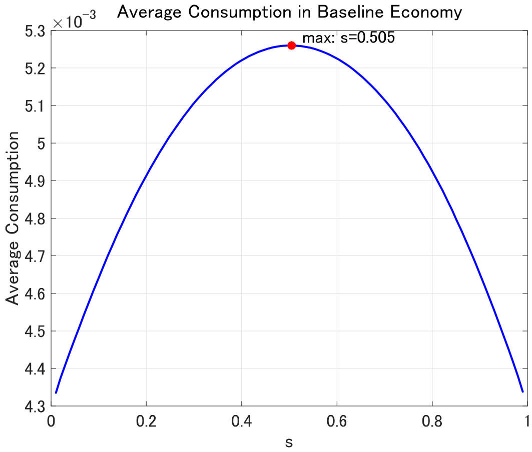
[23] Average common-language skill is highest at $s = 0.208$. Common-language (mother-tongue) skill is not maximized at very low (high) values of s because $e^*(\omega, a)$ is low in these ranges.



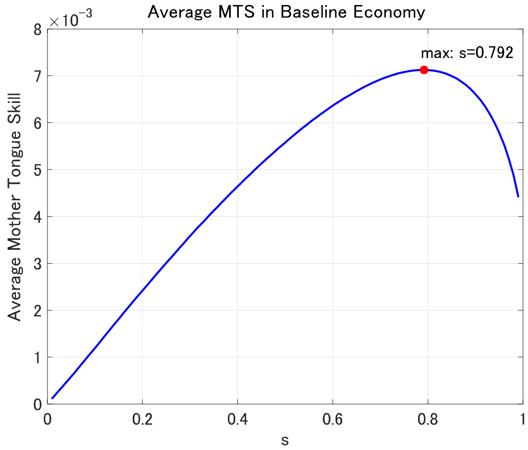
(a) Relationship between Social Welfare and s



(b) Relationship between Prop. of Wealth Constrained and s



(c) Relationship between Avg. Consumption and s



(d) Relationship between Mother Tongue Skill and s

Figure 2: Results for Baseline Economy

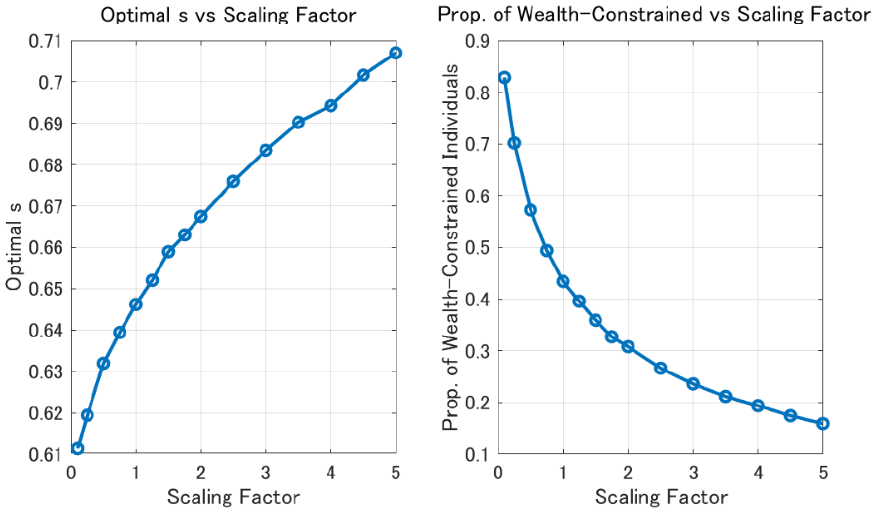


Figure 3: Results for Experiment on Mean a

Figure 3 presents the results of the experiment in which the average level of wealth is varied for a given degree of wealth inequality. The left panel illustrates the relationship between the scaling factor, which indicates how many times larger the mean a is relative to the baseline economy, and the socially optimal s. As the average level of wealth increases, the socially optimal s increases. For example, when average wealth is four times that of the baseline economy, the proportion of wealth constrained individuals falls to 0.194 from 0.434, and the optimal s increases to 0.694 from 0.644. This occurs because, as shown in the right panel, the proportion of wealth-constrained individuals, i.e., those with $a < a^{**}(\omega)$, whose optimal s is lower than those for unconstrained individuals, decreases with the average wealth.[24]

Figure 4 reports the results of the experiment in which the variance of wealth is varied with mean wealth held constant. The variance factor on the horizontal axes indicates many times larger the variance of a is relative to the baseline case. As shown in the right panel, the Gini coefficient of a increases with the variance. The wealth Gini is close to U.S. levels (between 0.77 and 0.82) reported by Kuhn and Ríos-Rull (2025) when the variance factor exceeds 2, while it is close to the Japanese value (0.647) reported by Kitao and Yamada (2025) when the variance factor is around 0.5. The central panel shows that the proportion of wealth-constrained individuals rises as wealth inequality increases. The left panel shows that the socially optimal s generally increases with the variance factor; however, the overall magnitude of the change is considerably smaller than in the first and next experiments. For example, when the variance of wealth is four times that of the baseline economy (when the wealth Gini increases to 0.814 from 0.718), the proportion of wealth constrained individuals rises to 0.549 from 0.434, but the optimal s increases only to 0.653 from 0.644. This result reflects two opposing forces: greater wealth variance increases the share of wealth-constrained individuals, which tends to lower the socially optimal s, while it also raises the share of wealthy individuals, which tends to raise the optimal s. Because neither force dominates across the entire range, the magnitude of the change is small.

Finally, Figure 5 shows the results of the experiment in which the correlation coefficent between

[24]The proportion of constrained individuals depends on s because $a^{**}(\omega)$ varies with s. The figure plots this proportion evaluated at the socially optimal value of s for each level of the scaling factor.

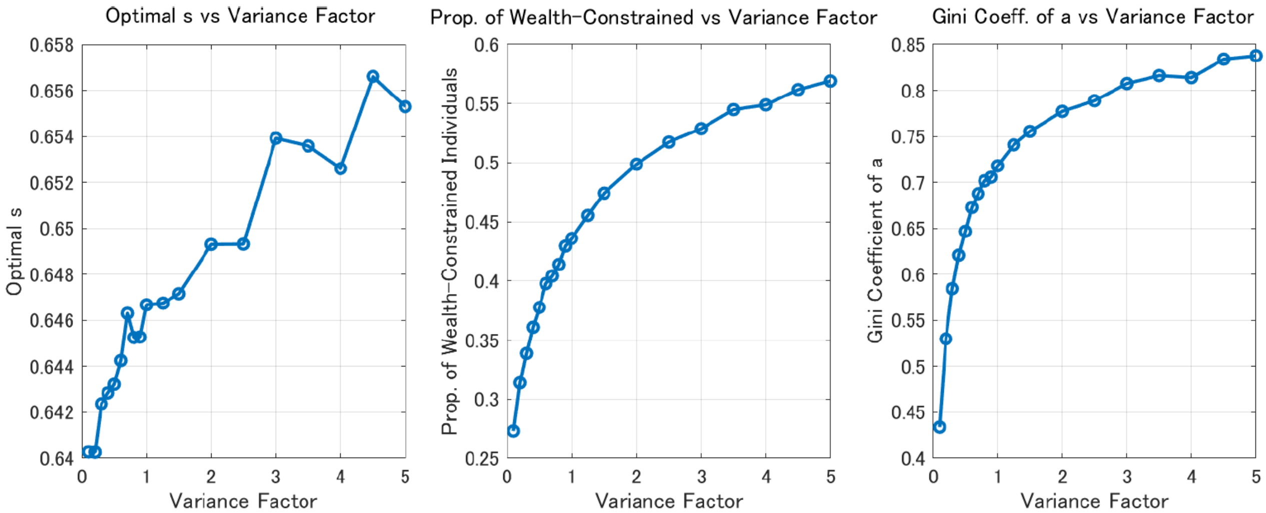


Figure 4: Results for Experiment on Variance of a

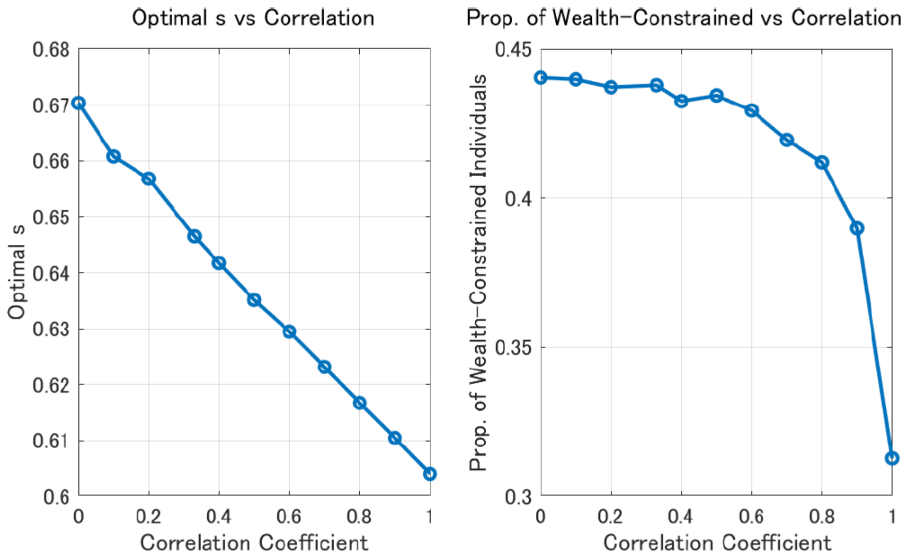
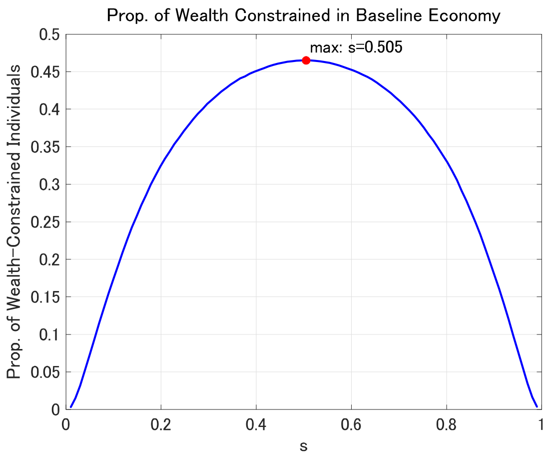
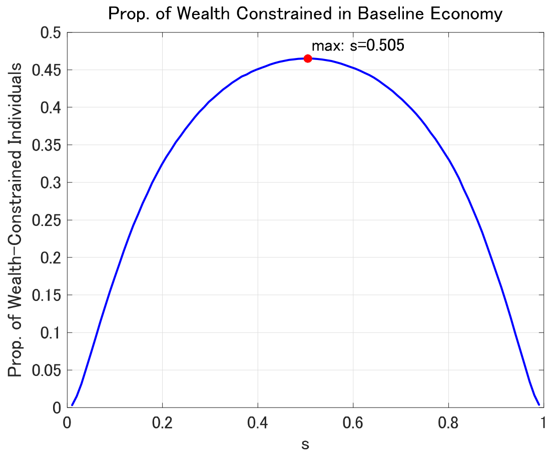


