Kyoto University,
Graduate School of Economics
Research Project Center Discussion Paper Series

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Discussion Paper No. E-12-012

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March 2013

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March 18, 2013

Abstract

This paper provides a general framework with which a dynamic problem with potential regime shifts can be analyzed in a strategic environment as well as from social planner's perspective. A typical situation described by such a game is the joint exploitation of a common-property resource such as lakes, forests, marine fish populations, and at a larger scale the global climate system. By applying the framework to a simple dynamic common-property resource problem, we show that when the risk is endogenous, potential of regime shifts can facilitate precautionary management of common-property resources even in a strategic environment. It is also shown that there exists a resource-depletion trap in which a regime shift, once it happens, triggers a reversal of resource accumulation dynamics, possibly leading to a collapse of resource base.

Keywords: Regime shift, Markov-perfect Nash equilibrium, common-property resource, tragedy of the commons

JEL classification: C72, C73, Q20

1 Introduction

In this paper, we study a non-cooperative dynamic game in which a system characterizing the game shifts from one regime to another at unpredictable timings. A typical situation described by such a game is the joint exploitation of a common-property resource such as lakes, forests, marine fish populations, and at a larger scale the global climate system. These resources have complex dynamic systems which are known to undergo sudden drastic changes of underlying regimes (Scheffer et al., 2001; Scheffer and Carpenter, 2003; Folke et al., 2004). Shallow lakes, for instance, tend to at some point suddenly lose transparency and vegetation due to a heavy use of fertilizers on surrounding land and increased inflow of waste water from human settlements and

*I would like to thank Ken-Ichi Akao for helpful comments.

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industries (Scheffer et al., 1993). A potential collapse of the West Antarctic Ice sheet or a change in the global ocean circulation caused by anthropogenic climate change are expected to be abrupt (Oppenheimer, 1998; Broecker, 1997). Aside from common-property resource problems, dynamic systems with potential regime shifts can also be found in financial markets (Cass and Shell, 1983; Guo et al., 2005).

The existing literature which studies potential regime shifts in dynamic systems has focused on optimal management of the systems. Cropper (1976) considers a regime shift in the form of catastrophic plunge of utility level triggered by pollution. Reed (1988) determines the socially optimal harvesting policy for a fishery subject to random catastrophic collapse. In a more general setting, Clarke and Reed (1994) considers the impact of a pollution stock-dependent risk of catastrophic environmental collapse on the optimal management of resource, which was later extended by Tsur and Zemel (1996). In these studies, catastrophic damage or collapse of resource stock are commonly used as a formulation of regime shifts. In a more recent paper, Polasky et al. (2011) considers changed system dynamics as a general type of regime shifts and shows that the optimal management with potential regime shifts can be precautionary. In this strand of literature, however, strategic aspects inherent in the management of common-property resources are not fully taken into account.

There exists a vast literature on non-cooperative dynamic games. A seminal paper by Levhari and Mirman (1980), for instance, examines the dynamic and steady-state properties of the fish population that results from strategic interaction among players. In this literature, issues of interest such as the existence, multiplicity and inefficiency of equilibria are discussed in Benhabib and Radner (1992), Dockner and Sorger (1996), and Sorger (1998). Sorger (2005) considers a commons problem with amenity value and extraction cost and presents condition under which the equilibrium is tractable. Only a few papers, however, incorporate the risk or uncertainty surrounding the joint exploitation of productive resources. Recently, Antoniadou et al. (2013) introduce a simple random shock into the growth function of common-property resource and identifies a class of dynamic games which supports a linear symmetric Markov-perfect Nash equilibrium in their setting. Their analysis shows that the existence of uncertainty can amplify or mitigate the commons problem, depending on preference and technology in the economy. Yet consideration of uncertain regime shifts is largely absent in the analysis of dynamic games.

The present paper provides a general framework with which a dynamic problem with potential regime shifts can be analyzed in a strategic environment as well as from social planner's perspective. In section 2, we explain the structure of the model and introduce basic assumptions. Based on a fairly general framework, necessary and sufficient conditions for the solution of agents' problem are discussed. We accordingly define a symmetric Markov-perfect Nash equilibria for the model of a general form.

Section 3 demonstrates how the framework presented in this paper can be used to investigate problems of interest. To this end, we focus on a simple dynamic common-property resource problem and derive an equilibrium in a tractable form. In order to clarify the implications of potential regime shifts in dynamic games, we consider three different cases: the standard model with no regime shift, the one with exogenous risk of regime shift, and the one with endogenous risk of regime shift. It is shown that when the risk is endogenous, potential of regime shifts can facilitate precautionary management of common-property resources even in a strategic environment. It is also shown that there exists a resource-depletion trap in which a regime shift, once it happens, triggers a reversal of resource accumulation dynamics in a direction of deterioration. Section 4 concludes the paper.

2 General framework

2.1 Regime

Consider an economy with $N \in \mathbb{N}$ identical agents sharing a productive resource. Let θ be a vector of parameters which characterize the current system of economy. We call θ a regime. We assume that utility of each agent in general depends both on flow x and on stock z of resource. So the utility function of agent $n \in \{1, 2, \ldots, N\}$ is given by

$$U(x_n(t), z(t); \theta). \tag{1}$$

Technology available in the economy is represented by a growth function G, and the dynamics is expressed as

$$\dot{z}(t) = G(z(t), \textstyle\sum_{n=1}^{N} x_n(t); \theta), \tag{2}$$

which governs the relationship between stock and flow of resource under a given regime. This also provides a channel through which agents strategically interact.

The economy experiences regime shifts, which we model as discontinuous changes in θ at unpredictable timings. Let Θ be the set of all possible values of θ. A regime shift, say from $\theta_1 \in \Theta$ to $\theta_2 \in \Theta$, might cause a sudden change in agents' taste or a sharp decline in productivity of resource, or both. Such a shift is triggered by stochastic events. The risk of regime shifts is thus captured by a hazard rate

$$\lambda(t) = \lambda(z(t); \theta) \tag{3}$$

so that conditional density of timing T of a regime shift is given by

$$f(T; \theta | T \geq s) = \lambda(z(T); \theta) e^{-\int_s^T \lambda(z(\tau); \theta) d\tau}. \tag{4}$$

Notice we allow for the possibility that the hazard rate is influenced by z. One could interpret this as representing the fact that the current level of resource stock directly affects the frequency of shift-triggering stochastic events. Alternatively, (3) could mean that while stochastic events themselves are exogenous, the current regime may or may not survive these events, depending on the state of resource stock at the timing of stochastic shocks.

The simplest way of modeling a regime shift is to assume that the shift occurs once and only once, as in Polasky et al. (2011). In reality, however, regime shifts are better modeled as an open-ended process. Also, not only the timing, but also the realized state of regime is usually unknown *ex ante*. So we assume θ is stochastic and obeys a Markov process. To be more precise, θ is a piecewise deterministic Markov process, which has a stationary distribution p over Θ and the conditional distribution $p(\cdot|\theta)$ of the next regime θ' at the timing of regime shift only depends on the preceding realization θ of regime. A game in this model is then represented by a list $\langle U, G, \lambda, \Theta, p\rangle$.

2.2 Agents' problem

We focus on Markov-perfect Nash equilibria or MPNE. Denote by Z the set of possible values of z and by $X(z)$ the set of admissible values of x given z. Let Θ_p be the support of p and $\phi : Z \times \Theta_p \to X$ be a stationary Markovian strategy. Suppose at period s, the current regime is given by $\theta \in \Theta_p$. The problem of each agent is then formulated as

$$V(z;\theta) = \max_{x(t)\in X(z(t))} \mathbb{E}\Bigg[\int_s^T e^{-\rho(t-s)}U(x(t),z(t);\theta)dt + e^{-\rho(T-s)}\mathbb{E}\left[V(z(T);\theta')|\theta\right]\Bigg| T \geq s\Bigg], \tag{5}$$

$$\text{s.t.} \quad \dot{z}(t) = G(z(t),(N-1)\phi(z(t);\theta)+x(t);\theta), \tag{6}$$

$$z(s) = z > 0 \text{ given}, \tag{7}$$

where

$$\mathbb{E}\left[V(z;\theta')|\theta\right] = \int_\Theta V(z;\theta')dp(\theta'|\theta), \tag{8}$$

and the expectation operation very outside of the objective function (5) is taken in terms of timing T of regime shifts.

It is worth noting that the objective function (5) may be rewritten as

$$\int_s^\infty e^{-\rho(t-s)-\int_s^t \lambda(z(\tau);\theta)d\tau}\Big\{U(x(t),z(t);\theta) + \lambda(z(t);\theta)\mathbb{E}\left[V(z(t);\theta')\Big|\theta\right]\Big\}dt, \tag{9}$$

so that the problem can be seen as a deterministic problem. One could even make it look more familiar by introducing another state variable

$$y(t) := e^{-\int_s^t \lambda(z(\tau);\theta)d\tau} \quad \text{and thus} \quad \dot{y}(t) = -\lambda(z(t);\theta)y(t) \ \text{ with } \ y(s) = 1, \tag{10}$$

with which the problem is a standard autonomous problem with two state variables.

2.3 Equilibrium

We here define equilibrium. Let us first state the necessary condition for the solutions of the problem.

