Appendix D: Proofs of Propositions A1 and A2

This Appendix presents proofs of Propositions A1 and A2 in Appendix A.2 of the paper.

Proof of Proposition A1. Net aggregate income is computed from L_h, L_m, and wages of Propositions 1 and 2 and (15), and average utility is from net aggregate income and (15).

(i) When $F_h+F_m<\frac{(1-\gamma_B)\theta A_T-\gamma_B(1+r)B}{\left[\gamma_B\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+(1-\gamma_B)\theta A_T\right]}$, $NI=\frac{1}{1-\gamma_B}\left[\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(F_h+F_m)+(1+r)B\right]$ and thus it increases with F_h+F_m and B. Average utility equals

$$(\gamma_B)^{\gamma_B}(\gamma_N)^{\gamma_N}(\gamma_b)^{\gamma_b}\left\{\frac{\frac{\gamma_B}{1-\gamma_B}[\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(F_h+F_m)+(1+r)B]}{A_T(1-F_h-F_m)}\right\}^{-\gamma_B}\frac{1}{1-\gamma_B}\left[\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(F_h+F_m)+(1+r)B\right]$$

$$=\frac{(\gamma_N)^{\gamma_N}(\gamma_b)^{\gamma_b}}{(1-\gamma_B)^{1-\gamma_B}}\left[A_T(1-F_h-F_m)\right]^{\gamma_B}\left[\widetilde{w_m}((\tfrac{F_h}{F_m})_{hm})(F_h+F_m)+(1+r)B\right]^{1-\gamma_B}, \quad \text{(D1)}$$

the derivative of which with respect to F_h+F_m equals the average utility times

$$-\frac{\gamma_B}{1-F_h-F_m}+\frac{(1-\gamma_B)\widetilde{w_m}((\frac{F_h}{F_m})_{hm})}{\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(F_h+F_m)+(1+r)B}=\frac{\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(1-\gamma_B-F_h-F_m)-\gamma_B(1+r)B}{(1-F_h-F_m)[\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(F_h+F_m)+(1+r)B]}, \quad \text{(D2)}$$

where, from $F_h+F_m<\frac{(1-\gamma_B)\theta A_T-\gamma_B(1+r)B}{\gamma_B\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+(1-\gamma_B)\theta A_T}$, the numerator of the expression is greater than

$$\frac{[(1-\gamma_B)\widetilde{w_m}((\frac{F_h}{F_m})_{hm})-\gamma_B(1+r)B][\gamma_B\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+(1-\gamma_B)\theta A_T]-\widetilde{w_m}((\frac{F_h}{F_m})_{hm})[(1-\gamma_B)\theta A_T-\gamma_B(1+r)B]}{\gamma_B\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+(1-\gamma_B)\theta A_T}$$

$$=\frac{[\widetilde{w_m}((\frac{F_h}{F_m})_{hm})-\theta A_T]\gamma_B(1-\gamma_B)[\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+(1+r)B]}{\gamma_B\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+(1-\gamma_B)\theta A_T}>0. \quad \text{(D3)}$$

Hence the average utility too increases with F_h+F_m and B. When $F_h+F_m\geq\frac{(1-\gamma_B)\theta A_T-\gamma_B(1+r)B}{\gamma_B\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+(1-\gamma_B)\theta A_T}$, $NI=\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(F_h+F_m)+\theta A_T(1-F_h-F_m)+(1+r)B$ and average utility equals ${\gamma_B}^{\gamma_B}{\gamma_N}^{\gamma_N}{\gamma_b}^{\gamma_b}(\theta)^{-\gamma_B}NI$. Thus, they increase with F_h+F_m and B.

(ii) (a) When $P(F_h,F_m,B)\leq\theta$, $NI=\frac{1}{1-\gamma_B}[A_M(F_h)^{\alpha}(F_m)^{1-\alpha}+(1+r)(B-e_hF_h-e_mF_m)]$ and thus it increases with F_h, F_m, and B. Average utility equals

$$\frac{(\gamma_N)^{\gamma_N}(\gamma_b)^{\gamma_b}}{(1-\gamma_B)^{1-\gamma_B}}\left[A_T(1-F_h-F_m)\right]^{\gamma_B}\left[A_M(F_h)^{\alpha}(F_m)^{1-\alpha}+(1+r)(B-e_hF_h-e_mF_m)\right]^{1-\gamma_B}, \quad \text{(D4)}$$

