Appendix C Proofs of Lemmas and Propositions

Proof of Proposition 1. (i) The statement is true iff the RHS of (mU) is greater than that of (mS). From these equations (multiplied by $\gamma\delta$),

$$\begin{aligned}&\gamma\delta\widetilde{S_2}+(\beta-\gamma)N_1(1-H_2)(1-\tau)(w_s-w_u)+[\rho\chi(1-\chi)+\beta\omega_q](\overline{q}-\overline{q}_2)[(\overline{q}_2-q_{2U}^i)+(\overline{q}-q_{2U}^i)]\\ >\ &\gamma\delta\widetilde{S_2}-(\beta+\gamma)N_1(1-H_2)(1-\tau)(w_s-w_u)+[\rho\chi(1-\chi)+\beta\omega_q](\overline{q}-\overline{q}_2)[-(q_{2S}^i-\overline{q}_2)+(\overline{q}-q_{2S}^i)]\\ \Leftrightarrow\ &2\beta N_1(1-H_2)(1-\tau)(w_s-w_u)+[\rho\chi(1-\chi)+\beta\omega_q](\overline{q}-\overline{q}_2)2(q_{2S}^i-q_{2U}^i)>0. \qquad \text{(C1)}\end{aligned}$$

Given the initial condition $q_{2S}^i=q_{2U}^i=0$, (C1) is true for the initial period and thus $q_{2S}^i\geq q_{2U}^i$ for the second period from (13), (14), and the initial condition $q_{1S}^i=1$. Then, (C1) holds for the second period and $q_{2S}^i\geq q_{2U}^i$ for the third period. Continuing in this way, one can prove (C1) for all periods.

(ii) (a) The statement is true iff the RHS of (M) is greater than that of (mS). From these equations (multiplied by $\gamma\delta$),

$$\begin{aligned}&\gamma\delta\widetilde{S_1}+(\beta+\gamma)(1-N_1)(1-H_2)(1-\tau)(w_s-w_u)+[\rho\chi(1-\chi)+\beta\omega_q](\overline{q}_1-\overline{q})^2\\ &>\gamma\delta\widetilde{S_2}-(\beta+\gamma)N_1(1-H_2)(1-\tau)(w_s-w_u)+[\rho\chi(1-\chi)+\beta\omega_q](\overline{q}-\overline{q}_2)[-(q_{2S}^i-\overline{q}_2)+(\overline{q}-q_{2S}^i)]\\ &\Leftrightarrow\gamma\delta\left(\widetilde{S_1}-\widetilde{S_2}\right)+(\beta+\gamma)(1-H_2)(1-\tau)(w_s-w_u)>-[\rho\chi(1-\chi)+\beta\omega_q]\left[(\overline{q}_1-\overline{q})^2-(\overline{q}-\overline{q}_2)[-(q_{2S}^i-\overline{q}_2)+(\overline{q}-q_{2S}^i)]\right], \qquad \text{(C2)}\end{aligned}$$

The RHS of (C2) equals $[\rho\chi(1-\chi)+\beta\omega_q]$ times

$$\begin{aligned}&-(\overline{q}_1-\overline{q}_2)\left\{(1-N_1)^2(\overline{q}_1-\overline{q}_2)-N_1\left[-2(q_{2S}^i-\overline{q}_2)+N_1(\overline{q}_1-\overline{q}_2)\right]\right\}\\ &=(\overline{q}_1-\overline{q}_2)\left[(2N_1-1)(\overline{q}_1-\overline{q}_2)-2N_1(q_{2S}^i-\overline{q}_2)\right]\leq 2N_1-1<(N_1)^2, \qquad \text{(C3)}\end{aligned}$$

where $q_{2S}^i\geq\overline{q}_2$ is used to prove the second last inequality. Hence, (C2) is true under Assumption 2.

(b) The majority are less likely to have a national identity than the minority's unskilled iff the RHS of (M) is greater than that of (mU). From these equations,

$$\begin{aligned}&\gamma\delta\widetilde{S_1}+(\beta+\gamma)(1-N_1)(1-H_2)(1-\tau)(w_s-w_u)+[\rho\chi(1-\chi)+\beta\omega_q](\overline{q}_1-\overline{q})^2\\ &>\gamma\delta\widetilde{S_2}+(\beta-\gamma)N_1(1-H_2)(1-\tau)(w_s-w_u)+[\rho\chi(1-\chi)+\beta\omega_q](\overline{q}-\overline{q}_2)[(\overline{q}_2-q_{2U}^i)+(\overline{q}-q_{2U}^i)]\\ &\Leftrightarrow\gamma\delta\left(\widetilde{S_1}-\widetilde{S_2}\right)+[\gamma-\beta(2N_1-1)](1-H_2)(1-\tau)(w_s-w_u)>-[\rho\chi(1-\chi)+\beta\omega_q]\left\{(\overline{q}_1-\overline{q})^2-(\overline{q}-\overline{q}_2)[(\overline{q}_2-q_{2U}^i)+(\overline{q}-q_{2U}^i)]\right\}, \qquad \text{(C4)}\end{aligned}$$

where the RHS equals $[\rho\chi(1-\chi)+\beta\omega_q]$ times

$$\begin{aligned}&(\overline{q}_1-\overline{q}_2)\left[(2N_1-1)(\overline{q}_1-\overline{q}_2)+2N_1(\overline{q}_2-q_{2U}^i)\right]\\ \leq\ &(1-\overline{q}_2)[(2N_1-1)(1-\overline{q}_2)+2N_1\overline{q}_2]=(1-\overline{q}_2)[(2N_1-1)+\overline{q}_2]\leq(N_1)^2, \qquad \text{(C5)}\end{aligned}$$

where the last inequality holds because the derivative of the second last expression with respect to $\overline{q}_2$ equals $-[(2N_1-1)+\overline{q}_2]+(1-\overline{q}_2)$ and thus the expression is highest at $\overline{q}_2=1-N_1$.

Hence, when $\gamma-\beta(2N_1-1)\geq 0\Leftrightarrow N_1\leq\frac{\beta+\gamma}{2\beta}$, (C4) is true under Assumption 2. When $N_1>\frac{\beta+\gamma}{2\beta}$, (C4) holds for large H_2, but it may not hold for small H_2. ■

Proof of Proposition 3.

To prove the results, we use Figures 1–3 that show the positions of (M) and (mU) (with $p_{1S}=0$ and with $p_{1S}=1$) in the initial period and in the steady state, and of (mS) in the initial period on

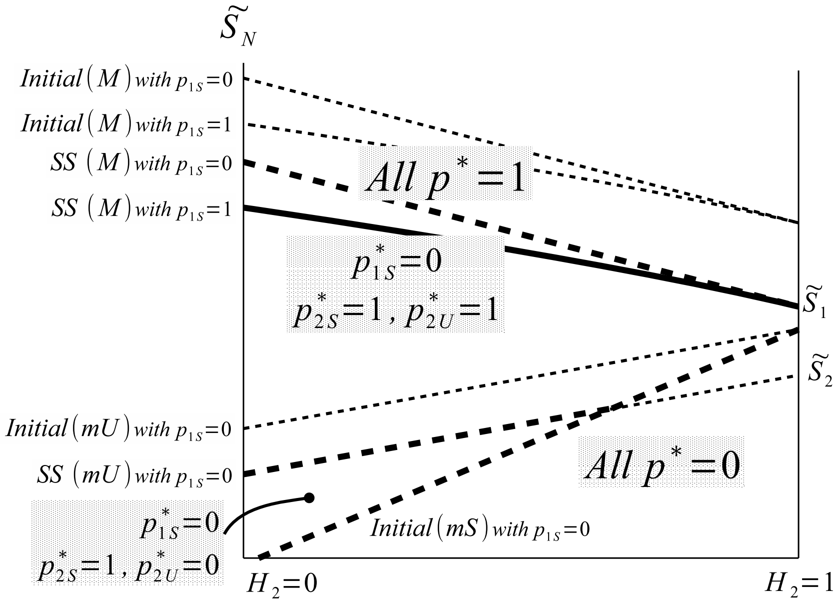

Figure C1: Relationship between initial $(H_2, \widetilde{S_N})$ and steady-state identity when $\beta \leq \gamma$ and H_2 is constant

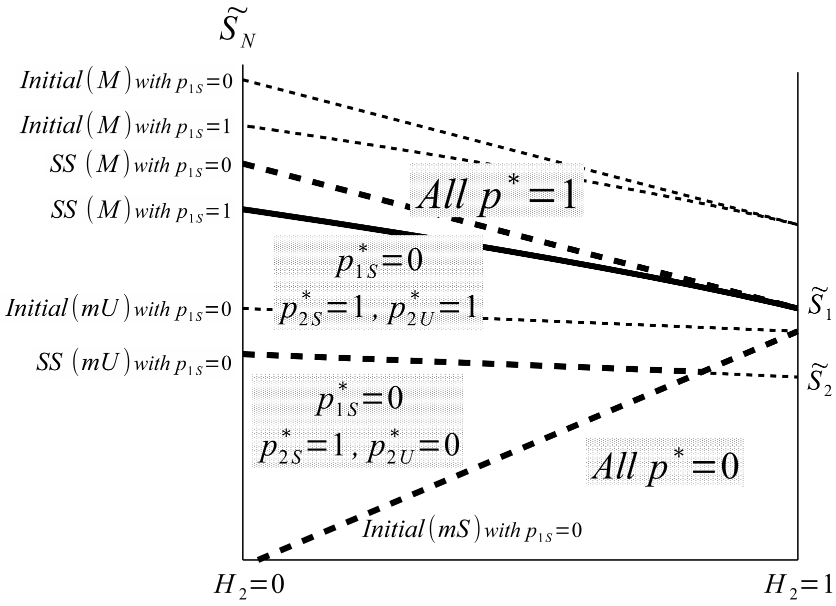

Figure C2: Relationship between initial $(H_2, \widetilde{S_N})$ and steady-state identity when $\beta > \gamma$, $N_1 \leq \frac{\beta+\gamma}{2\beta}$, and H_2 is constant

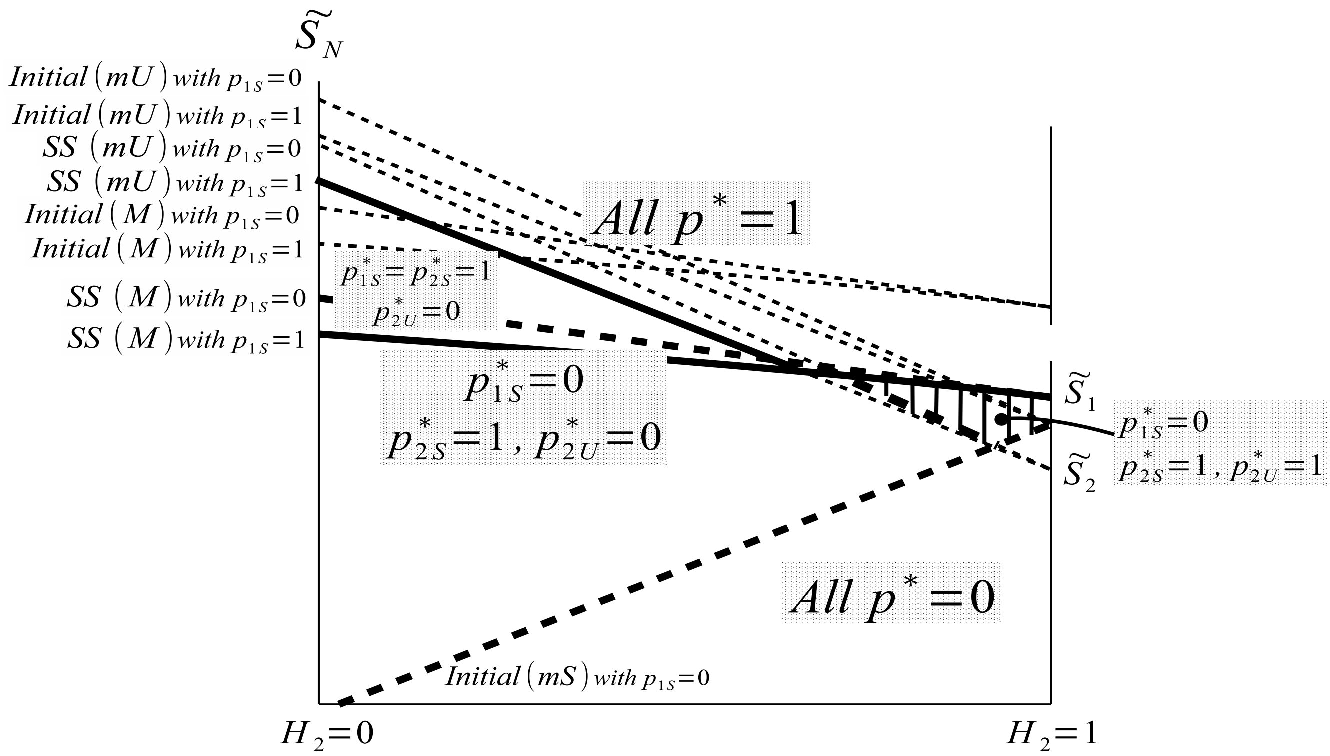

Figure C3: Relationship between initial $(H_2, \widetilde{S_N})$ and steady-state identity when $\beta > \gamma$, $N_1 > \frac{\beta+\gamma}{2\beta}$, and H_2 is constant

the $(H_2, \widetilde{S_N})$ plane. The steady-state dividing lines are for when $p^*_{1S} = p^*_{2S} = p^*_{2U} = 0$ does not hold, in which case $q^*_{1S} = q^*_{2S} = q^*_{2U}$ from Lemma A3.

Relative positions of the dividing lines for given period and p_{1S} are based on the following theoretical results. (mU) is located above (mS) from Proposition 1 (i), and unless $\beta > \gamma$ and $N_1 > \frac{\beta+\gamma}{2\beta}$, (M) is above (mU) from (ii)(b) of the proposition. When $\beta > \gamma$ and $N_1 > \frac{\beta+\gamma}{2\beta}$, (mU) may be above (M) for small H_2 from Proposition 1 (ii)(b), and Figure 3 illustrates such a case. (M) and (mU) with $p_{1S} = 0$ are located above (M) and (mU) with $p_{1S} = 1$ respectively, because $\tau > (=) 0$ when $p_{1S} = 1 (= 0)$. The dividing lines in the steady state are below the corresponding ones in the initial period from Lemma A2 (ii)−(iv). At $H_2 = 1$, the vertical level of initial (M) on the $(H_2, \widetilde{S_N})$ plane equals $\widetilde{S_1} + \frac{1}{\gamma\delta}[\rho\chi(1-\chi) + \beta\omega_q](1-N_1)^2$, that of initial (mU) and (mS) equals $\widetilde{S_2} + \frac{1}{\gamma\delta}[\rho\chi(1-\chi) + \beta\omega_q](N_1)^2$, while the vertical level of steady-state (M) is $\widetilde{S_1}$ and that of steady-state (mU) and (mS) is $\widetilde{S_2}$. Relative levels of these values are from Assumption 2.

(i) Given H_2, when $\widetilde{S_N}$ is very high so that $p_{1S} = p_{2S} = p_{2U} = 1$ in the initial period (i.e., $(H_2, \widetilde{S_N})$ is on or above initial (M) with $p_{1S} = 1$ and when $\beta > \gamma$ and $N_1 > \frac{\beta+\gamma}{2\beta}$ [Figure 3], also on or above initial (mU) with $p_{1S} = 1$ on the $(H_2, \widetilde{S_N})$ plane), $p_{1S} = p_{2S} = p_{2U} = 1$ holds in subsequent periods because (M) and (mU) shift downward over time on the $(H_2, \widetilde{S_N})$ plane from Lemma A2 (ii).