Proposition 1. *Consider a game $\langle U, G, \lambda, \Theta, p\rangle$. Suppose that the value function $V : Z \times \Theta_p \to \mathbb{R}$ in (5) is well defined and is continuously differentiable in z. Let $V_z := \partial V/\partial z$. Then V solves*

$$\begin{aligned}\rho V(z;\theta) =& U(\phi(z;\theta), z;\theta) + V_z(z;\theta)G(z, N\phi(z;\theta);\theta) \\ &+ \lambda(z;\theta)\left\{\mathbb{E}\left[V(z;\theta')\middle|\theta\right] - V(z;\theta)\right\} \quad \forall(z,\theta) \in Z \times \Theta_p,\end{aligned} \tag{11}$$

where $\phi : Z \times \Theta_p \to X$ satisfies

$$\phi(z;\theta) \in \operatorname*{argmax}_{x\in X(z)}\left\{U(x, z;\theta) - V_z(z;\theta)G(z,(N-1)\phi(z;\theta) + x;\theta)\right\} \tag{12}$$

for each $(z, \theta) \in Z \times \Theta_p$.

Proof. See Appendix A.1. □

The Hamilton-Jacobi-Bellman (HJB) equation (11) might seem a bit complicated. To interpret this, notice that (11) may be written as

$$\tilde{\rho}(z;\theta)V(z;\theta) = \max_{x\in X(z)}\left\{U(x, z;\theta) - V_z(z;\theta)G(z,(N-1)\phi(z;\theta) + x;\theta)\right\}, \tag{13}$$

where

$$\tilde{\rho}(z;\theta) := \rho + \frac{V(z;\theta) - \mathbb{E}[V(z;\theta')|\theta]}{V(z;\theta)}\lambda(z;\theta). \tag{14}$$

Comparing (13) with the standard HJB equation, we see that the risk of regime shifts in effect changes the discount rate from ρ to $\tilde{\rho}$. In fact, when there is no risk of regime shift, ρ and $\tilde{\rho}$ coincide because in that case $\mathbb{E}[V(z,\theta')|\theta] = V(z;\theta)$ in (14). In particular, if λ is independent of z and if $\mathbb{E}[V(z;\theta')|\theta] = 0$, which should be the case when the next regime is the 'end of the world,' then $\tilde{\rho} = \rho + \lambda$. This means that the effective discount rate is raised exactly to the extent of hazard rate, a well-known result since Yaari (1965). In such a case, the HJB equation can be solved just as in the case of no regime shift.

In general, however, how much agents effectively discount future value of resource stock depends on what kind of regimes are coming. As is clear from (14), the effective discount rate becomes higher than the original one if and only if $\mathbb{E}[V(z;\theta')|\theta] < V(z;\theta)$. When agents are expected to be better off by a regime shift, for instance, the risk of such a regime shift rather decreases the effective discount rate. And the better the coming regimes are, the lower the effective discount rate will be.

Since we assume that λ in general depends on z, however, solving the HJB equation involves extra work. In the standard model of no regime shift, a wide class of differential games have a linear equilibrium, in which V and ϕ are both linear in z. When, for example, U and G are both homogenous of degree one, it should be easy to see that (11) is satisfied by a linear equilibrium as long as $\tilde{\rho}$ is constant. In the case of endogenous risk of regime shifts, on the other hand, (14) suggests that $\tilde{\rho}$ may depend on z in a complicated way. Then even a nice combination of homogenous functions does not guarantee the existence of a linear equilibrium. Hence, if equilibria of tractable form are to be found, we need to assume that $\lambda(z;\theta)$ is also some well-behaved function as we will see in the next section.

For the sake of completeness, let us state the sufficient condition for the solution, which we will use to check a candidate equilibrium strategy in fact solves the original problem.

Proposition 2. *Consider a game characterized by $\langle U, G, \lambda, \Theta, p\rangle$. Suppose a continuously differentiable value function $V : Z \times \Theta_p \to \mathbb{R}$ and stationary Markovian strategy $\phi : Z \times \Theta_p \to X$ satisfy (11) and (12). If furthermore for each $\theta \in \Theta_p$*

$$\lim_{T\to\infty} e^{-\rho(T-s)} \left[V(\hat{z}(T);\theta)e^{-\int_s^T \lambda(\hat{z}(\tau);\theta)d\tau} - V(z(T);\theta)e^{-\int_s^T \lambda(z(\tau);\theta)d\tau} \right] \geq 0 \tag{15}$$

for any feasible path $\{\hat{z}(t)\}$ from $\hat{z}(s) = z$, then ϕ solves the problem (5).

Proof. See Appendix A.2. □

With the propositions above in mind, we define equilibrium as follows.

Definition 1. *A symmetric Markov-perfect Nash equilibrium of game $\langle U, G, \lambda, \Theta, p\rangle$ consists of continuously differentiable value function $V : Z \times \Theta_p \to \mathbb{R}$ and stationary Markovian strategy $\phi : Z \times \Theta_p \to X$ such that (11), (12), and (15) are satisfied.*

3 A common-property resource problem

In this section, we show the existence of a symmetric MPNE and characterize the equilibrium. To this end, we restrict ourselves to a specific class of games.

3.1 Specifications and benchmark results

The game we investigate here is a subclass of those analyzed by Sorger (2005), who considers a common-property resource with constant natural growth rate $R > 0$. The growth function is accordingly given by

$$G(z, \textstyle\sum_{n=1}^N x_n;\theta) := Rz - b\textstyle\sum_{n=1}^N x_n, \tag{16}$$

where we interpret $b > 0$ as the index of vulnerability of resource stock to human intervention[1]. Let $Z = \mathbb{R}_+$ and $X(z) = \mathbb{R}_+$ for $z > 0$ and $X(z) = \{0\}$ for $z = 0$. Agents are assumed to derive utility both from the flow of extracted resource and from the existing stock itself. To be more specific, instantaneous utility function is defined by

$$U(x, z;\theta) := x^\alpha (az)^{1-\alpha}, \quad \alpha \in (0, 1), \tag{17}$$

[1] In the absence of regime shifts, b can be normalized to unity by choosing the appropriate unit of x. But now that b changes upon a regime shift, b is not constant across different regimes and thus normalization is not appropriate.

where $a > 0$ captures the quality of amenity service associated with resource stock. For simplicity, we abstract cost of resource extraction. Notice that with this specification U and G are both homogenous of degree one. As for the hazard rate function, we specify

$$\lambda(z;\theta) := \frac{1}{cz^{\omega}} \quad \omega \geq 0, \tag{18}$$

where $c > 0$ represents the resilience of the system when $z = 1$[2].

In this class of games, a regime is represented by $\theta = (a, b, c, \rho, \alpha, \omega, R)$. Let Θ be defined by

$$\Theta := \{\theta \in \mathbb{R}^7_{++} \mid \alpha \in (0,1), \rho > R, \alpha N < 1, (1-\alpha)\omega R < \rho - R\}. \tag{19}$$

The restrictions $\rho > R$ and $\alpha N < 1$ are necessary so that a linear and finite extraction strategy constitutes an equilibrium in the absence of regime shift (Sorger, 2005). The last restriction $(1-\alpha)\omega R < \rho - R$ ensures the existence of an MPNE under the risk of regime shifts, which is shown in the appendix. To simplify the analysis, we assume the support $\Theta_p \subset \Theta$ of p is finite. This assumption will be made throughout.

Sorger (2005) studied this type of model in the absence of regime shifts, which corresponds to a special case of our model where Θ_p is a singleton. We take his result as a benchmark against which the role of regime shifts can be evaluated. For the benchmark case, the following proposition characterizes a symmetric MPNE.

Proposition 3. *Consider a game $\langle U, G, \lambda, \Theta, p \rangle$ specified above where $\Theta_p = \{\theta\}$ for some $\theta \in \Theta$. There exists a symmetric MPNE such that*

$$V(z;\theta) = \gamma^*(\theta)z \quad \textit{and} \quad \phi(z;\theta) = \beta^*(\theta)z, \tag{20}$$

where $\gamma^*(\theta)$ is a solution to

$$\rho - R = F(\gamma;\theta), \quad \textit{where} \quad F(\gamma;\theta) := (1-\alpha N)\alpha^{\frac{\alpha}{1-\alpha}} a b^{-\frac{\alpha}{1-\alpha}} \gamma^{-\frac{1}{1-\alpha}}, \tag{21}$$

and $\beta^*(\theta)$ is given by

$$\beta^*(\theta) := \frac{\alpha F(\gamma^*(\theta);\theta)}{(1-\alpha N)b} = \frac{\alpha(\rho - R)}{(1-\alpha N)b}. \tag{22}$$

There are no other linear symmetric MPNE.

Proof. See Appendix A.3 □

Note that $\gamma^*(\theta)$ is increasing with respect to R and a, and decreasing with respect to ρ and b. This makes sense since people are better off when the resource is more productive or when the amenity service from the resource stock is of high quality. On the other hand, the equilibrium level of welfare declines if the system become more vulnerable to human intervention or people become impatient. The equilibrium

[2] We could instead work on non-parametric model as in Sorger (2005) without specifying functional forms for U and λ. In that case we need to impose a bit more complicated restrictions on U and λ to ensure the existence of equilibria.

extraction rate $\beta^*(\theta)$ moves exactly in the opposite direction except for a, of which $\beta^*(\theta)$ is independent.

