Fertility, childbearing age, and education

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Abstract

This paper develops a simple, analytically tractable model of fertility and education in which individuals live for three periods. A distinguishing feature of the model is that adulthood consists of two periods, and individuals can have children in both periods. The periods differ in that parents can have a choice on their own education (corresponding to higher education) in the first period of adulthood, while in the second period, they do not have such an option and instead face higher costs of rearing and education of their children.

We examine how well the model can explain the major changes in fertility and education since the early 20th century in countries such as the U.S. and the U.K. qualitatively. The model effectively accounts for trends in fertility among younger and older mothers, total fertility, and the secondary and higher education of young cohorts, with parameter changes capturing possible factors that contributed to the evolution of fertility rates, as proposed in the literature or suggested by empirical evidence.

Keywords: fertility rate, childbearing age, timing of birth, education.

JEL-classification: I25, J13, O11.

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1 Introduction

Demographic transition refers to the historical shift from high fertility and infant mortality rates in societies with low levels of technology, education (especially among women), and economic development to low fertility and mortality rates in societies with advanced technology, education, and development. Actual fertility changes, however, are more complex: fertility can fluctuate greatly over relatively short periods (i.e., during the Baby Boom and "Baby Bust") and can exhibit different movements across age groups (i.e., increased fertility among older mothers and decreased fertility among younger mothers in recent years).

Theoretical analyses of the interactions between economic development and fertility behavior mainly focus on two aspects of fertility: the total fertility rate and childbearing age. Previous works develop models in which changes in the total fertility rate result from factors such as decreases in child mortality (Doepke, 2005), technological change that increases the returns to education (Galor and Weil, 2000), and innovations in contraceptive technology (Strulik, 2017). Changes in childbearing age are explained through channels such as changes in the opportunity cost of starting a family (Happel, Hill and Low, 1984) and career planning of women (Gustafsson, 2001). Several studies, including Cigno and Ermisch (1989) and Pestieau and Ponthiere (2015), consider the two aspects of fertility at the same time. Cigno and Ermisch (1989) build a microeconomic model with multiple childbearing periods and show numerically that women with more human capital at the time of marriage have children later and contribute to low fertility rate. Pestieau and Ponthiere (2015) develop a dynamic macroeconomic model, analytically examine the effects of parameter changes on steady-state fertility and capital accumulation, and show numerically that the model can explain the observed delay of childbearing through a rise in childbearing costs and a decline in family altruism.[1]

In this work, besides the two aspects of fertility, we also consider education and its interactions with the fertility variables. The main objective of this paper is to develop a simple, analytically tractable model and examine how well it can explain the major changes in total fertility, fertility among younger and older mothers, and education since the early 20th century in developed countries qualitatively. There are two reasons for considering the interactions between education and fertility.

First, empirical studies suggest that education affects fertility decisions. Murphy (2015), based on département-level panel data of France for the last quarter of the 19th century, finds that literacy, particularly of women, is negatively correlated with fertility. Hansen, Jensen, and Lønstrup (2018), using U.S. state-level panel data between 1850 and 1980, find that increased years of education lowers fertility. DeCicca and Krashinsky (2023), using census data from the U.S. and Canada, show that an increase in women's years of education raises their income and delays marriage, causing a substantial decline in the likelihood of having more than three children.

Second, theoretical works, such as Happel, Hill and Low (1984), Cigno and Ermisch (1989), and Gustafsson (2001), show how fertility decisions are related to education (human capital). A low level of education implies a low relative cost of early childbearing, leading to earlier births. As educational investment increases, the relative cost of early

[1] Momota and Horii (2013) develop an OLG model with exogenous fertility and show that sufficiently late childbearing causes fluctuations in the age composition of workers, generating cycles in the capital–labor ratio and lifetime welfare. Instead of examining how fertility and childbearing age change over time, given these variables, they analyze the impact of delayed childbearing on capital accumulation.

childbearing rises, delaying childbearing. When the level of education becomes sufficiently high, the decrease in early childbearing outweighs the increase in late childbearing, resulting in a fall in the total fertility rate.

Iyigun (2000) and d'Albis, Greulich, and Ponthiere (2018) examine education and its interactions with the total fertility rate and childbearing age. Iyigun (2000) focuses on the postponement of childbearing, while d'Albis, Greulich, and Ponthiere (2018) focus on the long-run decrease in the total fertility rate and the U-shaped trend of childbearing age (decreases first and then increases).

However, the observed fertility trends in developed countries are more complex. In countries such as the U.S. and the U.K., four distinct fertility phases can be identified: the period of large fertility decline from no later than the early 1920s until the early 1930s; the Baby Boom from the early 1940s until the early 1960s; the period of "Baby Bust" until the mid-1970s; and the period of rising fertility of older mothers and falling fertility of younger mothers afterwards. Although these works replicate some parts of these trends, they do not consider the remaining parts.

In order to explain the major trends of fertility and education since the early 20th century, this paper develops a simple, analytically tractable model of fertility and education in which individuals live for three periods. A distinguishing feature of the model is that adulthood consists of two periods, and individuals can have children in both periods. The periods differ in that parents can have a choice on their own education (corresponding to higher education) in the first period of adulthood, while in the second period, they do not have such an option and instead face higher costs of rearing and education of their children than in the first period. Because of these differences, parents choose different numbers of children in the two periods depending on exogenous variables and parameters, even though their utility does not directly depend on the timing of their children's births.

We examine how well the model can explain the major changes in fertility and education in countries such as the U.S. and the U.K. qualitatively. The model effectively accounts for trends in fertility among younger and older mothers, total fertility, and the secondary and higher education of young cohorts, with parameter changes capturing possible factors that contributed to the evolution of fertility rates, as proposed in the literature or suggested by empirical evidence.

First, during the period roughly from the early 1920s to the early 1930s, data show that age-specific and total fertility rates fell sharply. At the same time, the average years of secondary education of young generations rose greatly, while their average years of tertiary education (especially for women) changed little. Evidence suggests that the demand for skilled workers, corresponding to high school graduates at the time, increased significantly (Goldin and Katz, 1998). When the model captures the increased skill demand by an increase in the returns to childhood education, it can explain these facts.

Second, during the period of Baby Boom roughly from the early 1940s to the early 1960s, both age-specific and total fertility rates rose greatly, while the average years of secondary and tertiary education of young generations rose moderately. Researchers argue that the Baby Boom was caused by the increased productivity of the household sector from the diffusion of home appliances (Greenwood, Seshadri, and Vandenbroucke, 2005), a substantial decline in maternal mortality, which is a proxy for the overall maternal cost of having children (Albanesi and Olivetti, 2014), and a decrease in per-space housing costs due to suburbanization (Tamura and Simon, 2017). When the model captures the increased productivity of the household sector by a decrease in the fixed time spent on household chores, it can explain these facts except the moderate increase in years of secondary education. When the model captures the decreased maternal cost of childbearing mainly by a decrease in the relative cost of late childbearing, it can account for the observed patterns except the increases in the fertility of older mothers and years of secondary education. When the model represents the decreased housing costs by a decrease in the cost of childrearing, it can explain the facts except the increased years of secondary and tertiary education. Overall, the model can explain these facts except the moderate increase in years of secondary education, provided that the effects of the latter two factors are not very large.

Third, for the period of "Baby Bust" roughly from the early or mid-1960s to the mid-1970s, age-specific and total fertility rates fell greatly, while the average years of secondary and tertiary education among young cohorts rose moderately. Researchers argue that the labor demand for young women and their relative wage increased during this period, with WWII (Doepke, Hazan and Maoz, 2015) or the Great Depression (Bellou and Cardia, 2014) as the ultimate cause. When the model represents the increased relative wage of young women by a rise in the opportunity cost of childrearing, it can explain these facts.

Finally, after the mid-1970s, the fertility rate of older mothers rose, that of younger mothers fell, and the total fertility rate was relatively stable. Concurrently, the average years of secondary education changed little, while the average years of tertiary education, particularly among women, increased significantly. Empirical works show that the relative earnings of college graduate women to men increased significantly from the late 1960s (Goldin, 2014) and that the college wage premium increased sharply from the late 1970s (Goldin and Katz, 2008). When the model captures the increased relative earnings of educated workers, particularly women, by an increase in the returns to adulthood education, it can account for these facts, except the largely stable total fertility rate.

Our work is mainly related to Iyigun (2000) and d'Albis, Greulich, and Ponthiere (2018) that develop models with educational investment and two periods of childbearing. Iyigun (2000) develops an OLG model in which individual human capital depends positively on own education and parental human capital. He shows that if the complementarity between parental human capital and education is sufficiently strong, an increase in the parental human capital raises the opportunity cost of having children while young and induces individuals to delay childbearing. d'Albis, Greulich, and Ponthiere (2018) develop a unified growth model to rationalize the long-run decline in the total fertility rate and the U-shaped trend of childbearing age. They show that a traditional economy, where individuals do not invest in education and increased income lowers childbearing age, can progressively converge towards a modern economy, where individuals invest in education and increased income raises childbearing age.

Our work differs from Iyigun (2000) and d'Albis, Greulich, and Ponthiere (2018) mainly in three aspects. First, as mentioned above, Iyigun (2000) focuses on the postponement of childbearing, and d'Albis, Greulich, and Ponthiere (2018) focus on the decline in the total fertility rate and the U-shaped trend of childbearing age. By contrast, we examine all major fertility changes since the early 20th century in countries such as the U.S. and the U.K.: the large fertility decline in the early 20th century; the Baby boom; the “Baby Bust”; and the diverging fertility trends of older and younger mothers after the mid-1970s. Second, in addition to the decision on one's own education in adulthood (corresponding to higher education), we also consider decisions on education of their children (corresponding to primary and secondary education). This specification allows us to consider the interactions between adulthood education, education of children, and fertility. Finally, we examine how various factors, such as the cost of household chores and the costs of rearing and educating children for younger and older mothers, affect fertility and education, which are not incorporated in these works.

The rest of the paper is organized as follows. Section 2 presents the model, and Section 3 analyzes the model. Section 4 examines how the model can account for the observed changes in fertility and education in the actual economy. Section 5 concludes. Appendix A analyzes the case in which several parameters affecting fertility and education change over time. Appendix B contains the proofs of the propositions in the main text, and Online Appendix C provides the proofs of the propositions in Appendix A.

2 Model

Consider an economy in which individuals live for three periods, the first period as a child and the remaining two periods as an adult. Thus, three generations coexist in every period. In childhood, individuals do not make any decisions and just receive nurturing and education. In each period of adulthood, they decide on the number of children they have, how much time to spend on rearing and education of those children, and how much time to work. Further, in the first period of adulthood, which includes late adolescence in real life, they also decide how much time to spend on *their own* education. This investment, which corresponds to higher education in real life, raises earnings in the next period.

The utility function of an adult depends on consumption in two periods and total human capital of their children:

$$U = lnc_1 + lnc_2 + ln(n_1 h_{c1} + n_2 h_{c2}), \tag{1}$$

where c_i, n_i, and h_{ci} $(i = 1, 2)$ are, respectively, consumption in period i of adulthood, and the number and the human capital of children born in the period. Parents care about both the quantity and quality of their children, but not the timing of their births.

First, consider the first period of adulthood. The production function of consumption goods is linear in effective labor (human capital times labor supply) of workers. Let the total factor productivity (TFP) be A. Then, the wage rate per unit of time for an individual in the first period of adulthood equals:

$$w_1 = Ah_1, \tag{2}$$

where h_1 is her human capital in this period. Because of the linear production function, the wage rate does not depend on the human capital of other individuals, which makes the model analytically tractable.

She has total time of 1 to allocate among household chores (such as cleaning, cooking, and washing), her own education, the rearing and education of their children, and work. Thus, her consumption in this period is given by:

$$c_1 = w_1[1 - \delta - \eta e - (\theta_q + \theta_e e_{c1})n_1] = Ah_1[1 - \delta - \eta e - (\theta_q + \theta_e e_{c1})n_1], \tag{3}$$

where $\delta, \eta, \theta_q, \theta_e \in (0,1)$. In the equation, δ is the fixed time spent on household chores, e is the amount of education she takes in this period, η is the time cost of her education, θ_q is the per-child time cost of rearing, e_{c1} is the amount of education received by her children born in this period, and θ_e is the per-child time cost of the children's education. Note that the costs of educating herself and her children reflect not only the opportunity cost but also the cost of paying for teachers' time.[2] In particular, the latter would be the main component of the cost of educating children. She allocates her time to the various activities to maximize her utility.

[2] This is because the cost of hiring teachers is proportional to the wage rate. Other components of the cost of school education, such as study materials and school facilities, are abstracted from the model for analytical tractability.

The education she receives in this period raises her human capital in the next period, the second period of adulthood. The human capital production function is given by:

$$h_2 = (1 + \rho e)^{\alpha} h_1, \ \alpha \in (0,1), \tag{4}$$

where ρ is the effectiveness or productivity of adulthood (late adolescence) education, which corresponds to higher education in real life. In the real economy, ρ could increase not only due to the advancement of education technology but also from the increased economic value of workers with advanced education. When $e = 0$, human capital in the next period is the same as in this period.

Next, consider the second period of her adulthood. The wage rate per unit of time in this period equals

$$w_2 = gAh_2, \tag{5}$$

where g is the TFP growth rate.

In this period too, she has total time of 1 to allocate to various activities. The difference from the first period of adulthood is that she no longer spends time on her own education. It is assumed that the time costs of rearing and education of children in the second period of adulthood are weakly greater than the costs in the first period by the same proportion:

$$\theta_{q2} = \lambda \theta_q, \ \ \lambda \geq 1, \tag{6}$$

$$\theta_{e2} = \lambda \theta_e \ , \tag{7}$$

The proportionality assumption significantly simplifies the analysis. The assumption of greater costs reflects the fact that older parents have less energy and feel more tired from taking care of children, which implies less effective time left for other activities, than younger parents. In a poor health environment, the assumption also captures the fact that childbearing has long-lasting negative health effects on older mothers. Empirical evidence supports this: advanced maternal age has been shown to have a long-term negative impact on women's health. For example, it significantly increases the risk of severe maternal morbidity (Kennedy-Moulton et al., 2025) and the necessity of major surgical interventions, such as Cesarean sections (Grant, 2022).[3]

Hence, her consumption in this period equals

$$c_2 = w_2\left[1 - \delta - \left(\theta_{q2} + \theta_{e2} e_{c2}\right) n_2\right] = gA(1 + \rho e)^{\alpha} h_1 \left[1 - \delta - \lambda\left(\theta_q + \theta_e e_{c2}\right) n_2\right]. \tag{8}$$

Education received by children raises their human capital in the next period, the first period of their adulthood. Thus, their human capital production function is given by

$$h_{ci} = (1 + \rho_c e_{ci})^{\alpha_c}, i = 1,2 \ , \alpha_c \in (0,1), \tag{9}$$

where ρ_c is the effectiveness or productivity of childhood education, which corresponds to primary and secondary education, in raising human capital. Note that human capital in the next period is positive even without childhood education.

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[3] Kennedy-Moulton et al. (2025), using linked administrative data from California, find that rates of severe maternal morbidity (SMM) rise sharply with maternal age, showing that older mothers face a significantly higher risk of these complications compared to younger mothers. Grant (2022) provides evidence that delayed childbirth is a primary driver of the substantial long-term increase in C-section rates in the U.S.

A distinguishing feature of the model is that adulthood consists of two periods, and individuals can have children in both periods. The periods differ in that parents can have a choice on their own education in the first period of adulthood, while in the second period, they do not have such an option and instead face higher costs of rearing and education of their children. Because of these differences, parents choose different numbers of children in the two periods depending on exogenous variables and parameters, even though their utility does not directly depend on the timing of their children's births.

3 Analysis

The values of the endogenous variables are obtained by solving the utility maximization problem backward, starting with the maximization problem in the second period of adulthood. Because the derivation is lengthy and involved, a detailed analysis is provided in Appendix B. The following proposition summarizes the results, in particular, the number of children and the amount of education children receive in each period of their parents' adulthood, as well as the amount of education adults take in the first period of adulthood.

We assume the following condition:

$$\rho_c > \frac{\theta_e}{\alpha_c \theta_q}. \tag{10}$$

With this assumption, we can rule out the case $e_{c1} = e_{c2} = 0$, which is not realistic because it means that individuals do not receive any education during childhood.

The solutions for n_i $(i = 1, 2)$ and e are classified into four cases, depending on whether n_2 and e are positive or zero, as presented in the next proposition.

