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Positive and Negative Population Growth and Long-Run Trade Patterns: A Non-Scale Growth Model*

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Abstract

This paper builds a two-country, two-sector, non-scale growth model and investigates the relationship between trade patterns and the growth rate of per capita real consumption. We consider negative population growth as well as positive population growth. We show that, as long as the population growth rates of the two countries are different, if the country that accumulates capital stock has negative population growth, no trade patterns are sustainable in the long run. This is true irrespective of the population growth rate of the other country. Moreover, we show that, if the country that accumulates capital stock has positive population growth, two trade patterns are sustainable in the long run. In this case, either each country's per capita growth is determined by the population growth of the capital-accumulating country or the population growth of both countries, depending on which of the two trade patterns is realized.

Keywords: positive/negative population growth; trade patterns; non-scale growth model

JEL Classification: F10; F43; O11; O41

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1 Introduction

This study investigates the relationships between population growth, trade patterns, and per capita consumption growth. We consider negative population growth as well as positive population growth. For this purpose, we present a two-country, two-sector, two-factor, non-scale growth model.

In scale-growth models, the long-run growth rate of per capita income is proportional to the scale of the population, which seems counterfactual. On the other hand, in non-scale growth models, the long-run growth rate of per capita income is proportional to the growth rate of the population. A pioneering work on non-scale growth models was that of Jones (1995). Jones (1995) attempted to remove the scale effects, and presented a non-scale growth model in which the growth rate of output per capita depends positively on the population growth rate, and not on the size of the population. That is, the higher the population growth rate, the faster the country grows.[1)]

We use the non-scale growth model for two reasons. First, we can obtain sustainable income per capita growth, even when population growth is strictly positive. Second, we do not need to impose knife-edge conditions on the parameters of the model.

Other studies analyze the relationship between trade patterns and growth.[2)] Two good examples are the studies by Kaneko (2003) and Felbermayr (2007).

Kaneko (2003) builds a two-country, two-sector, AK growth model and endogenizes the terms of trade.[3)] He finds that, if an autarkic country with a lower growth rate than its trade partner has a comparative advantage in the consumption goods sector, then the country can narrow or even reverse the growth gap by opening trade.

Felbermayr (2007) describes a situation in which a capital-abundant North and a capital-scarce South trade with each other. In his model, the trade pattern is endogenously determined, and he analyzes the situation in which the North produces investment goods and the South produces consumption goods. The production technology of investment goods is AK and that of consumption goods is decreasing returns to scale. Along the balanced growth path, the Southern terms of trade are continuously improving such that, even with the de-

1) For a systematic exposition of scale effects and non-scale growth, see Jones (1999), Jones (2005), Aghion and Howitt (2005), and Dinopoulos and Sener (2007). For more sophisticated non-scale growth models, see also Kortum (1997), Dinopoulos and Thompson (1998), Peretto (1998), Segerstrom (1998), Young (1998), Howitt (1999), and Dinopoulos and Syropoulos (2007).

2) Wong and Yip (1999) present a small, open-economy, two-sector model of endogenous growth, including capital accumulation and learning by doing. They analyze the relationship between economic growth, industrialization, and international trade.

3) Kaneko (2000) builds a growth model with human capital accumulation and shows that the relationship between the terms of trade and growth depends on whether the country specializes in the consumption goods sector or the investment goods sector. However, Kaneko's (2000) model is a small, open-economy model, which means the terms of trade are exogenous.

creasing returns to scale, the South can grow at the same rate as the North. Therefore, the South can eliminate the growth gap by opening trade.

However, these are both scale growth models, and hence are subject to the aforementioned problems specific to these models.

In contrast, examples of studies that investigate the relationship between trade patterns and growth rates using non-scale growth models include Sasaki (2011a) and Sasaki (2012). Sasaki (2011a) derives a model to investigate which trade pattern is realized in the long run when both countries' population growth rates are equal. Then, Sasaki (2012) extends this model to include the case in which the countries' population growth rates are different.[4] However, both studies consider only positive population growth. Accordingly, we consider cases in which population growth is both negative and positive.

Many developed countries have stagnant population growth, and in some cases, a negative growth rate.[5] Existing economic growth theories assume positive population growth. However, given that population growth can be negative, we need to consider this case as well.

At first, it may seem easy to include negative population growth in economic growth theory, but this is not the case.[6] As Ferrara (2011) and Christiaans (2011) show, incorporating negative population growth in growth models is more complicated than simply replacing a positive value with a negative value.

Christiaans (2011) shows the importance of negative population growth using a simple model. Consider a Solow growth model with a production function that exhibits increasing, but relatively small returns to scale.[7] When the population growth rate is positive, per capita income growth is positive and increasing in the population growth rate. On the other hand, when the population growth rate is negative, contrary to expectations, per capita income growth remains positive, but is decreasing in the population growth rate.

Using a two-country model of international trade, we investigate the relationships between the size of exogenously given population growth rates, the sustainability of trade patterns, and the long-run growth rate of per capita consumption in each country, given that

4) In this respect, Sasaki (2011b) builds a non-scale growth, North-South economic development model, and shows that both countries grow at the same rate along the balanced growth path. However, their per capita incomes grow at different rates because the differences in population growth. Nevertheless, since the production pattern is fixed and given exogenously in this model, we cannot know whether the given trade pattern is sustainable over time.

5) For example, according to Japan's Ministry of Internal Affairs and Communications, as of March 2012, Japan has experienced its largest-ever decline in population.

6) Ritschel (1985) argues that, in the standard Solow growth model, a negative savings rate is necessary for the existence of a steady-state equilibrium with a negative population growth rate. See also Felderer (1988).

7) In the Solow model with a constant returns to scale production function, per capita income growth is zero when the population growth is positive, but positive when the population growth is negative. For details, see Christiaans (2011).

a specific trade pattern is sustainable.

The main results are as follows. We show that, as long as the population growth rates of the two countries are different, if the country that accumulates capital stock has negative population growth, no trade patterns are sustainable in the long run, irrespective of the population growth rate of the other country. Moreover, we show that, if the country that accumulates capital stock has positive population growth, two trade patterns are sustainable in the long run. Here, either each country's growth rate is determined by the population growth of the capital-accumulating country or the population growth of both countries, depending on which of the two trade patterns is realized.

The rest of the paper is organized as follows. Section 2 presents our model and determines the equilibrium and long-run growth rate of per capita consumption under autarky. Section 3 investigates the equilibrium under free trade. Section 4 investigates the long-run growth rate of per capita consumption under free trade. We compare the growth rates under free trade and autarky, and then compare our results to those of related studies. Finally, Section 5 concludes the paper.

2 The model

Consider a world that contains two countries: Home and Foreign. Both countries produce homogeneous manufactured and agricultural goods. The manufactured good is used for both consumption and investment, whereas the agricultural good is used only for consumption.

2.1 Production

Firms produce manufactured goods, X_i^M, with labor input, L_i^M, and capital stock, K_i, and produce agricultural goods, X_i^A, with only labor input, L_i^A. Here, $i = 1$ and $i = 2$ denote Home and Foreign, respectively. Both countries have the same production functions, which are specified as follows:

$$X_i^M = A_i K_i^{\alpha} (L_i^M)^{1-\alpha}, \quad \text{where } A_i = K_i^{\beta} \tag{1}$$

$$= K_i^{\alpha+\beta} (L_i^M)^{1-\alpha}, \quad 0 < \alpha < 1,\ 0 < \beta < 1,\ \alpha + \beta < 1, \tag{2}$$

$$X_i^A = L_i^A. \tag{3}$$

Here, A_i in equation (1) represents an externality associated with capital accumulation, which captures the learning-by-doing effect introduced by Arrow (1962). Substituting A_i into equation (1), we obtain equation (2), which shows that manufactured goods production has increasing returns to scale, with β corresponding to the extent of the increasing returns. Equation (3) shows that agricultural goods production has constant returns to scale.

We further suppose that the labor supply is equal to the population and that the population is fully employed. Moreover, the population grows at a constant rate, n_i, and the initial population is unity in each country: $L_i(t) = L_i^M(t) + L_i^A(t) = e^{n_i t}$, $n_i \gtrless 0$. Note that population growth can be negative.

Let p_i denote the price of manufactured goods relative to agricultural goods. Then, the profits of the manufacturing and agricultural firms are given by $\pi_i^M = p_i X_i^M - w_i L_i^M - p_i r_i K_i$ and $\pi_i^A = X_i^A - w_i L_i^A$, respectively, where w_i denotes the wage paid to produce agricultural goods and r_i denotes the rental rate of capital.

