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PRECAUTIONARY MEASURES FOR CREDIT RISK MANAGEMENT IN JUMP MODELS

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ABSTRACT. Sustaining efficiency and stability by properly controlling the equity to asset ratio is one of the most important and difficult challenges in bank management. Due to unexpected and abrupt decline of asset values, a bank must closely monitor its net worth as well as market conditions, and one of its important concerns is when to raise more capital so as not to violate capital adequacy requirements. In this paper, we model the tradeoff between avoiding costs of delay and premature capital raising, and solve the corresponding optimal stopping problem. In order to model defaults in a bank's loan/credit business portfolios, we represent its net worth by appropriate Lévy processes, and solve explicitly for the double exponential jump diffusion process. In particular, for the spectrally negative case, we generalize the formulation using the scale function, and obtain explicitly the optimal solutions for the exponential jump diffusion process.

Key words: Credit risk management; Double exponential jump diffusion; Spectrally negative Lévy processes; Scale functions; Optimal stopping
Mathematics Subject Classification (2000) : Primary: 60G40 Secondary: 60J75

1. INTRODUCTION

As an aftermath of the recent devastating financial crisis, more sophisticated risk management practices are now being required under the Basel II accord. In order to satisfy the capital adequacy requirements, a bank needs to closely monitor how much of its asset values has been damaged; it needs to examine whether it maintains sufficient equity values or needs to start enhancing its equity to asset ratio by raising more capital and/or selling its assets. Due to unexpected sharp declines in asset values as experienced in the fall of 2008, optimally determining when to undertake the action is an important and difficult problem. In this paper, we give a new framework for this problem and obtain its solutions explicitly.

We propose an alarm system that determines when a bank needs to start enhancing its own capital ratio. We use Lévy processes with jumps in order to model defaults in its loan/credit assets and sharp declines in their values under unstable market conditions. Because of their negative jumps and the necessity to allow time for completing its capital reinforcement plans, early practical action is needed to reduce the risk of violating the capital adequacy requirements. On the other hand, there is also a cost of premature undertaking. If the action is taken too quickly, it may run a risk of incurring a large amount of opportunity costs including burgeoning administrative and monitoring expenses. In other words, we need to solve this tradeoff in order to implement this alarm system.

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In this paper, we properly quantify the costs of delay and premature undertaking and set a well-defined objective function that models this tradeoff. Our problem is to obtain a stopping time that minimizes the objective function. We expect that this precautionary measure gives a new framework in risk management.

1.1. **Problem.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which a Lévy process $X = \{X_t; t \geq 0\}$ of the form

$$X_t = x + ct + \sigma B_t + J_t, \quad 0 \leq t < \infty \quad \text{and} \quad X_0 = x$$

is defined for some $c \in \mathbb{R}$ and $\sigma > 0$. Here $B = \{B_t; t \geq 0\}$ is a standard Brownian motion and $J = \{J_t; t \geq 0\}$ is a jump process independent of B.

We represent, by X, a bank's *net worth* or *equity capital* allocated to its loan/credit business. If a jump is negative, it is naturally thought of as a loss in capital. This jump term represents defaults in its credit portfolios, whereas the non-jump terms ct and σB_t represent, respectively, the growth of the capital (through the cash flows from its credit portfolio) and its fluctuations caused by non-default events (e.g., change in interest rates).

Since X is spacially homogeneous, we may assume, without loss of generality, that the first time X *goes below zero* signifies the event that the (BIS) net capital requirement is violated. We call this the *violation event* and denote it by

$$\theta := \inf\{t \geq 0 : X_t < 0\}$$

where we assume $\inf \varnothing = \infty$. Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by X. Then θ is an $\mathbb{F}$-stopping time taking values on $[0, \infty]$, and we denote by $\mathcal{S}$ the set of all stopping times *smaller than or equal to* the first time it *reaches or gets below zero*; namely,

$$\mathcal{S} := \{\tau \in \mathbb{F} : \tau \leq \tau_0 \ a.s.\}$$

where

$$\tau_0 := \inf\{t \geq 0 : X_t \leq 0\}.$$

We only need to consider stopping times in $\mathcal{S}$ because the violation event is observable and the game is over once it happens. The reason the stopping times are bounded by τ_0 (as opposed to θ) comes from the fact that, at zero, it is always optimal to stop in the formulation we shall consider. By taking advantage of this, we see that the problem can be reduced to a well-defined optimal stopping problem; see Section 2. Our goal is to obtain among $\mathcal{S}$ the *alarm time* that minimizes the two costs we describe below.

The first cost we consider is the risk that the alarm will be triggered *at or after* the violation event:

$$R_x^{(q)}(\tau) := \mathbb{E}^x \left[e^{-q\theta} 1_{\{\tau \geq \theta\}} \right], \quad \tau \in \mathcal{S}.$$

Here $q \in [0, \infty)$ is a discount rate and $\mathbb{P}^x$ is the probability measure under which the process starts at $X_0 = x$. We call this the *violation risk*. In particular, when $q = 0$, it can be reduced under a suitable condition to the probability of the event $\{\tau \geq \theta\}$; see Section 2.

The second cost relates to premature undertaking measured by

$$H_x^{(q,h)}(\tau) := \mathbb{E}^x \left[1_{\{\tau < \infty\}} \int_\tau^\theta e^{-qt} h(X_t) \mathrm{d}t \right], \quad \tau \in \mathcal{S}.$$

We shall call this the *regret*, and here we assume $h : [0, \infty) \mapsto \mathbb{R}$ to be *continuous* and *monotonically increasing*. The monotonicity assumption reflects the fact that, if a bank has a higher capital value X, then it naturally has better access to high quality assets and hence the opportunity cost $h(\cdot)$ becomes higher accordingly. In particular, when $h \equiv 1$ (i.e., $h(x) = 1$ for every $x \geq 0$), we have

$$H_x^{(0,1)}(\tau) = \mathbb{E}^x\theta - \mathbb{E}^x\tau \quad \text{and} \quad H_x^{(q,1)}(\tau) = \frac{1}{q}\left(\mathbb{E}^x\left[e^{-q\tau}\right] - \mathbb{E}^x\left[e^{-q\theta}\right]\right), \quad q > 0 \tag{1.1}$$

where the former is well-defined if $\mathbb{E}^x\tau < \infty$.

Now, using some fixed weight $\gamma > 0$, we consider a linear combination of these two costs described above:

$$U_x^{(q,h)}(\tau,\gamma) := R_x^{(q)}(\tau) + \gamma H_x^{(q,h)}(\tau), \quad \tau \in \mathcal{S}. \tag{1.2}$$

The form of this objective function has an origin from the *Bayes risk* in mathematical statistics. In the Bayesian formulation of *change-point detection*, the Bayes risk is defined as a linear combination of the expected detection delay and the false alarm probability. In *sequential hypothesis testing*, it is a linear combination of the expected sample size and the misdiagnosis probability. The optimal solutions in these problems are those stopping times that minimize the corresponding Bayes risks. Namely, the tradeoff between the promptness and accuracy are modeled in terms of the Bayes risk. Similarly, in our problem, we model the tradeoff between the violation risk and the regret by their linear combination U.

We consider the *double exponential jump diffusion* process, a Lévy process consisting of a Brownian motion (with constant drift and diffusion parameters) and a compound Poisson process with positive and negative exponentially-distributed jumps. Due to the memoryless property of the exponential distribution, the distributions of first passage times and overshoots by this process can be obtained explicitly (Kou and Wang [19]). It is this property that leads to analytical solutions in various models that would not be possible for other jump processes. Kou and Wang [20] used this process as the underlying asset and obtained a closed-form solution to the perpetual American option and the Laplace transforms of lookback and barrier options. According to them, due to the overshoot problems, it seems impossible to get the closed-form solutions for these options under other jump diffusion models. It is this fact that motivated us to first consider this process and solve the problem analytically.

It should be noted that the double exponential jump diffusion model has been recently extended to the hyper-exponential jump diffusion model (HEM); see Cai [7] and Cai et al. [8]. We expect that the results obtained in this paper may be extended to the the HEM. Due to the fact that our objective function depends on the overshoot distribution, it is hard to imagine that explicit solutions can be obtained in other Lévy models. However, existing analytical results for the phase-type Lévy model (see Asmussen et al. [1]) and stable processes with index $\alpha \in (1,2)$ (see Bernyk et al. [4] and Peskir [26]) maybe used to extend our problem to these Lévy processes.

There exist a large number of theoretical results concerning Lévy processes with one-sided jumps. We therefore study what additional results we can obtain if the process has only negative jumps (spectrally negative). Because we are interested in defaults, the restriction to negative jumps does not lose much reality in modeling. We also see that positive jumps do not have much influence on the solutions. We shall utilize the *scale function* to simplify the problem and obtain explicit solutions for the exponential jump diffusion case.

1.2. **Literature review.** Our model is original, and, to the best of our knowledge, the objective function defined in (1.2) cannot be found elsewhere. It is, however, relevant to the problem, arising in the optimal capital structure framework, of determining the endogenous bankruptcy levels. The original diffusion model was first proposed by Leland [24] and Leland and Toft [25], and it was extended, via the Wiener-Hopf factorization, to the model with jumps by Hilberink and Rogers [15]. In their problems, the *smooth fit* principle is a main tool in obtaining the optimal bankruptcy levels. Kyprianou and Surya [22] studied the case with the general spectrally negative Lévy process, showing that the smooth fit condition holds upon some path regularity conditions. Chen and Kou [10] and Dao and Jeanblanc [11] focus on the double exponential jump diffusion case; the former, in particular, showed that the smooth fit condition holds in their setting. It should be noted that the smooth fit principle is also an important tool in our problem.

In the *insurance* literature, as exemplified by the *Cramer-Lundberg* model, the compound Poisson process is commonly used to model the surplus of an insurance firm. Recently, more general forms of jump processes are also used (e.g., Huzak et al. [17] and Jang [18]). The literature also includes computations of ruin probabilities and extensions to jumps with heavy-tailed distributions; see Embrechts et al. [12] and references therein. See also Schmidli [28] for a survey on stochastic control problems in insurance.

Mathematical statistics problems as exemplified by sequential testing and change-point detection have a long history. It dates back to 1948 when Wald and Wolfowitz used the Bayesian approach and proved the optimality of the sequential probability ratio test (SPRT) (Wald and Wolfowitz [33, 34]). There are essentially two problems, the *Bayesian* and the *variational* (or the *fixed-error*) problems; the former minimizes the Bayes risk while the latter minimizes the expected detection delay (or the sample size) subject to the constraint that the error probability is smaller than some given threshold. For comprehensive surveys and references, we refer the reader to Peskir and Shiryaev [27] and Shiryaev [29]. Our problem was originally motivated by the Bayesian problem. However, it should be mentioned that it is also possible to consider its variational version where the regret needs to be minimized on constraint that the violation risk is bounded by some threshold.

Optimal stopping problems involving jumps (including the discrete-time model) are, in general, analytically intractable owing to the difficulty in obtaining the overshoot distribution. This is true in our problem and in the literatures introduced above. For example, in sequential testing and change-point detection, explicit solutions can be realized only in the Wiener case. For this reason, recent research focuses on obtaining asymptotically optimal solutions by utilizing renewal theory; see, for example, Baron and Tartakovsky [2], Baum and Veeravalli [3] and Lai [23]. Although we do not address in this paper, asymptotically optimal solutions to our problem may be pursued for other Lévy processes via renewal theory. We refer the reader to Gut [14] for the overshoot distribution of random walks and Siegmund [30] and Woodroofe [35] for more general cases in nonlinear renewal theory.

1.3. **Outline.** The rest of the paper is structured as in the following. We first give an optimal stopping model for the general Lévy process in the next section. Section 3 focuses on the double exponential jump diffusion process and solve for the case where $h \equiv 1$. Section 4 considers the case when the process is spectrally negative; we obtain the solution explicitly for the exponential jump diffusion case for the general h. We conclude with numerical results in Section 5. Some long proofs are deferred to the appendix.

2. Mathematical Model

2.1. **Lévy processes.** By the Lévy-Khintchine formula, there exists, for any Lévy process X, a triplet (c, σ, Π) where $c \in \mathbb{R}$, $\sigma \geq 0$ and Lévy measure Π such that

$$-\log \mathbb{E}^0\left[e^{\mathrm{i}\beta \mathrm{X}_1}\right] = \mathrm{i}c\beta + \frac{1}{2}\sigma^2\beta^2 + \int_{\mathbb{R}} (1 - \mathrm{e}^{\mathrm{i}\beta\mathrm{x}} + \mathrm{i}\beta \mathrm{x} 1_{\{|\mathrm{x}|<1\}})\Pi(\mathrm{dx}), \quad \beta \in \mathbb{R};$$

see, for example, Bertoin [5] and Kyprianou [21]. Here the Lévy measure Π is concentrated on $\mathbb{R}\backslash\{0\}$ and satisfies the standard integrability condition, $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$.

We also define the *Laplace exponent* of X in terms of

$$\psi(\beta) := \log \mathbb{E}^0 \left[e^{\beta X_1} \right], \quad \beta \in \mathbb{R}. \tag{2.1}$$

We see in later sections that the Laplace exponent and its inverse function are useful in simplifying the problem and characterizing the structure of the optimal solution; for their specific forms, see (3.2) for the double exponential jump diffusion process and (4.1) for the general spectrally negative Lévy process.