The symmetric MPNE described in the proposition above is inefficient. To see this, consider the case all agents employ the same linear strategy $x(t) = \beta z(t)$. Then $z(t) = z(0)e^{-(\rho - R + Nb\beta)}$ and the discounted value of total utility is given by

$$\int_s^\infty e^{-\rho(t-s)} z(t) U(\beta, 1; \theta) dt = \frac{z(0) a^{1-\alpha} \beta^\alpha}{\rho - R + Nb\beta}, \tag{23}$$

which is strictly concave and attains its maximum at

$$\beta^c(\theta) = \frac{\alpha(\rho - R)}{(1-\alpha)Nb}. \tag{24}$$

It should be easy to see $\beta^c(\theta) < \beta^*(\theta)$ as long as $N > 1$. This means that every agent can be better off by simultaneously reducing their equilibrium exploitation rate. In other words, agents overexploit the resource and end up with a lower level of welfare at MPNE. Hence, this game exemplifies the tragedy of the commons in a classical sense. A question of particular interest then is whether the risk of regime shifts can encourage agents to behave in a more precautionary manner and refrain from overexploitation. And if such a precautionary resource-use is possible, in what condition should it be the case?

3.2 Equilibrium under risk of regime shifts

We now turn to the case where Θ_p consists of multiple regimes. Then the system switches from one regime to another during the course of dynamic strategic interaction among agents. But some distinct regimes can be regarded as essentially the same since different combinations of parameter values can support the same equilibrium. These cases are exceptional and of no interest. In order to make occurrence of regime shifts substantial for agents, we define *strictly distinct regimes* as follows.

Definition 2. *Let $V^*(z;\theta)$ be the value function in the absence of regime shifts under a particular regime $\theta \in \Theta$. We say that two regimes $\theta_1 \in \Theta$ and $\theta_2 \in \Theta$ are strictly distinct from each other if $V^*(z;\theta_1) \neq V^*(z;\theta_2)$ for $z > 0$.*

To introduce the risk of regime shifts, consider first the case of $\omega = 0$ with probability one so that $\lambda(z;\theta) = 1/c > 0$. In this case the hazard rate is independent of existing level of resource stock.

Proposition 4. *Consider a game $\langle U, G, \lambda, \Theta, p \rangle$ specified above where Θ_p contains multiple and strictly distinct regimes and $\omega = 0$ for all $\theta \in \Theta_p$. There exists a symmetric MPNE such that*

$$V(z;\theta) = \gamma^x(\theta) z \quad \text{and} \quad \phi(z;\theta) = \beta^x(\theta) z, \tag{25}$$

where

$$\beta^x(\theta) > \beta^*(\theta) \quad \text{and} \quad \gamma^x(\theta) < \gamma^*(\theta) \tag{26}$$

for at least one $\theta \in \Theta_p$.

Proof. See Apendix A.4. □

This proposition shows there exists a linear equilibrium under the risk of regime shifts just as in the absence of regime shifts as long as the risk if exogenous. It furthermore states that the potential regime shifts always worsen the tragedy of the commons at least under one particular regime. It will be instructive if we express the equilibrium in a more explicit form. As shown in the appendix, $\gamma^x(\theta)$ is actually a solution to

$$\tilde{\rho}(\gamma;\theta) - R = F(\gamma;\theta), \tag{27}$$

in which $\tilde{\rho}$ is defined by

$$\tilde{\rho}(\gamma;\theta) := \rho + \left(1 - \frac{\mathbb{E}[\gamma^x(\theta')|\theta]}{\gamma}\right)\lambda(\theta) \tag{28}$$

and the extraction rate $\beta^x(\theta)$ is given by

$$\beta^x(\theta) := \frac{\alpha F(\gamma^x(\theta);\theta)}{(1-\alpha N)b} \tag{29}$$

for each $\theta \in \Theta_p$. Note that function F is decreasing in γ. So it follows from (29) and (22) that $\beta^x(\theta) > \beta^*(\theta)$ if and only if $\gamma^x(\theta) < \gamma^*(\theta)$. Comparing (27) with (21), we obtain

$$\gamma^x(\theta) < \gamma^*(\theta) \iff \tilde{\rho}(\gamma^*(\theta);\theta) > \rho \iff \mathbb{E}[\gamma^x(\theta')|\theta] < \gamma^*(\theta), \tag{30}$$

for each $\theta \in \Theta_p$. Hence, the exogenous risk of regime shifts accelerates the extraction if agents are expected to be worse off under the coming regimes than in the case of continuation of the current regime. And if there are strictly distinct regimes, there at least one regime which is unambiguously better than the other regimes. Under such a regime, any shift is undesirable, which provides an incentive for agents to accelerate their extraction.

To clarify this point, suppose there are only two regimes possible under p so that $\Theta_p = \{\theta_1, \theta_2\}$. Let us say that we are currently in regime θ_1 and the regime is expected to shift into θ_2 at an unpredictable timing. In many cases of concern such as regime shifts in an ecological system due to human intervention, regimes are expected to shift in a bad direction. Perhaps the growth rate declines ($R_2 < R_1$), the quality of resource amenity value decreases ($a_2 < a_1$), or the resource stock becomes more vulnerable to human intervention ($b_2 > b_1$). Once such a shift happens, people are likely to be worse off, which is naturally translated into $\mathbb{E}[\gamma^x(\theta')|\theta_1] < \gamma^*(\theta_1)$. As a result, agents accelerate strategic exploitation, anticipating an undesirable regime shift in the future. This argument is formalized by the next corollary.

Corollary 1. *Consider a game $\langle U, G, \lambda, \Theta, p\rangle$ specified above where $\Theta_p = \{\theta_1, \theta_2\}$ and $\omega_1 = \omega_2 = 0$. If $R_1 > R_2$, $a_1 > a_2$, $b_1 < b_2$, $c_1 < c_2$, or/and $\rho_1 < \rho_2$, then*

$$\beta^x(\theta_1) > \beta^*(\theta_1) \quad \textit{and} \quad \beta^x(\theta_2) < \beta^*(\theta_2). \tag{31}$$

Proof. See Apendix A.5. □

If the next regime is expected to be more preferable than the current one, which is the case when the current regime is θ_2, overexploitation is mitigated. When an undesirable regime shift is expected, however, our result provides a pessimistic prediction. It basically predicts that agents overexploit resource more aggressively exactly when such a reckless behavior should be avoided.

This alarming result might be altered when the risk of regime shift is rather endogenous and if people correctly recognize that their behavior affects the risk. To investigate this case, suppose $\omega > 0$ for sure so that the risk of regime shifts depends on z. In this case, it is easy to see there is no linear equilibrium. But a tractable solution exists, which we see in the next proposition.

Proposition 5. *Consider a game $\langle U, G, \lambda, \Theta, p \rangle$ specified above where Θ_p consists of multiple regimes and $\omega > 0$ for all $\theta \in \Theta_p$. There exists a symmetric MPNE such that*

$$V(z;\theta) = \gamma^l(z;\theta)z \quad \text{and} \quad \phi(z;\theta) = \beta^l(z;\theta)z \tag{32}$$

for each $\theta \in \Theta_p$, where $\beta^l(z;\theta)$ is strictly decreasing in z. Moreover, there exists z^* such that as long as $z \geq z^*$

$$\beta^l(z;\theta) < \beta^*(\theta) \quad \forall\theta \in \Theta_p. \tag{33}$$

Proof. See Appendix A.6 □

The equilibrium extraction rate $\phi(z;\theta)/z$ is not constant, but decreasing in z. This suggests that if the remaining resource stock is sufficiently large, endogenous risk of regime shifts may facilitate resource preservation. The proposition states that this is in fact the case. In other words, common-property resources with potential regime shifts could be better managed than in the absence of regime shift as long as the current resource base is in a good shape. It should be worth emphasizing here that this result is independent of how good or bad regimes are coming next. Hence, contrary to the case of exogenous regime shifts, the risk of regime shift does not necessarily imply acceleration of strategic resource exploitation, but rather encourages precautionary resource-use even if the regime is expected to shift in a bad direction.

This result relates to the recent work of Polasky et al. (2011). Based on a linear utility model combined with a fairly general growth function, they considered social planner's problem with potential regime shifts. They showed that when the risk of regime shift is increased by dwindling resource stock, the optimal management of productive resource must be precautionary in the sense that the optimal steady state level of resource stock is always higher. Our analysis suggests that a similar argument holds even in a decentralized and strategic environment. Also worth worth noting is that the results above hold in the case of social planer's problem as well since our model formulation includes the case of $N = 1$.

3.3 Equilibrium dynamics

Aside from the precautionary resource-use, the non-linearity of equilibrium extraction rate produces an interesting consequence in equilibrium dynamics. To highlight the point, consider first the case when the risk of regime shift is absent or exogenous. Then the equilibrium extraction rate is constant under each regime and agents' strategy can be expressed as $x = \beta z$. Then the dynamics of resource accumulation is given by

$$\dot{z}(t) = Rz(t) - bN\beta z(t) = (R - bN\beta)z(t), \tag{34}$$

implying that the equilibrium growth rate is constant, at least within the same regime. So $\dot{z}(t) > 0$ if and only if $R - bN\beta > 0$. Since β is a function of θ, the equilibrium dynamics is completely determined by regime θ itself, independent of the level of z.

When the risk of regime shift is endogenous, on the other hand, equilibrium extraction rate $\beta^l(z;\theta)$ is a decreasing function of the remaining resource stock z. This implies there exists a non-trivial steady state and the equilibrium dynamics depend on the level of resource stock, which we formalize by the following proposition.