When $p_{1S} = 0$, $p_{2S} = p_{2U} = 1$ initially (i.e., $(H_2, \widetilde{S_N})$ is on or above initial (mU) with $p_{1S} = 0$ and below initial (M) with $p_{1S} = 0$) and $\widetilde{S_N}$ is relatively high for given H_2 (i.e., $(H_2, \widetilde{S_N})$ is on or above steady-state (M) with $p_{1S} = 1$), society shifts to $p_{1S} = p_{2S} = p_{2U} = 1$ eventually (i.e., $(H_2, \widetilde{S_N})$ is on or above (M) with $p_{1S} = 1$, when $\beta > \gamma$ and $N_1 > \frac{\beta+\gamma}{2\beta}$ [Figure 3], also on or above (mU) with $p_{1S} = 1$) and stays in this state, because (M) and (mU) shift downward over time from Lemma A2 (iii).

When $p_{1S}=0$, $p_{2S}=1, p_{2U}=0$ initially (i.e., $(H_2, \widetilde{S_N})$ is on or above initial (mS) with $p_{1S}=0$ and below initial (M) and (mU) with $p_{1S}=0$) and $\widetilde{S_N}$ is relatively high (i.e., $(H_2, \widetilde{S_N})$ is on or above steady-state (M) and (mU) with $p_{1S}=1$), occuring only when $\beta>\gamma$ and $N_1>\frac{\beta+\gamma}{2\beta}$, society shifts to $p_{1S}=p_{2S}=p_{2U}=1$ eventually (typically, after shifting to $p_{1S}=0$, $p_{2S}=p_{2U}=1$) since (M) shifts downward over time and (mU) and (mS) shift downward in the long term from Lemma A2 (iv).

When $p_{1S}=1$, $p_{2S}=1$, $p_{2U}=0$ initially (i.e., $(H_2, \widetilde{S_N})$ is on or above initial (M) with $p_{1S}=1$ and below initial (mU) with $p_{1S}=1$), which may occur only when $\beta>\gamma$ and $N_1>\frac{\beta+\gamma}{2\beta}$ (Figure 3), and $\widetilde{S_N}$ is relatively high (i.e., $(H_2, \widetilde{S_N})$ is on or above steady-state (mU) with $p_{1S}=1$), society shifts to $p_{1S}=p_{2S}=p_{2U}=1$ because (M) and (mU) shift downward over time from Lemma A2 (iii).

To summarize, $p^*_{1S}=p^*_{2S}=p^*_{2U}=1$ when $(H_2, \widetilde{S_N})$ is located on or above steady-state (M) with $p_{1S}=1$, and when $\beta>\gamma$ and $N_1>\frac{\beta+\gamma}{2\beta}$ (Figure 3), also on or above steady-state (mU) with $p_{1S}=1$.

When $p_{1S}=1$, $p_{2S}=1$, $p_{2U}=0$ initially (i.e., $(H_2, \widetilde{S_N})$ is on or above initial (M) with $p_{1S}=1$ and below initial (mU) with $p_{1S}=1$), occuring only when $\beta>\gamma$ and $N_1>\frac{\beta+\gamma}{2\beta}$ (Figure 3), and $\widetilde{S_N}$ is not very high (i.e., $(H_2, \widetilde{S_N})$ is below steady-state (mU) with $p_{1S}=1$), society stays in $p_{1S}=1$, $p_{2S}=1$, $p_{2U}=0$ because (M) and (mU) shift downward over time from Lemma A2 (iii).

When $\beta>\gamma$, $N_1>\frac{\beta+\gamma}{2\beta}$ (Figure 3), $p_{1S}=0$, $p_{2S}=1$, $p_{2U}=0$ initially (i.e., $(H_2, \widetilde{S_N})$ is on or above initial (mS) with $p_{1S}=0$ and below initial (M) and (mU) with $p_{1S}=0$), and $\widetilde{S_N}$ is relatively, but not very, high (i.e., $(H_2, \widetilde{S_N})$ is on or above steady-state (M) with $p_{1S}=1$ and below steady-state (mU) with $p_{1S}=1$), society shifts to $p_{1S}=1$, $p_{2S}=1$, $p_{2U}=0$ because (M) and (mU) shift downward in the long term from Lemma A2 (iv).

To summarize, $p^*_{1S}=1, p^*_{2S}=1, p^*_{2U}=0$ when $\beta>\gamma$, $N_1>\frac{\beta+\gamma}{2\beta}$ (Figure 3), and $(H_2, \widetilde{S_N})$ is located on or above steady-state (M) with $p_{1S}=1$ and below steady-state (mU) with $p_{1S}=1$.

The result on the steady-state cultural composition is from Lemma A3 (i) and (ii). The negative relation between $\widetilde{S_N}$ and $\overline{q}^{\#}$ or $\overline{q}^{\dagger}$ holds because, as $\widetilde{S_N}$ is lower, the period during which cultural assimilation proceeds, i.e., $p_{1S}=0$, $p_{2S}=1, p_{2U}=0$ or 1, is longer. The proportion of the minority element in the integrated culture is highest when $p_{1S}=p_{2S}=p_{2U}=1$ always.

(ii) Given H_2, when $\widetilde{S_N}$ is low enough that $p_{1S}=p_{2S}=p_{2U}=0$ initially (i.e., $(H_2, \widetilde{S_N})$ is below initial (mS) with $p_{1S}=0$), $p_{1S}=p_{2S}=p_{2U}=0$ holds in subsequent periods, because $q_{2S}=q_{2U}=0$ continues to hold and thus (mS) does not shift from Lemma A2 (v) and (13).

Society does not shift to $p_{1S}=p_{2S}=p_{2U}=0$ from other combinations of p_{1S},p_{2S}, and p_{2U} because (mS) with $p_{1S}=0$ in the initial period is at a higher position than or the same position as those in subsequent periods on the $(H_2, \widetilde{S_N})$ plane from Lemma A2 (i). $q^*_{1S}=1$ and $q^*_{2S}=q^*_{2U}=0$ is from Lemma A3 (iv) and the result that $p_{1S}=p_{2S}=p_{2U}=0$ always holds.

(iii) When $p_{1S}=0$, $p_{2S}=p_{2U}=1$ initially (i.e., $(H_2, \widetilde{S_N})$ is on or above initial (mU) with $p_{1S}=0$ and below initial (M) with $p_{1S}=0$) and $\widetilde{S_N}$ is relatively low for given H_2 (i.e., $(H_2, \widetilde{S_N})$ is below steady-state (M) with $p_{1S}=0$), $p_{1S}=0$, $p_{2S}=p_{2U}=1$ holds in subsequent periods because (M) and (mU) shift downward over time on the $(H_2, \widetilde{S_N})$ plane from Lemma A2 (iii).

When $p_{1S}=0$, $p_{2S}=1, p_{2U}=0$ initially (i.e., $(H_2, \widetilde{S_N})$ is on or above initial (mS) with $p_{1S}=0$ and below initial (mU) with $p_{1S}=0$; when $\beta>\gamma$ and $N_1>\frac{\beta+\gamma}{2\beta}$ [Figure 3], also below initial (M) with $p_{1S}=0$) and $\widetilde{S_N}$ is relatively high (i.e., $(H_2, \widetilde{S_N})$ is on or above steady-state (mU) with $p_{1S}=0$; when $\beta>\gamma$ and $N_1>\frac{\beta+\gamma}{2\beta}$, also below steady-state (M) with $p_{1S}=0$), society shifts to $p_{1S}=0$, $p_{2S}=p_{2U}=1$ eventually and stays in this state, because (M) shifts downward over time, so does (mU) in the long run, from Lemma A2 (iv), and (mS)s in subsequent periods are not located above the one in the initial period from Lemma A2 (i).

When $p_{1S}=0$, $p_{2S}=1, p_{2U}=0$ initially and $\widetilde{S_N}$ is relatively low (i.e., $(H_2, \widetilde{S_N})$ is below steady-state (mU) with $p_{1S}=0$; when $\beta>\gamma$ and $N_1>\frac{\beta+\gamma}{2\beta}$, also below steady-state (M) with $p_{1S}=0$), society stays in this state for the same reasons as the previous case.

To summarize, $p^*_{1S}=0, p^*_{2S}=p^*_{2U}=1$ when $(H_2,\widetilde{S_N})$ is on or above initial (mS) with $p_{1S}=0$, as well as steady-state (mU) with $p_{1S}=0$, and below steady-state (M) with $p_{1S}=0$; $p^*_{1S}=0$, $p^*_{2S}=1, p^*_{2U}=0$ when $(H_2,\widetilde{S_N})$ is on or above initial (mS) with $p_{1S}=0$ and below steady-state (mU) with $p_{1S}=0$, when $\beta>\gamma$ and $N_1>\frac{\beta+\gamma}{2\beta}$ (Figure 3), also below steady-state (M) with $p_{1S}=0$.

$q^*_{1S}=q^*_{2S}=q^*_{2U}=1$ is from Lemma A3 (iii) and the result that only the society starting with $p_{1S}=0$ and never satisfying $p_{1S}=1$ ends up with $p^*_{1S}=0$, $p^*_{2S}=1, p^*_{2U}=0$ or 1.

(iv) The result on $\widetilde{S_1}$ ($\widetilde{S_2}$) holds because as $\widetilde{S_1}$ is lower ($\widetilde{S_2}$ is higher), (M) [(mS)] is located at a lower (higher) position on the $(H_2,\widetilde{S_N})$ plane. The result on ω_q holds because as ω_q is higher, (mS) in the initial period, whose last term equals $[\rho\chi(1-\chi)+\beta\omega_q](N_1)^2$, is located at a higher position. The level of ω_q does not affect the likelihood of universal national identity because steady-state (M) does not depend on ω_q when $q^*_{1S}=q^*_{2S}=q^*_{2U}$. The result on H_2 is from the figures. The result on $\overline{q}^{\#}$ or $\overline{q}^{\dagger}$ can be proved similarly to the corresponding result in (i). ■

Proof of Lemma 2. (i) The claim is proved if the difference in utility between when a group 1 individual takes education and when she does not is positive at $H_1=1$. To compute the utility when not taking education, the value of p_{1U} needs to be specified. It is reasonable to suppose $p_{1U}\geq p_{1S}$ since for a group 1 individual with cultural variable q^i_1, from (9), (10), (13), and (14),

$$\begin{aligned} &p_{1U}=1(=0)\Leftrightarrow u^i_{1UN}\geq(<)u^i_{1U1} \\ &\Leftrightarrow \gamma\delta\widetilde{S_N}\geq(<)\gamma\delta\widetilde{S_1}-(\beta-\gamma)(1-\tau)(\overline{w}_1-\overline{w})+[\rho\chi(1-\chi)+\beta\omega_q](\overline{q}_1-\overline{q})(2q^i_1-\overline{q}_1-\overline{q}) \end{aligned} \quad \text{(C6)}$$

and thus the RHS of the equation is smaller than that of (M). Hence, the cases to be examined are $p_{1S}=p_{1U}=1$, $p_{1S}=p_{1U}=0$, and $p_{1S}=0, p_{1U}=1$.

When $p_{1S}=p_{1U}=1$, for a group 1 individual with q^i_1, from (7), (9), (14), and the fact $v^i_{JCG}=u^i_{JCG}+(1+r)a$, the difference in utility between when taking education and not at $H_1=1$ equals (note $\tau=\frac{\beta-1}{1+\gamma}\frac{w_s-\overline{w}}{\overline{w}}$)

$$\begin{aligned} v^i_{1SN}-v^i_{1UN}&=(1-\tau)\{(w_s-w_u)-\beta[(w_s-\overline{w})-(\overline{w}-w_u)]\}-(1+r)\overline{e} \\ &=(1-\tau)(w_s-w_u)(1-\beta\{(1-N_1)(1-H_2)-[N_1+(1-N_1)H_2]\})-(1+r)\overline{e} \\ &=(1-\tau)(w_s-w_u)\{1+\beta[(2N_1-1)+2(1-N_1)H_2]\}-(1+r)\overline{e}\ >\ 0\ \text{ under Assumption 3.} \end{aligned} \quad \text{(C7)}$$

When $p_{1S}=p_{1U}=0$, from (8), (10), (13), $\tau=0$, and $\overline{w}_1=w_s$, the difference in utility between when taking education and when not at $H_1=1$ equals

$$\begin{aligned} v^i_{1S1}-v^i_{1U1}&=(w_s-w_u)-\beta[(w_s-\overline{w}_1)-(\overline{w}_1-w_u)]-(1+r)\overline{e} \\ &=(1+\beta)(w_s-w_u)-(1+r)\overline{e}>0\ \text{ under Assumption 3.} \end{aligned} \quad \text{(C8)}$$

When $p_{1S}=0, p_{1U}=1$, from (8), (9), (13), (14), and $\tau=0$, the difference in utility equals

$$\begin{aligned} v^i_{1S1}-v^i_{1UN}&=(w_s-w_u)-\beta[(w_s-\overline{w}_1)-(\overline{w}-w_u)]+\gamma\left[-\delta\left(\widetilde{S_N}-\widetilde{S_1}\right)+(\overline{w}_1-\overline{w})\right] \\ &\quad-[\rho\chi(1-\chi)+\beta\omega_q]\left[(q^i_1-\overline{q}_1)^2-(q^i_1-\overline{q})^2\right]-(1+r)\overline{e} \\ &>(w_s-w_u)-\beta[(w_s-\overline{w}_1)-(\overline{w}-w_u)]-(1+r)\overline{e}-\beta(\overline{w}_1-\overline{w})\ \text{(from (M) with ” < ”)} \\ &=(w_s-w_u)-\beta[(w_s-\overline{w})-(\overline{w}-w_u)]-(1+r)\overline{e}\ >\ 0\ \text{ from the first equation of (C7).} \end{aligned} \quad \text{(C9)}$$

The differences in utility are all positive and thus $H_1=1$.

(ii) From Propositions 1 and 2, the cases to be examined are $p_{2S}=p_{2U}=1$, $p_{2S}=1, p_{2U}=0$, and $p_{1S}=p_{2S}=p_{2U}=0$ when q_2^i is homogenous within each class. As shown below, q_{2S}^i can be heterogenous, in which case $p_{2S}\in(0,1), p_{2U}=0$ also occurs.

(a) When $p_{2S}=p_{2U}=1$, for a group 2 individual with q_2^i, from (18), (20), and $v_{JCG}^i=u_{JCG}^i+(1+r)a$, the difference in utility between when taking education and when not equals

$$\begin{aligned} v_{2SN}^i-v_{2UN}^i &= (1-\tau)\{(w_s-w_u)-\beta[(w_s-\overline{w})-(\overline{w}-w_u)]\}-(1+r)\overline{e}, \\ \text{where } \tau &= \frac{\beta-1}{1+\gamma}\frac{w_s-\overline{w}}{\overline{w}} \text{ when } p_{1S}=1 \text{ and } \tau=0 \text{ when } p_{1S}=0, \end{aligned} \tag{C10}$$

which is positive under Assumption 3 from the first equation of (C7).