the derivative of which with respect to $F_i(i=h,m)$ equals the average utility times

$$-\frac{\gamma_B}{1-F_h-F_m}+\frac{(1-\gamma_B)\widetilde{w}_i(\frac{F_h}{F_m})}{A_M(F_h)^{\alpha}(F_m)^{1-\alpha}+(1+r)(B-e_hF_h-e_mF_m)}\geq\frac{\gamma_B}{1-F_h-F_m}[-1+\frac{\widetilde{w}_i(\frac{F_h}{F_m})}{\theta A_T}]>0, \quad \text{(D5)}$$

where the first inequality is from $P(F_h,F_m,B)\leq\theta\Leftrightarrow\frac{\gamma_B}{1-\gamma_B}\frac{A_M(F_h)^{\alpha}(F_m)^{1-\alpha}+(1+r)(B-e_hF_h-e_mF_m)}{A_T[1-F_h-F_m]}\leq\theta$. Hence, the average utility too increases with F_h, F_m, and B. When $P(F_h,F_m,B)>\theta$ and thus $P=\theta$, $NI=A_M(F_h)^{\alpha}(F_m)^{1-\alpha}+(1+r)(B-e_hF_h-e_mF_m)+\theta A_T(1-F_h-F_m)$ and average utility equals $(\gamma_B)^{\gamma_B}(\gamma_N)^{\gamma_N}(\gamma_b)^{\gamma_b}(\theta)^{-\gamma_B}NI$. Thus, they increase with F_h, F_m, and B.

(b) 1. When $F_m\geq\phi(F_h,B)F_h$, $NI=\widetilde{w_h}([\phi(F_h,B)]^{-1})F_h+\widetilde{w_m}([\phi(F_h,B)]^{-1})(1-F_h)+(1+r)B$. The derivative of NI with respect to F_h equals

$$\widetilde{w_h}([\phi(F_h,B)]^{-1}) - \widetilde{w_m}([\phi(F_h,B)]^{-1}) - \frac{\widetilde{w_h}'([\phi(F_h,B)]^{-1})F_h + \widetilde{w_m}'([\phi(F_h,B)]^{-1})(1-F_h)}{[\phi(F_h,B)]^2}\frac{\partial\phi}{\partial F_h}, \quad \text{(D6)}$$

where $\widetilde{w_h}'([\phi(F_h,B)]^{-1})F_h + \widetilde{w_m}'([\phi(F_h,B)]^{-1})(1-F_h) = \alpha(1-\alpha)A_M([\phi(F_h,B)]^{-1})^{\alpha-1}[1-F_h-\phi(F_h,B)F_h] > 0$ (D7)

and thus the derivative is positive. Similarly, the derivative of NI with respect to B equals $-\left[\widetilde{w_h}'([\phi(F_h,B)]^{-1})F_h + \widetilde{w_m}'([\phi(F_h,B)]^{-1})(1-F_h)\right][\phi(F_h,B)]^{-2}\frac{\partial\phi}{\partial B} + (1+r) > 0$.

Since $P = \frac{\widetilde{w_m}([\phi(F_h,B)]^{-1})}{A_T}$, average utility equals

$$(\gamma_B A_T)^{\gamma_B}\gamma_N^{\gamma_N}\gamma_b^{\gamma_b}\left[\widetilde{w_m}([\phi(F_h,B)]^{-1})\right]^{-\gamma_B}\left\{\widetilde{w_h}([\phi(F_h,B)]^{-1})F_h + \widetilde{w_m}([\phi(F_h,B)]^{-1})(1-F_h) + (1+r)B\right\}. \quad \text{(D8)}$$

The derivative with respect to F_h equals the average utility times

$$-\left[-\gamma_B\frac{\widetilde{w_m}'([\phi(F_h,B)]^{-1})}{\widetilde{w_m}([\phi(F_h,B)]^{-1})} + \frac{\widetilde{w_h}'([\phi(F_h,B)]^{-1})F_h + \widetilde{w_m}'([\phi(F_h,B)]^{-1})(1-F_h)}{\widetilde{w_h}([\phi(F_h,B)]^{-1})F_h + \widetilde{w_m}([\phi(F_h,B)]^{-1})(1-F_h) + (1+r)B}\right][\phi(F_h,B)]^{-2}\frac{\partial\phi}{\partial F_h}$$
$$+ \frac{\widetilde{w_h}([\phi(F_h,B)]^{-1}) - \widetilde{w_m}([\phi(F_h,B)]^{-1})}{\widetilde{w_h}([\phi(F_h,B)]^{-1})F_h + \widetilde{w_m}([\phi(F_h,B)]^{-1})(1-F_h) + (1+r)B}, \quad \text{(D9)}$$