Figure 5: Results for Experiment on Correlation Coefficient between a and ω



(a) Relationship between Social Welfare and s



(b) Relationship between Prop. of Wealth Constrained and s

Figure 6: Results for Inequality-Averse SWF when $\sigma = 0.5$

a and ω is varied holding their marginal distributions fixed. The left panel shows that the socially optimal s decreases as the correlation coefficient increases. For example, when the correlation coefficent increases to 0.7 from 0.33 of the baseline economy, the optimal s decreases to 0.623 from 0.644. As wealth and exogenous ability become more strongly correlated, high-ability individual are more likely to be wealth-rich. Because the individually optimal level of s for unconstrained individuals, $s^*(\omega, a)$, increases with a and decreases with ω (Proposition 4 (iii)), the optimal s for these individuals would increase. At the same time, however, low-ability individual become more likely to be wealth-poor. For these individuals, the optimal s would decrease when they are unconstrained, because $s^*(\omega, a)$ increases with a.[25] The result suggests that the latter effect dominates and thus the socially optimal s decreases with the correlation coefficient.

5.4 Inequality-averse social welfare function

This section presents the results when the social weflare function accouting for inequality aversion (36) is employed. For the baseline analysis based on this function, the paramter capturing the degree of inequality aversion σ is set to 0.5. Values of other parameters are same as the previous analysis.

Figure 6 presents the baseline results under the inequality-averse social welfare function. Panel (a) shows that the socially optimal s decreases to 0.634 from 0.644 under the utilitarian social welfare function.[26] Proposition 5 shows that inter-wealth welfare inequality between two individuals—measured by the ratio of their utilities—decreases (increases) with s for $s < (>) \alpha$, or is independent of s, depending on their wealth levels. By contrast, Proposition 6 shows that inter-ability welfare inequality increases (decreases) with s for $s < (>) \alpha$, or is independent of s, again depending on their

[25]The middle panel shows that the proportion of wealth-constrained individuals generally decreases with the correlation coefficient. This decline contributes to raising the socially optimal s. However, the effect would be negligible over most of the range, since the change in the proportion is small except at very high levels of correlation.

[26]Panel (b) shows that the proportion of wealth-constrained individuals is close to the target value of 0.45 when s is in the intermediate range.

wealth levels. Since the socially optimal s under the utilitarian social welfare exceeds $\alpha = 0.5033$, inter-wealth welfare inequality is higher, and inter-ability welfare inequality is lower, than at $s = \alpha$. The finding that the socially optimal s under the inequality-averse social welfare is closer to α implies that inter-wealth welfare inequality is mitigated at the cost of greater inter-ability welfare inequality, compared to the utilitarian case. This would be because wealth inequality is much higher than ability inequality, with a Gini coefficient of 0.71 for a compared to 0.258 for ω.

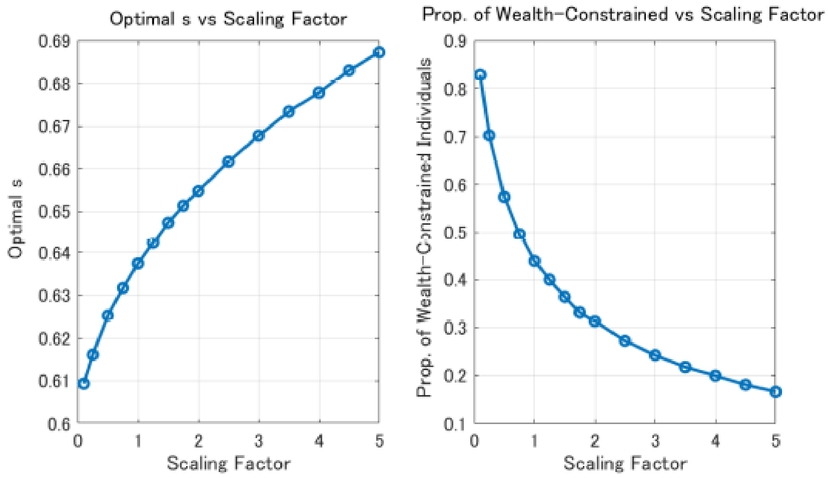


Figure 7: Results for Experiment on Mean a when $\sigma = 0.5$

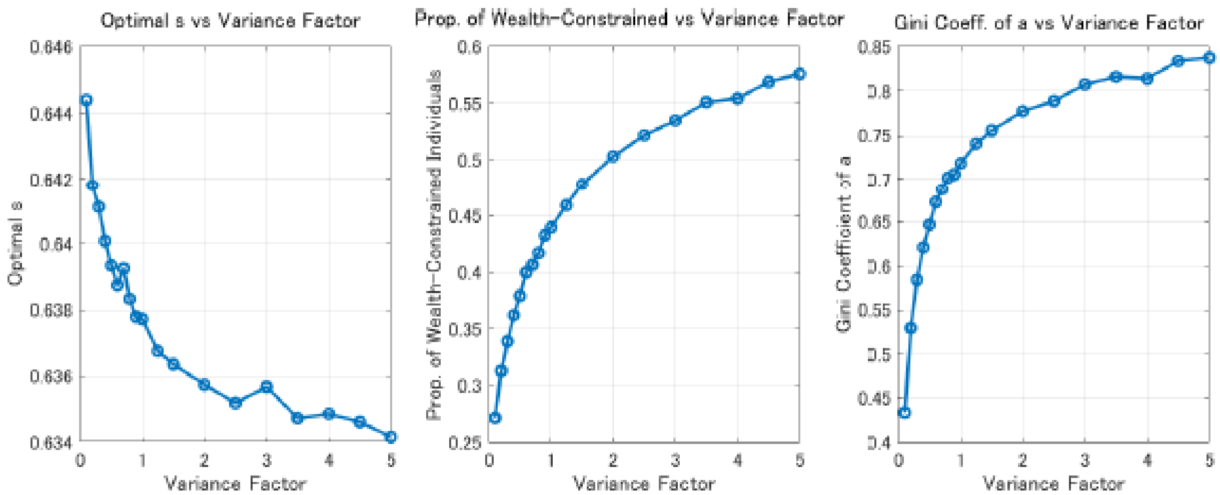


Figure 8: Results for Experiment on Variance of a when $\sigma = 0.5$

Figure 7, Figure 8, and Figure 9 presents the baseline results for the three experiments. As in the utilitarian case, Figure 7 shows that the socially optimal s increases with the average level of wealth. It is lower than in the utilitarian case for any value of the scaling factor. Also as in the previous case, Figure 9 shows that the socially optimal s decreases with the correlation coefficient.

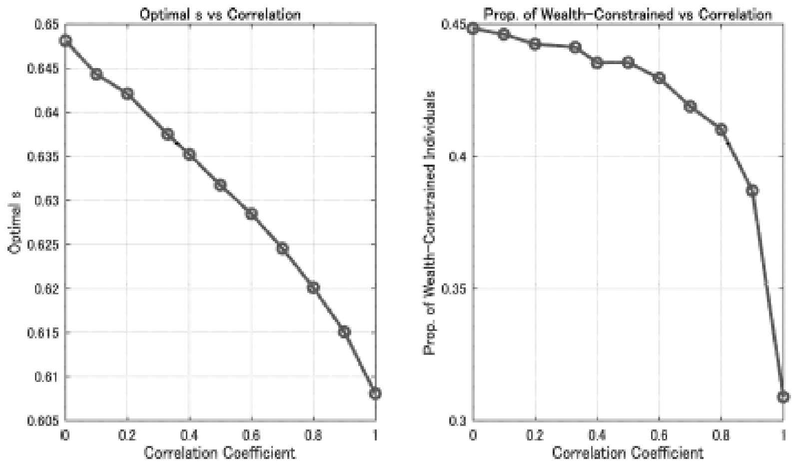


Figure 9: Results for Experiment on Correlation Coefficient between a and ω when $\sigma = 0.5$

	$\sigma = 0$ (Utilitarian case)	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 2$	$\sigma = 4$
Socially optimal s	0.644	0.634	0.631	0.622	0.609

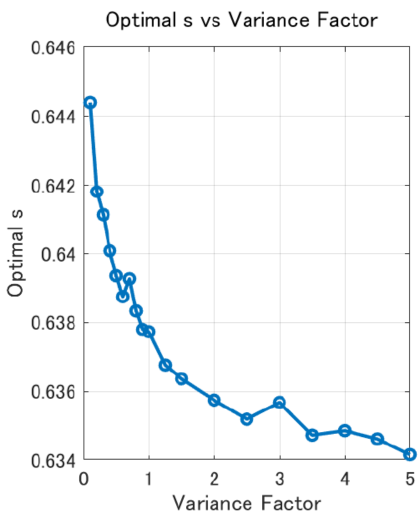
Table 1: Socially optimal s for different values of σ

The optimal s, however, is *higher* than in the utilitarian case when the correlation coefficient exceeds 0.7. When the correlation between wealth and exogenous market ability is high for a given level of inequality in each variable, disparities in consumption, which depend on both wealth and market ability, are large, while disparities in the mother-tongue skill, which do not depend on market ability, are relatively low. Hence, social welfare can be increased by reducing welfare inequality across individuals through narrowing consumption disparities. This can be achieved by lowering inter-ability welfare inequality by choosing s further from α.