When $p_{2S}=1, p_{2U}=0$, from (18) and (21), the difference in utility equals

$$\begin{aligned} v_{2SN}^i-v_{2U2}^i &= (1-\tau)\{(w_s-w_u)-\beta[(w_s-\overline{w})-(\overline{w}_2-w_u)]+\gamma(\overline{w}-\overline{w}_2)\}+\gamma\delta\left(\widetilde{S_N}-\widetilde{S_2}\right) \\ &\quad -[\rho\chi(1-\chi)+\beta\omega_q]\left[\left(\overline{q}-q_2^i\right)^2-\left(\overline{q}_2-q_2^i\right)^2\right]-(1+r)\overline{e} \\ &= (1-\tau)\{1-\beta[(1-N_1)(1-H_2)-H_2]+\gamma N_1(1-H_2)\}(w_s-w_u)+\gamma\delta\left(\widetilde{S_N}-\widetilde{S_2}\right) \\ &\quad -[\rho\chi(1-\chi)+\beta\omega_q]\left[\left(\overline{q}-q_2^i\right)^2-\left(\overline{q}_2-q_2^i\right)^2\right]-(1+r)\overline{e}, \\ \text{where } \tau &= \tfrac{\beta-1}{1+\gamma}\tfrac{w_s-\overline{w}}{\overline{w}} \text{ when } p_{1S}=1 \text{ and } \tau=0 \text{ when } p_{1S}=0. \end{aligned} \tag{C11}$$

When $p_{1S}=1$, from (M),

$$\begin{aligned} \text{RHS of (C11)} &\geq (1-\tau)[1+\beta H_2+\gamma(1-H_2)](w_s-w_u)+\gamma\delta\left(\widetilde{S_1}-\widetilde{S_2}\right) \\ &\quad +[\rho\chi(1-\chi)+\beta\omega_q]\left\{(\overline{q}_1-\overline{q})^2-\left[\left(\overline{q}-q_2^i\right)^2-\left(\overline{q}_2-q_2^i\right)^2\right]\right\}-(1+r)\overline{e} \\ &> (1-\tau)[1+\beta H_2+\gamma(1-H_2)](w_s-w_u) \\ &\quad +[\rho\chi(1-\chi)+\beta\omega_q]\left\{(N_1)^2+(\overline{q}_1-\overline{q})^2-\left[\left(\overline{q}-q_2^i\right)^2-\left(\overline{q}_2-q_2^i\right)^2\right]\right\}-(1+r)\overline{e} \quad \text{(from Assumption 2)}, \end{aligned} \tag{C12}$$

which is positive under Assumption 3 because

$$\begin{aligned} (N_1)^2+(\overline{q}_1-\overline{q})^2-\left[\left(\overline{q}-q_2^i\right)^2-\left(\overline{q}_2-q_2^i\right)^2\right] &= (N_1)^2+(\overline{q}_1-\overline{q})^2-(\overline{q}-\overline{q}_2)\left(\overline{q}-q_2^i+\overline{q}_2-q_2^i\right) \\ &\geq (N_1)^2+(\overline{q}_1-\overline{q})^2-[(\overline{q})^2-(\overline{q}_2)^2] \\ &= (N_1)^2+(\overline{q}_1)^2+(\overline{q}_2)^2-2\overline{q}_1\overline{q} \\ &= (N_1)^2+(\overline{q}_1-\overline{q}_2)^2-2N_1\overline{q}_1(\overline{q}_1-\overline{q}_2) \\ &= [(N_1)-(\overline{q}_1-\overline{q}_2)]^2+2N_1(1-\overline{q}_1)(\overline{q}_1-\overline{q}_2)>0. \end{aligned}$$

Hence, when $p_{2S}=p_{2U}=1$ ($p_{1S}=0$ or 1) and when $p_{1S}=1, p_{2S}=1, p_{2U}=0$, the utility return to education is positive and thus $H_2=F_2$ is an equilibrium for any F_2.

(b) When $p_{1S}=p_{2S}=p_{2U}=0$ is realized in adulthood if $H_2=F_2$ in childhood, from (19), (21), and $v_{JCG}^i=u_{JCG}^i+(1+r)a$, the difference in utility equals

$$v_{2S2}^i-v_{2U2}^i=[1-\beta(1-2H_2)](w_s-w_u)-(1+r)\overline{e}. \tag{C13}$$

Thus, $v_{2S2}^i-v_{2U2}^i<0$ when H_2 is close to 0 from $\beta>1$, whereas $v_{2S2}^i-v_{2U2}^i>0$ when $1-\beta(1-2H_2)\geq\frac{2}{3}\Leftrightarrow H_2\geq\frac{1}{2}(1-\frac{1}{3\beta})$ from Assumption 3. Hence, the unique $H_2^{\Diamond}\in(0,\frac{1}{2}(1-\frac{1}{3\beta}))$ exists such that $H_2=0$, or H_2 is smaller than the lowest H_2 satisfying $p_{1S}=p_{2S}=p_{2U}=0$, for $F_2<H_2^{\Diamond}$, and $H_2=F_2$ for greater F_2.

The remaining case is when $p_{1S}=0, p_{2S}\in(0,1], p_{2U}=0$ is realized in adulthood ($p_{2S}=1$ at least in the initial period) if $H_2=F_2$ in childhood. When $N_1\le\frac{2\beta}{\beta+\gamma}\Leftrightarrow\beta(2-N_1)-\gamma N_1\ge 0$, which is true when $\beta\ge\gamma$, $v^i_{2SN}-v^i_{2U2}$ increase with H_2 from (C11) (note $\tau=0$ from $p_{1S}=0$). (Figure 4 in the proof of Proposition 4 is helpful for understanding the proof for this case.) Because $\widetilde{S_N}-\widetilde{S_2}$ is greater than when $p_{1S}=p_{2S}=p_{2U}=0$ for given H_2 from (mS), the critical H_2 satisfying $v^i_{2SN}-v^i_{2U2}=0$ is smaller than $H^{\Diamond}_2$. In the initial period, q^i_2 is homogenous, thus the critical H_2 is common to everyone, which is denoted by $H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$, where $H^{\Diamond\Diamond\prime}_2(\widetilde{S_N}-\widetilde{S_2})<0$ from (C11). Then, $H_2=0$ for $F_2<H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$ and $H_2=F_2$ for greater F_2. Since $H^{\Diamond\Diamond\prime}_2(\widetilde{S_N}-\widetilde{S_2})<0$, there exists the unique $\widetilde{S_N}-\widetilde{S_2}$ satisfying $H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})=0$. When $\widetilde{S_N}-\widetilde{S_2}$ is greater than this level, $H_2=F_2$ always.

$H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$ for the initial period determines whether $H_2=0$ or $H_2=F_2$ in subsequent periods as well, as shown next. The second last term of (C11) equals $-[\rho\chi(1-\chi)+\beta\omega_q]$ times

$$\begin{aligned}(\overline{q}-q^i_2)^2-(\overline{q}_2-q^i_2)^2 &= (\overline{q}-\overline{q}_2)(\overline{q}-q^i_2+\overline{q}_2-q^i_2)\\ &= N_1(1-\overline{q}_2)\left[N_1(1-\overline{q}_2)-2(q^i_2-\overline{q}_2)\right],\end{aligned}\tag{C14}$$

where the second equation is from $(\overline{q}_1)'=\overline{q}_1=1$.

When $F_2<H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$ and thus $H_2=0$ hold in the initial period, $(q^i_2)'=\chi\overline{q}_2+(1-\chi)q^i_2=0$ from $p_{2U}=0$. Hence, $(\overline{q}-q^i_2)^2-(\overline{q}_2-q^i_2)^2$ and thus $H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$ are time-invariant. Therefore, $H_2=0$ for F_2 smaller than the initial $H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$ remains true in subsequent periods.

When $F_2\ge H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$ and thus $H_2=F_2$ initially, (C14) in the initial period, $(N_1)^2$, is greater than the values in subsequent periods because $q^i_{2S}\ge\overline{q}_2$. Hence, the initial $H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$ is greater than the critical values in subsequent periods. Therefore, when $F_2\ge H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$ initially, $H_2=F_2$ continues to be true subsequently. (When H_2 increases over time, after the initial period, the level of q^i_{2S} becomes different depending on when one becomes skilled, implying that $H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$ differs for those with different q^i_{2S}. The result remains unchanged because levels of $H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$ in subsequent periods are smaller than in the initial period for any q^i_{2S}. In this case, (mS) also differs for those with different q^i_{2S}, implying that $p_{2S}\in(0,1), p_{2U}=0$ can occur.)

When $N_1>\frac{2\beta}{\beta+\gamma}$, which occurs only when $\beta<\gamma$, (C11) decreases with H_2. (Figure 7 in the proof of Proposition 4 is helpful for understanding the proof for this case.) As before, $H_2=F_2$ for any $F_2\ge H^{\Diamond}_2$. This is because $v^i_{2SN}-v^i_{2U2}\ge 0$ for $F_2\ge H^{\Diamond}_2$ on the dividing line between $p_{2S}=1$ and $p_{2S}=0$, (mS), from (C13), and thus $v^i_{2SN}-v^i_{2U2}>0$ for greater $\widetilde{S_N}-\widetilde{S_2}$. For $F_2<H^{\Diamond}_2$, when $\widetilde{S_N}-\widetilde{S_2}$ is greater than the level at which (mS) and $H_2=H^{\Diamond}_2$ intersect, where $v^i_{2SN}-v^i_{2U2}=0$, $H_2=F_2$ because $v^i_{2SN}-v^i_{2U2}$ decreases with H_2. In the initial period, q^i_2 is homogenous, thus (mS) and the critical $\widetilde{S_N}-\widetilde{S_2}$ are common to everyone. Hence, when $\widetilde{S_N}-\widetilde{S_2}$ is smaller than this level, the unique $H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$ satisfying $H^{\Diamond\Diamond\prime}_2(\widetilde{S_N}-\widetilde{S_2})>0$ exists and $H_2=F_2$ for $F_2\le H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$, $H_2=H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$ for greater F_2 satisfying $p_{1S}=0$, $p_{2S}=1$, $p_{2U}=0$. Further, since $H^{\Diamond\Diamond\prime}_2(\widetilde{S_N}-\widetilde{S_2})>0$, the unique $\widetilde{S_N}-\widetilde{S_2}$ with $H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})=0$ exists, and $H_2=0$ when $\widetilde{S_N}-\widetilde{S_2}$ is smaller than this level.

In subsequent periods, $H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$ changes over time, since (C14) varies over time due to $H_2>0$ and $p_{2S}=1$, $p_{2U}=0$. (It can be shown that initial $H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$ is smallest and $H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$ increases in early periods.) Hence, when H_2 changes over time either because $F_2(\le$ initial $H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2}))$ increases or because F_2 is close to $H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$, q^i_{2S} and $H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$ become heterogenous, with $H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2})$ increasing in q^i_{2S}. Therefore, when H_2 evolves, H_2 in subsequent periods is given by: when $\widetilde{S_N}-\widetilde{S_2}$ is greater than the level at which (mS) for those with $\min_i\{q^i_{2S}\}$ and $F_2=H^{\Diamond}_2$ intersect, $H_2=F_2$; when $\widetilde{S_N}-\widetilde{S_2}$ is smaller than this level and greater than the level satisfying $H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2},\max_i\{q^i_{2S}\})=0$, $H_2=F_2$ for $F_2\le\max\left\{0,H^{\Diamond\Diamond}_2(\widetilde{S_N}-\widetilde{S_2},\min_i\{q^i_{2S}\})\right\}$, $H_2\in$

$\left(\max\left\{0, H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \min_i\{q_{2S}^i\})\right\}, F_2\right)$ for $F_2 \in \left(\max\left\{0, H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \min_i\{q_{2S}^i\})\right\}, \min\left\{H_2^{\lozenge}, H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \max_i\{q_{2S}^i\})\right\}\right)$, and $H_2 = \min\left\{H_2^{\lozenge}, H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \max_i\{q_{2S}^i\})\right\}$ for greater F_2, where time-variant $H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \max_i\{q_{2S}^i\})$ $(H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \min_i\{q_{2S}^i\}))$ is the value of $H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2})$ for those with highest (lowest) q_{2S}^i; when $\widetilde{S_N}-\widetilde{S_2}$ is smaller than the level satisfying $H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \max_i\{q_{2S}^i\}) = 0$, $H_2 = 0$. [(mS) differs depending on q_{2S}^i. (mS) with $q_{2S}^i = \max_i\{q_{2S}^i\}$ $(q_{2S}^i = \min_i\{q_{2S}^i\})$ intersects with $H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \max_i\{q_{2S}^i\})$ $(H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \min_i\{q_{2S}^i\}))$ at $F_2 = H_2^{\lozenge}$. $p_{2S} \in (0,1), p_{2U} = 0$ in the region between the two (mS)s.]

To summarize the results when $\beta \geq \gamma$ or $N_1 \leq \frac{2\beta}{\beta+\gamma}$, when $\widetilde{S_N}-\widetilde{S_2}$ is greater than the level such that $H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}) = 0$, $H_2 = F_2$; when $\widetilde{S_N}-\widetilde{S_2}$ is smaller than this level and greater than the level satisfying $H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}) = H_2^{\lozenge}$, $H_2 = 0$ for $F_2 \in \left[0, H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2})\right)$ and $H_2 = F_2$ for greater F_2; and when $\widetilde{S_N}-\widetilde{S_2}$ is smaller than the level satisfying $H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}) = H_2^{\lozenge}$, $H_2 = 0 (= F_2)$ for $F_2 < (\geq) H_2^{\lozenge}$, where $H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2})$ is the critical value in the initial period.

The results when $\beta < \gamma$ and $N_1 > \frac{2\beta}{\beta+\gamma}$ are summarized as follows. When $\widetilde{S_N}-\widetilde{S_2}$ is greater than the level at which (mS) for those with $\min_i\{q_{2S}^i\}$ and $H_2 = H_2^{\lozenge}$ intersect, $H_2 = F_2$; when $\widetilde{S_N}-\widetilde{S_2}$ is smaller than this level and greater than the level satisfying $H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \max_i\{q_{2S}^i\}) = 0$, $H_2 = F_2$ for $F_2 \leq \max\left\{0, H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \min_i\{q_{2S}^i\})\right\}$, $H_2 \in \left(\max\left\{0, H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \min_i\{q_{2S}^i\})\right\}, F_2\right)$ for $F_2 \in \left(\max\left\{0, H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \min_i\{q_{2S}^i\})\right\}, \min\left\{H_2^{\lozenge}, H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \max_i\{q_{2S}^i\})\right\}\right)$,
$H_2 = \min\left\{H_2^{\lozenge}, H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \max_i\{q_{2S}^i\})\right\}$ for $F_2 \in \left[\min\left\{H_2^{\lozenge}, H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \max_i\{q_{2S}^i\})\right\}, H_2^{\lozenge}\right)$, and $H_2 = F_2$ for $F_2 \geq H_2^{\lozenge}$; when $\widetilde{S_N}-\widetilde{S_2}$ is smaller than the level satisfying $H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \max_i\{q_{2S}^i\}) = 0$, $H_2 = 0 (= F_2)$ for $F_2 < (\geq) H_2^{\lozenge}$. ($H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \max_i\{q_{2S}^i\})$ and $H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \min_i\{q_{2S}^i\})$ change over time, and if H_2 is time-invariant, they are the same and equal to $H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2})$.) ■

Proof of Lemma 3. (i) The proof that H_2 non-decreases over time when $H_2 = F_2$ below applies to H_1 and F_1 as well. Then, the result follows from $F_1 = 1$ in the initial period and Lemma 2 (i).