From the profit-maximizing conditions, we obtain the following relations:

$$p_i \frac{\partial X_i^M}{\partial L_i^M} = w_i = 1, \tag{4}$$

$$\frac{\partial X_i^M}{\partial K_i} = r_i \text{ with } A_i \text{ given.} \tag{5}$$

From equation (4), we find that the wage is unity as long as agricultural production is positive. We assume a Marshallian externality in deriving equation (5); profit-maximizing firms regard A_i as exogenously given. Accordingly, firms do not internalize the effect of A_i.

2.2 Consumption

For simplification, we make the classical assumption that wage income and capital income are entirely devoted to consumption and saving, respectively.[8] In the canonical one-sector Solow model, under the golden rule steady state in which per capita consumption is maximized, consumption is equal to the total real wage and total capital income is saved and invested. Hence, our assumption has some rationality and can be interpreted as a simple rule of thumb for consumers with dynamic optimization (Christiaans, 2008). We define real consumption per capita, c_i, as $c_i = C_i/L_i = (C_i^M)^\gamma (C_i^A)^{1-\gamma}/L_i$, where C_i denotes economy-wide real consumption. In this case, a fraction, γ, of wage income is spent on C_i^M and the rest,

8) The same assumption is used in Uzawa (1961), which considers a two-sector growth model, and Krugman (1981), which considers a two-country, two-sector, North-South trade and development model. If dynamic optimization is used, then the Euler equation for consumption appears and the number of differential equations increases, significantly complicating the analysis. Therefore, an analysis with dynamic optimization will be left for future research. In addition, it is true that consumption smoothing with dynamic optimization is a standard tool in macroeconomics, although Mankiw (2000) states that, in reality, consumption behavior deviates from consumption smoothing.

$1-\gamma$, is spent on C_i^A.

$$p_i C_i^M = \gamma w_i L_i, \tag{6}$$

$$C_i^A = (1-\gamma) w_i L_i. \tag{7}$$

Moreover, the following relationship between real investment, I_i, and saving holds: $p_i I_i = p_i r_i K_i$. From this equation, we obtain the rate of capital accumulation:

$$g_{K_i} \equiv \frac{\dot{K}_i}{K_i} = r_i. \tag{8}$$

That is, the rate of capital accumulation is equal to the rental rate of capital. A dot over a variable denotes the time derivative of the variable (e.g., $\dot{K}_i \equiv dK_i/dt$).

2.3 Equilibrium under autarky and per capita consumption growth

Under autarky, both goods have to be produced. The market-clearing conditions are as follows: $X_i^M = C_i^M + I_i$ and $X_i^A = C_i^A$. Note that $w_i = 1$ under autarky. From the market-clearing condition for manufactured goods, we obtain p_i, which is used to derive each sector's employment share: $L_i^M/L_i = \gamma$ and $L_i^A/L_i = 1-\gamma$. Therefore, under autarky, each sector's employment share is constant.

Under autarky, the relative price of manufactured goods is given by

$$p_i = \frac{(\gamma L_i)^\alpha}{(1-\alpha) K_i^{\alpha+\beta}}. \tag{9}$$

First, we derive the balanced-growth path (BGP) under autarky when the population growth rate is positive, that is, $n_i > 0$. Along the BGP, the rate of capital accumulation is constant and equal to the rental rate of capital, which is given in equation (5) as $r_i = \alpha K_i^{\alpha+\beta-1}(\gamma L_i)^{1-\alpha}$. With $\dot{g}_{K_i}/g_{K_i} = (\alpha+\beta-1)g_{K_i} + (1-\alpha)n_i = 0$, the BGP growth rates of K_i and p_i are, respectively, given by

$$g_{K_i}^* = \frac{1-\alpha}{1-\alpha-\beta} n_i > 0, \tag{10}$$

$$g_{p_i}^* = -\frac{\beta}{1-\alpha-\beta} n_i < 0, \tag{11}$$

where $g_x \equiv \dot{x}/x$ denotes the growth rate of a variable x and an asterisk "$*$" denotes a BGP value. The rate of capital accumulation is positive and proportionate to population growth, and the relative price of manufactured goods is decreasing at a constant rate.

Consumption is defined as wages only, and hence, the growth rate of per capita real consumption is equal to the growth rate of the real wage.[9)]

$$g_{c_i} = g_{w_i} - \gamma g_{p_i}. \tag{12}$$

Here, the real wage is deflated by the consumer price index, p_i^{γ}.[10)] To obtain g_{c_i}, we must know g_{w_i} and g_{p_i}. Note that, as long as agricultural goods are produced, the nominal wage is equal to unity, that is, $w_i = 1$, which means that $g_{w_i} = 0$. Accordingly, we obtain the growth rate of per capita consumption under autarky as follows:

$$g_{c_i}^{AT} = \frac{\gamma\beta}{1-\alpha-\beta} n_i > 0, \tag{13}$$

where "AT" denotes autarky. Therefore, $g_{c_i}^{AT}$ is increasing in n_i.

Considering the BGP growth rate of capital stock, we introduce a new variable, scale-adjusted capital stock: $k_i \equiv K_i/L_i^{\phi}$, where $\phi \equiv \frac{1-\alpha}{1-\alpha-\beta}$. The dynamics of the scale-adjusted capital stock are given by

$$\dot{k}_i = \alpha\gamma^{1-\alpha} k_i^{\alpha+\beta} - \phi n_i k_i. \tag{14}$$

In the steady state, $\dot{k}_i = 0$, from which we obtain

$$k_i^* = \left(\frac{\alpha\gamma^{1-\alpha}}{\phi n_i} \right)^{\frac{1}{1-\alpha-\beta}}. \tag{15}$$

The steady state is stable because $d\dot{k}_i/k_i|_{k_i=k_i^*} = -k_i^*[(1-\alpha-\beta)\alpha\gamma^{1-\alpha}(k_i^*)^{\alpha+\beta-2} + \phi n_i] < 0$.

Next, we derive the long-run equilibrium under autarky when the population growth rate is negative, that is, $n_i < 0$. If $n_i < 0$, from equation (14), there never exists a $k_i > 0$ such that $\dot{k}_i = 0$, and we have $\dot{k}_i > 0$ for $k_i > 0$. That is, if the initial value of the capital stock is strictly greater than zero, $k_i(0) > 0$, then k_i diverges to infinity. However, even in this case, we can examine the long-run growth rate of per capita consumption. Considering that per capita consumption is equal to the real wage measured in terms of the consumer price index,

9) In every case in our model, the long-run growth rate of per capita real consumption is equal to that of per capita real income. Therefore, we use the growth rate of per capita real consumption.

10) Let p_c denote the consumer price index that is consistent with the expenditure minimization problem of consumers. Then, $p_c = \gamma^{-\gamma}(1-\gamma)^{-(1-\gamma)}p_i^{\gamma}$, and by definition, $p_c c_i = w_i$. Strictly speaking, the consumer price index is given by $\gamma^{-\gamma}(1-\gamma)^{-(1-\gamma)}p_i^{\gamma}$. However, we use p_i^{γ} because the constant terms have no effect on the results.

we have

$$
\begin{aligned}
g_{c_i}^{AT} &= -\gamma g_{p_i} = -\gamma[\alpha n_i - (\alpha+\beta)(g_{k_i} + \phi n_i)] \\
&= \alpha(\alpha+\beta)\gamma^{2-\alpha} k_i^{\alpha+\beta-1} - \gamma\alpha n_i.
\end{aligned} \tag{16}
$$

Since $\alpha + \beta - 1 < 0$ by assumption, $k_i^{\alpha+\beta-1}$ approaches zero when k_i approaches infinity. From this, we have

$$
\lim_{k_i \to +\infty} g_{c_i}^{AT} = -\gamma\alpha n_i > 0. \tag{17}
$$

Therefore, $g_{c_i}^{AT}$ is positive, even when $n_i < 0$ and is decreasing in n_i.

3 Equilibrium under free trade

Suppose that Home and Foreign engage in free trade at time zero. If $K_1(0) > K_2(0)$, then from equation (9), $p_1(0) < p_2(0)$ because $L_1(0) = L_2(0) = 1$. Thus, if $K_1(0) > K_2(0)$, Home has a comparative advantage in manufactured goods and Foreign has a comparative advantage in agricultural goods. In the following analysis, we assume that $K_1(0) > K_2(0)$ without loss of generality.