2.2. **Reduction to the optimal stopping problem.** Fix $\tau \in \mathcal{S}$ and $q > 0$. The violation risk is, for every $x \geq 0$,

$$R_x^{(q)}(\tau) := \mathbb{E}^x \left[e^{-q\theta} 1_{\{\tau \geq \theta\}} \right] = \mathbb{E}^x \left[e^{-q\theta} 1_{\{\tau \geq \theta,\, \theta < \infty\}} \right] = \mathbb{E}^x \left[e^{-q\theta} 1_{\{\tau = \theta, \theta < \infty\}} \right] = \mathbb{E}^x \left[e^{-q\tau} 1_{\{\tau = \theta, \theta < \infty\}} \right]$$

where the third equality follows because $\tau \leq \theta$ a.s. by definition. Moreover, we have

$$1_{\{\tau = \theta,\, \theta < \infty\}} = 1_{\{X_\tau < 0,\, \tau < \infty\}} \quad a.s.$$

because

$$\{\tau = \theta,\ \theta < \infty\} = \{X_\tau < 0,\ \tau = \theta,\ \theta < \infty\} = \{X_\tau < 0,\ \tau = \theta,\ \tau < \infty\}$$

and

$$\{X_\tau < 0,\ \tau < \infty\} = \{X_\tau < 0,\ \tau \leq \theta,\ \tau < \infty\} = \{X_\tau < 0,\ \tau = \theta,\ \tau < \infty\}$$

where the first equality holds because $\tau \in \mathcal{S}$ and the second equality holds by the definition of θ. Hence, we have

$$R_x^{(q)}(\tau) = \mathbb{E}^x \left[e^{-q\tau} 1_{\{X_\tau < 0,\, \tau < \infty\}} \right], \quad \tau \in \mathcal{S}. \tag{2.2}$$

On the other hand, for the regret, by the strong Markov property of X at time τ, we have

$$H_x^{(q,h)}(\tau) := \mathbb{E}^x \left[1_{\{\tau < \infty\}} \int_\tau^\theta e^{-qt} h(X_t) \mathrm{d}t \right] = \mathbb{E}^x \left[e^{-q\tau} Q^{(q,h)}(X_\tau) 1_{\{\tau < \infty\}} \right], \quad x \geq 0, \tag{2.3}$$

where $\left\{ Q^{(q,h)}(X_t); t \geq 0 \right\}$ is an $\mathbb{F}$-adapted Markov process such that

$$Q^{(q,h)}(x) := \mathbb{E}^x \left[\int_0^\theta e^{-qt} h(X_t) \mathrm{d}t \right], \quad x \geq 0.$$

Therefore, by (2.2) and (2.3), if we let

$$G(x) := 1_{\{x < 0\}} + \gamma Q^{(q,h)}(x) 1_{\{x \geq 0\}}, \quad x \in \mathbb{R} \tag{2.4}$$

denote the cost of stopping, we can rewrite the objective function as

$$U_x^{(q,h)}(\tau, \gamma) := R_x^{(q)}(\tau) + \gamma H_x^{(q,h)}(\tau) = \mathbb{E}^x \left[e^{-q\tau} G(X_\tau) 1_{\{\tau < \infty\}} \right], \quad \tau \in \mathcal{S}.$$

Our problem is to obtain a stopping time $\tau^* \in \mathcal{S}$ that attains

$$\min_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-q\tau} G(X_\tau) 1_{\{\tau < \infty\}} \right].$$

The problem can be naturally extended to the undiscounted case with $q = 0$. However, in this case, we will need to assume that $\mathbb{E}^x \theta < \infty$ in order to make the problem well-defined and non-trivial. To see this, if $\mathbb{E}^x \theta = \infty$, we have $G(x) = \infty$ and the optimal strategy must be $\tau^* = \tau_0$. Under the assumption that $\mathbb{E}^x\theta < \infty$, we have $\theta < \infty$ a.s and the violation risk reduces to the probability

$$R_x^{(0)}(\tau) = \mathbb{P}^x\{\tau \geq \theta\} = \mathbb{P}^x\{\tau = \theta\} = \mathbb{P}^x\{X_\tau < 0\}, \quad \tau \in \mathcal{S}.$$

2.3. **Obtaining optimal strategy via smooth fit.** Similarly to obtaining the optimal bankruptcy levels in Leland [24], Leland and Toft [25], Hilberink and Rogers [15] and Kyprianou and Surya [22], the smooth fit principle will be a useful tool in our problem in obtaining the optimal solution. Focusing on the set of *threshold strategies* defined by the first time the process reaches or goes below some fixed threshold, say A,

$$\tau_A := \inf\{t \geq 0 : X_t \leq A\}, \quad A \geq 0,$$

we choose the optimal threshold level A^* that satisfies the smooth fit condition and verify the optimality of the corresponding strategy over the set $\mathcal{S}$.

Fix $0 \leq A \leq x$. Let the continuation value corresponding to the threshold strategy τ_A be

$$V_A(x) := U_x^{(q,h)}(\tau_A, \gamma),$$

and the difference between the continuation and the stopping values be

$$\delta_A(x) := V_A(x) - G(x) = R_x^{(q)}(\tau_A) - \gamma\mathbb{E}^x\left[\int_0^{\tau_A} e^{-qt}h(X_t)\mathrm{d}t\right]. \tag{2.5}$$

We shall choose A such that the smooth fit condition holds, i.e., the right-derivative vanishes ($\delta'_A(A+) = 0$).

As observed by Kyprianou and Surya [22], we will need to approach the problem differently according to the path regularity of the underlying Lévy process X. In our problem, we assume regularity. Notice that it is guaranteed for the double exponential jump diffusion case because of its diffusion component. For necessary and sufficient conditions for regularity, see Theorem 6.5 of Kyprianou [21] for the general Lévy process and Section 3 of Kyprianou and Surya [22] for the spectrally negative Lévy process.

2.4. **Extension to the geometric model.** It should be noted that a version of this problem with the *geometric Lévy* process $Y = \{Y_t = \exp(X_t); t \geq 0\}$ and a slightly modified violation time

$$\theta = \inf\{t \geq 0 : Y_t < a\},$$

for some $a > 0$, can be modeled in the same framework. Indeed, defining $\widetilde{X} := \left\{\widetilde{X}_t = X_t - \log a; t \geq 0\right\}$, we have

$$\theta = \inf\left\{t \geq 0 : \widetilde{X}_t < 0\right\}.$$

Moreover, the regret function can be expressed in terms of $\widetilde{X}$ by replacing $h(x)$ with $\widetilde{h}(x) = h(\exp(x + \log a))$ for every $x \in \mathbb{R}$. The continuity and increasing properties remain valid because of the property of the exponential function.

3. Double Exponential Jump Diffusion

In this section, we consider the double exponential jump diffusion model that features exponential-type jumps in both positive and negative directions. We first summarize the results from Kou and Wang [19] and obtain the explicit representation of our violation risk and regret. We then find analytically the optimal strategy both when $q > 0$ and when $q = 0$. We assume throughout this section that $h \equiv 1$, i.e., the regret function reduces to (1.1).

3.1. **Double exponential jump diffusion.** The double exponential jump diffusion process is a Lévy process of the form

$$X_t := x + ct + \sigma B_t + \sum_{i=1}^{N_t} Z_i, \quad t \geq 0$$

where $c \in \mathbb{R}$, $\sigma > 0$, $B = \{B_t; t \geq 0\}$ is a standard Brownian motion, $N = \{N_t; t \geq 0\}$ is a Poisson process with parameter $\lambda > 0$ and $Z = \{Z_i; i \in \mathbb{N}\}$ is a sequence of i.i.d. random variables having a double exponential distribution with its density

$$f(z) := p\eta_- e^{-\eta_-(-z)} 1_{\{z \leq 0\}} + (1-p)\eta_+ e^{-\eta_+ z} 1_{\{z > 0\}}, \quad z \in \mathbb{R}, \tag{3.1}$$

for some $\eta_-, \eta_+ > 0$ and $p \in [0, 1]$. Here the stochastic processes B, N and Z are assumed to be mutually independent. Its Laplace exponent (2.1) is then

$$\psi(\beta) := c\beta + \frac{1}{2}\sigma^2\beta^2 + \lambda\left(\frac{p\eta_-}{\eta_- + \beta} + \frac{(1-p)\eta_+}{\eta_+ - \beta} - 1\right), \quad \beta \in \mathbb{R}, \tag{3.2}$$

and its infinitesimal generator denoted by $\mathcal{L}$ becomes

$$\mathcal{L}w(x) = cw'(x) + \frac{1}{2}\sigma^2 w''(x) + \lambda \int_{-\infty}^{\infty} \left[w(x+z) - w(x)\right] f(z)\mathrm{d}z, \quad x \in \mathbb{R}, \tag{3.3}$$

for any C^2-function $w : \mathbb{R} \mapsto \mathbb{R}$.

Fix $q > 0$. There are four roots of $\psi(\beta) = q$, and, in particular, we focus on $\xi_{1,q}$ and $\xi_{2,q}$ such that

$$0 < \xi_{1,q} < \eta_- < \xi_{2,q} < \infty \quad \text{and} \quad \psi(-\xi_{1,q}) = \psi(-\xi_{2,q}) = q.$$

Suppose that the overall drift is denoted by $\overline{u} := \mathbb{E}^0 X_1$, then it becomes

$$\overline{u} = c + \lambda\left(-\frac{p}{\eta_-} + \frac{1-p}{\eta_+}\right),$$

and

$$\xi_{1,q} \to \xi_{1,0} \left\{ \begin{array}{ll} = 0, & \overline{u} \leq 0 \\ > 0, & \overline{u} > 0 \end{array} \right\} \quad \text{and} \quad \xi_{2,q} \to \xi_{2,0} \quad \text{as } q \to 0 \tag{3.4}$$

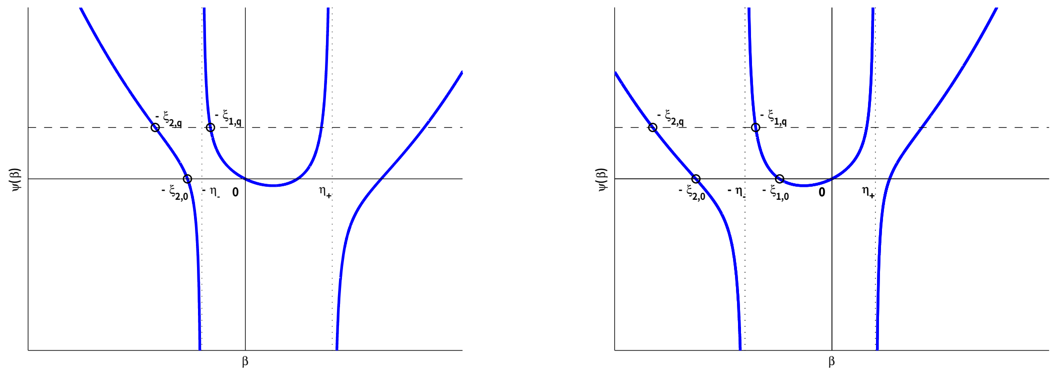
for some $\xi_{1,0}$ and $\xi_{2,0}$ such that

$$0 \leq \xi_{1,0} < \eta_- < \xi_{2,0} < \infty \quad \text{and} \quad \psi(-\xi_{1,0}) = \psi(-\xi_{2,0}) = 0;$$

see Figure 1 for an illustration. When $\overline{u} < 0$, by (3.4), l'Hôpital's rule and $\psi(-\xi_{1,q}) = q$, we have

$$\frac{\xi_{1,q}}{q} \xrightarrow{q \downarrow 0} -\frac{1}{\psi'(0)} = -\frac{1}{\overline{u}} = \frac{1}{|\overline{u}|}. \tag{3.5}$$

We will see that these roots characterize the optimal strategies; the optimal threshold levels can be expressed in terms of $\xi_{1,q}$ and $\xi_{2,q}$ when $q > 0$ and $\xi_{2,0}$ when $q = 0$.



(a) $\bar{u} < 0$ (b) $\bar{u} > 0$

FIGURE 1. Illustration of the Laplace exponent and its roots when the drift is negative and positive.

Due to the memoryless property of its jump-size distribution, the violation risk and regret can be obtained explicitly. The following two lemmas are due to Kou and Wang [19], Theorem 3.1 and its corollary. Here we let

$$l_{1,q} := \frac{\eta_- - \xi_{1,q}}{\xi_{2,q} - \xi_{1,q}} > 0 \quad \text{and} \quad l_{2,q} := \frac{\xi_{2,q} - \eta_-}{\xi_{2,q} - \xi_{1,q}} > 0, \quad q \geq 0,$$

where, in particular, when $q = 0$ and $\overline{u} < 0$,

$$l_{1,0} := \frac{\eta_-}{\xi_{2,0}} > 0 \quad \text{and} \quad l_{2,0} := \frac{\xi_{2,0} - \eta_-}{\xi_{2,0}} > 0.$$

Notice that $l_{1,q} + l_{2,q} = 1$ for every $q \geq 0$.

Lemma 3.1 (violation risk)**.** *For every $q \geq 0$ and $0 \leq A \leq x$, we have*

$$R_x^{(q)}(\tau_A) = \frac{e^{-\eta_- A}}{\eta_-} \left[(\xi_{2,q} - \eta_-) l_{1,q} e^{-\xi_{1,q}(x-A)} - (\eta_- - \xi_{1,q}) l_{2,q} e^{-\xi_{2,q}(x-A)} \right].$$

In particular, when $q = 0$ and $\overline{u} \leq 0$, this reduces to

$$R_x^{(0)}(\tau_A) = l_{2,0} e^{-\eta_- A} \left[1 - e^{-\xi_{2,0}(x-A)} \right].$$

Lemma 3.2 (functional associated with the regret when $h \equiv 1$)**.** *For every $q > 0$, we have*

$$\mathbb{E}^x \left[\int_0^{\tau_A} e^{-qt} \mathrm{d}t \right] = \frac{1}{q\eta_-} \left[l_{1,q} \xi_{2,q} \left(1 - e^{-\xi_{1,q}(x-A)} \right) + l_{2,q} \xi_{1,q} \left(1 - e^{-\xi_{2,q}(x-A)} \right) \right], \quad 0 \leq A \leq x.$$

Furthermore, it can be extended to the case $q = 0$, by taking $q \downarrow 0$ via (3.5) and the monotone convergence theorem;

$$\mathbb{E}^x \left[\tau_A \right] = \begin{cases} \frac{1}{|\overline{u}|} \left[(x - A) + \frac{\xi_{2,0} - \eta_-}{\eta_- \xi_{2,0}} (1 - e^{-\xi_{2,0}(x-A)}) \right], & \text{if } \overline{u} < 0, \\ \infty, & \text{if } \overline{u} \geq 0. \end{cases}$$

For every $q \geq 0$, we have, by Lemma 3.1,

$$\frac{\partial}{\partial x} R_x^{(q)}(\tau_A)\Big|_{x=A+} = e^{-\eta_- A}\frac{(\eta_- - \xi_{1,q})(\xi_{2,q}-\eta_-)}{\eta_-}, \quad A \geq 0, \tag{3.6}$$

$$\frac{\partial}{\partial A} R_x^{(q)}(\tau_A) = -e^{-\eta_- A}\frac{(\eta_- - \xi_{1,q})(\xi_{2,q}-\eta_-)}{\eta_-} \sum_{i=1,2} l_{i,q} e^{-\xi_{i,q}(x-A)}, \quad 0 \leq A \leq x, \tag{3.7}$$

and, by Lemma 3.2,

$$\frac{\partial}{\partial x}\mathbb{E}^x\left[\int_0^{\tau_A} e^{-qt}\mathrm{d}t\right]\Big|_{x=A+} = \left\{\begin{array}{ll} \frac{\xi_{1,q}\xi_{2,q}}{q\eta_-}, & q>0 \\ \frac{\xi_{2,0}}{|\overline{u}|\eta_-}, & q=0 \text{ and } \overline{u}<0 \end{array}\right\}, \quad A \geq 0, \tag{3.8}$$

$$\frac{\partial}{\partial A}\mathbb{E}^x\left[\int_0^{\tau_A} e^{-qt}\mathrm{d}t\right] = \left\{\begin{array}{ll} -\frac{\xi_{1,q}\xi_{2,q}}{q\eta_-}\sum_{i=1,2} l_{i,q}e^{-\xi_{i,q}(x-A)}, & q>0 \\ -\frac{\xi_{2,0}}{|\overline{u}|\eta_-}\left[l_{1,0}+l_{2,0}e^{-\xi_{2,0}(x-A)}\right], & q=0 \text{ and } \overline{u}<0 \end{array}\right\}, \quad 0 \leq A \leq x. \tag{3.9}$$

3.2. **Optimal strategy when $h \equiv 1$.** We shall obtain the optimal solution for $q \geq 0$. When $q = 0$, we shall focus on the case when $\overline{u} < 0$ because $\mathbb{E}^x\theta = \infty$ otherwise by Lemma 3.2 and the problem becomes trivial as we discussed in Section 2.