(ii) [Proof that H_2 usually non-decreases] When $H_2 = F_2$ for any F_2, which is the case when $\widetilde{S_N}-\widetilde{S_2}$ is sufficiently high from the proof of Lemma 2 (ii), H_2 non-decreases if $\lambda[(1-\tau)w_s + T] > \overline{e}$ for any H_2. It can be shown that $(1-\tau)w_s + T$ increases with H_2 and thus is lowest at $H_2 = 0$ from the equations similar to (C15) and (C16) below. Then, because $(1-\tau)w_s + T - (1+r)\overline{e} > (1-\tau)w_u + T$ for any H_2 from Assumption 3, $\lambda[(1-\tau)w_s + T] > \overline{e}$ from Assumption 4 (i).

When $\widetilde{S_N}-\widetilde{S_2}$ is very low, $H_2 = 0$ $(= F_2)$ for $F_2 < (\geq) H_2^{\lozenge}$ from the proof of Lemma 2 (ii)(b). In this case, H_2 does not decrease from $H_2 = F_2$ to $H_2 = 0$ because F_2 non-decreases when $H_2 = F_2$ and, as shown in the proof, $H_2^{\lozenge}$ is constant.

From the proof, when $\beta \geq \gamma$ or $N_1 \leq \frac{2\beta}{\beta+\gamma}$, and $\widetilde{S_N}-\widetilde{S_2}$ is not very, but relatively, low, $H_2 = 0$ $(= F_2)$ for $F_2 < (\geq) H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2})$. The shift from $H_2 = F_2$ to $H_2 = 0$ does not occur in this case as well, because, as shown in the proof, when $F_2 \geq H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2})$ holds in the initial period, the condition continues to hold in subsequent periods.

Finally, when $\beta < \gamma$, $N_1 > \frac{2\beta}{\beta+\gamma}$, and $\widetilde{S_N}-\widetilde{S_2}$ is not very, but relatively, low, $H_2 = F_2$ for $F_2 \leq \max\left\{0, H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \min_i\{q_{2S}^i\})\right\}$, $H_2 \in \left(\max\left\{0, H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \min_i\{q_{2S}^i\})\right\}, F_2\right)$ for $F_2 \in \left(\max\left\{0, H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \min_i\{q_{2S}^i\})\right\}, \min\left\{H_2^{\lozenge}, H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \max_i\{q_{2S}^i\})\right\}\right)$,
$H_2 = \min\left\{H_2^{\lozenge}, H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \max_i\{q_{2S}^i\})\right\}$ for $F_2 \in \left[\min\left\{H_2^{\lozenge}, H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \max_i\{q_{2S}^i\})\right\}, H_2^{\lozenge}\right)$, and $H_2 = F_2$ for $F_2 \geq H_2^{\lozenge}$, implying that $H_2 \in (0, F_2)$ for $F_2 \in \left(\max\left\{0, H_2^{\lozenge\lozenge}(\widetilde{S_N}-\widetilde{S_2}, \min_i\{q_{2S}^i\})\right\}, H_2^{\lozenge}\right)$.

In this case, H_2 could decrease over time under the following two situations where $p_{1S}=0, p_{2S}=1, p_{2U}=0$ holds. First, when $\frac{\lambda}{1-\lambda(1+r)}w_u<\overline{e}$ and $H_2\in(0,F_2)$, F_2 decreases over time and thus H_2 could decrease, because wealth holdings of those who can afford, but do not take, education and become unskilled workers decrease over time. Second, when $\frac{\lambda}{1-\lambda(1+r)}w_u\geq\overline{e}$ and thus F_2 is time-invariant, H_2 decreases when F_2 is slightly less than $H_2^{\diamondsuit\diamondsuit}(\widetilde{S_N}-\widetilde{S_2},\min_i\{q_{2S}^i\})$ or $H_2^{\diamondsuit\diamondsuit}(\widetilde{S_N}-\widetilde{S_2},\max_i\{q_{2S}^i\})$ and this critical H_2 decreases.

[Proof that H_2 increases over time when $p_{1S}=1$] When $p_{1S}=1$, $H_2=F_2$ from Lemma 2 (ii)(a). Then, the result is obvious when $\frac{\lambda}{1-\lambda(1+r)}w_u\geq\overline{e}$, thus the proof focuses on the case $\frac{\lambda}{1-\lambda(1+r)}w_u<\overline{e}$. When $p_{1S}=1$, from (2) and (16), the disposable labor income of unskilled workers is expressed as

$$(1-\tau)w_u+\left(\tau-\frac{\tau^2}{2}\right)\overline{w}=w_u+\frac{\beta-1}{1+\gamma}\frac{w_s-\overline{w}}{\overline{w}}\left[-w_u+\left(1-\frac{1}{2}\frac{\beta-1}{1+\gamma}\frac{w_s-\overline{w}}{\overline{w}}\right)\overline{w}\right]. \tag{C15}$$

The derivative of this equation with respect to H_2 equals

$$\begin{aligned}&\frac{\beta-1}{1+\gamma}\left\{-\frac{w_s}{(\overline{w})^2}(1-N_1)(w_s-w_u)\left[-w_u+\left(1-\frac{1}{2}\frac{\beta-1}{1+\gamma}\frac{w_s-\overline{w}}{\overline{w}}\right)\overline{w}\right]+\frac{w_s-\overline{w}}{\overline{w}}\left(1+\frac{1}{2}\frac{\beta-1}{1+\gamma}\right)(1-N_1)(w_s-w_u)\right\}\\ =\;&(1-N_1)(w_s-w_u)\frac{\beta-1}{1+\gamma}\left\{-\frac{w_s}{(\overline{w})^2}\left[\left(1+\frac{1}{2}\frac{\beta-1}{1+\gamma}\right)\overline{w}-\left(w_u+\frac{1}{2}\frac{\beta-1}{1+\gamma}w_s\right)\right]+\frac{w_s-\overline{w}}{\overline{w}}\left(1+\frac{1}{2}\frac{\beta-1}{1+\gamma}\right)\right\}\\ =\;&(1-N_1)(w_s-w_u)\frac{\beta-1}{1+\gamma}\left[\frac{w_s}{(\overline{w})^2}\left(w_u+\frac{1}{2}\frac{\beta-1}{1+\gamma}w_s\right)-\left(1+\frac{1}{2}\frac{\beta-1}{1+\gamma}\right)\right].\end{aligned} \tag{C16}$$

The second derivative is negative because $\overline{w}$ increases with H_2.

At $H_2=0$, (C15) equals

$$(1-\tau)w_u+\left(\tau-\frac{\tau^2}{2}\right)\overline{w}=w_u+\frac{\beta-1}{1+\gamma}\frac{(1-N_1)(w_s-w_u)^2}{N_1w_s+(1-N_1)w_u}\left[N_1-\frac{1}{2}\frac{\beta-1}{1+\gamma}(1-N_1)\right]. \tag{C17}$$

Thus, $\frac{\lambda}{1-\lambda(1+r)}[(1-\tau)w_u+T]>\overline{e}$ holds at $H_2=0$ from footnote 42 of Assumption 4 (i). Then, because $\frac{\lambda}{1-\lambda(1+r)}[(1-\tau)w_u+T]=\frac{\lambda}{1-\lambda(1+r)}w_u<\overline{e}$ at $H_2=1$ and the second derivative of (C15) is negative, there exists $\widetilde{H}_2\in(0,1)$ such that $\frac{\lambda}{1-\lambda(1+r)}[(1-\tau)w_u+T]>(<)\overline{e}$ for $H_2<(>)\widetilde{H}_2$.

From Assumption 4 (ii), a not-small proportion of group 2 individuals do not have wealth initially. If the proportion of such individuals is greater than $1-\widetilde{H}_2$, their descendants can accumulate wealth greater than $\overline{e}$ and thus H_2 jumps from a value less than $\widetilde{H}_2$ to 1 at some point in time because $\frac{\lambda}{1-\lambda(1+r)}[(1-\tau)w_u+T]>\overline{e}$ always holds for their lineages. If $\frac{\lambda}{1-\lambda(1+r)}[(1-\tau)w_u+T]$ at $H_2=0$ is sufficiently greater than $\overline{e}$ (Assumption 4 (i)), $\widetilde{H}_2$ is large enough that the initial proportion of those without wealth is greater than $1-\widetilde{H}_2$. This is the case if β or w_s is sufficiently large or γ is sufficiently small because $(1-\tau)w_u+T$ increases with β and w_s and decreases with γ, as shown next.

The derivative of the disposable income (C15) with respect to $\frac{\beta-1}{1+\gamma}$ equals $\frac{w_s-\overline{w}}{\overline{w}}$ times

$$\begin{aligned}-w_u+\left(1-\frac{1}{2}\frac{\beta-1}{1+\gamma}\frac{w_s-\overline{w}}{\overline{w}}\right)\overline{w}-\frac{1}{2}\frac{\beta-1}{1+\gamma}(w_s-\overline{w})&=(\overline{w}-w_u)-\frac{\beta-1}{1+\gamma}(w_s-\overline{w})\\ &=\left\{[N_1+(1-N_1)H_2]-\frac{\beta-1}{1+\gamma}(1-N_1)(1-H_2)\right\}(w_s-w_u)\\ &\geq\left[N_1-\frac{\beta-1}{1+\gamma}(1-N_1)\right](w_s-w_u)\geq\left[N_1-\frac{1}{3}(1-N_1)\right](w_s-w_u)>0,\end{aligned}$$

where the second last inequality is from Assumption 1.

The derivative of the disposable income with respect to w_s equals $\frac{\beta-1}{1+\gamma}$ times

$$
\begin{aligned}
& \frac{1}{\overline{w}}\left[-w_u+\left(1-\frac{1}{2}\frac{\beta-1}{1+\gamma}\frac{w_s-\overline{w}}{\overline{w}}\right)\overline{w}\right]-\frac{w_s-\overline{w}}{\overline{w}}\frac{1}{2}\frac{\beta-1}{1+\gamma} \\
& +[N_1+(1-N_1)H_2]\left\{-\frac{w_s}{(\overline{w})^2}\left[-w_u+\left(1-\frac{1}{2}\frac{\beta-1}{1+\gamma}\frac{w_s-\overline{w}}{\overline{w}}\right)\overline{w}\right]+\frac{w_s-\overline{w}}{\overline{w}}\left(1+\frac{1}{2}\frac{\beta-1}{1+\gamma}\right)\right\} \\
= & \frac{\overline{w}-w_u}{\overline{w}}-\frac{w_s-\overline{w}}{\overline{w}}\frac{\beta-1}{1+\gamma}+\frac{N_1+(1-N_1)H_2}{\overline{w}}\left\{\frac{w_s}{\overline{w}}\left[-(\overline{w}-w_u)+\frac{1}{2}\frac{\beta-1}{1+\gamma}(w_s-\overline{w})\right]+(w_s-\overline{w})\left(1+\frac{1}{2}\frac{\beta-1}{1+\gamma}\right)\right\} \\
= & \frac{1}{\overline{w}}\left\{\overline{w}-w_u+[N_1+(1-N_1)H_2]\left(\frac{w_s w_u}{\overline{w}}-\overline{w}\right)\right\}+\frac{w_s-\overline{w}}{\overline{w}}\frac{\beta-1}{1+\gamma}\left\{-1+\frac{1}{2}\frac{[N_1+(1-N_1)H_2]}{\overline{w}}(w_s+\overline{w})\right\} \\
= & \frac{1}{\overline{w}}\left[(1-N_1)(1-H_2)\overline{w}-(1-N_1)(1-H_2)\frac{(w_u)^2}{\overline{w}}\right]-\frac{w_s-\overline{w}}{\overline{w}}\frac{\beta-1}{1+\gamma}\frac{(1-N_1)(1-H_2)(w_u+\overline{w})}{2\overline{w}} \\
= & \frac{(1-N_1)(1-H_2)(w_u+\overline{w})}{(\overline{w})^2}(w_s-w_u)\left\{[N_1+(1-N_1)H_2]-\frac{\beta-1}{1+\gamma}\frac{(1-N_1)(1-H_2)}{2}\right\} \\
\geq & \frac{(1-N_1)(1-H_2)(w_u+\overline{w})}{(\overline{w})^2}(w_s-w_u)\left(N_1-\frac{\beta-1}{1+\gamma}\frac{1-N_1}{2}\right)>0,
\end{aligned}
$$

where the last inequality is from Assumption 1. ■

Proof of Proposition 4. Figures 4–7 would be helpful to understand the proof.

(i) The proof of Proposition 3 (i) applies for results on steady-state identity mostly, since $H_2=F_2$ from Lemma 2 (ii)(a) and Lemma A4 (ii) is same as Lemma A2 (ii). However, unlike the constant H_2 case, $p_{1S}=1$, $p_{2S}=1$, $p_{2U}=0$, which may be realized for low H_2 when $\beta>\gamma$ and $N_1>\frac{\beta+\gamma}{2\beta}$ (see Figure 3), shifts to $p_{1S}=p_{2S}=p_{2U}=1$ eventually because H_2 increases over time from Lemma 3 (ii). The result on cultural variables is from Lemma A5 (i) and Proposition 3 (i).

(ii) As long as $H_2=F_2$, F_2 is constant under $p_{2S}=0$ from $\tau=0$. When $(F_2,\widetilde{S_N})$ is located below initial (mS) with $p_{1S}=0$ and $F_2\geq H_2^{\Diamond}$, $H_2=F_2$ from the proof of Lemma 2 (ii)(b) and thus $p_{1S}=p_{2S}=p_{2U}=0$ initially. $p_{1S}=p_{2S}=p_{2U}=0$ holds in subsequent periods, because (mS) does not shift from Lemma A4 (v) and $H_2^{\Diamond}$ is constant from (C13).

When $\beta\geq\gamma$ or $N_1\leq\frac{\beta+\gamma}{2\beta}$ (Figures 4–6), if $(F_2,\widetilde{S_N})$ is below initial (mS) with $p_{1S}=0$ and $F_2<H_2^{\Diamond}$, or if $(F_2,\widetilde{S_N})$ is on or above initial (mS) with $p_{1S}=0$ and below initial (mU) with $p_{1S}=0$ and $F_2<H_2^{\Diamond\Diamond}(\widetilde{S_N}-\widetilde{S_2})$ (where $H_2^{\Diamond\Diamond\prime}(\widetilde{S_N}-\widetilde{S_2})\leq 0$), $H_2=0$ and $p_{1S}=p_{2U}=0$ initially from the proof of Lemma 2 (ii)(b). Similarly, when $\beta<\gamma$ and $N_1>\frac{\beta+\gamma}{2\beta}$ (Figure 7), if $F_2<H_2^{\Diamond}$ and $\widetilde{S_N}-\widetilde{S_2}$ is smaller than the level satisfying $H_2^{\Diamond\Diamond}(\widetilde{S_N}-\widetilde{S_2})=0$ (where $H_2^{\Diamond\Diamond\prime}(\widetilde{S_N}-\widetilde{S_2})>0$), $H_2=0$ and $p_{1S}=p_{2U}=0$ initially from the proof. $H_2=0$ and $p_{1S}=p_{2U}=0$ hold in subsequent periods. This is because (mS) and (mU) do not shift from Lemma A4 (vi), $H_2^{\Diamond}$ is constant, $H_2^{\Diamond\Diamond}(\widetilde{S_N}-\widetilde{S_2})$ does not change from (C11) and the fact that $q_{1S}=1, q_{2U}=0$ continues to hold from (13), and F_2 decreases over time.