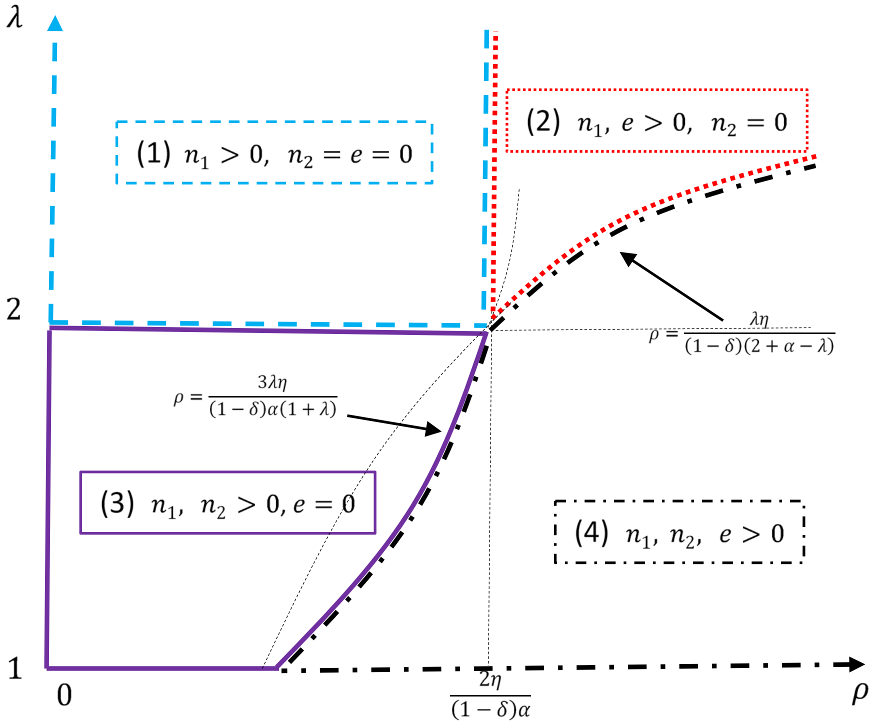

Figure 1 Fertility and education (Proposition 1)

Proposition 1 *Under the condition of* $\rho_c > \frac{\theta_e}{\alpha_c\theta_q}$, $e_{c1} = e_{c2} = \frac{\alpha_c\rho_c\theta_q-\theta_e}{(1-\alpha_c)\theta_e\rho_c}$ *if* $n_2 > 0$, *otherwise,* $e_{c1} = \frac{\alpha_c\rho_c\theta_q-\theta_e}{(1-\alpha_c)\theta_e\rho_c}$ *and* $e_{c2} = 0$.

(1) When $\rho \in \left(0, \frac{2\eta}{(1-\delta)\alpha}\right]$ *and* $\lambda \geq 2$, $n_1 = \frac{(1-\alpha_c)(1-\delta)\rho_c}{2(\rho_c\theta_q-\theta_e)}$, $n_2 = 0$, *and* $e = 0$.

(2) When $\rho \in \left[\frac{2\eta}{(1-\delta)\alpha}, \frac{\lambda\eta}{(1-\delta)(2+\alpha-\lambda)}\right]$ *and* $\lambda \geq 2$, $n_1 = \frac{(1-\alpha_c)\rho_c[(1-\delta)\rho+\eta]}{(2+\alpha)\rho(\rho_c\theta_q-\theta_e)}$, $n_2 = 0$, *and* $e = \frac{(1-\delta)\alpha\rho-2\eta}{(2+\alpha)\rho\eta}$.

(3) When $\rho \in \left(0, \frac{3\lambda\eta}{(1-\delta)\alpha(1+\lambda)}\right]$ *and* $\lambda \leq 2$, $n_1 = \frac{(1-\delta)\rho_c(1-\alpha_c)(2\lambda-1)}{3\lambda(\rho_c\theta_q-\theta_e)}$, $n_2 = \frac{(1-\delta)\rho_c(1-\alpha_c)(2-\lambda)}{3\lambda(\rho_c\theta_q-\theta_e)}$, *and* $e = 0$.

(4) When $\rho > \max\left[\frac{3\lambda\eta}{(1-\delta)\alpha(1+\lambda)}, \frac{\lambda\eta}{(1-\delta)(2+\alpha-\lambda)}\right]$, $n_1 = \frac{(1-\alpha_c)\rho_c\{(1-\delta)\rho[2\lambda-(1+\alpha)]+2\lambda\eta\}}{(3+\alpha)\lambda\rho(\rho_c\theta_q-\theta_e)}$, $n_2 = \frac{(1-\alpha_c)\rho_c[(1-\delta)\rho(2+\alpha-\lambda)-\lambda\eta]}{(3+\alpha)\lambda\rho(\rho_c\theta_q-\theta_e)}$, *and* $e = \frac{(1-\delta)\rho\alpha(1+\lambda)-3\lambda\eta}{(3+\alpha)\lambda\rho\eta}$.

Based on the proposition, Figure 1 illustrates the region corresponding to the four cases in the (ρ, λ) plane. When the productivity of adulthood (late adolescence) education ρ is low and the relative cost of childrearing and education borne by older parents λ is high (case (1)), the return to adulthood education is small and the cost of late childbearing is large. Thus, adults do not take education, i.e., $e = 0$, and do not have babies in the second period of adulthood, i.e., $n_2 = 0$. Equations in the proposition show how the parameters affect fertility and education. Increases in the productivity of childhood education ρ_c and in the per-child cost of rearing θ_q (an increase in the per-child cost of childhood education θ_e) reduce (increases) the number of children parents have in the first period of adulthood, n_1, and increase (reduces) the amount of childhood education their children receive, e_{c1}. That is, parents adjust the quantity and quality of children in response to the costs and benefit associated with these choices. An increase in the fixed time spent on household chores δ reduces individuals' disposable time and thus n_1.

When both ρ and λ are large (case (2)), that is, when the productivity of adulthood education and the cost of late childbearing are high, $e > 0$ and $n_2 = 0$ hold. In other words, individuals take education in the first period of adulthood and do not have children in the second period. An increase in ρ (an increase in the cost of adulthood education η) increases (reduces) the amount of education they acquire, e, and thus reduces (increases) n_1. That is, young adults adjust the amount of their own education and the number of children in response to the cost and benefit of advanced education. An increase in δ reduces disposable time and thus e. As in case (1), increases in ρ_c and θ_q (an increase in θ_e) raise (lowers) e_{c1} and lower (raises) n_1, and an increase in δ lowers n_1.

When both ρ and λ are small (case (3)), $e = 0$ and $n_2 > 0$ hold. Unlike in cases (1) and (2), individuals have babies in both periods. An increase in λ raises the relative cost of late childbearing, thereby raising n_1 and lowering n_2. As with e_{c1} and n_1, increases in ρ_c and θ_q (an increase in θ_e) raise (lowers) e_{c2} and lower (raises) n_2, and an increase in δ lowers n_2. Other results are the same as in the previous cases.

Finally, when ρ is large and λ is small (case (4)), $e > 0$ and $n_2 > 0$ hold. In other words, individuals acquire education in the first period of adulthood and have children in both periods. As in case (2), an increase in ρ (an increase in η) raises (lowers) e and lowers (raises) n_1. The higher benefit (cost) of advanced education also raises (lowers) n_2 because of intertemporal substitution in childbearing. As in case (3), an increase in λ raises n_1 and lowers n_2. The increased relative cost of late childbearing also lowers e, because it raises n_1, thereby reducing disposable time in the first period of adulthood. The remaining results are the same as in other cases.

In this model, $n_1 = 0 \text{ and } n_2 > 0$ —that is, individuals have children only in the second period of adulthood— never occurs. When ρ is sufficiently low that $e = 0$, this is because early childbearing ($n_1 > 0 \text{ and } n_2 = 0$) is preferred to late childbearing due to the higher cost of the latter ($\lambda \geq 1$). In contrast, when ρ is high enough that $e > 0$, *late* childbearing is preferred to early childbearing if the opportunity cost of early childrearing is high due to a large return to adulthood education, ρ, and the relative cost of late childrearing, λ, is low. However, late childbearing is dominated by balanced childbearing ($n_1, n_2 > 0$). While delaying all births to the second period of adulthood allows for greater educational investment and higher consumption in the first period, it leads to less work and more time devoted to childrearing in the second period. Given preferences for consumption smoothing (balanced consumption), individuals are reluctant to sacrifice second-period consumption significantly. As a result, the number of children and thus the utility from them are substantially lower under concentrated childbearing than under balanced childbearing.

The next corollary presents the solutions for the total number of children, $n_1 + n_2$. Since $n_2 = 0$ in cases (1) and (2), we consider only cases (3) and (4).

Corollary 1 *Under the condition* $\rho_c > \frac{\theta_e}{\alpha_c \theta_q}$,

(1) In case (3), i.e., when $\rho \in \left(0, \frac{3\lambda\eta}{(1-\delta)\alpha(1+\lambda)}\right]$ *and* $\lambda \leq 2$, $n_1 + n_2 = \frac{(1-\delta)\rho_c(1-\alpha_c)(\lambda+1)}{3\lambda(\rho_c\theta_q - \theta_e)}$.

(2) In case (4), i.e., when $\rho > max\left[\frac{\lambda\eta}{(1-\delta)(2+\alpha-\lambda)}, \frac{3\lambda\eta}{(1-\delta)\alpha(1+\lambda)}\right]$, $n_1 + n_2 = \frac{(1-\alpha_c)\rho_c[(1-\delta)\rho(\lambda+1)+\lambda\eta]}{(3+\alpha)\lambda\rho(\rho_c\theta_q - \theta_e)}$.

While an increase in λ raises n_1 and lowers n_2, it reduces $n_1 + n_2$. That is, the increased relative cost of late childbearing lowers the total number of children. As with n_1 and n_2, increases in ρ_c, θ_q and δ (an increase in θ_e) lower (raises) $n_1 + n_2$. In case (4), while an increase in ρ (an increase in η) lowers (raises) n_1 and raises (lowers) n_2, it lowers (raises) $n_1 + n_2$. That is, the higher benefit (cost) of advanced education reduces (increases) the total number of children.

4 Explaining the Evolution of Fertility and Education

4.1 Facts on Fertility and Education Trends

Figure 2 illustrates the total fertility rate, the fertility rate of women under age 29, and that of women over age 30 in the United States for the years 1895-2023.[4] After the relatively stable period of the early 20th century, the fertility rates declined sharply between the early 1920s and the early 1930s. From the early 1940s, particularly after World War II, they rose sharply and peaked at the end of the 1950s (the Baby Boom). Then, they declined sharply. Figure 2 illustrates the total fertility rate, the fertility rate of women under age 29, and that of women over age 30 in the United States for the years 1895-2023.[5] After the relatively stable period of the early 20th century, the fertility

[4] Data sources for Figure 2 are as follows. 1895-1899 and 1905-1910: Haines, M.R. (1990); 1917-1932: Office of Population Research at Princeton University (2013); 1933-2023: Human Fertility Database (2025).

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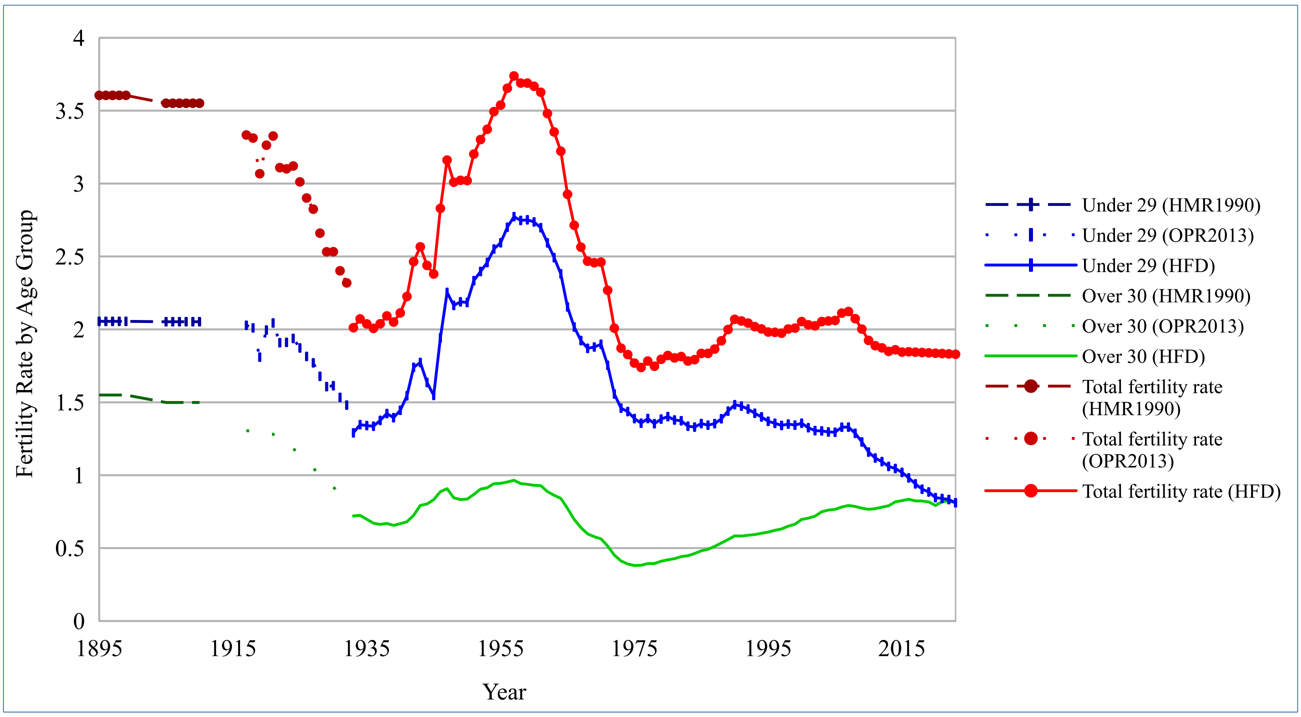

Figure 2 Fertility rate by age group (United States)

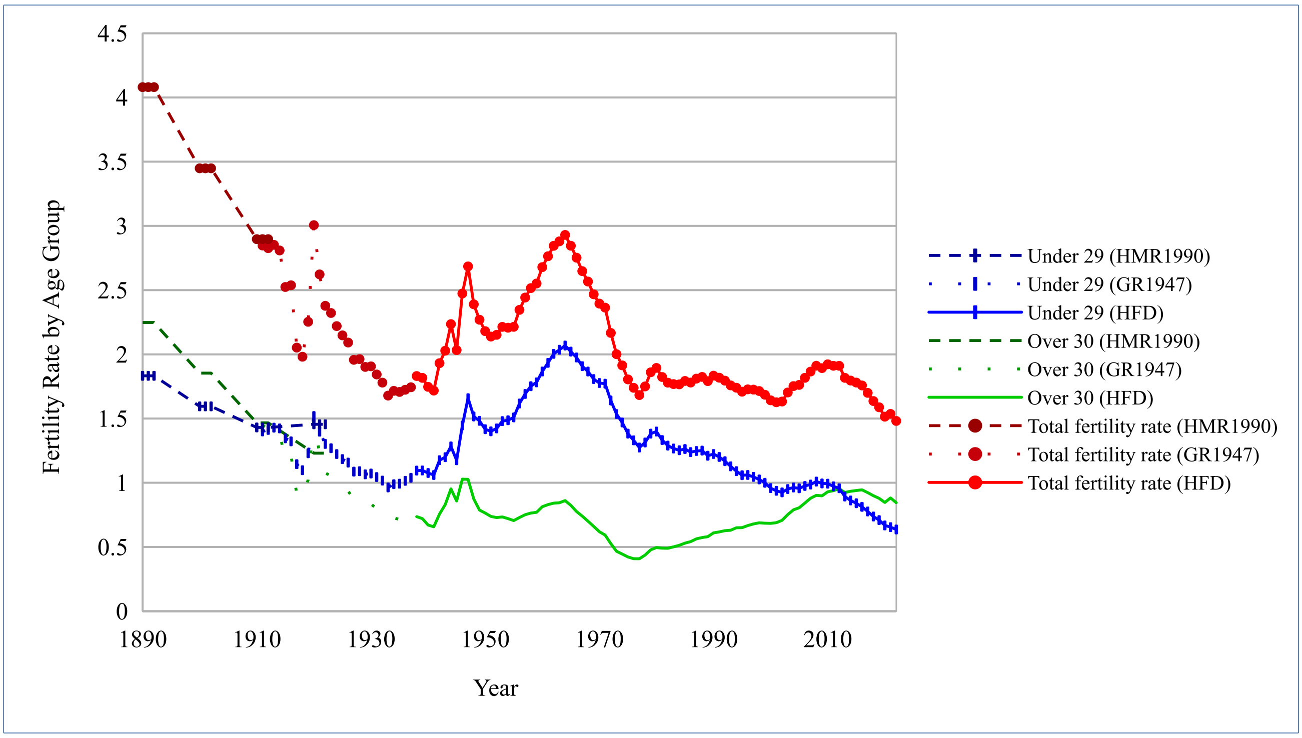

Figure 3 Fertility rate by age group (United Kingdom)

University (2013); 1933-2023: Human Fertility Database (2025).

rates declined sharply between the early 1920s and the early 1930s. From the early 1940s, particularly after World War II, they rose sharply and peaked at the end of the 1950s (the Baby Boom). Then, they declined sharply. After the mid-1970s, the three series showed divergent trends. Whereas the fertility rate of women under age 29 was relatively stable until the 1980s and then declined, particularly after the late 2000s, the fertility rate of women over age 30 increased consistently. The total fertility rate rose moderately until the mid-2000s and then fell slightly thereafter.