Proposition 6. *Consider a game $\langle U, G, \lambda, \Theta, p\rangle$ specified above where Θ_p consists of multiple regimes and $\omega > 0$ for all $\theta \in \Theta_p$. At the MPNE described in proposition 5, there exists a unique non-trivial steady state $z_{ss}(\theta)$ for each $\theta \in \Theta_p$ such that*

1. *$\dot{z}(t) < 0$ if $z(t) < z_{ss}(\theta)$ and*
2. *$\dot{z}(t) > 0$ if $z(t) > z_{ss}(\theta)$ as long as the current regime is θ.*

Moreover, there exist $\underline{z}_{ss}$ and $\overline{z}_{ss}$ such that

3. *$\lim_{t\to\infty} z(t) = 0$ if $z(t) < \underline{z}_{ss}$ for some $t \in [0, \infty)$ and*
4. *$\limsup_{t\to\infty} z(t) = \infty$ if $z(t) > \overline{z}_{ss}$ for some $t \in [0, \infty)$.*

Proof. See Appendix A.7 □

The equilibrium dynamics depend on the level of resource stock. When the resource stock is larger than the steady-state level, the stock continues to grow as long as the current regime persists. Since the hazard rate is decreasing in z, the risk of regime shift will decline over time, providing a further basis for sustainable growth of resource stock. If the stock level is smaller than the steady-state level, however, the logic is completely turned around. The resource stock will gradually diminish, followed by ever-more frequent occurrence of regime shifts.

What is critical then is the level of resource stock at the timing of regime shift. Suppose $z(t) > z_{ss}(\theta)$ under regime $\theta \in \Theta_p$. This implies that along the equilibrium path, the stock of resource continuously grows as long as regime θ persists. Let us say at some period $t' > t$, a regime shift happens and another regime $\theta' \in \Theta_p$ emerges. Then this shift changes the dynamics and the level of steady state stock is altered accordingly. If the level of remaining resource stock is 'overtaken' by the steady state under a new

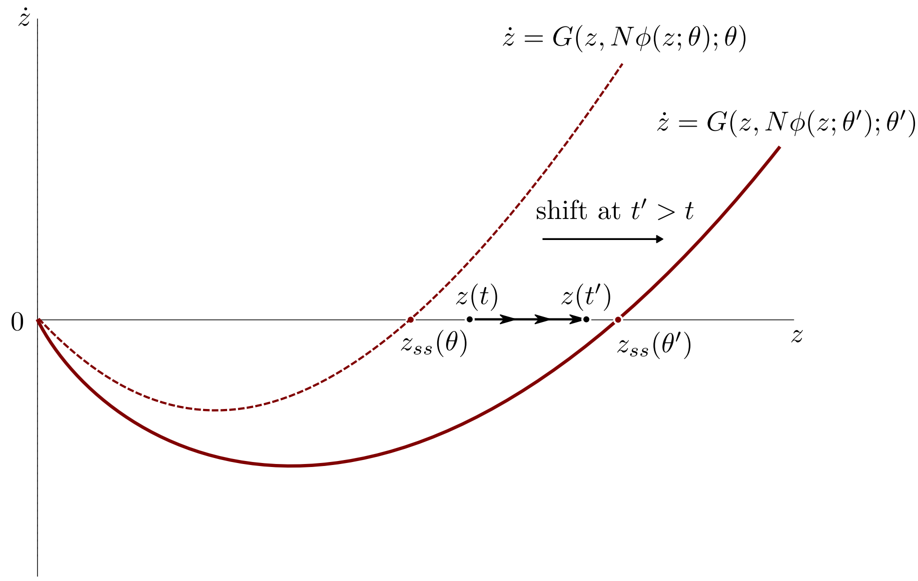

Figure 1: Equilibrium dynamics in the case of endogenous risk

regime (i.e., if $z(t') < z_{ss}(\theta')$), the resource stock declines thereafter, at least until the next regime shift occurs. The situation is depicted in Figure 1. Of course the logic can be the other way around. It might be possible that even if the equilibrium dynamics is characterized as a continuous decline of resource stock under a particular regime, subsequent regime shifts, once they happen, can reverse the dynamics into sustainable growth of resource.

But the proposition states that there are 'points of no return' in stock level, above or below which the equilibrium dynamics is never reversed. If at some period t the remaining resource stock $z(t)$ becomes smaller than the lower boundary $\underline{z}_{ss}$, the stock level declines thereafter no matter which regime emerges in the subsequent periods. So the process is irreversible, destined for collapse of resource base. Therefore, this implies on one hand that there exists a resource-depletion trap and once the system is caught in the trap, it will be impossible to escape from the trap. On the other hand, however, if the resource stock reaches the upper boundary $\overline{z}_{ss}$, then one can safely say that the equilibrium resource-use remains sustainable thereafter, regardless of future occurrence of regime shifts. Hence, the equilibrium dynamics can be characterized in terms of initial stock $z(0)$. When the initial resource stock $z(0)$ is smaller than $\underline{z}_{ss}$, then the resource management is not sustainable and the resource base diminishes over time whereas if $z(0)$ is larger than $\overline{z}_{ss}$, then the resource base never collapses. If $z(0)$ is in between $\underline{z}_{ss}$ and $\overline{z}_{ss}$, then the resource management may or may not be sustainable, depending on the realization of regimes in the course of strategic interaction among agents.

4 Conclusions

This paper presented a general framework with which a dynamic problem with potential regime shifts can be analyzed in a decentralized and strategic environment as well as from social planner's perspective. Based on a fairly general framework, we discussed necessary and sufficient conditions for the solution of agents' problem and defined a symmetric Markov-perfect Nash equilibria for the model of a general form.

We also demonstrated how the framework presented in this paper can be used to investigate problems of interest. To this end, we focused on a simple dynamic common-property resource problem and derive an equilibrium in a tractable form. Based on comparison of three different specifications of regime shifts, it was shown that when the risk is endogenous, potential of regime shifts can facilitate precautionary management of common-property resources even in a strategic environment. It was also shown that there exists a resource-depletion trap in which a regime shift, once it happens, triggers a reversal of resource accumulation dynamics in a direction of deterioration.

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A Proofs

A.1 Proof of Proposition 1

Let $J(s, z, \{x_n(t)\}; \theta)$ be the value of objective function along a feasible path $\{x_n(t)\}$ from $z(s) = z$, namely,

$$J(s, z, \{x_n(t)\}; \theta) := \int_s^{\infty} e^{-\rho(t-s)} \Big\{ U(x_n(t), z(t); \theta) + \lambda(z(t); \theta) \mathbb{E}[V(z(t); \theta') | \theta] \Big\} e^{-\int_s^t \lambda(z(\tau); \theta) d\tau} dt, \tag{35}$$

where $\{z(t)\}$ satisfies

$$\dot{z}(t) = G(z(t), (N-1)\phi(z(t); \theta) + x_n(t); \theta). \tag{36}$$

Then for any $\Delta s \geq 0$,

$$\begin{aligned} J(s, z, \{x_n(t)\}; \theta) = & \int_s^{s+\Delta s} e^{-\rho(t-s)} \Big\{ U(x_n(t), z(t); \theta) + \lambda(z(t); \theta) \mathbb{E}[V(z(t); \theta') | \theta] \Big\} e^{-\int_s^t \lambda(z(\tau); \theta) d\tau} dt \\ & + \int_{s+\Delta s}^{\infty} e^{-\rho(t-s)} \Big\{ U(x_n(t), z(t); \theta) + \lambda(z(t); \theta) \mathbb{E}[V(z(t); \theta') | \theta] \Big\} e^{-\int_s^t \lambda(z(\tau); \theta) d\tau} dt \qquad (37) \\ = & \int_s^{s+\Delta s} e^{-\rho(t-s)} \Big\{ U(x_n(t), z(t); \theta) + \lambda(z(t); \theta) \mathbb{E}[V(z(t); \theta') | \theta] \Big\} e^{-\int_s^t \lambda(z(\tau); \theta) d\tau} dt \\ & + e^{-\rho \Delta s} e^{-\int_s^{s+\Delta s} \lambda(z_\tau; \theta) d\tau} J(s + \Delta s, z + \Delta z, \{x_n(t)\}; \theta), \qquad (38) \end{aligned}$$

where we define Δz by

$$\Delta z := \int_s^{s+\Delta s} \dot{z}(t) dt = \int_s^{s+\Delta s} G(z(t), (N-1)\phi(z(t); \theta) + x(t); \theta) dt. \tag{39}$$

Note that by definition

$$V(z; \theta) = \max_{\{x_n(t)\}_{t \geq s}} J(s, z, \{x_n(t)\}; \theta), \tag{40}$$

and

$$V(z + \Delta z; \theta) = \max_{\{x_n(t)\}_{t \geq s + \Delta s}} J(s + \Delta s, z + \Delta z, \{x_n(t)\}; \theta). \tag{41}$$

So we have

$$V(z; \theta) = \max_{\{x_n(t)\}_{s+\Delta s \geq t \geq s}} \left\{ \int_s^{s+\Delta s} e^{-\rho(t-s)} \{ U(x_n(t), z(t); \theta) + \lambda(z(t); \theta) \mathbb{E}\left[V(z(t); \theta') | \theta \right] \} e^{-\int_s^t \lambda(z(\tau); \theta) d\tau} dt + e^{-\rho \Delta s} e^{-\int_s^{s+\Delta s} \lambda(z_\tau; \theta) d\tau} V(z + \Delta z; \theta) \right\}, \tag{42}$$

or equivalently

$$0 = \max_{\{x_n(t)\}_{s+\Delta s \geq t \geq s}} \left\{ \frac{1}{\Delta s} \int_s^{s+\Delta s} e^{-\rho(t-s)} \Big\{ U(x_n(t), z(t); \theta) + \lambda(z(t); \theta) \mathbb{E}[V(z(t); \theta') | \theta] \Big\} e^{-\int_s^t \lambda(z(\tau);\theta) d\tau} dt + \frac{e^{-\rho \Delta s} e^{-\int_s^{s+\Delta s} \lambda(z_\tau;\theta) d\tau} V(z + \Delta z; \theta) - V(z;\theta)}{\Delta s} \right\}. \tag{43}$$