Equilibria with other values of p_{1S}, p_{2S}, and p_{2U} do not shift to $p_{1S}=p_{2S}=p_{2U}=0$ or $H_2=0$, $p_{1S}=p_{2U}=0$, because (mS) with $p_{1S}=0$ in the initial period is located at a higher position than or the same position as those in subsequent periods on the $(F_2,\widetilde{S_N})$ plane from Lemma A4 (i) and initial $H_2^{\Diamond\Diamond}(\widetilde{S_N}-\widetilde{S_2})$ is greater (smaller) than those in subsequent periods when $\beta\geq\gamma$ or $N_1\leq\frac{2\beta}{\beta+\gamma}$(when $\beta<\gamma$ and $N_1>\frac{\beta+\gamma}{2\beta}$) from the proof of Lemma 2 (ii)(b).

$q_{1S}^*=1$ and $q_{2S}^*=q_{2U}^*=0$ ($q_{2U}^*=0$ when $H_2^*=0$) is from Lemma A5 (iii) and the result that only the society starting with $p_{1S}=p_{2S}=p_{2U}=0$ ($H_2=0$ and $p_{1S}=p_{2U}=0$) ends up with $p_{1S}^*=p_{2S}^*=p_{2U}^*=0$ ($H_2^*=0$ and $p_{1S}^*=p_{2U}^*=0$).

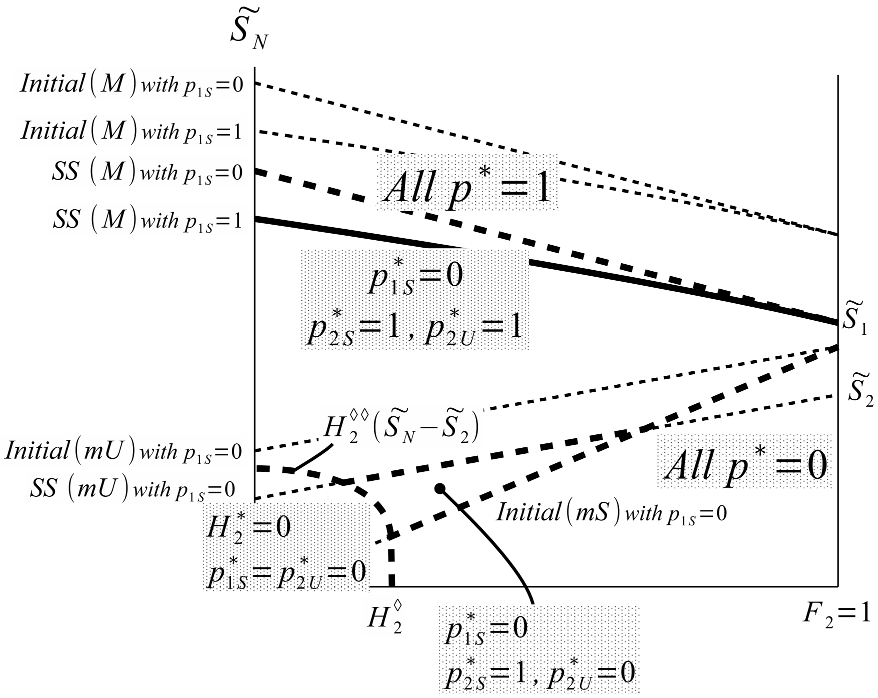

Figure C4: Relationship between initial $(F_2, \widetilde{S_N})$ and steady-state identity for the full-fledged model with $\frac{\lambda}{1-\lambda(1+r)} w_u \leq \overline{e}$ when $\beta \leq \gamma$ and $N_1 \leq \frac{2\beta}{\beta+\gamma}$

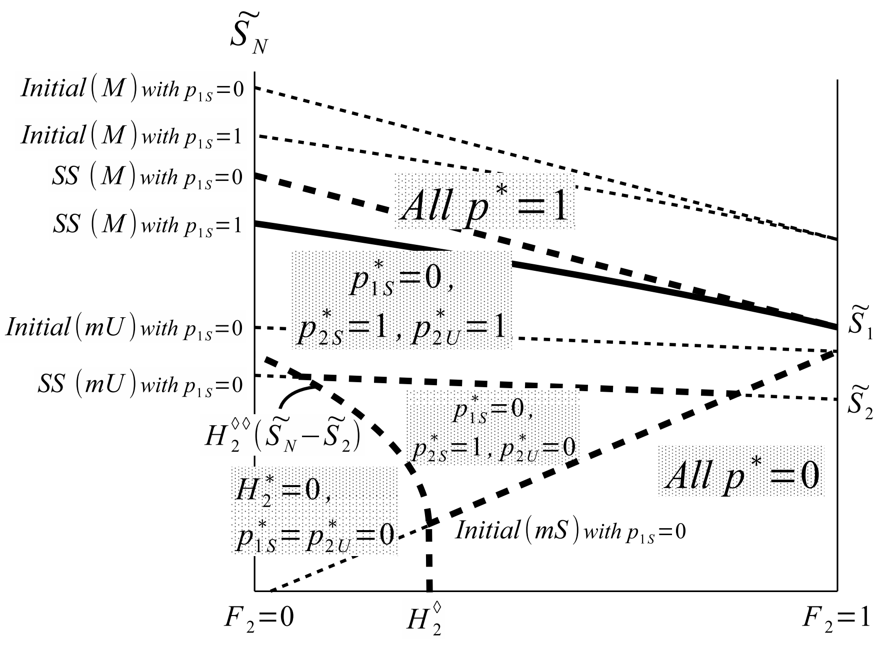

Figure C5: Relationship between initial $(F_2, \widetilde{S_N})$ and steady-state identity for the full-fledged model with $\frac{\lambda}{1-\lambda(1+r)} w_u \leq \overline{e}$ when $\beta > \gamma$ and $N_1 \leq \frac{\beta+\gamma}{2\beta}$

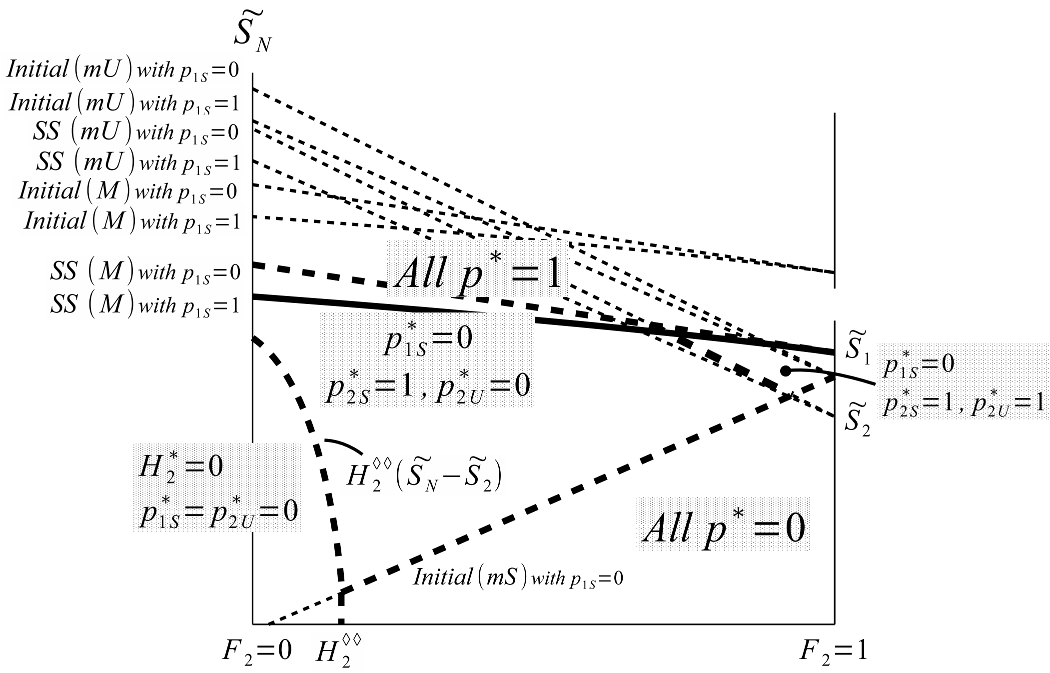

Figure C6: Relationship between initial $(F_2, \widetilde{S_N})$ and steady-state identity for the full-fledged model with $\frac{\lambda}{1-\lambda(1+r)}w_u \leq \overline{e}$ when $\beta > \gamma$ and $N_1 > \frac{\beta+\gamma}{2\beta}$

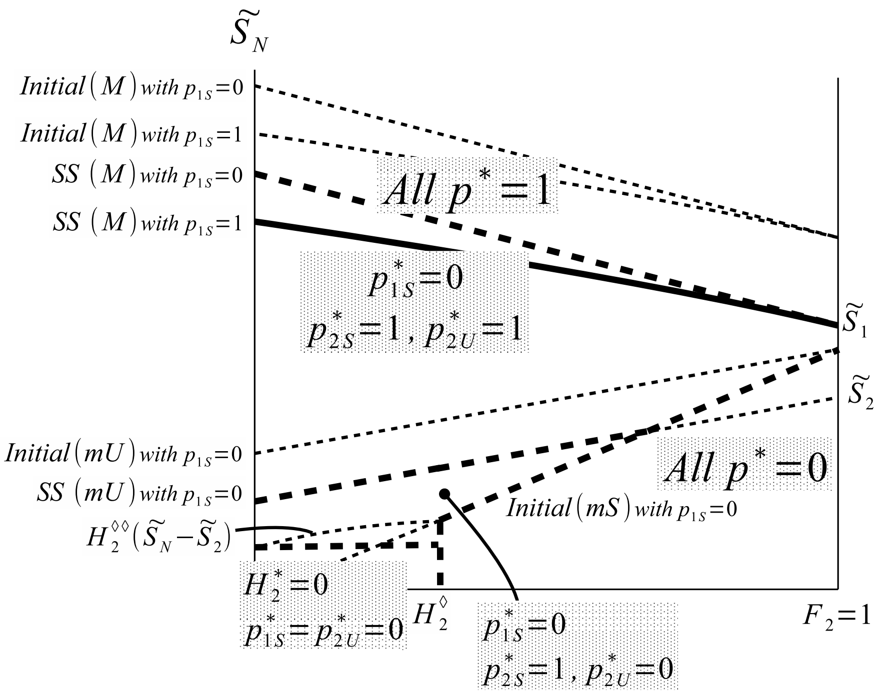

Figure C7: Relationship between initial $(F_2, \widetilde{S_N})$ and steady-state identity for the full-fledged model with $\frac{\lambda}{1-\lambda(1+r)}w_u \leq \overline{e}$ when $\beta < \gamma$ and $N_1 > \frac{2\beta}{\beta+\gamma}$

(iii) As long as $H_2 = F_2$, F_2 is constant under $p_{2S} = 0$ from $\tau = 0$. When $p_{1S} = 0$, $p_{2S} = p_{2U} = 1$ initially (i.e., $(F_2, \widetilde{S_N})$ is on or above initial (mU) with $p_{1S} = 0$ and below initial (M) with $p_{1S} = 0$), where $H_2 = F_2$ from Lemma 2 (ii)(a), and $\widetilde{S_N}$ is relatively low for given $H_2 = F_2$ (i.e., $(F_2, \widetilde{S_N})$ is below steady-state (M) with $p_{1S} = 0$), $p_{1S} = 0$, $p_{2S} = p_{2U} = 1$ holds in subsequent periods because (M) and (mU) shift downward over time on the $(F_2, \widetilde{S_N})$ plane from Lemma A4 (iii).

When $\beta \geq \gamma$ or $N_1 \leq \frac{2\beta}{\beta+\gamma}$ (Figures 4–6), $p_{1S} = 0$, $p_{2S} = 1, p_{2U} = 0$ initially (i.e., $(F_2, \widetilde{S_N})$ is on or above initial (mS) with $p_{1S} = 0$ and below initial (mU) with $p_{1S} = 0$; when $\beta > \gamma$ and $N_1 > \frac{\beta+\gamma}{2\beta}$ [Figure 6], also below initial (M) with $p_{1S} = 0$; *and* F_2 is greater than initial $H_2^{\Diamond\Diamond}(\widetilde{S_N} - \widetilde{S_2})$), where $H_2 = F_2$ from the proof of Lemma 2 (ii)(b), and $\widetilde{S_N}$ is relatively high (i.e., $(F_2, \widetilde{S_N})$ is on or above steady-state (mU) with $p_{1S} = 0$; when $\beta > \gamma$ and $N_1 > \frac{\beta+\gamma}{2\beta}$ [Figure 6], also below steady-state (M) with $p_{1S} = 0$), the society shifts to $p_{1S} = 0$, $p_{2S} = p_{2U} = 1$ eventually and stays in this state. This is because (M) shifts downward over time, so does (mU) in the long run, from Lemma A4 (iv), (mS)s in subsequent periods are not located above initial (mS) from Lemma A4 (i), and F_2 is greater than initial $H_2^{\Diamond\Diamond}(\widetilde{S_N} - \widetilde{S_2})$ in subsequent periods as well from the proof of Lemma 2 (ii)(b).

When $\beta < \gamma$, $N_1 > \frac{2\beta}{\beta+\gamma}$ (Figure 7), $p_{1S} = 0$, $p_{2S} = 1, p_{2U} = 0$ initially and $\widetilde{S_N}$ is relatively high (i.e., $(F_2, \widetilde{S_N})$ is on or above initial (mS) and steady-state (mU) with $p_{1S} = 0$, below initial (mU) with $p_{1S} = 0$), where $H_2 = F_2$ from the proof of Lemma 2 (ii)(b), the society shifts to $p_{1S} = 0$, $p_{2S} = p_{2U} = 1$ and stays in this state due to the reasons explained for the previous case.

To summarize, when $\beta \geq \gamma$ or $N_1 \leq \frac{2\beta}{\beta+\gamma}$ (Figures 4–6), $p_{1S}^* = 0, p_{2S}^* = p_{2U}^* = 1$ if $(F_2, \widetilde{S_N})$ is located on or above initial (mS) with $p_{1S} = 0$, as well as steady-state (mU) with $p_{1S} = 0$, and below steady-state (M) with $p_{1S} = 0$, and F_2 is greater than initial $H_2^{\Diamond\Diamond}(\widetilde{S_N} - \widetilde{S_2})$; when $\beta < \gamma$ and $N_1 > \frac{2\beta}{\beta+\gamma}$ (Figure 7), $p_{1S}^* = 0, p_{2S}^* = p_{2U}^* = 1$ if $(F_2, \widetilde{S_N})$ is located on or above initial (mS) with $p_{1S} = 0$, as well as steady-state (mU) with $p_{1S} = 0$, and below steady-state (M) with $p_{1S} = 0$.

When $\beta \geq \gamma$ or $N_1 \leq \frac{2\beta}{\beta+\gamma}$ (Figures 4–6), $p_{1S} = 0$, $p_{2S} = 1, p_{2U} = 0$ initially, and $\widetilde{S_N}$ is relatively low (i.e., $(F_2, \widetilde{S_N})$ is on or above initial (mS) with $p_{1S} = 0$, below steady-state (mU) with $p_{1S} = 0$; when $\beta > \gamma$ and $N_1 > \frac{\beta+\gamma}{2\beta}$ [Figure 6], also below steady-state (M) with $p_{1S} = 0$; and F_2 is greater than initial $H_2^{\Diamond\Diamond}(\widetilde{S_N} - \widetilde{S_2})$), where $H_2 = F_2$ from the proof of Lemma 2 (ii)(b), $p_{1S} = 0$, $p_{2S} = 1, p_{2U} = 0$ continues to hold due to the reasons explained for the case before the previous case.