Figure 3 shows the corresponding series for the United Kingdom for the years 1890-2022.[6] The U.K.'s fertility rates showed similar transitions as those of the U.S., although there are some differences: the declines started earlier, at least from 1890; rates fluctuated significantly associated with WWI; there were two peaks during the Baby Boom, the first just after WWII, and the second in the mid-1960s; and the fertility rate of women under age 29 declined consistently after the mid-1970s.

In sum, in both countries (and in countries such as France and Sweden), four distinct periods of fertility trends can be identified: the period of large fertility decline from no later than the early 1920s until the early 1930s; the period of Baby Boom from the early 1940s until the early 1960s; the period of "Baby Bust" until the mid-1970s; and the period of rising fertility among older mothers and falling fertility among younger mothers thereafter.

As for education, according to the Lee and Lee dataset (Lee and Lee, 2016), the average years of schooling among 15-24 year olds increased greatly in the 1920s and the 1930s (from 7.00 in 1920 to 9.68 in 1940 in the U.S., and from 5.48 to 8.07 in the U.K.). During this period, the average years of secondary education of the young generation increased significantly (from 1.75 to 3.78 in the U.S., and from 0.69 to 2.66 in the U.K.**)**, whereas the average years of tertiary education were very low (close to 0 for women in the U.K.) and changed little. During the Baby Boom period, the average years of secondary education and of tertiary education for 15-24 year olds increased moderately: in the U.S., the former increased from 3.78 in 1940 to 4.15 in 1960, and the latter from 0.2 to 0.35. In the period of the "Baby Bust", these series also increased moderately, according to the Barro and Lee dataset (Barro and Lee, 2013): in the U.S., the former for 15-24 year olds increased from 4.15 in 1960 to 4.99 in 1975, and the latter for 25-34 year olds from 0.71 to 1.25.[7] After the mid-1970s, the average years of secondary schooling changed little in the U.S. (from 4.99 in 1975 to 5.05 in 2015), while the increase in the average years of tertiary education of 25-34 year old women was large (from 1.09 in 1975 to 2.14 in 2015 in the U.S., and from 0.33 in 1975 to 1.67 in 2015 in the U.K.).

4.2 Possible Factors Contributing to the Evolution of Fertility Rates

For each of the four periods of fertility trends, possible factors contributing to the evolution of fertility rates have been proposed in the literature or can be discerned from empirical facts. In the following, we briefly describe these

[6] Data sources of Figure 3 are as follows. 1891, 1901, 1911, 1921: Haines, M.R. (1990); 1911-1937: General Register Office (1947); 1938-2022: Human Fertility Database (2025).

[7] Data for 25-34 year olds are used for the average years of tertiary education, as many people complete tertiary education after age 25. The Barro and Lee dataset is available only from 1950, so the Lee and Lee dataset, which does not provide data for 25-34 year olds, is used up to the Baby Boom period.

factors and explain how they can be interpreted as changes in specific parameters of the model. We then examine whether the model can capture the observed patterns of fertility and education in the actual economy.

Large fertility decline in the early 20th century: Becker, Murphy, and Tamura (1990) show theoretically that increased returns to human capital investment lead to substitution of child quality for quantity, and Galor and Weil (2000), based on a theoretical analysis, argue that increased demand for human capital leads to fertility decline and increased education in the later stages of economic development.[8] Goldin and Katz (1998), using data on U.S. manufacturing for the years 1909-40, show that the introduction of mass production methods during this period raised the relative demand for skilled workers, who corresponded to high school graduates at the time. In the model, the increased demand for high school graduates can be captured by an increase in the productivity of childhood education, ρ_c, of the human capital production function. This is because increased ρ_c implies a higher return to childhood education, which corresponds to primary and secondary education in the real world.

Baby Boom: Greenwood, Seshadri, and Vandenbroucke (2005), based on quantitative analysis of an OLG model, argue that increased productivity of the household sector—driven by the diffusion of home appliances such as automated washing machines and refrigerators—in the 1940s and 1950s (in the U.S.) was a main driver of the Baby Boom. In the model, this increase in household productivity can be captured by a decrease in the fixed time spent on household chores, δ, and to a lesser extent, a decrease in the per-child time cost of rearing, θ_q.

Albanesi and Olivetti (2014) find empirically that a large decline in maternal mortality—a proxy for the overall maternal cost of childbearing, including pregnancy-related diseases—during the mid-1930s to the 1950s in the U.S. is strongly associated with a surge in fertility during this period. In the model, the decreased cost of having children can be represented by decreases in θ_q and the relative cost of late childbearing, λ. This is because pregnancy-related diseases often had long-term negative effects on maternal health and thus labor supply, and the sharp decline in morbidity benefited particularly older mothers who faced higher risks of pregnancy-related troubles.

Tamura and Simon (2017), based on quantitative analysis of a model calibrated to many developed economies, argue that a sudden fall in per-space housing costs —largely due to suburbanization that started after World War II and ended by 1970—was a main contributor to the Baby Boom. In the model, a falling price for space can be captured by a decrease in the per-child cost of rearing, θ_q.

"Baby Bust": Doepke, Hazan and Maoz (2015), using a quantitative model calibrated to the U.S. economy, provide a unified explanation of the Baby Boom and the "Baby Bust". They argue that the entry of a large number of women into the labor force during WWII, many of whom continued to work after the war, lowered the demand for inexperienced young women and their relative wage in the 1940s and the 1950s, thereby leading them to have more children. In contrast, the retirement of these older women after the 1960s increased the demand for young women, encouraged their entry into the labor market, and contributed to the "Baby Bust". Bellou and Cardia (2014), based on econometric analysis, propose a similar mechanism with the Great Depression, rather than WWII, as the

[8] One hypothesis popular among demographers is that declining infant mortality is a major cause of the long-run fertility decline. The hypothesis receives mixed empirical support, although, as for gross fertility rate, there are somewhat more findings in favor of the thesis (such as Herzer, Strulik, and Vollmer, 2012; Ager, Hansen, and Jensen, 2018) than findings inconsistent with it (such as Murphy, 2015; Hansen, Jensen, and Lønstrup, 2018). Following Galor (2012), a rough examination of the effect of infant mortality rate can be conducted using a slightly modified version of the model.

ultimate cause.[9] In the model, increases in the demand for young women and their relative wage during the "Baby Bust" could be captured by an increase in θ_q, since higher demand raises the opportunity cost of childrearing.[10]

Diverging fertility trends of older and younger mothers after the mid-1970s: Because women bear the primary burdens of childbearing and childrearing, it is mainly women who face the trade-off between tertiary education and childbearing/rearing, Thus, it would be plausible to interpret e as the level of tertiary education of women and ρ as the effectiveness of or returns to the education they receive. During this period, ρ seems to have increased greatly. First, the earnings of female college graduates relative to those of male college graduates increased substantially.[11] Most of this increase occurred among cohorts born between 1938 and 1953, with the rise in relative earnings noticeable after age 30 and peaking in the early 40s—that is, between the late 1970s and the mid-1990s (Goldin, 2014). Second, since the late 1970s or early 1980s, the wage premium of college graduates has increased significantly (Goldin and Katz, 2008), although the premium stopped rising around 2015 (Autor, Goldin, and Katz, 2020).

4.3 How Well Does the Model Explain the Trends?

We now explore how well the model can explain the observed trends of fertility and education in the actual economy. As shown in Figures 2 and 3, the fertility rate of women over age 30 (late childbearing) was consistently well above 0. Thus, it is realistic to focus on cases (3) and (4), where $n_2 > 0$ holds. This corresponds to assuming $\lambda < 2$ (see Figure 1).

As mentioned above, it would be appropriate to interpret e as the level of higher education attained by women. At the beginning of the 20th century, the average years of tertiary education among young women were very low (close to 0 in the U.K.). So, we suppose that the model economy is in case (3) initially.

To apply the model to actual trends in fertility and education, we must allow the relevant parameters to change over time. In particular, this requires assuming that the fixed time spent on household chores δ and the productivity of childhood education ρ_c take different values in the first and second periods of adulthood.[12] Let δ and ρ_c be their values in the first period, and $g_\delta \delta$ and $g_{\rho_c} \rho_c$, where $g_\delta \leq 1$ and $g_{\rho_c} \geq 1$, be their values in the second period. A detailed analysis of the time-dependent specification is presented in Propositions A.1, A.2, and A.3 in Appendix A, and the proofs are provided in Online Appendix C.

Based on the analysis in the appendix, Proposition 2 summarizes how changes in the relevant parameters—which

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[9] Another potentially important factor contributing to the "Baby Bust" is examined by Bailey (2010), who shows empirically that the birth control pill accelerated the decline in fertility rate after 1960. This factor, however, cannot be analyzed in the present model.

[10] To be more precise, the increased relative wage of young women typically raises their household earnings and the opportunity costs of other activities as well. In the model, this can be represented by increases in the wage rates per unit of working time, w_1 and w_2, and hence in the TFP, A. Proposition 1 implies that the change in A does not affect fertility or education. Since the burdens of childrearing fall disproportionately on women compared to other activities, it is reasonable to suppose that the higher relative wage for young women raises the opportunity cost of childrearing θ_q, as well as the wage rates.

[11] Relative earnings of less educated female workers to corresponding male workers also increased from the 1980s, which may be captured as a decrease in θ_q in the model. However, this change cannot explain most of the observed changes in fertility and education, as shown in Proposition 2 below.

[12] It is also assumed that the TFP, A, and thus the wage rates per unit of working time, w_1 and w_2, increase over time. However, as suggested by Proposition 1, the increased TFP does not affect fertility or education.

represent the possible factors contributing to fertility evolution identified in Section 4.2—affect fertility and education when the economy is in case (3) or case (4).

Proposition 2 *Consider the time-dependent specification. Assume that* $\lambda < 2$, $g_{\rho_c} \geq 1$ *is not large and* $g_\delta \leq 1$ *is not small.*

(1) An increase in ρ_c *raises* e_{ci} $(i = 1,2)$ *and lowers* n_1, n_2, *and* $n_1 + n_2$*; in case (4), it also raises* e.

(2) (a) A decrease in δ *raises* n_1, n_2, *and* $n_1 + n_2$*; in case (4), it also raises* e.

(b) A decrease in λ *lowers* n_1 *and raises* n_2 *and* $n_1 + n_2$*; in case (4), it also raises* e.

(c) A decrease in θ_q *lowers* e_{ci} $(i = 1,2)$ *and raises* n_1, n_2, *and* $n_1 + n_2$*; in case (4), it also lowers* e.

(3) An increase in θ_q *raises* e_{ci} $(i = 1,2)$ *and lowers* n_1, n_2, *and* $n_1 + n_2$*; in case (4), it also raises* e.

(4) In case (4), an increase in ρ *lowers* n_1 *and* $n_1 + n_2$*, and raises* n_2 *and* e.

Under the assumptions in Proposition 2, the results for the time-dependent specification are mostly the same as the results for the time-invariant specification, which can be obtained from Proposition 1 (3), (4) and Corollary 1. The exception is that, in case (4), an increase in ρ_c raises e, and a decrease in θ_q lowers e in the time-dependent case, while these changes have no effect on e in the time-invariant case. The assumptions on g_{ρ_c} and g_δ are not very restrictive, because it seems unlikely that ρ_c and δ change substantially within 10 or 20 years (between the first and second periods of adulthood).

We now evaluate the model's performance across the four periods of fertility trends. For the period roughly from the early 1920s to the early 1930s, data show that both age-specific and total fertility rates fell sharply. Concurrently, the average years of secondary education of young generations rose greatly, while their average years of tertiary education (especially for women) changed little. Evidence suggests that the demand for skilled workers, corresponding to high school graduates at the time, increased significantly (Goldin and Katz, 1998). In the model, this can be captured by an increase in the returns to childhood education, ρ_c. Given that the average years of tertiary education among young women were very low (close to 0 in the U.K.) during this period, it would be reasonable to suppose that the economy is in case (3) and thus $e = 0$. Proposition 2 (1) shows that an increase in ρ_c raises e_{ci} $(i = 1,2)$ and lowers n_1, n_2 and $n_1 + n_2$ in case (3). Hence, the model can explain the above-mentioned facts qualitatively.

As for the period of Baby Boom roughly from the early 1940s to the early 1960s, age-specific and total fertility rates increased greatly, while the average years of secondary and tertiary education of young generations rose moderately. Researchers argue that the Baby Boom was caused by the increased productivity of the household sector from the diffusion of home appliances (Greenwood, Seshadri, and Vandenbroucke, 2005), a large fall in maternal mortality, which is a proxy for the overall maternal cost of having children (Albanesi and Olivetti, 2014), and a decrease in per-space housing costs due to suburbanization (Tamura and Simon, 2017). In the model, the increased productivity of the household sector can be represented by a decreased fixed time spent on household chores δ and, to a lesser extent, by a decreased cost of childrearing θ_q; the decreased maternal cost of childbearing can be captured by a decrease in θ_q and a decreased relative cost of late childbearing λ; the fall in price for space can also be captured by a decrease in θ_q.

Starting from a very low level, the average years of tertiary education of young women increased moderately during the period.[13] Thus, it is plausible that the economy shifted from case (3) to case (4) at some point during this period. In Figure 1 of Section 3, decreases in λ and δ shift the boundary line between case (3) and case (4) to leftward, and a decrease in λ shifts the position of the model economy downward. Hence, the model economy transits to case (4) when δ and λ become sufficiently small. Proposition 2 (2) shows that a decrease in δ raises n_1, n_2, $n_1 + n_2$, and, in case (4), e; a decrease in λ lowers n_1 and raises n_2, $n_1 + n_2$, and, in case (4), e; and a decrease in θ_q raises n_1, n_2 and $n_1 + n_2$, lowers e_{ci} $(i = 1,2)$ and, in case (4), e. Consequently, a decrease in δ can account for the observed facts except the increased years of secondary education (it does not affect years of secondary education); a decrease in λ can explain the facts except the increased fertility rate of young mothers (it counterfactually lowers the fertility rate) and the increased years of secondary education (it does not affect years of secondary education); and a decrease in θ_q can explain the facts except the increased years of secondary and tertiary education (it counterfactually decreases years of both education). Therefore, the model can explain the facts other than the moderate increase in years of secondary education, provided that the effects of the changes in λ and θ_q are not very large.[14]

As for the period of "Baby Bust" roughly from the early or middle 1960s to the middle 1970s, age-specific and total fertility rates declined sharply, while the average years of secondary and tertiary education among young cohorts rose moderately. Researchers argue that the labor demand for young women and their relative wage increased during this period, with WWII (Doepke, Hazan and Maoz, 2015) or the Great Depression (Bellou and Cardia, 2014) as the ultimate cause. These increases imply a higher opportunity cost of childrearing for young women and can thus be represented by an increase in θ_q in the model. Proposition 2 (3) shows that an increase in θ_q lowers n_1, n_2, and $n_1 + n_2$, while raising e_{ci} $(i = 1,2)$ and, in case (4), e. Hence, the model can qualitatively account for the facts.

Finally, as for the period after the mid-1970s, the fertility rate of older mothers rose, that of younger mothers fell (consistently in the U.K., after the 1990s in the U.S.), and the total fertility rate was relatively stable. Concurrently, the average years of secondary education changed little, while the average years of tertiary education, particularly among women, increased significantly. Empirical works show that the relative earnings of college graduate women to men increased greatly from the late 1960s, particularly between the late 1970s and the mid 1990s (Goldin, 2014), and that the college wage premium increased sharply from the late 1970s (Goldin and Katz, 2008), except after around 2010 (Autor, Goldin and Katz, 2020). In the model, these developments can be captured by an increase in the returns to adulthood (late adolescence) education, ρ. Proposition 2 (4) shows that an increase in ρ lowers n_1 and $n_1 + n_2$, raises n_2, and, in case (4), e. Hence, the model can explain these facts, except for the generally stable total fertility rate.

In summary, the model effectively accounts for the major changes in fertility and education since the early 20th century, with parameter changes capturing possible factors that contributed to the evolution of fertility rates as proposed in the literature or suggested by empirical evidence. In particular, the model can qualitatively explain the major trends in fertility among younger and older mothers, total fertility, and the secondary and higher education

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[13] The average years of tertiary education of 15-24 old women increased from 0.21 in 1940 to 0.39 in 1965 in the U.S. and from 0.01 to 0.13 in the U.K.