Fix $q > 0$. By Lemma 3.2, the stopping value (2.4) becomes, for every $x \geq 0$,

$$G(x) = \gamma\mathbb{E}^x\left[\int_0^{\theta} e^{-qh}\mathrm{d}t\right] = \gamma\mathbb{E}^x\left[\int_0^{\tau_0} e^{-qh}\mathrm{d}t\right] = \sum_{i=1,2} C_{i,q}\left(1-e^{-\xi_{i,q}x}\right) = \frac{\gamma}{q} - \sum_{i=1,2} C_{i,q}e^{-\xi_{i,q}x} \tag{3.10}$$

where

$$C_{1,q} := \frac{\gamma}{q}\frac{l_{1,q}\xi_{2,q}}{\eta_-}, \quad C_{2,q} := \frac{\gamma}{q}\frac{l_{2,q}\xi_{1,q}}{\eta_-} \quad \text{and} \quad \sum_{i=1,2} C_{i,q} = \sum_{i=1,2} C_{i,q}\frac{\eta_-}{\eta_- - \xi_{i,q}} = \frac{\gamma}{q}, \tag{3.11}$$

and the difference between the continuation and stopping values (defined in (2.5)) becomes, by Lemmas 3.1 and 3.2,

$$\begin{aligned}\delta_A(x) = {} & \frac{l_{1,q}}{\eta_-}\left[(\xi_{2,q}-\eta_-)e^{-\xi_{1,q}(x-A)-\eta_- A} - \frac{\gamma}{q}\xi_{2,q}\left(1-e^{-\xi_{1,q}(x-A)}\right)\right] \\ & + \frac{l_{2,q}}{\eta_-}\left[-(\eta_- - \xi_{1,q})e^{-\xi_{2,q}(x-A)-\eta_- A} - \frac{\gamma}{q}\xi_{1,q}\left(1-e^{-\xi_{2,q}(x-A)}\right)\right], \quad 0 \leq A \leq x;\end{aligned} \tag{3.12}$$

see Remark 3.3 for the case $q = 0$ and $\overline{u} < 0$.

Remark 3.1. We have $\delta_A(A) = 0$ for every $A \geq 0$; continuous fit holds whatever the choice of A is. □

We shall first obtain the threshold level A^* such that the smooth fit condition holds, i.e., $\delta'_{A^*}(A^*+) = 0$. By (3.6) and (3.8), we have

$$\delta'_A(A+) = \left\{\begin{array}{ll} \frac{1}{\eta_-}\left[(\eta_- - \xi_{1,q})(\xi_{2,q}-\eta_-)e^{-\eta_- A} - \frac{\gamma}{q}\xi_{1,q}\xi_{2,q}\right], & q>0, \\ (\xi_{2,0}-\eta_-)e^{-\eta_- A} - \frac{\gamma}{|\overline{u}|\eta_-}\xi_{2,0}, & q=0 \text{ and } \overline{u}<0. \end{array}\right.$$

Therefore, on condition that

$$\left\{\begin{array}{ll} (\eta_- - \xi_{1,q})(\xi_{2,q}-\eta_-) \geq \frac{\gamma}{q}\xi_{1,q}\xi_{2,q}, & q>0 \\ (\xi_{2,0}-\eta_-) \geq \frac{\gamma}{|\overline{u}|\eta_-}\xi_{2,0}, & q=0 \text{ and } \overline{u}<0 \end{array}\right\}, \tag{3.13}$$

the smooth fit condition $\delta'_A(A+) = 0$ is satisfied if and only if $A = A^* \geq 0$ where

$$A^* := \left\{ \begin{array}{ll} -\frac{1}{\eta_-} \log \left[\frac{\gamma}{q} \frac{\xi_{1,q}\xi_{2,q}}{(\eta_- - \xi_{1,q})(\xi_{2,q} - \eta_-)} \right], & q > 0, \\ -\frac{1}{\eta_-} \log \left[\frac{\gamma \xi_{2,0}}{|\overline{u}|\eta_-(\xi_{2,0} - \eta_-)} \right], & q = 0 \ \text{ and } \ \overline{u} < 0. \end{array} \right. \tag{3.14}$$

Notice that if (3.13) does not hold, we have $\delta'_A(A+) < 0$ for every $A \geq 0$; here, we set $A^* = 0$ when (3.13) does not hold.

Our objective is to show that the optimal stopping time is

$$\tau^* := \tau_{A^*},$$

and the *right-continuous* function representing the expected cost associated with this stopping time,

$$\phi(x) := \left\{ \begin{array}{ll} V(x) \equiv V_{A^*}(x), & A^* \leq x, \\ G(x), & 0 \leq x < A^*, \\ 1, & x < 0, \end{array} \right. \tag{3.15}$$

is indeed the value function. In particular, when (3.13) does not hold, we have $A^* = 0$ and therefore

$$\phi(x) := \left\{ \begin{array}{ll} V(x) = R_x^{(q)}(\tau_0), & x \geq 0, \\ 1, & x < 0 \end{array} \right. \tag{3.16}$$

where, by Lemma 3.1,

$$R_x^{(q)}(\tau_0) = \mathbb{E}^x \left[e^{-q\tau_0} 1_{\{X_{\tau_0} \neq 0\}} \right] = \frac{(\eta_- - \xi_{1,q})(\xi_{2,q} - \eta_-)}{\eta_-(\xi_{2,q} - \xi_{1,q})} \left(e^{-\xi_{1,q}x} - e^{-\xi_{2,q}x} \right).$$

Now we shall verify the optimality of the threshold strategy τ^*. Toward this end, we first show that our candidate $\phi(\cdot)$ is dominated from above by the stopping value $G(\cdot)$.

Lemma 3.3. *We have $\phi(x) \leq G(x)$ for every $x \geq 0$. In particular, when (3.13) holds, the equality is attained at $x = A^*$ due to Remark 3.1.*

Proof. Fix $0 \leq A \leq x$. By (3.7) and (3.9), we have $\partial V_A(x)/\partial A = \partial \delta_A(x)/\partial A$ equals

$$\left\{ \begin{array}{ll} \frac{1}{\eta_-} \left[l_{1,q} e^{-\xi_{1,q}(x-A)} + l_{2,q} e^{-\xi_{2,q}(x-A)} \right] \left[-(\eta_- - \xi_{1,q})(\xi_{2,q} - \eta_-) e^{-\eta_- A} + \frac{\gamma}{q} \xi_{1,q}\xi_{2,q} \right], & q > 0 \\ \frac{1}{\eta_-} \left[l_{1,0} + l_{2,0} e^{-\xi_{2,0}(x-A)} \right] \left[-\eta_-(\xi_{2,0} - \eta_-) e^{-\eta_- A} + \frac{\gamma}{|\overline{u}|} \xi_{2,0} \right], & q = 0 \text{ and } \overline{u} < 0 \end{array} \right\}.$$

Here, in both cases, the first term is strictly positive while the second term is increasing in A. Therefore, when (3.13) holds, A^* is the unique value that makes it vanish, and consequently $\partial V_A(x)/\partial A \geq 0$ if and only if $A \geq A^*$. On the other hand, if (3.13) does not hold, $\partial V_A(x)/\partial A \geq 0$ for every $A \geq 0$. These imply when $x \geq A^*$ that

$$V(x) = V_{A^*}(x) \leq V_x(x) = G(x) \quad \text{and} \quad V(x) = V_0(x) \leq V_x(x) = G(x) \tag{3.17}$$

when (3.13) holds and when it does not hold, respectively. On the other hand, when $x < A^*$, we have $\phi(x) = G(x)$ and hence the proof is complete. □

Remark 3.2. (1) Suppose $q > 0$. In view of (3.10), $G(\cdot)$ is bounded uniformly on $x \in [0, \infty)$ by $\frac{\gamma}{q}$.

(2) Suppose $\overline{u} < 0$ and fix $x \geq 0$. We have $G(x) \leq \mathbb{E}^x[\tau_0] < \infty$ uniformly for every $q \geq 0$; namely, $G(x)$ is bounded by $\mathbb{E}^x[\tau_0] < \infty$ uniformly on $q \in [0, \infty)$.

(3) Using the same argument as in the proof of Lemma 3.3, we have $\phi(x) = V(x) \leq V_0(x) = R_x^{(q)}(\tau_0) \leq 1$ for every $x \geq A^*$. When $0 \leq x < A^*$, using the monotonicity of $G(\cdot)$ and continuous fit, we have

$$\phi(x) = G(x) \leq G(A^*) = V(A^*) \leq V_0(A^*) = R_{A^*}^{(q)}(\tau_0) \leq 1.$$

Therefore, $\phi(\cdot)$ is uniformly bounded and it only takes values on $[0, 1]$. □

Fix $q > 0$ and let $\delta(x) := V(x) - G(x) = \delta_{A^*}(x)$ for every $x \geq A^*$. Simple algebra shows that, when (3.13) holds,

$$\delta(x) = \frac{\gamma}{q} \frac{1}{\xi_{2,q} - \xi_{1,q}} \left[\xi_{2,q} \left(e^{-\xi_{1,q}(x-A^*)} - 1 \right) - \xi_{1,q} \left(e^{-\xi_{2,q}(x-A^*)} - 1 \right) \right], \quad x \geq A^*, \tag{3.18}$$

and the continuation value becomes, by (3.10),

$$V(x) = \sum_{i=1,2} (L_{i,q} - C_{i,q}) e^{-\xi_{i,q} x}, \quad x \geq A^* \tag{3.19}$$

where $C_{1,q}$ and $C_{2,q}$ are defined in (3.11) and

$$L_{1,q} := \frac{\gamma}{q} \frac{\xi_{2,q}}{\xi_{2,q} - \xi_{1,q}} e^{\xi_{1,q} A^*}, \quad L_{2,q} := -\frac{\gamma}{q} \frac{\xi_{1,q}}{\xi_{2,q} - \xi_{1,q}} e^{\xi_{2,q} A^*}. \tag{3.20}$$

Remark 3.3. The function $\phi(\cdot)$ can be obtained for the case $q = 0$ and $\overline{u} < 0$. To see this, solely in this remark, let us emphasize the dependence on q and suppose that A_q^*, $\phi_q(\cdot)$, $G_q(\cdot)$ and $\delta_q(\cdot)$ are, respectively, the threshold level (3.14), the candidate value function, the stopping value and the difference between the stopping and continuation values corresponding to the problem with discount rate $q \geq 0$. Notice that $\phi_q(\cdot)$ can be reproduced by (3.10) and (3.18) when $q > 0$. Furthermore, in view of (3.14), $A_q^* \to A_0^*$ as $q \to 0$ and by Remark 3.2 (3), $\sup_{q \geq 0,\, x \in \mathbb{R}} |\phi_q(x)| \leq 1$. Therefore, by the dominated convergence theorem, we have pointwise for every fixed $x \in \mathbb{R}$

$$\phi_q(x) \to \phi_0(x) \quad \text{as } q \to 0.$$

In particular, taking $q \to 0$ on (3.10) and (3.18) and noticing that $A_q^* \to A_0^*$ as $q \to 0$, we have

$$G_0(x) = \frac{\gamma}{|\overline{u}|} \left[x + \frac{\xi_{2,0} - \eta_-}{\eta_- \xi_{2,0}} \left(1 - e^{-\xi_{2,0} x} \right) \right], \quad x \geq 0,$$

$$\delta_0(x) = -\frac{\gamma}{|\overline{u}|} \left[(x - A_0^*) - \frac{1 - e^{-\xi_{2,0}(x - A_0^*)}}{\xi_{2,0}} \right], \quad x \geq A_0^*.$$

The convergence of the first and second derivatives also hold; $\phi_q'(x) \to \phi_0'(x)$ and $\phi_q''(x) \to \phi_0''(x)$ as $q \to 0$ for every fixed $x \geq 0$; indeed, we have

$$G_q'(x) = \frac{\gamma}{q\eta_-} \xi_{1,q} \xi_{2,q} \sum_{i \in 1,2} l_{i,q} e^{-\xi_{i,q} x} \xrightarrow{q \downarrow 0} \frac{\gamma}{|\overline{u}|} \left[1 + \frac{\xi_{2,0} - \eta_-}{\eta_-} e^{-\xi_{2,0} x} \right] = G_0'(x),$$

$$G_q''(x) = -\frac{\gamma}{q\eta_-} \xi_{1,q} \xi_{2,q} \sum_{i \in 1,2} \xi_{i,q} l_{i,q} e^{-\xi_{i,q} x} \xrightarrow{q \downarrow 0} -\frac{\gamma}{|\overline{u}|} \left[\frac{(\xi_{2,0} - \eta_-)\xi_{2,0}}{\eta_-} e^{-\xi_{2,0} x} \right] = G_0''(x),$$

and

$$\delta_q'(x \vee A_q^*+) = \frac{\gamma}{q} \frac{\xi_{1,q}\xi_{2,q}}{\xi_{2,q}-\xi_{1,q}} \left[-e^{-\xi_{1,q}(x-A_q^*)^+} + e^{-\xi_{2,q}(x-A_q^*)^+} \right] \xrightarrow{q \downarrow 0} -\frac{\gamma}{|\overline{u}|} \left[1 - e^{-\xi_{2,0}(x-A_0^*)^+} \right] = \delta_0'(x \vee A_0^*+),$$

$$\delta_q''(x \vee A_q^*+) = \frac{\gamma}{q} \frac{\xi_{1,q}\xi_{2,q}}{\xi_{2,q}-\xi_{1,q}} \left[\xi_{1,q} e^{-\xi_{1,q}(x-A_q^*)^+} - \xi_{2,q} e^{-\xi_{2,q}(x-A_q^*)^+} \right] \xrightarrow{q \downarrow 0} -\frac{\gamma}{|\overline{u}|} \xi_{2,0} e^{-\xi_{2,0}(x-A_0^*)^+} = \delta_0''(x \vee A_0^*+).$$

□

Now we need to show that the stochastic process $M = \{M_t; t \geq 0\}$ with

$$M_t := e^{-q(t \wedge \tau_0)} \phi(X_{t \wedge \tau_0}), \quad t \geq 0 \tag{3.21}$$

is a submartingale.

Lemma 3.4. If (3.13) holds, we have

$$\delta''(A^*+) = \left\{ \begin{array}{ll} -\frac{\gamma}{q}\xi_{1,q}\xi_{2,q}, & q > 0 \\ -\frac{\gamma}{|\overline{u}|}\xi_{2,0}, & q = 0 \text{ and } \overline{u} < 0 \end{array} \right\} < 0,$$

and consequently, by the continuous and smooth fit conditions and the definition of $\mathcal{L}$ in (3.3), we have

$$\lim_{x \uparrow A^*-} \mathcal{L}\phi(x) > \lim_{x \downarrow A^*+} \mathcal{L}\phi(x).$$

Proof. As observed in Remark 3.3, for every $x \geq A^*+$,

$$\delta''(x) = \left\{ \begin{array}{ll} \frac{\gamma}{q} \frac{\xi_{1,q}\xi_{2,q}}{\xi_{2,q}-\xi_{1,q}} \left[\xi_{1,q} e^{-\xi_{1,q}(x-A^*)} - \xi_{2,q} e^{-\xi_{2,q}(x-A^*)} \right], & q > 0 \\ -\frac{\gamma}{|\overline{u}|}\xi_{2,0} e^{-\xi_{2,0}(x-A^*)}, & q = 0 \text{ and } \overline{u} < 0 \end{array} \right\},$$

and hence we have the claim after taking $x \downarrow A^*$. □

The proofs for the following two lemmas are lengthy and technical, and therefore relegated to the appendix.