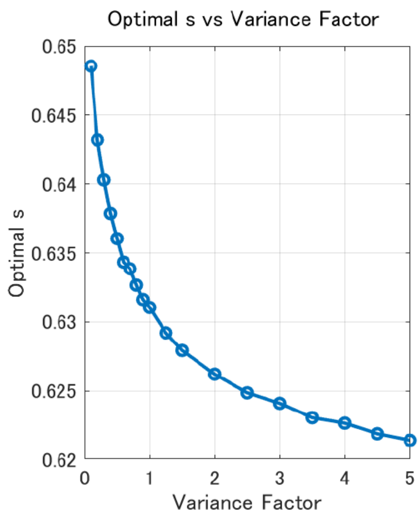
Differently from the utilitarian case, Figure 8 shows that the optimal s at high levels of wealth variance is *lower* than at low levels. When wealth inequality is high, the adverse effect of inter-wealth welfare inequality on inequality-averse social welfare is substantial. As a result, the optimal s moves closer to α to reduce inter-wealth inequality. As before, the overall magnitude of the change is substantially smaller than in the other experiments.

Table 1 presents the socially optimal s for different values of σ. It shows that as the degree of inequality aversion, σ, increases, the socially optimal s decreases. In other words, as social welfare places greater weight on inequality aversion, it becomes optimal to allocate a larger proportion of educational spending to common language education.

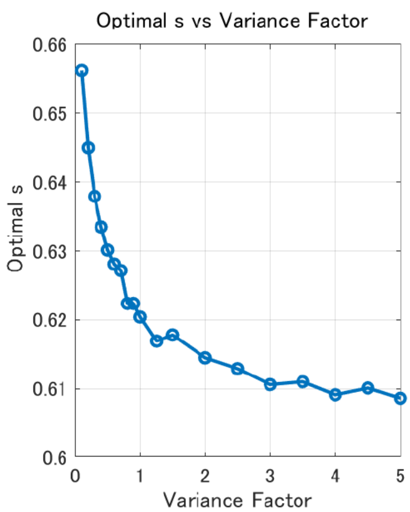
One noticeable difference from the utilitarian case ($\sigma = 0$) and the low–inequality-aversion case ($\sigma = 0.5$) is that, in the variance experiment, the magnitude of the change in the optimal s is large when inequality aversion is high. Figure 10 presents the results of the experiment for different values of σ (excluding the utilitarian case). When inequality aversion is high, i.e., $\sigma = 1, 2$, and 4, the optimal s decreases substantially with wealth inequality. That is, when social welfare prioritizes equity, the optimal weight on mother tongue education is *lower* in societies with large wealth inequality.



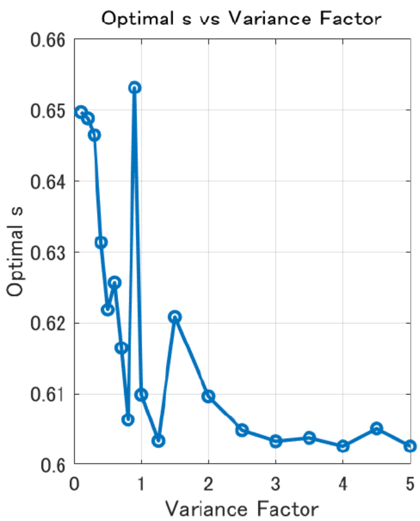
(a) $\sigma = 0.5$



(b) $\sigma = 1.0$



(c) $\sigma = 2.0$



(c) $\sigma = 4.0$

Figure 10: Results of Experiment on Variance for different values of σ

6 Conclusion

Determining the appropriate balance between mother-tongue and common-language education is a crucial policy issue in many countries. Despite the importance of the issue, little is known about the ideal combination of the two types of education in terms of future consumption and welfare, especially when the non-workplace benefits of the mother-tongue skill are taken into account.

This paper developed a simple model to examine these issues both theoretically and numerically. In particular, it analyzed how the desirable educational balance varies with individual wealth and ability, and how the socially-optimal weights on the two types of education differ depending on aggregate conditions of the economy, such as income level and wealth distribution. The main results can be summarized as follows.

In the standard model in which utility depends solely on goods consumption, individual consumption and welfare are maximized at the same intermediate level of s (the share of expenditure allocated to mother-tongue education) for all individuals. At this value of s, disparities in consumption and welfare between the poor (those who cannot afford optimal educational expenditure) and the wealthy (those who can), as well as among the wealthy, are minimized, while disparities among relatively wealthy individuals with different levels of non-language ability are maximized.

In the model in which utility depends directly on ethnic-language skill, as well as on consumption (thus, mother-tongue education have both consumption and investment roles), individual consumption is maximized at the same level of s as in the standard model. In contrast, the weight on mother-tongue education that maximizes individual welfare is higher than in the standard model and generally differs across individuals. While the optimal s is identical for the poor, it is heterogenous among the wealthy. Specifically, the welfare-maximizing weight on mother-tongue education is higher for those with *greater wealth* and lower for those with higher non-language ability, and it *exceeds* the level optimal for wealth-constrained individuals. At the same level of s as in the standard model, the inter-wealth inequalities in consumption and welfare are minimized, whereas the inter-ability inequalities in these outcomes are maximized.

Because the welfare-maximizing s differs across individuals, and the properties of the s that maximizes social welfare cannot be derived analytically, the socially optimal s is computed numerically by calibrating the model to the real economy. Under a utilitarian social welfare function and assuming a relatively weak consumption motive for mother-tongue education, the socially optimal s is found to be 0.644, which is greater than the optimal value of 0.5033 for the standard model with investment motive only. Holding wealth inequality fixed, as the average level of wealth increases, the socially optimal s increases, whereas it is relatively insensitive to changes in wealth inequality. As wealth and exogenous ability become more strongly correlated, which would be the case if the influence of family economic conditions on early-childhood rearing environments becomes stronger in the real economy, the optimal s decreases. These results suggest that the optimal weight on mother tongue education is *lower* for developing countries in which wealth accumulation (for given productivity level) is lower and the association between family wealth and early-life cognitive ability tends to be stronger. When a social welfare function incorporating inequality aversion is employed, the socially optimal s is lower than under the utilitarian function, and as the degree of inequality aversion increases, the optimal s declines. In other words, as social welfare places greater weight on inequality aversion, it becomes optimal to allocate a larger proportion of educational spending to *common language* education. Moreover, when inequality aversion is high, the optimal s decreases substantially with wealth inequality. That is, when social welfare prioritizes equity, the optimal weight on mother tongue education is *lower* in societies with large wealth inequality.

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Appendix A Proof of Propositions

Proof of Proposition 2. Consider two individuals with the same ω and different levels of a, where wealth for the wealthier (poorer) is denoted by a_h (a_l). When $a_h, a_l < e^\sharp(\omega)$, $\frac{e(\omega,a_h)}{e(\omega,a_l)} = \frac{a_h}{a_l}$ from (23) and $\frac{c(\omega,a_h)}{c(\omega,a_l)} = (\frac{a_h}{a_l})^\gamma$ from (24). When $a_h, a_l \geq e^\sharp(\omega)$, $\frac{e(\omega,a_h)}{e(\omega,a_l)} = 1$ from (23) and $\frac{c(\omega,a_h)}{c(\omega,a_l)} = \frac{(\frac{1}{\gamma}-1)e^\sharp(\omega)+a_h}{(\frac{1}{\gamma}-1)e^\sharp(\omega)+a_1} = 1 + \frac{a_h-a_l}{(\frac{1}{\gamma}-1)e^\sharp(\omega)+a_1}$ from (24). Thus, $\frac{c(\omega,a_h)}{c(\omega,a_l)}$ decreases (increases) with s for $s < (>)\alpha$ from Proposition 1. When $a_l < e^\sharp(\omega) \leq a_h$, $\frac{e(\omega,a_h)}{e(\omega,a_l)} = \frac{e^\sharp(\omega)}{a_l}$ from (23). Thus, it increases (decreases) with s for $s < (>)\alpha$ from Proposition 1. From (24) and (23),

$$\frac{c(\omega,a_h)}{c(\omega,a_l)} = \frac{(\frac{1}{\gamma}-1)(\gamma\Psi)^{\frac{1}{1-\gamma}} + a_h}{\Psi(a_l)^\gamma}, \text{ where } \Psi \equiv (\alpha T_L)^\alpha \left[(1-\alpha)T_N\right]^{1-\alpha} \omega\left\{(\delta_L s)^\alpha[\delta_N(1-s)]^{1-\alpha}\right\}^\gamma.$$