[14] To be precise, the effect of the change in λ on n_1 must not dominate the effects of the changes in δ and θ_q, and the effect of the change in θ_q on e must not dominate the effects of the changes in δ and λ.

of young cohorts, except for the increase in years of secondary education during the Baby Boom and the relatively stable total fertility after the mid-1970s.

5 Conclusions

This paper has developed a simple, analytically tractable model of fertility and education in which individuals live for three periods. A distinguishing feature of the model is that adulthood consists of two periods, and individuals can have children in both periods. The periods differ in that parents can have a choice on their own education (corresponding to higher education) in the first period of adulthood, while in the second period, they do not have such an option and instead face higher costs of rearing and education of their children.

We have examined how well the model can explain the long-run changes in fertility and education since the early 20th century in countries such as the U.S. and the U.K. qualitatively. First, during the period from the early 1920s to the early 1930s, fertility rates fell sharply, while the average years of secondary education of young generations rose greatly and their average years of tertiary education (especially for women) changed little. This period also experienced a significant increase in the demand for skilled workers, high school graduates. When the model captures the increased skill demand by an increase in the returns to childhood education, it can explain these facts.

Second, during the Baby Boom roughly from the early 1940s to the early 1960s, observed facts are that both age-specific and total fertility rates rose greatly, while the average years of secondary and tertiary education of young generations rose moderately. Researchers argue that the Baby Boom was caused by the increased productivity of the household sector, a substantial decline in the maternal cost of having children, and a decrease in housing costs. When the model captures these changes by decreases in the fixed time spent on household chores, the cost of childrearing, and the relative cost of late childbearing, it can explain the facts other than the moderate increase in years of secondary education, provided that the effects of the latter two factors are not very large.

Third, during the "Baby Bust" from the early or mid-1960s to the mid-1970s, age-specific and total fertility rates fell sharply, while the average years of secondary and tertiary education among young cohorts rose moderately. Researchers attribute the "Baby Bust" to increases in the labor demand for young women and their relative wage, with WWII or the Great Depression as the ultimate cause. When the model captures the increased relative wage of young women by a rise in the opportunity cost of childrearing, it can account for these facts.

Finally, after the mid-1970s, the fertility rate of older mothers rose, that of younger mothers fell, and the total fertility rate was relatively stable. Concurrently, the average years of secondary education changed little, while the average years of tertiary education, particularly among women, increased significantly. Empirical studies show that the relative earnings of college-educated workers, particularly women, increased significantly. When the model captures this change by an increase in the returns to adulthood education, it can account for these facts except the largely stable total fertility rate.

The present framework may be extended in several directions. First, the effects of childcare and education policies could be examined by incorporating such policies into the model. Second, a model in which individuals are heterogenous in innate ability could be developed to examine how the parameter changes considered in this paper differentially affect the education and fertility choices of individuals with different abilities and thus incomes. Third, a model in which each household consists of a husband and a wife could be developed to analyze how the parameter changes affect the education of men and females differently and, in turn, their fertility decisions. These extensions are left for future research.

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Appendix A Time-dependent Specification

Since the dynamic analysis focuses on cases (3) and (4), we consider these cases when the levels of the fixed time spent on household chores, δ, and of the productivity of childhood education, ρ_c, individuals face in the two periods of adulthood are different. Let δ and ρ_c be the variables in period 1 of adulthood, and let $g_\delta \delta$ and $g_{\rho_c}\rho_c$ be the variables in period 2, where $g_\delta \leq 1$ and $g_{\rho_c} \geq 1$.

The values and properties of e_{ci} and n_i $(i = 1, 2)$ in case (3) are summarized in the next proposition. Proofs of the propositions in this Appendix are provided in Online Appendix C.

Proposition A.1 *Assume* $\rho_c > \frac{\theta_e}{\alpha_c\theta_q}$, $g_\delta \leq 1$, *and* $g_{\rho_c} \geq 1$. *In Case (3) of the time-dependent specification, the following holds.* $e = 0$, $e_{c1} = \frac{\alpha_c\rho_c\theta_q-\theta_e}{(1-\alpha_c)\theta_e\rho_c}$, $e_{c2} = \frac{\alpha_c g_{\rho_c}\rho_c\theta_q-\theta_e}{(1-\alpha_c)\theta_e g_{\rho_c}\rho_c}$.

(1) $n_1 = \frac{(1-\alpha_c)\rho_c\left[(1-\delta)\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}2\lambda-(1-g_\delta\delta)g_{\rho_c}\left(\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}\right]}{3\lambda(\rho_c\theta_q-\theta_e)\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}}$. *It increases (decreases) with* λ, g_δ, *and* θ_e *(with* ρ_c, g_{ρ_c}, δ, *and* θ_q*).*

$n_2 = \frac{(1-\alpha_c)\rho_c\left[2(1-g_\delta\delta)g_{\rho_c}\left(\rho_{c1}\theta_q-\theta_e\right)^{1-\alpha_c}-(1-\delta)\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}\lambda\right]}{3\lambda\left(\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)}$. *It decreases with* λ *and* g_δ. *It decreases (increases) with* ρ_c *when* $\lambda < (\geq) \frac{2(1-g_\delta\delta)g_{\rho_c}\left(\rho_c\theta_q-\theta_e\right)^{2-\alpha_c}}{(1-\delta)\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}\left[\alpha_c(\rho_c\theta_q-\theta_e)+(1-\alpha_c)(g_{\rho_c}\rho_c\theta_q-\theta_e)\right]}$, *with* g_{ρ_c} *when* $\lambda < (\geq) \frac{2\theta_e(1-g_\delta\delta)\left(\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}}{(1-\delta)\alpha_c\rho_c\theta_q\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}}$, *with* δ *when* $\lambda < (\geq) \frac{2g_{\rho_c}g_\delta\left(\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}}{\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}}$, *with* θ_q *when* $\lambda < (\geq) \frac{2(1-g_\delta\delta)g_{\rho_c}^2\left(\rho_c\theta_q-\theta_e\right)^{2-\alpha_c}}{(1-\delta)\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}\left[\alpha_c g_{\rho_c}(\rho_c\theta_q-\theta_e)+(1-\alpha_c)(g_{\rho_c}\rho_c\theta_q-\theta_e)\right]}$, *and increases (decreases) with* θ_e *when* $\lambda < (\geq) \frac{2(1-g_\delta\delta)g_{\rho_c}\left(\rho_c\theta_q-\theta_e\right)^{2-\alpha_c}}{(1-\delta)\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}\left[\alpha_c(\rho_c\theta_q-\theta_e)+(1-\alpha_c)(g_{\rho_c}\rho_c\theta_q-\theta_e)\right]}$.

The values and properties of e_{ci}, n_i $(i = 1, 2)$ and e in case (4) are summarized in the next proposition.

Proposition A.2 *Assume* $\rho_c > \frac{\theta_e}{\alpha_c\theta_q}$, $g_\delta \leq 1$, *and* $g_{\rho_c} \geq 1$. *In Case (4) of the time-dependent specification, the following holds.* $e_{c1} = \frac{\alpha_c\rho_c\theta_q-\theta_e}{(1-\alpha_c)\theta_e\rho_c}$, $e_{c2} = \frac{\alpha_c g_{\rho_c}\rho_c\theta_q-\theta_e}{(1-\alpha_c)\theta_e g_{\rho_c}\rho_c}$.

(1) $n_1 = \frac{(1-\alpha_c)\rho_c\left\{2[(1-\delta)\rho+\eta]\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}\lambda-(1+\alpha)\rho(1-g_\delta\delta)g_{\rho_c}\left(\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}\right\}}{(3+\alpha)\lambda\rho(\rho_c\theta_q-\theta_e)\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}}$. *It increases (decreases) with* λ, η, g_δ, *and* θ_e *(with* ρ, ρ_c, g_{ρ_c}, *and* θ_q*). It decreases with* δ *when* g_{ρ_c} *is not large.*

(2) $n_2 = \frac{(1-\alpha_c)\rho_c\left[(2+\alpha)\rho(1-g_\delta\delta)g_{\rho_c}\left(\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}-[(1-\delta)\rho+\eta]\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}\lambda\right]}{(3+\alpha)\lambda\rho\left(\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)}$. *It increases (decreases) with* ρ *(with* λ, η, *and* g_δ*). It decreases (increases) with* ρ_c *when* $\lambda < (\geq) \frac{(2+\alpha)\rho(1-g_\delta\delta)g_{\rho_c}\left(\rho_c\theta_q-\theta_e\right)^{2-\alpha_c}}{[(1-\delta)\rho+\eta]\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}\left[\alpha_c(\rho_c\theta_q-\theta_e)+(1-\alpha_c)(g_{\rho_c}\rho_c\theta_q-\theta_e)\right]}$, *with* g_{ρ_c} *when* $\lambda < (\geq) \frac{(2+\alpha)\rho\theta_e(1-g_\delta\delta)\left(\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}}{[(1-\delta)\rho+\eta]\alpha_c\rho_c\theta_q\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}}$, *with* δ *when* $\lambda < (\geq) \frac{(2+\alpha)g_{\rho_c}g_\delta\left(\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}}{\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}}$, *with* θ_q *when* $\lambda < (\geq) \frac{(2+\alpha)\rho(1-g_\delta\delta)g_{\rho_c}^2\left(\rho_c\theta_q-\theta_e\right)^{2-\alpha_c}}{[(1-\delta)\rho+\eta]\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}\left[\alpha_c g_{\rho_c}(\rho_c\theta_q-\theta_e)+(1-\alpha_c)(g_{\rho_c}\rho_c\theta_q-\theta_e)\right]}$, *and increases (decreases) with* θ_e *when* $\lambda < (\geq) \frac{(2+\alpha)\rho(1-g_\delta\delta)g_{\rho_c}\left(\rho_c\theta_q-\theta_e\right)^{2-\alpha_c}}{[(1-\delta)\rho+\eta]\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}\left[\alpha_c(\rho_c\theta_q-\theta_e)+(1-\alpha_c)(g_{\rho_c}\rho_c\theta_q-\theta_e)\right]}$.

$e = \frac{\alpha\rho\left[(1-\delta)\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}\lambda+(1-g_\delta\delta)g_{\rho_c}\left(\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}\right]-3\lambda\eta\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}}{(3+\alpha)\lambda\rho\eta\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}}$. *It increases (decreases) with* ρ, ρ_c, g_{ρ_c}, *and* θ_q *(with* λ, η, δ, g_δ, *and* θ_e*).*

The values and properties of $n_1 + n_2$ in cases (3) and (4) are summarized in the next proposition.

Proposition A.3 *Assume* $\rho_c > \frac{\theta_e}{\alpha_c \theta_q}$, $g_\delta \le 1$ *and* $g_{\rho_c} \ge 1$. *In Case (3) and case (4) of the time-dependent specification, the following holds.*

(1) In case (3), $n_1 + n_2 = \frac{(1-\alpha_c)\rho_c}{3\lambda(\rho_c\theta_q-\theta_e)(g_{\rho_c}\rho_c\theta_q-\theta_e)}\left\{(1-\delta)\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}\left[2\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{\alpha_c}-\left(\rho_c\theta_q-\theta_e\right)^{\alpha_c}\right]\lambda+(1-g_\delta\delta)g_{\rho_c}\left(\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}\left[2\left(\rho_c\theta_q-\theta_e\right)^{\alpha_c}-\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{\alpha_c}\right]\right\}$. *It increases (decreases) with* θ_e *(with* λ, ρ_c, δ, g_δ, *and* θ_q*) when* g_{ρ_c} *is not large. It decreases (increases) with* g_{ρ_c} *when* $\lambda < (\ge) \frac{(1-g_\delta\delta)(\rho_c\theta_q-\theta_e)^{1-2\alpha_c}\left[(\alpha_c g_{\rho_c}\rho_c\theta_q-\theta_e)(g_{\rho_c}\rho_c\theta_q-\theta_e)^{\alpha_c}+2\theta_e(\rho_c\theta_q-\theta_e)^{\alpha_c}\right]}{(1-\delta)\alpha_c\rho_c\theta_q(g_{\rho_c}\rho_c\theta_q-\theta_e)^{1-\alpha_c}}$.

(2) In case (4), $n_1 + n_2 = \frac{(1-\alpha_c)\rho_c}{(3+\alpha)\lambda\rho(\rho_c\theta_q-\theta_e)(g_{\rho_c}\rho_c\theta_q-\theta_e)}\left\{[(1-\delta)\rho+\eta]\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}\left[2\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{\alpha_c}-\left(\rho_c\theta_q-\theta_e\right)^{\alpha_c}\right]\lambda+\rho(1-g_\delta\delta)g_{\rho_c}\left(\rho_c\theta_q-\theta_e\right)^{1-\alpha_c}\left[(2+\alpha)\left(\rho_c\theta_q-\theta_e\right)^{\alpha_c}-(1+\alpha)\left(g_{\rho_c}\rho_c\theta_q-\theta_e\right)^{\alpha_c}\right]\right\}$. *It increases (decreases) with* η *(with* ρ*). It increases (decreases) with* θ_e *(with* λ, ρ_c, δ, g_δ, *and* θ_q*) when* g_{ρ_c} *is not large. It decreases (increases) with* g_{ρ_c} *when* $\lambda < (\ge) \frac{(1-g_\delta\delta)(\rho_c\theta_q-\theta_e)^{1-2\alpha_c}\rho\left[(1+\alpha)(\alpha_c g_{\rho_c}\rho_c\theta_q-\theta_e)(g_{\rho_c}\rho_c\theta_q-\theta_e)^{\alpha_c}+(2+\alpha)\theta_e(\rho_c\theta_q-\theta_e)^{\alpha_c}\right]}{[(1-\delta)\rho+\eta]\alpha_c\rho_c\theta_q(g_{\rho_c}\rho_c\theta_q-\theta_e)^{1-\alpha_c}}$.

As in the time-invariant specification, e_{ci} increases (decreases) with ρ_c and θ_q (θ_e); n_1 increases (decreases) with λ and θ_e (with ρ_c, δ, and θ_q) and, in case (4), it increases (decreases) with η (with ρ);[15] n_2 decreases with λ and, in case (4), it increases (decreases) with ρ (with η); e increases (decreases) with ρ (with λ, η, and δ); and, in case (4), $n_1 + n_2$ increases (decreases) with η (with ρ).

By contrast, n_2 increases (decreases) with θ_e (with ρ_c, δ, and θ_q) when λ is small and decreases (increases) with θ_e (with ρ_c, δ, and θ_q) when λ is large in the time-dependent specification, while n_2 always increases (decreases) with θ_e (with ρ_c, δ, and θ_q) in the time-invariant specification. $n_1 + n_2$ increases (decreases) with θ_e (with λ, δ, ρ_c, and θ_q) when $g_{\rho_c} \ge 1$ is not large, while it always holds in the time-invariant specification. e increases (decreases) with ρ_c and θ_q (with θ_e), while these parameters do not affect e in the time-invariant specification. As for new parameters, e_{c2} increases with g_{ρ_c}; n_1 increases (decreases) with g_δ (with g_{ρ_c}); n_2 decreases with g_δ and decreases (increases) with g_{ρ_c} when λ is small (large); $n_1 + n_2$ decreases with g_δ when g_{ρ_c} is not large and decreases (increases) with g_{ρ_c} when λ is small (large); e increases (decreases) with g_{ρ_c} (with g_δ).

However, the qualitative results of the time-dependent specification are mostly the same as those of the time-invariant specification when $\lambda<2$, $g_{\rho_c} \ge 1$ is not large, and $g_\delta \le 1$ is not small. This can be explained as follows. When $g_{\rho_c} = g_\delta = 1$, the inequalities on λ (except those associated with g_{ρ_c}) in the above propositions reduce to $\lambda < (\ge) 2$ in case (3) and $\lambda < (\ge) \frac{(2+\alpha)\rho}{(1-\delta)\rho+\eta}$ (the one associated with δ becomes $\lambda < (\ge) 2+\alpha$) in case (4). Hence, in the region $\lambda < 2$ where cases (3) and (4) hold in the time-invariant specification, the results of the time-

[15] In case (4), the result on δ holds when g_{ρ_c} is not large.

dependent specification coincide with those of the time-invariant specification, which are obtained from Proposition 1 (3), (4) and Corollary 1, for parameters common to both. The difference from the time-invariant specification is that ρ_c, θ_q, and θ_e affect e. The same results hold for $g_{\rho_c} > 1$ and $g_\delta < 1$, as long as g_{ρ_c} is not large and g_δ is not small.

Appendix B Proofs of Propositions 1 and 2

Proof of Proposition 1

Necessary conditions of the utility maximization problem

The results of the proposition are obtained by solving the utility maximization problem.