Lemma 3.5. (1) *If (3.13) holds, then we have*

$$\mathcal{L}\phi(x) - q\phi(x) = 0, \quad x > A^*, \tag{3.22}$$

$$\mathcal{L}\phi(x) - q\phi(x) > 0, \quad 0 < x < A^*, \tag{3.23}$$

and at A^*

$$\lim_{x \uparrow A^*-} (\mathcal{L}\phi(x) - q\phi(x)) > \lim_{x \downarrow A^*+} (\mathcal{L}\phi(x) - q\phi(x)).$$

(2) *If (3.13) does not hold, then (3.22) holds for every $x \geq 0$.*

Lemma 3.6. (1) *If (3.13) holds, the stochastic process M is a submartingale.*

(2) *If (3.13) does not hold, the stochastic process M is a martingale.*

We now prove, using Lemmas 3.3 and 3.6, the optimality of τ^*.

Proposition 3.1. *We have*

$$\phi(x) = \inf_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-q\tau} G(X_\tau) 1_{\{\tau < \infty\}} \right]$$

and τ^* is the optimal stopping time.

Proof. Fix $\tau \in \mathcal{S}$. Recall by the definition of $\mathcal{S}$ that $\tau \wedge \tau_0 = \tau$ a.s. Suppose first that $q > 0$. Lemma 3.6 and the optional sampling theorem implies

$$\phi(x) \leq \mathbb{E}^x \left[e^{-q(\tau \wedge m)} \phi(X_{\tau \wedge m}) \right] = \mathbb{E}^x \left[e^{-q(\tau \wedge m)} \phi(X_{\tau \wedge m}) 1_{\{\tau \wedge m < \infty\}} \right], \quad m \geq 0.$$

Therefore, by the dominated convergence theorem (note that ϕ is bounded by Remark 3.2 (3)) and Lemma 3.3,

$$\phi(x) \leq \lim_{m \to \infty} \mathbb{E}^x \left[e^{-q(\tau \wedge m)} \phi(X_{\tau \wedge m}) 1_{\{\tau \wedge m < \infty\}} \right] = \mathbb{E}^x \left[e^{-q\tau} \phi(X_\tau) 1_{\{\tau < \infty\}} \right] \leq \mathbb{E}^x \left[e^{-q\tau} G(X_\tau) 1_{\{\tau < \infty\}} \right].$$

The same result holds for $q = 0$ and $\overline{u} < 0$ by the fact that $\mathbb{E}^x [\tau_0] < \infty$ and consequently $\mathbb{E}^x \tau < \infty$. Now because $\tau \in \mathcal{S}$ is arbitrary, we have

$$\phi(x) \leq \inf_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-q\tau} G(X_\tau) 1_{\{\tau < \infty\}} \right].$$

Moreover because by construction

$$\phi(x) = \mathbb{E}^x \left[e^{-q\tau^*} G(X_{\tau^*}) 1_{\{\tau^* < \infty\}} \right]$$

and $\tau^* \in \mathcal{S}$, we have the claim. □

4. Spectrally Negative Case

In this section, we analyze the case when X only has negative jumps, or the support of its Lévy measure Π is $(-\infty, 0)$. This is called the *spectrally negative* Lévy process, and we shall rewrite the regret function in terms of the *scale function* for the general h. For the (single) exponential jump diffusion case with only negative jumps, we obtain the explicit form of the scale function, and use it to obtain the optimal solution for any increasing and continuous function h.

4.1. **Scale functions.** In the class of spectrally negative Lévy processes, the Laplace exponent (2.1) reduces to

$$\psi(\beta) = c\beta + \frac{1}{2}\sigma^2\beta^2 + \int_{(-\infty,0)} (e^{\beta x} - 1 - \beta x 1_{\{x > -1\}}) \Pi(\mathrm{d}x), \quad \beta \in \mathbb{R}. \tag{4.1}$$

It is well-known that it is zero at the origin, convex on $\mathbb{R}_+$ and has the right continuous inverse:

$$\zeta_q := \sup\{\lambda > 0 : \psi(\lambda) = q\}, \quad q \geq 0.$$

Furthermore, there exists a *(q-)scale function*

$$W^{(q)} : [0, \infty) \mapsto \mathbb{R}; \quad q \geq 0,$$

that is continuous and differentiable with respect to x and satisfies

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) \mathrm{d}x = \frac{1}{\psi(\beta) - q}, \qquad \beta > \zeta_q,$$

and if τ_a^+ is the first time the process goes above $a > x$, we have

$$\mathbb{E}_x \left[e^{-q\tau_a^+} 1_{\{\tau_a^+ < \tau_0\}} \right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}.$$

It has been recently shown by Chan et al. [9] that if the process has a Gaussian component, the scale function $W^{(q)}$ belongs to the class $C^2(0, \infty)$.

The scale function $W^{(q)}$ increases exponentially in x; indeed, we have

$$W^{(q)}(x) \sim \frac{e^{\zeta_q x}}{\psi'(\zeta_q)} \quad \text{as } x \to \infty. \tag{4.2}$$

As is discussed in Kyprianou [21] and Surya [32], there exists a version of the scale function $W_{\zeta_q} = \{W_{\zeta_q}(x); x \geq 0\}$ that satisfies, for every fixed $q \geq 0$,

$$W^{(q)}(x) = e^{\zeta_q x} W_{\zeta_q}(x), \quad x \geq 0 \tag{4.3}$$

and

$$\int_0^\infty e^{-\beta x} W_{\zeta_q}(x) \mathrm{d}x = \frac{1}{\psi(\beta + \zeta_q) - q}, \quad \beta > 0.$$

In order to solve our problem for the general form of h, we express, in terms of the scale function, the regret function for every fixed threshold level $A \in [0, x]$. For a comprehensive account of the scale function, see Bertoin [5, 6], Kyprianou [21] and Kyprianou and Surya [22]. See Surya [32] for a numerical method for computing the scale function.

4.2. **Rewriting the regret function and the stopping value for the general h.** For given $q > 0$ and $0 \leq A \leq x$, we shall represent the regret function $H(\cdot)$ and the stopping value $G(\cdot)$ in terms of the scale function. We shall first represent them in terms of the following random measure

$$M^{(A,q)}(\omega, B) := \int_0^{\tau_A(\omega)} e^{-qt} 1_{\{X_t(\omega) \in B\}} \mathrm{d}t, \quad \omega \in \Omega,\ B \in \mathcal{B}[A, \infty).$$

Lemma 4.1. *For any $\omega \in \Omega$, we have*

$$\int_0^{\tau_A(\omega)} e^{-qt} h(X_t(\omega)) \mathrm{d}t = \int_{\mathbb{R}} M^{(A,q)}(\omega, \mathrm{d}y) h(y). \tag{4.4}$$

Proof. Take a sequence of functions $(h^{(n)})_{n \in \mathbb{N}}$ with

$$h^{(n)} = d_n \circ h, \quad n \geq 1$$

where we define

$$d_n(r) := \sum_{k=1}^{n2^n} \frac{k-1}{2^n} 1_{[\frac{k-1}{2^n}, \frac{k}{2^n})}(r) + n 1_{[n,\infty]}(r), \quad r \in \mathbb{R}_+ \cup \{\infty\}.$$

We then have $h^{(n)} \nearrow h$ (as $n \nearrow \infty$) pointwise. Since each $h^{(n)}$ is a simple function, by using the Borel measurable sets $\{B_{n,i}; n \geq 1, 0 \leq i \leq n\}$, we can rewrite

$$h^{(n)}(y) = \sum_{i=1}^{n} b_i^{(n)} 1_{\{y \in B_{n,i}\}}, \quad n \geq 1.$$

Then the right-hand side of (4.4) is

$$\begin{aligned}\int_{\mathbb{R}} M^{(A,q)}(\omega, \mathrm{d}y)h(y) &= \lim_{n\to\infty} \int_{\mathbb{R}} M^{(A,q)}(\omega, \mathrm{d}y) \sum_{i=1}^{n} b_i^{(n)} 1_{\{y\in B_{n,i}\}} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} b_i^{(n)} \int_{\mathbb{R}} M^{(A,q)}(\omega, \mathrm{d}y) 1_{\{y\in B_{n,i}\}} = \lim_{n\to\infty} \sum_{i=1}^{n} b_i^{(n)} M^{(A,q)}(\omega, B_{n,i}).\end{aligned}$$

This is indeed equal to the left-hand side of (4.4) because, by the monotone convergence theorem,

$$\begin{aligned}\int_0^{\tau_A(\omega)} e^{-qt} h(X_t(\omega))\mathrm{d}t &= \int_0^{\tau_A(\omega)} e^{-qt} \lim_{n\to\infty} \sum_{i=1}^{n} b_i^{(n)} 1_{\{X_t(\omega)\in B_{n,i}\}} \mathrm{d}t \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} b_i^{(n)} \int_0^{\tau_A(\omega)} e^{-qt} 1_{\{X_t(\omega)\in B_{n,i}\}} \mathrm{d}t = \lim_{n\to\infty} \sum_{i=1}^{n} b_i^{(n)} M^{(A,q)}(\omega, B_{n,i}).\end{aligned}$$

□

With this lemma and the property of the random measure, we have

$$\mathbb{E}^x \left[\int_0^{\tau_A} e^{-qt} h(X_t)\mathrm{d}t \right] = \mathbb{E}^x \left[\int_{\mathbb{R}} M^{(A,q)}(\omega, \mathrm{d}y)h(y) \right] = \int_{\mathbb{R}} \mu_x^{(A,q)}(\mathrm{d}y)h(y) \tag{4.5}$$

where

$$\mu_x^{(A,q)}(B) := \mathbb{E}^x \left[M^{(A,q)}(B) \right], \quad B \in \mathcal{B}[A, \infty)$$

is a version of the *q-resolvent kernel* that has a density owing to Radon-Nikodym theorem; see Bertoin [6]. By using Theorem 1 of Bertoin [6] (see also Emery [13] and Suprun [31]), if τ_a^+ is the first passage time it goes above $a > x$, we have, for every $B \in \mathcal{B}[A, \infty)$,

$$\mathbb{E}^x \left[\int_0^{\tau_A \wedge \tau_a^+} e^{-qt} 1_{\{X_t\in B\}} \mathrm{d}t \right] = \int_B \left[\frac{W^{(q)}(x-A)W^{(q)}(a-y)}{W^{(q)}(a-A)} - 1_{\{x\ge y\}} W^{(q)}(x-y) \right] \mathrm{d}y.$$

Moreover, when $q > 0$ or $\mathbb{E}^x[\tau_A] < \infty$ (which is satisfied when $\overline{u} < 0$ for the exponential jump diffusion case), by taking $a \uparrow \infty$ via the dominated convergence theorem, we have

$$\begin{aligned}\mu_x^{(A,q)}(B) &= \int_B \lim_{a\to\infty} \left[\frac{W^{(q)}(x-A)W^{(q)}(a-y)}{W^{(q)}(a-A)} - 1_{\{x\ge y\}} W^{(q)}(x-y) \right] \mathrm{d}y \\ &= \int_B \left[e^{-\zeta_q(y-A)} W^{(q)}(x-A) - 1_{\{x\ge y\}} W^{(q)}(x-y) \right] \mathrm{d}y\end{aligned}$$

where the second equality holds by (4.2). Hence, we have the following result.

Lemma 4.2. *For any $0 \le A \le x$ and if $q > 0$ or $\mathbb{E}^x[\tau_A] < \infty$, we have*

$$\mu_x^{(A,q)}(\mathrm{d}y) = \begin{cases} \left(e^{-\zeta_q(y-A)} W^{(q)}(x-A) - 1_{\{x\ge y\}} W^{(q)}(x-y) \right) \mathrm{d}y, & y \ge A, \\ 0, & y < A. \end{cases}$$

By Lemma 4.2 and (4.5), we have, for any arbitrary $0 \leq A \leq x$,

$$\mathbb{E}^x \left[\int_0^{\tau_A} e^{-qt} h(X_t) \mathrm{d}t \right] = W^{(q)}(x-A) \int_A^\infty e^{-\zeta_q (y-A)} h(y) \mathrm{d}y - \int_A^x W^{(q)}(x-y) h(y) \mathrm{d}y, \tag{4.6}$$

and this can be used to express the regret function $H(\cdot)$ and the stopping value $G(\cdot)$. Furthermore, its derivative calculated in the following lemma will be used later in this section.

Lemma 4.3. *Suppose X has a Gaussian component. For every $q \geq 0$, we have*

$$\frac{\partial}{\partial x} \mathbb{E}^x \left[\int_0^{\tau_A} e^{-qt} h(X_t) \mathrm{d}t \right] \Big|_{x=A+} = \frac{2}{\sigma^2} \int_0^\infty e^{-\zeta_q y} h(y+A) \mathrm{d}y, \quad A \geq 0,$$
$$\frac{\partial}{\partial A} \mathbb{E}^x \left[\int_0^{\tau_A} e^{-qt} h(X_t) \mathrm{d}t \right] = -W'_{\zeta_q}(x-A) e^{\zeta_q x} \int_A^\infty e^{-\zeta_q y} h(y) \mathrm{d}y > 0, \quad 0 \leq A \leq x.$$

Proof. Because $W^{(q)} \in C^2(0,\infty)$ by Chan et al. [9], the derivative of (4.6) with respect to x equals

$$W^{(q)\prime}(x-A) \int_A^\infty e^{-\zeta_q (y-A)} h(y) \mathrm{d}y - W^{(q)}(0) h(x) - \int_A^x W^{(q)\prime}(x-y) h(y) \mathrm{d}y.$$

Therefore, the first claim is immediate because

$$W^{(q)}(0) = 0 \quad \text{and} \quad W^{(q)\prime}(0+) = \frac{2}{\sigma^2}, \quad q \geq 0;$$

see Lemma 4.4 of Kyprianou and Surya [22]. For the second claim, notice that

$$\mathbb{E}^x \left[\int_0^{\tau_A} e^{-qt} h(X_t) \mathrm{d}t \right] = e^{\zeta_q x} W_{\zeta_q}(x-A) \int_A^\infty e^{-\zeta_q y} h(y) \mathrm{d}y - \int_A^x W^{(q)}(x-y) h(y) \mathrm{d}y,$$

and its derivative with respect to A becomes

$$-W'_{\zeta_q}(x-A) e^{\zeta_q x} \int_A^\infty e^{-\zeta_q y} h(y) - W_{\zeta_q}(x-A) e^{\zeta_q (x-A)} h(A) + W^{(q)}(x-A) h(A),$$

and therefore the second claim is immediate because the second and the third terms in the above equation cancel out. □

4.3. **Exponential jump diffusion case.** We now return back to the exponential jump diffusion case with only negative jumps, i.e., $p = 1$ in (3.1). We fix $q \geq 0$ and assume when $q = 0$ that $\overline{u} < 0$; this will ensure that $\zeta_0 > 0$ and $\xi_{1,0} = 0$; see page 211 of Kyprianou [21] and (3.4). In other words, we have $\zeta_q > 0$ in every case we consider in this section. We first represent ζ_q in terms of $\xi_{1,q}$ and $\xi_{2,q}$ when $q > 0$ and ζ_0 in terms of $\xi_{2,0}$ when $q = 0$, and then obtain the explicit representation of the scale function. Figure 2 displays how these look when the overall drift $\overline{u}$ is positive and negative.