where the expression inside the big square bracket of the first term equals ($\phi \equiv \phi(F_h,B)$) $\frac{1}{\widetilde{w_m}(\phi^{-1})[\widetilde{w_h}(\phi^{-1})F_h + \widetilde{w_m}([\phi^{-1})(1-F_h) + (1+r)B]}$ times

$$-\gamma_B\widetilde{w_m}'(\phi^{-1})\left[\widetilde{w_h}(\phi^{-1})F_h + \widetilde{w_m}(\phi^{-1})(1-F_h) + (1+r)B\right] + \widetilde{w_m}(\phi^{-1})\left[\widetilde{w_h}'(\phi^{-1})F_h + \widetilde{w_m}'(\phi^{-1})(1-F_h)\right]$$
$$= -\widetilde{w_m}'(\phi^{-1})\left[1-(1+\phi)F_h\right]\widetilde{w_m}(\phi^{-1}) + \left[\widetilde{w_h}'(\phi^{-1})F_h + \widetilde{w_m}'(\phi^{-1})(1-F_h)\right]\widetilde{w_m}(\phi^{-1}) \text{ (from eq. 14)}$$
$$= [\widetilde{w_h}'(\phi^{-1}) + \widetilde{w_m}'(\phi^{-1})\phi]F_h\widetilde{w_m}(\phi^{-1}) = 0.$$

Hence, the derivative is positive. The derivative with respect to B can be proved to be positive similarly. When $F_m < \phi(F_h,B)F_h$, the proof of (ii)(a) when $P(F_h,F_m,B) \le \theta$ applies.

2. $NI = \widetilde{w_h}((\frac{F_h}{F_m})_{ml,\theta})F_h + \theta A_T(1-F_h) + (1+r)B$ and average utility equals $\gamma_B^{\gamma_B}\gamma_N^{\gamma_N}\gamma_b^{\gamma_b}(\theta)^{-\gamma_B}NI$. Thus, they increase with F_h and B. ■

Proof of Proposition A2. Y and Y_M are computed from equilibrium L_h and L_m (Proposition 1), (6), and (16). Since $PC_B = \gamma_B NI$ and $C_{BM} = \gamma_B NI - \theta A_T[1-(L_h+L_m)]$, the result on $\frac{C_{BM}}{PC_B} = \gamma_B - \theta A_T\frac{1-(L_h+L_m)}{NI}$ is obtained from Propositions 1 and A1.

(i) When $F_h + F_m < \frac{(1-\gamma_B)\theta A_T - \gamma_B(1+r)B}{\left[\gamma_B\widetilde{w_m}((\frac{F_h}{F_m})_{hm}) + (1-\gamma_B)\theta A_T\right]}$,

$Y = A_M\frac{((\frac{F_h}{F_m})_{hm})^\alpha}{1+(\frac{F_h}{F_m})_{hm}}(F_h+F_m) + \frac{\gamma_B}{1-\gamma_B}\left[\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(F_h+F_m) + (1+r)B\right]$. Thus, Y increases with F_h+F_m and B, and $\frac{Y_M}{Y}$ increases with $\frac{F_h+F_m}{B}$. When $F_h+F_m \ge \frac{(1-\gamma_B)\theta A_T - \gamma_B(1+r)B}{\gamma_B\widetilde{w_m}((\frac{F_h}{F_m})_{hm}) + (1-\gamma_B)\theta A_T}$, $Y = A_M\frac{((\frac{F_h}{F_m})_{hm})^\alpha}{1+(\frac{F_h}{F_m})_{hm}}(F_h+F_m) + \theta A_T(1-F_h-F_m)$, where the first term is Y_M. Thus, Y and $\frac{Y_M}{Y}$ increase with F_h+F_m. $\frac{C_{BM}}{PC_B} = \gamma_B - \theta A_T\frac{1-(F_h+F_m)}{NI}$ and thus it increases with F_h+F_m and B.