Since $V(z;\theta)$ is differentiable by assumption, by passing to the limit for $\Delta s \to 0$, we obtain the Hamilton-Jacobi-Bellman equation

$$(\rho + \lambda(z;\theta))V(z;\theta) = \max_{x \in X(z)} \Big\{ U(x, z; \theta) + \lambda(z;\theta) \mathbb{E}[V(z;\theta')|\theta] + V_z(z;\theta) G(z, (N-1)\phi(z;\theta) + x; \theta) \Big\}. \tag{44}$$

A.2 Proof of Proposition 2

Since V and ϕ satisfy the HJB equation,

$$\begin{aligned} \rho V(z;\theta) &= U(\phi(z;\theta), z; \theta) + V_z(z;\theta) G(z, N\phi(z;\theta); \theta) \\ &\quad + \lambda(z;\theta) \left\{ \mathbb{E}[V(z;\theta')|\theta] - V(z;\theta) \right\} &(45) \\ &\geq U(x_n, z; \theta) + V_z(z;\theta) G(z, (N-1)\phi(z;\theta) + x_n; \theta) \\ &\quad + \lambda(z;\theta) \left\{ \mathbb{E}[V(z;\theta')|\theta] - V(z;\theta) \right\} &(46) \end{aligned}$$

for any $x \in X(z)$. Let $\{\hat{x}_n(t)\}$ be an arbitrary feasible path and $\{\hat{z}_n(t)\}$ be the associated path of state variable so that $\hat{x}(t) \in X(\hat{z}(t))$ for all $t \in [s, \infty)$ and

$$\dot{\hat{z}}(t) = G(\hat{z}(t), (N-1)\phi(\hat{z}(t);\theta) + \hat{x}_n(t); \theta) \tag{47}$$

with $\hat{z}(s) = z$. Then

$$\begin{aligned} \rho V(\hat{z}(t);\theta) \geq & U(\hat{x}_n(t), \hat{z}(t); \theta) + V_z(\hat{z}(t);\theta) G(\hat{z}(t), (N-1)\phi(\hat{z}(t);\theta) + \hat{x}_n(t); \theta) \\ & + \lambda(\hat{z}(t);\theta) \left\{ \mathbb{E}[V(\hat{z}(t);\theta')|\theta] - V(\hat{z}(t);\theta) \right\} \end{aligned} \tag{48}$$

while

$$\begin{aligned} \rho V(z(t);\theta) = & U(\phi(z(t)), z(t); \theta) + V_z(z(t);\theta) G(z(t), N\phi(z(t);\theta); \theta) \\ & + \lambda(z(t);\theta) \left\{ \mathbb{E}[V(z(t);\theta')|\theta] - V(z(t);\theta) \right\} \end{aligned} \tag{49}$$

where $\{z(t)\}$ satisfies

$$\dot{z}(t) = G(z(t), N\phi(z(t);\theta); \theta) \tag{50}$$

with $z(s) = z$.

Let $J_T(s,z,\{\hat{x}_n(t)\};\theta)$ and $J_T(s,z,\phi_\theta;\theta)$ be the values of objective function along $\{x_n(t)\}$ and $\{\phi(z(t);\theta)\}$, respectively, truncated up until $T > s$. Namely,

$$J_T(s,z,\{\hat{x}_n(t)\};\theta) := \int_s^T e^{-\rho(t-s)}\Big\{U(\hat{x}_n(t),\hat{z}(t);\theta) + \lambda(\hat{z}(t);\theta)\mathbb{E}[V(\hat{z}(t);\theta')|\theta]\Big\}e^{-\int_s^t \lambda(\hat{z}(\tau);\theta)d\tau}dt, \tag{51}$$

and

$$J_T(s,z,\phi_\theta;\theta) := \int_s^T e^{-\rho(t-s)}\Big\{U(\phi(z(t);\theta),z(t);\theta) + \lambda(z(t);\theta)\mathbb{E}[V(z(t);\theta')|\theta]\Big\}e^{-\int_s^t \lambda(z(\tau);\theta)d\tau}dt. \tag{52}$$

Then it follows from (48) and (47) that

$$J_T(s,z,\{\hat{x}_n(t)\};\theta) \leq \int_s^T e^{-\rho(t-s)}\Big\{\rho V(\hat{z}(t);\theta) - V_z(\hat{z}(t);\theta)\dot{\hat{z}}(t) + \lambda(\hat{z}(t);\theta)V(\hat{z}(t);\theta)\Big\}e^{-\int_s^t \lambda(\hat{z}(\tau);\theta)d\tau}dt \tag{53}$$

$$= -\int_s^T \frac{d}{dt}\left\{e^{-\rho(t-s)}V(\hat{z}(t);\theta)e^{-\int_s^t \lambda(\hat{z}(\tau);\theta)d\tau}\right\}dt \tag{54}$$

$$= V(z;\theta) - e^{-\rho(T-s)}V(\hat{z}(T);\theta)e^{-\int_s^T \lambda(\hat{z}(\tau);\theta)d\tau} \tag{55}$$

while (49) and (50) imply

$$J_T(s,z,\phi_\theta\};\theta) = \int_s^T e^{-\rho(t-s)}\Big\{\rho V(z(t);\theta) - V_z(z(t);\theta)\dot{z}(t) + \lambda(z(t);\theta)V(z(t);\theta)\Big\}e^{-\int_s^t \lambda(z(\tau);\theta)d\tau}dt \tag{56}$$

$$= -\int_s^T \frac{d}{dt}\left\{e^{-\rho(t-s)}V(z(t);\theta)e^{-\int_s^t \lambda(z(\tau);\theta)d\tau}\right\}dt \tag{57}$$

$$= V(z;\theta) - e^{-\rho(T-s)}V(z(T);\theta)e^{-\int_s^T \lambda(z(\tau);\theta)d\tau}. \tag{58}$$

Thus

$$J_T(s,z,\phi_\theta\};\theta) - J_T(s,z,\{\hat{x}_n(t)\};\theta) = e^{-\rho(T-s)}\left[V(\hat{z}(T);\theta)e^{-\int_s^T \lambda(\hat{z}(\tau);\theta)d\tau} - V(z(T);\theta)e^{-\int_s^T \lambda(z(\tau);\theta)d\tau}\right]. \tag{59}$$

Hence, if (15) is satisfied

$$V(z,\theta) - J(s,z,\{\hat{x}_n(t)\};\theta) = \lim_{T\to\infty} J_T(s,z,\phi_\theta\};\theta) - \lim_{T\to\infty} J_T(s,z,\{\hat{x}_n(t)\};\theta) \geq 0 \tag{60}$$

for any feasible path $\{\hat{x}_n(t)\}$.

A.3 Proof of Proposition 3

Fix $\theta \in \Theta$ and define $u : \mathbb{R}_+ \to \mathbb{R}$ by

$$u(y;\theta) := U(y,1;\theta) = y^\alpha a^{1-\alpha} \quad \forall y \in \mathbb{R}_+, \tag{61}$$

so that $U(x,z;\theta) = U(x/z,1;\theta)z = u(x/z;\theta)z$. Then it follows from the FOC that

$$u'(\phi(z;\theta)/z;\theta) = bV_z(z;\theta) \quad \text{or} \quad \phi(z;\theta) = v(bV_z(z;\theta);\theta)z, \tag{62}$$

where v is the inverse function of u'. This implies that $V(z;\theta)$ is linear in z if and only if $\phi(z,\theta)$ is linear in z. Suppose

$$V(z;\theta) = \gamma^*(\theta)z \tag{63}$$

for some constant $\gamma^*(\theta)$. Then $V_z(z;\theta) = \gamma^*(\theta)$ and the policy function is

$$\phi(z;\theta) = v(b\gamma^*(\theta);\theta)z = \beta^*(\theta)z \quad \text{where} \quad \beta^*(\theta) := \alpha^{\frac{1}{1-\alpha}} ab^{-\frac{1}{1-\alpha}} \gamma^*(\theta)^{-\frac{1}{1-\alpha}}. \tag{64}$$

It thus follows from the HJB equation that

$$\rho\gamma^*(\theta) = u(\beta^*(\theta);\theta) + \gamma^*(\theta)(R - Nb\beta^*(\theta)) \tag{65}$$

$$= R\gamma^*(\theta) + (1-\alpha N)\alpha^{\frac{\alpha}{1-\alpha}} ab^{-\frac{\alpha}{1-\alpha}} \gamma^*(\theta)^{-\frac{\alpha}{1-\alpha}}. \tag{66}$$

In other words, $\gamma^*(\theta)$ is a solution to the equation

$$\rho - R = F(\gamma;\theta) \quad \text{where} \quad F(\gamma;\theta) := (1-\alpha N)\alpha^{\frac{\alpha}{1-\alpha}} ab^{-\frac{\alpha}{1-\alpha}} \gamma^{-\frac{1}{1-\alpha}} \tag{67}$$

and

$$\beta^*(\theta) = \frac{\alpha F(\gamma^*(\theta);\theta)}{(1-\alpha N)b} = \frac{\alpha(\rho - R)}{(1-\alpha N)b}. \tag{68}$$

Note that $\rho - R$ is positive, and $\limsup_{\gamma\to 0} F(\gamma;\theta) = \infty$ and $\lim_{\gamma\to+\infty} F(\gamma;\theta) = 0$. Hence, (67) has a unique solution.