When $\beta < \gamma$, $N_1 > \frac{2\beta}{\beta+\gamma}$ (Figure 7), $p_{1S} = 0$, $p_{2S} = 1, p_{2U} = 0$ initially, and $\widetilde{S_N}$ is relatively low (i.e., $(F_2, \widetilde{S_N})$ is on or above initial (mS) with $p_{1S} = 0$ for $F_2 \geq H_2^{\Diamond\Diamond}$ and $\widetilde{S_N} - \widetilde{S_2}$ is greater than the level such that initial $H_2^{\Diamond\Diamond}(\widetilde{S_N} - \widetilde{S_2}) = 0$ for $F_2 < H_2^{\Diamond\Diamond}$, and $(F_2, \widetilde{S_N})$ is below steady-state (mU) with $p_{1S} = 0$), where $H_2 = H_2^{\Diamond\Diamond}(\widetilde{S_N} - \widetilde{S_2})$ if $F_2 < H_2^{\Diamond\Diamond}$ and $\widetilde{S_N} - \widetilde{S_2}$ is smaller than the level satisfying $H_2^{\Diamond\Diamond}(\widetilde{S_N} - \widetilde{S_2}) = F_2$, otherwise, $H_2 = F_2$, from the proof of Lemma 2 (ii)(b), $p_{1S} = 0$, $p_{2S} = (0, 1], p_{2U} = 0$ subsequently. This is because the graph of $H_2^{\Diamond\Diamond}(\widetilde{S_N} - \widetilde{S_2})$ in the initial period is located above those in subsequent periods on the $(F_2, \widetilde{S_N})$ plane from the proof of Lemma 2 (ii)(b), Lemma A4 (iv) holds, and F_2 decreases (is constant) when $H_2 < (=) F_2$. $p_{2S} \in (0, 1)$ is possible since $H_2^{\Diamond\Diamond}(\widetilde{S_N} - \widetilde{S_2})$ changes over time from the proof, but $p_{2S}^* = 1$ due to $q_{2S}^* = q_{2U}^* = 1$, as shown below.

To summarize, $p_{1S}^* = 0$, $p_{2S}^* = 1, p_{2U}^* = 0$ if $(F_2, \widetilde{S_N})$ is located above the region for $p_{1S}^* = p_{2S}^* = p_{2U}^* = 0$ and the one for $H_2^* = 0$ and $p_{1S}^* = p_{2U}^* = 0$ and below steady-state (mU) with $p_{1S} = 0$, when $\beta > \gamma$ and $N_1 > \frac{\beta+\gamma}{2\beta}$ (Figure 6), also below steady-state (M) with $p_{1S} = 0$.

$q_{1S}^* = q_{2S}^* = q_{2U}^* = 1$ is from Lemma A5 (ii) and the result that only the society starting with $p_{1S} = 1$ and never satisfying $p_{1S} = 0$ ends up with $p_{1S}^* = 0$, $p_{2S}^* = 1, p_{2U}^* = 0$ or 1.

(iv) The result can be proved similarly to Proposition 3 (iv). ■

Proof of Proposition 5. (i) (a) When $\widetilde{S_N}$ is very high so that $p_{1S}=p_{2S}=p_{2U}=1$ in the initial period (i.e., initial $(F_2,\widetilde{S_N})$ is on or above initial (M) with $p_{1S}=1$, when $\beta>\gamma$ and $N_1>\frac{\beta+\gamma}{2\beta}$, also on or above initial (mU) with $p_{1S}=1$ on the $(F_2,\widetilde{S_N})$ plane), $p_{1S}=p_{2S}=p_{2U}=1$ always because, as noted in the proof of Proposition 4 (i), the proof of Proposition 3 (i) applies. When $\beta>\gamma$ and $N_1>\frac{\beta+\gamma}{2\beta}$, and $p_{1S}=1$, $p_{2S}=1$, $p_{2U}=0$ initially (i.e., initial $(F_2,\widetilde{S_N})$ is on or above initial (M) with $p_{1S}=1$ and below initial (mU) with $p_{1S}=1$), the equilibrium shifts to $p_{1S}=p_{2S}=p_{2U}=1$ eventually from the proof of Proposition 4 (i). Hence, $p_{1S}=1$ always holds, thus H_2 increases over time and $H_2^*=1$ from Lemma 3 (ii). (b) The result holds because initial (M) with $p_{1S}=1$ is downward sloping, and as $\widetilde{S_1}$ and ω_q are lower, it is located at a lower position on the $(F_2,\widetilde{S_N})$ plane.

(ii)(a) When $\widetilde{S_N}$ is high enough that $p_{1S}=0$, $p_{2S}=p_{2U}=1$ initially (i.e., initial $(F_2,\widetilde{S_N})$ is on or above initial (mU) with $p_{1S}=0$ and below initial (M) with $p_{1S}=0$) or $p_{1S}=0$, $p_{2S}=1, p_{2U}=0$ initially (i.e., initial $(F_2,\widetilde{S_N})$ is on or above initial (mS) with $p_{1S}=0$ and below initial (M) and (mU) with $p_{1S}=0$; occurs only when $\beta>\gamma$ and $N_1>\frac{\beta+\gamma}{2\beta}$) *and* $\widetilde{S_N}$ is relatively high (i.e., $(F_2,\widetilde{S_N})$ is on or above steady-state (M) with $p_{1S}=1$, when $\beta>\gamma$ and $N_1>\frac{\beta+\gamma}{2\beta}$, also on or above steady-state (mU) with $p_{1S}=1$), the society shifts to $p_{1S}=p_{2S}=p_{2U}=1$ eventually from the proof of Proposition 3 (i). H_2 increases after the shift (when $p_{1S}=0$, $p_{2S}=1, p_{2U}=0$ initially, the shift to $p_{1S}=1$, $p_{2S}=1$, $p_{2U}=0$ may occur first; thus, the increase may start earlier) and $H_2^*=1$ from Lemma 3 (ii).

When $\beta>\gamma$, $N_1>\frac{\beta+\gamma}{2\beta}$, $p_{1S}=0$, $p_{2S}=1$, $p_{2U}=0$ initially (i.e., initial $(F_2,\widetilde{S_N})$ is on or above initial (mS) with $p_{1S}=0$ and below initial (M) and (mU) with $p_{1S}=0$), *and* $\widetilde{S_N}$ is relatively, but not very, high (i.e., initial $(F_2,\widetilde{S_N})$ is on or above steady-state (M) with $p_{1S}=1$ and below steady-state (mU) with $p_{1S}=1$), the society shifts to $p_{1S}=1$, $p_{2S}=1$, $p_{2U}=0$ first from the proof of Proposition 3 (i). H_2 starts increasing after the shift and $H_2^*=1$ from Lemma 3 (ii). (Eventually, the shift to $p_{1S}=p_{2S}=p_{2U}=1$ occurs.) The last result can be proved similary to (i)(b). Unlike (i)(b), ω_q does not have an effect since the last term of (M) disappears in the steady state from Proposition 4 (i).

(b) From the proof of Proposition 4 (ii) and (iii), when initial $(F_2,\widetilde{S_N})$ is located below steady-state (M) with $p_{1S}=0$, when $\beta>\gamma$ and $N_1>\frac{\beta+\gamma}{2\beta}$, also below steady-state (mU) with $p_{1S}=0$, $p_{1S}=0$ and thus $\tau=0$ always. Hence, when $H_2=F_2\in(0,1)$ initially, H_2 is time-invariant from $\frac{\lambda}{1-\lambda(1+r)}w_u\leq\overline{e}$. When $H_2=0$ initially, F_2 decreases over time. Since $H_2=0$ holds in subsequent periods from the proof of Proposition 4 (ii), $H_2^*=F_2^*=0$. When $H_2\in(0,F_2)$ initially, which can occur when $\beta<\gamma$, $N_1>\frac{2\beta}{\beta+\gamma}$ and initial F_2 is in the intermediate range from Lemma 2 (ii)(b), F_2 decreases from $\frac{\lambda}{1-\lambda(1+r)}w_u\leq\overline{e}$. After F_2 becomes low enough, $H_2=F_2$ holds and the decrease of F_2 stops from the proof of Lemma 2 (ii)(b). Hence, $H_2^*=F_2^*\in(0,1)$. Because $H_2^{\Diamond\Diamond}(\widetilde{S_N}-\widetilde{S_2})$ changes over time, if $H_2^{\Diamond\Diamond}(\widetilde{S_N}-\widetilde{S_2})$ decreases fast enough and F_2 decreases slow enough, H_2 increases, otherwise, H_2 decreases. The last result holds because steady-state (M) and (mU) (when $\beta>\gamma$ and $N_1>\frac{\beta+\gamma}{2\beta}$) are downward sloping, and as $\widetilde{S_1}$ [$\widetilde{S_2}$] is higher, steady-state (M) [(mU)] is located at a higher position on the $(F_2,\widetilde{S_N})$ plane. ■

Proof of Proposition 6. Figures 4−7 in the proof of Proposition 4 may be helpful to understand the proof. (i) Since F_2 increases even under $p_{2S}=0$, and $H_2=F_2$ from Lemma 2 (ii)(a), $p_{1S}^*=p_{2S}^*=p_{2U}^*=1$ when $\widetilde{S_N}$ is greater than the level at which steady-state (M) with $p_{1S}=1$ intersects with $H_2=1$, i.e., when $\widetilde{S_N}\geq\widetilde{S_1}$. The result on the cultural variable is from Lemma A5 (i).

(ii) When $(F_2,\widetilde{S_N})$ is below initial (mS) with $p_{1S}=0$ and $F_2\geq H_2^{\Diamond}$ in the initial period, i.e., $p_{1S}=p_{2S}=p_{2U}=0$ initially, $H_2=F_2$ from the proof of Lemma 2 (ii)(b). Then, the proof of Proposition 3 (ii) applies, because Lemma A4 (v) is same as Lemma A2 (v) and (mS) is upward-sloping on the $(F_2,\widetilde{S_N})$ plane.

Unlike the constant H_2 case, when $p_{1S}=0$, $p_{2S}=1$, $p_{2U}=0$ or 1 initially or when $H_2=0, p_{1S}=p_{2U}=0$ initially, and $\widetilde{S_N}$ is low, the shift to $p_{1S}=p_{2S}=p_{2U}=0$ and $p^*_{1S}=p^*_{2S}=p^*_{2U}=0$ can occur since F_2 increases over time. Such a shift is possible when $\widetilde{S_N}$ is strictly smaller than the level at which initial (mS) intersects with $H_2=1$, $\widetilde{S_2}+\frac{1}{\gamma\delta}[\rho\chi(1-\chi)+\beta\omega_q](N_1)^2$. When $\widetilde{S_N}<\widetilde{S_2}$, the shift to $p_{1S}=p_{2S}=p_{2U}=0$ occurs for certain because the level of $\widetilde{S_N}$ on steady-state (mS) with $p_{1S}=0$ at $H_2=1$ is weakly greater than $\widetilde{S_2}$, the corresponding level of $\widetilde{S_N}$ when $q^*_{1S}=q^*_{2S}=q^*_{2U}$.

The result on the cultural variable is from Lemma A5 (iii). Unlike the constant H_2 case, $p_{1S}=0$, $p_{2S}=1$, $p_{2U}=0$ or 1 can converge to $p^*_{1S}=p^*_{2S}=p^*_{2U}=0$, thus $q^*_{2S}=q^*_{2U}$ can be greater than 0.

(iii) When $\widetilde{S_N}\geq\widetilde{S_2}+[\frac{1}{\gamma\delta}[\rho\chi(1-\chi)+\beta\omega_q](N_1)^2$ and $\widetilde{S_N}<\widetilde{S_1}$, the proofs of (i) and (ii) do not apply. $p^*_{1S}=0$, $p^*_{2S}=p^*_{2U}=1$ holds, because F_2 increases over time and thus the society starting with $p_{1S}=0$, $p_{2S}=1, p_{2U}=0$ or $H_2=0, p_{1S}=p_{2U}=0$ transits to $p_{1S}=0$, $p_{2S}=p_{2U}=1$ eventually (Figure 6). When $\widetilde{S_N}\geq\widetilde{S_2}$ and $\widetilde{S_N}<\widetilde{S_2}+\frac{1}{\gamma\delta}[\rho\chi(1-\chi)+\beta\omega_q](N_1)^2$ and either $H_2=0, p_{1S}=p_{2U}=0$ or $p_{1S}=0$, $p_{2S}=1$, $p_{2U}=0$ or 1 initially, $p^*_{1S}=0$, $p^*_{2S}=p^*_{2U}=1$ holds if initial $(F_2,\widetilde{S_N})$ is located far above initial (mS) with $p_{1S}=0$ or an increase in H_2 is slow compared to the (long-term) downward shift of (mS) with $p_{1S}=0$. Otherwise, as shown in the proof of (ii), $p^*_{1S}=p^*_{2S}=p^*_{2U}=0$.

The result on the cultural variable is from Lemma A5 (ii). $q^*_{1S}=q^*_{2S}=q^*_{2U}=1$ because the states with $p_{1S}=1$ do not transit to $p_{1S}=0$, $p_{2S}=p_{2U}=1$.

(iv) The result can be proved similarly to Proposition 3 (iv). ■

Proof of Proposition 7. (i) When initial $(F_2,\widetilde{S_N})$ is located on or above initial (M) with $p_{1S}=1$ on the $(F_2,\widetilde{S_N})$ plane, $p_{1S}=1$ and $\tau>0$ initially from the proof of Proposition 3 (i). $p_{1S}=1$ and $\tau>0$ continue to hold from Lemma A4 (ii). Hence, the speed of convergence to $H^*_2=1$ is highest. (ii) (a) When initial $(F_2,\widetilde{S_N})$ is located below initial (M) with $p_{1S}=0$ and $\widetilde{S_N}\geq\widetilde{S_1}$, $p_{1S}=0$ and $\tau=0$ initially from the proof of Proposition 3 (i). $p_{1S}=1$ and $\tau>0$ (thus the convergence to $H^*_2=1$ accelerates) eventually from the proof of Proposition 6 (i). Given F_2, as $\widetilde{S_N}$ is higher, $p_{1S}=1$ is realized earlier and thus convergence to $H^*_2=1$ occurs faster. This is because (M) shifts downward over time or does not shift (when $H_2=0$, $p_{1S}=p_{2U}=0$) from Lemma A4 (iii)$-$(vi). (b) When $\widetilde{S_N}<\widetilde{S_1}$, $p_{1S}=0$ and thus $\tau=0$ always from the proof of Proposition 6 (ii) and (iii). (iii) The result holds because as $\widetilde{S_1}$ and ω_q are lower, (M) is located at a lower position on the $(F_2,\widetilde{S_N})$ plane and thus $\tau>0$ is more likely to hold. ■

Proof of Lemma A1. From (M), (mU), and (mS), the statement of the lemma holds iff $(1–\tau)(1–H_2)$ decreases with H_2 when $p_{1S}=1$. From (16), $(1–\tau)(1–H_2)=\frac{1}{1+\gamma}\left[\beta+\gamma-\frac{(\beta-1)w_s}{\overline{w}}\right](1–H_2)$, thus its derivative with respect to H_2 equals $\frac{1}{1+\gamma}$ times $-\left[\beta+\gamma-\frac{(\beta-1)w_s}{\overline{w}}\right]+(1–H_2)\frac{(\beta-1)w_s}{(\overline{w})^2}(1–N_1)(w_s–w_u)=-\left[\beta+\gamma-(\beta-1)(\frac{w_s}{\overline{w}})^2\right]$. Hence,

$$\frac{d[(1-\tau)(1-H_2)]}{dH_2}<0\Leftrightarrow\frac{1+\gamma}{\beta-1}>\left(\frac{w_s}{\overline{w}}\right)^2-1. \tag{C18}$$

From Assumption 1, the above condition holds if $3>\left(\frac{w_s}{\overline{w}}\right)^2-1\Leftrightarrow\frac{w_s}{\overline{w}}<2$, which is always true because $\frac{w_s}{\overline{w}}\leq\frac{w_s}{N_1w_s+(1–N_1)w_u}<2$ from $N_1>\frac{1}{2}$. ■

Proof of Lemma A2. Because $q^i_{2C}=\overline{q}_{2C}$ $(C=S,U)$ holds in any period under the initial condition $q^i_{2C}=0$ in the model with constant H_2, the notation q_{2C}, not q^i_{2C}, is used. In the following proofs, $q_{2S}\geq\overline{q}_2\geq q_{2U}$ is used, which is from Proposition 1 (i), (13), and (14).