It is sufficient for our purpose to consider the following four trade patterns from the viewpoint of Home:

Pattern 1: Both countries produce both goods; that is, both countries diversify.
Pattern 2: Home diversifies and Foreign completely specializes in agriculture.
Pattern 3: Home completely specializes in manufacturing and Foreign completely specializes in agriculture.
Pattern 4: Home completely specializes in manufacturing and Foreign diversifies.

In addition, we consider the following eight cases according to the size of the population growth: Case 1, $0 < n_1 = n_2$;[11)] Case 2, $n_1 = n_2 < 0$; Case 3, $0 < n_2 < n_1$; Case 4, $0 < n_1 < n_2$; Case 5, $n_2 < 0 < n_1$; Case 6, $n_1 < 0 < n_2$; Case 7, $n_2 < n_1 < 0$; and Case 8, $n_1 < n_2 < 0$.

11) Long-run growth rates and transitional dynamics in the case where $0 < n_1 = n_2$ are analyzed in detail in Sasaki (2011a). When $n_1 \neq n_2$, with regard to the state variable k_i, the locus of $\dot{k}_i = 0$ moves over time, which complicates the analysis. The analysis of transitional dynamics, including a numerical analysis when $n_1 \neq n_2$, is left for future research.

3.1 Equilibrium when both countries diversify: Pattern 1

The market-clearing conditions for both goods are given by

$$X_1^M + X_2^M = C_1^M + C_2^M + I_1 + I_2, \tag{18}$$

$$X_1^A + X_2^A = C_1^A + C_2^A. \tag{19}$$

From these, we obtain

$$p = \frac{[\gamma(L_1 + L_2)]^{\alpha}}{(1-\alpha)\left(K_1^{\frac{\alpha+\beta}{\alpha}} + K_2^{\frac{\alpha+\beta}{\alpha}}\right)^{\alpha}}. \tag{20}$$

Each country's employment share of the manufacturing sector, θ_i^M, is given by

$$\theta_1^M \equiv \frac{L_1^M}{L_1} = \frac{\gamma\left(1 + \frac{L_2}{L_1}\right)}{1 + \left(\frac{K_2}{K_1}\right)^{\frac{\alpha+\beta}{\alpha}}}, \quad \theta_2^M \equiv \frac{L_2^M}{L_2} = \frac{\gamma\left(1 + \frac{L_1}{L_2}\right)}{1 + \left(\frac{K_1}{K_2}\right)^{\frac{\alpha+\beta}{\alpha}}}. \tag{21}$$

The rates of capital accumulation in both countries are given by

$$g_{K_1} = \alpha K_1^{\alpha+\beta-1}(\theta_1^M L_1)^{1-\alpha}, \quad g_{K_2} = \alpha K_2^{\alpha+\beta-1}(\theta_2^M L_2)^{1-\alpha}. \tag{22}$$

First, if $n_1 = n_2$, so that $L_1 = L_2$,[12] then, after enough time has passed, we obtain

$$\lim_{t\to+\infty} \theta_1^M = 2\gamma, \quad \lim_{t\to+\infty} \theta_2^M = 0, \tag{23}$$

where $\gamma < 1/2$ is needed. Then, the manufacturing employment share in Foreign goes to zero, and Foreign asymptotically completely specializes in agriculture.[13] Hence, Pattern 1 is not sustainable when $n_1 = n_2$.

Second, if $n_1 > n_2$, then after enough time has passed, we obtain

$$\lim_{t\to+\infty} \theta_1^M = \gamma, \quad \lim_{t\to+\infty} \theta_2^M = 0. \tag{24}$$

In this case too, the manufacturing employment share in Foreign goes to zero, and Foreign asymptotically completely specializes in agriculture. Hence, Pattern 1 also is unsustainable

12) In addition, if $K_1(0) = K_2(0)$, the manufacturing employment share in each country is given by $\theta_i^M = \gamma$, which is constant. Pattern 1 is only sustainable in this case. However, the relative prices in both countries under autarky are equal, and therefore trade does not occur.

13) In our model, the agricultural output approaches zero, but it never vanishes because we assume that Foreign's capital stock is strictly positive. Therefore, the phrase "asymptotically" completely specializes in agriculture is more appropriate. For more information, see Christiaans (2008).

when $n_1 > n_2$.

Third, if $n_1 < n_2$, then after enough time has passed, we obtain

$$\lim_{t\to+\infty} \theta_1^M = +\infty, \quad \lim_{t\to+\infty} \theta_2^M = 0. \tag{25}$$

In this case, the manufacturing employment share in Home exceeds unity, the manufacturing employment share in Foreign goes to zero, and Foreign asymptotically completely specializes in agriculture. Hence, Pattern 1 also is unsustainable when $n_1 < n_2$.

Summarizing the above results, we obtain the following proposition.

Proposition 1. *Pattern 1, in which both Home and Foreign diversify, is unsustainable in the long run in every case.*

3.2 Equilibrium when Home diversifies and Foreign specializes in agriculture: Pattern 2

The market-clearing conditions for both goods are given by

$$X_1^M = C_1^M + C_2^M + I_1, \tag{26}$$

$$X_1^A + X_2^A = C_1^A + C_2^A. \tag{27}$$

Hence, we obtain

$$p = \frac{[\gamma(L_1 + L_2)]^\alpha}{(1-\alpha)K_1^{\alpha+\beta}}. \tag{28}$$

The manufacturing employment share in Home is given by

$$\theta_1^M = \gamma\left(1 + \frac{L_2}{L_1}\right). \tag{29}$$

First, if $n_1 = n_2$, the manufacturing employment share in Home becomes

$$\theta_1^M = 2\gamma. \tag{30}$$

In this case, we need $\gamma < 1/2$ for Pattern 2 to hold. Second, if $n_1 > n_2$, we obtain

$$\lim_{t\to+\infty} \theta_1^M = \gamma. \tag{31}$$

Here, the manufacturing employment share of Home converges to γ, and thus Pattern 2 is sustainable. Third, if $n_1 < n_2$, then θ_1^M continues to increase, becomes more than unity, and approaches infinity.

$$\lim_{t \to +\infty} \theta_1^M = +\infty. \tag{32}$$

In this case, Pattern 2 is again unsustainable.

The growth rate of capital stock is given by

$$g_{K_1} = \alpha\gamma^{1-\alpha}(L_1 + L_2)^{1-\alpha} K_1^{\alpha+\beta-1}. \tag{33}$$

Note that, in this case, we obtain $c_1 = c_2$ because $w_1 = w_2 = 1$ as long as agricultural goods are produced and both countries face the same relative price, p. Accordingly, in Pattern 2, the long-run growth rates of per capita consumption in Home and Foreign are equalized; that is, $g_{c_1}^{FT} = g_{c_2}^{FT}$, where "FT" denotes free trade.

We now examine in detail the conditions under which Pattern 2 holds. Following Wong and Yip (1999), we investigate whether the trade pattern is sustainable by comparing the size of the terms of trade and the size of the marginal rate of transformation (MRT) of the production possibilities frontier (PPF) at the corner point where a country completely specializes in manufacturing.

[Figure 1 around here]

The size of the MRT of the PPF in Home is given by

$$-\frac{dX_1^A}{dX_1^M} = \frac{[K_1^{\alpha+\beta}(\theta_1^M L_1)^{1-\alpha}]^{\frac{\alpha}{1-\alpha}}}{(1-\alpha)K_1^{\frac{\alpha+\beta}{1-\alpha}}}. \tag{34}$$

Substituting $\theta_1^M = 1$ into equation (34), the size of the MRT at the point where Home completely specializes in manufacturing is given by

$$\bar{\chi}_1 = \frac{L_1^{\alpha}}{(1-\alpha)K_1^{\alpha+\beta}}. \tag{35}$$

For Pattern 2 to be sustainable over time, we need $p < \bar{\chi}_1$; that is, from equations (28) and (35),

$$\frac{[\gamma(L_1 + L_2)]^{\alpha}}{(1-\alpha)K_1^{\alpha+\beta}} < \frac{L_1^{\alpha}}{(1-\alpha)K_1^{\alpha+\beta}}. \tag{36}$$

Rearranging this condition, we obtain

$$L_2 < \frac{1-\gamma}{\gamma} L_1. \tag{37}$$

From this, if $n_1 = n_2$, Pattern 2 is sustainable if $\gamma < 1/2$. However, if $n_1 > n_2$, Pattern 2 is sustainable in the long run irrespective of the size of γ. In contrast, if $n_1 < n_2$, Pattern 2 is unsustainable in the long run.