Since $\dot{z}(t)/z(t) \leq R$ for any feasible path,

$$e^{-\rho T} V(z(T);\theta) = e^{-\rho T}\gamma^*(\theta)z(T) \leq \gamma^*(\theta)z(0)e^{-(\rho-R)T} \tag{69}$$

for any $T \geq 0$. Noticing that $\rho > R$ by assumption, we see that the RHS of the inequality converges to 0 as $T \to \infty$. Therefore, the transversality condition is also satisfied.

A.4 Proof of Proposition 4

Suppose for each $\theta \in \Theta_p$ there exists some constant $\gamma^x(\theta)$ such that

$$V(z;\theta) = \gamma^x(\theta)z \quad z \in \mathbb{R}_+. \tag{70}$$

Then $V_z(z;\theta) = \gamma^x(\theta)$ and the policy function is

$$\phi(z;\theta) = v(b\gamma^x(\theta);\theta)z = \beta^x(\theta)z, \tag{71}$$

where

$$\beta^x(\theta) := \alpha^{\frac{1}{1-\alpha}} ab^{-\frac{1}{1-\alpha}} \gamma^x(\theta)^{-\frac{1}{1-\alpha}} = \frac{\alpha F(\gamma^x(\theta);\theta)}{(1-\alpha N)b}. \tag{72}$$

It thus follows from the HJB equation that

$$\rho\gamma^x(\theta) + \left(\gamma^x(\theta) - \mathbb{E}[\gamma^x(\theta')|\theta]\right)\lambda(c) = u(\beta^x(\theta);\theta) + \gamma^x(\theta)(R - Nb\beta^x(\theta)) \quad (73)$$

$$= F(\gamma^x(\theta);\theta)\gamma^x(\theta) + \gamma^x(\theta)R. \quad (74)$$

Note that $\dot{z}/z \leq R$ for any feasible path,

$$e^{-\rho T}V(z(T);\theta)e^{-\int_s^T \lambda(\theta)d\tau} \leq \gamma^x(\theta)z(0)e^{-(\rho-R+\lambda(\theta))T} \quad (75)$$

for any $T \geq 0$. Since $\rho > R$ by assumption and $\lambda(\theta) \geq 0$, the RHS of the inequality converges to 0 as $T \to \infty$. Therefore, the transversality condition is satisfied. Hence, if there exists a solution $\{\gamma^x(\theta)\}_{\theta\in\Theta_p}$ to the system of equations

$$\rho - R + \left(1 - \frac{\mathbb{E}[\gamma^x(\theta')|\theta]}{\gamma^x(\theta)}\right)\lambda(\theta) = F(\gamma^x(\theta);\theta) \quad \theta \in \Theta_p, \quad (76)$$

then (70) and (71) constitute an MPNE.

We shall show that there exists a solution to (76). By assumption, support Θ_p of p is finite. So we may write $\Theta_p = \{\theta_1, \theta_2, \ldots, \theta_M\}$ for some $M \in \mathbb{N}$. Let $\gamma_m^x := \gamma^x(\theta_m)$ for each $m \in \{1, 2, \ldots, M\}$ and $\pi_{m'|m} := \text{Prob}\{\theta' = \theta_{m'}|\theta = \theta_m\}$ so that

$$\mathbb{E}[\gamma^x(\theta')|\theta_m] = \sum_{m'=1}^{M} \gamma_{m'}^x \pi_{m'|m}. \quad (77)$$

Then (76) may be written as

$$(1 - \pi_{m|m})\gamma_m^x - \sum_{m'\neq m} \gamma_{m'}^x \pi_{m'|m} = H^x(\gamma_m^x;\theta_m), \quad (78)$$

where

$$H^x(\gamma;\theta) := \frac{1}{\lambda(\theta)}\left\{F(\gamma;\theta)\gamma - (\rho - R)\gamma\right\}. \quad (79)$$

Observe that

$$\frac{\partial H^x(\gamma;\theta)}{\partial\gamma} < 0, \quad \limsup_{\gamma\to 0} H^x(\gamma;\theta) = \infty, \quad \liminf_{\gamma\to\infty} H^x(\gamma;\theta) = -\infty, \quad (80)$$

which implies that for each $\gamma_{-m}^x := (\gamma_1^x, \ldots, \gamma_{m-1}^x, \gamma_{m+1}^x, \ldots, \gamma_M^x) \in \mathbb{R}_{++}^{M-1}$, there exists a unique $\gamma_m^x \in \mathbb{R}_{++}$ satisfying (78). Since F is continuously differentiable with respect to γ, so is H^x. Then by the implicit functions theorem, there exists a continuously differentiable function $\Gamma_m^x : \mathbb{R}_{++}^{M-1} \to \mathbb{R}_{++}$ such that

$$(1 - \pi_{m|m})\Gamma_m^x(\gamma_{-m}^x) - \sum_{m'\neq m} \gamma_{m'}^x \pi_{m'|m} = H^x(\Gamma_m^x(\gamma_{-m}^x);\theta_m), \quad (81)$$

for any $\gamma_{-m}^x \in \mathbb{R}_{++}^{M-1}$. Define $\mathbf{\Gamma}^x : \mathbb{R}_{++}^M \to \mathbb{R}_{++}^M$ by

$$\mathbf{\Gamma}^x(\gamma_1^x, \ldots, \gamma_M^x) := (\Gamma_1^x(\gamma_{-1}^x), \ldots, \Gamma_M^x(\gamma_{-M}^x)) \quad \text{for each } (\gamma_1^x, \ldots, \gamma_M^x) \in \mathbb{R}_{++}^M, \quad (82)$$

of which we shall find a fixed point.

Notice that

$$\frac{\partial \Gamma^x_m(\gamma^x_{-m})}{\partial \gamma^x_{m'}} > 0 \quad \forall m' \neq m. \tag{83}$$

So defining $\gamma^0_m := \Gamma^x_m(\mathbf{0}) > 0$ where $\mathbf{0} := (0, \ldots, 0) \in \mathbb{R}^{M-1}$, we obtain

$$\Gamma^x_m(\gamma^x_{-m}) \in [\gamma^0_m, \infty) \qquad \forall \gamma^x_{-m} \in \mathbb{R}^{M-1}_{++}. \tag{84}$$

Define $h^x_m : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$h^x_m(\gamma) := \Gamma^x_m(\mathbf{1} \cdot \gamma) \tag{85}$$

where $\mathbf{1} := (1, \ldots, 1) \in \mathbb{R}^{M-1}_{++}$. It follows from Proposition 3 and (79) that $H^x(\gamma^*_m; \theta_m) = 0$ where $\gamma^*_m := \gamma^*(\theta_m)$ and

$$(1 - \pi_{m|m})\gamma^*_m - \sum_{m' \neq m} \gamma^*_m \pi_{m'|m} = H^x(\gamma^*_m; \theta_m) = 0, \tag{86}$$

meaning that $h^x_m(\gamma^*_m) = \gamma^*_m$. Since $h^x_m(0) = \gamma^0_m > 0$ and γ^*_m is a unique solution to $H^x(\gamma; \theta_m) = 0$, it then must be the case that

$$h^x_m(\gamma) < \gamma \quad \forall \gamma > \gamma^*_m. \tag{87}$$

This, together with (83) and (84), implies

$$\Gamma^x_m(\gamma^x_{-m}) \in [\gamma^0_m, \gamma] \qquad \forall \gamma^x_{-m} \in (0, \gamma] \times (0, \gamma] \times \cdots \times (0, \gamma] \tag{88}$$

for any $\gamma \geq \gamma^*_m$, meaning

$$\mathbf{\Gamma}^x(\gamma^x_1, \ldots, \gamma^x_M) \in \bigtimes_{m=1}^{M} [\gamma^0_m, \overline{\gamma}^*] \quad \forall (\gamma^x_1, \ldots, \gamma^x_M) \in \bigtimes_{m=1}^{M} [\gamma^0_m, \overline{\gamma}^*]. \tag{89}$$

where $\overline{\gamma}^* := \max_m \{\gamma^*_m\}$. By Browder's fixed point theorem, there then exists $(\gamma^x_1, \ldots, \gamma^x_M) \in \bigtimes_{m=1}^{M} [\gamma^0_m, \overline{\gamma}^*]$ such that

$$\mathbf{\Gamma}^x(\gamma^x_1, \ldots, \gamma^x_M) = (\gamma^x_1, \ldots, \gamma^x_M). \tag{90}$$

This fixed point constitutes a solution to (78) and thus to (76).