(i) The last term of the RHS of (mS) in the initial period equals $\frac{1}{\gamma\delta}[\rho\chi(1-\chi)+\beta\omega_q](N_1)^2$ from $q_{1S}=1$ and $q_{2S}=q_{2U}=0$. In subsequent periods, $(\overline{q}-\overline{q}_2)[(\overline{q}_2-q_{2S})+(\overline{q}-q_{2S})]\leq(N_1)^2$, because when $(\overline{q}_2-q_{2S})+(\overline{q}-q_{2S})>0$ (the result is straightforward when $(\overline{q}_2-q_{2S})+(\overline{q}-q_{2S})\leq0$),

$(\overline{q}-\overline{q}_2)[(\overline{q}_2-q_{2S})+(\overline{q}-q_{2S})]\leq N_1(1-\overline{q}_2)[N_1(1-\overline{q}_2)-2(q_{2S}-\overline{q}_2)]\leq (N_1)^2$ from $\overline{q}_1\leq 1$ and thus $\overline{q}\leq N_1+(1-N_1)\overline{q}_2$. The last term of the RHS of (M) in the initial period equals $\frac{1}{\gamma\delta}[\rho\chi(1-\chi)+\beta\omega_q](1-N_1)^2$. The expression in subsequent periods is smaller because $\overline{q}_1-\overline{q}=(1-N_1)(\overline{q}_1-\overline{q}_2)\leq 1-N_1$.

(ii) When $p_{1S}=p_{2S}=p_{2U}=1$, $(\overline{q}_J)'=\overline{q}_J+\chi(\overline{q}-\overline{q}_J)$ from (14) and thus $(\overline{q})'=N_1[\overline{q}_1+\chi(\overline{q}-\overline{q}_1)]+(1-N_1)[\overline{q}_2+\chi(\overline{q}-\overline{q}_2)]=\overline{q}$. Hence, $(\overline{q})'-(\overline{q}_J)'=(1-\chi)(\overline{q}-\overline{q}_J)$ and thus (M) shifts downward over time. $(\overline{q}_2)'+(\overline{q})'-2(q_{2C})'=\overline{q}_2+\overline{q}+\chi(\overline{q}-\overline{q}_2)-2[\chi\overline{q}+(1-\chi)q_{2C}]=(1-\chi)(\overline{q}_2+\overline{q}-2q_{2C})$ $(C=S,U)$. Since $\overline{q}_2+\overline{q}-2q_{2U}>0$ from $q_{2S}\geq q_{2U}$, (mU) shifts downward, while (mS) shifts downward (upward) when $\overline{q}_2+\overline{q}-2q_{2S}>(<)0$. As long as $p_{1S}=p_{2S}=p_{2U}=1$ holds, the cultural distance between individuals becomes 0 in the long run from Lemma A3 (i), thus (mS) shifts downward in the long run.

(iii) [When $p_{1S}=0$, $p_{2S}=p_{2U}=1$] $(\overline{q}_1)'=\overline{q}_1$ from (13) and $(\overline{q}_2)'=\overline{q}_2+\chi(\overline{q}-\overline{q}_2)$ from (14). Thus, $(\overline{q})'=N_1\overline{q}_1+(1-N_1)[\overline{q}_2+\chi(\overline{q}-\overline{q}_2)]=\overline{q}+(1-N_1)\chi(\overline{q}-\overline{q}_2)$. Hence, $(\overline{q}_1)'-(\overline{q})'=\overline{q}_1-\overline{q}-(1-N_1)\chi(\overline{q}-\overline{q}_2)$ and thus (M) shifts downward. (mU) shifts downward because $(\overline{q})'-(\overline{q}_2)'=\overline{q}+(1-N_1)\chi(\overline{q}-\overline{q}_2)-[\overline{q}_2+\chi(\overline{q}-\overline{q}_2)]=(1-\chi N_1)(\overline{q}-\overline{q}_2)$ and

$$\begin{aligned}(\overline{q}_2)'+(\overline{q})'-2(q_{2U})' &= \overline{q}_2+\chi(\overline{q}-\overline{q}_2)+\overline{q}+(1-N_1)\chi(\overline{q}-\overline{q}_2)-2[\chi\overline{q}+(1-\chi)q_{2U}]\\ &= \overline{q}_2+\overline{q}-2q_{2U}+\chi[(2-N_1)(\overline{q}-\overline{q}_2)-2(\overline{q}-q_{2U})]\\ &= \overline{q}_2+\overline{q}-2q_{2U}-\chi[N_1(\overline{q}-\overline{q}_2)+2(\overline{q}_2-q_{2U})]\leq\overline{q}+\overline{q}_2-2q_{2U}.\end{aligned}$$

The result that (mS) shifts downward in the long run can be proved similarly to (i).

[When $p_{1S}=1$, $p_{2S}=1$, $p_{2U}=0$] $(\overline{q}_1)'=\chi\overline{q}+(1-\chi)\overline{q}_1$ from (14), $(\overline{q}_2)'=H_2[\chi\overline{q}+(1-\chi)\overline{q}_{2S}]+(1-H_2)[\chi\overline{q}_2+(1-\chi)\overline{q}_{2U}]=\chi H_2\overline{q}+(1-\chi H_2)\overline{q}_2$ from (14) and (13), thus $(\overline{q})'=N_1[\chi\overline{q}+(1-\chi)\overline{q}_1]+(1-N_1)[\chi H_2\overline{q}+(1-\chi H_2)\overline{q}_2]$. Hence,

$$\begin{aligned}(\overline{q}_1)'-(\overline{q})' &= (1-N_1)\{[\chi\overline{q}+(1-\chi)\overline{q}_1]-[\chi H_2\overline{q}+(1-\chi H_2)\overline{q}_2]\}\\ &= \overline{q}_1-\overline{q}-[1-(1-\chi)(1-N_1)](\overline{q}_1-\overline{q})+(1-N_1)(1-\chi H_2)(\overline{q}-\overline{q}_2)\\ &= \{(1-\chi)(1-N_1)+(1-\chi H_2)N_1\}(\overline{q}_1-\overline{q})\\ &= \{1-\chi[1-N_1(1-H_2)]\}(\overline{q}_1-\overline{q})<\overline{q}_1-\overline{q}.\end{aligned}$$

Thus, (M) shifts downward over time.

$$\begin{aligned}(\overline{q})'-(\overline{q}_2)' &= N_1\{[\chi\overline{q}+(1-\chi)\overline{q}_1]-[\chi H_2\overline{q}+(1-\chi H_2)\overline{q}_2]\}\\ &= \overline{q}-\overline{q}_2-(1-N_1+\chi N_1H_2)(\overline{q}-\overline{q}_2)+N_1(1-\chi)(\overline{q}_1-\overline{q})\\ &= \overline{q}-\overline{q}_2-[1-N_1(1-H_2)]\chi N_1(\overline{q}_1-\overline{q}_2).\\ (\overline{q})'+(\overline{q}_2)'-2(q_{2U})' &= N_1[\chi\overline{q}+(1-\chi)\overline{q}_1]+(2-N_1)[\chi H_2\overline{q}+(1-\chi H_2)\overline{q}_2]-2[\chi\overline{q}_2+(1-\chi)q_{2U}]\\ &= (\overline{q}+\overline{q}_2-2q_{2U})+N_1(1-\chi)(\overline{q}_1-\overline{q})-(1-N_1)(\overline{q}-\overline{q}_2)+(2-N_1)\chi H_2(\overline{q}-\overline{q}_2)-2\chi(\overline{q}_2-q_{2U})\\ &= (\overline{q}+\overline{q}_2-2q_{2U})+\{(1-\chi)(1-N_1)-[(1-N_1)-(2-N_1)\chi H_2]\}N_1(\overline{q}_1-\overline{q}_2)-2\chi(\overline{q}_2-q_{2U})\\ &= (\overline{q}+\overline{q}_2-2q_{2U})-\chi\{[(1-N_1)-(2-N_1)H_2]N_1(\overline{q}_1-\overline{q}_2)+2(\overline{q}_2-q_{2U})\}.\end{aligned}$$

From these equations,

$$[(\overline{q})'-(\overline{q}_2)'][(\overline{q})'+(\overline{q}_2)'-2(q_{2U})']$$

$$=(\overline{q}-\overline{q}_2)(\overline{q}+\overline{q}_2-2q_{2U})-\chi N_1(\overline{q}_1-\overline{q}_2)\begin{bmatrix}\{[(1-N_1)-(2-N_1)H_2]N_1(\overline{q}_1-\overline{q}_2)+2(\overline{q}_2-q_{2U})\}\\+[1-N_1(1-H_2)]\begin{pmatrix}(\overline{q}+\overline{q}_2-2q_{2U})\\-\chi\{[(1-N_1)-(2-N_1)H_2]N_1(\overline{q}_1-\overline{q}_2)+2(\overline{q}_2-q_{2U})\}\end{pmatrix}\end{bmatrix}$$

$$=(\overline{q}-\overline{q}_2)(\overline{q}+\overline{q}_2-2q_{2U})-\chi N_1(\overline{q}_1-\overline{q}_2)\begin{pmatrix}\{1-\chi[1-N_1(1-H_2)]\}\{[(1-N_1)-(2-N_1)H_2]N_1(\overline{q}_1-\overline{q}_2)+2(\overline{q}_2-q_{2U})\}\\+[1-N_1(1-H_2)](\overline{q}+\overline{q}_2-2q_{2U})\end{pmatrix}$$

$$=(\overline{q}-\overline{q}_2)(\overline{q}+\overline{q}_2-2q_{2U})-\chi N_1(\overline{q}_1-\overline{q}_2)\begin{pmatrix}\{1-\chi[1-N_1(1-H_2)]\}\{(1-N_1)(1-H_2)N_1(\overline{q}_1-\overline{q}_2)+2(\overline{q}_2-q_{2U})\}\\-\{1-\chi[1-N_1(1-H_2)]\}H_2N_1(\overline{q}_1-\overline{q}_2)\\+[1-N_1(1-H_2)][N_1(\overline{q}_1-\overline{q}_2)+2(\overline{q}_2-q_{2U})]\end{pmatrix}$$

$$=(\overline{q}-\overline{q}_2)(\overline{q}+\overline{q}_2-2q_{2U})-\chi N_1(\overline{q}_1-\overline{q}_2)\begin{pmatrix}\{1-\chi[1-N_1(1-H_2)]\}\{(1-N_1)(1-H_2)N_1(\overline{q}_1-\overline{q}_2)+2(\overline{q}_2-q_{2U})\}\\+\{(1-N_1)(1-H_2)+\chi H_2[1-N_1(1-H_2)]\}N_1(\overline{q}_1-\overline{q}_2)\\+[1-N_1(1-H_2)]2(\overline{q}_2-q_{2U})\end{pmatrix}.$$

Thus, (mU) shifts downward over time.

The result on (mS) can be proved similarly to the result when $p_{1S}=0$ and $p_{2S}=p_{2U}=1$.

(iv) Because $(\overline{q}_1)'=\overline{q}_1$ from (13) and $(\overline{q}_2)'=\chi H_2\overline{q}+(1-\chi H_2)\,\overline{q}_2$ from the proof of (iii) when $p_{1S}=1$, $p_{2S}=1$, $p_{2U}=0$, $(\overline{q})'=\overline{q}+\chi(1-N_1)H_2(\overline{q}-\overline{q}_2)$. Hence, $(\overline{q}_1)'-(\overline{q})'=\overline{q}_1-\overline{q}-\chi(1-N_1)H_2(\overline{q}-\overline{q}_2)$ and thus (M) shifts downward. The result on (mU) and (mS) can be proved similarly to the one on (mS) of (iii).

(v) When $p_{1S}=p_{2S}=p_{2U}=0$, $(\overline{q}_J)'=\overline{q}_J$ $(J=1,2)$ and $(\overline{q})'=\overline{q}$ from (13). Hence, $(\overline{q})'-(\overline{q}_J)'=\overline{q}-\overline{q}_J$ and thus (M) does not shift over time. From (13) and $q_{2S}\geq q_{2U}$,

$$\begin{aligned}(\overline{q})'+(\overline{q}_2)'-2(q_{2U})' &= \overline{q}_2+\overline{q}-2q_{2U}-2\chi(\overline{q}_2-q_{2U})\leq\overline{q}_2+\overline{q}-2q_{2U},\\(\overline{q})'+(\overline{q}_2)'-2(q_{2S})' &= \overline{q}_2+\overline{q}-2q_{2S}+2\chi(q_{2S}-\overline{q}_2)\geq\overline{q}_2+\overline{q}-2q_{2S},\end{aligned}$$

where the first (second) inequality holds with "$<$" ("$>$") unless $q_{2S}=q_{2U}$. Thus, the results on (mU) and (mS) hold. ■

Proof of Lemma A3. (i) When $p^*_{1S}=p^*_{2S}=p^*_{2U}=1$, $q^*_{1S}=q^*_{2S}=q^*_{2U}=\overline{q}^*$ from (14). $\overline{q}^*=\overline{q}^{\#}$ because $(\overline{q})'=\overline{q}$ when $p_{1S}=p_{2S}=p_{2U}=1$ from (14).