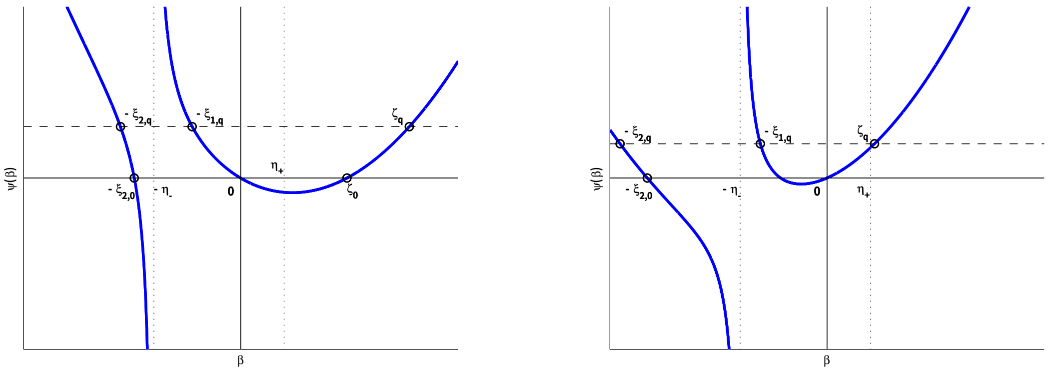
Lemma 4.4. *We have*

$$(i)\ \zeta_q = \frac{2q\eta_-}{\sigma^2 \xi_{1,q} \xi_{2,q}}, \quad q > 0 \quad \textit{and} \quad (ii)\ \zeta_0 = \frac{2|\overline{u}|\eta_-}{\sigma^2 \xi_{2,0}}, \quad q = 0 \textit{ and } \overline{u} < 0.$$

Proof. Substituting $A = 0$ and $h \equiv 1$ in Lemma 4.3, we obtain

$$\frac{\partial}{\partial x} \mathbb{E}^x \left[\int_0^{\tau_0} e^{-qt} \mathrm{d}t \right] \Big|_{x=0+} = \frac{2}{\sigma^2 \zeta_q}.$$

Matching with (3.8), we have the claim. □



(a) $\bar{u} < 0$ (b) $\bar{u} > 0$

FIGURE 2. Illustration of the Laplace exponent and its roots when $p = 1$.

Hubalek and Kyprianou [16] address that the scale function for this process can in principle be extracted from Kou and Wang [19]. In fact, it can be obtained explicitly in terms of ζ_q, $\xi_{1,q}$ and $\xi_{2,q}$. Notice that Lemma 4.4 and Lemma 4.5 below follow from Lemma 4.3, which holds for any spectrally negative Lévy process with a diffusion component. Therefore the scale function can be obtained in the same manner, in particular, for the hyper-exponential jump diffusion and the phase-type Lévy cases by using the results by Cai [7] and Asmussen et al. [1], respectively.

Lemma 4.5. *When (i) $q > 0$ or (ii) $q = 0$ and $\overline{u} < 0$, we have, for every $x \geq 0$,*

$$W_{\zeta_q}(x) = \frac{2}{\sigma^2} \sum_{i=1,2} \left(1 - e^{-(\xi_{i,q}+\zeta_q)x}\right) \frac{l_{i,q}}{\xi_{i,q}+\zeta_q} \quad \text{and} \quad W^{(q)}(x) = \frac{2}{\sigma^2} \sum_{i=1,2} \left(e^{\zeta_q x} - e^{-\xi_{i,q}x}\right) \frac{l_{i,q}}{\xi_{i,q}+\zeta_q}. \tag{4.7}$$

In particular, when (ii) $q = 0$ and $\overline{u} < 0$, we have $\xi_{1,0} = 0$ and therefore

$$\begin{aligned} W_{\zeta_0}(x) &= \frac{2}{\sigma^2} \left[\left(1 - e^{-\zeta_0 x}\right) \frac{l_{1,0}}{\zeta_0} + \left(1 - e^{-(\xi_{2,0}+\zeta_0)x}\right) \frac{l_{2,0}}{\xi_{2,0}+\zeta_0} \right], \\ W^{(q)}(x) &= \frac{2}{\sigma^2} \left[\left(e^{\zeta_0 x} - 1\right) \frac{l_{1,0}}{\zeta_0} + \left(e^{\zeta_0 x} - e^{-\xi_{2,0}x}\right) \frac{l_{2,0}}{\xi_{2,0}+\zeta_0} \right]. \end{aligned}$$

Proof. By substituting $h \equiv 1$ in Lemma 4.3, we obtain

$$\frac{\partial}{\partial A} \mathbb{E}^x \left[\int_0^{\tau_A} e^{-qt} \mathrm{d}t \right] = -W'_{\zeta_q}(x-A) \frac{e^{\zeta_q(x-A)}}{\zeta_q}, \quad 0 \leq A \leq x.$$

Matching with (3.9) and noticing that these equations hold for any x and A such that $x - A \geq 0$, we have by Lemma 4.4, for every $y \geq 0$,

$$\begin{aligned} W'_{\zeta_q}(y) &= \frac{\zeta_q}{q} \xi_{1,q}\xi_{2,q} \sum_{i=1,2} \frac{l_{i,q}}{\eta_-} e^{-(\xi_{i,q}+\zeta_q)y} = \frac{2}{\sigma^2} \sum_{i=1,2} l_{i,q} e^{-(\xi_{i,q}+\zeta_q)y}, \\ W'_{\zeta_0}(y) &= \frac{\zeta_0}{|\overline{u}|} \left[e^{-\zeta_0 y} + \frac{\xi_{2,0}-\eta_-}{\eta_-} e^{-(\xi_{2,0}+\zeta_0)y} \right] = \frac{2}{\sigma^2} \sum_{i=1,2} l_{i,0} e^{-(\xi_{i,0}+\zeta_0)y} \end{aligned} \tag{4.8}$$

when (i) $q > 0$ and (ii) $q = 0$ and $\overline{u} < 0$, respectively. Integrating the above and noting that $W_{\zeta_q}(0) = 0$, we have the first claim. The second claim is also immediate by (4.3). □

As we have studied in the previous section, a reasonable candidate for the optimal threshold level is A^* with which the smooth fit condition holds; $\delta'_{A^*}(A^*+) = 0$. Notice that continuous fit holds whatever the choice of A^* is. Lemma 4.3 together with (3.6) shows that

$$\delta'_A(A+) = e^{-\eta_- A}\frac{(\eta_- - \xi_{1,q})(\xi_{2,q} - \eta_-)}{\eta_-} - \frac{2\gamma}{\sigma^2}\int_0^\infty e^{-\zeta_q y} h(y+A)\mathrm{d}y, \quad A \geq 0.$$

Because $h(\cdot)$ is increasing, a threshold level $A \geq 0$ that attains smooth fit exists on condition that

$$\frac{(\eta_- - \xi_{1,q})(\xi_{2,q} - \eta_-)}{\eta_-} \geq \frac{2\gamma}{\sigma^2}\int_0^\infty e^{-\zeta_q y} h(y)\mathrm{d}y. \tag{4.9}$$

Lemma 4.6. *Suppose (4.9) holds. The smooth fit condition $\delta'_A(A+) = 0$ is attained if and only if*

$$e^{-\eta_- A}\frac{(\eta_- - \xi_{1,q})(\xi_{2,q} - \eta_-)}{\eta_-} - \frac{2\gamma}{\sigma^2}\int_0^\infty e^{-\zeta_q y} h(y+A)\mathrm{d}y = 0 \tag{4.10}$$

when (i) $q > 0$ and (ii) when $q = 0$ and $\overline{u} < 0$. Furthermore, because $h(\cdot)$ is monotonically increasing by assumption, we see that there exists a unique value of $A \geq 0$ that satisfies the smooth fit condition, $\delta'_A(A+) = 0$.

Similarly to the previous section, let us define A^* to be the unique root of (4.10) when (4.9) holds. We also set $A^* = 0$ when it does not hold. We write $\tau^* := \tau_{A^*}$, $V(\cdot) = V_{A^*}(\cdot)$ (the continuation value corresponding to the threshold strategy τ^*) and $\delta(\cdot) = \delta_{A^*}(\cdot)$ (the corresponding difference between the continuation and the stopping values). Now $\phi : \mathbb{R} \mapsto [0, \infty)$ defined by (3.15) is our candidate value function.

Lemma 4.7. *We have $\phi(x) \leq G(x)$ for every $x \geq 0$. In particular, when (4.9) holds, the equality is attained at $x = A^*$.*

Proof. We have, by Lemma 4.3 and (4.8),

$$\frac{\partial}{\partial A}\mathbb{E}^x\left[\int_0^{\tau_A} e^{-qt} h(X_t)\mathrm{d}t\right] = -\frac{2}{\sigma^2}\sum_{i=1,2} l_{i,q} e^{-\xi_{i,q}(x-A)}\int_0^\infty e^{-\zeta_q y} h(y+A)\mathrm{d}y,$$

and hence, by (3.7), we have $\partial V_A(x)/\partial A = \partial \delta_A(x)/\partial A$ equals

$$\left(\frac{1}{\eta_-}\sum_{i=1,2} l_{i,q} e^{-\xi_{i,q}(x-A)}\right)\left[-(\eta_- - \xi_{1,q})(\xi_{2,q} - \eta_-)e^{-\eta_- A} + \frac{2\gamma\eta_-}{\sigma^2}\int_0^\infty e^{-\zeta_q y} h(y+A)\mathrm{d}y\right].$$

Here, the first term is always strictly positive and the second term is identical to the left-hand side of (4.10). Therefore, the derivative is non-negative if and only if $A \geq A^*$. Hence the claim follows along the same line as Lemma 3.3.

□

Remark 4.1. *When $h \equiv 1$, the threshold level A^* reduces via Lemma 4.4 to the optimal threshold level that was obtained in the last section.*

4.4. **Verification of optimality.** We verify the optimality of τ^*. By Lemma 4.7, we only need to show the submartingale property of the discounted version of ϕ. Furthermore, when (4.9) does not hold, the value function has the same form as (3.16) and the optimality of τ^* holds along the same line as in the previous section. We therefore assume (4.9) for the rest of this section.

We decompose the scale function (4.7) as in the following:

$$W^{(q)}(x) = (c_1+c_2)e^{\zeta_q x} - c_1 e^{-\xi_{1,q}x} - c_2 e^{-\xi_{2,q}x}, \quad x \geq 0 \tag{4.11}$$

where

$$c_i := \frac{2}{\sigma^2}\frac{l_{i,q}}{\xi_{i,q}+\zeta_q} > 0, \quad i \in \{1,2\}.$$

This decomposition will later enable us to take advantage of the fact that

$$\psi(\zeta_q) = \psi(-\xi_{1,q}) = \psi(-\xi_{2,q}) = q, \quad q \geq 0. \tag{4.12}$$

Using the representation above of the scale function, we can express (4.6) explicitly as in the following.

Lemma 4.8. *Fix $0 \leq A \leq x$, we have*

$$\mathbb{E}^x\left[\int_0^{\tau_A} e^{-qt}h(X_t)\mathrm{d}t\right] = C_0(x)e^{\zeta_q x} - \sum_{i=1,2} C_{i,A}(x)e^{-\xi_{i,q}x},$$

$$\frac{\partial}{\partial x}\mathbb{E}^x\left[\int_0^{\tau_A} e^{-qt}h(X_t)\mathrm{d}t\right] = \zeta_q C_0(x)e^{\zeta_q x} + \sum_{i=1,2} \xi_{i,q}C_{i,A}(x)e^{-\xi_{i,q}x},$$

$$\frac{\partial^2}{\partial x^2}\mathbb{E}^x\left[\int_0^{\tau_A} e^{-qt}h(X_t)\mathrm{d}t\right] = (\zeta_q)^2 C_0(x)e^{\zeta_q x} - \sum_{i=1,2} (\xi_{i,q})^2 C_{i,A}(x)e^{-\xi_{i,q}x} - \frac{2}{\sigma^2}h(x)$$

where

$$C_0(x) := (c_1+c_2)\int_x^\infty e^{-\zeta_q y}h(y)\mathrm{d}y,$$

$$C_{i,A}(x) := c_i\left(e^{\xi_{i,q}A}\int_0^\infty e^{-\zeta_q y}h(y+A)\mathrm{d}y - \int_A^x e^{\xi_{i,q}y}h(y)\mathrm{d}y\right), \quad i \in \{1,2\}.$$

Proof. Substituting (4.11) in (4.6), we have

$$\mathbb{E}^x\left[\int_0^{\tau_A} e^{-qt}h(X_t)\mathrm{d}t\right] = \left[(c_1+c_2)e^{\zeta_q x}\int_A^\infty e^{-\zeta_q y}h(y)\mathrm{d}y - \sum_{i=1,2} c_i e^{-\xi_{i,q}(x-A)}\int_0^\infty e^{-\zeta_q y}h(y+A)\mathrm{d}y\right]$$
$$- \left[(c_1+c_2)e^{\zeta_q x}\int_A^x e^{-\zeta_q y}h(y)\mathrm{d}y - \sum_{i=1,2} c_i e^{-\xi_{i,q}x}\int_A^x e^{\xi_{i,q}y}h(y)\mathrm{d}y\right],$$

and this shows the first claim. Because

$$C_0'(x) = -(c_1+c_2)e^{-\zeta_q x}h(x) \quad \text{and} \quad C_{i,A}'(x) = -c_i e^{\xi_{i,q}x}h(x), \quad i \in \{1,2\}, \tag{4.13}$$

we have

$$C_0'(x)e^{\zeta_q x} - \sum_{i=1,2} C_{i,A}'(x)e^{-\xi_{i,q}x} = 0 \quad \text{and} \quad \zeta_q C_0'(x)e^{\zeta_q x} + \sum_{i=1,2} \xi_{i,q}C_{i,A}'(x)e^{-\xi_{i,q}x} = -\frac{2}{\sigma^2}h(x),$$

and hence the second and third claims are also immediate. □

The stopping value $G(\cdot)$ in (2.4) can be obtained by setting $A = 0$ in Lemma 4.8.