Thus,

$$\begin{aligned}\frac{d\left[\frac{c(\omega,a_h)}{c(\omega,a_l)}\right]}{d\Psi} &= \frac{\Psi\frac{1}{1-\gamma}(\frac{1}{\gamma}-1)\,(\gamma\Psi)^{\frac{1}{1-\gamma}}\,(\Psi)^{-1} - \left[(\frac{1}{\gamma}-1)\,(\gamma\Psi)^{\frac{1}{1-\gamma}} + a_h\right]}{(\Psi)^2\,(a_l)^\gamma} \\ &= \frac{e^\sharp(\omega) - a_h}{(\Psi)^2\,(a_l)^\gamma} \leq 0.\end{aligned}$$

Hence, $\frac{c(\omega,a_h)}{c(\omega,a_l)}$ decreases (increases) with s for $s < (>)\alpha$. ■

Proof of Proposition 3. Consider two individuals with the same a and different levels of ω, where the ω for those with higher (lower) ability is denoted by ω_h (ω_l). When $a \leq e^\sharp(\underline{\omega})$, where $\underline{\omega}$ is the lowest ω in the ability distribution, $e(\omega,a) = a$ from (23). Thus, $\frac{e(\omega_h,a)}{e(\omega_l,a)} = 1$ and $\frac{c(\omega_h,a)}{c(\omega_l,a)} = \frac{\omega_h}{\omega_l}$ from (24). When $a \geq e^\sharp(\overline{\omega})$, where $\overline{\omega}$ is the highest ω in the ability distribution, $e(\omega,a) = e^\sharp(\omega)$ and thus $\frac{e(\omega_h,a)}{e(\omega_l,a)} = (\frac{\omega_h}{\omega_l})^{\frac{1}{1-\gamma}}$ from (23). $\frac{c(\omega_h,a)}{c(\omega_l,a)}$ increases (decreases) with s for $s < (>)\alpha$ because, from (24),

$$\begin{aligned}\frac{c(\omega_h, a)}{c(\omega_l, a)} &= \frac{(\frac{1}{\gamma}-1)e^{\sharp}(\omega_h)+a}{(\frac{1}{\gamma}-1)e^{\sharp}(\omega_l)+a} \\ &= 1+\frac{(\frac{1}{\gamma}-1)\left[\frac{e^{\sharp}(\omega_h)}{e^{\sharp}(\omega_l)}-1\right]}{(\frac{1}{\gamma}-1)+\frac{a}{e^{\sharp}(\omega_l)}} \\ &= 1+\frac{(\frac{1}{\gamma}-1)\left[(\frac{\omega_h}{\omega_l})^{\frac{1}{1-\gamma}}-1\right]}{(\frac{1}{\gamma}-1)+\frac{a}{e^{\sharp}(\omega_l)}}.\end{aligned}$$

When $a \in (e^{\sharp}(\underline{\omega}), e^{\sharp}(\overline{\omega}))$, if $a \leq e^{\sharp}(\omega_l)$,the result is the same as the case $a \leq e^{\sharp}(\underline{\omega})$, and if $a \geq e^{\sharp}(\omega_h)$, the result is the same as the case $a \geq e^{\sharp}(\overline{\omega})$. If $a \in (e^{\sharp}(\omega_l), e^{\sharp}(\omega_h))$, $\frac{e(\omega_h,a)}{e(\omega_l,a)} = \frac{a}{e^{\sharp}(\omega)}$ from (23) and thus decreases (increases) with s for $s <(>)\alpha$. From (24) and (23),

$$\frac{c(\omega_h, a)}{c(\omega_l, a)} = \frac{\Psi_0 \omega_h (a)^{\gamma}}{(\frac{1}{\gamma}-1)(\gamma\Psi_0\omega_l)^{\frac{1}{1-\gamma}}+a}, \text{ where } \Psi_0 \equiv (\alpha T_L)^{\alpha}[(1-\alpha)T_N]^{1-\alpha}\left\{[\delta_N(1-s)]^{1-\alpha}(\delta_L s)^{\alpha}\right\}^{\gamma}.$$

Thus,

$$\begin{aligned}\frac{d\frac{c(\omega_h,a)}{c(\omega_l,a)}}{d\Psi_0} &= \omega_h(a)^{\gamma}\frac{\left[(\frac{1}{\gamma}-1)(\gamma\Psi_0\omega_l)^{\frac{1}{1-\gamma}}+a\right]-\Psi_0\left[\frac{1}{1-\gamma}(\frac{1}{\gamma}-1)(\gamma\Psi_0\omega_l)^{\frac{1}{1-\gamma}}(\Psi_0)^{-1}\right]}{\left[(\frac{1}{\gamma}-1)(\gamma\Psi_0\omega_l)^{\frac{1}{1-\gamma}}+a\right]^2} \\ &= \omega_h(a)^{\gamma}\frac{a-e^{\sharp}(\omega_l)}{\left[(\frac{1}{\gamma}-1)(\gamma\Psi_0\omega_l)^{\frac{1}{1-\gamma}}+a\right]^2} > 0.\end{aligned}$$

Hence, $\frac{c(\omega,a_h)}{c(\omega,a_l)}$ increases (decreases) with s for $s <(>)\alpha$. ■

Proof that $u_L = u_N$ and $e_L = e_N$ hold for everyone in the model with both educational motives. For those with $a \geq e_L^*(\omega,a), e_N^*(\omega,a)$, $u_L = (c_L)^{\theta}(h_L)^{1-\theta} = \left\{\frac{\theta}{\gamma}[\gamma a + (1-\gamma)e_L]\right\}^{\theta}(\delta_L s e_L)^{\gamma(1-\theta)}$ from (27) and $u_N = (c_N)^{\theta}(h_N)^{1-\theta} = \left\{\frac{\theta}{\gamma}[\gamma a + (1-\gamma)e_N]\right\}^{\theta}(\delta_L s e_N)^{\gamma(1-\theta)}$ from (29). If some of them are indifferent between the sectors, $\left\{\frac{\theta}{\gamma}[\gamma a + (1-\gamma)e_L]\right\}^{\theta}(\delta_L s e_L)^{\gamma(1-\theta)} = \left\{\frac{\theta}{\gamma}[\gamma a + (1-\gamma)e_N]\right\}^{\theta}(\delta_L s e_N)^{\gamma(1-\theta)}$ must hold for these individuals. Since both sides of the equation increase with educational expenditure, the condition holds only when $e_L = e_N \Leftrightarrow e_L^*(\omega,a) = e_N^*(\omega,a)$ for them. Then, $w_L(\delta_L s)^{\gamma} = T_N[\delta_N(1-s)]^{\gamma}$ from (27) and (29). Hence, $e_L^*(\omega,a) = e_N^*(\omega,a)$ and thus $u_L = u_N$ hold for all individuals with $a \geq e_L^*(\omega,a) = e_N^*(\omega,a)$. For those with $a < e_L^*(\omega,a) = e_N^*(\omega,a)$, $e = a$. Thus, $u_L = \left[\frac{w_L}{P}\omega(\delta_L s a)^{\gamma}\right]^{\theta}(\delta_L s a)^{\gamma(1-\theta)}$ from (25) and $u_N = \left\{\frac{T_N}{P}\omega[\delta_N(1-s)a]^{\gamma}\right\}^{\theta}(\delta_L s a)^{\gamma(1-\theta)}$ from (29). Hence, $u_L = u_N$ hold for those with $a < e_L^*(\omega,a) = e_N^*(\omega,a)$ as well.

Now suppose $u_L < u_N$ for those with $a \geq e_L^*(\omega,a), e_N^*(\omega,a)$. Then, $e_L^*(\omega,a) < e_N^*(\omega,a) \Leftrightarrow w_L(\delta_L s)^{\gamma} < T_N[\delta_N(1-s)]^{\gamma}$ must hold. Thus, $u_L < u_N$ for those with $a < e_L^*(\omega,a), e_N^*(\omega,a)$ as well. This implies that $u_L > u_N$ holds for some of those with $a \in [e_L^*(\omega,a), e_N^*(\omega,a))$, because both sectors are essential in the production of final goods. Since $u_L < u_N$ for both those with $a \geq e_N^*(\omega,a)$ and those with $a < e_L^*(\omega,a)$, there must exist at least two levels of $a \in [e_L^*(\omega,a), e_N^*(\omega,a))$ satisfying $u_L = u_N$. Denote such as by a_1 and a_2, where $a_1 > a_2$. The condition $u_L = u_N$ for these values of a can be expressed as,

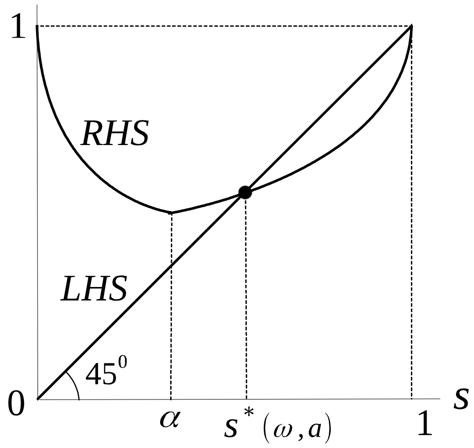

Figure A1: Determination of $s^*(\omega, a)$

$$\left\{\frac{\frac{w_L}{P}\omega(\delta_L s e_L^*(\omega,a_1))^\gamma + a_1 - e_L^*(\omega,a_1)}{(a_1)^{\frac{\gamma}{\theta}}}\right\}^\theta (\delta_L s e_L^*(\omega,a_1))^{\gamma(1-\theta)} = \left\{\frac{T_N}{P}\omega[\delta_N(1-s)]^\gamma\right\}^\theta (\delta_L s)^{\gamma(1-\theta)},$$

$$\left\{\frac{\frac{w_L}{P}\omega(\delta_L s e_L^*(\omega,a_2))^\gamma + a_2 - e_L^*(\omega,a_2)}{(a_2)^{\frac{\gamma}{\theta}}}\right\}^\theta (\delta_L s e_L^*(\omega,a_2))^{\gamma(1-\theta)} = \left\{\frac{T_N}{P}\omega[\delta_N(1-s)]^\gamma\right\}^\theta (\delta_L s)^{\gamma(1-\theta)}.$$