(ii) When $p^*_{1S}=p^*_{2S}=1$ and $p^*_{2U}=0$, $q^*_{1S}=q^*_{2S}=q^*_{2U}=\overline{q}^*_1=\overline{q}^*_2$ from (13) and (14). From (14), when $p_{1S}=p_{2S}=1$, $p_{2U}=0$,

$$\begin{aligned}(\overline{q}_1)'&=\chi\,[N_1\overline{q}_1+(1-N_1)\overline{q}_2]+(1-\chi)\overline{q}_1\\&=[\chi N_1+(1-\chi)]\overline{q}_1+\chi(1-N_1)\overline{q}_2,\end{aligned}\tag{C19}$$

and from (14) and (13),

$$\begin{aligned}(\overline{q}_2)'&=H_2[\chi\overline{q}+(1-\chi)\,\overline{q}_{2S}]+(1-H_2)[\chi\overline{q}_2+(1-\chi)\overline{q}_{2U}]\\&=\chi H_2[N_1\overline{q}_1+(1-N_1)\overline{q}_2]+(1-H_2)\overline{q}_2\\&=\chi H_2N_1\overline{q}_1+(1-\chi H_2N_1)\,\overline{q}_2.\end{aligned}\tag{C20}$$

These equations can be expressed as

$$\begin{pmatrix}(\overline{q}_1)'\\(\overline{q}_2)'\end{pmatrix}=\begin{pmatrix}a & 1-a\\ b & 1-b\end{pmatrix}\begin{pmatrix}\overline{q}_1\\\overline{q}_2\end{pmatrix},\tag{C21}$$

where $a\equiv\chi N_1+(1-\chi)$ and $b\equiv\chi H_2N_1$ in this proof, in which $a>b$.

Thus,

$$\begin{pmatrix}\overline{q}^*_1\\\overline{q}^*_2\end{pmatrix}=\lim_{n\to\infty}\begin{pmatrix}a & 1-a\\ b & 1-b\end{pmatrix}^n\begin{pmatrix}\overline{q}^\dagger_1\\\overline{q}^\dagger_2\end{pmatrix},\tag{C22}$$

where

$$
\begin{aligned}
\lim_{n\to\infty}\begin{pmatrix} a & 1-a \\ b & 1-b \end{pmatrix}^n &= \lim_{n\to\infty}\left[\begin{pmatrix} a & 1-a \\ b & 1-b \end{pmatrix}^2\begin{pmatrix} a & 1-a \\ b & 1-b \end{pmatrix}^{n-2}\right] \\
&= \lim_{n\to\infty}\left[\begin{pmatrix} a^2+(1-a)b & (1+a-b)(1-a) \\ (1+a-b)b & (1-b)^2+(1-a)b \end{pmatrix}\begin{pmatrix} a & 1-a \\ b & 1-b \end{pmatrix}^{n-2}\right] \\
&= \lim_{n\to\infty}\left[\begin{pmatrix} a^3+[1+(a-b)+a](1-a)b & \left[1+(a-b)+(a-b)^2\right](1-a) \\ \left[1+(a-b)+(a-b)^2\right]b & (1-b)^3+[1+(a-b)+(1-b)](1-a)b \end{pmatrix}\begin{pmatrix} a & 1-a \\ b & 1-b \end{pmatrix}^{n-3}\right] \\
&= \lim_{n\to\infty}\left[\begin{pmatrix} a^4+\left\{1+(a-b)+(a-b)^2+[1+(a-b)+a]a\right\}(1-a)b & \left[1+(a-b)+(a-b)^2+(a-b)^3\right](1-a) \\ \left[1+(a-b)+(a-b)^2+(a-b)^3\right]b & (1-b)^4+\left\{1+(a-b)+(a-b)^2+[1+(a-b)+(1-b)](1-b)\right\}(1-a)b \end{pmatrix} \times \begin{pmatrix} a & 1-a \\ b & 1-b \end{pmatrix}^{n-4}\right] \\
&= \lim_{n\to\infty}\begin{pmatrix} \sum_{t=0}^{n-1}(a-b)^t b & \sum_{t=0}^{n-1}(a-b)^t(1-a) \\ \sum_{t=0}^{n-1}(a-b)^t b & \sum_{t=0}^{n-1}(a-b)^t(1-a) \end{pmatrix} \quad (\text{because } \overline{q}_1^* = \overline{q}_2^*) \\
&= \frac{1}{1-a+b}\begin{pmatrix} b & 1-a \\ b & 1-a \end{pmatrix}. \qquad \text{(C23)}
\end{aligned}
$$

Hence, $q_{1S}^* = q_{2S}^* = q_{2U}^* = \frac{1}{1-(1-H_2)N_1}\left[H_2 N_1 \overline{q}_1^\dagger + (1-N_1)\overline{q}_2^\dagger\right]$.

(iii) When at least one of p_{2S}^* and p_{2U}^* equals 1, $q_{2C}^* = \overline{q}^*$ must hold for C such that $p_{2C}^* = 1$ from (14) and $q_{2C'}^* = \overline{q}_2^*$ must hold for C' such that $p_{2C'}^* = 0$ from (13). Thus, $q_{2C}^* = q_{2C'}^* = \overline{q}_2^* = \overline{q}_1^*$, which equals $q_{1S}^* = \overline{q}_1^\flat$ from (iv). (iv) When $p_{1S}^* = 0$, $q_{1S}^* = \overline{q}_1^*$ from (13). $\overline{q}_1^* = \overline{q}_1^\flat$ because $(\overline{q}_1)' = \overline{q}_1$ when $p_{1S} = 0$ from (13). The result for q_{2S}^* and q_{2U}^* can be proved similarly. ■

Proof of Lemma A4. As in the proof of Lemma A2, the fact $q_{2S}^i \geq \overline{q}_2 \geq q_{2U}^i$ is used in the proof. (i) The proof of Lemma A2 applies. (ii) When $p_{1S} = p_{2S} = p_{2U} = 1$, since H_2 increases over time (Lemma 3 (ii)),

$$
\begin{aligned}
(\overline{q}_2)' &= (H_2)'(\overline{q}_{2S})' + \left[1-(H_2)'\right](\overline{q}_{2U})' \\
&= H_2[\chi\overline{q}+(1-\chi)\overline{q}_{2S}] + \left[(H_2)'-H_2\right]\left[\chi\overline{q}+(1-\chi)E(q_{2U}^i|C'=S)\right] + \left[1-(H_2)'\right]\left[\chi\overline{q}+(1-\chi)E(q_{2U}^i|C'=U)\right] \\
&= H_2[\chi\overline{q}+(1-\chi)\overline{q}_{2S}] + (1-H_2)[\chi\overline{q}+(1-\chi)\overline{q}_{2U}] \\
&= \chi\overline{q}+(1-\chi)\overline{q}_2.
\end{aligned}
$$

The above equation is the same as in the model with constant H_2, hence the result can be proved as in Lemma A2 (ii). Unlike the model with constant H_2, q_{2S}^i is heterogenous among lineages with different periods of becoming skilled, so (mS) differs for those with different q_{2S}^i. The same result as in the previous model holds for each of the (mS)s. (As will be clear from the proof of Proposition 4, q_{2U}^i is homogenous due to the initial condition $q_{2U}^i = 0$ and the fact that only children of unskilled workers are unskilled.)

(iii) When $p_{1S} = p_{2S} = 1, p_{2U} = 0$, H_2 increases over time from Lemma 3 (ii) and when $p_{1S} = 0, p_{2S} = p_{2U} = 1$, H_2 non-decreases over time from Lemma 2 (ii)(a) and Lemma 3 (ii). Thus, as in the above proof for (ii), it can be shown that the result is same as in the model with constant H_2. (As in (ii), q_{2U}^i is homogenous, which will be clear from the proof of Proposition 4, while q_{2S}^i becomes heterogenous when H_2 increases over time.)

(iv) In the initial period, when $p_{1S} = 0$, $p_{2S} = 1$, $p_{2U} = 0$ is realized in adulthood if $H_2 = F_2$ in childhood (i.e., $(F_2, \widetilde{S_N})$ is on or above initial (mS) with $p_{1S} = 0$ and below initial (M) and (mU) with $p_{1S}=0$), from the proof of Lemma A2 (ii), the realized H_2 is positive and thus $p_{2S}=1$, $p_{2U}=0$ is true under the following cases: If $\widetilde{S_N}-\widetilde{S_2}$ is sufficiently large, $H_2=F_2$; otherwise, when $\beta \geq \gamma$ or $N_1 \leq \frac{2\beta}{\beta+\gamma}$, $H_2 = F_2$ for $F_2 \geq H_2^{\diamondsuit\diamondsuit}(\widetilde{S_N}-\widetilde{S_2})$; when $\beta < \gamma$, $N_1 > \frac{2\beta}{\beta+\gamma}$, and $\widetilde{S_N}-\widetilde{S_2}$ is not small, $H_2=F_2$ for $F_2 < H_2^{\diamondsuit\diamondsuit}(\widetilde{S_N}-\widetilde{S_2})$ and $H_2=H_2^{\diamondsuit\diamondsuit}(\widetilde{S_N}-\widetilde{S_2})$ for $F_2 \geq H_2^{\diamondsuit\diamondsuit}(\widetilde{S_N}-\widetilde{S_2})$. In subsequent periods, q^i_{2S} becomes heterogenous and $p_{2S} \in (0,1), p_{2U}=0$ may hold either when $H_2=F_2$ increases or when $\beta<\gamma$, $N_1 > \frac{2\beta}{\beta+\gamma}$, and $H_2^{\diamondsuit\diamondsuit}(\widetilde{S_N}-\widetilde{S_2})$ changes over time. Taking into account the possibility of becoming $p_{2S} \in (0,1)$,

$$\begin{aligned}
(\overline{q}_2)' &= H_2 p_{2S}[\chi\overline{q}+(1-\chi)\overline{q}_{2S}]+H_2(1-p_{2S})[\chi\overline{q}_2+(1-\chi)\overline{q}_{2S}]+(1-H_2)[\chi\overline{q}_2+(1-\chi)\overline{q}_{2U}] \\
&= \overline{q}_2+\chi[H_2 p_{2S}(\overline{q}-\overline{q}_{2S})+H_2(1-p_{2S})(\overline{q}_2-\overline{q}_{2S})+(1-H_2)(\overline{q}_2-\overline{q}_{2U})] \\
&= \overline{q}_2+\chi H_2 p_{2S}[\overline{q}-\overline{q}_{2S}+(1-H_2)(\overline{q}_{2S}-\overline{q}_{2U})] \\
&= \overline{q}_2+\chi H_2 p_{2S}(\overline{q}-\overline{q}_2)\,.
\end{aligned}$$

Thus, $(\overline{q})'=\overline{q}+\chi(1-N_1)H_2 p_{2S}(\overline{q}-\overline{q}_2)$. Hence, $(\overline{q}_1)'-(\overline{q})'=\overline{q}_1-\overline{q}-\chi(1-N_1)H_2 p_{2S}(\overline{q}-\overline{q}_2)$ and thus (M) shifts downward. The result on (mU) and (mS) can be proved similarly to the model with constant H_2. (As shown in the proof of Lemma 2 (ii)(b), when $\beta<\gamma$, $N_1>\frac{2\beta}{\beta+\gamma}$, and $H_2 \in (0,F_2)$, H_2 may decline, in which case q^i_{2U} too becomes heterogenous and (mU) differs for those with different q^i_{2U}, but $p_{2U}=0$ is always true.)

(v) When $p_{1S}=p_{2S}=p_{2U}=0$ is realized in adulthood if $H_2=F_2$ in childhood (i.e., $(F_2,\widetilde{S_N})$ is below initial (mS) with $p_{1S}=0$), $H_2=F_2$ is true for $F_2 \geq H_2^{\diamondsuit}$ from the proof of Lemma 2 (ii)(b). The result can be proved as in Lemma A2 (v). (As in (ii), q^i_{2U} is homogenous, while q^i_{2S} becomes heterogenous when H_2 increases over time.)

(vi) When $p_{1S}=p_{2S}=p_{2U}=0$ holds with $H_2=F_2$, $H_2=0$ and $p_{1S}=p_{2U}=0$ are true for $F_2<H_2^{\diamondsuit}$ if $\beta \geq \gamma$ or $N_1 \leq \frac{2\beta}{\beta+\gamma}$, otherwise, they are true for $F_2<H_2^{\diamondsuit}$ when $\widetilde{S_N}-\widetilde{S_2}$ is small from the proof of Lemma 2 (ii)(b). When $p_{1S}=0$, $p_{2S}=1$, $p_{2U}=0$ holds with $H_2=F_2$, $H_2=0$ and $p_{1S}=p_{2U}=0$ are realized for $F_2<H_2^{\diamondsuit\diamondsuit}(\widetilde{S_N}-\widetilde{S_2})$ when $\beta \geq \gamma$ or $N_1 \leq \frac{2\beta}{\beta+\gamma}$ from the proof of the lemma. As shown in the proof, shifts from other states to $H_2=0$ and $p_{1S}=p_{2U}=0$ do not occur. Hence, $q^i_1=1$ and $q^i_2=0$ for any i hold for any period, thus (M), (mU), and (mS) do not shift. ■

Proof of Lemma A5. Proofs are provided only for results different from Lemma A3. (i) When $\frac{\lambda}{1-\lambda(1+r)}w_u \leq \overline{e}$, $p^*_{1S}=p^*_{2S}=p^*_{2U}=1$ when $p_{1S}=p_{2S}=p_{2U}=1$ or $p_{1S}=p_{2S}=1, p_{2U}=0$ initially, and $p^*_{1S}=p^*_{2S}=p^*_{2U}=1$ may hold when $p_{1S}=0, p_{2S}=1, p_{2U}=0$ or 1 initially from the proof of Proposition 4 (i). When $p_{1S}=p_{2S}=p_{2U}=1$ initially, $p_{1S}=p_{2S}=p_{2U}=1$ always from the proof. Hence, $(\overline{q})'=\overline{q}$, thus, under the initial condition $q_{1S}=1, q_{2S}=q_{2U}=0$, $q^*_{1S}=q^*_{2S}=q^*_{2U}=\overline{q}^*=N_1$ holds. In other cases, after the society shifts to $p_{1S}=p_{2S}=p_{2U}=1$, $(\overline{q})'=N_1[\chi\overline{q}+(1-\chi)\overline{q}_1]+(1-N_1)(\chi\overline{q}+(1-\chi)\{H_2\overline{q}_{2S}+(1-H_2)\overline{q}_{2U}\})=\overline{q}$. Thus, $q^*_{1S}=q^*_{2S}=q^*_{2U}=\overline{q}^*=\overline{q}^{\#}\in(0,1)$.

When $\frac{\lambda}{1-\lambda(1+r)}w_u>\overline{e}$, $p^*_{1S}=p^*_{2S}=p^*_{2U}=1$ when $\widetilde{S_N}>\widetilde{S_1}$ from Proposition 6 (i). Thus, from Figures 4–7, unless $\beta>\gamma$ and $N_1>\frac{\beta+\gamma}{2\beta}$, $p_{2S}=p_{2U}=1$ always. When $\beta>\gamma$ and $N_1>\frac{\beta+\gamma}{2\beta}$, $p_{1S}=0$ or $1, p_{2S}=1, p_{2U}=0$ may converge to $p^*_{1S}=p^*_{2S}=p^*_{2U}=1$ from Figure 6. As with the case $\frac{\lambda}{1-\lambda(1+r)}w_u \leq \overline{e}$, if $p_{1S}=p_{2S}=p_{2U}=1$ initially, $q^*_{1S}=q^*_{2S}=q^*_{2U}=\overline{q}^*=N_1$, otherwise, $q^*_{1S}=q^*_{2S}=q^*_{2U}=\overline{q}^*=\overline{q}^{\#}\in(0,1)$.

(iii) When $p^*_{2C}=0$ $(C=S,U)$, $q^*_{2C}=\overline{q}^*_2$ from (13). $\overline{q}^*_2 \geq \overline{q}^{\flat}_2$ because $(\overline{q}_2)' \geq \overline{q}_2$ from the proof of Lemma A4 (v). ■