The maximization problem is solved backward. The objective function of the period 2 maximization problem is

$$ln\{gA(1+\rho e)^{\alpha}h_1[1-\delta-\lambda(\theta_q+\theta_e e_{c2})n_2]\} + ln[n_1(1+\rho_c e_{c1})^{\alpha_c} + n_2(1+\rho_c e_{c2})^{\alpha_c}] \quad (11)$$

The first order conditions are

$$\frac{\partial U}{\partial e_{c2}} = n_2\left[\frac{-\lambda\theta_e}{1-\delta-\lambda(\theta_q+\theta_e e_{c2})n_2} + \frac{\alpha_c\rho_c(1+\rho_c e_{c2})^{\alpha_c-1}}{n_1(1+\rho_c e_{c1})^{\alpha_c}+n_2(1+\rho_c e_{c2})^{\alpha_c}}\right] \leq 0, \quad (12)$$

$$\frac{\partial U}{\partial n_2} = \frac{-\lambda(\theta_q+\theta_e e_{c2})}{1-\delta-\lambda(\theta_q+\theta_e e_{c2})n_2} + \frac{(1+\rho_c e_{c2})^{\alpha_c}}{n_1(1+\rho_c e_{c1})^{\alpha_c}+n_2(1+\rho_c e_{c2})^{\alpha_c}} \leq 0. \quad (13)$$

When $n_2 > 0$, from $\frac{\partial U}{\partial n_2} = 0$,

$$n_1(1+\rho_c e_{c1})^{\alpha_c} + n_2(1+\rho_c e_{c2})^{\alpha_c} = \frac{(1+\rho_c e_{c2})^{\alpha_c}}{\lambda(\theta_q+\theta_e e_{c2})}\left[1-\delta-\lambda(\theta_q+\theta_e e_{c2})n_2\right]. \quad (14)$$

By substituting this equation into (12),

$$\frac{\partial U}{\partial e_{c2}} = n_2\left[\frac{-\lambda\theta_e}{1-\delta-\lambda(\theta_q+\theta_e e_{c2})n_2} + \frac{\alpha_c\rho_c(1+\rho_c e_{c2})^{\alpha_c-1}}{1-\delta-\lambda(\theta_q+\theta_e e_{c2})n_2}\frac{\lambda(\theta_q+\theta_e e_{c2})}{(1+\rho_c e_{c2})^{\alpha_c}}\right]$$

$$= \frac{\lambda n_2}{1-\delta-\lambda(\theta_q+\theta_e e_{c2})n_2}\left[-\theta_e + \frac{\alpha_c\rho_c(\theta_q+\theta_e e_{c2})}{1+\rho_c e_{c2}}\right]. \quad (15)$$

From $\rho_c > \frac{\theta_e}{\alpha_c\theta_q}$ (equation (10)), $e_{c2} > 0$ if $n_2 > 0$, otherwise $e_{c2} = 0$.

When $n_2 > 0$ from $\frac{\partial U}{\partial e_{c2}} = 0$,

$$e_{c2} = \frac{\alpha_c\rho_c\theta_q - \theta_e}{(1-\alpha_c)\theta_e\rho_c}. \quad (16)$$

When $n_2 = (>)0$, from (13),

$$\frac{\partial U}{\partial n_2}|_{n_2=0} = \frac{-\lambda(\theta_q+\theta_e e_{c2})}{1-\delta} + \frac{(1+\rho_c e_{c2})^{\alpha_c}}{n_1(1+\rho_c e_{c1})^{\alpha_c}} \leq (>)0. \quad (17)$$

Depending on whether n_2 is positive or zero, there are two cases to examine.

[1] When $n_2 = 0$

In this case, the objective function of the period 1 maximization problem becomes

$$U = ln\{Ah_1[1-\delta-\eta e-(\theta_q+\theta_e e_{c1})n_1]\} + ln[gA(1+\rho e)^{\alpha} h_1(1-\delta)]$$
$$+ ln[n_1(1+\rho_c e_{c1})^{\alpha_c}]. \tag{18}$$

The first order conditions are

$$\frac{\partial U}{\partial n_1} = \frac{-(\theta_q+\theta_e e_{c1})}{1-\delta-\eta e-(\theta_q+\theta_e e_{c1})n_1} + \frac{1}{n_1} \le 0, \tag{19}$$

$$\frac{\partial U}{\partial e_{c1}} = \frac{-\theta_e n_1}{1-\delta-\eta e-(\theta_q+\theta_e e_{c1})n_1} + \frac{\alpha_c \rho_c}{1+\rho_c e_{c1}} \le 0, \tag{20}$$

$$\frac{\partial U}{\partial e} = \frac{-\eta}{1-\delta-\eta e-(\theta_q+\theta_e e_{c1})n_1} + \frac{\alpha\rho}{1+\rho e} \le 0. \tag{21}$$

In this case, $n_1 > 0$ always holds because $\frac{\partial U}{\partial n_1} = \infty$ when $n_1 = 0$. Thus from $\frac{\partial U}{\partial n_1} = 0$,

$1-\delta-\eta e-(\theta_q+\theta_e e_{c1})n_1 = (\theta_q+\theta_e e_{c1})n_1$.

By substituting this equation into (20),

$$\frac{\partial U}{\partial e_{c1}} = \frac{-\theta_e}{\theta_q+\theta_e e_{c1}} + \frac{\alpha_c \rho_c}{1+\rho_c e_{c1}}. \tag{22}$$

Hence, $e_{c1} > 0$ from $\rho_c > \frac{\theta_e}{\alpha_c \theta_q}$ (equation (10)). From $\frac{\partial U}{\partial e_{c1}} = 0$,

$$e_{c1} = \frac{\alpha_c \rho_c \theta_q - \theta_e}{(1-\alpha_c)\theta_e \rho_c}. \tag{23}$$

When $e = (>)0$, from (21),

$$\frac{\partial U}{\partial e}\Big|_{e=0} = \frac{-\eta}{1-\delta-(\theta_q+\theta_e e_{c1})n_1} + \alpha\rho \le (>)0. \tag{24}$$

Depending on whether e is positive or zero, there are two cases to consider.

① When $n_1 > 0$ and $e = 0$

In this case, values of variables are $n_2 = e = e_{c2} = 0$, $e_{c1} = \frac{\alpha_c \rho_c \theta_q - \theta_e}{(1-\alpha_c)\theta_e \rho_c}$, and

$$n_1 = \frac{(1-\alpha_c)(1-\delta)\rho_c}{2(\rho_c \theta_q - \theta_e)}. \tag{25}$$

By substituting these values of n_1, e_{c1}, and e_{c2} into $\frac{\partial U}{\partial n_2} \le 0$ when $n_2 = 0$, (17),

$$\frac{-\lambda\theta_q}{1-\delta} + \frac{1}{\frac{(1-\alpha_c)(1-\delta)\rho_c}{2(\rho_c\theta_q-\theta_e)}\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{(1-\alpha_c)\theta_e}\right]^{\alpha_c}} \le 0.$$

By substituting the values of n_1 and e_{c1} into $\frac{\partial U}{\partial e} \le 0$ when $e = 0$, (24),

$$\frac{-\eta}{1-\delta-\frac{(\rho_c\theta_q-\theta_e)}{(1-\alpha_c)\rho_c}\frac{(1-\alpha_c)(1-\delta)\rho_c}{2(\rho_c\theta_q-\theta_e)}}+\alpha\rho\leq 0.$$

Hence, the necessary conditions for this case to be realized are

$$\lambda\geq\frac{2{\theta_e}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}{\rho_c\theta_q{\alpha_c}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}, \tag{26}$$

$$\rho\leq\frac{2\eta}{\alpha(1-\delta)}. \tag{27}$$

The right-hand side of (26) is less than 2, because,

$$\frac{2{\theta_e}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}{\rho_c\theta_q{\alpha_c}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}=\frac{2{\theta_e}^{\alpha_c}}{\theta_q{\alpha_c}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}\frac{(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}{\rho_c}<\frac{2{\theta_e}^{\alpha_c}}{\theta_q{\alpha_c}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}\frac{\left(\frac{\theta_e}{\alpha_c\theta_q}\theta_q-\theta_e\right)^{1-\alpha_c}}{\frac{\theta_e}{\alpha_c\theta_q}}=2, \tag{28}$$

where the inequality holds because, from $\rho_c>\frac{\theta_e}{\alpha_c\theta_q}$ (equation (10)),

$$\frac{\partial\frac{(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}{\rho_c}}{\partial\rho_c}=\frac{(\rho_c\theta_q-\theta_e)^{-\alpha_c}(\theta_e-\alpha_c\rho_c\theta_q)}{{\rho_c}^2}<0.$$

② When $n_1>0$ and $e>0$

In this case, values of variables are $n_2=e_{c2}=0$, $e_{c1}=\frac{\alpha_c\rho_c\theta_q-\theta_e}{(1-\alpha_c)\theta_e\rho_c}$,

$$n_1=\frac{(1-\alpha_c)\rho_c[(1-\delta)\rho+\eta]}{(2+\alpha)\rho(\rho_c\theta_q-\theta_e)}, \tag{29}$$

and $e=\frac{(1-\delta)\alpha\rho-2\eta}{(2+\alpha)\rho\eta}$. (30)

By substituting these values of n_1, e_{c1}, and e_{c2} into $\frac{\partial U}{\partial n_2}\leq 0$ when $n_2=0$, (17),

$$\frac{-\lambda\theta_q}{1-\delta}+\frac{(2+\alpha)\rho(\rho_c\theta_q-\theta_e)}{(1-\alpha_c)\rho_c[(1-\delta)\rho+\eta]\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{(1-\alpha_c)\theta_e}\right]^{\alpha_c}}\leq 0.$$

By substituting the values of n_1 and e_{c1} into $\frac{\partial U}{\partial e}>0$ when $e=0$, (24),

$$\frac{-\eta}{1-\delta-\frac{\rho_c\theta_q-\theta_e}{(1-\alpha_c)\rho_c}\frac{(1-\alpha_c)\rho_c[(1-\delta)\rho+\eta]}{(2+\alpha)\rho(\rho_c\theta_q-\theta_e)}}+\alpha\rho>0$$

Hence, the necessary conditions for this case to be realized are

$$\lambda\geq\frac{(2+\alpha)(1-\delta)\rho{\theta_e}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}{\rho_c\theta_q[(1-\delta)\rho+\eta]{\alpha_c}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}\Leftrightarrow\rho\leq\frac{\lambda\eta}{(1-\delta)\left[\frac{(2+\alpha){\theta_e}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}{\rho_c\theta_q{\alpha_c}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}-\lambda\right]}, \tag{31}$$

$$\rho>\frac{2\eta}{\alpha(1-\delta)}. \tag{32}$$

[2] When $n_2 > 0$

Since $n_2 > 0$, $e_{c2} = \frac{\alpha_c \rho_c \theta_q - \theta_e}{(1-\alpha_c)\theta_e \rho_c}$ from (16). Thus,

$$\theta_q + \theta_e e_{c2} = \frac{\rho_c \theta_q - \theta_e}{(1-\alpha_c)\rho_c},$$

$$(1+\rho_c e_{c2})^{\alpha_c} = \left[\frac{\alpha_c(\rho_c\theta_q - \theta_e)}{\theta_e(1-\alpha_c)}\right]^{\alpha_c}.$$

By substituting these equations into (14) and solving for n_2,

$$n_2 = \frac{(1-\delta)\rho_c(1-\alpha_c)\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{\theta_e(1-\alpha_c)}\right]^{\alpha_c} - \lambda(\rho_c\theta_q-\theta_e)(1+\rho_c e_{c1})^{\alpha_c} n_1}{2\lambda(\rho_c\theta_q-\theta_e)\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{\theta_e(1-\alpha_c)}\right]^{\alpha_c}}. \tag{33}$$

Hence,

$$1-\delta-\lambda(\theta_q+\theta_e e_{c2})n_2 = \frac{(1-\delta)\rho_c(1-\alpha_c)\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{\theta_e(1-\alpha_c)}\right]^{\alpha_c} + \lambda(\rho_c\theta_q-\theta_e)(1+\rho_c e_{c1})^{\alpha_c} n_1}{2\rho_c(1-\alpha_c)\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{\theta_e(1-\alpha_c)}\right]^{\alpha_c}},$$

$$n_1(1+\rho_c e_{c1})^{\alpha_c} + n_2(1+\rho_c e_{c2})^{\alpha_c} = \frac{(1-\delta)\rho_c(1-\alpha_c)\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{\theta_e(1-\alpha_c)}\right]^{\alpha_c} + \lambda(\rho_c\theta_q-\theta_e)(1+\rho_c e_{c1})^{\alpha_c} n_1}{2\lambda(\rho_c\theta_q-\theta_e)}.$$

By substituting these equations into (11), the objective function of the period 1 maximization problem becomes

$$U = ln\{Ah_1[1-\delta-\eta e-(\theta_q+\theta_e e_{c1})n_1]\}$$
$$+ln\left\{gA(1+\rho e)^{\alpha}h_1\frac{(1-\delta)\rho_c(1-\alpha_c)\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{\theta_e(1-\alpha_c)}\right]^{\alpha_c} + \lambda(\rho_c\theta_q-\theta_e)(1+\rho_c e_{c1})^{\alpha_c} n_1}{2\rho_c(1-\alpha_c)\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{\theta_e(1-\alpha_c)}\right]^{\alpha_c}}\right\}$$
$$+ln\left\{\frac{(1-\delta)\rho_c(1-\alpha_c)\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{\theta_e(1-\alpha_c)}\right]^{\alpha_c} + \lambda(\rho_c\theta_q-\theta_e)(1+\rho_c e_{c1})^{\alpha_c} n_1}{2\lambda(\rho_c\theta_q-\theta_e)}\right\} \tag{34}$$

The first order conditions are

$$\frac{\partial U}{\partial n_1} = \frac{-(\theta_q+\theta_e e_{c1})}{1-\delta-\eta e-(\theta_q+\theta_e e_{c1})n_1} + \frac{2\lambda(\rho_c\theta_q-\theta_e)(1+\rho_c e_{c1})^{\alpha_c}}{(1-\delta)\rho_c(1-\alpha_c)\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{\theta_e(1-\alpha_c)}\right]^{\alpha_c} + \lambda(\rho_c\theta_q-\theta_e)(1+\rho_c e_{c1})^{\alpha_c} n_1} \le 0, \tag{35}$$

$$\frac{\partial U}{\partial e_{c1}} = n_1\left\{\frac{-\theta_e}{1-\delta-\eta e-(\theta_q+\theta_e e_{c1})n_1} + \frac{2\lambda(\rho_c\theta_q-\theta_e)\alpha_c\rho_c(1+\rho_c e_{c1})^{\alpha_c-1}}{(1-\delta)\rho_c(1-\alpha_c)\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{\theta_e(1-\alpha_c)}\right]^{\alpha_c} + \lambda(\rho_c\theta_q-\theta_e)(1+\rho_c e_{c1})^{\alpha_c} n_1}\right\} \le 0, \tag{36}$$

$$\frac{\partial U}{\partial e} = \frac{-\eta}{1-\delta-\eta e-(\theta_q+\theta_e e_{c1})n_1} + \frac{\alpha\rho}{1+\rho e} \le 0. \tag{37}$$

When $n_1 > 0$, from $\frac{\partial U}{\partial n_1} = 0$,

$$(1-\delta)\rho_c(1-\alpha_c)\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{\theta_e(1-\alpha_c)}\right]^{\alpha_c} + \lambda(\rho_c\theta_q-\theta_e)(1+\rho_c e_{c1})^{\alpha_c} n_1$$

$= \frac{2\lambda(\rho_c\theta_q-\theta_e)(1+\rho_c e_{c1})^{\alpha_c}}{\theta_q+\theta_e e_{c1}}\left[1-\delta-\eta e-(\theta_q+\theta_e e_{c1})n_1\right].$

By substituting this equation into (36),

$$\frac{\partial U}{\partial e_{c1}} = \frac{n_1}{1-\delta-\eta e-(\theta_q+\theta_e e_{c1})n_1}\left[-\theta_e+\frac{\alpha_c\rho_c(\theta_q+\theta_e e_{c1})}{1+\rho_c e_{c1}}\right]. \quad (38)$$

Thus, $e_{c1} > 0$ if $n_1 > 0$, otherwise $e_{c1} = 0$ from $\rho_c > \frac{\theta_e}{\alpha_c\theta_q}$ (equation (10)).

When $n_1 > 0$, $e_{c1} = \frac{\alpha_c\rho_c\theta_q-\theta_e}{(1-\alpha_c)\theta_e\rho_c}$ from $\frac{\partial U}{\partial e_{c1}} = 0$.

When $n_1 = (>)0$, from (35),

$$\frac{\partial U}{\partial n_1}\Big|_{n_1=0} = \frac{-(\theta_q+\theta_e e_{c1})}{1-\delta-\eta e}+\frac{2\lambda(\rho_c\theta_q-\theta_e)(1+\rho_c e_{c1})^{\alpha_c}}{(1-\delta)\rho_c(1-\alpha_c)\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{\theta_e(1-\alpha_c)}\right]^{\alpha_c}} \le (>)0. \quad (39)$$

When $e = (>)0$, from (37),

$$\frac{\partial U}{\partial e}\Big|_{e=0} = \frac{-\eta}{1-\delta-(\theta_q+\theta_e e_{c1})n_1}+\alpha\rho \le (>)0. \quad (40)$$

Depending on whether n_1 and e are positive or zero, there are four cases to consider.