Corollary 4.1. *For every $x \geq 0$, we have*

$$\begin{aligned}
G(x)/\gamma &= C_0(x)e^{\zeta_q x} - \sum_{i=1,2} C_i(x)e^{-\xi_{i,q}x}, \\
G'(x)/\gamma &= \zeta_q C_0(x)e^{\zeta_q x} + \sum_{i=1,2} \xi_{i,q} C_i(x)e^{-\xi_{i,q}x}, \\
G''(x)/\gamma &= (\zeta_q)^2 C_0(x)e^{\zeta_q x} - \sum_{i=1,2} (\xi_{i,q})^2 C_i(x)e^{-\xi_{i,q}x} - \frac{2}{\sigma^2}h(x)
\end{aligned}$$

where

$$C_i(x) \equiv C_{i,0}(x) = c_i \left(\int_0^\infty e^{-\zeta_q y} h(y)\mathrm{d}y - \int_0^x e^{\xi_{i,q}y} h(y)\mathrm{d}y \right), \quad i \in \{1,2\}.$$

Remark 4.2. *In view of Corollary 4.1, we see that $G(x)$ is finite for every $x \geq 0$ when (4.9) holds.*

By (4.10), we can set

$$\varrho := \frac{\sigma^2}{2\gamma} e^{-\eta_- A^*} \frac{(\eta_- - \xi_{1,q})(\xi_{2,q} - \eta_-)}{\eta_-} = \int_0^\infty e^{-\zeta_q y} h(y + A^*)\mathrm{d}y.$$

We can then rewrite the violation risk associated with strategy τ^* in Lemma 3.1:

$$R_x^{(q)}(\tau^*) = \frac{2\gamma}{\sigma^2(\xi_{2,q} - \xi_{1,q})} \varrho \left(e^{-\xi_{1,q}(x-A^*)} - e^{-\xi_{2,q}(x-A^*)} \right). \tag{4.14}$$

Using (4.14) and Lemma 4.8, we obtain the following.

Lemma 4.9. *We have for every $x \geq A^*$*

$$\begin{aligned}
\delta(x)/\gamma &= -C_0(x)e^{\zeta_q x} + \sum_{i=1,2} K_i(x)e^{-\xi_{i,q}x}, \\
\delta'(x)/\gamma &= -\zeta_q C_0(x)e^{\zeta_q x} - \sum_{i=1,2} \xi_{i,q} K_i(x)e^{-\xi_{i,q}x}, \\
\delta''(x)/\gamma &= -(\zeta_q)^2 C_0(x)e^{\zeta_q x} + \sum_{i=1,2} (\xi_{i,q})^2 K_i(x)e^{-\xi_{i,q}x} + \frac{2}{\sigma^2}h(x)
\end{aligned}$$

where

$$\begin{aligned}
K_1(x) &:= \left(\frac{2}{\sigma^2(\xi_{2,q} - \xi_{1,q})} + c_1 \right) e^{\xi_{1,q}A^*} \varrho - c_1 \int_{A^*}^x e^{\xi_{1,q}y} h(y)\mathrm{d}y, \\
K_2(x) &:= \left(-\frac{2}{\sigma^2(\xi_{2,q} - \xi_{1,q})} + c_2 \right) e^{\xi_{2,q}A^*} \varrho - c_2 \int_{A^*}^x e^{\xi_{2,q}y} h(y)\mathrm{d}y.
\end{aligned}$$

Proof. By Lemma 4.8 and (4.14), we have

$$\begin{aligned}\frac{\delta(x)}{\gamma} &= \frac{R_x(\tau^*)}{\gamma} - \mathbb{E}^x\left[\int_0^{\tau_{A^*}} e^{-qt}h(X_t)\mathrm{d}t\right]\\ &= -C_0(x)e^{\zeta_q x} + \left[\frac{2\varrho}{\sigma^2(\xi_{2,q}-\xi_{1,q})}e^{\xi_{1,q}A^*} + C_{1,A^*}(x)\right]e^{-\xi_{1,q}x} + \left[-\frac{2\varrho}{\sigma^2(\xi_{2,q}-\xi_{1,q})}e^{\xi_{2,q}A^*} + C_{2,A^*}(x)\right]e^{-\xi_{2,q}x}\\ &= -C_0(x)e^{\zeta_q x} + \sum_{i=1,2} K_i(x)e^{-\xi_{i,q}x},\end{aligned}$$

which shows the first claim. Notice that because (4.13) and

$$K_i'(x) = -c_i e^{\xi_{i,q}x}h(x), \quad i \in \{1,2\},$$

we have

$$-C_0'(x)e^{\zeta_q x} + \sum_{i=1,2} K_i'(x)e^{-\xi_{i,q}x} = 0 \quad \text{and} \quad -\zeta_q C_0'(x)e^{\zeta_q x} - \sum_{i=1,2}\xi_{i,q}K_i'(x)e^{-\xi_{i,q}x} = \frac{2}{\sigma^2}h(x),$$

and hence the second and third claims are also immediate. □

Now the continuation value $V(x) = \delta(x) + G(x)$ becomes, by Corollary 4.1 and Lemma 4.9,

$$V(x) = \gamma\sum_{i=1,2}(K_i(x) - C_i(x))e^{-\xi_{i,q}x} = \gamma\sum_{i=1,2}L_i e^{-\xi_{i,q}x}, \quad x \geq A^* \tag{4.15}$$

where L_1 and L_2 are constants independent of x. To see why this is so, we have for, every $x \geq A^*$,

$$K_1(x) - C_1(x) = \left(\frac{2}{\sigma^2(\xi_{2,q}-\xi_{i,q})} + c_1\right)e^{\xi_{1,q}A^*}\varrho + c_1\left(\int_0^{A^*} e^{\xi_{1,q}y}h(y)\mathrm{d}y - \int_0^\infty e^{-\zeta_q y}h(y)\mathrm{d}y\right) =: L_1,$$

$$K_2(x) - C_2(x) = \left(-\frac{2}{\sigma^2(\xi_{2,q}-\xi_{i,q})} + c_2\right)e^{\xi_{2,q}A^*}\varrho + c_2\left(\int_0^{A^*} e^{\xi_{2,q}y}h(y)\mathrm{d}y - \int_0^\infty e^{-\zeta_q y}h(y)\mathrm{d}y\right) =: L_2.$$

We shall now show that the stochastic process $M = \{M_t; t \geq 0\}$ defined by (3.21) is a submartingale. The proof of the following lemma is lengthly and hence deferred to the appendix. Notice that the boundedness of $\phi(\cdot)$ by 1 in this case is also immediate by the same reason discussed in Remark 3.2 (3).

Lemma 4.10. *We have*

(1) $\mathcal{L}\phi(x) - q\phi(x) = 0$ *for every* $x > A^*$,
(2) $\mathcal{L}\phi(x) - q\phi(x) = -\gamma h(x) + \lambda e^{-\eta_- x}$ *for every* $0 < x < A^*$.

In the lemma above, we see that $\mathcal{L}\phi(x) - q\phi(x)$ is decreasing on $(0, A^*)$ because h is increasing. Therefore, in order to show its non-negativity, it is sufficient to show that it is non-negative at A^*- (see Lemma 3.4). This is indeed true by the continuous and smooth fit conditions, $\delta(A^*) = 0$ and $\delta'(A^*+) = 0$, and the following lemma.

Lemma 4.11. *We have* $\delta''(A^*+) < 0$.

Proof. By Lemma 4.9 and noticing that

$$C_0(A^*)e^{\zeta_q A^*} = (c_1+c_2)\varrho \quad \text{and} \quad K_i(A^*)e^{-\xi_{i,q}A^*} = \left\{\begin{array}{ll}\varrho\left(\frac{2}{\sigma^2(\xi_{2,q}-\xi_{1,q})} + c_1\right), & i=1\\ \varrho\left(-\frac{2}{\sigma^2(\xi_{2,q}-\xi_{1,q})} + c_2\right), & i=2\end{array}\right\},$$

we have

$$\begin{aligned}\frac{\delta''(A^*+)}{\gamma} &= \varrho\left[-(\zeta_q)^2(c_1+c_2)+(\xi_{1,q})^2\left(\frac{2}{\sigma^2(\xi_{2,q}-\xi_{1,q})}+c_1\right)+(\xi_{2,q})^2\left(-\frac{2}{\sigma^2(\xi_{2,q}-\xi_{1,q})}+c_2\right)\right]+\frac{2}{\sigma^2}h(A^*)\\
&= \varrho\left[c_1\left((\xi_{1,q})^2-(\zeta_q)^2\right)+c_2\left((\xi_{2,q})^2-(\zeta_q)^2\right)-\frac{2}{\sigma^2(\xi_{2,q}-\xi_{1,q})}\left((\xi_{2,q})^2-(\xi_{1,q})^2\right)\right]+\frac{2}{\sigma^2}h(A^*)\\
&= \varrho\left[\frac{2}{\sigma^2}l_{1,q}\left(\xi_{1,q}-\zeta_q\right)+\frac{2}{\sigma^2}l_{2,q}\left(\xi_{2,q}-\zeta_q\right)-\frac{2}{\sigma^2}\left(\xi_{2,q}+\xi_{1,q}\right)\right]+\frac{2}{\sigma^2}h(A^*)\\
&= \frac{2}{\sigma^2}\left(\varrho(-\zeta_q-\eta_-)+h(A^*)\right).\end{aligned}$$

Now because $h(\cdot)$ is increasing

$$\varrho=\int_0^\infty e^{-\zeta_q y}h(y+A^*)\mathrm{d}y \geq \int_0^\infty e^{-\zeta_q y}h(A^*)\mathrm{d}y=\frac{1}{\zeta_q}h(A^*),$$

and hence

$$\frac{\delta''(A^*+)}{\gamma}\leq -\eta_-\varrho\frac{2}{\sigma^2}<0,$$

as desired. □

Combining Lemmas 4.10 and 4.11, we have the following.

Lemma 4.12. *For every $0<x<A^*$, we have $\mathcal{L}\phi(x)-q\phi(x)\geq 0$.*

Combining the results above, we see that the stochastic process M is indeed a submartingale (see Lemma 3.6), and the optimality of τ^* is immediate using the same argument in Proposition 3.1.

Proposition 4.1. *We have*

$$\phi(x)=\inf_{\tau\in\mathcal{S}}\mathbb{E}^x\left[e^{-q\tau}G(X_\tau)1_{\{\tau<\infty\}}\right]$$

and τ^* is the optimal stopping time.

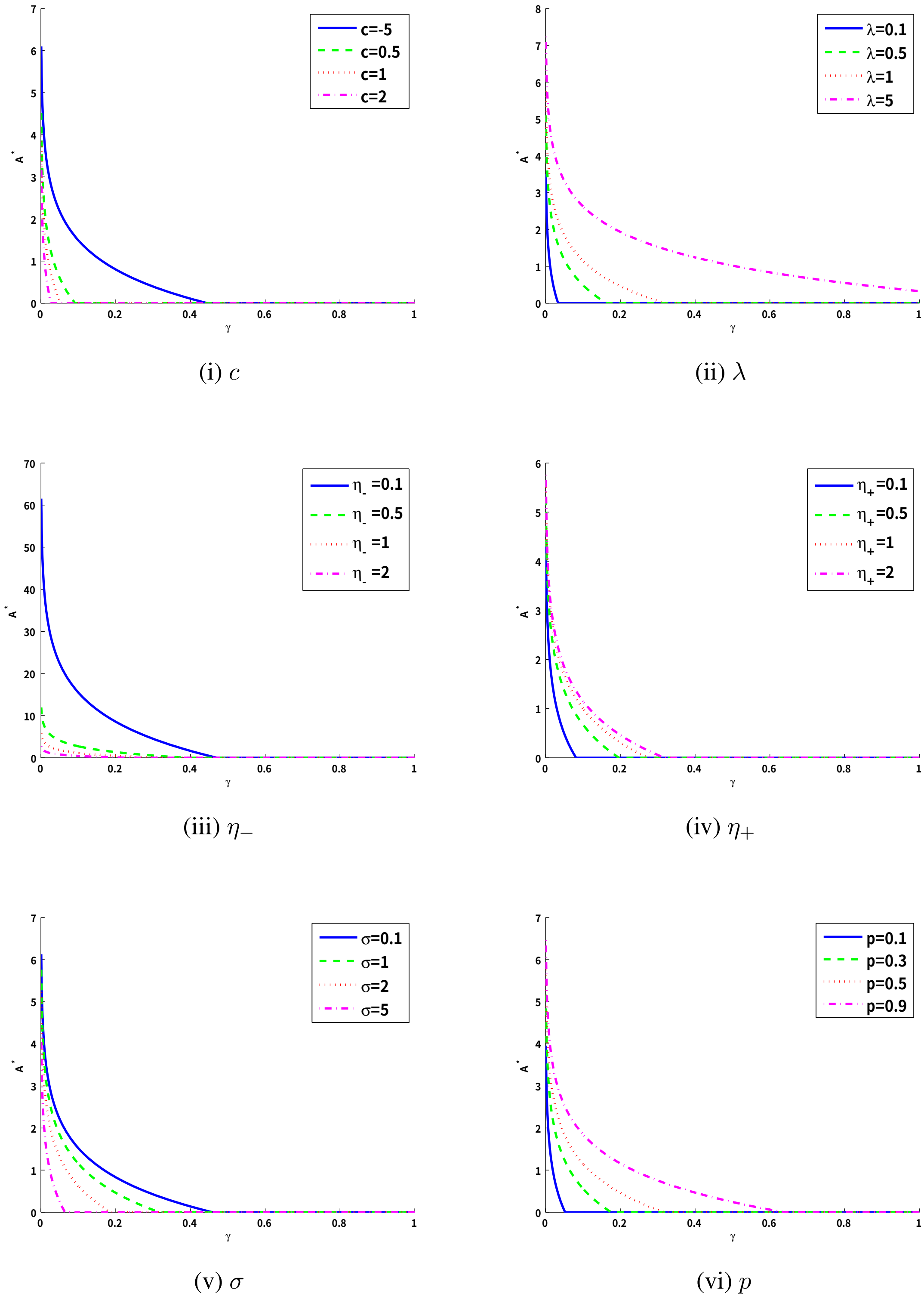
5. Numerical Results

We conclude this paper by providing numerical results on the models studied in Sections 3 and 4. We obtain optimal strategies numerically and study how the solution depends on each parameter. We first consider the case with jumps in both directions and $h\equiv 1$, and then consider the spectrally negative case with h in the form of the exponential utility function.

5.1. When there are jumps in both directions and $h\equiv 1$.

We evaluate the results obtained in Section 3 focusing on the case $h\equiv 1$. Here we plot the optimal threshold level A^* defined in (3.14) as a function of γ. The values of $\xi_{1,q}$ and $\xi_{2,q}$ are obtained via the bisection method with error bound 10^{-4}.

Figure 3 shows how the optimal threshold level changes with respect to each parameter when $q=0.1$. The results obtained in (i)-(iv) and (vi) are consistent with our intuition because these parameters determine the drift $\overline{u}$, and A^* is expected to decrease in $\overline{u}$. We show in (v) how it changes with respect to σ; although it does not play a part in determining the drift, we see that A^* is in fact decreasing in σ. This is related to the fact that, as



(i) c (ii) λ

(iii) η_- (iv) η_+

(v) σ (vi) p

FIGURE 3. The optimal threshold level A^* with respect to various parameters: note that the base case parameters are $c = -1$, $\sigma = 1$, $\eta_- = 1.0$, $\eta_+ = 2.0$, $p = 0.5$, $\lambda = 1$.