The left-hand sides of the equations decrease with a because

$$\begin{aligned} d\left[\frac{\frac{w_L}{P}\omega(\delta_L s e_L^*(\omega,a))^\gamma + a - e_L^*(\omega,a)}{(a)^{\frac{\gamma}{\theta}}}\right]/da &= \frac{1}{(a)^{\frac{2\gamma}{\theta}}}\left\{(a)^{\frac{\gamma}{\theta}} - \frac{\gamma}{\theta}(a)^{\frac{\gamma}{\theta}}\frac{1}{a}\left[\frac{w_L}{P}\omega(\delta_L s e_L^*(\omega,a))^\gamma + a - e_L^*(\omega,a)\right]\right\} \\ &= -\frac{1}{\theta}\frac{1}{(a)^{\frac{\gamma}{\theta}+1}}\left[\gamma\frac{w_L}{P}\omega(\delta_L s e_L^*(\omega,a))^\gamma + (\gamma-\theta)a - \gamma e_L^*(\omega,a)\right] \\ &= \frac{1-\gamma}{(a)^{\frac{\gamma}{\theta}+1}}\left[a - e_L^*(\omega,a)\right] > 0 \quad \text{(from (27))}. \end{aligned}$$

Thus, $u_L = u_N$ cannot hold for $a = a_1$ and $a = a_2$ simultaneously. Hence, the supposition that $u_L < u_N$ for those with $a \geq e_L^*(\omega,a), e_N^*(\omega,a)$ is not true. Similarly, $u_L > u_N$ for them also cannot be true. Therefore, $u_L = u_N$ holds for these individuals, which, as proved above, implies that the condition holds for every other individual as well. ■

Proof of Proposition 4. (i) is proved in the main text. (ii) and (iii) When $a < a^{**}(\omega)$, by substituting (35) into (34), $u(\omega,a) = \left\{(\alpha T_L)^\alpha[(1-\alpha)T_N]^{1-\alpha}\,\omega\right\}^\theta \left\{(\delta_L s)^{1-(1-\alpha)\theta}[\delta_N(1-s)]^{(1-\alpha)\theta}\right\}^\gamma (a)^\gamma$. The result follows from this equation.

When $a \geq a^{**}(\omega)$, from (34),

$$\frac{du(\omega,a)}{ds} = \frac{du(\omega,a)}{de^*(\omega,a)}\frac{de^*(\omega,a)}{ds} + u(\omega,a)\left[\frac{\theta}{c(\omega,a)}\frac{d\widehat{y}(\omega)}{ds}(e^*(\omega,a))^\gamma + \frac{\gamma(1-\theta)}{s}\right]$$
$$= u(\omega,a)\left[\frac{\gamma\theta}{c(\omega,a)}\widehat{y}(\omega)(e^*(\omega,a))^\gamma\frac{\alpha-s}{s(1-s)} + \frac{\gamma(1-\theta)}{s}\right] \text{ (since } \frac{du(\omega,a)}{de^*(\omega,a)} = 0)$$
$$= \frac{u(\omega,a)\gamma}{s(1-s)c(\omega,a)}\{\theta\widehat{y}(\omega)(e^*(\omega,a))^\gamma(\alpha-s) + (1-\theta)(1-s)[\widehat{y}(\omega)(e^*(\omega,a))^\gamma + a - e^*(\omega,a)]\}$$
$$= \frac{u(\omega,a)\gamma}{s(1-s)c(\omega,a)}\left(\frac{\theta}{\gamma}\{-\gamma(1-\theta)a + [\theta+\gamma(1-\theta)]e^*(\omega,a)\}(\alpha-s) + (1-\theta)(1-s)\theta\left[(\frac{1}{\gamma}-1)e^*(\omega,a)+a\right]\right) \text{ (from (31))}$$
$$= \frac{u(\omega,a)\theta}{s(1-s)c(\omega,a)}(\{-\gamma(1-\theta)a + [\theta+\gamma(1-\theta)]e^*(\omega,a)\}(\alpha-s) + (1-\theta)(1-s)[(1-\gamma)e^*(\omega,a)+\gamma a])$$
$$= \frac{u(\omega,a)\theta}{s(1-s)c(\omega,a)}((1-\alpha)\gamma(1-\theta)a + \{\alpha[\theta+\gamma(1-\theta)] + (1-\theta)(1-\gamma)\}\, e^*(\omega,a) - e^*(\omega,a)s)\,. \qquad \text{(A1)}$$

Thus,

$$\frac{du(\omega,a)}{ds} > (<) 0 \Leftrightarrow s < (>) (1-\alpha)\gamma(1-\theta)\frac{a}{e^*(\omega,a)} + \alpha[\theta+\gamma(1-\theta)] + (1-\theta)(1-\gamma), \qquad \text{(A2)}$$

where the RHS of the second equation is greater than $1-(1-\alpha)\theta$ from $\frac{a}{e^*(\omega,a)} \geq 1$. Thus, $\frac{du(\omega,a)}{ds} > 0$ at least for $s < 1-(1-\alpha)\theta$. From (31), $\frac{e^*(\omega,a)}{a}$ is a solution to

$$(\alpha T_L)^\alpha[(1-\alpha)T_N]^{1-\alpha}\,\omega\left\{(\delta_L s)^\alpha[\delta_N(1-s)]^{1-\alpha}\tfrac{e^*(\omega,a)}{a}\right\}^\gamma\left(\frac{1}{a}\right)^{1-\gamma} = \frac{1}{\gamma}\left\{-\gamma(1-\theta) + [\theta+\gamma(1-\theta)]\tfrac{e^*(\omega,a)}{a}\right\}. \qquad \text{(A3)}$$

The equation implies that $\frac{e^*(\omega,a)}{a}$ increases (decreases) with s for $s < (>) \alpha$ and as $s \to 0, 1$, $\frac{e^*(\omega,a)}{a} \to \frac{\gamma(1-\theta)}{\theta+\gamma(1-\theta)}$. Thus, the RHS of (A2) decreases (increases) with s for $s < (>) \alpha$ and as $s \to 0, 1$, $RHS \to (1-\alpha)\gamma(1-\theta)\left[1+\frac{\theta}{\gamma(1-\theta)}\right] + \alpha[\theta+\gamma(1-\theta)] + (1-\theta)(1-\gamma) = 1$. Further, as shown below, as $s \to 1$, $\frac{d(RHS\ of\ (A2))}{ds} \to \infty$, and $\frac{d(RHS\ of\ (A2))}{ds} < 1$ when s equals the RHS of (A2). Hence, as Figure A1 illustrates, there exists an $s^*(\omega,a) \in [1-(1-\alpha)\theta, 1)$ (when $a > a^{**}(\omega) \Leftrightarrow a > e^*(\omega,a)$, $s^*(\omega,a) > 1-(1-\alpha)\theta$) such that $\frac{du(\omega,a)}{ds} > (<) 0$ for $s < (>) s^*(\omega,a)$. (A3) implies that $\frac{e^*(\omega,a)}{a}$ decreases with a and increases with ω, thus $s^*(\omega,a)$ increases with a and decreases with ω.

$\frac{d(RHS\ of\ (A2))}{ds} = (1-\alpha)\gamma(1-\theta)\frac{d\left(\frac{a}{e^*(\omega,a)}\right)}{ds} \to \infty$ as $s \to 1$ is proved as follows. Since $\frac{e^*(\omega,a)}{a} = \left(\frac{a}{e^*(\omega,a)}\right)^{-1}$, $\frac{1}{a}de^*(\omega,a) = -\left(\frac{a}{e^*(\omega,a)}\right)^{-2} d\left(\frac{a}{e^*(\omega,a)}\right)$. By totally differentiating (31) (LHS is the left-hand side of the equation),

$$\left\{\gamma\frac{LHS}{e} - \frac{1}{\gamma}\,[\theta+\gamma(1-\theta)]\right\} de = -\gamma LHS\left(\frac{\alpha}{s} - \frac{1-\alpha}{1-s}\right) ds$$
$$\Leftrightarrow \frac{\gamma LHS - \frac{1}{\gamma}[\theta+\gamma(1-\theta)]e}{eLHS} de = -\gamma\frac{\alpha-s}{s(1-s)} ds$$
$$\Leftrightarrow \frac{\gamma\{-\gamma(1-\theta)a + [\theta+\gamma(1-\theta)]e\} - [\theta+\gamma(1-\theta)]e}{e\,\{-\gamma(1-\theta)a + [\theta+\gamma(1-\theta)]e\}} de = -\gamma\frac{\alpha-s}{s(1-s)} ds$$
$$\Leftrightarrow \frac{de^*(\omega,a)}{ds} = \gamma\frac{\alpha-s}{s(1-s)}e^*(\omega,a)\frac{-\gamma(1-\theta)a + [\theta+\gamma(1-\theta)]e^*(\omega,a)}{\gamma^2(1-\theta)a + (1-\gamma)[\theta+\gamma(1-\theta)]e^*(\omega,a)}. \qquad \text{(A4)}$$

Thus,

$$\frac{d\left(\frac{a}{e^*(\omega,a)}\right)}{ds} = -\left(\frac{a}{e^*(\omega,a)}\right)^2 \frac{1}{a}\frac{de^*(\omega,a)}{ds}$$
$$= \gamma \frac{s-\alpha}{s(1-s)} \frac{a}{e^*(\omega,a)} \frac{-\gamma(1-\theta)a+[\theta+\gamma(1-\theta)]e^*(\omega,a)}{\gamma^2(1-\theta)a+(1-\gamma)[\theta+\gamma(1-\theta)]e^*(\omega,a)}.$$