If $n_1 > 0$, then from equation (33), the growth rate of g_{K_1} is given by

$$\frac{\dot{g}_{K_1}}{g_{K_1}} = (1-\alpha)\left(\frac{L_1}{L_1+L_2} n_1 + \frac{L_2}{L_1+L_2} n_2\right) + (\alpha+\beta-1) g_{K_1}. \tag{38}$$

When $n_1 \geq n_2$, this leads to

$$\frac{\dot{g}_{K_1}}{g_{K_1}} = (1-\alpha) n_1 + (\alpha+\beta-1) g_{K_1}. \tag{39}$$

With $\dot{g}_{K_1}/g_{K_1} = 0$, the growth rates of capital stock, terms of trade, and per capita consumption are respectively given by

$$g_{K_1} = \frac{1-\alpha}{1-\alpha-\beta} n_1 > 0, \tag{40}$$

$$g_p = -\frac{\beta}{1-\alpha-\beta} n_1 < 0, \tag{41}$$

$$g_{c_1}^{FT} = g_{c_2}^{FT} = \frac{\gamma\beta}{1-\alpha-\beta} n_1. \tag{42}$$

Accordingly, both countries' growth rates are increasing in n_1.

On the other hand, if $n_1 < 0$, the growth rate of $k_1 = K_1/L_1^{\phi}$ is given by

$$\frac{\dot{k}_1}{k_1} = \alpha\gamma^{1-\alpha} k_1^{\alpha+\beta-1} - \phi n_1 > 0. \tag{43}$$

Here, k_i continues to increase and $\dot{k}_1/k_1$ approaches $-\phi n_1 > 0$. In this case, the growth rate of capital stock, the terms of trade, and per capita consumption are respectively given by

$$g_{K_1} = g_{k_1} - \phi n_1 = 0, \tag{44}$$

$$g_p = \alpha n_1 < 0, \tag{45}$$

$$g_{c_1}^{FT} = g_{c_2}^{FT} = -\gamma\alpha n_1 > 0. \tag{46}$$

Accordingly, both countries' growth rates are decreasing in n_1.

However, we must also consider another condition to ascertain the sustainability of Pattern 2, that is, the relationship between the MRT of the PPF in Foreign and the terms of trade. The size of the MRT at the point where Foreign completely specializes in manufacturing is given by

$$\bar{\chi}_2 = \frac{L_2^{\alpha}}{(1-\alpha)K_2^{\alpha+\beta}}. \tag{47}$$

For Pattern 2 to be sustainable in the long run, it is necessary that $p < \bar{\chi}_2$.

There exist five cases such that $n_1 \geq n_2$: $0 < n_1 = n_2$ (Case 1); $n_1 = n_2 < 0$ (Case 2); $0 < n_2 < n_1$ (Case 3); $n_2 < 0 < n_1$ (Case 5); and $n_2 < n_1 < 0$ (Case 7).

Case 1: If $0 < n_1 = n_2$, we find that $p < \bar{\chi}_2$ holds over time because, in the long run, $g_p = -\frac{\beta}{1-\alpha-\beta} n_1 < 0$ and $g_{\bar{\chi}_2} = \alpha n_1 > 0$. Therefore, Pattern 2 is sustainable.

Case 2: If $n_1 = n_2 < 0$, we find that $p < \bar{\chi}_2$ holds over time because, in the long run, $g_p = g_{\bar{\chi}_2} = \alpha n_1$. Therefore, Pattern 2 is sustainable.

Case 3: If $0 < n_2 < n_1$, we find that $p < \bar{\chi}_2$ holds over time because $g_p = -\frac{\beta}{1-\alpha-\beta} n_1 < 0$ and $g_{\bar{\chi}_2} = \alpha n_2 > 0$. Therefore, Pattern 2 is sustainable.

Case 5: If $n_2 < 0 < n_1$, we have $g_p = -\frac{\beta}{1-\alpha-\beta} n_1 < 0$ and $g_{\bar{\chi}_2} = \alpha n_2 < 0$. For the condition $p < \bar{\chi}_2$ to hold in the long run, we need $|g_p| > |g_{\bar{\chi}_2}|$.

$$|g_p| - |g_{\bar{\chi}_2}| = \frac{\beta}{1-\alpha-\beta} n_1 + \alpha n_2. \tag{48}$$

Accordingly, if $\frac{\beta}{1-\alpha-\beta} n_1 + \alpha n_2 > 0$, Pattern 2 is sustainable. In contrast, if $\frac{\beta}{1-\alpha-\beta} n_1 + \alpha n_2 < 0$, Pattern 2 is unsustainable.

Case 7: If $n_2 < n_1 < 0$, we have $g_p = \alpha n_1 < 0$ and $g_{\bar{\chi}_2} = \alpha n_2 < 0$, which imply that

$$|g_p| - |g_{\bar{\chi}_2}| = \alpha(n_2 - n_1) < 0. \tag{49}$$

This contradicts $|g_p| > |g_{\bar{\chi}_2}|$, and so Pattern 2 is unsustainable.

Summarizing the above results, we obtain the following propositions.

Proposition 2. *Pattern 2 (Home diversifies while Foreign asymptotically completely specializes in agriculture) is sustainable in the long run in Cases 1, 2, 3, and 5.*

Proposition 3. *If Pattern 2 is sustainable in the long run, both countries grow at the same per capita rate under free trade, irrespective of whether their population growth is positive or negative.*

Note that Cases 1 and 2 require $\gamma \leq 1/2$, while Case 5 requires $\frac{\beta}{1-\alpha-\beta} n_1 + \alpha n_2 > 0$.

The reason why this trade pattern is unsustainable in the long run when $n_1 < n_2$ is that demand for manufactured goods in Foreign grows faster than the supply from Home. As a result, Home cannot meet the world demand for manufactured goods on its own.

3.3 Equilibrium when Home specializes in manufacturing and Foreign specializes in agriculture: Pattern 3

The market-clearing conditions for both goods are given by

$$X_1^M = C_1^M + C_2^M + I_1, \tag{50}$$

$$X_2^A = C_1^A + C_2^A. \tag{51}$$

With $L_1^M = L_1$, because of complete specialization in manufacturing, we obtain

$$p = \frac{\gamma L_2}{(1-\alpha)(1-\gamma)K_1^{\alpha+\beta}L_1^{1-\alpha}}. \tag{52}$$

The growth rate of capital stock is given by

$$g_{K_1} = \alpha K_1^{\alpha+\beta-1}L_1^{1-\alpha}. \tag{53}$$

Pattern 3 is sustainable if $p > \bar{\chi}_1$, which, from equations (35) and (52), can be rewritten as

$$\frac{\gamma L_2}{(1-\alpha)(1-\gamma)K_1^{\alpha+\beta}L_1^{1-\alpha}} > \frac{L_1^{\alpha}}{(1-\alpha)K_1^{\alpha+\beta}}. \tag{54}$$

From this, we obtain

$$L_2 > \frac{1-\gamma}{\gamma}L_1. \tag{55}$$

Therefore, if $n_1 = n_2$, Pattern 3 is sustainable as long as $\gamma > 1/2$. If $n_2 > n_1$, Pattern 3 is sustainable in the long run, irrespective of the size of γ. On the other hand, if $n_2 < n_1$, Pattern 3 is unsustainable in the long run. In summary, $n_2 \geq n_1$ is necessary for Pattern 3 to be sustainable.

Note that, in this case, we obtain $c_1 > c_2$ because $w_1 = [\gamma/(1-\gamma)]\cdot(L_2/L_1) > 1 > w_2 = 1$ when $n_2 > n_1$, and both countries face the same relative price, p.