σ increases, the probability of creeping downward increases. Notice that the violation risk becomes zero when it creeps downward at 0.

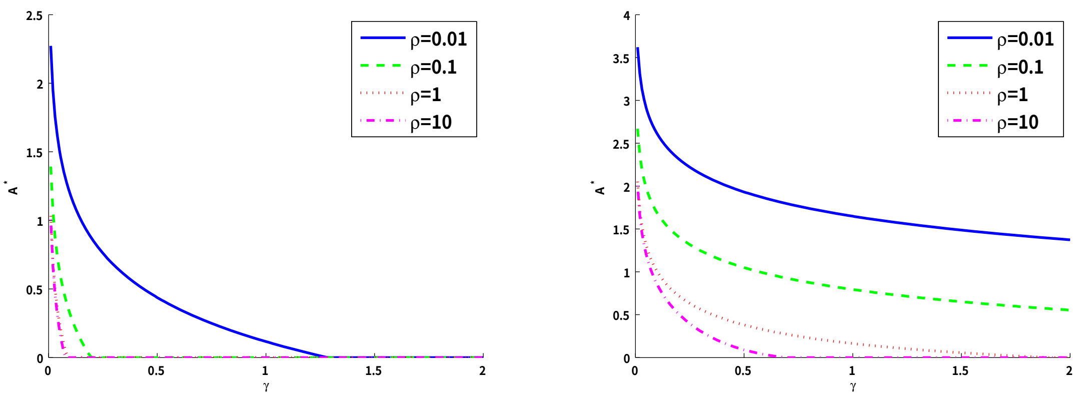
5.2. **When jumps are only downward and h is the exponential utility function.** We now consider the spectrally negative case, and evaluate the results obtained in Section 4. For the function h, we use the exponential utility function

$$h(x) = 1 - e^{-\rho x}, \quad x \geq 0.$$

Here $\rho > 0$ is called the *coefficient of absolute risk aversion*. It is well-known that $\rho = -h''(x)/h(x)$ for every x, and, in particular, $h \equiv 1$ when $\rho = \infty$. In this case, (4.10) reduces to

$$e^{-\eta_- A} \frac{(\eta_- - \xi_{1,q})(\xi_{2,q} - \eta_-)}{\eta_-} + e^{-\rho A} \frac{2\gamma}{\sigma^2(\zeta_q + \rho)} = \frac{2\gamma}{\sigma^2 \zeta_q}.$$

Figure 4 shows the optimal threshold level A^* as a function of γ with various values of ρ when $q > 0$ and $q = 0$. We see that it is indeed monotonically decreasing in ρ. This can be also analytically verified because, in view of the above equations, the left-hand side is decreasing in ρ for every fixed A, and consequently the root A^* must be decreasing in ρ. This is also clear because the regret function monotonically decreases in ρ.



(a) $q > 0$ and $\overline{u} > 0$ (b) $q = 0$ and $\overline{u} < 0$

FIGURE 4. The optimal threshold level A^* with various values of coefficients of absolute risk aversion ρ: (a) $q = 0.1$ and $\overline{u} = 0.5$ ($c = 1$, $\sigma = 1$, $\eta_- = 2$, $\eta_+ = 1$, $\lambda = 1$) and (b) $q = 0$ and $\overline{u} = -1.5$ ($c = -1$, $\sigma = 1$, $\eta_- = 2$, $\eta_+ = 1$, $\lambda = 1$).

APPENDIX A. PROOFS

A.1. **Proof of Lemma 3.5.** We shall prove for the case $q > 0$. This result can be extended to the case $q = 0$ if $\overline{u} < 0$. To see this, by Remark 3.3, we have $\phi_q'(x) \to \phi_0'(x)$ and $\phi_q''(x) \to \phi_0''(x)$ as $q \to 0$ for every $x \in \mathbb{R}$. Moreover, by Remark 3.2 (3), via the dominated convergence theorem,

$$\int_{-\infty}^{\infty} \phi_0(x+z) f(z) \mathrm{d}z = \lim_{q \to \infty} \int_{-\infty}^{\infty} \phi_q(x+z) f(z) \mathrm{d}z, \quad x \geq 0.$$

Consequently, $\lim_{q\to 0}(\mathcal{L}\phi_q(x) - q\phi_q(x)) = \mathcal{L}\phi_0(x)$, and hence if we can show for the case $q > 0$, the result holds automatically for the case $q = 0$ given $\overline{u} < 0$.

We first prove the following for the proof of Lemma 3.5.

Lemma A.1. *Fix $q > 0$ and consider a function $w : \mathbb{R} \mapsto \mathbb{R}$ and $x \in \mathbb{R}$. Suppose that, in a neighborhood of $x > 0$,*

$$w(x) = k + \sum_{i=1,2} k_i e^{-\xi_{i,q}x}$$

for some k, k_1 and k_2 in $\mathbb{R}$. Then we have

$$q(w(x) - k) = \mathcal{L}w(x) - \lambda\left[\int_{-\infty}^{\infty} w(x+z)f(z)\mathrm{d}z - \left(k + \sum_{i=1,2} k_i\left(\frac{p\eta_-}{\eta_- - \xi_{i,q}} + \frac{(1-p)\eta_+}{\eta_+ + \xi_{i,q}}\right)e^{-\xi_{i,q}x}\right)\right].$$

Proof. Because $\psi(-\xi_{1,q}) = \psi(-\xi_{2,q}) = q$, we have

$$q(w(x) - k) = \sum_{i=1,2} k_i \psi(-\xi_{i,q}) e^{-\xi_{i,q}x}.$$

Moreover, the right-hand side equals by (3.2)

$$\begin{aligned}
&\sum_{i=1,2} k_i\left[-c\xi_{i,q} + \frac{1}{2}\sigma^2(\xi_{i,q})^2 + \lambda\left(\frac{p\eta_-}{\eta_- - \xi_{i,q}} + \frac{(1-p)\eta_+}{\eta_+ + \xi_{i,q}} - 1\right)\right]e^{-\xi_{i,q}x} \\
&= -\lambda(w(x) - k) + \sum_{i=1,2} k_i\left[-c\xi_{i,q} + \frac{1}{2}\sigma^2(\xi_{i,q})^2 + \lambda\left(\frac{p\eta_-}{\eta_- - \xi_{i,q}} + \frac{(1-p)\eta_+}{\eta_+ + \xi_{i,q}}\right)\right]e^{-\xi_{i,q}x} \\
&= \frac{1}{2}\sigma^2 w''(x) + cw'(x) - \lambda w(x) + \lambda k + \lambda\sum_{i=1,2} k_i\left(\frac{p\eta_-}{\eta_- - \xi_{i,q}} + \frac{(1-p)\eta_+}{\eta_+ + \xi_{i,q}}\right)e^{-\xi_{i,q}x} \\
&= \mathcal{L}w(x) - \lambda\left[\int_{-\infty}^{\infty} w(x+z)f(z)\mathrm{d}z - \left(k + \sum_{i=1,2} k_i\left(\frac{p\eta_-}{\eta_- - \xi_{i,q}} + \frac{(1-p)\eta_+}{\eta_+ + \xi_{i,q}}\right)e^{-\xi_{i,q}x}\right)\right],
\end{aligned}$$

as desired. □

Proof of Lemma 3.5. (i) Suppose (3.13) holds. By Lemma A.1 above, we have $\mathcal{L}\phi(x) - q\phi(x)$ equals, for every $x > A^*$,

$$\lambda\left[\int_{-\infty}^{\infty} \phi(x+z)f(z)\mathrm{d}z - \sum_{i=1,2}(L_{i,q} - C_{i,q})\left(\frac{p\eta_-}{\eta_- - \xi_{i,q}} + \frac{(1-p)\eta_+}{\eta_+ + \xi_{i,q}}\right)e^{-\xi_{i,q}x}\right], \tag{A.1}$$

and, for every $0 \le x < A^*$,

$$-(q+\lambda)\frac{\gamma}{q} + \lambda\left[\int_{-\infty}^{\infty} \phi(x+z)f(z)\mathrm{d}z + \sum_{i=1,2} C_{i,q}\left(\frac{p\eta_-}{\eta_- - \xi_{i,q}} + \frac{(1-p)\eta_+}{\eta_+ + \xi_{i,q}}\right)e^{-\xi_{i,q}x}\right] \tag{A.2}$$

by using (3.10) and (3.20).

Proof of (3.22). We only need to show (A.1) equals zero. Notice that the integral can be split into four parts, and, by using (3.11), we have

$$\int_0^\infty \phi(x+z)f(z)\mathrm{d}z = \sum_{i=1,2}(L_{i,q}-C_{i,q})\frac{(1-p)\eta_+}{\eta_+ + \xi_{i,q}}e^{-\xi_{i,q}x},$$

$$\int_{-(x-A^*)}^0 \phi(x+z)f(z)\mathrm{d}z = \sum_{i=1,2}(L_{i,q}-C_{i,q})\frac{p\eta_-}{\eta_- - \xi_{i,q}}e^{-\xi_{i,q}x} - \sum_{i=1,2}(L_{i,q}-C_{i,q})\frac{p\eta_-}{\eta_- - \xi_{i,q}}e^{-\eta_-(x-A^*)-\xi_{i,q}A^*},$$

$$\int_{-x}^{-(x-A^*)} \phi(x+z)f(z)\mathrm{d}z = \frac{p\gamma}{q}e^{-\eta_-(x-A^*)} - \sum_{i=1,2}C_{i,q}\frac{p\eta_-}{\eta_- - \xi_{i,q}}e^{-\eta_-(x-A^*)-\xi_{i,q}A^*},$$

$$\int_{-\infty}^{-x} \phi(x+z)f(z)\mathrm{d}z = pe^{-\eta_- x}.$$

Putting altogether, we have (A.1) equals

$$-\sum_{i=1,2}L_{i,q}\frac{p\eta_-}{\eta_- - \xi_{i,q}}e^{-\eta_-(x-A^*)-\xi_{i,q}A^*}+\frac{p\gamma}{q}e^{-\eta_-(x-A^*)}+pe^{-\eta_- x} = pe^{-\eta_- x}\left[1-\frac{\gamma}{q}\frac{\xi_{1,q}\xi_{2,q}}{(\eta_- - \xi_{1,q})(\xi_{2,q}-\eta_-)}e^{\eta_- A^*}\right],$$

and this vanishes because of the way A^* is chosen.

Proof of (3.23). We shall show that (A.2) is decreasing in x. Note that $\int_{-\infty}^\infty \phi(x+z)f(z)\mathrm{d}z$ can be split into four parts where

$$\int_{A^*-x}^\infty \phi(x+z)f(z)\mathrm{d}z = (1-p)\sum_{i=1,2}(L_{i,q}-C_{i,q})\frac{\eta_+}{\xi_{i,q}+\eta_+}e^{-\eta_+(A^*-x)-\xi_{i,q}A^*},$$

$$\int_0^{A^*-x} \phi(x+z)f(z)\mathrm{d}z = (1-p)\left[\frac{\gamma}{q}\left(1-e^{-\eta_+(A^*-x)}\right)-\sum_{i=1,2}C_{i,q}\frac{\eta_+}{\xi_i+\eta_+}\left(e^{-x\xi_i}-e^{-\eta_+(A^*-x)-\xi_{i,q}A^*}\right)\right],$$

$$\int_{-x}^0 \phi(x+z)f(z)\mathrm{d}z = \frac{p\gamma}{q} - p\sum_{i=1,2}C_{i,q}\frac{\eta_-}{\eta_- - \xi_{i,q}}e^{-\xi_i x},$$

$$\int_{-\infty}^{-x} \phi(x+z)f(z)\mathrm{d}z = pe^{-\eta_- x},$$

and after some algebra, we have

$$\int_{-\infty}^\infty \phi(x+z)f(z)\mathrm{d}z + \sum_{i=1,2}C_{i,q}\left(\frac{p\eta_-}{\eta_- - \xi_{i,q}}+\frac{(1-p)\eta_+}{\eta_+ + \xi_{i,q}}\right)e^{-\xi_{i,q}x}$$
$$= \frac{\gamma}{q}+pe^{-\eta_- x}+(1-p)e^{\eta_+(x-A^*)}\left(-\frac{\gamma}{q}+\sum_{i=1,2}L_{i,q}\frac{\eta_+}{\xi_{i,q}+\eta_+}e^{-\xi_{i,q}A^*}\right),$$

which by (3.20) equals to

$$\frac{\gamma}{q}+pe^{-\eta_- x}+(1-p)e^{-\eta_+(A^*-x)}\frac{\gamma}{\xi_{2,q}-\xi_{1,q}}\frac{\xi_1\xi_2}{q}\left(\frac{1}{\xi_{2,q}+\eta_+}-\frac{1}{\xi_{1,q}+\eta_+}\right).$$

Hence we see that (A.2) or $\mathcal{L}\phi(x) - q\phi(x)$ equals

$$-\gamma + \lambda \left[p e^{-\eta_- x} - (1-p) e^{-\eta_+ (A^* - x)} \frac{\xi_1 \xi_2}{q} \frac{\gamma}{(\xi_{2,q} + \eta_+)(\xi_{1,q} + \eta_+)} \right]$$

and is therefore decreasing in x. This together with Lemma 3.4 shows $\mathcal{L}\phi(x) - q\phi(x) > 0$ on $(0, A^*)$.

(ii) Suppose (3.13) does not hold. By (3.16), we have

$$\phi(x) = C \left(e^{-\xi_{1,q} x} - e^{-\xi_{2,q} x} \right), \quad x \geq 0$$

where

$$C := \frac{(\eta_- - \xi_{1,q})(\xi_{2,q} - \eta_-)}{\eta_- (\xi_{2,q} - \xi_{1,q})}.$$

By Lemma A.1 above, we have $\mathcal{L}\phi(x) - q\phi(x)$ equals, for every $x \geq 0$,

$$\lambda \left[\int_{-\infty}^{\infty} \phi(x+z) f(z) \mathrm{d}z - \sum_{i=1,2} C \left(\frac{p\eta_-}{\eta_- - \xi_{i,q}} + \frac{(1-p)\eta_+}{\eta_+ + \xi_{i,q}} \right) e^{-\xi_{i,q} x} \right],$$

and this vanishes after some algebra. $\square$

A.2. **Proof of Lemma 3.6.** Notice that ϕ is discontinuous at 0 (in particular, $\phi(0) = 0$ whereas $\phi(0-) = 1$), and hence we need to adjust the proof accordingly. The proof is similar to that of Theorem 3.1 in Kou and Wang [19]. It is easy to verify that there exists a sequence of functions $\{\phi_n; n \geq 0\}$ each of which is C^2 everywhere except at A^* and

$$\phi_n(x) \begin{cases} = \phi(x), & x \geq 0, \\ \in (0,1), & -\frac{1}{n} < x < 0, \\ 1, & x \leq -\frac{1}{n}, \end{cases}$$

and $\phi_n(x) \to \phi(x)$ as $n \to \infty$ pointwise for every $x \in (-\infty, \infty)$; see Kou and Wang [19].