As $s \to 1, (1-\alpha)\gamma(1-\theta)\frac{d\left(\frac{a}{e^*(\omega,a)}\right)}{ds} \to \infty$ because

$$\frac{d\left(\frac{a}{e^*(\omega,a)}\right)}{ds} = \gamma \frac{s-\alpha}{s(1-s)} \frac{a}{e^*(\omega,a)} \frac{\gamma(\alpha T_L)^\alpha[(1-\alpha)T_N]^{1-\alpha}\,\omega\left\{(\delta_L s)^\alpha[\delta_N(1-s)]^{1-\alpha}e^*(\omega,a)\right\}^\gamma}{\gamma^2(1-\theta)a+(1-\gamma)[\theta+\gamma(1-\theta)]e^*(\omega,a)} \quad \text{(from (30))}$$
$$= \gamma \frac{s-\alpha}{s^{1-\alpha\gamma}(1-s)^{1-(1-\alpha)\gamma}} \frac{a}{e^*(\omega,a)} \frac{\gamma(\alpha T_L)^\alpha[(1-\alpha)T_N]^{1-\alpha}\,\omega\left[(\delta_L)^\alpha(\delta_N)^{1-\alpha}e^*(\omega,a)\right]^\gamma}{\gamma^2(1-\theta)a+(1-\gamma)[\theta+\gamma(1-\theta)]e^*(\omega,a)}.$$

$\frac{d(RHS\ of\ (A2))}{ds} = (1-\alpha)\gamma(1-\theta)\frac{d\left(\frac{a}{e^*(\omega,a)}\right)}{ds} < 1$ at $s = [RHS\ of\ (A2)] = (1-\alpha)\gamma(1-\theta)\frac{a}{e^*(\omega,a)} + \alpha[\theta+\gamma(1-\theta)] + (1-\theta)(1-\gamma)$ is proved as follows. At this value of s,

$$s-\alpha = (1-\alpha)\gamma(1-\theta)\frac{a}{e^*(\omega,a)} + \alpha[\theta+\gamma(1-\theta)] + (1-\theta)(1-\gamma) - \alpha$$
$$= (1-\theta)(1-\alpha)\left(\gamma\frac{a}{e^*(\omega,a)} + 1-\gamma\right).$$
$$1-s = 1-\left\{(1-\alpha)\gamma(1-\theta)\frac{a}{e^*(\omega,a)} + \alpha[\theta+\gamma(1-\theta)] + (1-\theta)(1-\gamma)\right\}$$
$$= (1-\alpha)\left\{\theta-\gamma(1-\theta)\left[\frac{a}{e^*(\omega,a)}-1\right]\right\}.$$

By using these equations,

$$(1-\alpha)\gamma(1-\theta)\frac{d\left(\frac{a}{e^*(\omega,a)}\right)}{ds} = (1-\alpha)\gamma(1-\theta)\gamma\frac{s-\alpha}{s(1-s)}\frac{a}{e^*(\omega,a)}\frac{-\gamma(1-\theta)\frac{a}{e^*(\omega,a)}+[\theta+\gamma(1-\theta)]}{\gamma^2(1-\theta)\frac{a}{e^*(\omega,a)}+(1-\gamma)[\theta+\gamma(1-\theta)]}$$
$$= (1-\alpha)\gamma(1-\theta)\gamma\frac{(1-\theta)\left[\gamma\frac{a}{e^*(\omega,a)}+(1-\gamma)\right]}{s\left\{\theta-\gamma(1-\theta)\left[\frac{a}{e^*(\omega,a)}-1\right]\right\}}\frac{a}{e^*(\omega,a)}\frac{-\gamma(1-\theta)\frac{a}{e^*(\omega,a)}+[\theta+\gamma(1-\theta)]}{\gamma^2(1-\theta)\frac{a}{e^*(\omega,a)}+(1-\gamma)[\theta+\gamma(1-\theta)]}$$
$$= \frac{(1-\alpha)\gamma(1-\theta)\frac{a}{e^*(\omega,a)}}{(1-\alpha)\gamma(1-\theta)\frac{a}{e^*(\omega,a)}+\alpha[\theta+\gamma(1-\theta)]+(1-\theta)(1-\gamma)}\frac{(1-\theta)\gamma\left[\gamma\frac{a}{e^*(\omega,a)}+(1-\gamma)\right]}{\gamma^2(1-\theta)\frac{a}{e^*(\omega,a)}+(1-\gamma)[\theta+\gamma(1-\theta)]} < 1.$$

■

Proof of Proposition 5. Consider two individuals with the same ω and different levels of a, where wealth for the wealthier (poorer) is denoted by a_h (a_l). When $a_h < a^{**}(\omega)$ and $a_l < a^{**}(\omega)$, $\frac{e(\omega,a_h)}{e(\omega,a_l)} = \frac{a_h}{a_l}$ from (30), and $\frac{c(\omega,a_h)}{c(\omega,a_l)} = \frac{u(\omega,a_h)}{u(\omega,a_l)} = (\frac{a_h}{a_l})^\gamma$ from (33) and (34).

When $a_h \geq a^{**}(\omega)$ and $a_l \geq a^{**}(\omega)$, $\frac{e(\omega,a_h)}{e(\omega,a_l)} = \frac{e^*(\omega,a_h)}{e^*(\omega,a_l)}$, which, from (31), satisfies

$$\left[\frac{e^*(\omega,a_h)}{e^*(\omega,a_l)}\right]^\gamma = \frac{-\gamma(1-\theta)a_h+[\theta+\gamma(1-\theta)]e^*(\omega,a_h)}{-\gamma(1-\theta)a_l+[\theta+\gamma(1-\theta)]e^*(\omega,a_l)}.$$

The derivative of the RHS of the equation with respect to s equals $\{-\gamma(1-\theta)a_l+[\theta+\gamma(1-\theta)]e^*(\omega,a_l)\}^{-2}$ times

$$[\theta+\gamma(1-\theta)]\Big(\{-\gamma(1-\theta)a_l+[\theta+\gamma(1-\theta)]e^*(\omega,a_l)\}\tfrac{de^*(\omega,a_h)}{ds}-\{-\gamma(1-\theta)a_h+[\theta+\gamma(1-\theta)]e^*(\omega,a_h)\}\tfrac{de^*(\omega,a_l)}{ds}\Big)$$

$$= [\theta+\gamma(1-\theta)]\,\gamma\frac{\alpha-s}{s(1-s)}\,\{-\gamma(1-\theta)a_h+[\theta+\gamma(1-\theta)]e^*(\omega,a_h)\}\{-\gamma(1-\theta)a_l+[\theta+\gamma(1-\theta)]e^*(\omega,a_l)\}$$

$$\times\left\{\frac{e^*(\omega,a_h)}{\gamma^2(1-\theta)a_h+(1-\gamma)[\theta+\gamma(1-\theta)]e^*(\omega,a_h)}-\frac{e^*(\omega,a_l)}{\gamma^2(1-\theta)a_l+(1-\gamma)[\theta+\gamma(1-\theta)]e^*(\omega,a_l)}\right\}$$

$$= [\theta+\gamma(1-\theta)]\,\gamma\frac{\alpha-s}{s(1-s)}\,\{-\gamma(1-\theta)a_h+[\theta+\gamma(1-\theta)]e^*(\omega,a_h)\}\{-\gamma(1-\theta)a_l+[\theta+\gamma(1-\theta)]e^*(\omega,a_l)\}$$

$$\times\left\{\frac{1}{\gamma^2(1-\theta)\frac{a_h}{e^*(\omega,a_h)}+(1-\gamma)[\theta+\gamma(1-\theta)]}-\frac{1}{\gamma^2(1-\theta)\frac{a_l}{e^*(\omega,a_l)}+(1-\gamma)[\theta+\gamma(1-\theta)]}\right\},$$

where the equation for $\frac{de^*(\omega,a)}{ds}$ is from (A4) in the proof of Proposition 4. Since $\frac{a}{e^*(\omega,a)}$ increases with a from the proof of the proposition, $\frac{a_h}{e^*(\omega,a_h)} > \frac{a_l}{e^*(\omega,a_l)}$ and thus the sign of the expression inside the large curly bracket is negative. Hence, $\frac{e^*(\omega,a_h)}{e^*(\omega,a_l)}$ decreases (increases) with s for $s<(>)\alpha$. From (33) and (31),

$$\frac{c(\omega,a_h)}{c(\omega,a_l)} = \frac{(\frac{1}{\gamma}-1)e^*(\omega,a_h)+a_h}{(\frac{1}{\gamma}-1)e^*(\omega,a_l)+a_l} = \frac{(\frac{1}{\gamma}-1)\frac{e^*(\omega,a_h)}{e^*(\omega,a_l)}+\frac{a_h}{e^*(\omega,a_l)}}{(\frac{1}{\gamma}-1)+\frac{a_l}{e^*(\omega,a_l)}}.$$