③ When $n_1 > 0$ and $e = 0$

In this case, values of variables are $e = 0, e_{c1} = e_{c2} = \frac{\alpha_c\rho_c\theta_q-\theta_e}{(1-\alpha_c)\theta_e\rho_c}$,

$$n_1 = \frac{(1-\delta)\rho_c(1-\alpha_c)(2\lambda-1)}{3\lambda(\rho_c\theta_q-\theta_e)}, \quad (41)$$

and $n_2 = \frac{(1-\delta)\rho_c(1-\alpha_c)(2-\lambda)}{3\lambda(\rho_c\theta_q-\theta_e)}$. (42)

By substituting these values of n_1, e_{c1}, and e_{c2} into $\frac{\partial U}{\partial n_2} > 0$ when $n_2 = 0$, (17),

$$\frac{-\lambda\frac{\rho_c\theta_q-\theta_e}{\rho_c(1-\alpha_c)}}{1-\delta}+\frac{3\lambda(\rho_c\theta_q-\theta_e)}{(1-\delta)\rho_c(1-\alpha_c)(2\lambda-1)} > 0.$$

By substituting the values of e and e_{c1} into $\frac{\partial U}{\partial n_1} > 0$ when $n_1 = 0$, (39),

$$\frac{-\frac{\rho_c\theta_q-\theta_e}{\rho_c(1-\alpha_c)}}{1-\delta}+\frac{2\lambda(\rho_c\theta_q-\theta_e)}{(1-\delta)\rho_c(1-\alpha_c)} > 0 \Leftrightarrow \lambda > \frac{1}{2},$$

which always holds because of the assumption of $\lambda \ge 1$.

By substituting the values of n_1 and e_{c1} into $\frac{\partial U}{\partial e} \le 0$ when $e = 0$, (40),

$$\frac{-\eta}{1-\delta-\frac{\rho_c\theta_q-\theta_e}{\rho_c(1-\alpha_c)}\frac{(1-\delta)\rho_c(1-\alpha_c)(2\lambda-1)}{3\lambda(\rho_c\theta_q-\theta_e)}}+\alpha\rho \le 0.$$

Hence, the necessary conditions for this case to be realized are

$\lambda < 2,$ (43)

$\rho \leq \frac{3\lambda\eta}{(1-\delta)\alpha(1+\lambda)}.$ (44)

④ When $n_1 > 0$ and $e > 0$

In this case, values of variables are $e_{c1} = e_{c2} = \frac{\alpha_c\rho_c\theta_q-\theta_e}{(1-\alpha_c)\theta_e\rho_c}$,

$n_1 = \frac{(1-\alpha_c)\rho_c\{(1-\delta)\rho[2\lambda-(1+\alpha)]+2\lambda\eta\}}{(3+\alpha)\lambda\rho(\rho_c\theta_q-\theta_e)},$ (45)

$n_2 = \frac{(1-\alpha_c)\rho_c[(1-\delta)\rho(2+\alpha-\lambda)-\lambda\eta]}{(3+\alpha)\lambda\rho(\rho_c\theta_q-\theta_e)},$ (46)

and $e = \frac{(1-\delta)\rho\alpha(1+\lambda)-3\lambda\eta}{(3+\alpha)\lambda\rho\eta}.$ (47)

By substituting these values of n_1, e_{c1}, and e_{c2} into $\frac{\partial U}{\partial n_2} > 0$ when $n_2 = 0$, (17),

$$\frac{-\lambda\frac{\rho_c\theta_q-\theta_e}{\rho_c(1-\alpha_c)}}{1-\delta} + \frac{(3+\alpha)\lambda\rho(\rho_c\theta_q-\theta_e)}{(1-\alpha_c)\rho_c\{(1-\delta)\rho[2\lambda-(1+\alpha)]+2\lambda\eta\}} > 0.$$

By substituting the values of e and e_{c1} into $\frac{\partial U}{\partial n_1} > 0$ when $n_1 = 0$, (39),

$$\frac{-\frac{\rho_c\theta_q-\theta_e}{\rho_c(1-\alpha_c)}}{1-\delta-\eta\frac{(1-\delta)\rho\alpha(1+\lambda)-3\lambda\eta}{(3+\alpha)\lambda\rho\eta}} + \frac{2\lambda(\rho_c\theta_q-\theta_e)}{(1-\delta)(1-\alpha_c)\rho_c} > 0 \Leftrightarrow \lambda > \frac{(1+\alpha)(1-\delta)\rho}{2[(1-\delta)\rho+\eta]},$$

which always holds because of the assumptions of $0 < \alpha < 1$ and $\lambda \geq 1$.

By substituting the values of n_1 and e_{c1} into $\frac{\partial U}{\partial e} > 0$ when $e = 0$, (40),

$$\frac{-\eta}{1-\delta-\frac{\rho_c\theta_q-\theta_e}{\rho_c(1-\alpha_c)}\frac{(1-\alpha_c)\rho_c\{(1-\delta)\rho[2\lambda-(1+\alpha)]+2\lambda\eta\}}{(3+\alpha)\lambda\rho(\rho_c\theta_q-\theta_e)}} + \alpha\rho > 0.$$

Hence, the necessary conditions for this case to be realized are

$\lambda < \frac{(1-\delta)\rho(2+\alpha)}{(1-\delta)\rho+\eta} \Leftrightarrow \rho > \frac{\lambda\eta}{(1-\delta)(2+\alpha-\lambda)},$ (48)

$\lambda < \frac{(1-\delta)\rho\alpha}{3\eta-(1-\delta)\rho\alpha} \Leftrightarrow \rho > \frac{3\lambda\eta}{(1-\delta)\alpha(1+\lambda)}.$ (49)

The right-hand side of (48) is smaller than that of (31),

$$\frac{\lambda\eta}{(1-\delta)(2+\alpha-\lambda)} < \frac{\lambda\eta}{(1-\delta)\left[(2+\alpha)\frac{{\theta_e}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}{\rho_c\theta_q{\alpha_c}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}-\lambda\right]},$$

because $\frac{{\theta_e}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}{\rho_c\theta_q{\alpha_c}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}} < 1$ from (10) and (28).

⑤ When $n_1 = 0$ and $e = 0$

In this case, values of variables are $n_1 = e_{c1} = e = 0$, $e_{c2} = \frac{\alpha_c \rho_c \theta_q - \theta_e}{(1-\alpha_c)\theta_e \rho_c}$, and

$$n_2 = \frac{(1-\delta)(1-\alpha_c)\rho_c}{2\lambda(\rho_c\theta_q - \theta_e)}. \quad (50)$$

By substituting these values of n_1, e_{c1}, and e_{c2} into $\frac{\partial U}{\partial n_2} > 0$ when $n_2 = 0$, (17),

$$\frac{-\lambda\frac{\rho_c\theta_q - \theta_e}{\rho_c(1-\alpha_c)}}{1-\delta} + \frac{1}{0} = \infty > 0.$$

By substituting the values of e and e_{c1} into $\frac{\partial U}{\partial n_1} \leq 0$ when $n_1 = 0$, (39),

$$\frac{-\theta_q}{1-\delta} + \frac{2\lambda(\rho_c\theta_q - \theta_e)}{(1-\delta)(1-\alpha_c)\rho_c\left[\frac{\alpha_c(\rho_c\theta_q - \theta_e)}{\theta_e(1-\alpha_c)}\right]^{\alpha_c}} \leq 0.$$

By substituting the values of n_1 and e_{c1} into $\frac{\partial U}{\partial e} \leq 0$ when $e = 0$, (40),

$$\frac{-\eta}{1-\delta} + \alpha\rho \leq 0.$$

Hence, the necessary conditions for this case to be realized are

$$\lambda \leq \left[\frac{2\theta_e{}^{\alpha_c}(\rho_c\theta_q - \theta_e)^{1-\alpha_c}}{\rho_c\theta_q\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}\right]^{-1}, \quad (51)$$

$$\rho \leq \frac{\eta}{(1-\delta)\alpha}. \quad (52)$$

When ρ_c is small, the right-hand side of (51) becomes smaller than 1, which means that this case does not exist. Especially when ρ_c converges to $\frac{\theta_e}{\alpha_c\theta_q}$, the right-hand side of (51) converges to $\frac{1}{2}$.

⑥ When $n_1 = 0$ and $e > 0$

In this case, values of variables are, $n_1 = e_{c1} = 0$, $e_{c2} = \frac{\alpha_c \rho_c \theta_q - \theta_e}{(1-\alpha_c)\theta_e \rho_c}$

$$n_2 = \frac{(1-\delta)(1-\alpha_c)\rho_c}{2\lambda(\rho_c\theta_q - \theta_e)}, \quad (53)$$

and $e = \frac{(1-\delta)\rho\alpha - \eta}{(1+\alpha)\rho\eta}$. (54)

By substituting these values of n_1, e_{c1}, and e_{c2} into $\frac{\partial U}{\partial n_2} > 0$ when $n_2 = 0$, (17),

$$\frac{-\lambda\frac{\rho_c\theta_q - \theta_e}{\rho_c(1-\alpha_c)}}{1-\delta} + \frac{1}{0} = \infty > 0.$$

By substituting the values of e and e_{c1} into $\frac{\partial U}{\partial n_1} \leq 0$ when $n_1 = 0$, (39),

$$\frac{-\theta_q}{1-\delta-\eta\frac{(1-\delta)\rho\alpha-\eta}{(1+\alpha)\rho\eta}}+\frac{2\lambda(\rho_c\theta_q-\theta_e)}{(1-\delta)(1-\alpha_c)\rho_c\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{\theta_e(1-\alpha_c)}\right]^{\alpha_c}}\le 0.$$

By substituting the values of n_1 and e_{c1} into $\frac{\partial U}{\partial e}>0$ when $e=0$, (40),

$$\frac{-\eta}{1-\delta}+\alpha\rho>0.$$

Hence, the necessary conditions for this case to be realized are

$$\lambda\le\frac{\rho_c\theta_q}{2}\frac{(1+\alpha)(1-\delta)\rho}{[(1-\delta)\rho+\eta]}\left(\frac{\alpha_c}{\theta_e}\right)^{\alpha_c}\left(\frac{1-\alpha_c}{\rho_c\theta_q-\theta_e}\right)^{1-\alpha_c}\Leftrightarrow\rho\ge\frac{\lambda\eta}{(1-\delta)\left[\frac{(1+\alpha)\rho_c\theta_q\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}{2\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}-\lambda\right]},\qquad(55)$$

$$\rho>\frac{\eta}{(1-\delta)\alpha}.\qquad(56)$$

When $\frac{\rho_c\theta_q\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}{2\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}>(<)\frac{2+\alpha}{1+\alpha}$, the right-hand side of (55) is smaller (greater) than that of (48), i.e.,

$$\frac{\lambda\eta}{(1-\delta)\left[\frac{(1+\alpha)\rho_c\theta_q\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}{2\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}-\lambda\right]}<(>)\frac{\lambda\eta}{(1-\delta)(2+\alpha-\lambda)}.$$

When $\frac{\rho_c\theta_q\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}{2\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}>(<)\frac{3+2\alpha}{3(1+\alpha)}$, the right-hand side of (55) is smaller (greater) than that of (49) at $\lambda=1$, i.e., $\frac{\lambda\eta}{(1-\delta)\left[\frac{(1+\alpha)\rho_c\theta_q\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}{2\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}-\lambda\right]}<(>)\frac{3\lambda\eta}{(1-\delta)\alpha(1+\lambda)}$.

Solutions of the utility maximization problem

Figure A.1[16] and Figure A.2[17] illustrate necessary conditions for the existence of the six cases on the (ρ,λ) plane when $\frac{2\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}{\rho_c\theta_q\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}>1$ and when $\frac{2\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}{\rho_c\theta_q\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}<1$, respectively. From these figures, we can see that some cases overlap. This does not mean multiple solutions because the utilities obtained from these cases are different as shown below. To find the correct solutions, we need to compare the utilities of the overlapping areas.

Since $e_{ci}=\frac{\alpha_c\rho_c\theta_q-\theta_e}{(1-\alpha_c)\theta_e\rho_c}$ if $n_i>0$, otherwise $e_{ci}=0$, $i=1,2$, from (9), we have $h_{ci}=\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{(1-\alpha_c)\theta_e}\right]^{\alpha_c}$ if $n_i>0$, otherwise $h_{ci}=1$. The utilities of the six cases are computed as follows.

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16 To be precise, Figure A.1 shows the case when $\frac{2\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}{\rho_c\theta_q\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}<\frac{3(1+\alpha)}{3+2\alpha}$. When $\frac{2\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}{\rho_c\theta_q\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}>\frac{3(1+\alpha)}{3+2\alpha}$, $\rho=\frac{\lambda\eta}{(1-\delta)\left[\frac{(1+\alpha)\rho_c\theta_q\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}{2\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}-\lambda\right]}$ is located below $\rho=\frac{3\lambda\eta}{\alpha(1-\delta)(1+\lambda)}$, implying that case ③ and case ⑥ do not overlap.

17 To be precise, Figure A.2 shows the case when $\frac{2\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}{\rho_c\theta_q\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}<\frac{1}{2}$. When $\frac{2\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}{\rho_c\theta_q\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}>\frac{1}{2}$, Case ⑤ exists only for $\lambda<2$. When $\frac{2\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}{\rho_c\theta_q\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}>\frac{1+\alpha}{2+\alpha}>\frac{1}{2}$, the positions of the two boundary lines $\rho=\frac{\lambda\eta}{(1-\delta)\left[\frac{(1+\alpha)\rho_c\theta_q\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}{2\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}-\lambda\right]}$ and $\rho=\frac{\lambda\eta}{(1-\delta)(2+\alpha-\lambda)}$ are reversed, so $\rho=\frac{\lambda\eta}{(1-\delta)\left[\frac{(1+\alpha)\rho_c\theta_q\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}{2\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}-\lambda\right]}$ and $\rho=\frac{3\lambda\eta}{\alpha(1-\delta)(1+\lambda)}$ intersect.

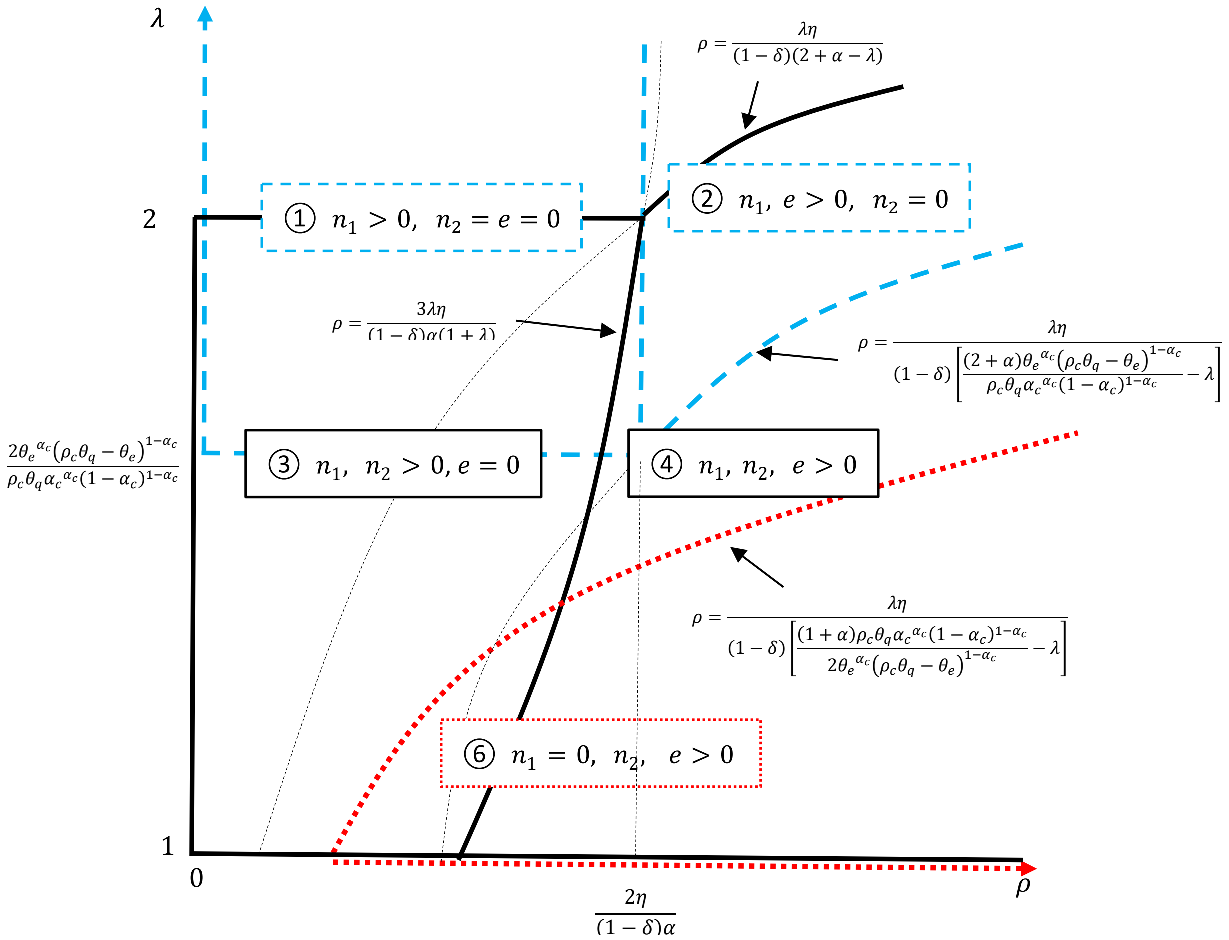

Figure A.1 Necessary conditions for the existence of the six cases when $\frac{2\theta_e^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}{\rho_c\theta_q\alpha_c^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}} > 1$