We notice that

$$|\phi(x) - \phi_n(x)| \leq 1, \quad \text{uniformly on } (-\infty, \infty). \tag{A.3}$$

This implies after some calculation that

$$|(\mathcal{L}\phi(x) - q\phi(x)) - (\mathcal{L}\phi_n(x) - q\phi_n(x))| \leq \frac{\lambda p \eta_-}{n}, \quad \text{uniformly on } (-\infty, \infty), \tag{A.4}$$

and it converges to 0 uniformly on $(-\infty, \infty)$ as $n \to 0$. Fix $t \geq 0$ and $n \geq 0$. Notice that, although ϕ_n is not C^2 at A^*, the Lebesque measure of ϕ_n at which $X = A^*$ is zero and hence $\phi_n''(A^*)$ can be chosen arbitrarily. By applying Ito's lemma to $\left\{ e^{-q(t \wedge \tau_0)} \phi_n(X_{t \wedge \tau_0}); t \geq 0 \right\}$, we see that

$$M_t^{(n)} := e^{-q(t \wedge \tau_0)} \phi_n(X_{t \wedge \tau_0}) - \int_0^{t \wedge \tau_0} e^{-qs} \left(\mathcal{L}\phi_n(X_s) - q\phi_n(X_s) \right) \mathrm{d}s$$

is a local martingale. Because $\phi(x) = \phi_n(x)$ for every $x \geq 0$, we have

$$\mathbb{E}^x \left[e^{-q(t \wedge \tau_0)} \phi_n(X_{t \wedge \tau_0}) \right] = \phi(x) + \mathbb{E}^x \left[\int_0^{t \wedge \tau_0} e^{-qs} \left(\mathcal{L}\phi_n(X_s) - q\phi_n(X_s) \right) \mathrm{d}s \right].$$

By (A.3) and the dominated convergence theorem thanks to Remark 3.2 (3), we have

$$\lim_{n\to\infty} \mathbb{E}^x \left[e^{-q(t\wedge\tau_0)} \phi_n(X_{t\wedge\tau_0}) \right] = \mathbb{E}^x \left[e^{-q(t\wedge\tau_0)} \phi(X_{t\wedge\tau_0}) \right],$$

and, by (A.4) and Lemma 3.5,

$$\lim_{n\to\infty} \mathbb{E}^x \left[\int_0^{t\wedge\tau_0} e^{-qs} \left(\mathcal{L}\phi_n(X_s) - q\phi_n(X_s) \right) \mathrm{d}s \right] \geq 0$$

where the equality holds for (2). Therefore, we have $\mathbb{E}^x \left[e^{-q(t\wedge\tau_0)} \phi(X_{t\wedge\tau_0}) \right]$ is greater than or equal to $\phi(x)$ for (1) and is equal to $\phi(x)$ for (2), which shows the (sub-)martingale property. Finally, the integrability condition holds by Remark 3.2 (3).

A.3. **Proof of Lemma 4.10.** Proof of (1). In view of (4.15), by Lemma A.1, we only need to show

$$\frac{1}{\gamma} \int_0^\infty \phi(x-z)\eta_- e^{-\eta_- z} \mathrm{d}z = \sum_{i=1,2} L_i e^{-\xi_{i,q} x} \frac{\eta_-}{\eta_- - \xi_{i,q}}.$$

Here, the left-hand side equals

$$\frac{1}{\gamma} \left(\int_0^{x-A^*} V(x-z)\eta_- e^{-\eta_- z} \mathrm{d}z + \int_{x-A^*}^x G(x-z)\eta_- e^{-\eta_- z} \mathrm{d}z + e^{-\eta_- x} \right)$$

where, in particular, we have

$$\frac{1}{\gamma} \int_0^{x-A^*} V(x-z)\eta_- e^{-\eta_- z} \mathrm{d}z = \sum_{i=1,2} L_i \frac{\eta_-}{\eta_- - \xi_{i,q}} \left[e^{-\xi_{i,q} x} - e^{-\eta_-(x-A^*) - \xi_{i,q} A^*} \right].$$

Therefore, it is sufficient to show that

$$\sum_{i=1,2} L_i \frac{\eta_-}{\eta_- - \xi_{i,q}} e^{-\eta_-(x-A^*) - \xi_{i,q} A^*} = \frac{1}{\gamma} \left(\int_{x-A^*}^x G(x-z)\eta_- e^{-\eta_- z} \mathrm{d}z + e^{-\eta_- x} \right),$$

or equivalently, because $L_i = K_i(x) - C_i(x)$ for any arbitrary $x \geq A^*$,

$$\sum_{i=1,2} (K_i(A^*) - C_i(A^*)) \frac{\eta_-}{\eta_- - \xi_{i,q}} e^{-\eta_-(x-A^*) - \xi_{i,q} A^*} = \frac{1}{\gamma} \left(\int_{x-A^*}^x G(x-z)\eta_- e^{-\eta_- z} \mathrm{d}z + e^{-\eta_- x} \right).$$

Because

$$K_1(A^*) = \left(\frac{2}{\sigma^2(\xi_{2,q} - \xi_{1,q})} + c_1 \right) e^{\xi_{1,q} A^*} \varrho \quad \text{and} \quad K_2(A^*) = \left(-\frac{2}{\sigma^2(\xi_{2,q} - \xi_{1,q})} + c_2 \right) e^{\xi_{2,q} A^*} \varrho,$$

we have

$$\begin{aligned}
&\sum_{i=1,2} K_i(A^*) \frac{\eta_-}{\eta_- - \xi_{i,q}} e^{-\eta_-(x-A^*) - \xi_{i,q} A^*} \\
&= \varrho e^{-\eta_-(x-A^*)} \left[\left(\frac{2}{\sigma^2(\xi_{2,q} - \xi_{1,q})} + c_1 \right) \frac{\eta_-}{\eta_- - \xi_{1,q}} + \left(-\frac{2}{\sigma^2(\xi_{2,q} - \xi_{1,q})} + c_2 \right) \frac{\eta_-}{\eta_- - \xi_{2,q}} \right] \\
&= \frac{2}{\sigma^2} \varrho e^{-\eta_-(x-A^*)} \frac{\eta_-}{(\eta_- - \xi_{1,q})(\xi_{2,q} - \eta_-)} + \varrho e^{-\eta_-(x-A^*)} \sum_{i=1,2} \frac{\eta_-}{\eta_- - \xi_{i,q}} c_i \\
&= \frac{1}{\gamma} e^{-\eta_- x} + \varrho e^{-\eta_-(x-A^*)} \sum_{i=1,2} \frac{\eta_-}{\eta_- - \xi_{i,q}} c_i
\end{aligned}$$

where the last equality holds by the definition of ϱ. Therefore, the problem reduces to showing the following equality:

$$\text{(A.5)}\quad \frac{1}{\gamma}\int_{x-A^*}^{x} G(x-z)\eta_- e^{-\eta_- z}\mathrm{d}z = \left(\sum_{i=1,2}\frac{\eta_-}{\eta_- - \xi_{i,q}}c_i\right)\varrho e^{-\eta_-(x-A^*)} - \sum_{i=1,2} C_i(A^*)\frac{\eta_-}{\eta_- - \xi_{i,q}} e^{-\eta_-(x-A^*)-\xi_{i,q}A^*}.$$

By Corollary 4.1, the left-hand side equals

$$\int_{x-A^*}^{x} C_0(x-z)e^{\zeta_q(x-z)}\eta_- e^{-\eta_- z}\mathrm{d}z - \sum_{i=1,2}\int_{x-A^*}^{x} C_i(x-z)e^{-\xi_{i,q}(x-z)}\eta_- e^{-\eta_- z}\mathrm{d}z.$$

In particular, by integration by parts and (4.13), we have

$$\begin{aligned}
&\int_{x-A^*}^{x} C_0(x-z)e^{\zeta_q(x-z)}\eta_- e^{-\eta_- z}\mathrm{d}z = e^{\zeta_q x}\int_{x-A^*}^{x} C_0(x-z)e^{-(\zeta_q+\eta_-)z}\eta_-\mathrm{d}z \\
&= e^{\zeta_q x}\left(-\left[C_0(x-z)e^{-(\zeta_q+\eta_-)z}\frac{\eta_-}{\zeta_q+\eta_-}\right]_{z=x-A^*}^{z=x} + \frac{\eta_-}{\zeta_q+\eta_-}\int_{x-A^*}^{x}\frac{\partial C_0(x-z)}{\partial z}e^{-(\zeta_q+\eta_-)z}\mathrm{d}z\right) \\
&= \frac{\eta_-}{\zeta_q+\eta_-}\left(C_0(A^*)e^{-\eta_-(x-A^*)+\zeta_q A^*} - C_0(0)e^{-\eta_- x} + (c_1+c_2)\int_{x-A^*}^{x} e^{-\eta_- z}h(x-z)\mathrm{d}z\right) \\
&= \frac{\eta_-}{\zeta_q+\eta_-}(c_1+c_2)\left(e^{-\eta_-(x-A^*)}\int_{A^*}^{\infty} e^{-\zeta_q(y-A^*)}h(y)\mathrm{d}y - e^{-\eta_- x}\int_0^{\infty} e^{-\zeta_q y}h(y)\mathrm{d}y + \int_{x-A^*}^{x} e^{-\eta_- z}h(x-z)\mathrm{d}z\right) \\
&= \frac{\eta_-}{\zeta_q+\eta_-}(c_1+c_2)e^{-\eta_-(x-A^*)}\varrho + \frac{\eta_-}{\zeta_q+\eta_-}(c_1+c_2)\left(-e^{-\eta_- x}\int_0^{\infty} e^{-\zeta_q y}h(y)\mathrm{d}y + \int_{x-A^*}^{x} e^{-\eta_- z}h(x-z)\mathrm{d}z\right),
\end{aligned}$$

and similarly, again by integration by parts and (4.13), for every $i \in \{1,2\}$,

$$\begin{aligned}
&\int_{x-A^*}^{x} C_i(x-z)e^{-\xi_{i,q}(x-z)}\eta_- e^{-\eta_- z}\mathrm{d}z = e^{-\xi_{i,q}x}\int_{x-A^*}^{x} C_i(x-z)e^{-(\eta_- -\xi_{i,q})z}\eta_-\mathrm{d}z \\
&= e^{-\xi_{i,q}x}\left(-\left[C_i(x-z)e^{-(\eta_- -\xi_{i,q})z}\frac{\eta_-}{\eta_- -\xi_{i,q}}\right]_{z=x-A^*}^{z=x} + \frac{\eta_-}{\eta_- -\xi_{i,q}}\int_{x-A^*}^{x}\frac{\partial C_i(x-z)}{\partial z}e^{-(\eta_- -\xi_{i,q})z}\mathrm{d}z\right) \\
&= \frac{\eta_-}{\eta_- -\xi_{i,q}}\left(C_i(A^*)e^{-\eta_-(x-A^*)-\xi_{i,q}A^*} - C_i(0)e^{-\eta_- x} + c_i\int_{x-A^*}^{x} e^{-\eta_- z}h(x-z)\mathrm{d}z\right) \\
&= \frac{\eta_-}{\eta_- -\xi_{i,q}}C_i(A^*)e^{-\eta_-(x-A^*)-\xi_{i,q}A^*} + \frac{\eta_-}{\eta_- -\xi_{i,q}}c_i\left(-e^{-\eta_- x}\int_0^{\infty} e^{-\zeta_q y}h(y)\mathrm{d}y + \int_{x-A^*}^{x} e^{-\eta_- z}h(x-z)\mathrm{d}z\right).
\end{aligned}$$

Furthermore, noting that

$$\text{(A.6)}\quad \frac{\eta_-}{\zeta_q+\eta_-}(c_1+c_2) = \sum_{i=1,2}\frac{\eta_-}{\eta_- -\xi_{i,q}}c_i = \frac{2\eta_-}{\sigma^2(\xi_{1,q}+\zeta_q)(\xi_{2,q}+\zeta_q)},$$

we have (A.5) as desired.

Proof of (2). Fix $x \in (0, A^*)$. In view of (4.12), Corollary 4.1 and the argument similar to Lemma A.1 show

$$\mathcal{L}\phi(x) - q\phi(x) = -\gamma h(x) + \lambda \left[-\gamma \left(\frac{\eta_-}{\zeta_q + \eta_-} e^{\zeta_q x} C_0(x) - \sum_{i=1,2} \frac{\eta_-}{\eta_- - \xi_{i,q}} e^{-\xi_{i,q} x} C_i(x) \right) + \int_{-\infty}^{\infty} \phi(x+z) f(z) \mathrm{d}z \right].$$

Moreover, we have

$$\int_{-\infty}^{\infty} \phi(x+z) f(z) \mathrm{d}z = \int_0^x G(x-z) \eta_- e^{-\eta_- z} \mathrm{d}z + e^{-\eta_- x}. \tag{A.7}$$

Therefore, the proof is complete once we show that

$$\int_0^x G(x-z) \eta_- e^{-\eta_- z} \mathrm{d}z = \gamma \left(\frac{\eta_-}{\zeta_q + \eta_-} e^{\zeta_q x} C_0(x) - \sum_{i=1,2} \frac{\eta_-}{\eta_- - \xi_{i,q}} e^{-\xi_{i,q} x} C_i(x) \right).$$

By Corollary 4.1, we have

$$\frac{1}{\gamma} \int_0^x G(x-z) \eta_- e^{-\eta_- z} \mathrm{d}z = \int_0^x C_0(x-z) e^{\zeta_q (x-z)} \eta_- e^{-\eta_- z} \mathrm{d}z - \sum_{i=1,2} \int_0^x C_i(x-z) e^{-\xi_{i,q}(x-z)} \eta_- e^{-\eta_- z} \mathrm{d}z.$$

Similarly to the calculation above, we have

$$\begin{aligned} &\int_0^x C_0(x-z) e^{\zeta_q (x-z)} \eta_- e^{-\eta_- z} \mathrm{d}z \\ &\qquad = \frac{\eta_-}{\zeta_q + \eta_-} e^{\zeta_q x} C_0(x) + \frac{\eta_-}{\zeta_q + \eta_-} (c_1 + c_2) \left(-e^{-\eta_- x} \int_0^{\infty} e^{-\zeta_q y} h(y) dy + \int_0^x e^{-\eta_- z} h(x-z) \mathrm{d}z \right), \end{aligned}$$

and, for every $i \in \{1, 2\}$,

$$\begin{aligned} &\int_0^x C_i(x-z) e^{-\xi_{i,q}(x-z)} \eta_- e^{-\eta_- z} \mathrm{d}z \\ &\qquad = \frac{\eta_-}{\eta_- - \xi_{i,q}} e^{-\xi_{i,q} x} C_i(x) + \frac{\eta_-}{\eta_- - \xi_{i,q}} c_i \left[-e^{-\eta_- x} \int_0^{\infty} e^{-\zeta_q y} h(y) \mathrm{d}y + \int_0^x e^{-\eta_- z} h(x-z) \mathrm{d}z \right], \end{aligned}$$

and hence (A.7) is indeed true by (A.6). This completes the proof.

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