We now show that if Θ_p consists of strictly distinct regimes, then $\gamma^x_m < \gamma^*_m$ for some $m \in \{1, 2, \ldots, M\}$. Suppose on the contrary that $\gamma^x_m \geq \gamma^*_m$ for all m. By construction

$$\gamma^x_m \leq \overline{\gamma}^* = \max_{m'} \{\gamma^*(\theta_{m'})\} \tag{91}$$

for each $m \in \{1, 2, \ldots, M\}$ and $\gamma^x_{\bar{m}} = \gamma^*_{\bar{m}}$ for $\bar{m} := \underset{m'}{\operatorname{argmax}} \{\gamma^*_{m'}\}$. If $\gamma^x_m = \gamma^x_{\bar{m}}$ for all m,

$$\begin{aligned}
0 &= (1 - \pi_{m|m})\gamma^*_{\bar{m}} - \sum_{m' \neq m} \gamma^*_{\bar{m}} \pi_{m'|m} && (92)\\
&= (1 - \pi_{m|m})\gamma^x_m - \sum_{m' \neq m} \gamma^x_{m'} \pi_{m'|m} && (93)\\
&= H^x(\gamma^x_m; \theta_m) && (94)\\
&= H^x(\gamma^*_{\bar{m}}; \theta_m) && (95)\\
&< H^x(\gamma^*_m; \theta_m) = 0 \quad \forall m \neq \bar{m}, && (96)
\end{aligned}$$

where the strict inequality follows the fact that regimes are strictly distinct, which in turn implies $\gamma^*_m < \gamma^*_{\bar{m}}$. Since this is impossible, there must exist some m such that $\gamma^x_m < \gamma^x_{\bar{m}} = \gamma^*_{\bar{m}}$. But this then implies

$$
\begin{aligned}
0 &= H^x(\gamma^*_{\bar{m}};\theta_{\bar{m}}) && (97)\\
&= H^x(\gamma^x_{\bar{m}};\theta_{\bar{m}}) && (98)\\
&= (1-\pi_{\bar{m}|\bar{m}})\gamma^x_{\bar{m}} - \sum_{m'\neq\bar{m}} \gamma^x_{m'}\pi_{m'|\bar{m}} && (99)\\
&> (1-\pi_{\bar{m}|\bar{m}})\gamma^*_{\bar{m}} - \sum_{m'\neq\bar{m}} \gamma^*_{\bar{m}}\pi_{m'|\bar{m}} && (100)\\
&= 0, && (101)
\end{aligned}
$$

a contradiction. Therefore $\gamma^x_m < \gamma^*_m$ for some $m \in \{1, 2, \ldots, M\}$. For such m, (29) and (22) imply $\beta^x(\theta_m) > \beta^*(\theta_m)$.

A.5 Proof of corollary 1

Let $\gamma^*_1 := \gamma^*(\theta_1)$ and $\gamma^*_2 := \gamma^*(\theta_2)$. Notice first that

$$
\begin{aligned}
H^x(\gamma;\theta) &= \frac{1}{\lambda(\theta)}\left\{F(\gamma;\theta)\gamma - (\rho - R)\gamma\right\} && (102)\\
&= c\left\{(1-\alpha N)\alpha^{\frac{\alpha}{1-\alpha}} a b^{-\frac{\alpha}{1-\alpha}}\gamma^{-\frac{\alpha}{1-\alpha}} - (\rho - R)\gamma\right\}, && (103)
\end{aligned}
$$

which, given $\gamma > 0$, is increasing in R, a, and c, and decreasing in b and ρ. Since $H^x(\gamma^*_1;\theta_1) = H^x(\gamma^*_2;\theta_2) = 0$ and $H^x(\gamma;\theta)$ is strictly decreasing with respect to γ, this means $\gamma^*_2 < \gamma^*_1$.

Let $\gamma^x_1 := \gamma^x(\theta_1)$, $\gamma^x_2 := \gamma^x(\theta_2)$. Then

$$(1-\pi_{1|1})\gamma^x_1 - \pi_{2|1}\gamma^x_2 = H^x(\gamma^x_1;\theta_1) \tag{104}$$

$$(1-\pi_{2|2})\gamma^x_2 - \pi_{1|2}\gamma^x_1 = H^x(\gamma^x_2;\theta_2), \tag{105}$$

where $\pi_{2|1} = 1 - \pi_{1|1}$ and $\pi_{1|2} = 1 - \pi_{2|2}$, and thus

$$\gamma^x_1 = \gamma^x_2 + \frac{1}{1-\pi_{1|1}} H^x(\gamma^x_1;\theta_1) \tag{106}$$

$$\gamma^x_2 = \gamma^x_1 + \frac{1}{1-\pi_{2|2}} H^x(\gamma^x_2;\theta_2). \tag{107}$$

If either $\gamma^x_1 = \gamma^*_1$ or $\gamma^x_2 = \gamma^*_2$, then $\gamma^*_1 = \gamma^x_1 = \gamma^x_2 = \gamma^*_2$, a contradiction. Therefore $\gamma^*_2 < \gamma^x_2 < \gamma^x_1 < \gamma^*_1$, which together with (29) and (22) implies $\beta^x(\theta_1) > \beta^*(\theta_1)$ and $\beta^x(\theta_2) < \beta^*(\theta_2)$.

A.6 Proof of Proposition 5

Fix $z > 0$ and suppose

$$V(z;\theta) = \gamma^l(z;\theta)z \quad \text{where} \quad \gamma^l(z;\theta) := \chi(\theta)\left[\lambda(z;\theta)\right]^{-(1-\alpha)} \tag{108}$$

for some constant $\chi(\theta)$ for each $\theta \in \Theta_p$. Then

$$V_z(z;\theta) = \left[1 - (1-\alpha)\frac{\lambda'(z;\theta)z}{\lambda(z;\theta)}\right]\frac{V(z;\theta)}{z} = (1 + (1-\alpha)\omega)\gamma^l(z;\theta) \tag{109}$$

and the FOC implies

$$\begin{aligned}
\phi(z;\theta)/z &= v(b(1+(1-\alpha)\omega)\gamma^l(z;\theta);\theta) && (110)\\
&= \alpha^{\frac{1}{1-\alpha}}(1+(1-\alpha)\omega)^{-\frac{1}{1-\alpha}}ab^{-\frac{1}{1-\alpha}}\chi(\theta)^{-\frac{1}{1-\alpha}}\lambda(z;\theta) && (111)\\
&= \frac{\alpha F(\gamma^l(z;\theta);\theta)}{(1-\alpha N)b(1+(1-\alpha)\omega)^{1/(1-\alpha)}} && (112)\\
&=: \beta^l(z;\theta). && (113)
\end{aligned}$$

Notice that $\beta^l(z;\theta)$ is strictly decreasing in z. The HJB equation is

$$\begin{aligned}
\rho =& \frac{u(\beta^l(z;\theta);\theta)}{\gamma^l(z;\theta)} + \left(\frac{\mathbb{E}\left[V(z;\theta')|\theta\right]}{\gamma^l(z;\theta)z} - 1\right)\lambda(z;\theta) + \frac{V_z(z;\theta)z}{V(z;\theta)}\left[R - bN\beta^l(z;\theta)\right] && (114)\\
=& R + (1-\alpha)\omega R + \left(\frac{\mathbb{E}[\gamma^l(z;\theta)|\theta]}{\gamma^l(z;\theta)} - 1\right)\lambda(z;\theta)\\
&+ \left(\frac{\alpha}{1+(1-\alpha)\omega}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha N)ab^{-\frac{\alpha}{1-\alpha}}(\gamma^l(z;\theta))^{-\frac{1}{1-\alpha}}, && (115)
\end{aligned}$$

or

$$\rho - [1 + (1-\alpha)\omega]R + \left(1 - \frac{\mathbb{E}\left[\gamma^l(z;\theta')|\theta\right]}{\gamma^l(z;\theta)}\right)\lambda(z;\theta) = [1+(1-\alpha)\omega]^{-\frac{\alpha}{1-\alpha}}F(\gamma^l(z;\theta);\theta) \tag{116}$$

for all z.

Note that $\dot{z}/z \leq R$ for any feasible path,

$$e^{-\rho T}V(z(T);\theta)e^{-\int_s^T \lambda(z;\theta)d\tau} \leq \chi^l(\theta)c^{-1}z(0)e^{-(\rho - R - (1-\alpha)\varphi R)T - \int_s^T \lambda(z;\theta)d\tau} \tag{117}$$

for any $T \geq 0$. Since $\rho - R - (1-\alpha)\varphi R > 0$ by assumption and $\lambda(z;\theta) \geq 0$ for any z, the RHS of the inequality converges to 0 as $T \to \infty$, meaning that the transversality condition is satisfied. Hence, if there exists a solution to (116), then (108) and (110) constitutes an MPNE.