① When $n_1 > 0$ and $n_2 = e = 0$

$n_1 = \frac{(1-\alpha_c)(1-\delta)\rho_c}{2(\rho_c\theta_q-\theta_e)}$ from (25). Thus, from (3), (4) and (8), $c_1 = Ah_1\frac{1-\delta}{2}$, $c_2 = gAh_1(1-\delta)$, and $n_1 h_{c1} + n_2 h_{c2} = \frac{(1-\alpha_c)(1-\delta)\rho_c}{2(\rho_c\theta_q-\theta_e)}\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{(1-\alpha_c)\theta_e}\right]^{\alpha_c}$. Hence, from (1), the utility of case ① is

$$U_1 = ln\left[\frac{gA^2h_1^2\rho_c(1-\delta)\alpha_c^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}{\theta_e^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}\right] + 2ln\left(\frac{1-\delta}{2}\right). \tag{57}$$

② When $n_1 > 0,\ n_2 = 0$ and $e > 0$

$n_1 = \frac{(1-\alpha_c)\rho_c[(1-\delta)\rho+\eta]}{(2+\alpha)\rho(\rho_c\theta_q-\theta_e)}$ and $e = \frac{(1-\delta)\alpha\rho-2\eta}{(2+\alpha)\rho\eta}$ from (29) and (30). Thus, from (3), (4) and (8), $c_1 = Ah_1\frac{(1-\delta)\rho+\eta}{(2+\alpha)\rho}$, $c_2 = gAh_1(1-\delta)\left\{\frac{\alpha[(1-\delta)\rho+\eta]}{(2+\alpha)\eta}\right\}^{\alpha}$, and $n_1 h_{c1} + n_2 h_{c2} = \frac{(1-\alpha_c)\rho_c[(1-\delta)\rho+\eta]}{(2+\alpha)\rho(\rho_c\theta_q-\theta_e)}\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{(1-\alpha_c)\theta_e}\right]^{\alpha_c}$. Hence, the utility of case ② is

$$U_2 = ln\left[\frac{gA^2h_1^2\rho_c(1-\delta)\alpha_c^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}{\theta_e^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}\right] + 2ln\left[\frac{(1-\delta)\rho+\eta}{(2+\alpha)\rho}\right] + \alpha ln\left\{\frac{\alpha[(1-\delta)\rho+\eta]}{(2+\alpha)\eta}\right\}. \tag{58}$$

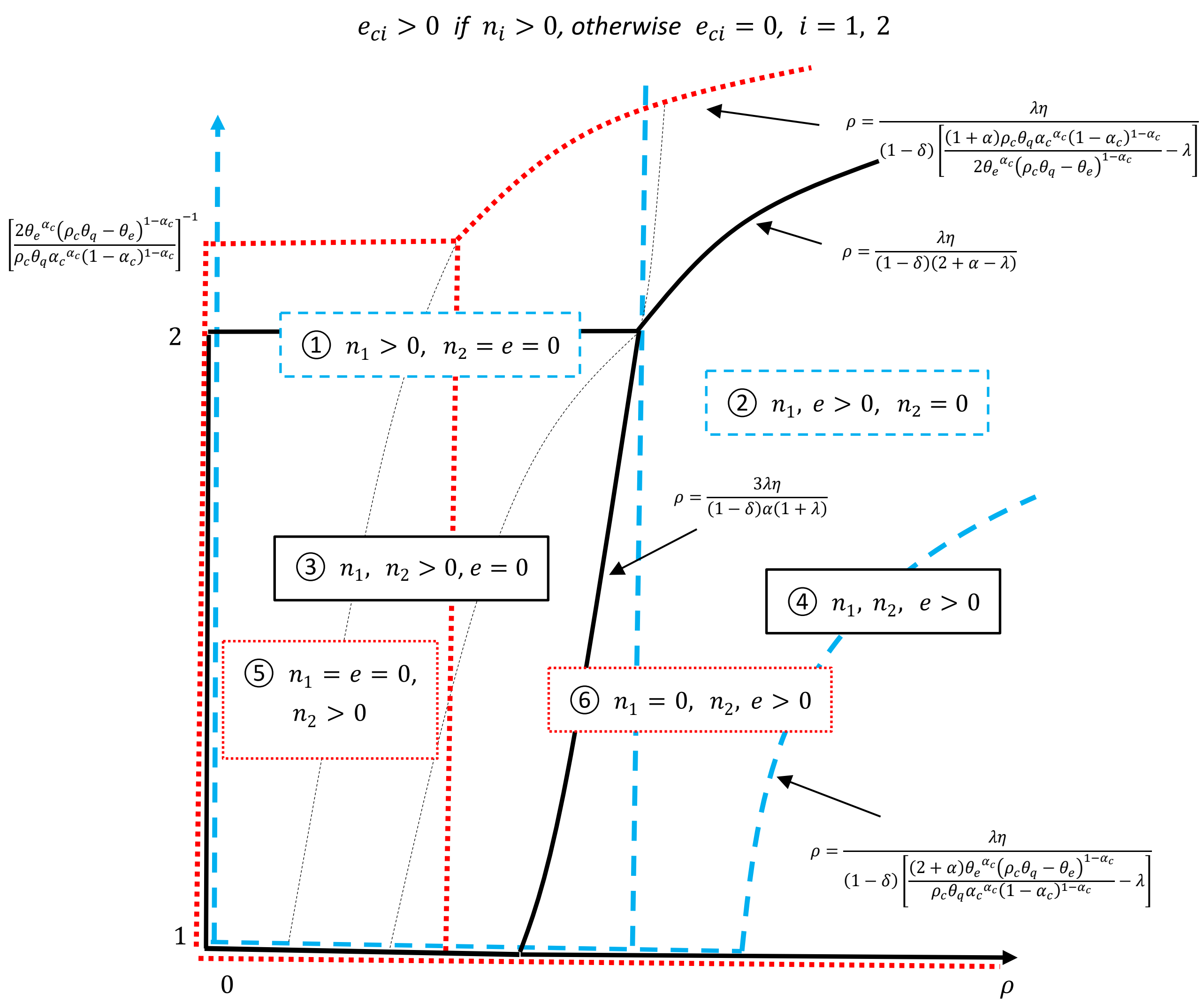

Figure A.2 Necessary conditions for the existence of the six cases when $\frac{2\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}{\rho_c\theta_q\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}} < 1$

③ When $n_1 > 0,\ n_2 > 0$ and $e = 0$

$n_1 = \frac{(1-\delta)\rho_c(1-\alpha_c)(2\lambda-1)}{3\lambda(\rho_c\theta_q-\theta_e)}$ and $n_2 = \frac{(1-\delta)\rho_c(1-\alpha_c)(2-\lambda)}{3\lambda(\rho_c\theta_q-\theta_e)}$ from (41) and (42). Thus, from (3), (4) and (8), $c_1 = Ah_1(1-\delta)\frac{1+\lambda}{3\lambda}$, $c_2 = gAh_1(1-\delta)\frac{1+\lambda}{3}$, and $n_1h_{c1} + n_2h_{c2} = \frac{(1-\delta)\rho_c(1-\alpha_c)(\lambda+1)}{3\lambda(\rho_c\theta_q-\theta_e)}\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{(1-\alpha_c)\theta_e}\right]^{\alpha_c}$. Hence, the utility of case ③ is

$$U_3 = ln\left[\frac{gA^2h_1^2\rho_c(1-\delta)\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}{\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}\right] + 2ln\left[\frac{(1-\delta)(1+\lambda)}{3\lambda}\right] + ln\left(\frac{1+\lambda}{3}\right). \tag{59}$$

④ When $n_1 > 0,\ n_2 > 0$ and $e > 0$

$n_1 = \frac{(1-\alpha_c)\rho_c\{(1-\delta)\rho[2\lambda-(1+\alpha)]+2\lambda\eta\}}{(3+\alpha)\lambda\rho(\rho_c\theta_q-\theta_e)}$, $n_2 = \frac{(1-\alpha_c)\rho_c[(1-\delta)\rho(2+\alpha-\lambda)-\lambda\eta]}{(3+\alpha)\lambda\rho(\rho_c\theta_q-\theta_e)}$ and $e = \frac{(1-\delta)\rho\alpha(1+\lambda)-3\lambda\eta}{(3+\alpha)\lambda\rho\eta}$ from (45), (46) and (47). Thus, from (3), (4) and (8), $c_1 = Ah_1\frac{(1-\delta)\rho(1+\lambda)+\eta\lambda}{(3+\alpha)\lambda\rho}$, $c_2 = gAh_1\left\{\frac{\alpha[(1-\delta)\rho(1+\lambda)+\eta\lambda]}{(3+\alpha)\eta\lambda}\right\}^{\alpha}\frac{(1-\delta)\rho(1+\lambda)+\eta\lambda}{(3+\alpha)\rho}$, and

$n_1h_{c1}+n_2h_{c2}=\frac{(1-\alpha_c)\rho_c[(1-\delta)\rho(\lambda+1)+\lambda\eta]}{(3+\alpha)\lambda\rho(\rho_c\theta_q-\theta_e)}\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{(1-\alpha_c)\theta_e}\right]^{\alpha_c}$. Hence, the utility of case ④ is

$$U_4=ln\left[\frac{gA^2h_1^2\rho_c(1-\delta)\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}{\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}\right]+2ln\left[\frac{(1-\delta)\rho(1+\lambda)+\eta\lambda}{(3+\alpha)\lambda\rho}\right]+ln\left[\frac{(1-\delta)\rho(1+\lambda)+\eta\lambda}{(1-\delta)(3+\alpha)\rho}\right]$$

$$+\alpha ln\left\{\frac{\alpha[(1-\delta)\rho(1+\lambda)+\eta\lambda]}{(3+\alpha)\eta\lambda}\right\}. \quad (60)$$

⑤ When $n_1=0,\ n_2>0$ and $e=0$

$n_2=\frac{(1-\delta)(1-\alpha_c)\rho_c}{2\lambda(\rho_c\theta_q-\theta_e)}$ from (50). Thus, from (3), (4) and (8), $c_1=Ah_1(1-\delta)$, $c_2=gAh_1\frac{1-\delta}{2}$, and $n_1h_{c1}+n_2h_{c2}=\frac{(1-\delta)(1-\alpha_c)\rho_c}{2\lambda(\rho_c\theta_q-\theta_e)}\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{(1-\alpha_c)\theta_e}\right]^{\alpha_c}$. Hence, the utility of case ⑤ is

$$U_5=ln\left[\frac{gA^2h_1^2\rho_c(1-\delta)\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}{\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}\right]+2ln\left(\frac{1-\delta}{2}\right)+ln\left(\frac{1}{\lambda}\right). \quad (61)$$

⑥ When $n_1=0,\ n_2>0$ and $e>0$

$n_2=\frac{(1-\delta)(1-\alpha_c)\rho_c}{2\lambda(\rho_c\theta_q-\theta_e)}$ and $e=\frac{(1-\delta)\rho\alpha-\eta}{(1+\alpha)\rho\eta}$ from (53) and (54). Thus, from (3), (4) and (8), $c_1=Ah_1\frac{(1-\delta)\rho+\eta}{(1+\alpha)\rho}$, $c_2=gAh_1\left\{\frac{\alpha[(1-\delta)\rho+\eta]}{(1+\alpha)\eta}\right\}^{\alpha}\frac{1-\delta}{2}$, and $n_1h_{c1}+n_2h_{c2}=\frac{(1-\delta)(1-\alpha_c)\rho_c}{2\lambda(\rho_c\theta_q-\theta_e)}\left[\frac{\alpha_c(\rho_c\theta_q-\theta_e)}{(1-\alpha_c)\theta_e}\right]^{\alpha_c}$. Hence, the utility of case ⑥ is

$$U_6=ln\left[\frac{gA^2h_1^2\rho_c(1-\delta)\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}{\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}\right]+2ln\left(\frac{1}{2}\right)+ln\left[\frac{(1-\delta)[(1-\delta)\rho+\eta]}{(1+\alpha)\lambda\rho}\right]+\alpha ln\left\{\frac{\alpha[(1-\delta)\rho+\eta]}{(1+\alpha)\eta}\right\}. \quad (62)$$

The case of Figure A.1

The figure shows that we need to compare the utilities of the following pairs of cases: case ① and case ③, case ① and case ④, case ② and case ④, case ③ and case ⑥, case ④ and case ⑥.

[1] case ① and case ③

From (57) and(59), we have

$$U_3-U_1=2ln\left[\frac{2(1+\lambda)}{3\lambda}\right]+ln\left(\frac{1+\lambda}{3}\right)=ln\left[\frac{4(1+\lambda)^3}{27\lambda^2}\right]. \quad (63)$$

Since

$\frac{d\left[\frac{4(1+\lambda)^3}{27\lambda^2}\right]}{d\lambda}=\frac{4}{27}\frac{(1+\lambda)^2}{\lambda^3}(\lambda-2)<0$ from $\lambda<2$ (equation (43)), $U_3-U_1>ln\left[\frac{4(1+2)^3}{27(2)^2}\right]=0$ and thus $U_3>U_1$ holds.

[2] case ① and case ④

From (57) and(60), we have

$$U_4-U_1=2ln\left\{\frac{2[(1-\delta)\rho(1+\lambda)+\eta\lambda]}{(3+\alpha)(1-\delta)\lambda\rho}\right\}+ln\left[\frac{(1-\delta)\rho(1+\lambda)+\eta\lambda}{(3+\alpha)(1-\delta)\rho}\right]+\alpha ln\left\{\frac{\alpha[(1-\delta)\rho(1+\lambda)+\eta\lambda]}{(3+\alpha)\eta\lambda}\right\}$$

$$=ln\left\{\frac{2^2\alpha^{\alpha}[(1-\delta)\rho(1+\lambda)+\eta\lambda]^{3+\alpha}}{(3+\alpha)^{3+\alpha}\eta^{\alpha}(1-\delta)^3\lambda^{2+\alpha}\rho^3}\right\}.$$

Because

$$\frac{d\left\{\frac{2^2\alpha^\alpha[(1-\delta)\rho(1+\lambda)+\eta\lambda]^{3+\alpha}}{(3+\alpha)^{3+\alpha}\eta^\alpha(1-\delta)^3\lambda^{2+\alpha}\rho^3}\right\}}{d\rho} = \frac{2^2\alpha^\alpha}{(3+\alpha)^{3+\alpha}\eta^\alpha(1-\delta)^3\lambda^{2+\alpha}}\frac{[(1-\delta)\rho(1+\lambda)+\eta\lambda]^{2+\alpha}}{\rho^4}[\alpha(1-\delta)(1+\lambda)\rho-3\eta\lambda] > 0$$ from $\rho > \frac{3\eta\lambda}{(1-\delta)\alpha(1+\lambda)}$ (equation (49)),

$$U_4 - U_1 > \ ln\left\{\frac{2^2\alpha^\alpha\left[(1-\delta)\frac{3\eta\lambda}{(1-\delta)\alpha(1+\lambda)}(1+\lambda)+\eta\lambda\right]^{3+\alpha}}{(3+\alpha)^{3+\alpha}\eta^\alpha(1-\delta)^3\lambda^{2+\alpha}\left[\frac{3\eta\lambda}{(1-\delta)\alpha(1+\lambda)}\right]^3}\right\} = ln\left[\frac{4(1+\lambda)^3}{27\lambda^2}\right].$$

Since $\lambda < 2$ from (27) and (49), $\frac{d\left[\frac{4(1+\lambda)^3}{27\lambda^2}\right]}{d\lambda} = \frac{4}{27}\frac{(1+\lambda)^2}{\lambda^3}(\lambda-2) < 0$. Hence, $U_4 - U_1 > ln\left[\frac{4(1+2)^3}{27(2)^2}\right] = 0$ and thus $U_4 > U_1$ holds.