If $n_1 > 0$, the growth rates of the capital stock, terms of trade, and per capita consumption are given by

$$g_{K_1} = \frac{1-\alpha}{1-\alpha-\beta} n_1 > 0, \tag{56}$$

$$g_p = n_2 - \frac{1-\alpha}{1-\alpha-\beta} n_1, \tag{57}$$

$$g_{c_1}^{FT} = (n_2 - n_1) - \gamma \left(n_2 - \frac{1-\alpha}{1-\alpha-\beta} n_1 \right) = \frac{\beta - (1-\gamma)(1-\alpha)}{1-\alpha-\beta} n_1 + (1-\gamma) n_2, \tag{58}$$

$$g_{c_2}^{FT} = \frac{\gamma(1-\alpha)}{1-\alpha-\beta} n_1 - \gamma n_2. \tag{59}$$

Accordingly, $g_{c_1}^{FT}$ is increasing in n_1 if $\beta > (1-\gamma)(1-\alpha)$, decreasing in n_1 if $\beta < (1-\gamma)(1-\alpha)$, and increasing in n_2. In addition, $g_{c_2}^{FT}$ is increasing in n_1 and decreasing in n_2. In standard non-scale growth models, the growth rate of per capita consumption (income) is increasing in the growth rate of population. However, in our model, Home's per capita consumption growth can be increasing or decreasing in its population growth, and Foreign's per capita consumption growth is decreasing in its population growth.[14)]

On the other hand, if $n_1 < 0$, the growth rate of k_1 is given by

$$\frac{\dot{k}_1}{k_1} = \alpha k_1^{\alpha+\beta-1} - \phi n_1 > 0. \tag{60}$$

Accordingly, k_1 continues to increase over time. When k_1 increases, $\dot{k}_1/k_1$ approaches $-\phi n_1 > 0$. From this, we obtain

$$g_{K_1} = 0, \tag{61}$$

$$g_p = n_2 - (1-\alpha) n_1, \tag{62}$$

$$g_{c_1}^{FT} = (n_2 - n_1) - \gamma[n_2 - (1-\alpha)n_1] = -[1 - \gamma(1-\alpha)]n_1 + (1-\gamma)n_2, \tag{63}$$

$$g_{c_2}^{FT} = -\gamma[n_2 - (1-\alpha)n_1] = \gamma(1-\alpha)n_1 - \gamma n_2. \tag{64}$$

Accordingly, $g_{c_1}^{FT}$ is decreasing in n_1 and increasing in n_2. In addition, $g_{c_2}^{FT}$ is increasing in n_1 and decreasing in n_2.

However, as in Pattern 2, we must also consider another condition. For Pattern 3 to be sustainable in the long run, we need $p < \bar{\chi}_2$.

There exist five cases such that $n_1 \leq n_2$: $0 < n_1 = n_2$ (Case 1); $n_1 = n_2 < 0$ (Case 2); $0 < n_1 < n_2$ (Case 4); $n_1 < 0 < n_2$ (Case 6); and $n_1 < n_2 < 0$ (Case 8).

14) Sasaki (2011b) presents empirical evidence indicating that, in developed countries, the correlation between per capita income growth and population growth is ambiguous, whereas in developing countries, the correlation is negative.

Case 1: If $0 < n_1 = n_2$, we obtain $g_p = -\frac{\beta}{1-\alpha-\beta} n_1 < 0$ and $g_{\bar{X}_2} = \alpha n_1 > 0$, which show that $p < \bar{X}_2$ holds over time, and hence, Pattern 3 is sustainable.
Case 2: If $n_1 = n_2 < 0$, we obtain $g_p = \alpha n_1 < 0$ and $g_{\bar{X}_2} = \alpha n_1 < 0$, which show that $p < \bar{X}_2$ holds over time, and hence, Pattern 3 is sustainable.
Case 4: If $0 < n_1 < n_2$, we have $g_p = n_2 - \frac{1-\alpha}{1-\alpha-\beta} n_1$ and $g_{\bar{X}_2} = \alpha n_2 > 0$. In this case, we obtain

$$|g_{\bar{X}_2}| - |g_p| = -(1-\alpha)\left(n_2 - \frac{1}{1-\alpha-\beta} n_1\right). \tag{65}$$

If $n_2 < \frac{1-\alpha}{1-\alpha-\beta} n_1$, that is, $g_p < 0$, Pattern 3 is sustainable. If $n_2 > \frac{1-\alpha}{1-\alpha-\beta} n_1$, that is, $g_p > 0$, we need $n_2 < \frac{1}{1-\alpha-\beta} n_1$ for $|g_{\bar{X}_2}| > |g_p|$ to hold. Therefore, if $\frac{1-\alpha}{1-\alpha-\beta} n_1 < n_2 < \frac{1}{1-\alpha-\beta} n_1$, Pattern 3 is sustainable. On the other hand, if $\frac{1}{1-\alpha-\beta} n_1 < n_2$, Pattern 3 is unsustainable. In summary, if $n_2 < \frac{1}{1-\alpha-\beta} n_1$, Pattern 3 is sustainable.
Case 6: If $n_1 < 0 < n_2$, we have $g_p = n_2 - (1-\alpha)n_1 > 0$ and $g_{\bar{X}_2} = \alpha n_2 > 0$. For Pattern 3 to be sustainable, we need $|g_{\bar{X}_2}| > |g_p|$. However, we obtain

$$|g_{\bar{X}_2}| - |g_p| = -(1-\alpha)(n_2 - n_1) < 0. \tag{66}$$

Accordingly, Patterns 3 is unsustainable in this case.
Case 8: If $n_1 < n_2 < 0$, we have $g_p = n_2 - (1-\alpha)n_1$ and $g_{\bar{X}_2} = \alpha n_2 < 0$. If $n_2 - (1-\alpha)n_1 > 0$, that is, $g_p > 0$, Pattern 3 is clearly unsustainable. If $n_2 - (1-\alpha)n_1 < 0$, that is, $g_p < 0$, we need $|g_p| > |g_{\bar{X}_2}|$ for Pattern 3 to be sustainable. However, we obtain

$$|g_p| - |g_{\bar{X}_2}| = -(1-\alpha)(n_2 - n_1) < 0. \tag{67}$$

Therefore, Pattern 3 is unsustainable in this case.

Summarizing the above results, we obtain the following propositions.

Proposition 4. *Pattern 3 (Home completely specializes in manufacturing while Foreign asymptotically completely specializes in agriculture) is sustainable in the long run in Cases 1, 2, and 4.*

Proposition 5. *If Pattern 3 is sustainable in the long run and if both countries' population growth rates are different, both countries grow at different per capita rates under free trade.*

Note that Cases 1 and 2 require $1/2 < \gamma$, while Case 4 requires $n_2 < \frac{1}{1-\alpha-\beta} n_1$.

The reason why this trade pattern is unsustainable in the long run when $n_1 > n_2$ is that demand for agricultural goods in Home grows faster than they are supplied by Foreign. Hence, Foreign cannot meet the world demand for agricultural goods on its own.

3.4 Equilibrium when Home specializes in manufacturing and Foreign diversifies: Pattern 4

The market-clearing conditions for both goods are given by

$$X_1^M + X_2^M = C_1^M + C_2^M + I_1 + I_2, \tag{68}$$

$$X_2^A = C_1^A + C_2^A. \tag{69}$$

From equations (68) and (69), we find that the terms of trade satisfy the following equation:

$$(1-\alpha)^{\frac{1}{\alpha}} p^{\frac{1}{\alpha}} K_2^{\frac{\alpha+\beta}{\alpha}} = \gamma L_2 - (1-\alpha)(1-\gamma) p K_1^{\alpha+\beta} L_1^{1-\alpha}. \tag{70}$$

Here, p is implicitly and uniquely determined from equation (70), and hence, p is a function of K_1, K_2, L_1, and L_2: $p = p(K_1, K_2, L_1, L_2)$.[15)]

The growth rate of the capital stock in each country is given by

$$g_{K_1} = \alpha K_1^{\alpha+\beta-1} L_1^{1-\alpha}, \quad g_{K_2} = \alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}} p^{\frac{1-\alpha}{\alpha}} K_2^{\frac{\beta}{\alpha}}, \tag{71}$$

where p is endogenously determined by equation (70).

The employment share of manufacturing in Foreign is given by

$$\theta_2^M = \frac{(1-\alpha)^{\frac{1}{\alpha}} p^{\frac{1}{\alpha}} K_2^{\frac{\alpha+\beta}{\alpha}}}{L_2}. \tag{72}$$

In this case, analytical solutions are difficult to obtain, so we conduct numerical simulations.[16)] Using equation (70), we obtain the time derivative of p as follows:

$$\dot{p} = \frac{\gamma \dot{L}_2 - (1-\alpha)^2(1-\gamma) p K_1^{\alpha+\beta} L_1^{-\alpha} \dot{L}_1 - (1-\alpha)(1-\gamma)(\alpha+\beta) p K_1^{\alpha+\beta-1} L_1^{1-\alpha} \dot{K}_1 - \frac{\alpha+\beta}{\alpha}(1-\alpha)^{\frac{1}{\alpha}} K_2^{\frac{\beta}{\alpha}} \dot{K}_2}{\frac{1}{\alpha}(1-\alpha)^{\frac{1}{\alpha}} p^{\frac{1-\alpha}{\alpha}} K_2^{\frac{\alpha+\beta}{\alpha}} + (1-\alpha)(1-\gamma) K_1^{\alpha+\beta} L_1^{1-\alpha}}. \tag{73}$$

Substituting $\dot{L}_1 = n_1 L_1$, $\dot{L}_2 = n_2 L_2$, and the two equations from (71) into equation (73), we obtain the differential equation of p.