(ii)(a) When $P(F_h,F_m,B) \le \theta$, $Y = A_M(F_h)^\alpha(F_m)^{1-\alpha} + \frac{\gamma_B}{1-\gamma_B}[A_M(F_h)^\alpha(F_m)^{1-\alpha} + (1+r)(B - e_hF_h - e_mF_m)]$, where the first term is Y_M. Thus, Y increases with F_h, F_m, and B, and $\frac{Y_M}{Y}$ increases with F_h and F_m and decreases with B. When $P(F_h,F_m,B) > \theta$ and thus $P = \theta$, $Y = A_M(F_h)^\alpha(F_m)^{1-\alpha} + \theta A_T(1-F_h-F_m)$, where the first term is Y_M. Thus, Y and $\frac{Y_M}{Y}$ increase with F_h and F_m. $\frac{C_{BM}}{PC_B} = \gamma_B - \theta A_T\frac{1-(F_h+F_m)}{NI}$ and thus it increases with F_h, F_m, and B.

(b) 1. $Y = A_M(\phi(F_h,B))^{1-\alpha}F_h + \frac{\gamma_B}{1-\gamma_B}\{A_M(\phi(F_h,B))^{1-\alpha}F_h + (1+r)[B-(e_h+\phi(F_h,B)e_m)F_h]\}$, where the first term is Y_M. The derivative of Y with respect to F_h equals ($\phi \equiv \phi(F_h,B)$)

$$\begin{aligned}
&\frac{1}{1-\gamma_B}[A_M(\phi)^{1-\alpha} - \gamma_B(1+r)(e_h+\phi e_m)] + \frac{1}{1-\gamma_B}[(1-\alpha)A_M(\phi)^{-\alpha} - \gamma_B(1+r)e_m]F_h\frac{\partial\phi}{\partial F_h} \\
&= \frac{1}{1-\gamma_B}[(1-\alpha)A_M(\phi)^{-\alpha} - \gamma_B(1+r)e_m](\phi + F_h\frac{\partial\phi}{\partial F_h}) + \frac{1}{1-\gamma_B}[\alpha A_M(\phi)^{1-\alpha} - \gamma_B(1+r)e_h] \\
&> \frac{1}{1-\gamma_B}[\widetilde{w_m}(\phi^{-1})(\phi + F_h\frac{\partial\phi}{\partial F_h}) + \widetilde{w_h}(\phi^{-1})]. \qquad \text{(D10)}
\end{aligned}$$

In the above equation, from $(1-\alpha)A_M(\phi)^{-\alpha} - (1+r)e_m = \frac{\gamma_B}{1-\gamma_B}\frac{A_M(\phi)^{1-\alpha}F_h + (1+r)[B-(e_h+\phi e_m)F_h]}{1-(1+\phi)F_h}$ (eq. 42 in the proof of Lemma A1),

$$\begin{aligned}
\frac{\partial\phi}{\partial F_h} &= -\frac{(1+\phi)\left[(1-\alpha)A_M(\phi)^{-\alpha} - (1+r)e_m\right] + \frac{\gamma_B}{1-\gamma_B}\left[A_M(\phi)^{1-\alpha} - (1+r)(e_h+\phi e_m)\right]}{\frac{1}{1-\gamma_B}\left[(1-\alpha)A_M(\phi)^{-\alpha} - (1+r)e_m\right]F_h + [\alpha(1-\alpha)A_M(\phi)^{-\alpha-1}]\left[1-(1+\phi)F_h\right]} \\
&= -\frac{(1+\phi)\widetilde{w_m}(\phi^{-1}) + \frac{\gamma_B}{1-\gamma_B}\left[\widetilde{w_h}(\phi^{-1}) + \phi\widetilde{w_m}(\phi^{-1})\right]}{\frac{1}{1-\gamma_B}\widetilde{w_m}(\phi^{-1})F_h + [\alpha(1-\alpha)A_M(\phi)^{-\alpha-1}]\left[1-(1+\phi)F_h\right]}. \qquad \text{(D11)}
\end{aligned}$$

$\widetilde{w_m}(\phi^{-1})(\phi + F_h\frac{\partial\phi}{\partial F_h}) + \widetilde{w_h}(\phi^{-1})$ in (D10) thus equals $\frac{1}{\frac{\widetilde{w_m}(\phi^{-1})F_h}{1-\gamma_B} + [\alpha(1-\alpha)A_M(\phi)^{-\alpha-1}][1-(1+\phi)F_h]}$ times