[3] case ② and case ④

From (58) and(60), we have

$$U_4 - U_2 = (2+\alpha)ln\left\{\frac{(2+\alpha)[(1-\delta)\rho(1+\lambda)+\eta\lambda]}{(3+\alpha)[(1-\delta)\rho+\eta]\lambda}\right\} + ln\left[\frac{(1-\delta)\rho(1+\lambda)+\eta\lambda}{(3+\alpha)(1-\delta)\rho}\right]$$

$$= ln\left\{\frac{(2+\alpha)^{2+\alpha}[(1-\delta)\rho(1+\lambda)+\eta\lambda]^{3+\alpha}}{(3+\alpha)^{3+\alpha}(1-\delta)\lambda^{2+\alpha}\rho[(1-\delta)\rho+\eta]^{2+\alpha}}\right\}.$$

Because

$$\frac{d\left\{\frac{(2+\alpha)^{2+\alpha}[(1-\delta)\rho(1+\lambda)+\eta\lambda]^{3+\alpha}}{(3+\alpha)^{3+\alpha}(1-\delta)\lambda^{2+\alpha}\rho[(1-\delta)\rho+\eta]^{2+\alpha}}\right\}}{d\rho} = \frac{(2+\alpha)^{2+\alpha}}{(3+\alpha)^{3+\alpha}(1-\delta)\lambda^{2+\alpha}}\frac{\eta[(1-\delta)\rho(1+\lambda)+\eta\lambda]^{2+\alpha}}{\rho^2[(1-\delta)\rho+\eta]^{3+\alpha}}[(1-\delta)\rho(2+\alpha-\lambda)-\eta\lambda] > 0$$ from $\rho > \frac{\lambda\eta}{(1-\delta)(2+\alpha-\lambda)}$ (equation (48)),

$$U_4 - U_2 > \ ln\left\{\frac{(2+\alpha)^{2+\alpha}\left[(1-\delta)\frac{\lambda\eta}{(1-\delta)(2+\alpha-\lambda)}(1+\lambda)+\eta\lambda\right]^{3+\alpha}}{(3+\alpha)^{3+\alpha}(1-\delta)\lambda^{2+\alpha}\frac{\lambda\eta}{(1-\delta)(2+\alpha-\lambda)}\left[(1-\delta)\frac{\lambda\eta}{(1-\delta)(2+\alpha-\lambda)}+\eta\right]^{2+\alpha}}\right\} = 0$$ and thus $U_4 > U_2$ holds.

[4] case③ and case ⑥

From (59) and (62), we have

$$U_6 - U_3 = 2ln\left[\frac{3\lambda}{2(1-\delta)(1+\lambda)}\right] + ln\left[\frac{3(1-\delta)[(1-\delta)\rho+\eta]}{(1+\alpha)\lambda(1+\lambda)\rho}\right] + \alpha ln\left\{\frac{\alpha[(1-\delta)\rho+\eta]}{(1+\alpha)\eta}\right\}$$

$$= ln\left\{\frac{3^3\alpha^\alpha[(1-\delta)\rho+\eta]^{1+\alpha}\lambda}{2^2(1+\alpha)^{1+\alpha}\eta^\alpha(1-\delta)\rho(1+\lambda)^3}\right\}.$$

Because

$$\frac{d\left\{\frac{3^3\alpha^\alpha[(1-\delta)\rho+\eta]^{1+\alpha}\lambda}{2^2(1+\alpha)^{1+\alpha}\eta^\alpha(1-\delta)\rho(1+\lambda)^3}\right\}}{d\rho} = \frac{3^3\alpha^\alpha\lambda}{2^2(1+\alpha)^{1+\alpha}\eta^\alpha(1-\delta)(1+\lambda)^3}\frac{[(1-\delta)\rho+\eta]^\alpha}{\rho^2}[\alpha(1-\delta)\rho-\eta] > 0$$ from $\rho > \frac{\eta}{(1-\delta)\alpha}$ (equation (56)), from (44),

$$U_6 - U_3 \le \ ln\left\{\frac{3^3\alpha^\alpha\left[(1-\delta)\frac{3\eta\lambda}{(1-\delta)\alpha(1+\lambda)}+\eta\right]^{1+\alpha}\lambda}{2^2(1+\alpha)^{1+\alpha}\eta^\alpha(1-\delta)\frac{3\eta\lambda}{(1-\delta)\alpha(1+\lambda)}(1+\lambda)^3}\right\} = \ ln\left[\frac{3^2(3\lambda+\alpha\lambda+\alpha)^{1+\alpha}}{2^2(1+\alpha)^{1+\alpha}(1+\lambda)^{3+\alpha}}\right].$$

Further, since $\frac{d\left[\frac{3^2(3\lambda+\alpha\lambda+\alpha)^{1+\alpha}}{2^2(1+\alpha)^{1+\alpha}(1+\lambda)^{3+\alpha}}\right]}{d\lambda} = \frac{3^2(3+\alpha)}{2^2(1+\alpha)^{1+\alpha}}\frac{(3\lambda+\alpha\lambda+\alpha)^\alpha}{(1+\lambda)^{4+\alpha}}(1-2\lambda) < 0$ from $\lambda > 1$, $U_6 - U_3 < ln\left[\frac{3^2(3+\alpha+\alpha)^{1+\alpha}}{2^2(1+\alpha)^{1+\alpha}(1+1)^{3+\alpha}}\right] = ln\left[\left(\frac{3}{4}\right)^2\left(\frac{3+2\alpha}{2+2\alpha}\right)^{1+\alpha}\right]$.

Because

$\frac{d\left(\frac{3+2\alpha}{2+2\alpha}\right)^{1+\alpha}}{d\alpha} = \left(\frac{3+2\alpha}{2+2\alpha}\right)^{1+\alpha}\left[ln\left(\frac{3+2\alpha}{2+2\alpha}\right) - \frac{1}{3+2\alpha}\right] > \left(\frac{3+2\alpha}{2+2\alpha}\right)^{1+\alpha}\left[ln\left(\frac{5}{4}\right) - \frac{1}{5}\right] > \left(\frac{3+2\alpha}{2+2\alpha}\right)^{1+\alpha}[0.22 - 0.2] > 0$ from $\alpha < 1$ and $\frac{d\left[ln\left(\frac{3+2\alpha}{2+2\alpha}\right) - \frac{1}{3+2\alpha}\right]}{d\alpha} = \frac{2}{3+2\alpha}\left(\frac{1}{3+2\alpha} - \frac{1}{2+2\alpha}\right) < 0$, $U_6 - U_3 < ln\left[\left(\frac{3}{4}\right)^2\left(\frac{5}{4}\right)^2\right] < 0$ and thus $U_6 < U_3$ holds.

[5] case ④ and case ⑥

From (60) and (62), we have

$$U_4 - U_6 = 2ln\left\{\frac{2[(1-\delta)\rho(1+\lambda)+\eta\lambda]}{(3+\alpha)(1-\delta)\rho}\right\} + (1+\alpha)ln\left\{\frac{(1+\alpha)[(1-\delta)\rho(1+\lambda)+\eta\lambda]}{(3+\alpha)\lambda[(1-\delta)\rho+\eta]}\right\}$$

$$= ln\left\{\frac{2^2(1+\alpha)^{1+\alpha}[(1-\delta)\rho(1+\lambda)+\eta\lambda]^{3+\alpha}}{(3+\alpha)^{3+\alpha}(1-\delta)^2\rho^2[(1-\delta)\rho+\eta]^{1+\alpha}\lambda^{1+\alpha}}\right\}.$$

$\frac{d\left\{\frac{2^2(1+\alpha)^{1+\alpha}[(1-\delta)\rho(1+\lambda)+\eta\lambda]^{3+\alpha}}{(3+\alpha)^{2+\alpha}(1-\delta)^2\rho^2[(1-\delta)\rho+\eta]^{1+\alpha}\lambda^{1+\alpha}}\right\}}{d\rho} = \frac{2^2(1+\alpha)^{1+\alpha}}{(3+\alpha)^{2+\alpha}(1-\delta)^2\lambda^{1+\alpha}}\frac{\eta[(1-\delta)\rho(1+\lambda)+\eta\lambda]^{2+\alpha}}{\rho^3[(1-\delta)\rho+\eta]^{2+\alpha}}[(1-\delta)\rho(1+\alpha-2\lambda) - 2\eta\lambda] < 0$ and $\frac{d\left\{\frac{2^2(1+\alpha)^{1+\alpha}[(1-\delta)\rho(1+\lambda)+\eta\lambda]^{3+\alpha}}{(3+\alpha)^{2+\alpha}(1-\delta)^2\rho^2[(1-\delta)\rho+\eta]^{1+\alpha}\lambda^{1+\alpha}}\right\}}{d\lambda} = \frac{2^2(1+\alpha)^{1+\alpha}}{(3+\alpha)^{2+\alpha}(1-\delta)^2\rho^2[(1-\delta)\rho+\eta]^{1+\alpha}}\frac{[(1-\delta)\rho(1+\lambda)+\eta\lambda]^{2+\alpha}}{\lambda^{2+\alpha}}\{2[(1-\delta)\rho+\eta]\lambda - (1+\alpha)(1-\delta)\rho\} > 0$ from $\lambda > 1$.

Hence, $U_4 - U_6 > (U_4 - U_6)|_{\lambda=1,\rho=\infty} = ln\left[\frac{4^2(2+2\alpha)^{1+\alpha}}{(3+\alpha)^{3+\alpha}}\right] > ln\left[\frac{4^2 4^2}{4^4}\right] = 0$ from $\alpha < 1$ and $\frac{d\left[\frac{(2+2\alpha)^{1+\alpha}}{(3+\alpha)^{3+\alpha}}\right]}{d\alpha} = \frac{(2+2\alpha)^{1+\alpha}}{(3+\alpha)^{3+\alpha}}ln\left(\frac{2+2\alpha}{3+\alpha}\right) < 0$ and thus $U_4 > U_6$ holds.

Based on these results, solutions of the maximization problem and the conditions for the existence of these cases are obtained as presented in Proposition 1 and as illustrated in Figure 1.

The case of Figure A.2

The figure shows that, besides the comparisons above, we need to compare the utilities of the following pairs of cases: case ① and case ⑤, case ① and case ⑥, case ② and case ⑥, case ③ and case ⑤.

[6] case ① and case ⑤

From (57) and (61), we have

$U_5 - U_1 = ln\left(\frac{1}{\lambda}\right) < 0$ from $\lambda > 1$ and thus $U_5 < U_1$ holds.

[7] case ① and case ⑥

From (57) and (62), we have

$$U_6 - U_1 = ln\left(\frac{(1-\delta)\rho+\eta}{(1+\alpha)(1-\delta)\rho\lambda}\right) + \alpha ln\left\{\frac{\alpha[(1-\delta)\rho+\eta]}{(1+\alpha)\eta}\right\} = ln\left\{\frac{\alpha^\alpha[(1-\delta)\rho+\eta]^{1+\alpha}}{(1+\alpha)^{1+\alpha}(1-\delta)\eta^\alpha\rho\lambda}\right\}.$$

From Figure A.2, case ① and case ⑥ overlap in the area where $\rho \in \left[\frac{\eta}{(1-\delta)\alpha}, \frac{2\eta}{(1-\delta)\alpha}\right]$ and $\lambda \in \left[1, \frac{(1-\delta)\rho}{(1-\delta)\rho+\eta}\frac{(1+\alpha)\rho_c\theta_q{\alpha_c}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}{2{\theta_e}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}\right]$ hold. We have already shown that $U_3 > U_1$ and $U_3 > U_6$ in the sub-area where $\rho \leq \frac{3\lambda\eta}{(1-\delta)\alpha(1+\lambda)}$ and $\lambda < 2$ hold, and $U_4 > U_1$ and $U_4 > U_6$ (implying that neither case ① nor case ⑥ is optimal) in the sub-area where $\rho > \frac{3\lambda\eta}{(1-\delta)\alpha(1+\lambda)}$ holds. Hence, there is only the area where $\rho < \frac{2\eta}{(1-\delta)\alpha}$ and $\lambda \geq$

2 hold left to consider (see Figure A.2).

Because $\frac{d\left\{\frac{\alpha^{\alpha}[(1-\delta)\rho+\eta]^{1+\alpha}}{(1+\alpha)^{1+\alpha}(1-\delta)\eta^{\alpha}\rho\lambda}\right\}}{d\rho} = \frac{\alpha^{\alpha}}{(1+\alpha)^{1+\alpha}(1-\delta)\eta^{\alpha}\lambda}\frac{[(1-\delta)\rho+\eta]^{\alpha}}{\rho^{2}}[\alpha(1-\delta)\rho-\eta] > 0$ from $\rho > \frac{\eta}{(1-\delta)\alpha}$ (equation (56)) and $\frac{\alpha^{\alpha}[(1-\delta)\rho+\eta]^{1+\alpha}}{(1+\alpha)^{1+\alpha}(1-\delta)\eta^{\alpha}\rho\lambda}$ decreases with λ, $U_6 - U_1 < (U_6 - U_1)|_{\lambda=2,\rho=\frac{2\eta}{(1-\delta)\alpha}} = ln\frac{(2+\alpha)^{1+\alpha}}{4(1+\alpha)^{1+\alpha}} < 0$ and thus $U_6 < U_1$ holds.

[8] case ② and case ⑥

From (58) and (62), we have

$$U_6 - U_2 = ln\left\{\frac{(2+\alpha)(1-\delta)\rho}{4\lambda[(1-\delta)\rho+\eta]}\right\} + (1+\alpha)ln\left(\frac{2+\alpha}{1+\alpha}\right).$$

Case ② and case ⑥ overlap in the area where $\rho \in \left[\frac{2\eta}{(1-\delta)\alpha}, \frac{\lambda\eta}{(1-\delta)\left[\frac{(2+\alpha)\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}{\rho_c\theta_q\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}} - \lambda\right]}\right]$ and $\lambda \in \left[1, \frac{(1-\delta)\rho}{(1-\delta)\rho+\eta}\frac{(1+\alpha)\rho_c\theta_q\alpha_c{}^{\alpha_c}(1-\alpha_c)^{1-\alpha_c}}{2\theta_e{}^{\alpha_c}(\rho_c\theta_q-\theta_e)^{1-\alpha_c}}\right]$ hold from Figure A.2 and footnote 18. We have already shown that $U_4 > U_6$ and $U_4 > U_2$ (implying that neither case ② nor case ⑥ is optimal) in the sub-area where $\rho > \frac{\lambda\eta}{(1-\delta)(2+\alpha-\lambda)}$ holds. Hence, there is only the area $\rho < \frac{\lambda\eta}{(1-\delta)(2+\alpha-\lambda)}$ left to consider.

Because $\frac{(2+\alpha)(1-\delta)\rho}{4\lambda[(1-\delta)\rho+\eta]}$ increases with ρ and decreases with λ,

$U_6 - U_2 < (U_6 - U_2)|_{\lambda=1,\rho=\frac{\lambda\eta}{(1-\delta)(2+\alpha-\lambda)}} = ln\frac{(2+\alpha)^{1+\alpha}}{4(1+\alpha)^{1+\alpha}} < 0$ and thus $U_6 < U_2$ holds.

[9] case ③ and case ⑤

We have already shown that $U_3 > U_1$ and $U_1 > U_5$, thus $U_3 > U_5$ holds.

Based on these results, solutions of the maximization problem and the conditions for the existence of these cases are obtained as presented in Proposition 1 and as illustrated in Figure 1. ■

Proof of Proposition 2

When $g_{\rho_c} = g_{\delta} = 1$, all the inequalities on λ except those associated with g_{ρ_c} of Propositions A.1, A.2, and A.3 become $\lambda < (\geq) 2$ in case (3) and $\lambda < (\geq) \frac{(2+\alpha)\rho}{(1-\delta)\rho+\eta}$ (the one associated with δ becomes $\lambda < (\geq) 2+\alpha$) in case (4). Hence, the inequalities in case (3) and the one associated with δ in case (4) hold with "<" when $\lambda < 2$. The remaining inequalities of case (4) also hold with "<" because one of the conditions for Case 4, (88) of Web Appendix C, implies $\lambda < \frac{(2+\alpha)\rho}{(1-\delta)\rho+\eta}$. Further, these inequalities hold with "<" when $g_{\rho_c} > 1$ is not large and $g_{\delta} < 1$ is not small, because values of the expressions of the right-hand sides of the inequalities are not very different from the ones when $g_{\rho_c} = g_{\delta} = 1$. Hence, the results of the time-dependent case for ρ_c, δ and θ_q are the same as those of the time-invariant case, which are obtained from Proposition 1 (3), (4) and Corollary 1. The remaining results are straightforward from Propositions A.1, A.2 and A.3. ■