Using the initial conditions $K_1(0)$, $K_2(0)$, $L_1(0)$, $L_2(0)$, as well as the parameters, we can

15) The left-hand side of equation (70) is an increasing function of p, whereas the right-hand side of this equation is a decreasing function of p for given values of K_1, K_2, L_1, and L_2. Plotting both functions, we find that their intersection is unique and gives an instantaneous equilibrium value of p.

16) We use the following values for the parameters and initial capital stocks: $\alpha = 0.3$, $\beta = 0.2$, $\gamma = 0.6$, $K_1(0) = 1.2$, $K_2(0) = 1$.

obtain the initial value of the terms of trade, $p(0)$, using equation (70). Using this initial value, $p(0)$, and equation (73), we obtain the time path of $p(t)$.

From the numerical simulation, we find that, regardless of whether $n_1 \gtreqless n_2$, the manufacturing employment share in Foreign tends to zero in finite time; that is, $\theta_2^M \to 0$. Therefore, Pattern 4 is unsustainable in the long run.

Proposition 6. *Pattern 4 (Home completely specializes in manufacturing while Foreign diversifies) is unsustainable in the long run in every case.*

4 Per capita consumption growth under free trade

From the above analysis, we find that the sustainable trade patterns are Patterns 2 and 3. In this section, we summarize the long-run growth rate of per capita consumption under free trade according to the rate of the population growth.

4.1 Case 1: $0 < n_1 = n_2$

If $0 < \gamma < 1/2$, only Pattern 2 is sustainable, and if $1/2 < \gamma < 1$, only Pattern 3 is sustainable. In both cases, the BGP growth rates of per capita consumption are given by

$$g_{c_1}^{FT} = g_{c_2}^{FT} = \frac{\gamma\beta}{1-\alpha-\beta} n_1 > 0. \tag{74}$$

4.2 Case 2: $n_1 = n_2 < 0$

If $0 < \gamma < 1/2$, only Pattern 2 is sustainable, and if $1/2 < \gamma < 1$, only Pattern 3 is sustainable. In both cases, balanced growth is impossible, but the long-run growth rates of per capita consumption are given by

$$g_{c_1}^{FT} = g_{c_2}^{FT} = -\gamma\alpha n_1 > 0. \tag{75}$$

4.3 Case 3: $0 < n_2 < n_1$

Only Pattern 2 is sustainable, and the BGP growth rates of per capita consumption are given by

$$g_{c_1}^{FT} = g_{c_2}^{FT} = \frac{\gamma\beta}{1-\alpha-\beta} n_1 > 0. \tag{76}$$

4.4 Case 4: $0 < n_1 < n_2$

Only Pattern 3 is sustainable. The BGP growth rates of per capita consumption are given by

$$g_{c_1}^{FT} = (n_2 - n_1) - \gamma\left(n_2 - \frac{1-\alpha}{1-\alpha-\beta}n_1\right), \tag{77}$$

$$g_{c_2}^{FT} = \frac{\gamma(1-\alpha)}{1-\alpha-\beta}n_1 - \gamma n_2. \tag{78}$$

Note that we need the condition that $n_2 < \frac{1}{1-\alpha-\beta}n_1$.

If $\frac{1-\alpha}{1-\alpha-\beta}n_1 < n_2 < \frac{1}{1-\alpha-\beta}n_1$, we have $g_p > 0$, $g_{c_1}^{FT} > 0$, and $g_{c_2}^{FT} < 0$.[17] If $n_2 < \frac{1-\alpha}{1-\alpha-\beta}n_1$, we have $g_p < 0$, $g_{c_1}^{FT} > 0$, and $g_{c_2}^{FT} > 0$.

In either case, we find that

$$g_{c_1}^{FT} - g_{c_2}^{FT} = n_2 - n_1 > 0, \tag{79}$$

from which we obtain $g_{c_1}^{FT} > g_{c_2}^{FT}$.

This case is realistic because we can regard Home and Foreign as a developed country and a developing country, respectively: (1) the rate of population growth in developed countries is lower than that in developing countries; (2) the per capita income growth in developed countries is higher than that in developing countries; and (3) developed countries are industrialized countries while developing countries are agricultural countries. Chamon and Kremer (2009) also point out the importance of relative population growth for the development of developing countries. Population growth in developing countries is considered a problem, although since it is declining, it may not be an obstacle to development. Nevertheless, if the population growth in developed countries declines more rapidly than that in developing countries, the size of the relative population growth also declines. This decline will be an obstacle for developing countries.

Then, as stated above, when $\frac{1-\alpha}{1-\alpha-\beta}n_1 \leq n_2 \leq \frac{1}{1-\alpha-\beta}n_1$, the per capita consumption growth of the developing country is negative ($g_{c_2}^{FT} \leq 0$). To obtain positive growth, other things being equal, a developing country needs to decrease its population growth and satisfy the condition that $n_2 < \frac{1-\alpha}{1-\alpha-\beta}n_1$. However, since the population growth in developed countries is currently decreasing (i.e., n_1 is decreasing), even though some policies aim to decrease n_2, the above inequality will not be satisfied as long as n_1 decreases. Therefore, a decrease in

17) The necessary and sufficient condition for $g_{c_1}^{FT} > 0$ is given by $\frac{\beta-(1-\gamma)(1-\alpha)}{1-\alpha-\beta}n_1 + (1-\gamma)n_2 > 0$, which can be rewritten as $n_2 > \left[\frac{1-\alpha}{1-\alpha-\beta} - \frac{\beta}{(1-\gamma)(1-\alpha)}\right]n_1$. Note that the coefficient of n_1 is less than unity. Then, if $0 < n_1 < n_2$, the condition $n_2 > \left[\frac{1-\alpha}{1-\alpha-\beta} - \frac{\beta}{(1-\gamma)(1-\alpha)}\right]n_1$ is always satisfied. Therefore, if $0 < n_1 < n_2$, we necessarily have $g_{c_1}^{FT} > 0$.

the population growth in developing countries does not necessarily narrow the growth gap between them and developed countries.

4.5 Case 5: $n_2 < 0 < n_1$

Only Pattern 2 is sustainable, and the BGP growth rates of per capita consumption are given by

$$g_{c_1}^{FT} = g_{c_2}^{FT} = \frac{\gamma\beta}{1-\alpha-\beta} n_1 > 0. \tag{80}$$

Note that, for Pattern 2 to be sustainable, we need the condition that $\frac{\beta}{1-\alpha-\beta} n_1 + \alpha n_2 > 0$.

4.6 Case 6: $n_1 < 0 < n_2$

Only Pattern 3 is sustainable. The long-run growth rates of per capita consumption are given by

$$g_{c_1}^{FT} = (n_2 - n_1) - \gamma[n_2 - (1-\alpha)n_1] = (1-\gamma)n_2 - [1-\gamma(1-\alpha)]n_1 > 0, \tag{81}$$

$$g_{c_2}^{FT} = -\gamma[n_2 - (1-\alpha)n_1] < 0. \tag{82}$$

Hence, we have $g_{c_1}^{FT} > g_{c_2}^{FT}$. Note that, in this case, Pattern 3 is sustainable for a while, but is unsustainable in the long run.

4.7 Case 7: $n_2 < n_1 < 0$

In this case, the long-run growth rates of per capita consumption are given by

$$g_{c_1}^{FT} = g_{c_2}^{FT} = -\gamma\alpha n_1 > 0. \tag{83}$$

Here, Pattern 2 is sustainable for a while, but unsustainable in the long run.

4.8 Case 8: $n_1 < n_2 < 0$

When $n_2 > (1-\alpha)n_1$, the long-run growth rates of per capita consumption are given by

$$g_{c_1}^{FT} = (1-\gamma)n_2 - [1-\gamma(1-\alpha)]n_1, \tag{84}$$

$$g_{c_2}^{FT} = -\gamma[n_2 - (1-\alpha)n_1] < 0. \tag{85}$$

To obtain $g_{c_1}^{FT} > 0$, we must have

$$n_2 < \frac{1 - \gamma(1 - \alpha)}{1 - \gamma} n_1. \tag{86}$$

The coefficient of n_1 is less than unity, and accordingly, there exist combinations of n_1 and n_2 that simultaneously satisfy $n_1 < n_2 < 0$ and equation (86). Therefore, $g_{c_1}^{FT} > 0$ is possible. In addition, we have $g_{c_1}^{FT} > g_{c_2}^{FT}$, irrespective of whether $g_{c_1}^{FT} > 0$ or $g_{c_1}^{FT} < 0$.