$$\begin{aligned}
&\left[\widetilde{w_h}(\phi^{-1}) + \phi\widetilde{w_m}(\phi^{-1})\right]\left\{\frac{1}{1-\gamma_B}\widetilde{w_m}(\phi^{-1})F_h + \left[\alpha(1-\alpha)A_M(\phi)^{-\alpha-1}\right]\left[1-(1+\phi)F_h\right]\right\} \\
&- \left\{(1+\phi)\widetilde{w_m}(\phi^{-1}) + \frac{\gamma_B}{1-\gamma_B}\left[\widetilde{w_h}(\phi^{-1}) + \phi\widetilde{w_m}(\phi^{-1})\right]\right\}\widetilde{w_m}(\phi^{-1})F_h \\
&= \left[\widetilde{w_h}(\phi^{-1}) + \phi\widetilde{w_m}(\phi^{-1})\right]\left\{\widetilde{w_m}(\phi^{-1})F_h + \left[\alpha(1-\alpha)A_M(\phi)^{-\alpha-1}\right]\left[1-(1+\phi)F_h\right]\right\} - (1+\phi)\widetilde{w_m}(\phi^{-1})\widetilde{w_m}(\phi^{-1})F_h \\
&= \left[\widetilde{w_h}(\phi^{-1}) + \phi\widetilde{w_m}(\phi^{-1})\right]\alpha(1-\alpha)A_M(\phi)^{-\alpha-1}\left[1-(1+\phi)F_h\right] + (\widetilde{w_h}(\phi^{-1}) - \widetilde{w_m}(\phi^{-1}))\widetilde{w_m}(\phi^{-1})F_h > 0.
\end{aligned}$$

The derivative of Y with respect to B equals

$$\frac{\gamma_B(1+r)}{1-\gamma_B} + \frac{1}{1-\gamma_B}[(1-\alpha)A_M\phi^{-\alpha} - \gamma_B(1+r)e_m]F_h\frac{\partial\phi}{\partial B} > \frac{1}{1-\gamma_B}\left[\widetilde{w_m}(\phi^{-1})F_h\frac{\partial\phi}{\partial B} + \gamma_B(1+r)\right]. \qquad \text{(D12)}$$

In the above equation, from $(1-\alpha)A_M(\phi)^{-\alpha} - (1+r)e_m = \frac{\gamma_B}{1-\gamma_B}\frac{A_M(\phi)^{1-\alpha}F_h + (1+r)[B-(e_h+\phi e_m)F_h]}{1-(1+\phi)F_h}$,

$$\frac{\partial\phi}{\partial B} = -\frac{\frac{\gamma_B}{1-\gamma_B}(1+r)}{\frac{1}{1-\gamma_B}\widetilde{w_m}(\phi^{-1})F_h + [\alpha(1-\alpha)A_M(\phi)^{-\alpha-1}]\left[1-(1+\phi)F_h\right]}. \qquad \text{(D13)}$$

Thus,

$$\widetilde{w_m}(\phi^{-1})F_h\frac{\partial\phi}{\partial B} + \gamma_B(1+r) = \frac{\gamma_B(1+r)}{1-\gamma_B}\frac{\frac{\gamma_B}{1-\gamma_B}\widetilde{w_m}(\phi^{-1})F_h + [\alpha(1-\alpha)A_M(\phi)^{-\alpha-1}][1-(1+\phi)F_h]}{\frac{1}{1-\gamma_B}\widetilde{w_m}(\phi^{-1})F_h + [\alpha(1-\alpha)A_M(\phi)^{-\alpha-1}][1-(1+\phi)F_h]} > 0. \qquad \text{(D14)}$$

Hence, Y increases with F_h and B. Since $\frac{Y_M}{Y} = \left(1 + \frac{\gamma_B}{1-\gamma_B}\left\{1 + (1+r)\frac{B-(e_h+\phi(F_h,B)e_m)F_h}{A_M(\phi(F_h,B))^{1-\alpha}F_h}\right\}\right)^{-1}$, $\frac{Y_M}{Y}$ decreases with B, but the effect of F_h is ambiguous.

2. $Y = A_M[(\frac{F_h}{F_m})_{ml,\theta}]^{\alpha-1}F_h + \theta A_T(1 - \{1 + [(\frac{F_h}{F_m})_{ml,\theta}]\}^{-1}F_h)$. Thus, Y and $\frac{Y_M}{Y}$ increase with F_h. $\frac{C_{BM}}{PC_B} = \gamma_B - \frac{\theta A_T}{NI}(1 - \{1 + [(\frac{F_h}{F_m})_{ml,\theta}]\}^{-1}F_h)$, which increases with F_h and B. ■