Thus, $d\Big[\frac{c(\omega,a_h)}{c(\omega,a_l)}\Big]/ds$ equals $\Big[(\frac{1}{\gamma}-1)+\frac{a_l}{e^*(\omega,a_l)}\Big]^{-2}$ times

$$\Big[(\tfrac{1}{\gamma}-1)+\tfrac{a_l}{e^*(\omega,a_l)}\Big]\left\{(\tfrac{1}{\gamma}-1)\frac{d\Big[\frac{e^*(\omega,a_h)}{e^*(\omega,a_l)}\Big]}{ds}-\frac{a_h}{[e^*(\omega,a_l)]^2}\frac{de^*(\omega,a_l)}{ds}\right\}+\Big\{(\tfrac{1}{\gamma}-1)\tfrac{e^*(\omega,a_h)}{e^*(\omega,a_l)}+\tfrac{a_h}{e^*(\omega,a_l)}\Big\}\frac{a_l}{[e^*(\omega,a_l)]^2}\frac{de^*(\omega,a_l)}{ds}$$

$$=(\tfrac{1}{\gamma}-1)\left\{\Big[(\tfrac{1}{\gamma}-1)+\tfrac{a_l}{e^*(\omega,a_l)}\Big]\frac{d\Big[\frac{e^*(\omega,a_h)}{e^*(\omega,a_l)}\Big]}{ds}+\Big[-\frac{a_h}{a_l}+\frac{e^*(\omega,a_h)}{e^*(\omega,a_l)}\Big]\frac{a_l}{[e^*(\omega,a_l)]^2}\frac{de^*(\omega,a_l)}{ds}\right\},$$

where $\frac{d\Big[\frac{e^*(\omega,a_h)}{e^*(\omega,a_l)}\Big]}{ds}<(>)0$ for $s<(>)\alpha$, $-\frac{a_h}{a_l}+\frac{e^*(\omega,a_h)}{e^*(\omega,a_l)}<0$, and $\frac{de^*(\omega,a_l)}{ds}>(<)0$ for $s<(>)\alpha$ from the equation defining $e^*(\omega,a)$ just after (30). Hence, $\frac{c^*(\omega,a_h)}{c^*(\omega,a_l)}$ decreases (increases) with s for $s<(>)\alpha$. Since $\frac{u(\omega,a_h)}{u(\omega,a_l)}=\Big(\frac{c^*(\omega,a_h)}{c^*(\omega,a_l)}\Big)^{\theta}\Big(\frac{e^*(\omega,a_h)}{e^*(\omega,a_l)}\Big)^{\gamma(1-\theta)}$, $\frac{u(\omega,a_h)}{u(\omega,a_l)}$ too decreases (increases) with s for $s<(>)\alpha$.

When $a_l<a^{**}(\omega)\le a_h$, $\frac{e(\omega,a_h)}{e(\omega,a_l)}=\frac{e^*(\omega,a_h)}{a_l}$, thus it increases (decreases) with s for $s<(>)\alpha$. From (33), (31), and (35)

$$\frac{c(\omega,a_h)}{c(\omega,a_l)}=\frac{\theta\Big[(\frac{1}{\gamma}-1)e^*(\omega,a_h)+a_h\Big]}{(\alpha T_L)^{\alpha}[(1-\alpha)T_N]^{1-\alpha}\,\omega\{(\delta_L s)^{\alpha}[\delta_N(1-s)]^{1-\alpha}a_l\}^{\gamma}}.$$

$d\Big[\frac{c(\omega,a_h)}{c(\omega,a_l)}\Big]/ds$ equals $\Big\{(\alpha T_L)^{\alpha}[(1-\alpha)T_N]^{1-\alpha}\,\omega\big[(\delta_L)^{\alpha}(\delta_N)^{1-\alpha}a_l\big]^{\gamma}\Big\}^{-1}\theta$ times

$$\frac{(\frac{1}{\gamma}-1)\frac{de^*(\omega,a_h)}{ds}-\Big[(\frac{1}{\gamma}-1)e^*(\omega,a_h)+a_h\Big]\gamma(\frac{\alpha}{s}-\frac{1-\alpha}{1-s})}{[s^{\alpha}(1-s)^{1-\alpha}]^{\gamma}}$$

$$=\frac{\alpha-s}{s(1-s)}\frac{(1-\gamma)e^*(\omega,a_h)\frac{-\gamma(1-\theta)a_h+[\theta+\gamma(1-\theta)]e^*(\omega,a_h)}{\gamma^2(1-\theta)a_h+(1-\gamma)[\theta+\gamma(1-\theta)]e^*(\omega,a_h)}-[(1-\gamma)e^*(\omega,a_h)+\gamma a_h]}{[s^{\alpha}(1-s)^{1-\alpha}]^{\gamma}},$$

where the numerator of the second fraction satisfies

$$\begin{aligned}
& (1-\gamma)e^*(\omega,a_h)\frac{-\gamma(1-\theta)a_h+[\theta+\gamma(1-\theta)]e^*(\omega,a_h)}{\gamma^2(1-\theta)a_h+(1-\gamma)[\theta+\gamma(1-\theta)]e^*(\omega,a_h)}-[(1-\gamma)e^*(\omega,a_h)+\gamma a_h]\\
\le\ & (1-\gamma)e^*(\omega,a_h)\frac{-\gamma(1-\theta)e^*(\omega,a_h)+[\theta+\gamma(1-\theta)]e^*(\omega,a_h)}{\gamma^2(1-\theta)e^*(\omega,a_h)+(1-\gamma)[\theta+\gamma(1-\theta)]e^*(\omega,a_h)}-e^*(\omega,a_h)\\
=\ & e^*(\omega,a_h)\left[\frac{(1-\gamma)\theta}{(1-\gamma)\theta+\gamma(1-\theta)}-1\right]\\
=\ & -e^*(\omega,a_h)\frac{\gamma(1-\theta)}{(1-\gamma)\theta+\gamma(1-\theta)}<0.
\end{aligned}$$

Hence, $\frac{c(\omega,a_h)}{c(\omega,a_l)}$ decreases (increases) with s for $s<(>)\alpha$.

$$\frac{u(\omega,a_h)}{u(\omega,a_l)}=\left(\frac{\theta\left[(\frac{1}{\gamma}-1)e^*(\omega,a_h)+a_h\right]}{(\alpha T_L)^{\alpha}[(1-\alpha)T_N]^{1-\alpha}\,\omega\{(\delta_L s)^{\alpha}[\delta_N(1-s)]^{1-\alpha}a_l\}^{\gamma}}\right)^{\theta}\left(\frac{e^*(\omega,a_h)}{a_l}\right)^{\gamma(1-\theta)}$$

$$=\left\{\frac{\theta}{(\alpha T_L)^{\alpha}[(1-\alpha)T_N]^{1-\alpha}\,\omega[(\delta_L)^{\alpha}(\delta_N)^{1-\alpha}a_l]^{\gamma}}\right\}^{\theta}\frac{1}{(a_l)^{\gamma(1-\theta)}}\frac{\left[(\frac{1}{\gamma}-1)e^*(\omega,a_h)+a_h\right]^{\theta}[e^*(\omega,a_h)]^{\gamma(1-\theta)}}{[s^{\alpha}(1-s)^{1-\alpha}]^{\gamma\theta}}.$$

$d\left[\frac{u(\omega,a_h)}{u(\omega,a_l)}\right]/ds$ equals $\frac{u(\omega,a_h)}{u(\omega,a_l)}$ times

$$\left[\frac{\theta(\frac{1}{\gamma}-1)}{(\frac{1}{\gamma}-1)e^*(\omega,a_h)+a_h}+\frac{\gamma(1-\theta)}{e^*(\omega,a_h)}\right]\frac{de^*(\omega,a_h)}{ds}-\gamma\theta\left(\frac{\alpha}{s}-\frac{1-\alpha}{1-s}\right)$$

$$=\gamma\frac{\alpha-s}{s(1-s)}\left\{\left[\frac{\theta(\frac{1}{\gamma}-1)}{(\frac{1}{\gamma}-1)e^*(\omega,a_h)+a_h}+\frac{\gamma(1-\theta)}{e^*(\omega,a_h)}\right]e^*(\omega,a_h)\frac{-\gamma(1-\theta)a_h+[\theta+\gamma(1-\theta)]e^*(\omega,a_h)}{\gamma^2(1-\theta)a_h+(1-\gamma)[\theta+\gamma(1-\theta)]e^*(\omega,a_h)}-\theta\right\},$$

where the expression inside the large curly bracket satisfies

$$\begin{aligned}
& \left[\frac{\theta(\frac{1}{\gamma}-1)}{(\frac{1}{\gamma}-1)e^*(\omega,a_h)+a_h}+\frac{\gamma(1-\theta)}{e^*(\omega,a_h)}\right]e^*(\omega,a_h)\frac{-\gamma(1-\theta)a_h+[\theta+\gamma(1-\theta)]e^*(\omega,a_h)}{\gamma^2(1-\theta)a_h+(1-\gamma)[\theta+\gamma(1-\theta)]e^*(\omega,a_h)}-\theta\\
\le\ & [(1-\gamma)\theta+\gamma(1-\theta)]\,\frac{\theta}{(1-\gamma)\theta+\gamma(1-\theta)}-\theta=0.
\end{aligned}$$

Hence, $\frac{u(\omega,a_h)}{u(\omega,a_l)}$ decreases (increases) with s for $s<(>)\alpha$. ■

Proof of Proposition 6. Consider two individuals with the same a and different levels of ω, where the ω for one with higher (lower) ability is denoted by ω_h (ω_l). When $a\le a^{**}(\underline{\omega})$, $e(\omega,a)=a$ for any ω from (30). Thus, $\frac{e(\omega_h,a)}{e(\omega_l,a)}=1$, $\frac{c(\omega_h,a)}{c(\omega_l,a)}=\frac{\omega_h}{\omega_l}$ from (33), and $\frac{u_2(\omega_h,a)}{u_2(\omega_l,a)}=(\frac{\omega_h}{\omega_l})^{\theta}$ from (34).