When $n_2 < (1 - \alpha)n_1$, the long-run growth rates of per capita consumption are given by

$$g_{c_1}^{FT} = (1 - \gamma)n_2 - [1 - \gamma(1 - \alpha)]n_1 = (n_2 - n_1) - \gamma[n_2 - (1 - \alpha)n_1] > 0, \tag{87}$$

$$g_{c_2}^{FT} = -\gamma[n_2 - (1 - \alpha)n_1] > 0. \tag{88}$$

Therefore, we clearly have $g_{c_1}^{FT} > g_{c_2}^{FT}$.

Note that Pattern 3 is sustainable for a while, but unsustainable in the long run.

4.9 Comparisons of autarky and free trade growth rates

In this section, we compare $g_{c_1}^{AT}$ to $g_{c_1}^{FT}$ and $g_{c_2}^{AT}$ to $g_{c_2}^{FT}$. The results are summarized in Table 1.[18)]

[Table 1 around here]

With regard to per capita consumption growth, we obtain the following propositions.

Proposition 7. *In Cases 1, 2, 3, and 5, both countries' per capita consumption growth under free trade can be equal to or more than per capita consumption growth under autarky* ($g_{c_i}^{AT} \leq g_{c_i}^{FT}$, $i = 1, 2$)*. Other than in Cases 1 and 2, which are special cases, Pattern 2 is realized in Cases 3 and 5.*

Proposition 8. *In Cases 3 and 5, the growth gap between Home and Foreign under autarky narrows, and both countries' growth rates are equalized by switching from autarky to free trade* ($g_{c_1}^{AT} > g_{c_2}^{AT}$ *to* $g_{c_1}^{FT} = g_{c_2}^{FT}$)*. In these cases, Pattern 2 holds.*

Proposition 9. *In Case 4, the growth gap between Home and Foreign under autarky reverses (from* $g_{c_1}^{AT} < g_{c_2}^{AT}$ *to* $g_{c_1}^{FT} > g_{c_2}^{FT}$*). In this case, Pattern 3 holds. In addition, in Case 4, switching from autarky to free trade decreases per capita consumption growth* ($g_{c_2}^{AT} > g_{c_2}^{FT}$)*.*

18) For derivation, see Appendix A.

5 Conclusions

In this study, we built a two-country, two-sector, non-scale growth model and investigated the relationship between trade patterns and per capita consumption growth. In addition, we considered both negative and positive population growth. Our analysis yielded some interesting results with regard to trade patterns and per capita consumption growth.

First, when the population growth of Home is higher than that of Foreign, a trade pattern such that Home diversifies while Foreign specializes in agriculture is sustainable in the long run. In this case, after switching from autarky to free trade, the growth gap between the countries disappears, and both countries grow at the same per capita rate.

Second, when the population growth of Home is lower than that of Foreign, a trade pattern such that Home specializes in manufacturing while Foreign specializes in agriculture is sustainable in the long run. In this case, after switching from autarky to free trade, the growth gap between the two countries reverses, and Home grows faster than Foreign in per capita terms.

Third, no trade patterns are unsustainable when the population growth of a country that produces manufactured goods is negative. This is true irrespective of whether the population growth of the other country, which does not produce manufactured goods, is positive or negative.

Finally, under autarky, even if the population growth of Home is negative, the per capita growth rate is positive in the long run. However, under free trade, if the population growth of Home is negative, it can neither specialize in manufacturing nor diversify. Therefore, we can say that whether the economy is sustainable in the long run when the population growth is negative depends on trade openness.

References

Aghion, H. and Howitt, P. (2005) “Growth with quality-improving innovations: an integrated framework,” in P. Aghion and S. N. Durlauf (eds.) *Handbook of Economic Growth*, ch. 2, Amsterdam: North-Holland, pp. 67–110.

Arrow, K. J. (1962) “The economic implications of learning by doing,” *Review of Economic Studies* 29, pp. 155–173.

Chamon, M. and Kremer, M. (2009) “Economic transformation, population growth and the long-run world income distribution,” *Journal of International Economics* 79, pp. 20–30.

Christiaans, T. (2008) "International trade and industrialization in a non-scale model of economic growth," *Structural Change and Economic Dynamics* 19, pp. 221–236.

Christiaans, T. (2011) "Semi-endogenous growth when population is decreasing," *Economics Bulletin* 31 (3), pp. 2667–2673.

Dinopoulos, E. and Thompson, P. (1998) "Schumpeterian growth without scale effects," *Journal of Economic Growth* 3, pp. 313–335.

Dinopoulos, E. and Sener, F. (2007) "New directions in Schumpeterian growth theory," in H. Hanush and A. Pyca (eds.) *Edgar Companion to Neo-Schumpeterian Economics*, ch. 42, New York: Edward Elgar.

Dinopoulos, E. and Syropoulos, C. (2007) "Rent protection as a barrier to innovation and growth," *Economic Theory* 32 (2), pp. 309–332.

Felbermayr, G. J. (2007) "Specialization on a technologically stagnant sector need not be bad for growth," *Oxford Economic Papers* 59, pp. 682–701.

Felderer, B. (1988) "The existence of a neoclassical steady state when population growth is negative," *Journal of Economics* 48 (4), pp. 413–418.

Ferrara, M. (2011) "An AK Solow model with a non-positive rate of population growth," *Applied Mathematical Sciences* 5 (25), pp. 1241–1244.

Howitt, P. (1999) "Steady endogenous growth with population and R&D inputs growth," *Journal of Political Economy* 107, pp. 715–30.

Jones, C. I. (1995) "R&D-based models of economic growth," *Journal of Political Economy* 103, pp. 759–784.

Jones, C. I. (1999) "Growth: with or without scale effects?" *American Economic Review* 89, pp. 139–44.

Jones, C. I. (2005) "Growth and ideas," in P. Aghion and S. N. Durlauf (eds.) *Handbook of Economic Growth*, ch. 16, Amsterdam: North Holland, pp. 1063–1111.

Kaneko, A. (2000) "Terms of trade, economic growth, and trade patterns: a small open-economy case," *Journal of International Economics* 52, pp. 169–181.

Kaneko, A. (2003) "The long-run trade pattern in a growing economy," *Journal of Economics* 79 (1), pp. 1–17.

Kortum, S. (1997) "Research, patenting, and technological change," *Econometrica* 65, pp. 1389–1419.

Krugman, P. R. (1981) "Trade, accumulation, and uneven development," *Journal of Development Economics* 8, pp. 149–161.

Mankiw, N. G. (2000) "The savers-spenders theory of fiscal policy," *American Economic Review* 90, pp. 120–125.

Peretto, P. F. (1998) "Technological change and population growth," *Journal of Economic Growth* 3, pp. 283–311.

Ritschel, A. (1985) "On the stability of the steady state when population is decreasing," *Journal of Economics* 45 (2), pp. 161–170.

Sasaki, H. (2011a) "Trade, non-scale growth, and uneven development," *Metroeconomica* 62 (4), pp. 691–711.

Sasaki, H. (2011b) "Population growth and north-south uneven development," *Oxford Economic Papers* 63, pp. 307–330.

Sasaki, H. (2012) "International trade and industrialization with negative population growth," Kyoto University, Graduate School of Economics Research Project Center Discussion Paper Series, No. E-12-009.

Segerstrom, P. (1998) "Endogenous growth without scale effects," *American Economic Review* 88, pp. 1290–310.

Uzawa, H. (1961) "On a two-sector model of economic growth," *Review of Economic Studies* 29 (1), pp. 40–47.

Wong, K.-Y., and Yip, C. K. (1999) "Industrialization, economic growth, and international trade," *Review of International Economics* 7, pp. 522–540.

Young, A. (1998) "Growth without scale effects," *Journal of Political Economy* 106, pp. 41–63.