We shall show that there exists a solution to (116). By assumption, support Θ_p of p is finite. So we may write $\Theta_p = \{\theta_1, \theta_2, \ldots, \theta_M\}$ for some $M \in \mathbb{N}$. Let $\gamma^l_m(z) := \gamma^l(z;\theta_m)$ for each $m \in \{1, 2, \ldots, M\}$ and $\pi_{m'|m} := \text{Prob}\{\theta' = \theta_{m'}|\theta = \theta_m\}$ so that

$$\mathbb{E}[\gamma^l(z;\theta')|\theta_m] = \sum_{m'=1}^{M}\gamma^l_{m'}(z)\pi_{m'|m}. \tag{118}$$

Then (116) may be written as

$$(1-\pi_{m|m})\gamma_m^l(z) - \sum_{m'\neq m}\gamma_{m'}^l(z)\pi_{m'|m} = H^l(\gamma_m^l(z);\theta_m), \tag{119}$$

where

$$H^l(\gamma;z,\theta) := \frac{1}{\lambda(z;\theta)}\left\{[1+(1-\alpha)\omega]^{-\frac{\alpha}{1-\alpha}}F(\gamma;\theta)\gamma - (\rho-[1+(1-\alpha)\omega]R)\gamma\right\}. \tag{120}$$

Observe that since $\rho-[1+(1-\alpha)\omega]R>0$ by construction of Θ,

$$\frac{\partial H^l(\gamma;z,\theta)}{\partial\gamma}<0, \quad \limsup_{\gamma\to 0} H^l(\gamma;z,\theta)=\infty, \quad \liminf_{\gamma\to\infty} H^l(\gamma;z,\theta)=-\infty, \tag{121}$$

which implies that for each $\gamma_{-m}^l := (\gamma_1^l,\ldots,\gamma_{m-1}^l,\gamma_{m+1}^l,\ldots,\gamma_M^l)\in\mathbb{R}_{++}^{M-1}$, there exists a unique $\gamma_m^l\in\mathbb{R}_{++}$ satisfying (119). Since F is continuously differentiable with respect to γ, so is H^l. Then by the implicit functions theorem, there exists a continuously differentiable function $\Gamma_m^l(\cdot;z):\mathbb{R}_{++}^{M-1}\to\mathbb{R}_{++}$ such that

$$(1-\pi_{m|m})\Gamma_m^l(\gamma_{-m}^l;z) - \sum_{m'\neq m}\gamma_{m'}^l\pi_{m'|m} = H^l(\Gamma_m^l(\gamma_{-m}^l;z);z,\theta_m), \tag{122}$$

for any $\gamma_{-m}^l\in\mathbb{R}_{++}^{M-1}$. Define $\mathbf{\Gamma}^l(\cdot;z):\mathbb{R}_{++}^M\to\mathbb{R}_{++}^M$ by

$$\mathbf{\Gamma}^l(\gamma_1^l,\ldots,\gamma_M^l;z) := (\Gamma_1^l(\gamma_{-1}^l;z),\ldots,\Gamma_M^l(\gamma_{-M}^l;z)) \quad \text{for each } (\gamma_1^l,\ldots,\gamma_M^l)\in\mathbb{R}_{++}^M, \tag{123}$$

of which again we shall find a fixed point.

Notice that

$$\frac{\partial\Gamma_m^l(\gamma_{-m}^l;z)}{\partial\gamma_{m'}^l}>0 \quad \forall m'\neq m. \tag{124}$$

So defining $\gamma_m^0(z):=\Gamma_m^l(\mathbf{0};z)>0$ where $\mathbf{0}:=(0,\ldots,0)\in\mathbb{R}^{M-1}$, we obtain

$$\Gamma_m^l(\gamma_{-m}^l;z)\in[\gamma_m^0,\infty) \qquad \forall\gamma_{-m}^l\in\mathbb{R}_{++}^{M-1}. \tag{125}$$

Define $h_m^l(\cdot;z):\mathbb{R}_+\to\mathbb{R}_+$ by

$$h_m^l(\gamma;z) := \Gamma_m^l(\mathbf{1}\cdot\gamma;z) \tag{126}$$

where $\mathbf{1}:=(1,\ldots,1)\in\mathbb{R}_{++}^{M-1}$. Let $\gamma_m^\star(z)>0$ be the unique solution to $H^l(\gamma;z,\theta_m)=0$, which exists since $\rho-[1+(1-\alpha)\omega]R>0$ by construction of Θ. Then

$$(1-\pi_{m|m})\gamma_m^\star(z) - \sum_{m'\neq m}\gamma_m^\star(z)\pi_{m'|m} = H^l(\gamma_m^\star(z);z,\theta_m)=0, \tag{127}$$

meaning that $h_m^l(\gamma_m^\star(z))=\gamma_m^\star(z)$. Since $h_m^l(0;z)=\gamma_m^0(z)>0$ and $\gamma_m^\star(z)$ is a unique solution to $H^l(\gamma;z,\theta_m)=0$, it then must be the case that

$$h_m^l(\gamma;z)<\gamma \quad \forall\gamma>\gamma_m^\star(z). \tag{128}$$

This, together with (124) and (125), implies

$$\Gamma^l_m(\gamma^l_{-m}; z) \in [\gamma^0_m(z), \gamma] \qquad \forall \gamma^l_{-m} \in (0,\gamma] \times (0,\gamma] \times \cdots \times (0,\gamma] \tag{129}$$

for any $\gamma \geq \gamma^\star_m(z)$ and (129) implies

$$\mathbf{\Gamma}^l(\gamma^l_1, \ldots, \gamma^l_M; z) \in \bigtimes_{m=1}^{M} [\gamma^0_m(z), \overline{\gamma}^\star(z)] \quad \forall (\gamma^l_1, \ldots, \gamma^l_M) \in \bigtimes_{m=1}^{M} [\gamma^0_m(z), \overline{\gamma}^\star(z)]. \tag{130}$$

where $\overline{\gamma}^\star(z) := \max_m\{\gamma^\star_m(z)\}$. By Browder's fixed point theorem, there then exists $(\gamma^l_1(z), \ldots, \gamma^l_M(z)) \in \bigtimes_{m=1}^{M}[\gamma^0_m(z), \overline{\gamma}^\star(z)]$ such that

$$\mathbf{\Gamma}^l(\gamma^l_1(z), \ldots, \gamma^l_M(z); z) = (\gamma^l_1(z), \ldots, \gamma^l_M(z)), \tag{131}$$

which constitutes a solution to (119) and thus to (116) for given z.

To find the sufficient condition for precautionary resource-use, notice that for each $m \in \{1, \ldots, M\}$

$$\gamma^l_m(z) \geq \gamma^0_m(z) \tag{132}$$

and $\gamma^0_m(z)$ is the unique solution to

$$\rho_m - [1 + (1-\alpha_m)\omega_m]R_m + (1 - \pi_{m|m})\lambda(z; \theta_m) = [1 + (1-\alpha_m)\omega_m]^{-\frac{\alpha_m}{1-\alpha_m}} F(\gamma; \theta_m). \tag{133}$$

Since F is decreasing in γ, this means

$$\beta^l(z; \theta_m) = \frac{\alpha_m F(\gamma^l_m(z); \theta_m)}{(1-\alpha_m N) b_m (1 + (1-\alpha_m)\omega_m)^{1/(1-\alpha_m)}} \tag{134}$$

$$\leq \frac{\alpha_m F(\gamma^0_m(z); \theta_m)}{(1-\alpha_m N) b_m (1 + (1-\alpha_m)\omega_m)^{1/(1-\alpha_m)}} \tag{135}$$

$$= \frac{\alpha_m}{(1-\alpha_m N) b_m} \frac{\rho_m - R_m - (1-\alpha_m)\omega_m R_m + (1-\pi_{m|m})\lambda(z;\theta_m)}{1 + (1-\alpha_m)\omega_m} \tag{136}$$

for each z. Comparing (136) with (68) yields

$$\beta^l(z; \theta_m) < \beta^*(\theta_m) \tag{137}$$

if

$$(1 - \pi_{m|m})\lambda(z; \theta_m) = \frac{(1 - \pi_{m|m})}{c_m z^{\omega_m}} < (1-\alpha_m)\omega_m \rho_m. \tag{138}$$

Define z^* by

$$z^* := \max_m \left\{ \left[\frac{c_m (1-\alpha_m)\omega_m \rho_m}{1 - \pi_{m|m}} \right]^{-1/\omega_m} \right\}, \tag{139}$$

then as long as $z \geq z^*$, (138) is satisfied for any $m \in \{1, 2, \ldots, M\}$ and thus $\beta^l(z; \theta) < \beta^*(\theta)$ for any $\theta \in \Theta_p$.

A.7 Proof of Proposition 6

Proof of Proposition 5 shows that for each $\theta \in \Theta_p$ there exists a constant $\chi(\theta) > 0$ such that

$$\beta^l(z;\theta) = \alpha^{\frac{1}{1-\alpha}}(1+(1-\alpha)\omega)^{-\frac{1}{1-\alpha}} ab^{-\frac{1}{1-\alpha}} \chi(\theta)^{-\frac{1}{1-\alpha}} c^{-1} z^{-\omega}. \tag{140}$$

Notice that $\limsup_{z\to 0} \beta^l(z;\theta) = \infty$ and $\lim_{z\to\infty} \beta^l(z;\theta) = 0$. Since $\beta^l(z;\theta)$ is continuous, there then exists a unique $z_{ss}(\theta)$ such that

$$R - bN\beta^l(z_{ss}(\theta);\theta) = 0. \tag{141}$$

Since $\beta^l(z;\theta)$ is strictly decreasing in z, this implies

$$G(z, N\phi(z;\theta);\theta) = R - bN\beta^l(z;\theta) < 0 \quad \text{if} \quad z < z_{ss}(\theta) \tag{142}$$

while

$$G(z, N\phi(z;\theta);\theta) = R - bN\beta^l(z;\theta) > 0 \quad \text{if} \quad z > z_{ss}(\theta), \tag{143}$$

which proves the first part of the theorem. Sine Θ_p is finite, we can define

$$\bar{z}_{ss} := \max_{\theta\in\Theta_p}\{z_{ss}(\theta)\} \quad \text{and} \quad \underline{z}_{ss} := \min_{\theta\in\Theta_p}\{z_{ss}(\theta)\}. \tag{144}$$

Once $z(t) > \bar{z}_{ss}$ for some $t \in [0,\infty)$, then $z(t) > z_{ss}(\theta)$ for all $\theta \in \Theta_p$, and thus $\dot{z}(t) > 0$ no matter which regime characterizes the system at period t. Therefore $\limsup_{t\to\infty} z(t) = \infty$. The proof for the case of $z(t) < \underline{z}_{ss}$ is similar.