When $a\ge a^{**}(\overline{\omega})$, $e(\omega,a)=e^*(\omega,a)$ for any ω. Thus, $\frac{e(\omega_h,a)}{e(\omega_l,a)}=\frac{e^*(\omega_h,a)}{e^*(\omega_l,a)}$, which, from (30), satisfies for $s>0$

$$\begin{aligned}
\left[\frac{e^*(\omega_h,a)}{e^*(\omega_l,a)}\right]^{\gamma}\frac{\omega_h}{\omega_l} &= \frac{-\gamma(1-\theta)a+[\theta+\gamma(1-\theta)]e^*(\omega_h,a)}{-\gamma(1-\theta)a+[\theta+\gamma(1-\theta)]e^*(\omega_l,a)}\\
&= 1+\frac{[\theta+\gamma(1-\theta)]\left[\frac{e^*(\omega_h,a)}{e^*(\omega_l,a)}-1\right]}{-\gamma(1-\theta)\frac{a}{e^*(\omega_l,a)}+[\theta+\gamma(1-\theta)]}.
\end{aligned}$$

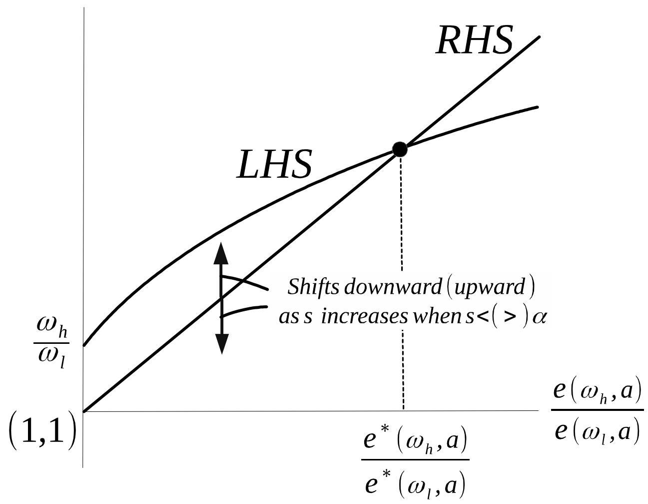

Figure A2: Determination of $\frac{e^*(\omega_h,a)}{e^*(\omega_l,a)}$

Because $e^*(\omega_l,a)$ increases (decreases) with s for $s<(>)\alpha$, so does $\frac{e^*(\omega_h,a)}{e^*(\omega_l,a)}$ from this equation (see Figure A2). From (33),

$$\frac{c(\omega_h,a)}{c(\omega_l,a)} = \frac{(\frac{1}{\gamma}-1)e^*(\omega_h,a)+a}{(\frac{1}{\gamma}-1)e^*(\omega_l,a)+a}$$
$$= 1+\frac{(1-\gamma)\left[\frac{e^*(\omega_h,a)}{e^*(\omega_l,a)}-1\right]}{(1-\gamma)+\gamma\frac{a}{e^*(\omega_l,a)}}.$$

Thus, from the above results on $e^*(\omega_l,a)$ and $\frac{e^*(\omega_h,a)}{e^*(\omega_l,a)}$, $\frac{c(\omega_h,a)}{c(\omega_l,a)}$ increases (decreases) with s for $s<(>)\alpha$. Hence, $\frac{u(\omega_h,a)}{u(\omega_l,a)} = \left(\frac{c(\omega_h,a)}{c(\omega_l,a)}\right)^{\theta}\left(\frac{e^*(\omega_h,a)}{e^*(\omega_l,a)}\right)^{\gamma(1-\theta)}$ also increases (decreases) with s for $s<(>)\alpha$.

When $a\in(a^{**}(\underline{\omega}),a^{**}(\overline{\omega}))$, the result for $a\le a^{**}(\omega_l)$ is the same as in the corresponding case above, as is the result for $a\ge a^{**}(\omega_h)$. When $a\in(a^{**}(\omega_l),a^{**}(\omega_h))$, $e(\omega_h,a)=a$ and $e(\omega_l,a)=e^*(\omega_l,a)$. Thus, $\frac{e(\omega_h,a)}{e(\omega_l,a)}=\frac{a}{e^*(\omega_l,a)}$, which decreases (increases) with s for $s<(>)\alpha$. From (33), (31), and (35),

$$\frac{c(\omega_h,a)}{c(\omega_l,a)} = \frac{(\alpha T_L)^{\alpha}[(1-\alpha)T_N]^{1-\alpha}\,\omega_h\left\{(\delta_L s)^{\alpha}[\delta_N(1-s)]^{1-\alpha}a\right\}^{\gamma}}{\theta\left[(\frac{1}{\gamma}-1)e^*(\omega_l,a)+a\right]}$$
$$= \frac{(\alpha T_L)^{\alpha}[(1-\alpha)T_N]^{1-\alpha}\,\omega_h\left[(\delta_L)^{\alpha}(\delta_N)^{1-\alpha}a\right]^{\gamma}}{\theta}\frac{\left[s^{\alpha}(1-s)^{1-\alpha}\right]^{\gamma}}{(\frac{1}{\gamma}-1)e^*(\omega_l,a)+a}.$$

$d\left(\frac{c(\omega_h,a)}{c(\omega_l,a)}\right)/ds$ equals $\frac{c(\omega_h,a)}{c(\omega_l,a)}$ times ((A4) in the proof of Proposition 4 is substituted for $\frac{de^*(\omega_l,a)}{ds}$)

$$\gamma\left(\frac{\alpha}{s}-\frac{1-\alpha}{1-s}\right)-\frac{(\frac{1}{\gamma}-1)}{(\frac{1}{\gamma}-1)e^*(\omega_l,a)+a}\frac{de^*(\omega_l,a)}{ds}$$
$$=\gamma\frac{\alpha-s}{s(1-s)}\left\{1-\frac{(\frac{1}{\gamma}-1)e^*(\omega_l,a)}{(\frac{1}{\gamma}-1)e^*(\omega_l,a)+a}\frac{-\gamma(1-\theta)a+[\theta+\gamma(1-\theta)]e^*(\omega_l,a)}{\gamma^2(1-\theta)a+(1-\gamma)[\theta+\gamma(1-\theta)]e^*(\omega_l,a)}\right\},$$

where

$$
\begin{aligned}
&1-\frac{(\frac{1}{\gamma}-1)e^*(\omega_l,a)}{(\frac{1}{\gamma}-1)e^*(\omega_l,a)+a}\frac{-\gamma(1-\theta)a+[\theta+\gamma(1-\theta)]e^*(\omega_l,a)}{\gamma^2(1-\theta)a+(1-\gamma)[\theta+\gamma(1-\theta)]e^*(\omega_l,a)}\\
&>1-\frac{(\frac{1}{\gamma}-1)e^*(\omega_l,a)}{(\frac{1}{\gamma}-1)e^*(\omega_l,a)+e^*(\omega_l,a)}\frac{-\gamma(1-\theta)e^*(\omega_l,a)+[\theta+\gamma(1-\theta)]e^*(\omega_l,a)}{\gamma^2(1-\theta)e^*(\omega_l,a)+(1-\gamma)[\theta+\gamma(1-\theta)]e^*(\omega_l,a)}\\
&=1-\frac{(1-\gamma)\theta}{\gamma(1-\theta)+(1-\gamma)\theta}>0.
\end{aligned}
$$

Hence $\frac{c(\omega_h,a)}{c(\omega_l,a)}$ increases (decreases) with s for $s<(>)\alpha$. From (34),

$$
\frac{u(\omega_h,a)}{u(\omega_l,a)}=\left\{\frac{(\alpha T_L)^\alpha[(1-\alpha)T_N]^{1-\alpha}\,\omega_h\big[(\delta_L)^\alpha(\delta_N)^{1-\alpha}a\big]^\gamma}{\theta}\frac{\big[s^\alpha(1-s)^{1-\alpha}\big]^\gamma}{(\frac{1}{\gamma}-1)e^*(\omega_l,a)+a}\right\}^\theta\left(\frac{a}{e^*(\omega_l,a)}\right)^{\gamma(1-\theta)}.
$$

$d\left(\frac{u(\omega_h,a)}{u(\omega_l,a)}\right)/ds$ equals $\frac{u(\omega_h,a)}{u(\omega_l,a)}$ times

$$
\begin{aligned}
&\theta\gamma\left(\frac{\alpha}{s}-\frac{1-\alpha}{1-s}\right)-\left[\frac{\theta(\frac{1}{\gamma}-1)}{(\frac{1}{\gamma}-1)e^*(\omega_l,a)+a}+\frac{\gamma(1-\theta)}{e^*(\omega_l,a)}\right]\frac{de^*(\omega_l,a)}{ds}\\
&=\gamma\frac{\alpha-s}{s(1-s)}\left\{\theta-\left[\frac{\theta(\frac{1}{\gamma}-1)}{(\frac{1}{\gamma}-1)e^*(\omega_l,a)+a}+\frac{\gamma(1-\theta)}{e^*(\omega_l,a)}\right]e^*(\omega_l,a)\frac{-\gamma(1-\theta)a+[\theta+\gamma(1-\theta)]e^*(\omega_l,a)}{\gamma^2(1-\theta)a+(1-\gamma)[\theta+\gamma(1-\theta)]e^*(\omega_l,a)}\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
&\theta-\left[\frac{\theta(\frac{1}{\gamma}-1)}{(\frac{1}{\gamma}-1)e^*(\omega_l,a)+a}+\frac{\gamma(1-\theta)}{e^*(\omega_l,a)}\right]e^*(\omega_l,a)\frac{-\gamma(1-\theta)a+[\theta+\gamma(1-\theta)]e^*(\omega_l,a)}{\gamma^2(1-\theta)a+(1-\gamma)[\theta+\gamma(1-\theta)]e^*(\omega_l,a)}\\
&>\theta-[\theta(1-\gamma)+\gamma(1-\theta)]\,\frac{\theta}{\gamma(1-\theta)+(1-\gamma)\theta}=0.
\end{aligned}
$$

Hence, $\frac{u(\omega_h,a)}{u(\omega_l,a)}$ increases (decreases) with s for $s<(>)\alpha$. ■