A Appendix: Comparison between autarky and free trade

Cases 1 and 2: We have $g_{c_1}^{AT} = g_{c_1}^{FT}$ and $g_{c_2}^{AT} = g_{c_2}^{FT}$.
Case 3: We have $g_{c_1}^{AT} = g_{c_1}^{FT}$ and $g_{c_2}^{AT} < g_{c_2}^{FT}$.
Case 4: We have

$$g_{c_1}^{AT} - g_{c_1}^{FT} = (1-\gamma)(n_1 - n_2) < 0, \tag{A-1}$$

$$g_{c_2}^{AT} - g_{c_2}^{FT} = -\frac{\gamma(1-\alpha)}{1-\alpha-\beta}(n_1 - n_2) > 0. \tag{A-2}$$

Therefore, we obtain $g_{c_1}^{AT} < g_{c_1}^{FT}$ and $g_{c_2}^{AT} > g_{c_2}^{FT}$.
Case 5: We have

$$g_{c_2}^{AT} - g_{c_2}^{FT} = -\gamma\left(\frac{\beta}{1-\alpha-\beta} n_1 + \alpha n_2\right). \tag{A-3}$$

Therefore, we obtain $g_{c_2}^{AT} < g_{c_2}^{FT}$ if $\frac{\beta}{1-\alpha-\beta} n_1 + \alpha n_2 > 0$.
Case 6: We have

$$g_{c_1}^{AT} - g_{c_1}^{FT} = (1-\gamma)(n_1 - n_2) < 0, \tag{A-4}$$

$$g_{c_2}^{AT} - g_{c_2}^{FT} = -\gamma(1-\alpha)\left(n_1 - \frac{1}{1-\alpha-\beta} n_2\right) > 0. \tag{A-5}$$

Therefore, we obtain $g_{c_1}^{AT} < g_{c_1}^{FT}$ and $g_{c_2}^{AT} > g_{c_2}^{FT}$.
Case 7: We have

$$g_{c_2}^{AT} - g_{c_2}^{FT} = \gamma\alpha(n_1 - n_2) > 0. \tag{A-6}$$

Therefore, we obtain $g_{c_2}^{AT} > g_{c_2}^{FT}$.
Case 8: We have

$$g_{c_1}^{AT} - g_{c_1}^{FT} = -(1-\gamma)(n_2 - n_1) < 0. \tag{A-7}$$

Therefore, we obtain $g_{c_1}^{AT} < g_{c_1}^{FT}$.

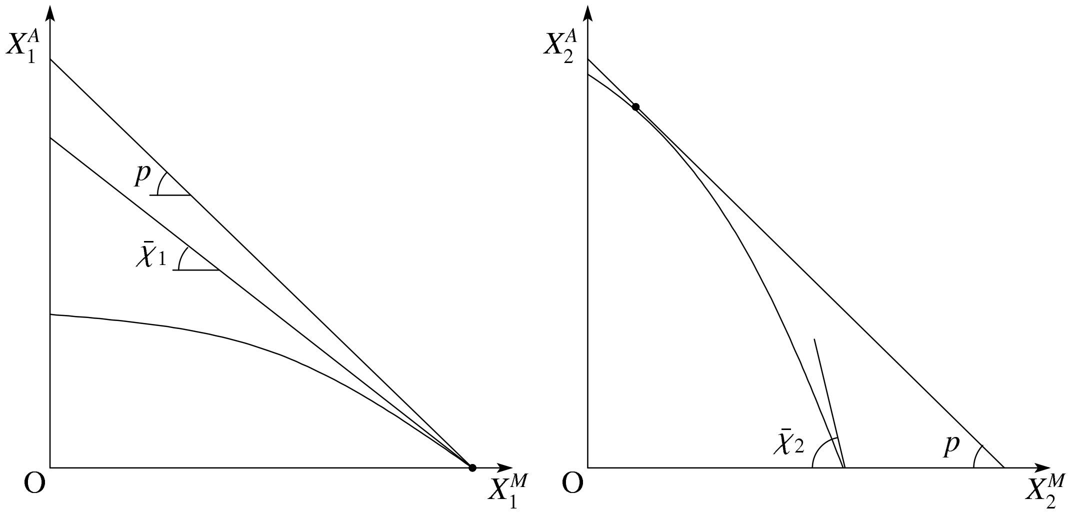

Figure 1: Production possibility frontiers in Home and Foreign

Table 1: Comparisons of autarky and free trade growth rates

Case	Case 1	Case 2	Case 3	Case 4$^{\diamond}$	Case 5	Case 6*	Case 7*	Case 8*
Population growth	$0 < n_1 = n_2$	$n_1 = n_2 < 0$	$0 < n_2 < n_1$	$0 < n_1 < n_2$	$n_2 < 0 < n_1$	$n_1 < 0 < n_2$	$n_2 < n_1 < 0$	$n_1 < n_2 < 0$
Trade pattern**	2 and 3†	2 and 3†	2	3	2	3	2	3
Relationship	$g_{c_1}^{AT} = g_{c_2}^{AT}$	$g_{c_1}^{AT} = g_{c_2}^{AT}$	$g_{c_1}^{AT} > g_{c_2}^{AT}$	$g_{c_1}^{AT} < g_{c_2}^{AT}$	$g_{c_1}^{AT} > g_{c_2}^{AT\bullet}$	$g_{c_1}^{AT} > g_{c_2}^{AT\star}$	$g_{c_1}^{AT} < g_{c_2}^{AT}$	$g_{c_1}^{AT} > g_{c_2}^{AT}$
between g_1 and g_2	$g_{c_1}^{FT} = g_{c_2}^{FT}$	$g_{c_1}^{FT} = g_{c_2}^{FT}$	$g_{c_1}^{FT} = g_{c_2}^{FT}$	$g_{c_1}^{FT} > g_{c_2}^{FT}$	$g_{c_1}^{AT} < g_{c_2}^{AT\bullet\bullet}$	$g_{c_1}^{AT} < g_{c_2}^{AT\star\star}$	$g_{c_1}^{FT} = g_{c_2}^{FT}$	$g_{c_1}^{FT} > g_{c_2}^{FT}$
	$g_{c_1}^{AT} = g_{c_1}^{FT}$	$g_{c_1}^{AT} = g_{c_1}^{FT}$	$g_{c_1}^{AT} = g_{c_1}^{FT}$	$g_{c_1}^{AT} < g_{c_1}^{FT}$	$g_{c_1}^{FT} = g_{c_2}^{FT}$	$g_{c_1}^{FT} > g_{c_2}^{FT}$	$g_{c_1}^{AT} = g_{c_1}^{FT}$	$g_{c_1}^{AT} < g_{c_1}^{FT}$
	$g_{c_2}^{AT} = g_{c_2}^{FT}$	$g_{c_2}^{AT} = g_{c_2}^{FT}$	$g_{c_2}^{AT} < g_{c_2}^{FT}$	$g_{c_2}^{AT} > g_{c_2}^{FT}$	$g_{c_1}^{AT} = g_{c_1}^{FT}$	$g_{c_1}^{AT} < g_{c_1}^{FT}$	$g_{c_2}^{AT} > g_{c_2}^{FT}$	$g_{c_2}^{AT} > g_{c_2}^{FT}$
					$g_{c_2}^{AT} < g_{c_2}^{FT\bullet}$	$g_{c_2}^{AT} > g_{c_2}^{FT}$		
					$g_{c_2}^{AT} > g_{c_2}^{FT\bullet\bullet}$			
Negative growth								$g_{c_1}^{FT} < 0$
				$g_{c_2}^{FT} < 0^{\spadesuit}$	n/a	$g_{c_2}^{FT} < 0$		$g_{c_2}^{FT} < 0$

$^{\diamond}$ $n_2 < \frac{1}{1-\alpha-\beta} n_1$ is imposed.

* These cases are unsustainable in the long run.

** Pattern 2 is the case in which Home diversifies and Foreign specializes in agriculture. Pattern 3 is the case in which Home and Foreign specialize in manufacturing and agriculture, respectively.

† Pattern 2 is obtained if $\gamma \leq 1/2$, while Pattern 3 is obtained if $1/2 < \gamma$.

$^{\spadesuit}$ $\frac{1-\alpha}{1-\alpha-\beta} n_1 < n_2 < \frac{1}{1-\alpha-\beta} n_1$.

$^{\bullet}$ $\frac{\beta}{1-\alpha-\beta} n_1 + \alpha n_2 > 0$.

$^{\bullet\bullet}$ $\frac{\beta}{1-\alpha-\beta} n_1 + \alpha n_2 < 0$. However, this case is unsustainable in the long run.

* $\alpha n_1 + \frac{\beta}{1-\alpha-\beta} n_2 < 0$.

** $\alpha n_1 + \frac{\beta}{1-\alpha-\beta} n_2 < 0$